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Sandra Pinelas · Tomás Caraballo
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Differential and Difference Equations with Applications

ICDDEA, Amadora, Portugal, June 2017

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Preface

For the five days 5–9 June 2017, more than 230 mathematicians from 52 countries attended the 3rd International Conference on Differential & Difference Equations and Applications, held at the Military Academy, Amadora, Portugal.

The scientific aim of this conference was to bring together mathematicians working in various disciplines of differential and difference equations and their applications. There were 12 plenary lectures, 14 main lectures and 175 communications about the current research in this field. This volume contains 50 selected original papers which are connected to research lectures given at the conference. Each paper has been carefully reviewed.

We take this opportunity to thank all the participants of the conference and the contributors to these proceedings. Our special thanks belong to the Military Academy for the sincere hospitality. We are also grateful to the Scientific and Organizing Committees for all the effort in the preparation of the conference.

The conference was dedicated at memory of Prof. Ondřej Došlý (1956–2016). Professor Ondřej Došlý had been invited to the ICDDEA 2011 as a plenary speaker, to the ICDDEA 2015 as Scientific Committee and main speaker and to the ICDDEA 2017 plenary speaker, but was unable to come and died shortly afterwards.

We hope that this volume will serve researchers in all fields of differential and difference equations.

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Wuhan, China	Peter Kloeden
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Contents

On Asymptotic Behavior of Blow-Up Solutions to Higher-Order Differential Equations with General Nonlinearity

Irina V. Astashova

Abstract New results are proved on the asymptotic behavior of blow-up solutions to a higher-order equation with general potential are proved. Several author's results are presented concerning both positive and oscillatory solutions to equations with regular and singular nonlinearities. Some applications of the results obtained are proposed.

Keywords Nonlinear equations · Blow-up · Asymptotic behavior · Oscillatory solutions

1 Introduction

Consider the equation

$$y^{(n)} = P(x, y, y', \dots, y^{(n-1)})|y|^k_{\pm} \tag{1}$$

where $|y|^k_{\pm}$ denotes $|y|^k \operatorname{sgn} y$, $n \geq 2$, $k \in (0, 1) \cup (1, \infty)$, P is a continuous and Lipschitz continuous in the last n variables function satisfying the inequalities

$$0 < P_* < P < P^*. \tag{2}$$

Consider also a special case of (1), namely

$$y^{(n)} = p_0\, |y|^k_{\pm} \tag{3}$$

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with $p_0 > 0,\ k \neq 1$. Hereafter, we put

$$\alpha = \frac{n}{k-1}. \tag{4}$$

The main purpose of this article is to collect together and to present recent and new author's results on asymptotic properties of solutions to Eq. (1).

Equation (1) has been investigated by a lot of mathematicians from different points of view because it is a generalization of the well-known Emden–Fowler equation (see, for example, [1, 2]). The first asymptotic classification of solutions to the Emden–Fowler equation of the second order appears in [3]. Asymptotic classification of solutions to Eq. (1) in the case $P = P(x)$, $n = 2$ is presented in [4]. Higher order generalizations of the equation were investigated later in [4, 5] (see also references in these books) and in a great number of articles of different authors. In particular, sufficient conditions are given for the existence of some special types of solutions to these equations (see, for example, [4–10, 13, 19]).

In this article some new results on asymptotic behavior of "blow-up" solution and results on the existence of oscillatory quasi-periodic solutions are formulated and the methods of proof are done (see also [23]).

Qualitative properties of solutions to third- and fourth-order equations of this type were investigated in [10–17]. In [25] an asymptotic classification of solutions to Eq. (3) is given in the cases of regular ($k > 1$) and singular ($0 < k < 1$) nonlinearities for $n = 3, 4$. Proofs for different cases see in [22, 24, 29]. A more precise result for the behavior of oscillatory solutions to Eq. (3) for $n = 3$ is presented in [31]. The results on the behavior of oscillatory solutions to higher-order equations see in [20, 28].

2 Asymptotic Behavior of Blow-Up Solutions

Definition 1 A solution $y(x)$ of Eq. (1) is said to be **n-positive** if it is maximally extended in both directions and eventually satisfies the inequalities

$$y(x) > 0,\ y'(x) > 0, \ldots,\ y^{(n-1)}(x) > 0.$$

Note that if the above inequalities are satisfied by a solution to (1) at some point x_0, then they are also satisfied at any point $x > x_0$ in the domain of the solution. Moreover, such a solution to (1) with (2), if maximally extended, must be a so-called **blow-up solution**, i.e. must have a vertical asymptote at the right endpoint of its domain.

Immediate calculations show that Eq. (3) has n-positive solutions with the exact power-law behavior, namely

$$y(x) = C(x^* - x)^{-\alpha} \tag{5}$$

defined on $(-\infty,\ x^*)$ with

$$C = \left(\frac{\alpha(\alpha + 1) \dots (\alpha + n - 1)}{p_0} \right)^{\frac{1}{k-1}} \tag{6}$$

and arbitrary $x^* \in \mathbf{R}$.

For $n = 1$ all n-positive solutions to (3) are defined by (5). For $n \in \{2, 3, 4\}$ it is known that any n-positive solution to (3) and even to more general Eq. (1) under some assumptions is asymptotically equivalent, near the right endpoint of its domain, to the solution defined by (5) with appropriate x^*:

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)), \quad x \to x^* - 0, \tag{7}$$

where C is defined by (6) with p_0 equal, in the case of Eq. (1), to the limit of $P(x,\ y_0,\ \dots,\ y_{n-1})$ as $x \to x^* - 0$, $y_0 \to \infty,\ \dots,\ y_{n-1} \to \infty$. See [4] for $n = 2$, and [5, 11, 14], for $n \in \{3, 4\}$.

For Eq. (1) with some additional assumptions on the function P the existence of solutions with power-law asymptotic behavior (7) is proved in [5, 11]. For $5 \le n \le 11$, the existence of an $(n - 1)$-parametrical family of such solutions is obtained (see [5, 11]).

2.1 Existence of Positive Solutions with Non-power-law Asymptotic Behavior

In [4], Problem 16.4, the question was posed whether (7) is satisfied for all positive blow-up solutions to (1) with the vertical asymptote $x = x^*$. The natural hypothesis that they all satisfy (7) for any $n > 4$ appears to be wrong even for Eq. (3). It was proved [18] that for any N and $K > 1$ there exist an integer $n > N$ and a real number $k \in (1, K)$ such that Eq. (3) has a solution of the form

$$y = p_0^{-\frac{1}{k-1}} (x^* - x)^{-\alpha}\, h(\log\,(x^* - x)), \tag{8}$$

where h is a positive periodic non-constant function on $\mathbf{R}$.

Some informations on possible values of n for such solutions is given by the following

Theorem 1 *([21]) If $12 \le n \le 14$, then there exists $k > 1$ such that Eq. (3) has a solution $y(x)$ with*

$$y^{(j)}(x) = p_0^{-\frac{1}{k-1}} (x^* - x)^{-\alpha - j}\, h_j(\,\log(x^* - x)\,),$$

$$j = 0, 1, \dots, n - 1,$$

where h_j are periodic positive non-constant functions on $\mathbf{R}$.

Sketch of the proof of Theorem 1. For investigation of blow-up solutions to Eq. (3) having the vertical asymptote $x = x^*$, the substitutions

$$x^* - x = e^{-t}, \qquad y = (C + v)\, e^{\alpha t} \tag{9}$$

with C defined by (6) transform equation (3) with $p_0 = 1$ to another one, which can be reduced to the first-order system

$$\frac{dV}{dt} = A_\alpha V + F_\alpha(V), \tag{10}$$

where A_α is a constant $n \times n$ matrix with eigenvalues satisfying the equation

$$\prod_{j=0}^{n-1}(\lambda + \alpha + j) = \prod_{j=0}^{n-1}(1 + \alpha + j) \tag{11}$$

and F_α is a mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$ satisfying $||F_\alpha(V)|| = O\left(||V||^2\right)$ and $||F'_{\alpha,V}(V)|| = O\,(||V||)$ as $V \to 0$.

The Hopf Bifurcation theorem [27] provides the existence, for some α, of a periodic non-constant solution to system (10), which can be transformed to the solution needed in Theorem 1.

To apply the Hopf Bifurcation theorem, we need to proof the existence of the family λ_α of complex simple roots of Eq. (11) such that for some $\tilde{\alpha}$ we have $\operatorname{Re} \lambda_{\tilde{\alpha}} = 0$ and

$$\operatorname{Re} \frac{d\lambda_\alpha}{d\alpha}\,(\tilde{\alpha}) \neq 0. \tag{12}$$

All roots of Eq. (11) are simple, which can be proved for any $n > 1$.

The existence of pure imaginary roots for some $\tilde{\alpha}$ can be proved for any $n > 11$. To do this, consider the positive C^1-functions $\rho_n(\alpha)$ and $\sigma_n(\alpha)$ defined for all $\alpha > 0$ via the equations

$$\prod_{j=0}^{n-1}\left(\rho_n(\alpha)^2 + (\alpha + j)^2\right) = \prod_{j=0}^{n-1}(1 + \alpha + j)^2$$

and

$$\sum_{j=0}^{n-1} \arg\left(\sigma_n(\alpha) i + \alpha + j\right) = 2\pi$$

supposing $\arg z \in [0, 2\pi)$ for all $z \in \mathbf{C} \setminus \{0\}$.

One can show that $\rho_n(\alpha)/\alpha \to 0$ as $\alpha \to +\infty$, while $\sigma_n(\alpha)/\alpha \to \tan 2\pi/n > 0$, whence for sufficiently large α we have $\rho_n(\alpha) < \sigma_n(\alpha)$.

For sufficiently small $\alpha > 0$, one can prove that $\sigma_{12}(\alpha) < 2 < \rho_{12}(\alpha)$ and, for any $\alpha > 0$, that $\rho_{n+1}(\alpha) > \rho_n(\alpha)$ and $\sigma_{n+1}(\alpha) < \sigma_n(\alpha)$.

So, for any $n \geq 12$ there exists $\tilde{\alpha} > 0$ such that $\rho_n(\tilde{\alpha}) = \sigma_n(\tilde{\alpha})$ producing the pure imaginary root $\lambda_{\tilde{\alpha}} = \rho_n(\tilde{\alpha})\, i$ of Eq. (11).

As for inequality (12), it was successfully proved only for $n \in \{12, 13, 14\}$, and the greater is n, the more cumbersome the proof turns out.

2.2 *On Power-Law Asymptotic Behavior of Solutions to Weakly Super-Linear Emden–Fowler Type Equations with General Nonlinearity*

It appears that a weaker version of the I.T. Kiguradze's hypothesis about power-law asymptotic behavior of blow-up solutions for higher-order equations (3) is correct.

Theorem 2 ([30]) *For any integer $n > 4$ there exists $K > 1$ such that for any real $k \in (1, K)$, all n-positive solutions to Eq. (3) have the power-law asymptotic behavior (7) near the right endpoint of their domains.*

More general result concerning (1) is following:

Theorem 3 *Suppose $n > 4$, $P \in C(\mathbf{R}^{n+1}) \bigcap Lip_{y_0,\ldots,y_{n-1}}(\mathbf{R}^n)$, $P \to p_0 > 0$ as $x \to x^*$, $y_0 \to \infty, \ldots, y_{n-1} \to \infty$, and satisfies (2). Then there exists $K > 1$ such that for any real $k \in (1, K)$, any solution to Eq. (1) tending to $+\infty$ as $x \to x^* - 0$ has the power-law asymptotic behavior given by (7) with C defined by (6).*

Proof As well as in the proof of Theorem 2 (see [30]) we put

$$m = n - 1, \quad \gamma = \frac{1}{\alpha} = \frac{k-1}{n} \tag{13}$$

and consider an auxiliary γ-parameterized dynamical system on the m-dimensional sphere S^m. This sphere is considered as the quotient space of $\mathbf{R}^n \setminus \{0\}$ with respect to the equivalence relation

$$(z_0, \ldots, z_m) \sim (\lambda z_0, \ldots, \lambda z_m)\,, \quad \lambda > 0.$$

The equivalence class of the point $(z_0, \ldots, z_m) \in \mathbf{R}^n \setminus \{0\}$ is denoted by

$$(z_0 : \cdots : z_m)\,.$$

Any non-trivial solution $y(x)$ to Eq. (3) with $p_0 = 1$ generates a curve in S^m consisting of the points

$$\left(y(x) : \left|\frac{y'(x)}{a_1}\right|_{\pm}^{\frac{1}{1+\gamma}} : \ldots : \left|\frac{y^{(j)}(x)}{a_j}\right|_{\pm}^{\frac{1}{1+\gamma j}} : \ldots : \left|\frac{y^{(m)}(x)}{a_m}\right|_{\pm}^{\frac{1}{1+\gamma m}} \right), \quad x \in \operatorname{dom} y,$$

$$a_1 = \left(\prod_{l=1}^{m} (1+\gamma l) \right)^{-1/n}, \tag{14}$$

$$a_{j+1} = (1+\gamma j)\, a_1 a_j = a_1^{j+1} \prod_{l=1}^{j} (1+\gamma l), \quad j \in \{1, \dots, m-1\}.$$

This curve locally parameterized with

$$\tau = a_1 \int_{x_0}^{x} y(\xi)^{\gamma}\, d\xi$$

can be described within the chart that covers the part S_+^m with all positive z_j and has the coordinate functions $v_j : (z_0 : \cdots : z_m) \mapsto \left(\frac{z_j}{z_0}\right)^{1+\gamma j}$, $j \in 1, ..., m$, as follows:

$$\begin{cases} \dfrac{dv_1}{d\tau} = (1+\gamma)\,(v_2 - v_1^2), \\ \dfrac{dv_j}{d\tau} = (1+\gamma j)\,(v_{j+1} - v_1 v_j), \quad j \in \{2, \dots, m-1\}, \\ \dfrac{dv_m}{d\tau} = (1+\gamma m)\,(1 - v_1 v_m). \end{cases} \tag{15}$$

Any such trajectory, if entered S_+^m, newer leaves it. The only equilibrium point in S_+^m, which has all v_j coordinates equal to 1, is denoted by v^*. Similar formulas describe the curve in other charts covering the whole sphere. Different variables parameterizing the curve in different charts can be combined into a single one by using a partition of unity. Thus we obtain a global γ-parameterized dynamical system $\mathfrak{S}$ in the whole S^m.

In [30] the following lemma is proved.

Lemma 1 *There exist $\gamma_2 > 0$ and an open neighborhood U of the point v^* such that for any positive $\gamma < \gamma_2$, any trajectory of the global dynamical system passing through the closure $\overline{U}$ tends to v^*. If such a trajectory does not coincide with v^*, then it passes transversally, at some time, through the boundary ∂U.*

Now consider a solution $y(x)$ to Eq. (1) assuming $P \to 1$ as $x \to x^*$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$. This solution generates in S^m a curve described in the same chart by

$$\begin{cases} \dfrac{dv_1}{d\tau} = (1+\gamma)\,(v_2 - v_1^2), \\ \dfrac{dv_j}{d\tau} = (1+\gamma j)\,(v_{j+1} - v_1 v_j), \quad j \in \{2, \dots, m-1\}, \\ \dfrac{dv_m}{d\tau} = (1+\gamma m)\,(q(\tau) - v_1 v_m), \end{cases} \tag{16}$$

here $q(\tau)$ is obtained by the related substitution to the function P and tends to 1 as $\tau \to \infty$.

Lemma 2 *The set of all ω-limit points of the trajectory described by (16) with $q(\tau)$ tending to 1 as $\tau \to \infty$ is the union of some whole trajectories of the system $\mathfrak{S}$.*

Proof of this lemma is similar to that of Lemma 5.6 in [5].

Since the sphere S^m is compact, any trajectory $s(\tau)$ on it has at least one limit point. If this limit point is unique then it is the limit of the trajectory, so if the trajectory does not tend to v^*, then it must have at least one ω-limit point $w \neq v^*$. If the trajectory $s(\tau)$ generated by a solution to Eq. (1) tending to $+\infty$ as $x \to x^* - 0$, then we may assume that $w \in S^m_+$. According to Lemma 1, the trajectory $s_1(\tau)$ of the system $\mathfrak{S}$ passes transversally through ∂U whenever $\gamma \in (0,\ \gamma_2)$. When the function $q(\tau)$ is sufficiently close to 1, then the trajectory $s(\tau)$ also passes transversally through ∂U. In this case it can enter U and cannot leave it. Hence the points of $s_1(\tau)$ which are outside of U cannot be ω-limit points of $s(\tau)$. This contradiction with Lemma 2 shows that $s(\tau) \to v^*$ as $\tau \to \infty$. In particular,

$$v_1 = \left(\frac{z_1}{z_0}\right)^{1+\gamma} \to 1 \quad \text{as} \quad \tau \to \infty.$$

This yields that the related solution $y(x)$ to Eq. (1) satisfies

$$\frac{y'}{a_1\ y^{1+\gamma}} \to 1 \quad \text{as } x \to x^* - 0,$$

whence

$$y' \sim a_1\, y^{1+\gamma} \text{ as } x \to x^* - 0,$$

and

$$y \sim (a_1\, \gamma)^{-\frac{1}{\gamma}}\, (x^* - x)^{-\frac{1}{\gamma}},$$

hence, by (13) and (14) we obtain

$$y \sim (\alpha(\alpha+1)\dots(\alpha+n-1))^{\frac{1}{k-1}}\, (x^* - x)^{-\alpha},\ \ x \to x^* - 0. \tag{17}$$

This completes the proof of Theorem 3 with $p_0 = 1$.

If $y(x)$ is a solution of (1) with P tending to arbitrary p_0, then $y\, p_0^{\frac{1}{k-1}}$ is a solution to (1) with a similar function P tending to 1. Hence $y\, p_0^{\frac{1}{k-1}}$ satisfies (17) whence

$$y = \left(\frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{p_0}\right)^{\frac{1}{k-1}} (x^* - x)^{-\alpha}(1 + o(1)) \text{ as } x \to x^* - 0.$$

Theorem 3 is proved.

3 Asymptotic Behavior of Oscillatory Solutions

This section is devoted to the existence of oscillatory quasi-periodic in some sense solutions to a higher-order Emden–Fowler type differential equation

$$y^{(n)} + p_0\,|y|^k_{\pm} = 0,\; p_0 \neq 0. \tag{18}$$

with $n > 2$ and $k \in (0, 1) \cup (1, \infty)$.

Theorem 4 *For any integer $n > 2$ and real $k > 1$ there exists a periodic oscillatory function h on $\mathbf{R}$ such that for any $p_0 > 0$ and $x^* \in \mathbf{R}$ the function*

$$y(x) = p_0^{-\frac{1}{k-1}}(x^* - x)^{-\alpha}\, h(\log(x^* - x)) \tag{19}$$

is a solution to Eq. (18) on $(-\infty, x^*)$.

Definition 2 A solution having the form (19) is called **quasi-periodic**.

Sketch of the proof of Theorem 4. For $0 \le j < n$ put

$$B_j = \frac{nk}{n + j(k-1)} > 1, \qquad \beta_j = \frac{1}{B_j}.$$

For any $q = (q_0, \dots, q_{n-1}) \in \mathbf{R}^n$ let $y_q(x)$ be the maximally extended solution to the equation

$$y^{(n)}(x) + |y(x)|^k = 0 \tag{20}$$

with the initial data $y^{(j)}(0) = q_j,\;\; 0 \le j < n$.

Consider also the function $N : \mathbf{R}^n \to \mathbf{R}$ and the mapping $\tilde{N} : \mathbf{R}^n \setminus \{0\} \to \mathbf{R}^n \setminus \{0\}$ defined by

$$N(q) = \sum_{j=0}^{n-1} \left|q_j\right|^{B_j}, \qquad \tilde{N}(q)_j = N(q)^{-\beta_j}\, q_j$$

and satisfying $N\left(\tilde{N}(q)\right) = 1$ for all $q \in \mathbf{R}^n \setminus \{0\}$.

Next, consider the subset $Q \subset \mathbf{R}^n$ consisting of all $q \in \mathbf{R}^n$ satisfying the following conditions:

(1) $q_0 = 0$,
(2) $q_j \ge 0, \quad 0 < j < n$,
(3) $N(q) = 1$.

The restriction of the projection

$$(q_0, \dots, q_{n-1}) \mapsto (q_1, \dots, q_{n-2})$$

to the set Q is a homeomorphism of Q onto the convex compact subset of $\mathbf{R}^{n-2}$ consisting of all its points with non-negative coordinates satisfying the inequality $\sum_{j=1}^{n-2} \left|q_j\right|^{B_j} \le 1$.

Lemma 3 *For any* $q \in Q$ *there exists* $a_q > 0$ *satisfying* $y_q(a_q) = 0$ *and* $y_q^{(j)}(a_q) < 0$ *for* $0 < j < n$.

Note that a_q is not only the first positive zero of $y_q(x)$, but the only positive one.

To continue the proof of Theorem 4, consider the function $\xi : q \mapsto a_q$ taking each $q \in Q$ to the first positive zero of the function y_q. Due to the implicit function theorem, the function ξ is continuous.

