Discrete Heat Equation with Shift Values

G. Britto Antony Xavier, S. John Borg and M. Meganathan

Abstract In this paper, we investigate the generalized partial difference operator and propose a model of it in discrete heat equation with several parameters and shift values. The diffusion of heat is studied by the application of Fourier's law of heat conduction in dimensions up to three and several solutions are postulated for the same. Through numerical simulations using MATLAB, solutions are validated and applications are derived.

Keywords Generalized partial difference equation · Partial difference operator and discrete heat equation.

Mathematics Subject Classication (2010) 35K05 · 39A10 · 39A14 · 58J35

1 Introduction

In 1984, Jerzy Popenda [6] introduced the difference operator $\underset{\alpha}{\Delta}$defined on $u(k)$ as $\underset{\alpha}{\Delta} u(k) = u(k+1) - \alpha u(k)$. In 1989, Miller and Rose [9] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the inverse fractional difference operator Δ_{ℓ}^{-1} ([3, 4]). Several formula on higher order partial sums on arithmetic, geometric progressions and products of n-consecutive terms of arithmetic progression have been derived in [10].

G. B. A. Xavier · S. J. Borg (✉) · M. Meganathan
Department of Mathematics, Sacred Heart College, Tirupattur, India
e-mail: sjborg@gmail.com

G. B. A. Xavier
e-mail: brittoshc@gmail.com

M. Meganathan
e-mail: meganathanmath@gmail.com

S. Pinelas et al. (eds.), *Differential and Difference Equations with Applications*,
Springer Proceedings in Mathematics & Statistics 230,
https://doi.org/10.1007/978-3-319-75647-9_2

In 2011, M. Maria Susai Manuel, et al. [8], extended the definition of Δ_α to $\underset{\alpha(\ell)}{\Delta}$ defined as $\underset{\alpha(\ell)}{\Delta} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function v(k), $\ell > 0$. In 2014, the authors in [2], have applied q-difference operator defined as $\Delta_q v(k) = v(qk) - v(k)$ and obtained finite series formula for logarithmic function. The difference operator $\underset{k(l)}{\Delta}$ with variable coefficients defined as $\underset{k(\ell)}{\Delta} v(k) = v(k+\ell) - kv(k)$ is established in [2].

The theory of difference and generalized difference equations using the forward difference operator Δ and generalized difference operators $\underset{\ell}{\Delta}, \underset{\alpha}{\Delta}, \underset{\alpha(\ell)}{\Delta}, \underset{L}{\Delta}, \underset{q(\ell)}{\Delta}$ are developed in [1, 2, 4, 7, 8, 10]. Partial difference and differential equations play a vital role in heat equations [1, 3, 5, 6]. Generalized difference operator with n-shift values $\ell = (\ell_1, \ell_2, ..., \ell_n) \neq 0$ on a real valued function $v(k) : R^n \to R$ is defined as

$$\underset{(\ell)}{\Delta} v(k) = v(k_1 + \ell_1, k_2 + \ell_2, ..., k_n + \ell_n) - v(k_1, k_2, ..., k_n). \tag{1}$$

This operator $\underset{(\ell)}{\Delta}$ becomes generalized partial difference operator if some $\ell_i = 0$. In this paper, we formulate the heat equation for medium in R^3 and obtain the solution using the operator defined in (1).

2 Preliminaries

Consider the difference operator defined in (1). Equations involving $\underset{(\ell)}{\Delta}$ with atleast one $\ell_i = 0$ is called generalized partial difference equation. A linear generalized partial difference equation is of the form,

$$\underset{(\ell)}{\Delta} v(k) = u(k), \tag{2}$$

where $\underset{(\ell)}{\Delta}$ is as given in (1), $\ell_i = 0$ for some i and $u(k) : R^n \to R$ is a given function.

A function $v(k) : R^n \to R$ satisfying (2) is called a solution of the Eq. (2). The Eq. (2) has a numerical solution of the form,

$$v(k) - v(k - m\ell) = \sum_{r=1}^{m} u(k - r\ell) = \underset{(\ell)}{\overset{-1}{\Delta}} u(k)\Big|_{k-m\ell}^{k}, \tag{3}$$

where $k - r\ell = (k_1 - r\ell_1, k_2 - r\ell_2, ..., k_n - r\ell_n)$, m is any positive integer. Relation (3) is the basic inverse principle with respect to $\underset{(\ell)}{\Delta}$ [2, 8, 10].

