A Note on the Existence for a Model of Turbulent Flows Through Porous Media

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Abstract In this work, turbulent flows through porous media are considered. We begin by making a historical review of the equations governing laminar flows in porous media, from Darcy's law to Darcy–Brinkman–Forchheimer's more general model. Using the double averaging concept (in time and in space) we explain how to obtain the more general system of equations that governs turbulent flows through porous media. For the one-equation turbulent problem in the steady-state we show that the known existence results can be generalized to any space dimension $d \geq 2$ and for a more general function of turbulence production.

Keywords Turbulence · $k-$epsilon modelling · Porous media · General existence

1 Turbulent Flows Through Porous Media

Fluid flows through porous media are usually described by Darcy's law [1], an empirical flow model that represents a simple linear relationship between flow rate and the pressure drop in a porous media. Today, Darcy's law reads

$$\mathbf{u} = -\frac{\mathbf{K}}{\mu}\nabla p \quad \Leftrightarrow \quad \mathbf{0} = -\mathbf{K}\nabla p - \mu\mathbf{u}\,, \tag{1}$$

where $\mathbf{u}$ is the fluid velocity field, p is the pressure and μ is the fluid (dynamic) viscosity that was only observed and included in Darcy's law later on by Hazen [2]. The tensor $\mathbf{K}$, called permeability, is independent of the nature of the fluid but it depends on the pore size, the porosity, and also on the geometry of the medium. In particular, $\mathbf{K}$ reduces to a scalar K if the medium is isotropic. The Darcy law assumes no effect of boundaries and the fluid velocity in Darcy's equation is determined by the permeability of the matrix. If the boundary is impermeable, then the usual assumption is that the normal component of the velocity must vanish: $\mathbf{u} \cdot \mathbf{n} = \mathbf{0}$ on the solid-fluid

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interface, where $\mathbf{n}$ is the unit normal. At a solid wall boundary, the fluid velocity will not reduce to the no-slip condition when the Darcy law is enforced. In this situation, the Brinkman law [3] may be employed, which is an extension of the Darcy law and facilitates the matching of boundary conditions,

$$\nabla p = -\frac{\mu}{K}\mathbf{u} + \mu_e \Delta \mathbf{u} \quad \Leftrightarrow \quad \mathbf{0} = -\nabla p - \frac{\mu}{K}\mathbf{u} + \mu_e \Delta \mathbf{u}\,, \tag{2}$$

where μ_e is the effective fluid viscosity, a function of the fluid viscosity and of the geometry of the medium. Equations (1) and (2) describe well porous media flows at sufficiently small velocities. But, for larger values of $\mathbf{u}$ there is a breakdown in the linearity of these equations which is owing to the fact that the form-drag due to solid obstacles is now comparable with the surface drag due to friction. In this case, Dupuit–Forchheimer's law [4, 5] remedies the situation by stating that the relationship between the flow rate and pressure gradient is nonlinear at sufficiently high velocity and that this nonlinearity increases with the flow rate. According to many authors (see e.g. Joseph et al. [6]), the appropriate modification of Darcy's law, to take into account high flow rates, is to replace (1) by the following Dupuit-Forchheimer equation,

$$\nabla p = \rho \mathbf{g} - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} \quad \Leftrightarrow \quad \mathbf{0} = \rho \mathbf{g} - \nabla p - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u}\,, \tag{3}$$

where ρ is the fluid density and $c_{\mathbf{F}}$ is a dimensionless form-drag constant. Several authors (see e.g. Nakayama [7] and Kuznetsov [8]) have added, in their studies, a diffusion term to (3) in order to form a Brinkman–Dupuit–Forchheimer model,

$$\nabla p = -\frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} + \mu_e \Delta \mathbf{u} \quad \Leftrightarrow \quad \mathbf{0} = -\nabla p - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} + \mu_e \Delta \mathbf{u}. \tag{4}$$

Drawing a parallel between Eqs. (2) and (4) and the Navier–Stokes equations for creep flow may led to misleading interpretations. For instance, the pressure in Eqs. (2) and (4) represents a force per unit of permeable area, including solid and fluid, while the pressure in the Navier–Stokes equations is a force per unit area of fluid only – the same is true also for the fluid velocities. However, if we confine ourselves to the pore scale (microscopic scale), the flow quantities can be determined by the incompressible Navier–Stokes equations (for homogeneous fluids)

$$\operatorname{div} \mathbf{u} = 0, \tag{5}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu \mathbf{div}(\mathbf{D}(\mathbf{u}))\,, \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right), \tag{6}$$

where ν is the kinematic viscosity and $\mathbf{g}$ is the gravity forces field. If the boundary is impermeable, then, as we already have seen, $\mathbf{u} \cdot \mathbf{n} = \mathbf{0}$ on the solid-fluid interface. But, contrary to the Darcy flow model (the maximum velocity occurs at the impermeable surface), the no-slip boundary condition can be used in this case: $\mathbf{u} \cdot \boldsymbol{\tau} = \mathbf{0}$

on the solid-fluid interface, where $\boldsymbol{\tau}$ is the unit tangent. The problem of considering (5)–(6) is that, due to the complexity of internal geometries and interfacial structures, it is impractical to solve the microscopic Eqs. (5)–(6) inside the pores. A common approach is to average the microscopic equations inside porous medium over a representative elementary volume (REV). REV is the smallest volume over which a measurement can be made that will yield a value representative of the whole domain (including fluid and solid). The volumetric average of the microscopic Eqs. (5)–(6), under the assumption of a rigid, isotropic and fixed porous matrix, results (cf. Hsu and Cheng [9]) on the following macroscopic equations,

$$\operatorname{div} \mathbf{u}_f = 0; \tag{7}$$

$$\frac{\partial \mathbf{u}_f}{\partial t} + \mathbf{div}\left(\frac{1}{\phi}\mathbf{u}_f \otimes \mathbf{u}_f\right) = \mathbf{g}_f - \frac{1}{\rho_f}\nabla p_f + \frac{\mu_f}{\rho_f}\mathbf{div}\left(\mathbf{D}\left(\mathbf{u}_f\right)\right) + \frac{1}{\rho_f}(\mathbf{H}+\mathbf{R})s, \tag{8}$$

where $\mathbf{u}_f = \phi\,\langle\mathbf{u}\rangle^i$, $p_f = \phi\,\langle p\rangle^i$, $\mathbf{g}_f = \phi\,\langle\mathbf{g}\rangle^i$, $\rho_f = \phi\,\langle\rho\rangle^i$ and $\mu_f = \phi\,\langle\mu\rangle^i$ are (fluid) phase averages and $\phi = \frac{V_f}{V}$ is the local medium porosity. For instance, $\langle\mathbf{u}\rangle^i := \frac{1}{V_f}\int_{V_f}\mathbf{u}\,dV$ is the intrinsic (fluid) average of the fluid phase velocity $\mathbf{u}$ over the fluid domain V_f contained in the representative elementary volume V. Fluid velocities $\mathbf{u}$ and $\langle\mathbf{u}\rangle^i$ are related through $\mathbf{u} = \langle\mathbf{u}\rangle^i +{}^i\,\mathbf{u}$, where ${}^i\mathbf{u}$ is the spatial deviation of $\mathbf{u}$ with respect to $\langle\mathbf{u}\rangle^i$. In the momentum equation (8), $\mathbf{H}$ and $\mathbf{R}$ represent, respectively, the hydrodynamic dispersion due to spatial deviations and the total drag force per unit volume due to the presence of the porous matrix,

$$\mathbf{H} = -\mathbf{div}\left(\phi\left\langle{}^i\mathbf{u}\otimes{}^i\,\mathbf{u}\right\rangle^i\right), \qquad \mathbf{R} = -\frac{\mu_f}{K}\mathbf{u}_f - \frac{c_F}{\sqrt{K}}\rho_f|\mathbf{u}_f|\mathbf{u}_f.$$