Consider the C^1 "solution" mapping

$$S : (q, x) \mapsto= \left(y_q(x), y_q'(x), \dots, y_q^{(n-1)}(x)\right)$$

defined on a domain including $\mathbf{R}^n \times \{0\}$ and the continuous mapping $\tilde{S} : q \mapsto \tilde{N}\left(-S\left(q, \xi(q)\right)\right)$, which maps Q into itself.

By the Brouwer fixed-point theorem, there exists $\hat{q} \in Q$ such that $\tilde{S}\left(\hat{q}\right) = \hat{q}$. In other words, there exists a non-negative solution $\hat{y}(x) = y_{\hat{q}}(x)$ to Eq. (20) defined on a segment $[0, a_1]$ with $a_1 = a_{\hat{q}}$, positive on the open interval $(0, a_1)$, and such that

$$\lambda^{-\beta_j}\, \hat{y}^{(j)}(a_1) = -\hat{y}^{(j)}(0), \quad 0 \le j < n, \tag{21}$$

with

$$\lambda = N\left(S\left(\hat{q}, \xi(\hat{q})\right)\right) = \sum_{j=0}^{n-1} \left|\hat{y}^{(j)}(a_1)\right|^{B_j} > 0.$$

Since $\hat{y}(x)$ is non-negative, it is also a solution to the equation

$$y^{(n)}(x) + |y(x)|_{\pm}^{k} = 0.$$

Due to property (21), the solution $\hat{y}(x)$ can be smoothly extended onto some segment $[a_1, a_2]$, then onto $[a_2, a_3]$, etc., as well as in the opposite direction, with the following relation between the lengths of the neighboring segments and the values of $\hat{y}(x)$ at their points:

$$\frac{a_s - a_{s-1}}{a_{s+1} - a_s} = b = \lambda^{\frac{k-1}{nk}},$$

$$\hat{y}(x) = -b^{\alpha}\, \hat{y}\left(b(x - a_s) + a_{s-1}\right),$$

where $x \in \left[a_s, a_{s+1}\right],\ b(x - a_s) + a_{s-1} \in \left[a_{s-1},\ a_s\right]$.

It can be proved that $b > 1$ whenever $k > 1$, which yields $a^* = \sum_{s=0}^{\infty} < \infty$, and that the function

$$h(t) = e^{t\alpha}\,\hat{y}\left(a^* - e^t\right)$$

is just a periodic function needed for Theorem 4.

Corollary 1 *For any integer $n > 2$ and real $k > 1$ there exists a periodic oscillatory function h on* **R** *such that for any $p_0 \in \boldsymbol{R}$ satisfying $(-1)^n p_0 > 0$ and any $x^* \in \boldsymbol{R}$ the function*

$$y(x) = |p_0|^{-\frac{1}{k-1}}\,(x - x^*)^{-\alpha}\,h(\log(x - x^*))$$

is a solution to Eq. (18) on (x^*, ∞).

Theorem 5 *For any integer $n > 2$ and real positive $k < 1$ there exists a non-constant oscillatory periodic function h such that for any p_0 with $(-1)^n p_0 > 0$ and any real x^* the function*

$$y(x) = |p_0|^{\frac{1}{1-k}}\,(x^* - x)^{|\alpha|} h(\log(x^* - x)),$$

is a solution to Eq. (18) on $(-\infty,\ x^*)$.

Part of these results are included in [23], its application can be found in [26].

4 Some Applications of These Results

Equation (3) it is a model of nonlinear equations. Methods developed for its research can be applied to study of more complex non-linear equations of the form (1). (See [4, 5]). The equation of type (1) also appears in investigation of some spectral problems (see [5] **IV**).

5 Open Problems Connected with Equation (3)

1. Does positive blow-up solution with non-power law asymptotic behavior exist for $5 \le n \le 11$ and $n \ge 15$? **2.** Does positive blow-up solutions with different from power-law (7) and non-power law (8) behavior exist for $n \ge 4$? **3.** What is the meaning of K in Theorems 2 and 3?

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Discrete Heat Equation with Shift Values

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Abstract In this paper, we investigate the generalized partial difference operator and propose a model of it in discrete heat equation with several parameters and shift values. The diffusion of heat is studied by the application of Fourier's law of heat conduction in dimensions up to three and several solutions are postulated for the same. Through numerical simulations using MATLAB, solutions are validated and applications are derived.

Keywords Generalized partial difference equation · Partial difference operator and discrete heat equation.

Mathematics Subject Classication (2010) 35K05 · 39A10 · 39A14 · 58J35

1 Introduction

In 1984, Jerzy Popenda [6] introduced the difference operator $\underset{\alpha}{\Delta}$defined on $u(k)$ as $\underset{\alpha}{\Delta} u(k) = u(k+1) - \alpha u(k)$. In 1989, Miller and Rose [9] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the inverse fractional difference operator Δ_{ℓ}^{-1} ([3, 4]). Several formula on higher order partial sums on arithmetic, geometric progressions and products of n-consecutive terms of arithmetic progression have been derived in [10].

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In 2011, M. Maria Susai Manuel, et al. [8], extended the definition of Δ_α to $\underset{\alpha(\ell)}{\Delta}$ defined as $\underset{\alpha(\ell)}{\Delta} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function v(k), $\ell > 0$. In 2014, the authors in [2], have applied q-difference operator defined as $\Delta_q v(k) = v(qk) - v(k)$ and obtained finite series formula for logarithmic function. The difference operator $\underset{k(l)}{\Delta}$ with variable coefficients defined as $\underset{k(\ell)}{\Delta} v(k) = v(k+\ell) - kv(k)$ is established in [2].

The theory of difference and generalized difference equations using the forward difference operator Δ and generalized difference operators $\underset{\ell}{\Delta}, \underset{\alpha}{\Delta}, \underset{\alpha(\ell)}{\Delta}, \underset{L}{\Delta}, \underset{q(\ell)}{\Delta}$ are developed in [1, 2, 4, 7, 8, 10]. Partial difference and differential equations play a vital role in heat equations [1, 3, 5, 6]. Generalized difference operator with n-shift values $\ell = (\ell_1, \ell_2, ..., \ell_n) \neq 0$ on a real valued function $v(k) : R^n \to R$ is defined as

$$\underset{(\ell)}{\Delta} v(k) = v(k_1 + \ell_1, k_2 + \ell_2, ..., k_n + \ell_n) - v(k_1, k_2, ..., k_n). \tag{1}$$

This operator $\underset{(\ell)}{\Delta}$ becomes generalized partial difference operator if some $\ell_i = 0$. In this paper, we formulate the heat equation for medium in R^3 and obtain the solution using the operator defined in (1).

2 Preliminaries

Consider the difference operator defined in (1). Equations involving $\underset{(\ell)}{\Delta}$ with atleast one $\ell_i = 0$ is called generalized partial difference equation. A linear generalized partial difference equation is of the form,

$$\underset{(\ell)}{\Delta} v(k) = u(k), \tag{2}$$

where $\underset{(\ell)}{\Delta}$ is as given in (1), $\ell_i = 0$ for some i and $u(k) : R^n \to R$ is a given function.

A function $v(k) : R^n \to R$ satisfying (2) is called a solution of the Eq. (2). The Eq. (2) has a numerical solution of the form,

$$v(k) - v(k - m\ell) = \sum_{r=1}^{m} u(k - r\ell) = \underset{(\ell)}{\overset{-1}{\Delta}} u(k)\Big|_{k-m\ell}^{k}, \tag{3}$$

where $k - r\ell = (k_1 - r\ell_1, k_2 - r\ell_2, ..., k_n - r\ell_n)$, m is any positive integer. Relation (3) is the basic inverse principle with respect to $\underset{(\ell)}{\Delta}$ [2, 8, 10].

For example, the basic inverse principle with respect to $\underset{(0,\ell_2)}{\Delta}$ is given by

$$v(k_1,k_2)-v(k_1,k_2-m\ell_2)=\sum_{r=1}^{m}u(k_1,k_2-r\ell_2)=\underset{(0,\ell_2)}{\Delta^{-1}}u(k)\big|_{k-m\ell}^{k}, \tag{4}$$

where $v(k_1,k_2)=\underset{(0,\ell_2)}{\Delta^{-1}}u(k_1,k_2)$. From the theory of generalized difference equation, we have two types of solutions to (2), namely closed form and summation form solutions [2, 8, 10]. Similarly, the partial difference equation (2) has two types of solutions. Here we form partial difference equation for the heat flow and apply Fourier cooling law and obtain solution of heat equation with several variables and shift values.

3 Heat Equation for Medium, When γ is Constant

Consider homogeneous diffusion medium in $\Re^3$. Let γ be heat diffusion constant and $v(k_1,k_2,k_3,k_4,k_5)$ be the temperature at position (k_1,k_2,k_3), at time k_4 with density (or pressure) k_5. The proportional amount of heat flows from left to right at (k_1,k_2,k_3,k_4,k_5) is $\underset{(-\ell_1,0,0)}{\Delta}v(k)$, right to left $\underset{(\ell_1,0,0)}{\Delta}v(k)$, top to bottom $\underset{(0,\ell_2,0)}{\Delta}v(k)$, bottom to top $\underset{(0,-\ell_2,0)}{\Delta}v(k)$, front to rear $\underset{(0,0,\ell_3)}{\Delta}v(k)$, rear to front $\underset{(0,0,-\ell_3)}{\Delta}v(k)$. By the Fourier law of cooling, the heat equation for medium in $\Re^3$ is

$$\underset{(\ell_4,\ell_5)}{\Delta}v(k)=\gamma\underset{\pm\ell_{(1,2,3)}}{\Delta}v(k), \tag{5}$$

where $\underset{\pm\ell_{(1,2,3)}}{\Delta}=\underset{(\ell_1)}{\Delta}+\underset{(-\ell_1)}{\Delta}+\underset{(\ell_2)}{\Delta}+\underset{(-\ell_2)}{\Delta}+\underset{(\ell_3)}{\Delta}+\underset{(-\ell_3)}{\Delta}$ and $k=(k_1,k_2,k_3,k_4,k_5)$.

Theorem 1 *Assume that $v(k_1,k_2,k_3,k_4-m\ell_4,k_4-m\ell_5)$ and the partial differences $\underset{\pm\ell_{(1,2,3)}}{\Delta}v(k)=\underset{\pm\ell(1,2,3)}{u}(k)$ are known functions. Then the heat equation (5) has a solution of the form*

$$v(k)=v(k_1,k_2,k_3,k_4-m\ell_4,k_5-m\ell_5)+\gamma\sum_{r=1}^{m}\underset{\pm\ell_{(1,2,3)}}{u}(k_1,k_2,k_3,k_4-r\ell_4,k_5-m\ell_5). \tag{6}$$

Proof Taking $\underset{\pm\ell_{(1,2,3)}}{\Delta}v(k)=\underset{\pm\ell_{(1,2,3)}}{u}(k)$ in (5), we get

$$v(k)=\gamma\underset{(\ell_4,\ell_5)}{\Delta^{-1}}\underset{(\pm\ell_{(1,2,3)})}{u}(k). \tag{7}$$

The proof follows by applying inverse principle (4) in (7).

In the following theorem, we use the following notations:

$$v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, *, *) = v(k_1 + \ell_1, k_2, k_3, *, *) + v(k_1 - \ell_1, k_2, k_3, *, *) + v(k_1, k_2 + \ell_2, k_3, *, *) + v(k_1, k_2 - \ell_2, k_3, *, *) + v(k_1, k_2, k_3 + \ell_3, *, *) + v(k_1, k_2, k_3 - \ell_3, *, *).$$

$$v(*, k_{(2,3)} \pm \ell_{(2,3)}, *, *) = v(*, k_2 + \ell_2, k_3, *, *) + v(*, k_2 - \ell_2, k_3, *, *) + v(*, k_2, k_3 + \ell_3, *, *) + v(*, k_2, k_3 - \ell_3, *, *).$$

Theorem 2 *If $v(k)$ is a solution of the Eq. (5) and m is apositive integer then the following relations are equivalent:*

(a)
$$v(k) =(1 - 6\gamma)^m v(k_1, k_2, k_3, k_4 - m\ell_4, k_5 - m\ell_5) + \sum_{r=0}^{m-1} \gamma(1 - 6\gamma)^r \Big[v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4 - (r + 1)\ell_4, k_5 - (r + 1)\ell_5) \Big], \tag{8}$$

(b)
$$v(k) = \frac{1}{(1 - 6\gamma)^m} v(k_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5) - \sum_{r=1}^{m} \frac{\gamma}{(1 - 6\gamma)^r} \Big[v(k_{(1,2,3)} \pm l_{(1,2,3)}, k_4 + (r - 1)\ell_4, k_5 + (r - 1)\ell_5) \Big], \tag{9}$$

(c)
$$v(k) =\frac{1}{\gamma^m} v(k_1 - m\ell_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5) - \sum_{r=1}^{m} \frac{1 - 6\gamma}{\gamma^r} v(k_1 - r\ell_1, k_2, k_3, k_4 + (r - 1)\ell_4, k_5 + (r - 1)\ell_5) - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 - (r + 1)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5), \tag{10}$$

(d)
$$v(k) =\frac{1}{\gamma^m} v(k_1 + m\ell_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5) - \sum_{r=1}^{m} \frac{1 - 6\gamma}{\gamma^r} v(k_1 + r\ell_1, k_2, k_3, k_4 + (r - 1)\ell_4, k_5 + (r - 1)\ell_5) - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 + (r + 1)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5). \tag{11}$$

Proof From (5) and (1), we arrive

(i)

$$v(k) = (1-6\gamma)v(k_1,k_2,k_3,k_4-\ell_4,k_5-\ell_5)+ \\ \gamma[v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4-\ell_4, k_5-\ell_5),$$

(ii)

$$v(k) = \frac{1}{(1-6\gamma)}v(k_1,k_2,k_3,k_4+\ell_4,k_5+\ell_5) \\ - \frac{\gamma}{(1-6\gamma)}[v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4, k_5)],$$

(iii)

$$v(k) = \frac{1}{\gamma}v(k_1-\ell_1,k_2,k_3,k_4+\ell_4,k_5+\ell_5) - \frac{1-6\gamma}{\gamma}v(k_1-\ell_1,k_2,k_3,k_4,k_5) \\ - v(k_1-2\ell_1,k_2,k_3,k_4,k_5) - v(k_1-\ell_1,k_{(2,3)} \pm \ell_{(2,3)},k_4,k_5)\text{and}$$

(iv)

$$v(k) = \frac{1}{\gamma}v(k_1+\ell_1,k_2,k_3,k_4+\ell_4,k_5+\ell_5) - \frac{1-6\gamma}{\gamma}v(k_1+\ell_1,k_2,k_3,k_4,k_5) \\ - v(k_1+2\ell_1,k_2,k_3,k_4,k_5) - v(k_1+\ell_1,k_{(2,3)} \pm \ell_{(2,3)},k_4,k_5).$$

Now the proof of (a), (b), (c), (d) follows by replacing
k_4 and k_5 by $k_4-\ell_4, k_4-2\ell_4, ..., k_m-m\ell_4, k_5-\ell_5, k_5-2\ell_5, ..., k_m-m\ell_5$,
k_4 and k_5 by $k_4+\ell_4, k_4+2\ell_4, ..., k_m+m\ell_4, k_5+\ell_5, k_5+2\ell_5, ..., k_m-m\ell_5$,
k_1 by $k_1-\ell_1, k_1-2\ell_1, ..., k_m-m\ell_1$, k_4 by $k_4+\ell_4, k_4+2\ell_4, ..., k_m+m\ell_4$ and
k_5 by $k_5+\ell_5, k_5+2\ell_5, ..., k_m-m\ell_5$,
k_1 by $k_1+\ell_1, k_1+2\ell_1, ..., k_m+m\ell_1$, k_4 by $k_4+\ell_4, k_4+2\ell_4, ..., k_m+m\ell_4$ and
k_5 by $k_5+\ell_5, k_5+2\ell_5, ..., k_m-m\ell_5$ in (i), (ii), (iii) and (iv) respectively.

Example 1 The following example shows that the diffusion of medium in three dimensional system can be identified if the solution $v(k_1,k_2,k_3,k_4,k_5)$ of (5) is known and vice versa. Suppose that $v(k_1,k_2,k_3,k_4,k_5) = e^{k_1+k_2+k_3+k_4+k_5}$ is a closed form solution of (5), then we have the relation

$\underset{(\ell_4,\ell_5)}{\Delta} e^{k_1+k_2+k_3+k_4+k_5} = \gamma\big[\underset{\pm\ell_{(1,2,3)}}{\Delta} e^{k_1+k_2+k_3+k_4+k_5}]$, which yields

$$e^{k_1+k_2+k_3+k_4+k_5}(e^{\ell_4+\ell_5}-1) = \\ \gamma\big[e^{k_1+k_2+k_3+k_4+k_5}(e^{\ell_1}+e^{-\ell_1}+e^{\ell_2}+e^{-\ell_2}+e^{\ell_3}+e^{-\ell_3}-6\big].$$

Cancelling $e^{k_1+k_2+k_3+k_4+k_5}$ on both sides derives

$$\gamma = \frac{e^{\ell_4+\ell_5}-1}{e^{\ell_1}+e^{-\ell_1}+e^{\ell_2}+e^{-\ell_2}+e^{\ell_3}+e^{-\ell_3}-6}. \tag{12}$$

For numerical verification, if we assume that $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5$, $\ell_1 = 1, \ell_2 = 2, \ell_3 = 3, \ell_4 = 4, \ell_5 = 5, m = 1$ then $v(k_1, k_2, k_3, k_4, k_5) = e^{15}$,

$$\gamma = \frac{e^{4+5}-1}{e^{1}+e^{-1}+e^{2}+e^{-2}+e^{3}+e^{-3}-6}.$$

LHS and RHS of (a), (b) of Theorem 2 are given below respectively.

(a) $3269017.37 = (-1963.46857)403.42879 + 4061136.705.$

(b) $3269017.37 = -13490909.90 + 16759999.00.$

If we assume that $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5, \ell_1 = 1, \ell_2 = 2, \ell_3 = 3$, $\ell_4 = 4, \ell_5 = 5, m = 1$ then $v(k_1, k_2, k_3, k_4, k_5) = e^{15}$,

$$\gamma = \frac{e^{1+2}-1}{e^{3}+e^{-3}+e^{4}+e^{-4}+e^{5}+e^{-5}-6}.$$

LHS and RHS of (c), (d) of Theorem 2 are as similar as above (a), (b).
For MATLAB coding, if we assume that $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$, $k_5 = 5$, $\ell_1 = 1$ and $\ell_2 = 2$, $\ell_3 = 3$, $\ell_4 = 4$, $\ell_5 = 5$, $m = 5$ then
$exp(15) = (1 - 6. * (327.4114733)). \wedge (5). * exp(-30) + symsum$ $((327.4114733). *(1 - 6. * (327.4114733)). \wedge r. * ((exp(16 - (r + 1). * 4 - (r + 1). * 5)) + (exp(14 - (r + 1). * 4 - (r + 1). * 5)) + (exp(17 - (r + 1). * 4 - (r + 1). * 5)) + (exp(13 - (r + 1). * 4 - (r + 1). * 5)) + (exp(18 - (r + 1). * 4 - (r + 1). * 5)) + (exp(12 - (r + 1). * 4 - (r + 1). * 5))), r, 0, 4).$

4 Conclusion

The study of partial difference operator has wide applications in discrete fields and heat equation is one such. The nature of propagation of heat through materials of dimensions(up to three) can be postulated.

The core Theorem 2 provides the possibility of predicting the temperature either for the past or the future after getting to know the temperature at few finite points on the material at the present time.

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A Note on the Existence for a Model of Turbulent Flows Through Porous Media

Hermenegildo Borges de Oliveira

Abstract In this work, turbulent flows through porous media are considered. We begin by making a historical review of the equations governing laminar flows in porous media, from Darcy's law to Darcy–Brinkman–Forchheimer's more general model. Using the double averaging concept (in time and in space) we explain how to obtain the more general system of equations that governs turbulent flows through porous media. For the one-equation turbulent problem in the steady-state we show that the known existence results can be generalized to any space dimension $d \geq 2$ and for a more general function of turbulence production.