For example, the basic inverse principle with respect to $\underset{(0,\ell_2)}{\Delta}$ is given by

$$v(k_1,k_2) - v(k_1,k_2 - m\ell_2) = \sum_{r=1}^{m} u(k_1,k_2 - r\ell_2) = \underset{(0,\ell_2)}{\Delta}^{-1} u(k)\big|_{k-m\ell}^{k}, \tag{4}$$

where $v(k_1,k_2) = \underset{(0,\ell_2)}{\Delta}^{-1} u(k_1,k_2)$. From the theory of generalized difference equation, we have two types of solutions to (2), namely closed form and summation form solutions [2, 8, 10]. Similarly, the partial difference equation (2) has two types of solutions. Here we form partial difference equation for the heat flow and apply Fourier cooling law and obtain solution of heat equation with several variables and shift values.

3 Heat Equation for Medium, When γ is Constant

Consider homogeneous diffusion medium in $\Re^3$. Let γ be heat diffusion constant and $v(k_1,k_2,k_3,k_4,k_5)$ be the temperature at position (k_1,k_2,k_3), at time k_4 with density (or pressure) k_5. The proportional amount of heat flows from left to right at (k_1,k_2,k_3,k_4,k_5) is $\underset{(-\ell_1,0,0)}{\Delta} v(k)$, right to left $\underset{(\ell_1,0,0)}{\Delta} v(k)$, top to bottom $\underset{(0,\ell_2,0)}{\Delta} v(k)$, bottom to top $\underset{(0,-\ell_2,0)}{\Delta} v(k)$, front to rear $\underset{(0,0,\ell_3)}{\Delta} v(k)$, rear to front $\underset{(0,0,-\ell_3)}{\Delta} v(k)$. By the Fourier law of cooling, the heat equation for medium in $\Re^3$ is

$$\underset{(\ell_4,\ell_5)}{\Delta} v(k) = \gamma \underset{\pm\ell_{(1,2,3)}}{\Delta} v(k), \tag{5}$$

where $\underset{\pm\ell_{(1,2,3)}}{\Delta} = \underset{(\ell_1)}{\Delta} + \underset{(-\ell_1)}{\Delta} + \underset{(\ell_2)}{\Delta} + \underset{(-\ell_2)}{\Delta} + \underset{(\ell_3)}{\Delta} + \underset{(-\ell_3)}{\Delta}$ and $k = (k_1,k_2,k_3,k_4,k_5)$.

Theorem 1 *Assume that $v(k_1,k_2,k_3,k_4 - m\ell_4,k_4 - m\ell_5)$ and the partial differences $\underset{\pm\ell_{(1,2,3)}}{\Delta} v(k) = \underset{\pm\ell(1,2,3)}{u}(k)$ are known functions. Then the heat equation (5) has a solution of the form*

$$v(k) = v(k_1,k_2,k_3,k_4 - m\ell_4,k_5 - m\ell_5) + \gamma \sum_{r=1}^{m} \underset{\pm\ell_{(1,2,3)}}{u}(k_1,k_2,k_3,k_4 - r\ell_4,k_5 - m\ell_5). \tag{6}$$

Proof Taking $\underset{\pm\ell_{(1,2,3)}}{\Delta} v(k) = \underset{\pm\ell_{(1,2,3)}}{u}(k)$ in (5), we get

$$v(k) = \gamma \underset{(\ell_4,\ell_5)}{\Delta}^{-1} \underset{(\pm\ell_{(1,2,3)})}{u}(k). \tag{7}$$

The proof follows by applying inverse principle (4) in (7).

In the following theorem, we use the following notations:

$$v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, *, *) = v(k_1 + \ell_1, k_2, k_3, *, *) + v(k_1 - \ell_1, k_2, k_3, *, *) \\ + v(k_1, k_2 + \ell_2, k_3, *, *) + v(k_1, k_2 - \ell_2, k_3, *, *) \\ + v(k_1, k_2, k_3 + \ell_3, *, *) + v(k_1, k_2, k_3 - \ell_3, *, *).$$

$$v(*, k_{(2,3)} \pm \ell_{(2,3)}, *, *) = v(*, k_2 + \ell_2, k_3, *, *) + v(*, k_2 - \ell_2, k_3, *, *) \\ + v(*, k_2, k_3 + \ell_3, *, *) + v(*, k_2, k_3 - \ell_3, *, *).$$

Theorem 2 *If $v(k)$ is a solution of the Eq. (5) and m is apositive integer then the following relations are equivalent:*

(a)
$$v(k) = (1-6\gamma)^m v(k_1, k_2, k_3, k_4 - m\ell_4, k_5 - m\ell_5) \\ + \sum_{r=0}^{m-1} \gamma(1-6\gamma)^r \Big[v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4 - (r+1)\ell_4, k_5 - (r+1)\ell_5) \Big], \tag{8}$$