In the applications, the choice of the flow equations to model porous media flows, within similar flow conditions, is usually based on the pore Reynolds number $Re_p := \frac{\rho q D}{\mu}$, where q is the specific discharge and D is some representative (microscopic) diameter characterizing the void space (see e.g. Dybbs and Edwards [10]). In particular, $Re_p \leq 1$ holds when $\mathbf{u}$ is sufficiently small and therefore the flow equation is linear in the velocity. In this case, the flow is well described by one of the Eq. (1) or (2) and the dominated flow regime is called Darcy or viscous-drag. As $\mathbf{u}$ increases, the transition to nonlinear drag is quite smooth as long as $1 < Re_p \leq 10$ and the breakdown in the linearity of $\mathbf{u}$ occurs when $Re_p > 10$. If $1 \sim 10 < Re_p < 150$, the dominated flow regime is called Forchheimer or form-drag and the flow can be described by one of the models (3) or (4). By using the local volume averaging, some authors (e.g. Vafai and Kim [11]) have added to the Eq. (4) the advective inertia terms of the Navier–Stokes equations to model some situations of form-drag flows. For $Re_p > 150$, the flow regime is called post-Forchheimer and almost works in the literature consider, in this case, the local volume average of the Navier–Stokes equations to form what is now known as the Brinkman–Forchheimer-extended Darcy model (or generalized model). In particular, if $150 < Re_p < 300$ the flow regime is still laminar but unsteady and the time inertia terms need to be considered. If $Re_p > 300$, the flow becomes fully turbulent and therefore turbulence modelling is required. With this regard, it should be mentioned that two main differences exist between turbulent flow through porous media and turbulent flow in the absence of a porous matrix. By one hand, the size of the turbulent eddies within the pores is limited by the pore size. On the other, the presence of a porous matrix induces additional drag while preventing motion of larger size eddies. To model turbulent flows through porous media, it is usually considered the turbulent $k-$epsilon model which is obtained by time-averaging the incompressible Navier–Stokes equations (5) and (6),

$$\operatorname{div}\overline{\mathbf{u}} = 0, \tag{9}$$

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \mathbf{div}(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) = \overline{\mathbf{g}} - \frac{1}{\rho}\nabla \overline{p} + \mathbf{div}\left((\nu + \nu_T(k,\varepsilon))\mathbf{D}(\overline{\mathbf{u}})\right), \tag{10}$$

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k = \operatorname{div}\left(\nu_D(k,\varepsilon)\nabla k\right) + \nu_T(k,\varepsilon)|\mathbf{D}(\overline{\mathbf{u}})|^2 - \varepsilon, \tag{11}$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \varepsilon = \operatorname{div}\left(\nu_D(k,\varepsilon)\nabla \varepsilon\right) + C_1 k|\mathbf{D}(\overline{\mathbf{u}})|^2 + C_2\frac{\varepsilon^2}{k}. \tag{12}$$

Here, $\overline{\mathbf{u}}$, $\overline{p}$ and $\overline{\mathbf{g}}$ denote the time averaged velocity, pressure and external forces, whereas k is the turbulent kinetic energy and ε expresses the turbulent dissipation. The averaged quantities result from their Reynolds decomposition, for instance $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, where $\overline{\mathbf{u}} = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathbf{u}\, dt$ is the time averaged velocity, being Δt small when compared with the magnitude of fluctuations $\mathbf{u}'$ of $\overline{\mathbf{u}}$. The functions ν_T and $\nu_D = \frac{\nu_T}{\sigma_k}$ in (9)–(12) account for the turbulent viscosity and turbulent diffusivity, where σ_k is the Schmidt-Prandtl number, and C_1, C_2 are positive constants that can be determined from the experiments. The consideration of one-equation models is acceptable in the sense that the equation for ε may be discarded by prescribing an appropriate length scale l,

$$\operatorname{div}\overline{\mathbf{u}} = 0, \tag{13}$$

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \mathbf{div}(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) = \overline{\mathbf{g}} - \frac{1}{\rho}\nabla \overline{p} + \mathbf{div}\left((\nu + \nu_T(k)\mathbf{D})(\overline{\mathbf{u}})\right), \tag{14}$$

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k = \operatorname{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)|\mathbf{D}(\overline{\mathbf{u}})|^2 - \varepsilon(k), \quad \varepsilon(k) = \frac{C_D}{l}k^{\frac{3}{2}}. \tag{15}$$

See e.g. Chacón-Rebollo and Lewandowski [12] and Lemos [13] for the derivation of the turbulent k-epsilon model (see also Oliveira and Paiva [14]). From a broad perspective, for high pore Reynolds number ($Re_p > 300$), turbulent models presented in the literature follow two different approaches. In both developments, the porous medium is considered to be rigid, fixed, isotropic and saturated by an incompressible fluid, and both techniques aim to derive suitable macroscopic transport equations. The first method (see Getachewa et al. [15] and the references cited therein), starts with the volume average of the microscopic equations and then the macroscopic equations are averaged in time. However, some works (see e.g. Antohe and Lage [16]) have concluded that turbulent models derived directly from the general macroscopic equations do not accurately characterize turbulence induced by the porous matrix. The second approach (see Nakayama and Kuwahara [17] and the references cited therein), makes use, first, of the time averaged equations, and then proceeds with volume averaging. In this case, the governing equations are obtained by volume averaging the microscopic Reynolds-averaged equations (13)–(15),

$$\operatorname{div}\,\langle\overline{\mathbf{u}}\rangle^i = 0, \tag{16}$$

$$\frac{\partial\,\langle\overline{\mathbf{u}}\rangle^i}{\partial t} + \mathbf{div}\big(\langle\overline{\mathbf{u}}\rangle^i \otimes \langle\overline{\mathbf{u}}\rangle^i\big) = \langle\overline{\mathbf{g}}\rangle^i - \frac{1}{\rho_f}\nabla\overline{\pi} + \mathbf{div}\big[\big(\nu_f + \nu_T\big)\,\mathbf{D}\left(\langle\overline{\mathbf{u}}\rangle^i\right)\big] + \overline{\mathbf{R}}, \tag{17}$$

$$\frac{\partial\,\langle k\rangle^i}{\partial t} + \langle\overline{\mathbf{u}}\rangle^i\cdot\nabla\langle k\rangle^i = \operatorname{div}\left[\big(\nu_f + \nu_D\big)\,\nabla\langle k\rangle^i\right] + 2\nu_T\left|\mathbf{D}\left(\langle\overline{\mathbf{u}}\rangle^i\right)\right|^2 - \langle\varepsilon\rangle^i + P. \tag{18}$$

Here, $\overline{\pi} = \overline{\langle p\rangle^i} + \frac{2}{3}\rho_f\,\langle k\rangle^i, \nu_f = \frac{\mu_f}{\rho_f}$, $\overline{\mathbf{R}}$ represents the time averaged total drag forces and P accounts for the production of turbulence due to solid obstacles inside the porous domain. The main features of Nakayama and Kuwahara's model are that the hydrodynamic dispersion was incorporated in the drag forces and the additional term P appearing in the governing equation for $\langle k\rangle^i$ (and also in the equation for $\langle\varepsilon\rangle^i$), is determined by using two unknown model constants,

$$\overline{\mathbf{R}} = -\phi\left(\frac{\nu_f}{K}\,\langle\overline{\mathbf{u}}\rangle^i - \frac{c_F}{\sqrt{K}}\phi|\,\langle\overline{\mathbf{u}}\rangle^i\,|\,\langle\overline{\mathbf{u}}\rangle^i\right), \quad P = \frac{39\phi^2\sqrt[5]{(1-\phi)^2}}{d}|\langle\overline{\mathbf{u}}\rangle^i|^3.$$