Keywords Turbulence · k−epsilon modelling · Porous media · General existence

1 Turbulent Flows Through Porous Media

Fluid flows through porous media are usually described by Darcy's law [1], an empirical flow model that represents a simple linear relationship between flow rate and the pressure drop in a porous media. Today, Darcy's law reads

$$\mathbf{u} = -\frac{\mathbf{K}}{\mu}\nabla p \quad \Leftrightarrow \quad \mathbf{0} = -\mathbf{K}\nabla p - \mu\mathbf{u}\,, \tag{1}$$

where $\mathbf{u}$ is the fluid velocity field, p is the pressure and μ is the fluid (dynamic) viscosity that was only observed and included in Darcy's law later on by Hazen [2]. The tensor $\mathbf{K}$, called permeability, is independent of the nature of the fluid but it depends on the pore size, the porosity, and also on the geometry of the medium. In particular, $\mathbf{K}$ reduces to a scalar K if the medium is isotropic. The Darcy law assumes no effect of boundaries and the fluid velocity in Darcy's equation is determined by the permeability of the matrix. If the boundary is impermeable, then the usual assumption is that the normal component of the velocity must vanish: $\mathbf{u} \cdot \mathbf{n} = \mathbf{0}$ on the solid-fluid

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interface, where $\mathbf{n}$ is the unit normal. At a solid wall boundary, the fluid velocity will not reduce to the no-slip condition when the Darcy law is enforced. In this situation, the Brinkman law [3] may be employed, which is an extension of the Darcy law and facilitates the matching of boundary conditions,

$$\nabla p = -\frac{\mu}{K}\mathbf{u} + \mu_e \Delta\mathbf{u} \quad \Leftrightarrow \quad \mathbf{0} = -\nabla p - \frac{\mu}{K}\mathbf{u} + \mu_e \Delta\mathbf{u}\,, \tag{2}$$

where μ_e is the effective fluid viscosity, a function of the fluid viscosity and of the geometry of the medium. Equations (1) and (2) describe well porous media flows at sufficiently small velocities. But, for larger values of $\mathbf{u}$ there is a breakdown in the linearity of these equations which is owing to the fact that the form-drag due to solid obstacles is now comparable with the surface drag due to friction. In this case, Dupuit–Forchheimer's law [4, 5] remedies the situation by stating that the relationship between the flow rate and pressure gradient is nonlinear at sufficiently high velocity and that this nonlinearity increases with the flow rate. According to many authors (see e.g. Joseph et al. [6]), the appropriate modification of Darcy's law, to take into account high flow rates, is to replace (1) by the following Dupuit-Forchheimer equation,

$$\nabla p = \rho\mathbf{g} - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} \quad \Leftrightarrow \quad \mathbf{0} = \rho\mathbf{g} - \nabla p - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u}\,, \tag{3}$$

where ρ is the fluid density and $c_{\mathbf{F}}$ is a dimensionless form-drag constant. Several authors (see e.g. Nakayama [7] and Kuznetsov [8]) have added, in their studies, a diffusion term to (3) in order to form a Brinkman–Dupuit–Forchheimer model,

$$\nabla p = -\frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} + \mu_e \Delta\mathbf{u} \quad \Leftrightarrow \quad \mathbf{0} = -\nabla p - \frac{\mu}{K}\mathbf{u} - \frac{c_F \rho}{\sqrt{K}}|\mathbf{u}|\mathbf{u} + \mu_e \Delta\mathbf{u}. \tag{4}$$

Drawing a parallel between Eqs. (2) and (4) and the Navier–Stokes equations for creep flow may led to misleading interpretations. For instance, the pressure in Eqs. (2) and (4) represents a force per unit of permeable area, including solid and fluid, while the pressure in the Navier–Stokes equations is a force per unit area of fluid only – the same is true also for the fluid velocities. However, if we confine ourselves to the pore scale (microscopic scale), the flow quantities can be determined by the incompressible Navier–Stokes equations (for homogeneous fluids)

$$\operatorname{div}\mathbf{u} = 0, \tag{5}$$

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{g} - \frac{1}{\rho}\nabla p + \nu\,\mathbf{div}(\mathbf{D}(\mathbf{u}))\,, \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \tag{6}$$

where ν is the kinematic viscosity and $\mathbf{g}$ is the gravity forces field. If the boundary is impermeable, then, as we already have seen, $\mathbf{u}\cdot\mathbf{n} = \mathbf{0}$ on the solid-fluid interface. But, contrary to the Darcy flow model (the maximum velocity occurs at the impermeable surface), the no-slip boundary condition can be used in this case: $\mathbf{u}\cdot\boldsymbol{\tau} = \mathbf{0}$

on the solid-fluid interface, where $\boldsymbol{\tau}$ is the unit tangent. The problem of considering (5)–(6) is that, due to the complexity of internal geometries and interfacial structures, it is impractical to solve the microscopic Eqs. (5)–(6) inside the pores. A common approach is to average the microscopic equations inside porous medium over a representative elementary volume (REV). REV is the smallest volume over which a measurement can be made that will yield a value representative of the whole domain (including fluid and solid). The volumetric average of the microscopic Eqs. (5)–(6), under the assumption of a rigid, isotropic and fixed porous matrix, results (cf. Hsu and Cheng [9]) on the following macroscopic equations,

$$\operatorname{div} \mathbf{u}_f = 0; \tag{7}$$

$$\frac{\partial \mathbf{u}_f}{\partial t} + \mathbf{div}\left(\frac{1}{\phi}\mathbf{u}_f \otimes \mathbf{u}_f\right) = \mathbf{g}_f - \frac{1}{\rho_f}\nabla p_f + \frac{\mu_f}{\rho_f}\mathbf{div}\left(\mathbf{D}\left(\mathbf{u}_f\right)\right) + \frac{1}{\rho_f}(\mathbf{H}+\mathbf{R})s, \tag{8}$$

where $\mathbf{u}_f = \phi\,\langle \mathbf{u}\rangle^i$, $p_f = \phi\,\langle p\rangle^i$, $\mathbf{g}_f = \phi\,\langle \mathbf{g}\rangle^i$, $\rho_f = \phi\,\langle \rho\rangle^i$ and $\mu_f = \phi\,\langle \mu\rangle^i$ are (fluid) phase averages and $\phi = \frac{V_f}{V}$ is the local medium porosity. For instance, $\langle \mathbf{u}\rangle^i := \frac{1}{V_f}\int_{V_f} \mathbf{u}\, dV$ is the intrinsic (fluid) average of the fluid phase velocity $\mathbf{u}$ over the fluid domain V_f contained in the representative elementary volume V. Fluid velocities $\mathbf{u}$ and $\langle \mathbf{u}\rangle^i$ are related through $\mathbf{u} = \langle \mathbf{u}\rangle^i +^i \mathbf{u}$, where ${}^i\mathbf{u}$ is the spatial deviation of $\mathbf{u}$ with respect to $\langle \mathbf{u}\rangle^i$. In the momentum equation (8), $\mathbf{H}$ and $\mathbf{R}$ represent, respectively, the hydrodynamic dispersion due to spatial deviations and the total drag force per unit volume due to the presence of the porous matrix,

$$\mathbf{H} = -\mathbf{div}\left(\phi\left\langle {}^i\mathbf{u} \otimes^i \mathbf{u}\right\rangle^i\right), \qquad \mathbf{R} = -\frac{\mu_f}{K}\mathbf{u}_f - \frac{c_F}{\sqrt{K}}\rho_f|\mathbf{u}_f|\mathbf{u}_f.$$

In the applications, the choice of the flow equations to model porous media flows, within similar flow conditions, is usually based on the pore Reynolds number $Re_p := \frac{\rho q D}{\mu}$, where q is the specific discharge and D is some representative (microscopic) diameter characterizing the void space (see e.g. Dybbs and Edwards [10]). In particular, $Re_p \leq 1$ holds when $\mathbf{u}$ is sufficiently small and therefore the flow equation is linear in the velocity. In this case, the flow is well described by one of the Eq. (1) or (2) and the dominated flow regime is called Darcy or viscous-drag. As $\mathbf{u}$ increases, the transition to nonlinear drag is quite smooth as long as $1 < Re_p \leq 10$ and the breakdown in the linearity of $\mathbf{u}$ occurs when $Re_p > 10$. If $1 \sim 10 < Re_p < 150$, the dominated flow regime is called Forchheimer or form-drag and the flow can be described by one of the models (3) or (4). By using the local volume averaging, some authors (e.g. Vafai and Kim [11]) have added to the Eq. (4) the advective inertia terms of the Navier–Stokes equations to model some situations of form-drag flows. For $Re_p > 150$, the flow regime is called post-Forchheimer and almost works in the literature consider, in this case, the local volume average of the Navier–Stokes equations to form what is now known as the Brinkman–Forchheimer-extended Darcy model (or generalized model). In particular, if $150 < Re_p < 300$ the flow regime is still laminar but unsteady and the time inertia terms need to be considered. If $Re_p > 300$, the flow becomes fully turbulent and therefore turbulence modelling is required. With this regard, it should be mentioned that two main differences exist between turbulent flow through porous media and turbulent flow in the absence of a porous matrix. By one hand, the size of the turbulent eddies within the pores is limited by the pore size. On the other, the presence of a porous matrix induces additional drag while preventing motion of larger size eddies. To model turbulent flows through porous media, it is usually considered the turbulent $k-$epsilon model which is obtained by time-averaging the incompressible Navier–Stokes equations (5) and (6),

$$\operatorname{div}\overline{\mathbf{u}} = 0, \tag{9}$$

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + \mathbf{div}(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}) = \overline{\mathbf{g}} - \frac{1}{\rho}\nabla\overline{p} + \mathbf{div}\left((\nu + \nu_T(k,\varepsilon))\mathbf{D}(\overline{\mathbf{u}})\right), \tag{10}$$

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}}\cdot\nabla k = \operatorname{div}\left(\nu_D(k,\varepsilon)\nabla k\right) + \nu_T(k,\varepsilon)|\mathbf{D}(\overline{\mathbf{u}})|^2 - \varepsilon, \tag{11}$$

$$\frac{\partial\varepsilon}{\partial t} + \overline{\mathbf{u}}\cdot\nabla\varepsilon = \operatorname{div}\left(\nu_D(k,\varepsilon)\nabla\varepsilon\right) + C_1 k|\mathbf{D}(\overline{\mathbf{u}})|^2 + C_2\frac{\varepsilon^2}{k}. \tag{12}$$

Here, $\overline{\mathbf{u}}$, $\overline{p}$ and $\overline{\mathbf{g}}$ denote the time averaged velocity, pressure and external forces, whereas k is the turbulent kinetic energy and ε expresses the turbulent dissipation. The averaged quantities result from their Reynolds decomposition, for instance $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, where $\overline{\mathbf{u}} = \frac{1}{\Delta t}\int_t^{t+\Delta t}\mathbf{u}\,dt$ is the time averaged velocity, being Δt small when compared with the magnitude of fluctuations $\mathbf{u}'$ of $\overline{\mathbf{u}}$. The functions ν_T and $\nu_D = \frac{\nu_T}{\sigma_k}$ in (9)–(12) account for the turbulent viscosity and turbulent diffusivity, where σ_k is the Schmidt-Prandtl number, and C_1, C_2 are positive constants that can be determined from the experiments. The consideration of one-equation models is acceptable in the sense that the equation for ε may be discarded by prescribing an appropriate length scale l,

$$\operatorname{div}\overline{\mathbf{u}} = 0, \tag{13}$$

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + \mathbf{div}(\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}) = \overline{\mathbf{g}} - \frac{1}{\rho}\nabla\overline{p} + \mathbf{div}\left((\nu + \nu_T(k)\mathbf{D})(\overline{\mathbf{u}})\right), \tag{14}$$

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}}\cdot\nabla k = \operatorname{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)|\mathbf{D}(\overline{\mathbf{u}})|^2 - \varepsilon(k), \quad \varepsilon(k) = \frac{C_D}{l}k^{\frac{3}{2}}. \tag{15}$$

See e.g. Chacón-Rebollo and Lewandowski [12] and Lemos [13] for the derivation of the turbulent k-epsilon model (see also Oliveira and Paiva [14]). From a broad perspective, for high pore Reynolds number ($Re_p > 300$), turbulent models presented in the literature follow two different approaches. In both developments, the porous medium is considered to be rigid, fixed, isotropic and saturated by an incompressible fluid, and both techniques aim to derive suitable macroscopic transport equations. The first method (see Getachewa et al. [15] and the references cited therein), starts with the volume average of the microscopic equations and then the macroscopic equations are averaged in time. However, some works (see e.g. Antohe and Lage [16]) have concluded that turbulent models derived directly from the general macroscopic equations do not accurately characterize turbulence induced by the porous matrix. The second approach (see Nakayama and Kuwahara [17] and the references cited therein), makes use, first, of the time averaged equations, and then proceeds with volume averaging. In this case, the governing equations are obtained by volume averaging the microscopic Reynolds-averaged equations (13)–(15),

$$\operatorname{div}\, \langle \overline{\mathbf{u}} \rangle^i = 0, \tag{16}$$

$$\frac{\partial \langle \overline{\mathbf{u}} \rangle^i}{\partial t} + \mathbf{div}\big(\langle \overline{\mathbf{u}} \rangle^i \otimes \langle \overline{\mathbf{u}} \rangle^i\big) = \langle \overline{\mathbf{g}} \rangle^i - \frac{1}{\rho_f} \nabla \overline{\pi} + \mathbf{div}\big[\big(\nu_f + \nu_T\big)\, \mathbf{D}\left(\langle \overline{\mathbf{u}} \rangle^i\right)\big] + \overline{\mathbf{R}}, \tag{17}$$

$$\frac{\partial \langle k \rangle^i}{\partial t} + \langle \overline{\mathbf{u}} \rangle^i \cdot \nabla \langle k \rangle^i = \operatorname{div}\left[\big(\nu_f + \nu_D\big)\, \nabla \langle k \rangle^i\right] + 2\nu_T \left|\mathbf{D}\left(\langle \overline{\mathbf{u}} \rangle^i\right)\right|^2 - \langle \varepsilon \rangle^i + P. \tag{18}$$

Here, $\overline{\pi} = \overline{\langle p \rangle^i} + \frac{2}{3}\rho_f \langle k \rangle^i, \nu_f = \frac{\mu_f}{\rho_f}$, $\overline{\mathbf{R}}$ represents the time averaged total drag forces and P accounts for the production of turbulence due to solid obstacles inside the porous domain. The main features of Nakayama and Kuwahara's model are that the hydrodynamic dispersion was incorporated in the drag forces and the additional term P appearing in the governing equation for $\langle k \rangle^i$ (and also in the equation for $\langle \varepsilon \rangle^i$), is determined by using two unknown model constants,

$$\overline{\mathbf{R}} = -\phi \left(\frac{\nu_f}{K} \langle \overline{\mathbf{u}} \rangle^i - \frac{c_F}{\sqrt{K}} \phi |\, \langle \overline{\mathbf{u}} \rangle^i \,|\, \langle \overline{\mathbf{u}} \rangle^i \right), \quad P = \frac{39\phi^2 \sqrt[5]{(1-\phi)^2}}{d} |\langle \overline{\mathbf{u}} \rangle^i|^3.$$

Following a slight different approach, Pedras and Lemos [18] obtained

$$\operatorname{div}\, \langle \overline{\mathbf{u}} \rangle^i = 0, \tag{19}$$

$$\frac{\partial \langle \overline{\mathbf{u}} \rangle^i}{\partial t} + \mathbf{div}\big(\langle \overline{\mathbf{u}} \rangle^i \otimes \langle \overline{\mathbf{u}} \rangle^i\big) = \langle \overline{\mathbf{g}} \rangle^i - \frac{1}{\rho_f} \nabla \overline{\pi} + \mathbf{div}\big[\big(\nu_f + \nu_{T_\phi}\big)\, \mathbf{D}\left(\langle \overline{\mathbf{u}} \rangle^i\right)\big] + \overline{\mathbf{R}}, \tag{20}$$

$$\frac{\partial \langle k \rangle^i}{\partial t} + \langle \overline{\mathbf{u}} \rangle^i \cdot \nabla \langle k \rangle^i = \operatorname{div}\left[\big(\nu_f + \nu_{D_\phi}\big) \nabla \langle k \rangle^i\right] + 2\nu_T \left|\mathbf{D}\left(\langle \overline{\mathbf{u}} \rangle^i\right)\right|^2 - \langle \varepsilon \rangle^i + P\,. \tag{21}$$

In this case, the total drag term $\overline{\mathbf{R}}$ is only closed after all the equations are obtained and the additional term that is included in the equation for $\langle k \rangle^i$ to account for the porous structure is defined through

$$P = \frac{c_k \phi^3}{\sqrt{K}} \langle k \rangle^i \,|\, \langle \overline{\mathbf{u}} \rangle^i |.$$

Moreover, to model the Reynolds stresses it is proposed a macroscopic Boussinesq assumption: $\left\langle \overline{\mathbf{u}}' \otimes \mathbf{u}' \right\rangle_i = \frac{2}{3} \langle k \rangle_i \mathbf{I} - \nu_{T_\phi} \langle \mathbf{D}(\overline{\mathbf{u}}) \rangle_i$, where ν_{T_ϕ} and ν_{D_ϕ} denote the macroscopic turbulent viscosity and the macroscopic turbulent diffusivity, which satisfy to $\nu_{T_\phi} \mathbf{D}\left(\langle \overline{\mathbf{u}} \rangle^i\right) = \langle \nu_T \mathbf{D}(\overline{\mathbf{u}}) \rangle^i$ and $\nu_{D_\phi} = \frac{\nu_{T_\phi}}{\sigma_k}$. From the mathematical point of view, the main difference between systems (16)–(18) and (19)–(21) relies on the production of turbulence term, denoted by P at Eqs. (18) and (21). This term, that appears as an output of the averaging process, is a production term of turbulent kinetic energy and gives account of the solids inside the fluid. Note that different approaches or distinct assumptions led to different diffusivity functions between Eqs. (18), (21) and (15).

2 The Problem Under Consideration

Motivated by the systems of equations (16)–(18) and (19)–(21), we study, in this work, a one-equation turbulent model for the description of incompressible fluids within a fluid-saturated and rigid porous medium, which for simplicity is also assumed to be fixed, with a constant porosity function ϕ, and isotropic. The problem is assumed to be governed by the following general set of equations in the steady-state,

$$\mathbf{div}\, \mathbf{u} = 0 \quad \text{in } \Omega, \tag{22}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \nabla p + \mathbf{div}\left((\nu + \nu_T(k))\, \mathbf{D}(\mathbf{u})\right) \quad \text{in } \Omega, \tag{23}$$

$$\mathbf{u} \cdot \nabla k = \operatorname{div}\left((\nu + \nu_D(k))\nabla k\right) + \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 + P(\mathbf{u}, k) - \varepsilon(k) \quad \text{in } \Omega\,. \tag{24}$$

Here, Ω denotes the porous domain in consideration and the velocity field $\mathbf{u}$, the pressure p and the external forces field $\mathbf{g}$ are, in fact, averages that result by the application of the averaging procedures that lead us to (16)–(18) and (19)–(21). The feedback terms $\mathbf{f}(\mathbf{u})$ and $P(\mathbf{u}, k)$ (up to the minus sign in the first case) represent the total drag $\overline{\mathbf{R}}$ and the turbulence production considered in these systems: $\mathbf{f}(\mathbf{u}) = C_D \mathbf{u} + C_F |\mathbf{u}|\mathbf{u}$ and $P(\mathbf{u}, k) = C_1 |\mathbf{u}|^3$ in (16)–(18), or $P(\mathbf{u}, k) = C_2 |\mathbf{u}| k$ in (19)–(21), where C_D, C_F, C_1 and C_2 are the correspondingly multiplicative constants in the mentioned turbulent models. We supplement Eqs. (22)–(24) with Dirichlet homogeneous boundary conditions,

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega, \tag{25}$$

where $\partial\Omega$ denotes the rigid boundary of the porous domain Ω. Problems (22)–(25) with the smaller difference of the term $\operatorname{div}\left((\nu + \nu_D(k))\nabla k\right)$ replaced by $\operatorname{div}\left(\nu_D(k)\nabla k\right)$, was considered by Oliveira and Paiva [19, 20], where it was proved the existence and uniqueness of weak solutions in the dimensions of physics interest $d = 2,\ 3$ and also for $d = 4$. Due to the mathematical interest, we shall consider now the problems (22)–(25) in a general dimension d, i.e. we assume that Ω is a bounded subdomain of $\mathbb{R}^d$ for a general $d \geq 2$. Our aim in the rest of the paper, is to show that the existence results of [19] can be suitable adapted to hold for any dimension $d \geq 2$. In the mathematical treatment of the turbulence problems (22)–(25), there is a set of usual assumptions that although do not follow from the real situation they are physically admissible,

$$\mathbf{f} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \quad \varepsilon,\ \nu_T,\ \nu_D : \Omega \times \mathbb{R} \to \mathbb{R}, \quad P : \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \quad (26)$$

are Carathéodory functions. Observe that we are considering the possibility that all the functions $\mathbf{f}$, P, ε, ν_T and ν_D may also depend on the space variable. In particular, assumption (26) fits with turbulent dissipation, turbulent viscosity and turbulent diffusion functions involved in realistic models (see e.g. [12]). There is another set of assumptions that impose some restrictions on the physics of the problem, but are mathematically needed. We assume the existence of positive constants C_T and C_D such that

$$0 \leq \nu_T(k) \leq C_T, \quad 0 \leq \nu_D(k) \leq C_D \quad \text{for all } k \in \mathbb{R} \text{ and a.e. in } \Omega. \quad (27)$$

Definition 1 Let the conditions (26) and (27) be fulfilled and assume that $\mathbf{g} \in \mathbf{V}'$. We say a pair $(\mathbf{u}, k)$ is a weak solution to the problems (22)–(25), if: (1) $\mathbf{u} \in \mathbf{V}$ and for every $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^d(\Omega)$ there hold $\mathbf{f}(\mathbf{u}) \cdot \mathbf{v} \in \mathbf{L}^1(\Omega)$ and

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k))\, \mathbf{D}(\mathbf{u}) : \nabla \mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}) \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v}\, d\mathbf{x}; \quad (28)$$

(2) $k \in \mathrm{W}_0^{1,q}(\Omega)$, with $\frac{2d}{d+2} \leq q < d'$, and for every $\varphi \in \mathrm{W}_0^{1,q'}(\Omega)$ there hold $\varepsilon(k)$, φ, $P(\mathbf{u}, k)\, \varphi \in \mathrm{L}^1(\Omega)$ and

$$\begin{aligned} &\int_\Omega (\mathbf{u} \cdot \nabla k)\varphi\, d\mathbf{x} + \int_\Omega (\nu + \nu_D(k)) \nabla k \cdot \nabla \varphi\, d\mathbf{x} + \int_\Omega \varepsilon(k)\, \varphi\, d\mathbf{x} = \\ &\int_\Omega \nu_T(k) |\mathbf{D}(\mathbf{u})|^2 \varphi\, d\mathbf{x} + \int_\Omega P(\mathbf{u}, k)\, \varphi\, d\mathbf{x}; \end{aligned} \quad (29)$$

(3) $k \geq 0$ and $\varepsilon(k) \geq 0$ a.e. in Ω.