(b)
$$v(k) = \frac{1}{(1-6\gamma)^m} v(k_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5) \\ - \sum_{r=1}^{m} \frac{\gamma}{(1-6\gamma)^r} \Big[v(k_{(1,2,3)} \pm l_{(1,2,3)}, k_4 + (r-1)\ell_4, k_5 + (r-1)\ell_5) \Big], \tag{9}$$

(c)
$$v(k) = \frac{1}{\gamma^m} v(k_1 - m\ell_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5) \\ - \sum_{r=1}^{m} \frac{1-6\gamma}{\gamma^r} v(k_1 - r\ell_1, k_2, k_3, k_4 + (r-1)\ell_4, k_5 + (r-1)\ell_5) \\ - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 - (r+1)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5), \tag{10}$$

(d)
$$v(k) = \frac{1}{\gamma^m} v(k_1 + m\ell_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5) \\ - \sum_{r=1}^{m} \frac{1-6\gamma}{\gamma^r} v(k_1 + r\ell_1, k_2, k_3, k_4 + (r-1)\ell_4, k_5 + (r-1)\ell_5) \\ - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 + (r+1)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5). \tag{11}$$

Proof From (5) and (1), we arrive

(i)

$$v(k) = (1-6\gamma)v(k_1,k_2,k_3,k_4-\ell_4,k_5-\ell_5)+ \\ \gamma[v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4-\ell_4, k_5-\ell_5),$$

(ii)

$$v(k) = \frac{1}{(1-6\gamma)}v(k_1,k_2,k_3,k_4+\ell_4,k_5+\ell_5) \\ - \frac{\gamma}{(1-6\gamma)}[v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4, k_5)],$$

(iii)

$$v(k) = \frac{1}{\gamma}v(k_1-\ell_1,k_2,k_3,k_4+\ell_4,k_5+\ell_5) - \frac{1-6\gamma}{\gamma}v(k_1-\ell_1,k_2,k_3,k_4,k_5) \\ - v(k_1-2\ell_1,k_2,k_3,k_4,k_5) - v(k_1-\ell_1,k_{(2,3)} \pm \ell_{(2,3)},k_4,k_5)\text{and}$$

(iv)

$$v(k) = \frac{1}{\gamma}v(k_1+\ell_1,k_2,k_3,k_4+\ell_4,k_5+\ell_5) - \frac{1-6\gamma}{\gamma}v(k_1+\ell_1,k_2,k_3,k_4,k_5) \\ - v(k_1+2\ell_1,k_2,k_3,k_4,k_5) - v(k_1+\ell_1,k_{(2,3)} \pm \ell_{(2,3)},k_4,k_5).$$

Now the proof of (a), (b), (c), (d) follows by replacing
k_4 and k_5 by $k_4-\ell_4, k_4-2\ell_4, ..., k_m-m\ell_4, k_5-\ell_5, k_5-2\ell_5, ..., k_m-m\ell_5$,
k_4 and k_5 by $k_4+\ell_4, k_4+2\ell_4, ..., k_m+m\ell_4, k_5+\ell_5, k_5+2\ell_5, ..., k_m-m\ell_5$,
k_1 by $k_1-\ell_1, k_1-2\ell_1, ..., k_m-m\ell_1$, k_4 by $k_4+\ell_4, k_4+2\ell_4, ..., k_m+m\ell_4$ and
k_5 by $k_5+\ell_5, k_5+2\ell_5, ..., k_m-m\ell_5$,
k_1 by $k_1+\ell_1, k_1+2\ell_1, ..., k_m+m\ell_1$, k_4 by $k_4+\ell_4, k_4+2\ell_4, ..., k_m+m\ell_4$ and
k_5 by $k_5+\ell_5, k_5+2\ell_5, ..., k_m-m\ell_5$ in (i), (ii), (iii) and (iv) respectively.

Example 1 The following example shows that the diffusion of medium in three dimensional system can be identified if the solution $v(k_1,k_2,k_3,k_4,k_5)$ of (5) is known and vice versa. Suppose that $v(k_1,k_2,k_3,k_4,k_5) = e^{k_1+k_2+k_3+k_4+k_5}$ is a closed form solution of (5), then we have the relation

$\underset{(\ell_4,\ell_5)}{\Delta} e^{k_1+k_2+k_3+k_4+k_5} = \gamma\big[\underset{\pm\ell_{(1,2,3)}}{\Delta} e^{k_1+k_2+k_3+k_4+k_5}]$, which yields

$$e^{k_1+k_2+k_3+k_4+k_5}(e^{\ell_4+\ell_5}-1) = \\ \gamma\big[e^{k_1+k_2+k_3+k_4+k_5}(e^{\ell_1}+e^{-\ell_1}+e^{\ell_2}+e^{-\ell_2}+e^{\ell_3}+e^{-\ell_3}-6\big].$$