Following a slight different approach, Pedras and Lemos [18] obtained

$$\operatorname{div}\,\langle\overline{\mathbf{u}}\rangle^i = 0, \tag{19}$$

$$\frac{\partial\,\langle\overline{\mathbf{u}}\rangle^i}{\partial t} + \mathbf{div}\big(\langle\overline{\mathbf{u}}\rangle^i \otimes \langle\overline{\mathbf{u}}\rangle^i\big) = \langle\overline{\mathbf{g}}\rangle^i - \frac{1}{\rho_f}\nabla\overline{\pi} + \mathbf{div}\big[\big(\nu_f + \nu_{T_\phi}\big)\,\mathbf{D}\left(\langle\overline{\mathbf{u}}\rangle^i\right)\big] + \overline{\mathbf{R}}, \tag{20}$$

$$\frac{\partial\,\langle k\rangle^i}{\partial t} + \langle\overline{\mathbf{u}}\rangle^i\cdot\nabla\langle k\rangle^i = \operatorname{div}\,\left[\big(\nu_f + \nu_{D_\phi}\big)\nabla\langle k\rangle^i\right] + 2\nu_T\left|\mathbf{D}\left(\langle\overline{\mathbf{u}}\rangle^i\right)\right|^2 - \langle\varepsilon\rangle^i + P\,. \tag{21}$$

In this case, the total drag term $\overline{\mathbf{R}}$ is only closed after all the equations are obtained and the additional term that is included in the equation for $\langle k\rangle^i$ to account for the porous structure is defined through

$$P = \frac{c_k\phi^3}{\sqrt{K}}\,\langle k\rangle^i\,|\,\langle\overline{\mathbf{u}}\rangle^i|.$$

Moreover, to model the Reynolds stresses it is proposed a macroscopic Boussinesq assumption: $\left\langle \overline{\mathbf{u}}' \otimes \mathbf{u}' \right\rangle_i = \frac{2}{3} \langle k \rangle_i \mathbf{I} - \nu_{T_\phi} \langle \mathbf{D}(\overline{\mathbf{u}}) \rangle_i$, where ν_{T_ϕ} and ν_{D_ϕ} denote the macroscopic turbulent viscosity and the macroscopic turbulent diffusivity, which satisfy to $\nu_{T_\phi} \mathbf{D}\left(\langle \overline{\mathbf{u}} \rangle^i\right) = \langle \nu_T \mathbf{D}(\overline{\mathbf{u}}) \rangle^i$ and $\nu_{D_\phi} = \frac{\nu_{T_\phi}}{\sigma_k}$. From the mathematical point of view, the main difference between systems (16)–(18) and (19)–(21) relies on the production of turbulence term, denoted by P at Eqs. (18) and (21). This term, that appears as an output of the averaging process, is a production term of turbulent kinetic energy and gives account of the solids inside the fluid. Note that different approaches or distinct assumptions led to different diffusivity functions between Eqs. (18), (21) and (15).

2 The Problem Under Consideration

Motivated by the systems of equations (16)–(18) and (19)–(21), we study, in this work, a one-equation turbulent model for the description of incompressible fluids within a fluid-saturated and rigid porous medium, which for simplicity is also assumed to be fixed, with a constant porosity function ϕ, and isotropic. The problem is assumed to be governed by the following general set of equations in the steady-state,

$$\mathbf{div}\,\mathbf{u} = 0 \quad \text{in } \Omega, \tag{22}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \nabla p + \mathbf{div}\left((\nu + \nu_T(k))\,\mathbf{D}(\mathbf{u})\right) \quad \text{in } \Omega, \tag{23}$$

$$\mathbf{u} \cdot \nabla k = \operatorname{div}\left((\nu + \nu_D(k))\nabla k\right) + \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 + P(\mathbf{u}, k) - \varepsilon(k) \quad \text{in } \Omega\,. \tag{24}$$

Here, Ω denotes the porous domain in consideration and the velocity field $\mathbf{u}$, the pressure p and the external forces field $\mathbf{g}$ are, in fact, averages that result by the application of the averaging procedures that lead us to (16)–(18) and (19)–(21). The feedback terms $\mathbf{f}(\mathbf{u})$ and $P(\mathbf{u}, k)$ (up to the minus sign in the first case) represent the total drag $\overline{\mathbf{R}}$ and the turbulence production considered in these systems: $\mathbf{f}(\mathbf{u}) = C_D\mathbf{u} + C_F|\mathbf{u}|\mathbf{u}$ and $P(\mathbf{u}, k) = C_1|\mathbf{u}|^3$ in (16)–(18), or $P(\mathbf{u}, k) = C_2|\mathbf{u}|k$ in (19)–(21), where C_D, C_F, C_1 and C_2 are the correspondingly multiplicative constants in the mentioned turbulent models. We supplement Eqs. (22)–(24) with Dirichlet homogeneous boundary conditions,

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega, \tag{25}$$

where $\partial\Omega$ denotes the rigid boundary of the porous domain Ω. Problems (22)–(25) with the smaller difference of the term $\operatorname{div}\left((\nu + \nu_D(k))\nabla k\right)$ replaced by $\operatorname{div}\left(\nu_D(k)\nabla k\right)$, was considered by Oliveira and Paiva [19, 20], where it was proved the existence and uniqueness of weak solutions in the dimensions of physics interest $d = 2,\ 3$ and also for $d = 4$. Due to the mathematical interest, we shall consider now the problems (22)–(25) in a general dimension d, i.e. we assume that Ω is a bounded subdomain of $\mathbb{R}^d$ for a general $d \geq 2$. Our aim in the rest of the paper, is to show that the existence results of [19] can be suitable adapted to hold for any dimension $d \geq 2$. In the mathematical treatment of the turbulence problems (22)–(25), there is a set of usual assumptions that although do not follow from the real situation they are physically admissible,

$$\mathbf{f} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \quad \varepsilon,\ \nu_T,\ \nu_D : \Omega \times \mathbb{R} \to \mathbb{R}, \quad P : \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \quad (26)$$

are Carathéodory functions. Observe that we are considering the possibility that all the functions $\mathbf{f}$, P, ε, ν_T and ν_D may also depend on the space variable. In particular, assumption (26) fits with turbulent dissipation, turbulent viscosity and turbulent diffusion functions involved in realistic models (see e.g. [12]). There is another set of assumptions that impose some restrictions on the physics of the problem, but are mathematically needed. We assume the existence of positive constants C_T and C_D such that

$$0 \leq \nu_T(k) \leq C_T, \quad 0 \leq \nu_D(k) \leq C_D \quad \text{for all } k \in \mathbb{R} \text{ and a.e. in } \Omega. \quad (27)$$

Definition 1 Let the conditions (26) and (27) be fulfilled and assume that $\mathbf{g} \in \mathbf{V}'$. We say a pair $(\mathbf{u}, k)$ is a weak solution to the problems (22)–(25), if: (1) $\mathbf{u} \in \mathbf{V}$ and for every $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^d(\Omega)$ there hold $\mathbf{f}(\mathbf{u}) \cdot \mathbf{v} \in \mathbf{L}^1(\Omega)$ and

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k))\, \mathbf{D}(\mathbf{u}) : \nabla \mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}) \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v}\, d\mathbf{x}; \quad (28)$$

(2) $k \in \mathrm{W}_0^{1,q}(\Omega)$, with $\frac{2d}{d+2} \leq q < d'$, and for every $\varphi \in \mathrm{W}_0^{1,q'}(\Omega)$ there hold $\varepsilon(k)$, φ, $P(\mathbf{u}, k)\, \varphi \in \mathrm{L}^1(\Omega)$ and

$$\begin{aligned} &\int_\Omega (\mathbf{u} \cdot \nabla k)\varphi\, d\mathbf{x} + \int_\Omega (\nu + \nu_D(k))\nabla k \cdot \nabla \varphi\, d\mathbf{x} + \int_\Omega \varepsilon(k)\, \varphi\, d\mathbf{x} = \\ &\int_\Omega \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \varphi\, d\mathbf{x} + \int_\Omega P(\mathbf{u}, k)\, \varphi\, d\mathbf{x}; \end{aligned} \quad (29)$$

(3) $k \geq 0$ and $\varepsilon(k) \geq 0$ a.e. in Ω.