The notation and the function spaces we use in this work are well known (see e.g. Galdi [21]). In particular, $\mathcal{V} := \{\mathbf{V} \in \mathbf{C}_0^\infty(\Omega) : \mathbf{div}\, \mathbf{v} = 0\}$, $\mathbf{H} :=$ closure of $\mathcal{V}$ in $\mathbf{L}^2(\Omega)$, $\mathbf{V} :=$ closure of $\mathcal{V}$ in $\mathbf{H}^1(\Omega)$, $\mathbf{V}'$ denotes the dual space of $\mathbf{V}$ and $\mathrm{v} :=$ closure of $\mathrm{C}_0^\infty(\Omega)$ in $\mathrm{H}^1(\Omega)$. Observe that, in the case of $d \leq 4$, the Sobolev imbedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^d(\Omega)$ holds and therefore it is only needed to require the test functions of (28) are in the function space $\mathbf{V}$. In this case ($d \leq 4$), it were proved, in [19, 20], existence results to the problems (22)–(25) under different conditions on the feedback functions $\mathbf{f}(\mathbf{u})$, $\varepsilon(k)$ and $P(\mathbf{u}, k)$. However, in the case of $d > 4$, requiring the test functions are also in $\mathbf{L}^d(\Omega)$ will cause the conditions to prove these existence results to be improved. In this section, we assume for any space dimension $d \geq 2$ the existence of nonnegative constants C_f and C_ε such that the following growth conditions are satisfied a.e. in Ω,

$$|\mathbf{f}(\mathbf{u})| \leq C_f |\mathbf{u}|^\alpha \quad \text{for } 0 \leq \alpha \leq \max\left\{\frac{d+2}{d-2}, \frac{2d-2}{d-2}\right\} \text{ if } d \neq 2, \quad \text{or } \alpha \geq 0 \text{ if } d = 2, \tag{30}$$

$$|\varepsilon(k)| \leq C_\varepsilon |k|^\theta \quad \text{for } 0 \leq \theta < \frac{d}{d-2} \text{ if } d \neq 2, \quad \text{or } \theta \geq 0 \text{ if } d = 2. \tag{31}$$

On the production term $P(\mathbf{u}, k)$, we assume the existence of a positive constant C_P such that

$$|P(\mathbf{u}, k)| \leq C_P |\mathbf{u}|^\beta |k|^\vartheta \quad \text{a.e. in } \Omega \tag{32}$$

for

$$\left.\begin{array}{l} \vartheta = 0 \quad \text{and} \quad \beta \leq \dfrac{d+2}{d-2}, \quad \text{or} \\ 0 < \vartheta \leq 1 \quad \text{and} \quad \beta + \vartheta \leq \dfrac{d+2}{d-2} \quad \text{and} \quad \beta + 2\vartheta < \dfrac{2d}{d-2} \end{array}\right\} \text{ if } d \neq 2, \qquad \text{or} \quad \beta \in [0, \infty), \ \vartheta \in [0, 1] \quad \text{if } d = 2. \tag{33}$$

In the sequel we shall consider our analysis only for the cases $d \neq 2$, because for $d = 2$ the reasoning is easier. In this case, observe that $\frac{d+2}{d-2} \geq \frac{2d-2}{d-2}$ holds in (30) as long as $d \leq 4$. Taking this into account, we note that in the particular case of $d \leq 4$ and of only $\vartheta = 0$ or $\vartheta = 1$, we fall in the exact growth conditions of the existence result established in [19, Thorem 3.1]. Additionally to the growth conditions (30)–(33), we assume the following sign conditions,

$$\mathbf{f}(\mathbf{u}) \cdot \mathbf{u} \geq 0 \quad \text{and} \quad \varepsilon(k)\, k \geq 0 \quad \text{a.e. in } \Omega \tag{34}$$

for all $\mathbf{u} \in \mathbb{R}^d$ and all $k \in \mathbb{R}$, respectively. We consider, in this work, that our general turbulent dissipation function can be written in such a way that

$$\varepsilon(k) = k e(k) \qquad \text{where} \quad e : \Omega \times \mathbb{R} \to \mathbb{R}_0 \text{ is a Carathéeodory function.} \tag{35}$$

Gathering the information of (34) and (35) it follows immediately that $e(k) \geq 0$ for all $k \in \mathbb{R}$ and a.e. in Ω. To avoid the trivial solution $k = 0$, we shall assume in the sequel, and in addition to (27), that $\nu_T(k) \neq 0$ when $k = 0$.

3 Existence

Theorem 1 *Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume all the conditions (26), (27), (30), (31), (34) and (35) hold. If*

$$\mathbf{g} \in \mathbf{L}^2(\Omega), \tag{36}$$

and if (32) and (33) hold but, in the case of $0 < \vartheta \leq 1$, with the extra assumption that

$$\nu > C\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}^{\frac{\beta}{1+\beta}}, \quad \text{with } C \text{ defined at (48)}, \tag{37}$$

then there exists, at least, a weak solution to the problems (22)–(25).

The rest of the section is devoted to prove Theorems 1. We start by considering, for each $n \in \mathbb{N}$, the following regularized problem

$$\mathbf{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{38}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \nabla p + \mathbf{div}\,((\nu + \nu_T(k))\,\mathbf{D}(\mathbf{u})) \quad \text{in } \Omega, \tag{39}$$

$$\mathbf{u}\cdot\nabla k = \operatorname{div}\,((\nu + \nu_D(k))\nabla k) + \nu_T(k)\,\mathcal{R}_n\big(|\mathbf{D}(\mathbf{u})|^2\big) + P(\mathbf{u},k) - \varepsilon(k) \quad \text{in } \Omega, \tag{40}$$

$$\mathbf{u} = 0 \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega, \tag{41}$$

where $\mathcal{R}_n\big(|\mathbf{D}(\mathbf{u})|^2\big) := \frac{|\mathbf{D}(\mathbf{u})|^2}{1+\frac{1}{n}|\mathbf{D}(\mathbf{u})|^2}$. Under the assumptions of Definition 1, we say a pair $(\mathbf{u}, k)$ is a weak solution to the regularized problem (38)–(41) if, for each $n \in \mathbb{N}$, (1) and (3) of Definition 1 hold, and: (2') $k \in \mathrm{H}_0^1(\Omega)$ and for every $\varphi \in \mathrm{H}_0^1(\Omega) \cap \mathrm{L}^d(\Omega)$ there holds (29'), i.e. (29) with $\mathcal{R}_n\big(|\mathbf{D}(\mathbf{u})|^2\big)$ in the place of $|\mathbf{D}(\mathbf{u})|^2$. Observe again that, as we have mentioned for the test functions in (28), due to the Sobolev imbedding $\mathrm{H}_0^1(\Omega) \hookrightarrow \mathrm{L}^d(\Omega)$ it would only be needed to require the test functions of (29') are in the function space $\mathrm{H}_0^1(\Omega)$ in the case of $d \leq 4$. The existence of a weak solution to the problem (38)–(41) is established in the following proposition.

Proposition 1 *Let the conditions of Theorem 1 be fulfilled. Then (for each $n \in \mathbb{N}$) there exists, at least, a weak solution to the problems (38)–(41).*

Proof For each $j \in \mathbb{N}$, we search for the Galerkin approximations $\mathbf{u}_j = \sum_{i=1}^{j} c_{ij}\mathbf{v}_i$ and $k_j = \sum_{i=1}^{j} d_{ij}\upsilon_i$, solutions to the system formed by (28) and of (29'), where $c_{ij},\ d_{ij} \in \mathbb{R}$, $\mathbf{v}_i \in \mathbf{V}^j$, $\upsilon_i \in V^j$, and $\mathbf{V}^j$, V^j are j-dimensional subspaces of $\mathbf{V}^s :=$ closure of $\mathcal{V}$ in $\mathrm{W}^{s,2}(\Omega)$ and of $V^r :=$ closure of $C_0^\infty(\Omega)$ in $\mathrm{W}^{r,2}(\Omega)$, being s and r the smallest integers such that $s,\ r \geq \frac{d}{2}$. Note that in the case of $d \leq 4$, we may let $r,\ s = 1$ and replace $\mathbf{V}^s$ and V^r by the function spaces $\mathbf{V}$ and V defined above. Functions $\mathbf{u}_j$ and k_j are found by solving the following system of $2j$ nonlinear algebraic equations, with respect to the $2j$ unknowns $c_{1j},\ c_{2j},\ \ldots,\ c_{jj}$ and $d_{1j},\ d_{2j},\ \ldots,\ d_{jj}$,

$$\int_\Omega ((\mathbf{u}_j \cdot \nabla)\mathbf{u}_j) \cdot \mathbf{v}_i \, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k_j))\mathbf{D}(\mathbf{u}_j) : \nabla \mathbf{v}_i \, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_j) \cdot \mathbf{v}_i \, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v}_i \, d\mathbf{x}, \tag{42}$$

$$\int_\Omega (\mathbf{u}_j \cdot \nabla k_j)\upsilon_i \, d\mathbf{x} + \int_\Omega (\nu + \nu_D(k_j))\nabla k_j \cdot \nabla \upsilon_i \, d\mathbf{x} + \int_\Omega \varepsilon(k_j)\upsilon_i \, d\mathbf{x} =$$
$$\int_\Omega \nu_T(k_j)\,\mathcal{R}_n\big(|\mathbf{D}(\mathbf{u}_j)|^2\big)\,\upsilon_i \, d\mathbf{x} + \int_\Omega P(\mathbf{u}_j, k_j)\,\upsilon_i \, d\mathbf{x}, \tag{43}$$

for $i = 1, \ldots, j$. To prove the existence of, at least, a solution to the system (42) and (43), we consider a function $\mathcal{P}$, from $\mathbf{V}^j \times V^j$ into itself defined in such a way that

$$\mathcal{P}(\mathbf{v}, \upsilon) \cdot (\mathbf{v}, \upsilon) = I_1 + \cdots - I_4 + \cdots - I_8 - I_9 := \int_\Omega ((\mathbf{v} \cdot \nabla)\mathbf{v}) \cdot \mathbf{v} \, d\mathbf{x} +$$
$$\int_\Omega (\nu + \nu_T(\upsilon))\mathbf{D}(\mathbf{v}) : \nabla \mathbf{v} \, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{v}) \cdot \mathbf{v} \, d\mathbf{x} - \int_\Omega \mathbf{g} \cdot \mathbf{v} \, d\mathbf{x} + \int_\Omega (\mathbf{v} \cdot \nabla \upsilon)\upsilon \, d\mathbf{x} +$$
$$\int_\Omega (\nu + \nu_D(\upsilon))|\nabla \upsilon|^2 \, d\mathbf{x} + \int_\Omega \varepsilon(\upsilon)\upsilon \, d\mathbf{x} - \int_\Omega \nu_T(\upsilon)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{v})|^2\Big)\,\upsilon \, d\mathbf{x} - \int_\Omega P(\mathbf{v}, \upsilon)\,\upsilon \, d\mathbf{x}$$

for all $(\mathbf{v}, \upsilon) \in \mathbf{V}^j \times V^j$ and where the scalar product is induced by $\mathbf{V} \times V$. Reasoning as we did in the proof of [19, Theorem 3.1], it can be proved that $I_1 = 0$ and $I_5 = 0$, $I_2 \geq \nu\, C_K^2 \|\nabla \mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}$, I_3, $I_7 \geq 0$ and $I_6 \geq \nu \|\nabla \upsilon\|^2_{\mathrm{L}^2(\Omega)}$, $I_4 \leq \Lambda_P(d)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)}$ and $I_8 \leq C_T\, n\sqrt{\mathcal{L}^d(\Omega)}\lambda(2, d)\|\nabla \upsilon\|_{\mathrm{L}^2(\Omega)}$, where C_K is the Korn's inequality constant, $\lambda(2, d)$ and $\Lambda_P(d)$ are the best constants of the scalar and vectorial Sobolev inequalities. For the term I_9, we argue similarly as in the previous reference, to show that

$$I_9 \leq C_P\, \lambda(2, d)^{1+\vartheta}\, \Lambda(2, d)^\beta \|\nabla \mathbf{v}\|^\beta_{\mathbf{L}^2(\Omega)} \|\nabla \upsilon\|^{1+\vartheta}_{\mathrm{L}^2(\Omega)}, \quad \beta + \vartheta \leq \frac{d+2}{d-2}. \tag{44}$$

Then, gathering the information of the estimates of $I_1, \ldots, I_9$, it can be proved that $\mathcal{P}(\mathbf{v}, \upsilon) \cdot (\mathbf{v}, \upsilon) > 0$ for $\|\mathbf{v}\|_{\mathbf{V}} = \rho$ and $\|\upsilon\|_V = \varsigma$, and ρ and ς suitably chosen (see again the aforementioned reference). Due to this and to assumptions (27), (34) and (36), we can use a variant of Brower's theorem to prove the existence of a solution $(\mathbf{c}_j, \mathbf{d}_j)$, with $\mathbf{c}_j := (c_{1j}, c_{2j}, \ldots, c_{jj})$ and $\mathbf{d}_j := (d_{1j}, d_{2j}, \ldots, d_{jj})$ to the system (42) and (43).

Arguing as we did in [19], we can also prove that

$$\|\nabla \mathbf{u}_j\|_{\mathbf{L}^2(\Omega)} \leq \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{45}$$

Consequently, we have (up to some subsequences) that $\mathbf{u}_j \to \mathbf{u}$ weakly in $\mathbf{H}^1_0(\Omega)$, $\mathbf{u}_j \to \mathbf{u}$ strongly in $\mathbf{L}^\gamma(\Omega)$ for $\gamma \in \left[1, \frac{2d}{d-2}\right)$, and $\mathbf{u}_j \to \mathbf{u}$ a.e. in Ω, all as $j \to \infty$. Proceeding again as we did in [19], we have

$$\int_{\Omega} (\nu + \nu_D(k_j))|\nabla k_j|^2\, d\mathbf{x} \leq \int_{\Omega} \nu_T(k_j)\, \mathcal{R}_n\big(|\mathbf{D}(\mathbf{u}_j)|^2\big)\, k_j\, d\mathbf{x} + \int_{\Omega} P(\mathbf{u}_j, k_j)k_j\, d\mathbf{x}.$$

If $0 = \vartheta < 1$, we can argue as we did for I_6, I_8 above and, in particular for (44), to prove that

$$\|\nabla k_j\|^2_{\mathrm{L}^2(\Omega)} \leq C \quad \text{for } \beta + \vartheta \leq \frac{d+2}{d-2}, \tag{46}$$

for some positive constant C not depending on j. In the case of $\vartheta = 1$, we argue as we did for (46) to obtain

$$\|\nabla k_j\|_{\mathrm{L}^2(\Omega)} \leq \frac{C_T\, n\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)}{\nu - C_P\, \lambda(2,d)^2 \Lambda(2,d)^{\beta} \left(\frac{\Lambda_P(d)}{\nu C_K^2}\right)^{\beta} \|\mathbf{g}\|^{\beta}_{\mathbf{L}^2(\Omega)}}, \quad \beta \leq \frac{4}{d-2}. \tag{47}$$

By using assumption (37), with C defined by

$$C := \left(C_P\, \lambda(2,d)^2 \Lambda(2,d)^{\beta} C_K^{-2\beta} \Lambda_P(d)^{\beta}\right)^{\frac{1}{1+\beta}}, \tag{48}$$

we can readily see that the right-hand side of (47) is a positive constant independent of j. Then by a usual reasoning, we have (up to some subsequences) that $k_j \to k$ weakly in $\mathrm{H}^1_0(\Omega)$, $k_j \to k$ strongly in $\mathrm{L}^{\gamma}(\Omega)$ for $\gamma \in \left[1, \frac{2d}{d-2}\right)$, and $k_j \to k$ a.e. in Ω, all as $j \to \infty$.

Now we pass to the limit $j \to \infty$ the integral equality (42). The convergence of the last term of (42) follows from the weak convergence of $\mathbf{u}_j$ and assumption (36). The convergence of the first and third terms of (42) follows a reasoning a little bit different from the one used in [19], because now $d \geq 4$. For the convergence of the third, we observe that since $\mathbf{f}$ is continuous on $\mathbf{u}$ (see (26)), we have by virtue of the a.e. convergence of $\mathbf{u}_j$,

$$\mathbf{f}(\mathbf{u}_j) \to \mathbf{f}(\mathbf{u}) \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty. \tag{49}$$

On the other hand, using Sobolev's inequality together with (30) and (45), it can be proved that

$$\|\mathbf{f}(\mathbf{u}_j)\|_{\mathbf{L}^{\gamma}(\Omega)} \leq C \quad \text{for } \gamma = \frac{2d}{d+2} \text{ and } \alpha \leq \frac{d+2}{d-2}, \quad \text{or } \gamma = d' \text{ and } \alpha \leq \frac{2d-2}{d-2}, \tag{50}$$

for some positive constant C independent of j. Note that conditions on α given by (50) are responsible for the assumption (30). Owing to (49) and (50), $\mathbf{f}(\mathbf{u}_j) \to \mathbf{f}(\mathbf{u})$ weakly in $\mathbf{L}^{\gamma}(\Omega)$, as $j \to \infty$. As a consequence,

$$\int_{\Omega} \mathbf{f}(\mathbf{u}_j) \cdot \mathbf{v}_i \to \int_{\Omega} \mathbf{f}(\mathbf{u}) \cdot \mathbf{v}_i, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1.$$

Note that in the case of $\gamma = \frac{2d}{d+2}$, we may use the fact that $\mathbf{v}_i \in \mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{L}^{\frac{2d}{d-2}}(\Omega)$. For the first term of (42), we observe that, due to (22) and (25), we can write

$$\int_{\Omega} ((\mathbf{u}_j \cdot \nabla)\mathbf{u}_j) \cdot \mathbf{v}_i \, d\mathbf{x} = -\int_{\Omega} \mathbf{u}_j \otimes \mathbf{u}_j : \nabla \mathbf{v}_i \, d\mathbf{x}.$$

From (45), this used together with the Sobolev imbedding $\mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{L}^{\frac{2d}{d-2}}(\Omega)$, and with the a.e. convergence of $\mathbf{u}_j$, we have

$$\|\mathbf{u}_j \otimes \mathbf{u}_j\|_{\mathbf{L}^{\frac{d}{d-2}}(\Omega)} \leq C \quad \text{and} \quad \mathbf{u}_j \otimes \mathbf{u}_j \to \mathbf{u} \otimes \mathbf{u} \ \text{ a.e. in } \Omega, \quad \text{as } j \to \infty, \tag{51}$$

where C is a positive constant not depending on j. Consequently, (51) yields

$$\mathbf{u}_j \otimes \mathbf{u}_j \to \mathbf{u} \otimes \mathbf{u} \text{ weakly in } \mathbf{L}^{\frac{d}{d-2}}(\Omega), \text{ as } j \to \infty. \tag{52}$$

Then, since, by the Sobolev imbedding, $\nabla \mathbf{v}_i \in \mathbf{H}^{s-1}(\Omega) \hookrightarrow \mathbf{L}^{\frac{d}{2}}(\Omega)$ for $s \geq \frac{d}{2} - 1$, which is guaranteed by the choice of $s \geq \frac{d}{2}$, we have, by virtue of (52) and once that $\left(\frac{d}{d-2}\right)^{-1} + \left(\frac{d}{2}\right)^{-1} = 1$,

$$\int_{\Omega} \mathbf{u}_j \otimes \mathbf{u}_j : \nabla \mathbf{v}_i \, d\mathbf{x} \to \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{53}$$

Let us now show the convergence of the second term of (42). We first observe that (26) and (27) and the a.e. convergence of k_j imply $\left|(\nu + \nu_T(k_j))\nabla \mathbf{v}_i\right| \leq (\nu + C_T)|\nabla \mathbf{v}_i|$ and $(\nu + \nu_T(k_j))\nabla \mathbf{v}_i \to (\nu + \nu_T(k))\nabla \mathbf{v}_i$ a.e. in Ω, as $j \to \infty$. Then, since, by the Sobolev imbedding, $\nabla \mathbf{v}_i \in \mathbf{H}^{s-1}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ for $s \geq 1$, which again is guaranteed by the choice of $s \geq \frac{d}{2}$, we have, by Lebesgue's dominated convergence theorem,

$$(\nu + \nu_T(k_j))\nabla \mathbf{v}_i \to (\nu + \nu_T(k))\nabla \mathbf{v}_i \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } j \to \infty. \tag{54}$$