Cancelling $e^{k_1+k_2+k_3+k_4+k_5}$ on both sides derives

$$\gamma = \frac{e^{\ell_4+\ell_5} - 1}{e^{\ell_1} + e^{-\ell_1} + e^{\ell_2} + e^{-\ell_2} + e^{\ell_3} + e^{-\ell_3} - 6}. \tag{12}$$

For numerical verification, if we assume that $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5,$ $\ell_1 = 1, \ell_2 = 2, \ell_3 = 3, \ell_4 = 4, \ell_5 = 5, m = 1$ then $v(k_1, k_2, k_3, k_4, k_5) = e^{15}$,

$$\gamma = \frac{e^{4+5} - 1}{e^1 + e^{-1} + e^2 + e^{-2} + e^3 + e^{-3} - 6}.$$

LHS and RHS of (a), (b) of Theorem 2 are given below respectively.

(a) $3269017.37 = (-1963.46857)403.42879 + 4061136.705.$

(b) $3269017.37 = -13490909.90 + 16759999.00.$

If we assume that $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5, \ell_1 = 1, \ell_2 = 2, \ell_3 = 3,$ $\ell_4 = 4, \ell_5 = 5, m = 1$ then $v(k_1, k_2, k_3, k_4, k_5) = e^{15}$,

$$\gamma = \frac{e^{1+2} - 1}{e^3 + e^{-3} + e^4 + e^{-4} + e^5 + e^{-5} - 6}.$$

LHS and RHS of (c), (d) of Theorem 2 are as similar as above (a), (b).
For MATLAB coding, if we assume that $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$, $k_5 = 5$, $\ell_1 = 1$ and $\ell_2 = 2$, $\ell_3 = 3$, $\ell_4 = 4$, $\ell_5 = 5$, $m = 5$ then
$exp(15) = (1 - 6. * (327.4114733)). \wedge (5). * exp(-30) + symsum$ $((327.4114733). *(1 - 6. * (327.4114733)). \wedge r. * ((exp(16 - (r + 1). * 4 - (r + 1). * 5)) + (exp(14 - (r + 1). * 4 - (r + 1). * 5)) + (exp(17 - (r + 1). * 4 - (r + 1). * 5)) + (exp(13 - (r + 1). * 4 - (r + 1). * 5)) + (exp(18 - (r + 1). * 4 - (r + 1). * 5)) + (exp(12 - (r + 1). * 4 - (r + 1). * 5))), r, 0, 4).$

4 Conclusion

The study of partial difference operator has wide applications in discrete fields and heat equation is one such. The nature of propagation of heat through materials of dimensions(up to three) can be postulated.

The core Theorem 2 provides the possibility of predicting the temperature either for the past or the future after getting to know the temperature at few finite points on the material at the present time.

References

1. Ablowitz, M.J., Ladik, J.F.: On the solution of a class of nonlinear partial difference equations. Stud. Appl. Math. **57**, 1–12 (1977)
2. Xavier, G.B.A., Gerly, T.G., Begum, H.N.: Finite Series of Polynomials and Polynomial Factorials arising from Generalized q-Difference operator. Far East J. Math. Sci. **94**(1), 47–63 (2014)
3. Bastos, N.R.O., Ferreira, R.A.C., Torres, D.F.M.: Discrete-time fractional variational problems. Signal Process. **91**(3), 513–524 (2011)

4. Ferreira, R.A.C., Torres, D.F.M.: Fractional h-difference equations arising from the calculus of variations. Appl. Anal. Discret. Math. **5**(1), 110–121 (2011)
5. Cheng, S.S.: Sturmian comparison theorems for three term recurrence equations. J. Math. Anal. Appl. **111**, 464–474 (1985)
6. Popenda, J., Szmanda, B.: On the oscillation of solutions of certain difference equations. Demonstratio Mathematica **XVII**(1), 153–164 (1984)
7. Koshy, T.: Fibonacci and Lucas Numbers with Applications. Wiley-Interscience, New York (2001)
8. Manuel, M.M.S., Chandrasekar, V., Xavier, G.B.A.: Solutions and applications of certain class of α- difference equations. Int. J. Appl. Math. **24**(6), 943–954 (2011)
9. Miller, K.S., Ross, B.: Fractional Difference Calculus in Univalent Functions, pp. 139–152. Horwood, UK (1989)
10. Manuel, M.S., Xavier, G.B.A., Chandrasekar, V., Pugalarasu, R.: Theory and application of the Generalized Difference Operator of the n^{th} kind(Part I). Demonstratio Mathematica **45**(1), 95–106 (2012)