The notation and the function spaces we use in this work are well known (see e.g. Galdi [21]). In particular, $\mathcal{V} := \{\mathbf{V} \in \mathbf{C}_0^\infty(\Omega) : \mathbf{div}\, \mathbf{v} = 0\}$, $\mathbf{H} :=$ closure of $\mathcal{V}$ in $\mathbf{L}^2(\Omega)$, $\mathbf{V} :=$ closure of $\mathcal{V}$ in $\mathbf{H}^1(\Omega)$, $\mathbf{V}'$ denotes the dual space of $\mathbf{V}$ and $\mathrm{v} :=$ closure of $\mathrm{C}_0^\infty(\Omega)$ in $\mathrm{H}^1(\Omega)$. Observe that, in the case of $d \leq 4$, the Sobolev imbedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^d(\Omega)$ holds and therefore it is only needed to require the test functions of (28) are in the function space $\mathbf{V}$. In this case ($d \leq 4$), it were proved, in [19, 20], existence results to the problems (22)–(25) under different conditions on the feedback functions $\mathbf{f}(\mathbf{u})$, $\varepsilon(k)$ and $P(\mathbf{u}, k)$. However, in the case of $d > 4$, requiring the test functions are also in $\mathbf{L}^d(\Omega)$ will cause the conditions to prove these existence results to be improved. In this section, we assume for any space dimension $d \geq 2$ the existence of nonnegative constants C_f and C_ε such that the following growth conditions are satisfied a.e. in Ω,

$$|\mathbf{f}(\mathbf{u})| \leq C_f |\mathbf{u}|^\alpha \quad \text{for } 0 \leq \alpha \leq \max\left\{\frac{d+2}{d-2}, \frac{2d-2}{d-2}\right\} \text{ if } d \neq 2, \quad \text{or } \alpha \geq 0 \text{ if } d = 2, \tag{30}$$

$$|\varepsilon(k)| \leq C_\varepsilon |k|^\theta \quad \text{for } 0 \leq \theta < \frac{d}{d-2} \text{ if } d \neq 2, \quad \text{or } \theta \geq 0 \text{ if } d = 2. \tag{31}$$

On the production term $P(\mathbf{u}, k)$, we assume the existence of a positive constant C_P such that

$$|P(\mathbf{u}, k)| \leq C_P |\mathbf{u}|^\beta |k|^\vartheta \quad \text{a.e. in } \Omega \tag{32}$$

for

$$\left.\begin{array}{l} \vartheta = 0 \quad \text{and} \quad \beta \leq \dfrac{d+2}{d-2}, \quad \text{or} \\ 0 < \vartheta \leq 1 \quad \text{and} \quad \beta + \vartheta \leq \dfrac{d+2}{d-2} \quad \text{and} \quad \beta + 2\vartheta < \dfrac{2d}{d-2} \end{array}\right\} \text{ if } d \neq 2, \\ \text{or} \quad \beta \in [0, \infty), \ \vartheta \in [0, 1] \qquad \text{if } d = 2. \tag{33}$$

In the sequel we shall consider our analysis only for the cases $d \neq 2$, because for $d = 2$ the reasoning is easier. In this case, observe that $\frac{d+2}{d-2} \geq \frac{2d-2}{d-2}$ holds in (30) as long as $d \leq 4$. Taking this into account, we note that in the particular case of $d \leq 4$ and of only $\vartheta = 0$ or $\vartheta = 1$, we fall in the exact growth conditions of the existence result established in [19, Thorem 3.1]. Additionally to the growth conditions (30)–(33), we assume the following sign conditions,

$$\mathbf{f}(\mathbf{u}) \cdot \mathbf{u} \geq 0 \quad \text{and} \quad \varepsilon(k)\, k \geq 0 \quad \text{a.e. in } \Omega \tag{34}$$

for all $\mathbf{u} \in \mathbb{R}^d$ and all $k \in \mathbb{R}$, respectively. We consider, in this work, that our general turbulent dissipation function can be written in such a way that

$$\varepsilon(k) = k e(k) \qquad \text{where} \quad e : \Omega \times \mathbb{R} \to \mathbb{R}_0 \text{ is a Carathéeodory function.} \tag{35}$$

Gathering the information of (34) and (35) it follows immediately that $e(k) \geq 0$ for all $k \in \mathbb{R}$ and a.e. in Ω. To avoid the trivial solution $k = 0$, we shall assume in the sequel, and in addition to (27), that $\nu_T(k) \neq 0$ when $k = 0$.

3 Existence

Theorem 1 *Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume all the conditions (26), (27), (30), (31), (34) and (35) hold. If*

$$\mathbf{g} \in \mathbf{L}^2(\Omega), \tag{36}$$

and if (32) and (33) hold but, in the case of $0 < \vartheta \leq 1$, with the extra assumption that

$$\nu > C\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}^{\frac{\beta}{1+\beta}}, \quad \textit{with C defined at (48)}, \tag{37}$$

then there exists, at least, a weak solution to the problems (22)–(25).

The rest of the section is devoted to prove Theorems 1. We start by considering, for each $n \in \mathbb{N}$, the following regularized problem

$$\mathbf{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{38}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \nabla p + \mathbf{div}\,((\nu + \nu_T(k))\,\mathbf{D}(\mathbf{u})) \quad \text{in } \Omega, \tag{39}$$

$$\mathbf{u}\cdot\nabla k = \operatorname{div}\,((\nu + \nu_D(k))\nabla k) + \nu_T(k)\,\mathcal{R}_n\big(|\mathbf{D}(\mathbf{u})|^2\big) + P(\mathbf{u},k) - \varepsilon(k) \quad \text{in } \Omega, \tag{40}$$

$$\mathbf{u} = 0 \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega, \tag{41}$$

where $\mathcal{R}_n\big(|\mathbf{D}(\mathbf{u})|^2\big) := \frac{|\mathbf{D}(\mathbf{u})|^2}{1+\frac{1}{n}|\mathbf{D}(\mathbf{u})|^2}$. Under the assumptions of Definition 1, we say a pair $(\mathbf{u}, k)$ is a weak solution to the regularized problem (38)–(41) if, for each $n \in \mathbb{N}$, (1) and (3) of Definition 1 hold, and: (2') $k \in \mathrm{H}_0^1(\Omega)$ and for every $\varphi \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^d(\Omega)$ there holds (29'), i.e. (29) with $\mathcal{R}_n\big(|\mathbf{D}(\mathbf{u})|^2\big)$ in the place of $|\mathbf{D}(\mathbf{u})|^2$. Observe again that, as we have mentioned for the test functions in (28), due to the Sobolev imbedding $\mathrm{H}_0^1(\Omega) \hookrightarrow \mathrm{L}^d(\Omega)$ it would only be needed to require the test functions of (29') are in the function space $\mathrm{H}_0^1(\Omega)$ in the case of $d \leq 4$. The existence of a weak solution to the problem (38)–(41) is established in the following proposition.