Then, from the weak convergence of $\mathbf{u}_j$ and (54), we can prove that

$$\int_{\Omega} (\nu + \nu_T(k_j))\mathbf{D}(\mathbf{u}_j) : \nabla \mathbf{v}_i \, d\mathbf{x} \to \int_{\Omega} (\nu + \nu_T(k))\mathbf{D}(\mathbf{u}) : \nabla \mathbf{v}_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \tag{55}$$

for all $i \geq 1$. The convergence of third and last terms of (42) (see [19]) together with (53) and (55) imply that we can pass to the limit $j \to \infty$ in the approximate system (42) and thus we obtain

$$\int_{\Omega} ((\mathbf{u} \cdot \nabla)\mathbf{u}) \cdot \mathbf{v}_i \, d\mathbf{x} + \int_{\Omega} (\nu + \nu_T(k))\mathbf{D}(\mathbf{u}) : \nabla \mathbf{v}_i \, d\mathbf{x} + \int_{\Omega} \mathbf{f}(\mathbf{u}) \cdot \mathbf{v}_i \, d\mathbf{x} = \int_{\Omega} \mathbf{g} \cdot \mathbf{v}_i \, d\mathbf{x} \tag{56}$$

for all $i \geq 1$. Using the linearity of (56) in $\mathbf{v}_i$ and the density of the finite linear combinations of the system $\{\mathbf{v}_i\}_{i=1}^{\infty}$ in $\mathbf{V} \cap \mathbf{L}^d(\Omega)$, we deduce that (56) holds true in the whole space $\mathbf{V}$, that is

$$\int_{\Omega} ((\mathbf{u} \cdot \nabla)\mathbf{u}) \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Omega} (\nu + \nu_T(k))\mathbf{D}(\mathbf{u}) : \nabla \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \mathbf{f}(\mathbf{u}) \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{g} \cdot \mathbf{v} \, d\mathbf{x} \tag{57}$$

for all $\mathbf{v} \in \mathbf{v} \cap \mathbf{L}^d(\Omega)$. This allows us to take $\mathbf{v} = \mathbf{u}$ as a test function in (57), which yields

$$\int_{\Omega} (\nu + \nu_T(k))|\mathbf{D}(\mathbf{u})|^2 \, d\mathbf{x} + \int_{\Omega} \mathbf{f}(\mathbf{u}) \cdot \mathbf{u} \, d\mathbf{x} = \int_{\Omega} \mathbf{g} \cdot \mathbf{u} \, d\mathbf{x}.$$

Taking $\mathbf{v}_i = \mathbf{u}_j$ in (42), we also have the equality

$$\int_{\Omega} (\nu + \nu_T(k_j))|\mathbf{D}(\mathbf{u}_j)|^2 \, d\mathbf{x} + \int_{\Omega} \mathbf{f}(\mathbf{u}_j) \cdot \mathbf{u}_j \, d\mathbf{x} = \int_{\Omega} \mathbf{g} \cdot \mathbf{u}_j \, d\mathbf{x}.$$

Then, proceeding as in [19], we obtain (eventually up to some subsequence) that

$$\mathbf{D}(\mathbf{u}_j) \to \mathbf{D}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^2(\Omega) \qquad \text{and} \qquad \mathbf{D}(\mathbf{u}_j) \to \mathbf{D}(\mathbf{u}) \quad \text{a.e. in } \Omega, \tag{58}$$

as $j \to \infty$. We will now pass to the limit $j \to \infty$ the integral equality (43). To pass the first term of this equality to the limit, we can argue as we did for the convective term of the Navier–Stokes equations (see (53)). The convergence of the second and third terms of (43) follows as in the proof of [19, Theorem 3.1]. Due to assumption (26) and to the a.e. convergence of k_j, we have

$$\varepsilon(k_j) \to \varepsilon(k) \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty. \tag{59}$$

Using Sobolev's inequality together with (31) and (46), it can be proved that

$$\|\varepsilon(k_j)\|_{\mathrm{L}^{\gamma}(\Omega)} \leq C \quad \text{for } \gamma = \frac{2d}{d+2} \text{ and } \theta \leq \frac{d+2}{d-2}, \quad \text{or } \gamma = d' \text{ and } \theta \leq \frac{2d-2}{d-2}, \tag{60}$$

for some positive constant C not depending on j. Owing to (59) and (60), $\varepsilon(k_j) \to \varepsilon(k)$ weakly in $\mathrm{L}^{\gamma}(\Omega)$, as $j \to \infty$. Thus

$$\int_{\Omega} \varepsilon(k_j) \upsilon_i \, d\mathbf{x} \to \int_{\Omega} \varepsilon(k) \upsilon_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{61}$$

Note that in the case of $\gamma = \frac{2d}{d+2}$, we use the fact that $\upsilon_i \in \mathrm{H}^1_0(\Omega) \hookrightarrow \mathrm{L}^{\frac{2d}{d-2}}(\Omega)$. Let us now focus our attention on the last term of (43). Here, we first observe that (26) together with the a.e. convergence of $\mathbf{u}_j$ and k_j imply that

$$P(\mathbf{u}_j, k_j) \to P(\mathbf{u}, k) \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty. \tag{62}$$

By using assumption (32), Hölder's inequality (in the case of $\vartheta \neq 0$) and Sobolev's inequality together with (45) and (46), or (47), it can be proved that

$$\|P(\mathbf{u}_j, k_j)\|_{\mathrm{L}^\gamma} \leq C \text{ for } \gamma = \frac{2d}{d+2} \text{ and } \beta + \vartheta \leq \frac{d+2}{d-2}, \text{ or } \gamma = d' \text{ and } \beta + \vartheta \leq \frac{2d-2}{d-2}, \tag{63}$$

for some positive constant C not depending on j. Thus, (62) and (63) imply that $P(\mathbf{u}_j, k_j) \to P(\mathbf{u}, k)$ weakly in $\mathrm{L}^\gamma(\Omega)$, as $j \to \infty$, and consequently, as we did for (61), we obtain

$$\int_\Omega P(\mathbf{u}_j, k_j)\upsilon_i \, d\mathbf{x} \to \int_\Omega P(\mathbf{u}, k)\upsilon_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{64}$$

The convergence of the first four terms of (43) together with (64), assure us that we can pass to the limit $j \to \infty$ in the approximate system (43) to obtain

$$\begin{aligned} &\int_\Omega (\mathbf{u} \cdot \nabla k)\upsilon_i \, d\mathbf{x} + \int_\Omega (\nu + \nu_D(k))\nabla k \cdot \nabla \upsilon_i \, d\mathbf{x} + \int_\Omega \varepsilon(k)\upsilon_i \, d\mathbf{x} \\ &= \int_\Omega \nu_T(k)\, \mathcal{R}_n\big(|\mathbf{D}(\mathbf{u})|^2\big)\, \upsilon_i \, d\mathbf{x} + \int_\Omega P(\mathbf{u}, k)\, \upsilon_i \, d\mathbf{x} \quad \text{for all } i \geq 1. \end{aligned}$$

We have thus proved that, for each $n \in \mathbb{N}$, there exists a weak solution $(\mathbf{u}_n, k_n) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ to the problems (38)–(41) and such that

$$\int_\Omega (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n \cdot \mathbf{v} \, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k_n))\mathbf{D}(\mathbf{u}_n) : \nabla \mathbf{v} \, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_n) \cdot \mathbf{v} \, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v} \, d\mathbf{x}, \tag{65}$$

$$\begin{aligned} &\int_\Omega (\mathbf{u}_n \cdot \nabla k_n)v \, d\mathbf{x} + \int_\Omega (\nu + \nu_D(k_n))\nabla k_n \cdot \nabla v \, d\mathbf{x} + \int_\Omega \varepsilon(k_n)v \, d\mathbf{x} \\ &= \int_\Omega \nu_T(k_n)\, \mathcal{R}_n\big(|\mathbf{D}(\mathbf{u}_n)|^2\big)\, v \, d\mathbf{x} + \int_\Omega P(\mathbf{u}_n, k_n)\, v \, d\mathbf{x} \end{aligned} \tag{66}$$

hold for all $(\mathbf{v}, v) \in \mathbf{V}^j \times V^j$ and all $j \geq 1$. By linearity and density these relations hold for all $(\mathbf{v}, v) \in \mathbf{V}^s \times \mathrm{V}^r$, and by continuity they hold for all $(\mathbf{v}, v) \in (\mathbf{V} \cap \mathbf{L}^d(\Omega)) \times (\mathrm{H}^1_0(\Omega) \cap \mathrm{L}^d(\Omega))$ due to the ranges of α, θ, β and ϑ set forth at (30)–(32).

The proof that $k \geq 0$ and $\varepsilon(k) \geq 0$ a.e. in Ω follows as in the proof of [19, Theorem 3.1], in particular by using (35) for the expression of the turbulent dissipation function. The proof of Proposition 1 is now concluded. □

From Proposition 1, we know that, for each $n \in \mathbb{N}$, there exists a weak solution $(\mathbf{u}_n, k_n) \in \mathbf{V} \times \mathrm{H}_0^1(\Omega)$ to the problems (38)–(41) and such that (65) and (66) hold. Arguing as in [19], it can be proved that

$$\|\nabla \mathbf{u}_n\|_{\mathbf{L}^2(\Omega)} \leq \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{67}$$

As a consequence, it follows (up to some subsequences) that $\mathbf{u}_n \to \mathbf{u}$ weakly in $\mathbf{H}_0^1(\Omega)$, $\mathbf{u}_n \to \mathbf{u}$ strongly in $\mathbf{L}^\gamma(\Omega)$ for $\gamma \in \left[1, \frac{2d}{d-2}\right)$, and $\mathbf{u}_n \to \mathbf{u}$ a.e. in Ω, all as $n \to \infty$. To achieve an a priori estimate for k_n, independent of n, we consider the special test function $\varphi(k_n) := 1 - \frac{1}{(1+k_n)^\delta}$, where δ is a positive constant such that $\varphi \in \mathrm{W}^{1,q'}(\Omega) \hookrightarrow C^{0,\delta}(\Omega)$. Taking $v = \varphi(k_n)$ in (66) and proceeding as we did in [19], we have

$$\delta \int_\Omega (\nu + \nu_D(k_n)) \frac{|\nabla k_n|^2}{(1+k_n)^{1+\delta}}\, d\mathbf{x} \leq \int_\Omega \nu_T(k_n)\, |\mathbf{D}(\mathbf{u}_n)|^2\, d\mathbf{x} + \int_\Omega |P(\mathbf{u}_n, k_n)|\, d\mathbf{x}. \tag{68}$$

With respect to the last term of (68), we firstly observe that, since $q < d'$, by the Sobolev imbedding we have $\mathrm{W}_0^{1,q}(\Omega) \hookrightarrow \mathrm{L}^\gamma(\Omega)$ for $\gamma < \frac{d}{d-2}$. Therefore, in view of (32) and (33),

$$\int_\Omega |P(\mathbf{u}_n, k_n)|\, d\mathbf{x} \leq \begin{cases} C_1 \|\nabla \mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)} & \text{for } \beta \leq \frac{2d}{d-2} \text{ if } \vartheta = 0, \quad \text{or} \\ C_2 \|\nabla \mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)} \|\nabla k_n\|^\vartheta_{\mathrm{L}^q(\Omega)} & \text{for } \beta + 2\vartheta < \frac{2d}{d-2} \text{ if } \vartheta > 0, \end{cases} \tag{69}$$

where C_1 and C_2 are independent of n positive constants. Then, using the assumption (27) together with (69), and arguing as in [19], we can prove, in the most difficult case of $\vartheta \neq 0$,

$$\int_\Omega |\nabla k_n|^q\, d\mathbf{x} \leq \frac{C_1}{\delta}\|\nabla \mathbf{u}_n\|^2_{\mathbf{L}^2(\Omega)} + \frac{C_2}{\delta}\|\nabla \mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)}\|\nabla k_n\|^\vartheta_{\mathrm{L}^q(\Omega)} + C_3\|\nabla k_n\|^{\frac{(1+\delta)q}{2}}_{\mathrm{L}^q(\Omega)} + C_4, \tag{70}$$

where C_1, C_2, C_3 and C_4 are positive constants not depending on n. In this case, we need also to apply Young's inequality to the third term of (70) which is possible as long as $\vartheta < \frac{2d}{d+2}$, condition that is satisfied due to (32) and (33). The case $\vartheta = 0$ is easier. All this reasoning together with (67) and assumption (36), yield

$$\int_\Omega |\nabla k_n|^q\, d\mathbf{x} \leq C, \quad C = C(\nu, \beta, C_T, C_P, d, q, \Omega, \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}), \tag{71}$$

where C is a positive constant not depending on n. Then, in view of (71) and up to some subsequences, we have $k_n \to k$ weakly in $\mathrm{W}_0^{1,q}(\Omega)$ for $q < d'$, $k_n \to k$ strongly in $\mathrm{L}^{\gamma}(\Omega)$ for all $\gamma \in [1, q^*)$ and $k_n \to k$ a.e. in Ω, all as $n \to \infty$. Now, we can pass to the limit $n \to \infty$ all the integral terms of (65) by arguing analogously as we did in the proof of Proposition 1. With respect to the convergence of the integral terms of (66), we first observe that since $q < d'$, we have $\mathrm{W}_0^{1,q'}(\Omega) \hookrightarrow \mathrm{C}^{0,\delta}(\bar{\Omega})$ for $\delta = 1 - \frac{d}{q'}$. As a consequence $v \in \mathrm{W}_0^{1,q'}(\Omega)$ implies that $v \in \mathrm{L}^{\gamma'}(\Omega)$ for any $\gamma' \geq 1$. With minor modifications, the convergence of all the integral terms of (66) follows as in the proof of Proposition 1, with the exception of the one involving $\mathcal{R}_n$, because we do not know whether if this term remains bounded as $n \to \infty$. The convergence of the third and fifth terms of (66) needs also some comments. Due to assumption (26) and to the a.e. convergence of $\mathbf{u}_n$ and k_n, we have

$$\varepsilon(k_n) \to \varepsilon(k) \quad \text{and} \quad P(\mathbf{u}_n, k_n) \to P(\mathbf{u}, k) \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty. \tag{72}$$

Since $k \in \mathrm{W}_0^{1,q}(\Omega)$ for $q < d'$, we have, by virtue of (67) and (71), and for any $\gamma \geq 1$,

$$\|\varepsilon(k_n)\|_{\mathrm{L}^{\gamma}(\Omega)} \leq C_1 \quad \text{for} \quad \theta < \frac{d}{d-2}, \tag{73}$$

$$\|P(\mathbf{u}_n, k_n)\|_{\mathrm{L}^{\gamma}} \leq C_2 \quad \text{for} \quad \beta \leq \frac{2d}{d-2} \text{ if } \vartheta = 0, \quad \text{or} \quad \beta + 2\vartheta < \frac{2d}{d-2} \text{ if } \vartheta > 0. \tag{74}$$

for some positive constants C_1 and C_2 not depending on n. Note that the conjunction of conditions on θ given by (60) and (73) are responsible for the assumption (31). On the other hand, the conjunction of all the conditions on β and ϑ given by (44), (63), (69) and (74) are responsible for the assumptions (32) and (33). Then, owing to (72), (73) and (74), $\varepsilon(k_n) \to \varepsilon(k)$ and $P(\mathbf{u}_n, k_n) \to P(\mathbf{u}, k)$ weakly in $\mathrm{L}^{\gamma}(\Omega)$, as $n \to \infty$ and for possible distinct γ. Thus, the convergence of the correspondingly integral terms follows. Let us now look to the fourth term of (66). Firs we observe that we can readily justify that

$$\begin{aligned} &\int_{\Omega} \left| \left(\nu_T(k_n) \mathcal{R}_n\left(|\mathbf{D}(\mathbf{u}_n)|^2\right) - \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \right) v \right| \, d\mathbf{x} \\ \leq &\int_{\Omega} \left| \nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 - \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \right| \, |v| \, d\mathbf{x} + \int_{\Omega} \frac{1}{n} \frac{\nu_T(k)|\mathbf{D}(\mathbf{u})|^2 |\mathbf{D}(\mathbf{u}_n)|^2}{1 + \frac{1}{n}|\mathbf{D}(\mathbf{u}_n)|^2} |v| \, d\mathbf{x}. \end{aligned} \tag{75}$$

Then, we observe that, by reasoning similarly as we did to prove (58), we also have

$$\mathbf{D}(\mathbf{u}_n) \to \mathbf{D}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^2(\Omega) \quad \text{and} \quad \mathbf{D}(\mathbf{u}_n) \to \mathbf{D}(\mathbf{u}) \quad \text{a.e. in } \Omega, \tag{76}$$

as $n \to \infty$. Thus, the last integral of (75) converges to zero by the application of Lebesgue's dominated convergence theorem, due to (76) and to assumption (27).

With respect to the first of the two last integrals, we can argue as in [19] to prove that

$$\nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 \to \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \quad \text{strongly in } \mathbf{L}^1(\Omega), \quad \text{as } m \to \infty$$

and, consequently, that the first integral of the right-hand side of (75) also converges to zero. Finally, we can pass to the limit $n \to \infty$ the equations (65) and (66) to obtain (28) and (29) for any $(\mathbf{v}, \varphi) \in \mathbf{V} \times \mathrm{W}_0^{1,q'}(\Omega)$. The proof of Theorem 1 is now concluded.

Remark 1 The existence result established in [20, Theorem 3.1] for the case of considering strong nonlinear functions $\mathbf{f}(\mathbf{u})$ and $\varepsilon(k)$, i.e. when no upper restrictions on the growth of these functions with respect to $\mathbf{u}$ and k are required, can also be generalized to any space dimension $d \geq 2$ and for a general function of turbulence production. In this case, besides the sign conditions (34), we just need to assume that (32) and (33) hold together with

$$\exists\, \tau > 0 : |\text{angle}(\mathbf{f}(\mathbf{u}), \mathbf{u})| \notin \left(\frac{\pi}{2} - \tau, \frac{\pi}{2} + \tau\right) \quad \forall\, \mathbf{u} : |\mathbf{u}| \geq L, \quad \forall\, L > 0,$$

$$H_L \in \mathbf{L}^1(\Omega),\ G_M \in \mathrm{L}^1(\Omega) \quad \forall\, L,\ M > 0, \quad H_L := \sup_{|\mathbf{u}| \leq L} |\mathbf{f}(\mathbf{u})|, \quad G_M := \sup_{|k| \leq M} |\varepsilon(k)|.$$

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Critical Point Approaches to Difference Equations of Kirchhoff-Type

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Abstract In this paper, using variational methods and critical point theory we discuss the existence of at least three solutions for nonlinear Kirchhoff-type difference equations with Dirichlet boundary conditions. We also provide examples in order to illustrate the main results.

Keywords Three solutions · Difference equation · Kirchhoff-type problem Variational methods

1 Introduction

The aim of this paper is to establish the existence of at least three solutions for the following Kirchhoff-type discrete boundary-value problem

$$\begin{cases} \mathscr{T}(u) = \lambda f(k, u(k)) + \mu g(k, u(k)) + h(u(k)), & k \in [1, T], \\ u(0) = u(T+1) = 0 \end{cases} \tag{1}$$

where

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$$\mathscr{T}(u) = M\Big(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p + q_k|u(k)|^p\Big)\Big(-\Delta(\phi_p(\Delta u(k-1))) + q_k\phi_p(u(k))\Big),$$

$1 < p < +\infty$, $\lambda > 0$, $\mu \geq 0$, $M : [0, +\infty[\to R$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \leq M(t) \leq m_1$ for all $t \geq 0$, $\phi_p(s) = |s|^{p-2}s$, $T \geq 2$ is a fixed integer, $[1, T]$ is the discrete interval $\{1, \dots, T\}$, $f, g : [1, T] \times R \to R$ are two continuous functions, $h : R \to R$ is a Lipschitz continuous function of order $p-1$ with Lipschitz constant $L \geq 0$ and $h(0) = 0$, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator and $q_k = q(k) \in R_0^+$ for all $k \in [1, T]$.

There is an increasing interest in the existence of solutions to boundary value problems for finite difference equations with p-Laplacian operator. Their applications in many fields such as biological neural networks, economics, optimal control and other areas of study have led to the rapid development of the theory of difference equations; see the monograph of Agarwal [1]. Recently, the study of discrete problems subject to various boundary value conditions has been widely approached by using different abstract methods as fixed point theory, lower and upper solutions method, critical point theory, variational methods, Morse theory and the mountain-pass theorem. For background and recent results, we refer the reader to [3, 4, 8–11, 13, 19, 21, 25, 26] and the references therein for details.