Proposition 1 *Let the conditions of Theorem 1 be fulfilled. Then (for each $n \in \mathbb{N}$) there exists, at least, a weak solution to the problems (38)–(41).*

Proof For each $j \in \mathbb{N}$, we search for the Galerkin approximations $\mathbf{u}_j = \sum_{i=1}^{j} c_{ij}\mathbf{v}_i$ and $k_j = \sum_{i=1}^{j} d_{ij}\upsilon_i$, solutions to the system formed by (28) and of (29'), where $c_{ij},\ d_{ij} \in \mathbb{R}$, $\mathbf{v}_i \in \mathbf{V}^j$, $\upsilon_i \in V^j$, and $\mathbf{V}^j$, V^j are j-dimensional subspaces of $\mathbf{V}^s :=$ closure of $\mathcal{V}$ in $\mathrm{W}^{s,2}(\Omega)$ and of $V^r :=$ closure of $C_0^\infty(\Omega)$ in $\mathrm{W}^{r,2}(\Omega)$, being s and r the smallest integers such that $s,\ r \geq \frac{d}{2}$. Note that in the case of $d \leq 4$, we may let $r,\ s = 1$ and replace $\mathbf{V}^s$ and V^r by the function spaces $\mathbf{V}$ and V defined above. Functions $\mathbf{u}_j$ and k_j are found by solving the following system of $2j$ nonlinear algebraic equations, with respect to the $2j$ unknowns $c_{1j},\ c_{2j},\ \ldots,\ c_{jj}$ and $d_{1j},\ d_{2j},\ \ldots,\ d_{jj}$,

$$\int_\Omega ((\mathbf{u}_j \cdot \nabla)\mathbf{u}_j) \cdot \mathbf{v}_i \, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k_j))\mathbf{D}(\mathbf{u}_j) : \nabla \mathbf{v}_i \, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_j) \cdot \mathbf{v}_i \, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v}_i \, d\mathbf{x}, \tag{42}$$

$$\int_\Omega (\mathbf{u}_j \cdot \nabla k_j)\upsilon_i \, d\mathbf{x} + \int_\Omega (\nu + \nu_D(k_j))\nabla k_j \cdot \nabla \upsilon_i \, d\mathbf{x} + \int_\Omega \varepsilon(k_j)\upsilon_i \, d\mathbf{x} =$$
$$\int_\Omega \nu_T(k_j)\, \mathcal{R}_n\big(|\mathbf{D}(\mathbf{u}_j)|^2\big)\, \upsilon_i \, d\mathbf{x} + \int_\Omega P(\mathbf{u}_j, k_j)\, \upsilon_i \, d\mathbf{x}, \tag{43}$$

for $i = 1, \ldots, j$. To prove the existence of, at least, a solution to the system (42) and (43), we consider a function $\mathcal{P}$, from $\mathbf{V}^j \times V^j$ into itself defined in such a way that

$$\mathcal{P}(\mathbf{v}, \upsilon) \cdot (\mathbf{v}, \upsilon) = I_1 + \cdots - I_4 + \cdots - I_8 - I_9 := \int_\Omega ((\mathbf{v} \cdot \nabla)\mathbf{v}) \cdot \mathbf{v} \, d\mathbf{x} +$$
$$\int_\Omega (\nu + \nu_T(\upsilon))\mathbf{D}(\mathbf{v}) : \nabla \mathbf{v} \, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{v}) \cdot \mathbf{v} \, d\mathbf{x} - \int_\Omega \mathbf{g} \cdot \mathbf{v} \, d\mathbf{x} + \int_\Omega (\mathbf{v} \cdot \nabla \upsilon)\upsilon \, d\mathbf{x} +$$
$$\int_\Omega (\nu + \nu_D(\upsilon))|\nabla \upsilon|^2 \, d\mathbf{x} + \int_\Omega \varepsilon(\upsilon)\upsilon \, d\mathbf{x} - \int_\Omega \nu_T(\upsilon)\, \mathcal{R}_n\Big(|\mathbf{D}(\mathbf{v})|^2\Big)\, \upsilon \, d\mathbf{x} - \int_\Omega P(\mathbf{v}, \upsilon)\, \upsilon \, d\mathbf{x}$$

for all $(\mathbf{v}, \upsilon) \in \mathbf{V}^j \times V^j$ and where the scalar product is induced by $\mathbf{V} \times V$. Reasoning as we did in the proof of [19, Theorem 3.1], it can be proved that $I_1 = 0$ and $I_5 = 0$, $I_2 \geq \nu\, C_K^2 \|\nabla \mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}$, I_3, $I_7 \geq 0$ and $I_6 \geq \nu \|\nabla \upsilon\|^2_{\mathrm{L}^2(\Omega)}$, $I_4 \leq \Lambda_P(d)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)}$ and $I_8 \leq C_T\, n\sqrt{\mathcal{L}^d(\Omega)}\lambda(2, d)\|\nabla \upsilon\|_{\mathrm{L}^2(\Omega)}$, where C_K is the Korn's inequality constant, $\lambda(2, d)$ and $\Lambda_P(d)$ are the best constants of the scalar and vectorial Sobolev inequalities. For the term I_9, we argue similarly as in the previous reference, to show that

$$I_9 \leq C_P\, \lambda(2, d)^{1+\vartheta}\, \Lambda(2, d)^\beta \|\nabla \mathbf{v}\|^\beta_{\mathbf{L}^2(\Omega)} \|\nabla \upsilon\|^{1+\vartheta}_{\mathrm{L}^2(\Omega)}, \quad \beta + \vartheta \leq \frac{d+2}{d-2}. \tag{44}$$

Then, gathering the information of the estimates of $I_1, \ldots, I_9$, it can be proved that $\mathcal{P}(\mathbf{v}, \upsilon) \cdot (\mathbf{v}, \upsilon) > 0$ for $\|\mathbf{v}\|_{\mathbf{V}} = \rho$ and $\|\upsilon\|_V = \varsigma$, and ρ and ς suitably chosen (see again the aforementioned reference). Due to this and to assumptions (27), (34) and (36), we can use a variant of Brower's theorem to prove the existence of a solution $(\mathbf{c}_j, \mathbf{d}_j)$, with $\mathbf{c}_j := (c_{1j}, c_{2j}, \ldots, c_{jj})$ and $\mathbf{d}_j := (d_{1j}, d_{2j}, \ldots, d_{jj})$ to the system (42) and (43).

Arguing as we did in [19], we can also prove that

$$\|\nabla \mathbf{u}_j\|_{\mathbf{L}^2(\Omega)} \leq \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{45}$$

Consequently, we have (up to some subsequences) that $\mathbf{u}_j \to \mathbf{u}$ weakly in $\mathbf{H}^1_0(\Omega)$, $\mathbf{u}_j \to \mathbf{u}$ strongly in $\mathbf{L}^\gamma(\Omega)$ for $\gamma \in \big[1, \frac{2d}{d-2}\big)$, and $\mathbf{u}_j \to \mathbf{u}$ a.e. in Ω, all as $j \to \infty$. Proceeding again as we did in [19], we have

$$\int_{\Omega}(\nu+\nu_D(k_j))|\nabla k_j|^2\,d\mathbf{x} \le \int_{\Omega}\nu_T(k_j)\,\mathcal{R}_n\big(|\mathbf{D}(\mathbf{u}_j)|^2\big)\,k_j\,d\mathbf{x}+\int_{\Omega}P(\mathbf{u}_j,k_j)k_j\,d\mathbf{x}.$$

If $0=\vartheta<1$, we can argue as we did for I_6, I_8 above and, in particular for (44), to prove that

$$\|\nabla k_j\|^2_{\mathrm{L}^2(\Omega)} \le C \quad \text{for } \beta+\vartheta \le \frac{d+2}{d-2}, \tag{46}$$

for some positive constant C not depending on j. In the case of $\vartheta=1$, we argue as we did for (46) to obtain

$$\|\nabla k_j\|_{\mathrm{L}^2(\Omega)} \le \frac{C_T\, n\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)}{\nu - C_P\,\lambda(2,d)^2\Lambda(2,d)^\beta\left(\frac{\Lambda_P(d)}{\nu C_K^2}\right)^\beta\|\mathbf{g}\|^\beta_{\mathbf{L}^2(\Omega)}}, \quad \beta\le\frac{4}{d-2}. \tag{47}$$

By using assumption (37), with C defined by

$$C := \left(C_P\,\lambda(2,d)^2\Lambda(2,d)^\beta C_K^{-2\beta}\Lambda_P(d)^\beta\right)^{\frac{1}{1+\beta}}, \tag{48}$$

we can readily see that the right-hand side of (47) is a positive constant independent of j. Then by a usual reasoning, we have (up to some subsequences) that $k_j\to k$ weakly in $\mathrm{H}^1_0(\Omega)$, $k_j\to k$ strongly in $\mathrm{L}^\gamma(\Omega)$ for $\gamma\in\left[1,\frac{2d}{d-2}\right)$, and $k_j\to k$ a.e. in Ω, all as $j\to\infty$.

Now we pass to the limit $j\to\infty$ the integral equality (42). The convergence of the last term of (42) follows from the weak convergence of $\mathbf{u}_j$ and assumption (36). The convergence of the first and third terms of (42) follows a reasoning a little bit different from the one used in [19], because now $d\ge 4$. For the convergence of the third, we observe that since $\mathbf{f}$ is continuous on $\mathbf{u}$ (see (26)), we have by virtue of the a.e. convergence of $\mathbf{u}_j$,

$$\mathbf{f}(\mathbf{u}_j)\to\mathbf{f}(\mathbf{u}) \quad \text{a.e. in } \Omega, \quad \text{as } j\to\infty. \tag{49}$$

On the other hand, using Sobolev's inequality together with (30) and (45), it can be proved that

$$\|\mathbf{f}(\mathbf{u}_j)\|_{\mathbf{L}^\gamma(\Omega)} \le C \quad \text{for } \gamma=\frac{2d}{d+2} \text{ and } \alpha\le\frac{d+2}{d-2}, \quad \text{or } \gamma=d' \text{ and } \alpha\le\frac{2d-2}{d-2}, \tag{50}$$

for some positive constant C independent of j. Note that conditions on α given by (50) are responsible for the assumption (30). Owing to (49) and (50), $\mathbf{f}(\mathbf{u}_j)\to\mathbf{f}(\mathbf{u})$ weakly in $\mathbf{L}^\gamma(\Omega)$, as $j\to\infty$. As a consequence,

$$\int_{\Omega} \mathbf{f}(\mathbf{u}_j) \cdot \mathbf{v}_i \to \int_{\Omega} \mathbf{f}(\mathbf{u}) \cdot \mathbf{v}_i, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1.$$

Note that in the case of $\gamma = \frac{2d}{d+2}$, we may use the fact that $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^{\frac{2d}{d-2}}(\Omega)$. For the first term of (42), we observe that, due to (22) and (25), we can write

$$\int_{\Omega} ((\mathbf{u}_j \cdot \nabla)\mathbf{u}_j) \cdot \mathbf{v}_i \, d\mathbf{x} = -\int_{\Omega} \mathbf{u}_j \otimes \mathbf{u}_j : \nabla \mathbf{v}_i \, d\mathbf{x}.$$

From (45), this used together with the Sobolev imbedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^{\frac{2d}{d-2}}(\Omega)$, and with the a.e. convergence of $\mathbf{u}_j$, we have

$$\|\mathbf{u}_j \otimes \mathbf{u}_j\|_{\mathbf{L}^{\frac{d}{d-2}}(\Omega)} \leq C \quad \text{and} \quad \mathbf{u}_j \otimes \mathbf{u}_j \to \mathbf{u} \otimes \mathbf{u} \;\text{ a.e. in } \Omega, \quad \text{as } j \to \infty, \tag{51}$$

where C is a positive constant not depending on j. Consequently, (51) yields

$$\mathbf{u}_j \otimes \mathbf{u}_j \to \mathbf{u} \otimes \mathbf{u} \text{ weakly in } \mathbf{L}^{\frac{d}{d-2}}(\Omega), \text{ as } j \to \infty. \tag{52}$$

Then, since, by the Sobolev imbedding, $\nabla \mathbf{v}_i \in \mathbf{H}^{s-1}(\Omega) \hookrightarrow \mathbf{L}^{\frac{d}{2}}(\Omega)$ for $s \geq \frac{d}{2} - 1$, which is guaranteed by the choice of $s \geq \frac{d}{2}$, we have, by virtue of (52) and once that $\left(\frac{d}{d-2}\right)^{-1} + \left(\frac{d}{2}\right)^{-1} = 1$,

$$\int_{\Omega} \mathbf{u}_j \otimes \mathbf{u}_j : \nabla \mathbf{v}_i \, d\mathbf{x} \to \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{53}$$

Let us now show the convergence of the second term of (42). We first observe that (26) and (27) and the a.e. convergence of k_j imply $\left|(\nu + \nu_T(k_j))\nabla \mathbf{v}_i\right| \leq (\nu + C_T)|\nabla \mathbf{v}_i|$ and $(\nu + \nu_T(k_j))\nabla \mathbf{v}_i \to (\nu + \nu_T(k))\nabla \mathbf{v}_i$ a.e. in Ω, as $j \to \infty$. Then, since, by the Sobolev imbedding, $\nabla \mathbf{v}_i \in \mathbf{H}^{s-1}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ for $s \geq 1$, which again is guaranteed by the choice of $s \geq \frac{d}{2}$, we have, by Lebesgue's dominated convergence theorem,

$$(\nu + \nu_T(k_j))\nabla \mathbf{v}_i \to (\nu + \nu_T(k))\nabla \mathbf{v}_i \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } j \to \infty. \tag{54}$$

Then, from the weak convergence of $\mathbf{u}_j$ and (54), we can prove that

$$\int_{\Omega} (\nu + \nu_T(k_j))\mathbf{D}(\mathbf{u}_j) : \nabla \mathbf{v}_i \, d\mathbf{x} \to \int_{\Omega} (\nu + \nu_T(k))\mathbf{D}(\mathbf{u}) : \nabla \mathbf{v}_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \tag{55}$$

for all $i \geq 1$. The convergence of third and last terms of (42) (see [19]) together with (53) and (55) imply that we can pass to the limit $j \to \infty$ in the approximate system (42) and thus we obtain

$$\int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{v}_i\,d\mathbf{x}+\int_\Omega(\nu+\nu_T(k))\mathbf{D}(\mathbf{u}):\nabla\mathbf{v}_i\,d\mathbf{x}+\int_\Omega\mathbf{f}(\mathbf{u})\cdot\mathbf{v}_i\,d\mathbf{x}=\int_\Omega\mathbf{g}\cdot\mathbf{v}_i\,d\mathbf{x} \tag{56}$$

for all $i\geq 1$. Using the linearity of (56) in $\mathbf{v}_i$ and the density of the finite linear combinations of the system $\{\mathbf{v}_i\}_{i=1}^{\infty}$ in $\mathbf{V}\cap\mathbf{L}^d(\Omega)$, we deduce that (56) holds true in the whole space $\mathbf{V}$, that is

$$\int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x}+\int_\Omega(\nu+\nu_T(k))\mathbf{D}(\mathbf{u}):\nabla\mathbf{v}\,d\mathbf{x}+\int_\Omega\mathbf{f}(\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x}=\int_\Omega\mathbf{g}\cdot\mathbf{v}\,d\mathbf{x} \tag{57}$$

for all $\mathbf{v}\in\mathbf{v}\cap\mathbf{L}^d(\Omega)$. This allows us to take $\mathbf{v}=\mathbf{u}$ as a test function in (57), which yields

$$\int_\Omega(\nu+\nu_T(k))|\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x}+\int_\Omega\mathbf{f}(\mathbf{u})\cdot\mathbf{u}\,d\mathbf{x}=\int_\Omega\mathbf{g}\cdot\mathbf{u}\,d\mathbf{x}.$$

Taking $\mathbf{v}_i=\mathbf{u}_j$ in (42), we also have the equality

$$\int_\Omega(\nu+\nu_T(k_j))|\mathbf{D}(\mathbf{u}_j)|^2\,d\mathbf{x}+\int_\Omega\mathbf{f}(\mathbf{u}_j)\cdot\mathbf{u}_j\,d\mathbf{x}=\int_\Omega\mathbf{g}\cdot\mathbf{u}_j\,d\mathbf{x}.$$