Problems like (1) are usually called nonlocal problem because of the presence of the integral over the entire domain, and this implies that the first equation in (1) is no longer a point-wise identity. In fact, such kind of problem can be traced back to the work of Kirchhoff. In [22], Kirchhoff proposed the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The problem (1) is related to the stationary analogue of the problem (2). Kirchhoff's changes in length of the string produced by transverse vibrations. Similar nonlocal problems also model several physical and biological systems where u describes a process which depends on the average of itself, for example the population density. Lion in [27] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions, various equations of Kirchhoff-type have been studied extensively, for instance see [14, 28].

In recent years, the existence of solutions to discrete boundary value problems of Kirchhoff-type have been studied in many papers and we refer the reader to the papers [12, 16, 17, 23] and the references therein for details.

To the best of our knowledge, for discrete problems of Kirchhoff type, there has so far been few papers concerning its existence of solutions.

Motivated by the above facts, in the present paper, using two kind of three critical points theorems due to Bonanno and Candito [5, 6] which we recall in the next section (Theorems 1 and 2), we establish the existence of at least three solutions for the problem parameters are involved. Precise estimates of these two parameters λ and μ will be given, see Theorems 3 and 5. Theorem 4 is a consequence of Theorem 3. We present Examples 1 and 2 in which the hypotheses of Theorems 4 and 5 are fulfilled, respectively. As a special case of Theorem 4, we obtain Theorem 6 which under suitable conditions on f at zero and at infinity, ensures two positive solutions for the autonomous case of the problem. Finally, we point out Theorem 7, a simple consequence of Theorem 6.

2 Preliminaries

In the present paper X denotes a finite dimensional real Banach space and $I_\lambda : X \to R$ is a functional satisfying the following structure hypothesis:
$I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$ where $\Phi, \Psi : X \to R$ are two functions of class C^1 on X with Φ coercive, i.e. $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$, and λ is a positive real parameter.

In this framework a finite dimensional variant of Theorem 3.3 of [6] (see also Corollary 3.1 and Remark 3.9 of [6]) is the following:

Let X be a nonempty set and $\Phi, \Psi : X \to R$ be two functions. For all $r,\ r_1,\ r_2$, with $r_2 > r_1$ and $r_2 > \inf_X \Phi$, and all $r_3 > 0$, we define

$$\varphi(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1, r_2) = \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v)-\Psi(u)}{\Phi(v)-\Phi(u)},\ \gamma(r_2, r_3) = \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3},$$

$$\alpha(r_1, r_2, r_3) = \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

Theorem 1 ([6, Theorem 3.3]) *Assume that*

(a_1) Φ *is convex and* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
(a_2) *for every* $u_1, u_2 \in X$ *such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$, *one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1 - s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 *with* $r_1 < r_2$, *such that*

(a_3) $\varphi(r_1) < \beta(r_1, r_2)$;
(a_4) $\varphi(r_2) < \beta(r_1, r_2)$;
(a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in]\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}[$ *the functional* $\Phi - \lambda\Psi$ *admits three distinct critical points* u_1, u_2, u_3 *such that* $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$ *and* $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.

We refer the interested reader to the papers [7, 15, 18, 24] in which Theorem 1 has been successfully employed to obtain the existence of at least three solutions for boundary value problems. Now, put $\varphi^{(1)}(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u))-\Psi(u)}{r-\Phi(u)}$ for all $r > \inf_X \Phi$, and $\lambda^* = \frac{1}{\inf_{r>\inf_X \Phi} \varphi^{(1)}(r)}$ where we read $\frac{1}{0} = +\infty$ if this case occurs.

Theorem 2 ([5, Theorem 2.3]) *Let X be a finite dimensional real Banach space. Assume that for each* $\lambda \in]0, \lambda^*[$ *one has*

(*b*) $\lim_{\|u\|\to+\infty} I_\lambda(u) = -\infty$.

Then, for each $\lambda \in]0, \lambda^*[$ *the functional* I_λ *admits at least three distinct critical points.*

Theorem 2 has been successfully used to ensure the existence of at least three solutions for a discrete boundary value problem in [9].

Remark 1 It is worth noticing that whenever X is a finite dimensional Banach space, a careful reading of the proofs of Theorems 1 and 2 shows that regarding to Φ and Ψ, it is enough to require only that Φ' and Ψ' are two continuous functionals on X^*.

Now, consider the T-dimensional Banach space $X := \{u : [0, T+1] \to R : u(0) = u(T+1) = 0\}$ equipped with the norm $\|u\| := \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p + q_k|u(k)|^p\right)^{1/p}$. In the sequel, we will use the following inequality

$$\max_{k\in[1,T]} |u(k)| \leq \frac{(T+1)^{(p-1)/p}}{2}\|u\| \tag{3}$$

for every $u \in X$. The inequality immediately follows, for instance, from Lemma 2.2 of [20].

Let $T \geq 2$ be a fixed positive integer and $h : R \to R$ be a Lipschitz continuous function of order $p-1$ with Lipschitzian constant $L \geq 0$, i.e., $|h(t_1) - h(t_2)| \leq L|t_1 - t_2|^{p-1}$ for every $t_1, t_2 \in R$, and $h(0) = 0$. Suppose that the constant $L \geq 0$ satisfies $LT(T+1)^{p-1} < 2^p m_0$. Put $F(k,t) := \int_0^t f(k,\xi)d\xi$ for all $(k,t) \in [1,T] \times R$, $G(k,t) := \int_0^t g(k,\xi)d\xi$ for all $(k,t) \in [1,T] \times R$, $\widetilde{M}(t) = \int_0^t M(\xi)d\xi$ for all $t \geq 0$ and $H(t) := \int_0^t h(\xi)d\xi$ for all $t \in R$. We state the following consequence of the strong comparison principle [2, Lemma 2.3] (see also [4, Theorem 2.2]) which we will use in the sequel in order to obtain positive solutions to the problem (1), i.e. $u(k) > 0$ for each $k \in [1, T]$.

Lemma 1 *If* $\mathscr{T}(u) \geq 0$, $k \in [1, T]$, $u(0) \geq 0$, $u(k+1) \geq 0$, *then either u is positive or* $u \equiv 0$.

3 Main Results

In this section, we formulate our main results on the existence of at least three solutions for the problem (1). For our convenience, set $G^{\theta} := \sum_{k=1}^{T} \max_{|\xi|\le\theta} G(k,\xi)$ for all $\theta > 0$ and $G_{\eta} := T \inf_{[1,T]\times[0,\eta]} G(k,t)$ for all $\eta > 0$. If g is sign-changing, then clearly $G^{\theta} \ge 0$ and $G_{\eta} \le 0$. Fixing four positive constants $\theta_1, \theta_2, \theta_3$ and η put

$$\delta_{\lambda,G} := \min\Bigg\{ \frac{1}{p(T+1)^{p-1}} \min\Bigg\{ \frac{\kappa_1\theta_1^p - \lambda p(T+1)^{p-1}\sum_{k=1}^{T} F(k,\theta_1)}{G^{\theta_1}},$$

$$\frac{\kappa_1\theta_2^p - \lambda p(T+1)^{p-1}\sum_{k=1}^{T} F(k,\theta_2)}{G^{\theta_2}}, \frac{\kappa_1(\theta_3^p-\theta_2^p) - \lambda p(T+1)^{p-1}\sum_{k=1}^{T} F(k,\theta_3)}{G^{\theta_3}} \Bigg\},$$

$$\frac{\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)\eta^p - \lambda\Big(\sum_{k=1}^{T} F(k,\eta) - \sum_{k=1}^{T} F(k,\theta_1)\Big)}{G_{\eta}-G^{\theta_1}} \Bigg\} \tag{4}$$

where $\kappa_1 = 2^p m_0 - LT(T+1)^{p-1}$ and $\kappa_2 = 2^p m_1 + LT(T+1)^{p-1}$.

Theorem 3 *Assume that there exist positive constants $\theta_1, \theta_2, \theta_3$ and η with $\theta_1 < \frac{(T+1)^{(p-1)/p}}{2}(2+\sum_{k=1}^{T} q_k)^{\frac{1}{p}}\eta$, $\frac{(T+1)^{(p-1)/p}}{2}\big(\frac{\kappa_2}{\kappa_1}(2+\sum_{k=1}^{T} q_k)\big)^{\frac{1}{p}}\eta < \theta_2$ and $\theta_2 < \theta_3$ such that*

(A_1) $\quad f(k,t) \ge 0$ *for each* $(k,t) \in [1,T]\times[0,\theta_3]$;

(A_2) $\quad \max\left\{ \frac{\sum_{k=1}^{T} F(k,\theta_1)}{\theta_1^p}, \frac{\sum_{k=1}^{T} F(k,\theta_2)}{\theta_2^p}, \frac{\sum_{k=1}^{T} F(k,\theta_3)}{\theta_3^p-\theta_2^p} \right\} < \frac{\kappa_1\Big(\sum_{k=1}^{T} F(k,\eta) - \sum_{k=1}^{T} F(k,\theta_1)\Big)}{p(T+1)^{p-1}\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)\eta^p}$.

Then, for every

$$\lambda \in \Lambda := \Bigg(\frac{\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)\eta^p}{\sum_{k=1}^{T} F(k,\eta) - \sum_{k=1}^{T} F(k,\theta_1)},$$

$$\frac{\kappa_1}{p(T+1)^{p-1}} \min\left\{ \frac{\theta_1^p}{\sum_{k=1}^{T} F(k,\theta_1)}, \frac{\theta_2^p}{\sum_{k=1}^{T} F(k,\theta_2)}, \frac{\theta_3^p-\theta_2^p}{\sum_{k=1}^{T} F(k,\theta_3)} \right\} \Bigg)$$

and for every non-negative continuous function $g : [1,T]\times R \to R$, there exists $\delta_{\lambda,G} > 0$ given by (4) such that, for each $\mu \in [0, \delta_{\lambda,G})$, the problem (1) has at least three non-negative solutions u_1, u_2 and u_3 such that $\max_{k\in[1,T]} |u_1(k)| < \theta_1$, $\max_{k\in[1,T]} |u_2(k)| < \theta_2$ *and* $\max_{k\in[1,T]} |u_3(k)| < \theta_3$.

Proof Our goal is to apply Theorem 1 to the problem (1). We consider the auxiliary problem

$$\begin{cases} \mathscr{T}(u) = \lambda \hat{f}(k, u(k)) + \mu g(k, u(k)) + h(u(k)), & k \in [1, T], \\ u(0) = u(T+1) = 0 \end{cases} \tag{5}$$

where $\hat{f} : [1, T] \times R \to R$ is a continuous function defined putting

$$\hat{f}(k, \xi) = \begin{cases} f(k, 0), & \text{if } \xi < 0, \\ f(k, \xi), & \text{if } 0 \le \xi \le \theta_3, \\ f(k, \theta_3), & \text{if } \xi > \theta_3. \end{cases}$$

From (A_1) owing to Lemma 1, any solution of the problem (5) is non-negative. In addition, if it satisfies also the condition $0 \le u(k) \le \theta_3$, and for every $k \in [1, T]$, clearly it turns to be also a non-negative solution of (1). Therefore, for our goal, it is enough to show that our conclusion holds for (1). Let the functionals Φ, Ψ for every $u \in X$, defined by $\Phi(u) = \frac{1}{p}\widetilde{M}(\|u\|^p) - \sum_{k=1}^{T} H(u(k))$ and $\Psi(u) = \sum_{k=1}^{T} \left[F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))\right]$. Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \sum_{k=1}^{T} \left[f(k, u(k)) - \frac{\mu}{\lambda} g(k, u(k) \right] v(k)$$

for every $v \in X$, as well as is sequentially weakly upper semicontinuous. Recalling (3), taking into account that h is a $(p-1)$-Lipschitz continues function with Lipschizian constant $L \ge 0$ and $h(0) = 0$, we have

$$\frac{\kappa_1}{p2^p}\|u\|^p \le \Phi(u) \le \frac{\kappa_2}{p2^p}\|u\|^p, \tag{6}$$

which due to the condition $LT(T+1)^{p-1} < 2^p m_0$, it follows that Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is $\Phi'(u)(v) = M\Big(\|u\|^p\Big)\Big(-\sum_{k=1}^{T}\big[\Delta(\phi_p(\Delta u(k-1))) - q_k|u(k)|^{p-2}u(k) + h(u(k))\big]\, v(k)\Big)$ for every $v \in X$. Furthermore, Φ is sequentially weakly lower semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested in Theorem 1, are verified. Note that the critical points of the functional $\Phi - \lambda\Psi$ are the solutions of the problem (1). Define w by setting $w(k) = \begin{cases} \eta, \ k \in [1, T], \\ 0, \ k = 0, \ k = T+1. \end{cases}$ Clearly, $w \in X$ and one has $\|w\|^p = (2 + \sum_{k=1}^{T} q_k)\eta^p$. By using (6), we have $\frac{\kappa_1}{p2^p}(2 + \sum_{k=1}^{T} q_k)\eta^p \le \Phi(w) \le \frac{\kappa_2}{p2^p}(2 + \sum_{k=1}^{T} q_k)\eta^p$. Choose $r_1 = \frac{\kappa_1}{p(T+1)^{p-1}}\theta_1^p$, $r_2 = \frac{\kappa_1}{p(T+1)^{p-1}}\theta_2^p$ and $r_3 = \frac{\kappa_1}{p(T+1)^{p-1}}(\theta_3^p - \theta_2^p)$. From the conditions $\theta_3 > \theta_2$, $\theta_1 < \frac{(T+1)^{(p-1)/p}}{2}(2 + \sum_{k=1}^{T} q_k)^{\frac{1}{p}}\eta$ and $\frac{(T+1)^{(p-1)/p}}{2}\left(\frac{\kappa_2}{\kappa_1}(2 + \sum_{k=1}^{T} q_k)\right)^{\frac{1}{p}}$ $\eta < \theta_2$, we achieve $r_3 > 0$ and $r_1 < \Phi(w) < r_2$. From the definition of Φ and (3), the estimate $\Phi(u) \le r_1$ implies that

$$|u(k)|^p \leq \|u\|_\infty^p \leq \frac{(T+1)^{(p-1)}}{2^p}\|u\|^p \leq \frac{p(T+1)^{(p-1)}}{\kappa_1}\Phi(u) \leq \theta_1^p, \quad \forall k \in [1,T].$$

From the definition of r_1, it follows that $\Phi^{-1}(-\infty, r_1) \subseteq \{u \in X; |u| \leq \theta_1\}$. Hence, by using the assumption (A_1), one has

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)} \sum_{k=1}^{T} F(k,u(k)) \leq \sum_{k=1}^{T} \max_{|t|\leq\theta_1} F(k,t) \leq \sum_{k=1}^{T} F(k,\theta_1).$$

In a similar way, we have $\sup_{u\in\Phi^{-1}(-\infty,r_2)} \sum_{k=1}^{T} F(k,u(k)) \leq \sum_{k=1}^{T} F(k,\theta_2)$ and $\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \sum_{k=1}^{T} F(k,u(k)) \leq \sum_{k=1}^{T} F(k,\theta_3)$. Therefore, since $0 \in \Phi^{-1}(-\infty, r_1)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\varphi(r_1) \leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u)}{r_1} \leq \frac{p(T+1)^{p-1}}{\kappa_1}\left(\frac{\sum_{k=1}^{T} F(k,\theta_1)}{\theta_1^p} + \frac{\mu}{\lambda}\frac{G^{\theta_1}}{\theta_1^p}\right),$$

$$\varphi(r_2) \leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u)}{r_2} \leq \frac{p(T+1)^{p-1}}{\kappa_1}\left(\frac{\sum_{k=1}^{T} F(k,\theta_2)}{\theta_2^p} + \frac{\mu}{\lambda}\frac{G^{\theta_2}}{\theta_2^p}\right)$$

and

$$\gamma(r_2,r_3) \leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3} \leq \frac{p(T+1)^{p-1}}{\kappa_1}\left(\frac{\sum_{k=1}^{T} F(k,\theta_3)}{\theta_3^p-\theta_2^p} + \frac{\mu}{\lambda}\frac{G^{\theta_3}}{\theta_3^p-\theta_2^p}\right).$$

On the other hand, for each $u \in \Phi^{-1}(-\infty, r_1)$, one has

$$\begin{aligned}\beta(r_1,r_2) &\geq \frac{\sum_{k=1}^{T} F(k,\eta) - \sum_{k=1}^{T} F(k,\theta_1) + \frac{\mu}{\lambda}(G_\eta - G^{\theta_1})}{\Phi(w)-\Phi(u)} \\ &\geq \frac{\sum_{k=1}^{T} F(k,\eta) - \sum_{k=1}^{T} F(k,\theta_1) + \frac{\mu}{\lambda}(G_\eta - G^{\theta_1})}{\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)\eta^p}.\end{aligned}$$

Due to (A_2) we get $\alpha(r_1,r_2,r_3) < \beta(r_1,r_2)$. Therefore, (a_1) and (a_2) of Theorem 1 are verified. Finally, we verify that $\Phi - \lambda\Psi$ satisfies the second assumption of Theorem 1. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are solutions for the problem (1). Then, due to Lemma 1, we deduce u_1 and u_2 are non-negative. Thus, it follows that $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0, 1]$, and that $(\lambda f + \mu g)(k, su_1 + (1-s)u_2) \geq 0$, and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0, 1]$. Hence, Theorem 1 implies that for every

$$\lambda \in \Big(\frac{\frac{\kappa_2}{p2^p m_1}(2+\sum_{k=1}^T q_k)\eta^p}{\sum_{k=1}^T F(k,\eta)-\sum_{k=1}^T F(k,\theta_1)},$$

$$\frac{\kappa_1}{p(T+1)^{p-1}}\min\Big\{\frac{\theta_1^p}{\sum_{k=1}^T F(k,\theta_1)},\frac{\theta_2^p}{\sum_{k=1}^T F(k,\theta_2)},\frac{\theta_3^p-\theta_2^p}{\sum_{k=1}^T F(k,\theta_3)}\Big\}\Big)$$

and $\mu \in [0, \delta_{\lambda,G})$, the functional $\Phi - \lambda\Psi$ has three critical points u_i, $i = 1, 2, 3$, in X such that $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$ and $\Phi(u_3) < r_2 + r_3$, that is $\max_{k\in[1,T]} |u_1(k)| < \theta_1$, $\max_{k\in[1,T]} |u_2(k)| < \theta_2$ and $\max_{k\in[1,T]} |u_3(k)| < \theta_3$. Then, taking into account the fact that the solutions of the problem (1) are exactly critical points of the functional $\Phi - \lambda\Psi$ we have the desired conclusion.

For positive constants θ_1, θ_4 and η, set

$$\delta'_{\lambda,G} := \min\Big\{\frac{1}{p(T+1)^{p-1}}\min\Big\{\frac{\kappa_1\theta_1^p - p(T+1)^{p-1}\lambda\sum_{k=1}^T F(k,\theta_1)}{G^{\theta_1}},$$

$$\frac{\kappa_1\theta_4^p - 2p(T+1)^{p-1}\lambda\sum_{k=1}^T F(k,\frac{1}{\sqrt[p]{2}}\theta_4)}{2G^{\frac{1}{\sqrt[p]{2}}\theta_4}},\frac{\kappa_1\theta_4^p - 2p(T+1)^{p-1}\lambda\sum_{k=1}^T F(k,\theta_4)}{2G^{\theta_4}}\Big\},$$

$$\frac{\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^T q_k)\eta^p - \lambda\big(\sum_{k=1}^T F(k,\eta)-\sum_{k=1}^T F(k,\theta_1)\big)}{G_\eta - G^{\theta_1}}\Big\}. \tag{7}$$

Now, we deduce the following straightforward consequence of Theorem 3.

Theorem 4 *Assume that there exist positive constants θ_1, θ_4 and η with $\theta_1 < \min\{\eta, \frac{(T+1)^{(p-1)/p}}{2}(2+\sum_{k=1}^T q_k)^{\frac{1}{p}}\eta\}$ and $\frac{\sqrt[p]{2}(T+1)^{(p-1)/p}}{2}\big(\frac{\kappa_2}{\kappa_1}(2+\sum_{k=1}^T q_k)\big)^{\frac{1}{p}}\eta < \theta_4$ such that*

- (A_3) *$f(k,t) \geq 0$ for each $(k,t) \in [1,T]\times[0,\theta_4]$;*
- (A_4) $\max\Big\{\frac{\sum_{k=1}^T F(k,\theta_1)}{\theta_1^p}, \frac{2\sum_{k=1}^T F(k,\theta_4)}{\theta_4^p}\Big\} < \frac{\kappa_1\sum_{k=1}^T F(k,\eta)}{(\kappa_1 + p(T+1)^{p-1}\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^T q_k))\eta^p}$.