Then, proceeding as in [19], we obtain (eventually up to some subsequence) that

$$\mathbf{D}(\mathbf{u}_j)\to\mathbf{D}(\mathbf{u})\quad\text{strongly in}\,\mathbf{L}^2(\Omega)\qquad\text{and}\qquad\mathbf{D}(\mathbf{u}_j)\to\mathbf{D}(\mathbf{u})\quad\text{a.e. in}\,\Omega, \tag{58}$$

as $j\to\infty$. We will now pass to the limit $j\to\infty$ the integral equality (43). To pass the first term of this equality to the limit, we can argue as we did for the convective term of the Navier–Stokes equations (see (53)). The convergence of the second and third terms of (43) follows as in the proof of [19, Theorem 3.1]. Due to assumption (26) and to the a.e. convergence of k_j, we have

$$\varepsilon(k_j)\to\varepsilon(k)\quad\text{a.e. in}\,\Omega,\quad\text{as}\ j\to\infty. \tag{59}$$

Using Sobolev's inequality together with (31) and (46), it can be proved that

$$\|\varepsilon(k_j)\|_{\mathrm{L}^\gamma(\Omega)}\leq C\quad\text{for}\ \gamma=\frac{2d}{d+2}\ \text{and}\ \theta\leq\frac{d+2}{d-2},\quad\text{or}\ \gamma=d'\ \text{and}\ \theta\leq\frac{2d-2}{d-2}, \tag{60}$$

for some positive constant C not depending on j. Owing to (59) and (60), $\varepsilon(k_j)\to\varepsilon(k)$ weakly in $\mathrm{L}^\gamma(\Omega)$, as $j\to\infty$. Thus

$$\int_\Omega\varepsilon(k_j)\upsilon_i\,d\mathbf{x}\to\int_\Omega\varepsilon(k)\upsilon_i\,d\mathbf{x},\quad\text{as}\ j\to\infty,\quad\text{for all}\,i\geq 1. \tag{61}$$

Note that in the case of $\gamma = \frac{2d}{d+2}$, we use the fact that $\upsilon_i \in \mathrm{H}^1_0(\Omega) \hookrightarrow \mathrm{L}^{\frac{2d}{d-2}}(\Omega)$. Let us now focus our attention on the last term of (43). Here, we first observe that (26) together with the a.e. convergence of $\mathbf{u}_j$ and k_j imply that

$$P(\mathbf{u}_j, k_j) \to P(\mathbf{u}, k) \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty. \tag{62}$$

By using assumption (32), Hölder's inequality (in the case of $\vartheta \neq 0$) and Sobolev's inequality together with (45) and (46), or (47), it can be proved that

$$\|P(\mathbf{u}_j, k_j)\|_{\mathrm{L}^\gamma} \leq C \text{ for } \gamma = \frac{2d}{d+2} \text{ and } \beta + \vartheta \leq \frac{d+2}{d-2}, \text{ or } \gamma = d' \text{ and } \beta + \vartheta \leq \frac{2d-2}{d-2}, \tag{63}$$

for some positive constant C not depending on j. Thus, (62) and (63) imply that $P(\mathbf{u}_j, k_j) \to P(\mathbf{u}, k)$ weakly in $\mathrm{L}^\gamma(\Omega)$, as $j \to \infty$, and consequently, as we did for (61), we obtain

$$\int_\Omega P(\mathbf{u}_j, k_j)\upsilon_i \, d\mathbf{x} \to \int_\Omega P(\mathbf{u}, k)\upsilon_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{64}$$

The convergence of the first four terms of (43) together with (64), assure us that we can pass to the limit $j \to \infty$ in the approximate system (43) to obtain

$$\begin{aligned} &\int_\Omega (\mathbf{u} \cdot \nabla k)\upsilon_i \, d\mathbf{x} + \int_\Omega (\nu + \nu_D(k))\nabla k \cdot \nabla \upsilon_i \, d\mathbf{x} + \int_\Omega \varepsilon(k)\upsilon_i \, d\mathbf{x} \\ = &\int_\Omega \nu_T(k)\, \mathcal{R}_n\big(|\mathbf{D}(\mathbf{u})|^2\big)\, \upsilon_i \, d\mathbf{x} + \int_\Omega P(\mathbf{u}, k)\, \upsilon_i \, d\mathbf{x} \quad \text{for all } i \geq 1. \end{aligned}$$

We have thus proved that, for each $n \in \mathbb{N}$, there exists a weak solution $(\mathbf{u}_n, k_n) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ to the problems (38)–(41) and such that

$$\int_\Omega (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n \cdot \mathbf{v} \, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k_n))\mathbf{D}(\mathbf{u}_n) : \nabla \mathbf{v} \, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_n) \cdot \mathbf{v} \, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v} \, d\mathbf{x}, \tag{65}$$

$$\begin{aligned} &\int_\Omega (\mathbf{u}_n \cdot \nabla k_n)v \, d\mathbf{x} + \int_\Omega (\nu + \nu_D(k_n))\nabla k_n \cdot \nabla v \, d\mathbf{x} + \int_\Omega \varepsilon(k_n)v \, d\mathbf{x} \\ = &\int_\Omega \nu_T(k_n)\, \mathcal{R}_n\big(|\mathbf{D}(\mathbf{u}_n)|^2\big)\, v \, d\mathbf{x} + \int_\Omega P(\mathbf{u}_n, k_n)\, v \, d\mathbf{x} \end{aligned} \tag{66}$$

hold for all $(\mathbf{v}, v) \in \mathbf{V}^j \times V^j$ and all $j \geq 1$. By linearity and density these relations hold for all $(\mathbf{v}, v) \in \mathbf{V}^s \times \mathrm{V}^r$, and by continuity they hold for all $(\mathbf{v}, v) \in (\mathbf{V} \cap \mathbf{L}^d(\Omega)) \times (\mathrm{H}^1_0(\Omega) \cap \mathrm{L}^d(\Omega))$ due to the ranges of α, θ, β and ϑ set forth at (30)–(32).

The proof that $k \geq 0$ and $\varepsilon(k) \geq 0$ a.e. in Ω follows as in the proof of [19, Theorem 3.1], in particular by using (35) for the expression of the turbulent dissipation function. The proof of Proposition 1 is now concluded. □

From Proposition 1, we know that, for each $n \in \mathbb{N}$, there exists a weak solution $(\mathbf{u}_n, k_n) \in \mathbf{V} \times \mathrm{H}_0^1(\Omega)$ to the problems (38)–(41) and such that (65) and (66) hold. Arguing as in [19], it can be proved that

$$\|\nabla \mathbf{u}_n\|_{\mathbf{L}^2(\Omega)} \leq \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{67}$$

As a consequence, it follows (up to some subsequences) that $\mathbf{u}_n \to \mathbf{u}$ weakly in $\mathbf{H}_0^1(\Omega)$, $\mathbf{u}_n \to \mathbf{u}$ strongly in $\mathbf{L}^\gamma(\Omega)$ for $\gamma \in \left[1, \frac{2d}{d-2}\right)$, and $\mathbf{u}_n \to \mathbf{u}$ a.e. in Ω, all as $n \to \infty$. To achieve an a priori estimate for k_n, independent of n, we consider the special test function $\varphi(k_n) := 1 - \frac{1}{(1+k_n)^\delta}$, where δ is a positive constant such that $\varphi \in \mathrm{W}^{1,q'}(\Omega) \hookrightarrow C^{0,\delta}(\Omega)$. Taking $v = \varphi(k_n)$ in (66) and proceeding as we did in [19], we have

$$\delta \int_\Omega (\nu + \nu_D(k_n))\frac{|\nabla k_n|^2}{(1+k_n)^{1+\delta}}\,d\mathbf{x} \leq \int_\Omega \nu_T(k_n)\,|\mathbf{D}(\mathbf{u}_n)|^2\,d\mathbf{x} + \int_\Omega |P(\mathbf{u}_n, k_n)|\,d\mathbf{x}. \tag{68}$$

With respect to the last term of (68), we firstly observe that, since $q < d'$, by the Sobolev imbedding we have $\mathrm{W}_0^{1,q}(\Omega) \hookrightarrow \mathrm{L}^\gamma(\Omega)$ for $\gamma < \frac{d}{d-2}$. Therefore, in view of (32) and (33),

$$\int_\Omega |P(\mathbf{u}_n, k_n)|\,d\mathbf{x} \leq \begin{cases} C_1\|\nabla \mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)} & \text{for } \beta \leq \frac{2d}{d-2} \text{ if } \vartheta = 0, \quad \text{or} \\ C_2\|\nabla \mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)}\|\nabla k_n\|^\vartheta_{\mathrm{L}^q(\Omega)} & \text{for } \beta + 2\vartheta < \frac{2d}{d-2} \text{ if } \vartheta > 0, \end{cases} \tag{69}$$

where C_1 and C_2 are independent of n positive constants. Then, using the assumption (27) together with (69), and arguing as in [19], we can prove, in the most difficult case of $\vartheta \neq 0$,

$$\int_\Omega |\nabla k_n|^q\,d\mathbf{x} \leq \frac{C_1}{\delta}\|\nabla \mathbf{u}_n\|^2_{\mathbf{L}^2(\Omega)} + \frac{C_2}{\delta}\|\nabla \mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)}\|\nabla k_n\|^\vartheta_{\mathrm{L}^q(\Omega)} + C_3\|\nabla k_n\|^{\frac{(1+\delta)q}{2}}_{\mathrm{L}^q(\Omega)} + C_4, \tag{70}$$

where C_1, C_2, C_3 and C_4 are positive constants not depending on n. In this case, we need also to apply Young's inequality to the third term of (70) which is possible as long as $\vartheta < \frac{2d}{d+2}$, condition that is satisfied due to (32) and (33). The case $\vartheta = 0$ is easier. All this reasoning together with (67) and assumption (36), yield

$$\int_\Omega |\nabla k_n|^q\,d\mathbf{x} \leq C, \quad C = C(\nu, \beta, C_T, C_P, d, q, \Omega, \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}), \tag{71}$$

where C is a positive constant not depending on n. Then, in view of (71) and up to some subsequences, we have $k_n \to k$ weakly in $\mathrm{W}_0^{1,q}(\Omega)$ for $q < d'$, $k_n \to k$ strongly in $\mathrm{L}^\gamma(\Omega)$ for all $\gamma \in [1, q^*)$ and $k_n \to k$ a.e. in Ω, all as $n \to \infty$. Now, we can pass to the limit $n \to \infty$ all the integral terms of (65) by arguing analogously as we did in the proof of Proposition 1. With respect to the convergence of the integral terms of (66), we first observe that since $q < d'$, we have $\mathrm{W}_0^{1,q'}(\Omega) \hookrightarrow \mathrm{C}^{0,\delta}(\bar{\Omega})$ for $\delta = 1 - \frac{d}{q'}$. As a consequence $v \in \mathrm{W}_0^{1,q'}(\Omega)$ implies that $v \in \mathrm{L}^{\gamma'}(\Omega)$ for any $\gamma' \geq 1$. With minor modifications, the convergence of all the integral terms of (66) follows as in the proof of Proposition 1, with the exception of the one involving $\mathcal{R}_n$, because we do not know whether if this term remains bounded as $n \to \infty$. The convergence of the third and fifth terms of (66) needs also some comments. Due to assumption (26) and to the a.e. convergence of $\mathbf{u}_n$ and k_n, we have

$$\varepsilon(k_n) \to \varepsilon(k) \quad \text{and} \quad P(\mathbf{u}_n, k_n) \to P(\mathbf{u}, k) \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty. \tag{72}$$

Since $k \in \mathrm{W}_0^{1,q}(\Omega)$ for $q < d'$, we have, by virtue of (67) and (71), and for any $\gamma \geq 1$,

$$\|\varepsilon(k_n)\|_{\mathrm{L}^\gamma(\Omega)} \leq C_1 \quad \text{for} \quad \theta < \frac{d}{d-2}, \tag{73}$$

$$\|P(\mathbf{u}_n, k_n)\|_{\mathrm{L}^\gamma} \leq C_2 \quad \text{for} \quad \beta \leq \frac{2d}{d-2} \text{ if } \vartheta = 0, \quad \text{or} \quad \beta + 2\vartheta < \frac{2d}{d-2} \text{ if } \vartheta > 0. \tag{74}$$

for some positive constants C_1 and C_2 not depending on n. Note that the conjunction of conditions on θ given by (60) and (73) are responsible for the assumption (31). On the other hand, the conjunction of all the conditions on β and ϑ given by (44), (63), (69) and (74) are responsible for the assumptions (32) and (33). Then, owing to (72), (73) and (74), $\varepsilon(k_n) \to \varepsilon(k)$ and $P(\mathbf{u}_n, k_n) \to P(\mathbf{u}, k)$ weakly in $\mathrm{L}^\gamma(\Omega)$, as $n \to \infty$ and for possible distinct γ. Thus, the convergence of the correspondingly integral terms follows. Let us now look to the fourth term of (66). Firs we observe that we can readily justify that

$$\begin{aligned} &\int_\Omega \left| \left(\nu_T(k_n)\,\mathcal{R}_n\big(|\mathbf{D}(\mathbf{u}_n)|^2\big) - \nu_T(k)|\mathbf{D}(\mathbf{u})|^2\right) v \right| d\mathbf{x} \\ \leq &\int_\Omega \left| \nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 - \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \right| \, |v|\, d\mathbf{x} + \int_\Omega \frac{1}{n} \frac{\nu_T(k)|\mathbf{D}(\mathbf{u})|^2|\mathbf{D}(\mathbf{u}_n)|^2}{1 + \frac{1}{n}|\mathbf{D}(\mathbf{u}_n)|^2} \, |v|\, d\mathbf{x}. \end{aligned} \tag{75}$$

Then, we observe that, by reasoning similarly as we did to prove (58), we also have

$$\mathbf{D}(\mathbf{u}_n) \to \mathbf{D}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^2(\Omega) \quad \text{and} \quad \mathbf{D}(\mathbf{u}_n) \to \mathbf{D}(\mathbf{u}) \quad \text{a.e. in } \Omega, \tag{76}$$

as $n \to \infty$. Thus, the last integral of (75) converges to zero by the application of Lebesgue's dominated convergence theorem, due to (76) and to assumption (27).

With respect to the first of the two last integrals, we can argue as in [19] to prove that

$$\nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 \to \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \quad \text{strongly in } \mathbf{L}^1(\Omega), \quad \text{as } m \to \infty$$

and, consequently, that the first integral of the right-hand side of (75) also converges to zero. Finally, we can pass to the limit $n \to \infty$ the equations (65) and (66) to obtain (28) and (29) for any $(\mathbf{v}, \varphi) \in \mathbf{V} \times \mathrm{W}_0^{1,q'}(\Omega)$. The proof of Theorem 1 is now concluded.

Remark 1 The existence result established in [20, Theorem 3.1] for the case of considering strong nonlinear functions $\mathbf{f}(\mathbf{u})$ and $\varepsilon(k)$, i.e. when no upper restrictions on the growth of these functions with respect to $\mathbf{u}$ and k are required, can also be generalized to any space dimension $d \geq 2$ and for a general function of turbulence production. In this case, besides the sign conditions (34), we just need to assume that (32) and (33) hold together with

$$\exists\, \tau > 0 : |\mathrm{angle}(\mathbf{f}(\mathbf{u}), \mathbf{u})| \notin \left(\frac{\pi}{2} - \tau, \frac{\pi}{2} + \tau\right) \quad \forall\, \mathbf{u} : |\mathbf{u}| \geq L, \quad \forall\, L > 0,$$

$$H_L \in \mathbf{L}^1(\Omega),\ G_M \in \mathrm{L}^1(\Omega) \quad \forall\, L,\ M > 0, \quad H_L := \sup_{|\mathbf{u}| \leq L} |\mathbf{f}(\mathbf{u})|, \quad G_M := \sup_{|k| \leq M} |\varepsilon(k)|.$$

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