Then, for every

$$\lambda \in \Lambda' := \Big]\frac{(\kappa_1 + p(T+1)^{p-1}\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^T q_k))\eta^p}{p(T+1)^{p-1}\sum_{k=1}^T F(k,\eta)},$$

$$\frac{\kappa_1}{p(T+1)^{p-1}}\min\Big\{\frac{\theta_1^p}{\sum_{k=1}^T F(k,\theta_1)},\frac{\theta_4^p}{2\sum_{k=1}^T F(k,\theta_4)}\Big\}\Big[$$

and for every non-negative continuous function $g : [1, T] \times R \to R$*, there exists* $\delta'_{\lambda,G} > 0$ *given by (7) such that, for each* $\mu \in [0, \delta'_{\lambda,G})$*, the problem (1) has at least three non-negative solutions* u_1*,* u_2 *and* u_3 *such that* $\max_{k\in[1,T]} |u_1(k)| < \theta_1$*,* $\max_{k\in[1,T]} |u_2(k)| < \frac{1}{\sqrt[p]{2}}\theta_4$ *and* $\max_{k\in[1,T]} |u_3(k)| < \theta_4$*.*

Proof Choose $\theta_2 = \frac{1}{\sqrt[p]{2}}\theta_4$ and $\theta_3 = \theta_4$. So, by using (A_4), one has

$$\frac{\sum_{k=1}^{T} F(k, \theta_2)}{\theta_2^p} \le \frac{2\sum_{k=1}^{T} F(k, \theta_4)}{\theta_4^p} < \frac{\kappa_1 \sum_{k=1}^{T} F(k, \eta)}{(\kappa_1 + p(T+1)^{p-1} \frac{\kappa_2}{p2^p}(2 + \sum_{k=1}^{T} q_k))\eta^p} \tag{8}$$

and

$$\frac{\sum_{k=1}^{T} F(k,\theta_3)}{\theta_3^p - \theta_2^p} = \frac{2\sum_{k=1}^{T} F(k,\theta_4)}{\theta_4^p} < \frac{\kappa_1 \sum_{k=1}^{T} F(k,\eta)}{(\kappa_1 + p(T+1)^{p-1} \frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k))\eta^p}. \tag{9}$$

Moreover, taking into account that $\theta_1 < \eta$, by using (A_4) we have

$$\begin{aligned}
&\frac{\kappa_1}{p(T+1)^{p-1}} \frac{\sum_{k=1}^{T} F(k,\eta) - \sum_{k=1}^{T} F(k,\theta_1)}{\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)\eta^p} > \frac{\kappa_1}{p(T+1)^{p-1}} \frac{\sum_{k=1}^{T} F(k,\eta)}{\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)\eta^p} \\
&- \frac{\kappa_1}{p(T+1)^{p-1}} \frac{\sum_{k=1}^{T} F(k,\theta_1)}{\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)\theta_1^p} > \frac{\kappa_1}{p(T+1)^{p-1}} \frac{\sum_{k=1}^{T} F(k,\eta)}{\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)\eta^p} \\
&- \frac{\kappa_1^2}{p(T+1)^{p-1} \frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)} \times \frac{\sum_{k=1}^{T} F(k,\eta)}{\left(\kappa_1 + p(T+1)^{p-1} \frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)\right)\eta^p} \\
&= \frac{\kappa_1}{\kappa_1 + p(T+1)^{p-1} \frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)} \frac{\sum_{k=1}^{T} F(k,\eta)}{\eta^p}.
\end{aligned}$$

Hence, from (A_4), (8) and (9), it is easy to see that the assumption (A_2) of Theorem 3 is satisfied, and since the critical points of the functional $\Phi - \lambda\Psi$ are the solutions of the problem (1) we have the conclusion.

We point out the following consequence of Theorem 2.

Theorem 5 *Let* $p \ge 2$ *and* $f : [1, T] \times R \to R$ *be a continuous function. Assume that there exist four constants* c_1*,* c_2*,* s *and* β *with* $c_1 > 0$*,* $s > p$ *and* $0 \le \beta < s$ *such that*

(B_1) $F(k, \xi) \ge c_1|\xi|^s - c_2|\xi|^\beta$ *for every* $(k, \xi) \in [1, T] \times R$*.*

Then, for every $\lambda \in]0, \bar{\lambda}[$ *where* $\bar{\lambda} = \frac{\kappa_1}{p(T+1)^{p-1}} \sup_{\theta>0} \frac{\theta^p}{\sum_{k=1}^{T} \max_{|t|\le\theta} F(k,t)}$*, and for every continuous function* $g : [1, T] \times R \to R$*, there exists* $\tilde{\delta}_{\lambda,g} > 0$ *such that for each* $\mu \in [0, \tilde{\delta}_{\lambda,g})$*, the problem (1) possesses at least three solutions.*

Proof Our aim is to apply Theorem 2. Fix $\lambda \in]0, \bar{\lambda}[$ and take Φ and Ψ as in the proof of Theorem 3, and put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in X$. Then, there is $\bar{\theta} > 0$ such that $\lambda < \frac{\kappa_1}{p(T+1)^{p-1}} \frac{\bar{\theta}^p}{\sum_{k=1}^{T} \max_{|t|\le\bar{\theta}} F(k,t)}$. Setting $\bar{r} = \frac{\kappa_1}{p(T+1)^{p-1}}\bar{\theta}^p$ and arguing as in the proof of Theorem 3, one has

$$\frac{1}{\lambda^*} \le \varphi^{(1)}(\bar{r}) < \frac{\sup_{u\in\Phi^{-1}(-\infty,\bar{r})}\Psi(u)}{\bar{r}} \\ \le \frac{p(T+1)^{p-1}}{\kappa_1}\frac{\sum_{k=1}^{T}\max_{|t|\le\bar{\theta}}F(k,t)+\frac{\mu}{\lambda}G^{\bar{\theta}}}{\bar{\theta}^{p^-}} < \frac{1}{\lambda}, \tag{10}$$

that is $\lambda < \lambda^*$. Moreover, it is easy to show that there exist two positive constants ι_1 and ι_2 such that, for each $u \in X$, one has $\sum_{k=1}^{T}|u(k)|^s \ge \iota_1^s\|u\|^s$ and $\sum_{k=1}^{T}|u(k)|^\beta \ge \iota_2^\beta\|u\|^\beta$. Hence, from ($B_1$), for each $u \in X$, we get

$$I_\lambda(u) \le \frac{\kappa_2}{p2^p}\|u\|^p - \lambda c_1\iota_1^s\|u\|^s + \lambda c_2\iota_2^\beta\|u\|^\beta \tag{11}$$

as $\|u\| \to +\infty$. Therefore, since $s > p$ and $s > \beta$, the condition (b) is verified. Hence, from Theorem 2 the functional I_λ admits three critical points, which are three solutions for (1) and the conclusion is proved.

Corollary 1 *Let $p \ge 2$ and $f : [1,T]\times R \to R$ be a continuous function. Assume that there exist four constants c_1, c_2, s and β with $c_1 > 0$ and $0 \le \beta < p$ such that*

(B_2) $\frac{\sum_{k=1}^{T}\max_{|t|\le\theta}F(k,t)}{\theta^p} < \frac{2^p c_1\iota_1^p\kappa_1}{\kappa_2(T+1)^{p-1}}$;

(B_3) $F(k,\xi) \ge c_1|\xi|^s - c_2|\xi|^\beta$ *for every* $(k,\xi)\in[1,T]\times R$.

Then, for every $\lambda \in \left]\frac{\kappa_2}{p2^p c_1\iota_1^p}, \frac{\kappa_1}{p(T+1)^{p-1}}\frac{\theta^p}{\sum_{k=1}^{T}\max_{|t|\le\theta}F(k,t)}\right[$, *and for every continuous function* $g : [1,T]\times R \to R$, *there exists* $\tilde{\delta}_{\lambda,g} > 0$ *such that for each* $\mu \in [0,\tilde{\delta}_{\lambda,g})$, *the problem (1) possesses at least three solutions.*

Proof Our claim is to prove that condition (b) of Theorem 2 holds for every $\lambda \in \left]\frac{\kappa_2}{p2^p c_1\iota_1^p}, \frac{\kappa_1}{p(T+1)^{p-1}}\frac{\theta^p}{\sum_{k=1}^{T}\max_{|t|\le\theta}F(k,t)}\right[\subset]0,\lambda^*[$. Indeed, from ($B_2$), arguing as in (10), one has that $\lambda < \lambda^*$. Moreover, by (B_3), from (11) with $s = p$, for every $u \in X$, we have $I_\lambda(u) \le \left(\frac{\kappa_2}{p2^p} - \lambda c_1\iota_1^p\right)\|u\|^p + \lambda c_2\iota_2^\beta\|u\|^\beta$ where $\frac{\kappa_2}{p2^p} - \lambda c_1\iota_1^p < 0$, which implies condition (b).

Remark 2 If in Theorems 3 and 5, either $f(k,0) \ne 0$ for some $k \in [1,T]$ or $g(k,0) \ne 0$ for some $k \in [1,T]$, or both hold true, then the ensured solutions are obviously non-trivial.

We now present the following examples to illustrate Theorems 4 and 5, respectively.

Example 1 Consider the following problem

$$\begin{cases} M\Big(\sum_{k=1}^{4}|\Delta u(k-1)|^3 + q_k|u(k)|^3\Big)\Big(-\Delta(\phi_3(\Delta u(k-1))) + q_k\phi_3(u(k))\Big) \\ = \lambda f(u) + \mu g(u) + h(u), \quad k\in[1,3], \\ u(0) = u(4) = 0 \end{cases} \tag{12}$$

where $M(t) = \frac{3}{2} + \frac{\sin(t)}{2}$ for all $t \geq 0$, $q(k) = 10^{-3k}$ for $k = [1, 3]$, $f(t) = \begin{cases} 17t^{16}, & \text{if } t \leq 1, \\ \frac{17}{t}, & \text{if } t > 1 \end{cases}$ and $h(t) = \frac{1}{10^8}(1 - \cos(t))$ for every $t \in R$. By the expressions of f and h, we have $F(t) = \begin{cases} t^{17}, & \text{if } t \leq 1, \\ 1 + 17\ln(t), & \text{if } t > 1 \end{cases}$ and $H(t) = \frac{1}{10^8}(t - \sin(t))$ for every $t \in R$. Choosing $\theta_1 = 10^{-8}$, $\theta_4 = 10^8$ and $\eta = 1$, we clearly see that all assumptions of Theorem 4 are satisfied. Then, for every $\lambda \in \left]\frac{8 - \frac{48}{10^8} + (32 + \frac{96}{10^8})(2 + 10^{-3} + 10^{-6} + 10^{-9})}{144}, \frac{8 - \frac{48}{10^6}}{48} \frac{10^{24}}{6 + 102\ln(10^8)}\right[$ and for every non-negative continuous function $g : [1, T] \times R \to R$, there exists $\bar{\delta}_{\lambda,G} > 0$ such that, for each $\mu \in [0, \bar{\delta}_{\lambda,G})$, the problem (12) has at least three non-negative solutions u_1, u_2 and u_3 such that $\max_{k \in [1,3]} |u_1(k)| < 10^{-8}$, $\max_{k \in [1,3]} |u_2(k)| < \frac{10^8}{\sqrt[3]{2}}$ and $\max_{k \in [1,3]} |u_3(k)| < 10^8$.

Example 2 We consider the following problem

$$\begin{cases} M\Big(\sum_{k=1}^{5} |\Delta u(k-1)|^4 + q_k|u(k)|^4\Big)\Big(-\Delta(\phi_4(\Delta u(k-1))) + q_k\phi_4(u(k))\Big) \\ = \lambda f(u) + \mu g(u), \quad k \in [1, 4], \\ u(0) = u(5) = 0 \end{cases} \tag{13}$$

where $M(t) = 1 + e^{\cos(t)+1}$ for all $t \geq 0$, $q(k) = 1 + \ln(e^k)$ for $k = [1, 4]$, $f(t) = 6t^5 + 4t^3 + 2t + e^{2t}$ for each $t \in R$ and $h(t) = \frac{1}{10^2}\sin(t)$ for every $t \in R$. By the expression of f, we have $F(t) = t^6 + t^4 + t^2 + e^{2t} - 1$ for each $t \in R$. Direct calculations give $m_0 = 2$ and $L = \frac{1}{10^2}$. By choosing $s = 6$, $\beta = 2$, $c_1 = 1$, $c_2 = 2$ and $\theta = 10^{-3}$, then all conditions in Theorem 5 are satisfied. Then, for every $\lambda \in \left(0, \frac{13}{300}\right)$ and for every continuous function $g : R \to R$, there exists $\delta_{\lambda,g} > 0$ such that for each $\mu \in [0, \delta_{\lambda,g})$, the problem (13) possesses at least three non-trivial solutions.

Now, we deduce the following straightforward consequence of Theorem 4.

Theorem 6 *Let f be a non-negative continuous and non-zero function such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = \lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = 0 \tag{14}$$

for every $\lambda > \lambda^*$ where $\lambda^* = \inf\left\{\frac{\kappa_1 + p(T+1)^{p-1}\frac{\kappa_2}{p2^p}(2 + \sum_{k=1}^{T} q_k)\eta^p}{Tp(T+1)^{p-1}F(\eta)} : \eta > 0, F(\eta) > 0\right\}$, $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in R$, the problem

$$\begin{cases} \mathscr{T}(u) = \lambda f(u(k)) + h(u(k)), \ k \in [1, T], \\ u(0) = u(T+1) = 0 \end{cases} \tag{15}$$

has at least two distinct positive solutions.

Proof Fix $\lambda > \lambda^*$ and let $\eta > 0$ such that $F(\eta) > 0$ and $\lambda > \frac{\kappa_1 + p(T+1)^{p-1}\frac{\kappa_2}{p2^p}(2 + \sum_{k=1}^{T} q_k)\eta^p}{Tp(T+1)^{p-1}F(\eta)}$. From (14) there is $\theta_1 > 0$ such that $\theta_1 < \min$

$\{\eta, \frac{(T+1)^{(p-1)/p}}{2}(2+\sum_{k=1}^{T} q_k)^{\frac{1}{p}}\eta\}$ and $\frac{F(\theta_1)}{\theta_1^p} < \frac{\kappa_1}{\lambda T p(T+1)^{p-1}}$, and $\theta_4 > 0$ such that $\frac{\sqrt[p]{2}(T+1)^{(p-1)/p}}{2}\left(\frac{\kappa_2}{\kappa_1}(2+\sum_{k=1}^{T} q_k)\right)^{\frac{1}{p}}\eta < \theta_4$ and $\frac{F(\theta_4)}{\theta_4} < \frac{\kappa_1}{2\lambda T p(T+1)^{p-1}}$. Therefore, Theorem 4 ensures the conclusion.

Finally, we point out the following simple consequence of Theorem 6.

Theorem 7 *Let $f : R \to R$ be a continuous function such that $tf(t) > 0$ for all $t \neq 0$ and $\lim_{t\to 0} \frac{f(t)}{t^{p-1}} = \lim_{|t|\to+\infty} \frac{f(t)}{t^{p-1}} = 0$. Then, for every $\lambda > \underline{\lambda}$ where $\underline{\lambda} = \frac{\kappa_1 + p(T+1)^{p-1}\frac{\kappa_2}{p2^p}(2+\sum_{k=1}^{T} q_k)}{Tp(T+1)^{p-1}} \times \max\left\{\inf_{\eta>0}\frac{\eta^p}{F(\eta)};\ \inf_{\eta<0}\frac{(-\eta)^p}{F(\eta)}\right\}$, the problem (15), in the case $h \equiv 0$ has at least four distinct non-trivial solutions.*

Proof Putting $f_1(t) = \begin{cases} 0, & \text{if } t < 0, \\ f(t), & \text{if } t \geq 0, \end{cases}$ and $f_2(t) = \begin{cases} 0, & \text{if } t < 0, \\ -f(-t), & \text{if } t \geq 0 \end{cases}$ and applying Theorem 6 to f_1 and f_2 the desired result follows.

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Continuous Selections of Solution Sets of a Second-Order Integro-Differential Inclusion

Aurelian Cernea

Abstract We study a Cauchy problem associated to a second-order integro-differential inclusion. The general framework of evolution operators that define the problem that we consider has been developed by Kozak and, afterwards, improved by Henriquez. Our aim is to show the existence of mild solutions continuously depending on a parameter for the problem studied in the case when the set-valued map is Lipschitz in state variables. Moreover, as a consequence, we deduce the existence of a continuous selection of the set of all mild solutions of the problem considered. The proof our main result is based on a result of Bressan and Colombo concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values.

Keywords Measurable multifunction · Differential inclusion · Selection
Decomposable set

1 Introduction

In this paper we study the following problem

$$x'' \in A(t)x + F(t, x, V(x)(t)), \quad x(0) = x_0, \quad x'(0) = y_0, \tag{1}$$

where $F : [0, T] \times X \times X \to \mathscr{P}(X)$ is a set-valued map, X is a separable Banach space, $x_0, y_0 \in X$, $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{\mathscr{U}(t, s)\}_{t,s\in[0,T]}$ and $V : C(I, X) \to C(I, X)$ is a nonlinear Volterra integral operator. The general framework of evolution

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operators $\{A(t)\}_{t\geq 0}$ that define problem (1) has been developed by Kozak [12] and improved by Henriquez [10].

In the case when F does not depend on the last variable, i.e., without Volterra integral operators existence results and qualitative properties of mild solutions for problem (1) have been obtained by using fixed point techniques in several recent papers [2–5, 10, 11] etc.

In the present paper we consider the more general problem (1) and our aim is to study problem (1) when the set-valued map is Lipschitz in the second and third variable. We show first that Filippov's ideas [9] can be suitably adapted in order to obtain the existence of mild solutions of problems (1). We recall that for a first order differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem [9] consists in proving the existence of o solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. Afterwards, we obtain a continuous variant of this result; namely, we show the existence of mild solutions continuously depending on a parameter for problems (1), under Filippov type hypotheses. The key tool in the proof of this theorem is a result of Bressan and Colombo [4] concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values. As a consequence we deduce the existence of a continuous selection of the set of all mild solutions of problem (1).

We note that similar results for other classes of differential inclusions may be found in [7, 8, 13].

The paper is organized as follows: in Sect. 2 we recall some preliminary results that we use in the sequel, in Sect. 3 we obtain our Filippov type existence results and in Sect. 4 we treat the parameterized situation.

2 Preliminaries

Let denote by I the interval $[0, T]$, $T > 0$ and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$. As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|\mathrm{d}t$. With $B(X)$ we denote the Banach space of linear nounded operators on X.

In the sequel $V : C(I, X) \to C(I, X)$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t, s, x(s))ds$ where $k(.,.,.) : I \times X \times X \to X$ is a given function and $F(.,.,.) : I \times X \times X \to \mathscr{P}(X)$ is a set-valued map.

In what follows $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{\mathscr{U}(t, s)\}_{t,s\in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t.

Definition 1 ([10, 12]) A family of bounded linear operators $\mathscr{U}(t, s) : X \to X$, $(t, s) \in \Delta := \{(t, s) \in I \times I; s \leq t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t) \tag{2}$$

if

(i) For any $x \in X$, the map $(t, s) \to \mathcal{U}(t, s)x$ is continuously differentiable and

(a) $\mathcal{U}(t, t) = 0, t \in I$.
(b) If $t \in I$, $x \in X$ then $\frac{\partial}{\partial t}\mathcal{U}(t, s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t, s)x|_{t=s} = -x$.

(ii) If $(t, s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t, s)x \in D(A(t))$, the map $(t, s) \to \mathcal{U}(t, s)x$ is of class C^2 and

(a) $\frac{\partial^2}{\partial t^2}\mathcal{U}(t, s)x \equiv A(t)\mathcal{U}(t, s)x$.
(b) $\frac{\partial^2}{\partial s^2}\mathcal{U}(t, s)x \equiv \mathcal{U}(t, s)A(t)x$.
(c) $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t, s)x|_{t=s} = 0$.

(iii) If $(t, s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t, s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t, s)x$ and

(a) $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t, s)x \equiv A(t)\frac{\partial}{\partial s}\mathcal{U}(t, s)x$ and the map $(t, s) \to A(t)\frac{\partial}{\partial s}\mathcal{U}(t, s)x$ is continuous.
(b) $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t, s)x \equiv \frac{\partial}{\partial t}\mathcal{U}(t, s)A(s)x$.

As an example for Eq. (2) one may consider the problem (e.g., [10])

$$\frac{\partial^2 z}{\partial t^2}(t, \tau) = \frac{\partial^2 z}{\partial \tau^2}(t, \tau) + a(t)\frac{\partial z}{\partial t}(t, \tau), \quad t \in [0, T], \tau \in [0, 2\pi],$$

$$z(t, 0) = z(t, \pi) = 0, \quad \frac{\partial z}{\partial \tau}(t, 0) = \frac{\partial z}{\partial \tau}(t, 2\pi), \quad t \in [0, T],$$

where $a(.) : I \to \mathbf{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbf{R}, \mathbf{C})$ of 2π-periodic 2-integrable functions from $\mathbf{R}$ to $\mathbf{C}$, $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$ with domain $H^2(\mathbf{R}, \mathbf{C})$ the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbf{R}, \mathbf{C})$. It is well known that A_1 is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2, n \in \mathbf{Z}$ with associated eigenvectors $z_n(\tau) = \frac{1}{\sqrt{2\pi}}e^{in\tau}$, $n \in \mathbf{N}$. The set z_n, $n \in \mathbf{N}$ is an orthonormal basis of X. In particular, $A_1 z = \sum_{n \in \mathbf{Z}} -n^2 < z, z_n > z_n$, $z \in D(A_1)$. The cosine function is given by $C(t)z = \sum_{n \in \mathbf{Z}} \cos(nt) < z, z_n > z_n$ with the associated sine function $S(t)z = t < z, z_0 > z_0 + \sum_{n \in \mathbf{Z}^*} \frac{\sin(nt)}{n} < z, z_n > z_n$.

For $t \in I$ define the operator $A_2(t)z = a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R}, \mathbf{C})$. Set $A(t) = A_1 + A_2(t)$. It has been proved in [10] that this family generates an evolution operator as in the above definition.

Definition 2 A continuous mapping $x(.) \in C(I, X)$ is called a mild solution of problem (1) if there exists a (Bochner) integrable function $f(.) \in L^1(I, X)$ such that

$$f(t) \in F(t, x(t), V(x)(t)) \quad a.e.\,(I), \tag{3}$$

$$x(t) = -\frac{\partial}{\partial s}\mathscr{U}(t,0)x_0 + \mathscr{U}(t,0)y_0 + \int_0^t \mathscr{U}(t,s)f(s)ds,\ t \in I. \tag{4}$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1) if $f(.)$ verifies (3) and $x(.)$ is defined by (4).

We shall use the following notations for the solution sets of (1).

$$\mathscr{S}(x_0, y_0) = \{x(.); \quad x(.) \text{ is a mild solution of (1)}\}. \tag{5}$$

Finally, we recall several preliminary results we shall use in the sequel.

Lemma 1 *Let X be a separable Banach space, let $H : I \to \mathscr{P}(X)$ be a measurable set-valued map with nonempty closed values and $g, h : I \to X$, $L : I \to (0, \infty)$ measurable functions. Then one has.*

(i) The function $t \to d(h(t), H(t)$ is measurable.

(ii) If $H(t) \cap (g(t) + L(t)B) \neq \emptyset$ a.e. (I) then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

Its proof may be found in [1].

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathscr{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$. We denote by $\mathscr{D}(I, X)$ the family of all decomposable closed subsets of $L^1(I, X)$.

Next (S, d) is a separable metric space; we recall that a set-valued map $G(\cdot) : S \to \mathscr{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed. The proof of the next two lemmas may be found in [6].

Lemma 2 *Let $F^*(.,.) : I \times S \to \mathscr{P}(X)$ be a closed-valued $\mathscr{L}(I) \otimes \mathscr{B}(S)$ measurable set-valued map such that $F^*(t, .)$ is l.s.c. for any $t \in I$.*

Then the set-valued map $G(.) : S \to \mathscr{D}(I, X)$ defined by

$$G(s) = \{v \in L^1(I, X); \quad v(t) \in F^*(t, s) \quad a.e.\ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p(.) : S \to L^1(I, X)$ such that

$$d(0, F^*(t, s)) \le p(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

Lemma 3 *Let $G(.) : S \to \mathscr{D}(I, X)$ be a l.s.c. set-valued map with closed decomposable values and let $\phi(.) : S \to L^1(I, X)$, $\psi(.) : S \to L^1(I, \mathbf{R})$ be continuous such that the set-valued map $H(.) : S \to \mathscr{D}(I, X)$ defined by*

$$H(s) = cl\{v \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad a.e. \ (I)\}$$

has nonempty values.

Then H has a continuous selection, i.e. there exists a continuous mapping $h : S \to L^1(I, X)$ such that $h(s) \in H(s) \quad \forall s \in S$.

3 A Filippov Type Result

In order to establish our existence result for problem (1) we need the following hypotheses.

Hypothesis H1. (i) There exists an evolution operator $\{\mathcal{U}(t,s)\}_{t,s\in I}$ associated to the family $\{A(t)\}_{t\geq 0}$.

(ii) There exist $M, M_0 \geq 0$ such that $|\mathcal{U}(t,s)|_{B(X)} \leq M$, $|\frac{\partial}{\partial s}\mathcal{U}(t,s)| \leq M_0$, for all $(t,s) \in \Delta$.

(iii) $F(.,.,.) : I \times X \times X \to \mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(X \times X)$ measurable.

(iv) There exists $L(.) \in L^1(I, \mathbf{R}_+)$ such that, for almost all $t \in I$, $F(t,.,.)$ is $L(t)$-Lipschitz in the sense that for almost $t \in I$

$$d(F(t,x_1,y_1), F(t,x_2,y_2)) \leq L(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall\, x_1, x_2, y_1, y_2 \in X,$$

where $d(A, B)$ is the Hausdorff distance

$$d(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\}.$$

(v) $k(.,.,.) : I \times X \times X \to X$ satisfy: $\forall x \in X$, $(t,s) \to k(t,s,x)$ is measurable and $|k(t,s,x) - k(t,s,y)| \leq L(t)|x - y| \ \ a.e.(t,s) \in I \times I$, $\forall x, y \in X$.

We shall use next the following notations

$$m(t) = \int_0^t L(u)du, \quad \alpha(x) = \frac{(x+1)^2 - 1}{2}, \quad x \in \mathbf{R}.$$

In what follows we consider $u_0, v_0 \in X$, $g(.) \in L^1(I, X)$ and $y(.) \in C(I, X)$ is a mild solution of the Cauchy problem

$$y'' = A(t)y + g(t) \quad y(0) = u_0, \quad y'(0) = v_0,$$

Hypothesis H2. (i) Hypothesis H1 is satisfied.

(ii) The function $t \to p(t) := d(g(t), F(t, y(t), V(y)(t))$ is integrable on I.

Theorem 1 *Consider $\delta \geq 0$ and assume that Hypothesis H2 is satisfied. Then for any $x_0, y_0 \in X$ with $M_0|x_0 - u_0| + M|y_0 - v_0| \leq \delta$ and any $\varepsilon > 0$ there exists*

$(x(.), f(.))$ *a trajectory-selection pair of (1) such that*

$$|x(t) - y(t)| \leq \xi(t) \quad \forall\, t \in I,$$

$$|f(t) - g(t)| \leq L(t)(\xi(t) + \int_0^t L(u)\xi(u)du) + \gamma(t) + \varepsilon \quad a.e.\ (I),$$

where

$$\xi(t) = \delta e^{M\alpha(m(t))} + \int_0^t p(u)e^{M\alpha(m(t)-m(u))}du + Mt\varepsilon.$$

Proof Let $\varepsilon > 0$ and set $x_0(t) \equiv y(t)$, $f_0(t) \equiv g(t)$, $t \in I$ and for $n \geq 1$ define

$$p_n(t) = \int_0^t p(u)\frac{(\alpha(m(t) - m(u))^{n-1}}{(n-1)!}du + \frac{(\alpha(m(t))^{n-1}}{(n-1)!}(M_0|x_0 - u_0| + M|y_0 - v_0|).$$

We claim that is enough to construct the sequences $x_n(.) \in C(I, X)$, $f_n(.) \in L^1(I, X)$, $n \geq 1$ with the following properties

$$x_n(t) = -\frac{\partial}{\partial s}\mathcal{U}(t, 0)x_0 + \mathcal{U}(t, 0)y_0 + \int_0^t \mathcal{U}(t, s)f_n(s)ds, \quad \forall\, t \in I, \tag{6}$$

$$|x_1(t) - x_0(t)| \leq \delta + M(\int_0^t p(u)du + \varepsilon t) =: p_0(t) \quad \forall\, t \in I, \tag{7}$$

$$|f_1(t) - f_0(t)| \leq p(t) + \varepsilon \quad a.e.\ (I), \tag{8}$$

$$f_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e.\ (I),\ n \geq 1, \tag{9}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)(|x_n(t) - x_{n-1}(t)| + \int_0^t L(u)|x_n(u) - x_{n-1}(u)|du)\ a.e., \tag{10}$$

$$|x_n(t) - x_{n-1}(t)| \leq M^{n-1}p_n(t) \quad \forall\, t \in I. \tag{11}$$

Indeed, from (11) $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I, X)$. Thus, from (10) for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in X. Moreover, from (7) and the last inequality we have

$$|x_n(t) - y(t)| \leq \sum_{i=0}^{n-1}|x_{i+1}(t) - x_i(t)| \leq \sum_{i=0}^{n-1}M^i p_{i+1}(t) \leq \xi(t) \tag{12}$$

On the other hand, from (8), (10) and (11) we obtain for almost all $t \in I$

$$\begin{aligned} &|f_n(t) - g(t)| \le \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - g(t)| \le \\ &L(t)(\xi(t) + \int_0^t L(u)\xi(u)du) + \gamma(t) + \varepsilon. \end{aligned} \tag{13}$$

Let $x(.) \in C(I, X)$ be the limit of the Cauchy sequence $x_n(.)$. From (13) the sequence $f_n(.)$ is integrably bounded and we have already proved that for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in X. Take $f(.) \in L^1(I, X)$ with $f(t) = \lim_{n\to\infty} f_n(t)$.

Passing to the limit in (9) and using the fact that the values of F are closed we get (3); passing to the limit in (6) and using Lebesgue's dominated convergence theorem we get (4). Finally, passing to the limit in (12) and (13) we obtained the desired estimations.

It remains to construct the sequences $x_n(.)$, $f_n(.)$ with the properties in (6)–(11). The construction will be done by induction.

The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \{g(t) + (p(t) + \varepsilon)B\} \neq \emptyset \quad a.e.\ (I).$$

From Lemma 1 we find $f_1(.)$ a measurable selection of the set-valued map $H_1(t) := F(t, y(t), V(y)(t)) \cap \{g(t) + (p(t) + \varepsilon)B\}$. Obviously, $f_1(.)$ satisfy (8). Define $x_1(.)$ as in (6) with $n = 1$. Therefore, we have

$$\begin{aligned} &|x_1(t) - y(t)| \le | - \tfrac{\partial}{\partial s}\mathcal{U}(t, 0)(x_0 - u_0)| + |\mathcal{U}(t, 0)(y_0 - v_0)| + \\ &|\int_0^t \mathcal{U}(t, s)(f_1(s) - g(s))ds| \le \delta + M\int_0^t (p(s) + \varepsilon)ds = p_0(t). \end{aligned}$$

Assume that for some $N \ge 1$ we already constructed $x_n(.) \in C(I, X)$ and $f_n(.) \in L^1(I, X)$, $n = 1, 2, ...N$ satisfying (6)–(11). We define the set-valued map

$$\begin{aligned} &H_{N+1}(t) := F(t, x_N(t), V(x_N)(t)) \cap \{f_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| + \\ &\int_0^t L(u)|x_N(u) - x_{N-1}(u)|duB\}, \quad t \in I. \end{aligned}$$

The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable and from the lipschitzianity of $F(t, ., .)$ we have that for almost all $t \in I$ $H_{N+1}(t) \neq \emptyset$. We apply Lemma 1 and find a measurable selection $f_{N+1}(.)$ of $F(., x_N(.), V(x_N)(.))$ such that for almost $t \in I$

$$|f_{N+1}(t) - f_N(t)| \le L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(u)|x_N(u) - x_{N-1}(u)|du).$$

We define $x_{N+1}(.)$ as in (6) with $n = N + 1$ and we get

$|x_{N+1}(t) - x_N(t)| \le M_1 \int_0^t |f_{N+1}(u) - f_N(u)|du \le M_1 \int_0^t L(u)(|x_N(u) - x_{N-1}(u)| + \int_0^u L(s)|x_N(s) - x_{N-1}(s)|ds)du \le M_1 \int_0^t L(u)(M_1^{N-1} p_N(u) + \int_0^u L(s)M_1^{N-1} p_N(r)dr)du.$

We shall prove next that

$$\int_0^t L(u)(p_n(u) + \int_0^u L(r)p_n(r)dr)du \le p_{n+1}(t) \tag{14}$$

and therefore (11) holds true with $n = N + 1$ which completes the proof.

One has

$$\int_0^t L(u)(p_n(u) + \int_0^u L(r)p_n(r)dr)du = \int_0^t (1 + m(t) - m(u))L(u)p_n(u)du$$

$$= \int_0^t (1 + m(t) - m(u))L(u)\frac{(\alpha(m(u))^{n-1}}{(n-1)!}|x_0 - u_0|du +$$

$$\int_0^t (1 + m(t) - m(u))L(u)(\int_0^u p(r)\frac{(\alpha(m(t) - m(r))^{n-1}}{(n-1)!}dr)du \le$$

$$|x_0 - u_0| \int_0^t (1 + m(t) - m(u))L(u)\frac{(\alpha(m(u)))^{n-1}}{(n-1)!} +$$

$$\int_0^t (\int_r^t \frac{(\alpha(m(u) - m(r)))^{n-1}}{(n-1)!}(1 + m(t) - m(u))L(u))p(r)drdu.$$

According to the definition of $\alpha(.)$ we have

$$\int_0^t (1 + m(t) - m(u))L(u)\frac{\alpha(m(u)))^{n-1}}{(n-1)!}du = \int_0^t (2 + m(t))L(u)\frac{(\alpha(m(u))^{n-1}}{(n-1)!}du -$$

$$\frac{(\alpha(m(t))^n}{n!} \le (m(t) + 2)\frac{(m(t)/2 + 1)^{n-1}}{(n-1)!}\int_0^t (m(u))^{n-1}L(u)du - \frac{(\alpha(m(t)))^n}{n!}$$

$$= \frac{(\alpha(m(t)))^n}{n!}.$$

As above we deduce that

$$\int_r^t \frac{(\alpha(m(u)-m(r)))^{n-1}}{(n-1)!}(1+m(t)-m(u))L(u)du \le \frac{(\alpha(m(t)-m(r)))^n}{n!}$$

and inequality (14) is proved.

4 Continuous Family of Solutions

In order to establish our continuous version of Filippov theorem for problem (1) we need the following hypotheses.

Hypothesis H3. (i) S is a separable metric space and $a(.), b(.) : S \to X, c(.) : S \to (0, \infty)$ are continuous mappings.

(ii) There exists the continuous mappings $g(.) : S \to L^1(I, X)$, $p(.) : S \to \mathbf{R}$, $y(.) : S \to C(I, X)$ such that

$$(y(s))''(t) = A(t)y(s)(t) + g(s)(t) \quad \forall s \in S,\ t \in I$$

and

$$d(g(s)(t), F(t, y(s), V(y(s))(t)) \le p(s)(t) \quad a.e.\ (I),\ \forall\, s \in S.$$

Theorem 2 *Assume that Hypotheses H2 and H3 are satisfied.*

Then there exist the continuous mappings $x(.) : S \to C(I, X)$, $f(.) : S \to L^1(I, X)$ *such that for any* $s \in S$, $(x(s)(.), f(s)(.))$ *is a trajectory-selection pair of*

$$x'' \in A(t)x + F(t, x, V(x)(t)), \quad x(0) = a(s), \quad x'(0) = b(s)$$

and

$$|x(s)(t) - y(s)(t)| \le \xi(s)(t) \quad \forall (t, s) \in I \times S,$$

$$|f(s)(t) - g(s)(t)| \le L(t)\xi(s)(t) + p(s)(t) + c(s) \quad a.e.\ (I),\ \forall s \in S,$$

where

$$\xi(s)(t) = Me^{M\alpha(m(t))}[tc(s) + M_0|a(s) - y(s)(0)| + M|b(s) - (y(s))'(0)|] + \int_0^t p(s)(u)e^{M\alpha(m(t)-m(u))}du.$$

Proof Denote $\varepsilon_n(s) = c(s)\frac{n+1}{n+2}$, $n \ge 0$, $d(s) = M_0|a(s) - y(s)(0)| + M|b(s) - (y(s))'(0)|$ and for $n \ge 1$

$$p_n(s)(t) = M^n[\int_0^t p(s)(u)\frac{(m(t)-m(u))^{n-1}}{(n-1)!}du + \frac{(m(t))^{n-1}}{(n-1)!}t\varepsilon_n(s)] + M^{n-1}d(s).$$

Set also $x_0(s)(t) = y(s)(t)$, $f_0(s)(t) = g(s)(t)$, $\forall s \in S$.

We consider the set-valued maps $G_0(.)$, $H_0(.)$ defined, respectively, by

$$G_0(s) = \{v \in L^1(I, X); \quad v(t) \in F(t, y(s)(t), V(y(s))(t)) \quad a.e. (I)\},$$

$$H_0(s) = \text{cl}\{v \in G_0(s); \quad |v(t) - g(s)(t)| < p(s)(t) + \varepsilon_0(s)\}.$$

Since $d(g(s)(t), F(t, y(s)(t), V(y(s))(t)) \leq p(s)(t) < p(s)(t) + \varepsilon_0(s)$, according with Lemma 1, the set $H_0(s)$ is not empty.

Set $F_0^*(t,s) = F(t, y(s)(t), V(y(s))(t))$ and note that

$$d(0, F_0^*(t,s)) \leq |g(s)(t)| + p(s)(t) = p^*(s)(t)$$

and $p^*(.) : S \to L^1(I, X)$ is continuous.

Applying now Lemmas 2 and 3 we obtain the existence of a continuous selection f_0 of H_0, i.e. such that

$$f_0(s)(t) \in F(t, y(s)(t), V(y(s))(t)) \quad a.e. (I), \ \forall s \in S,$$

$$|f_0(s)(t) - g(s)(t)| \leq p_0(s)(t) = p(s)(t) + \varepsilon_0(s) \quad \forall s \in S, \ t \in I.$$

We define $x_1(s)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)a(s) + \mathcal{U}(t,0)b(s) + \int_0^t \mathcal{U}(t,u)f_0(s)(u)du$ and one has

$$|x_1(s)(t) - x_0(s)(t)| \leq M_0|a(s) - y(s)(0)| + M|b(s) - (y(s))'(0)| +$$
$$M\int_0^t |f_0(s)(u) - g(s)(u)|du \leq d(s) + M\int_0^t (p(s)(u) + \varepsilon_0(s))du = p_1(s)(t).$$

We shall construct two sequences of approximations $f_n(.) : S \to L^1(I, X), x_n(.) : S \to C(I, X)$ with the following properties

(a) $f_n(.) : S \to L^1(I, X)$, $x_n(.) : S \to C(I, X)$ are continuous.

(b) $f_n(s)(t) \in F(t, x_n(s)(t), V(x_n(s))(t))$, a.e. (I), $s \in S$.

(c) $|f_n(s)(t) - f_{n-1}(s)(t)| \leq L(t)(p_n(s)(t) + \int_0^t L(u)p_n(s)(u)du)$, a.e. (I), $s \in S$.

(d) $x_{n+1}(s)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)a(s) + \mathcal{U}(t,0)b(s) + \int_0^t \mathcal{U}(t,u)f_n(s)(u)du, \forall t \in I, s \in S$.

Suppose we have already constructed $f_i(.), x_i(.)$, $i = 1, ..., n$ satisfying (a)–(c) and define $x_{n+1}(.)$ as in (d). As in the proof of inequality (14) we have

$$\int_0^t L(u)(p_n(s)(u) + \int_0^u L(r)p_n(s)(r)dr)du \leq p_{n+1}(s)(t) - \frac{c(s)(\alpha(m(t)))^n t}{(n+2)(n+3)n!}. \tag{15}$$

From (c) and (d) one has

$$|x_{n+1}(s)(t) - x_n(s)(t)| \le M \int_0^t |f_n(s)(u) - f_{n-1}(s)(u)|du \le \\ M \int_0^t L(u)(p_n(s)(u) + \int_0^u L(r)p_n(s)(r)dr)du < p_{n+1}(s)(t). \quad (16)$$

Consider the following set-valued maps, for any $s \in S$,

$$G_{n+1}(s) = \{v \in L^1(I, X); \quad v(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) \quad a.e.\ (I)\},$$

$$H_{n+1}(s) = \mathrm{cl}\{v \in G_{n+1}(s);\ |v(t) - f_n(s)(t)| < L(t)(p_n(s)(t) + \int_0^t L(u)p_n(s)(u)du) \\ a.e.\ (I)\}.$$

To prove that $H_{n+1}(s)$ is nonempty we note first that the real function $t \to r_n(s)(t) = c(s)\frac{(MT)^{n+1}tL(t)(m(t))^n}{(n+2)(n+3)n!}$ is measurable and strictly positive for any s. From (15) we get

$$d(f_n(s)(t), F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) \le L(t)(|x_n(s)(t) - x_{n+1}(s)(t)|+ \\ \int_0^t L(u)|x_n(s)(u) - x_{n+1}(s)(u)|du) \le L(t)(p_n(s)(t) + \int_0^t L(u)p_n(s)(u)du) - r_n(s)(t)$$

and therefore according to Lemma 1 there exists $v(.) \in L^1(I, X)$ such that $v(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t))$ a.e. (I) and

$$|v(t) - f_n(s)(t)| < d(f_n(s)(t), F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) + r_n(s)(t)$$

and hence $H_{n+1}(s)$ is not empty.

Set $F^*_{n+1}(t, s) = F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t))$ and note that we may write

$$d(0, F^*_{n+1}(t, s)) \le |f_n(s)(t)| + L(t)(p_{n+1}(s)(t) + \int_0^t L(u)p_{n+1}(s)(u)du) = \\ p^*_{n+1}(s)(t) \quad a.e.\ (I)$$

and $p^*_{n+1}(.) : S \to L^1(I, X)$ is continuous.

By Lemmas 2 and 3 there exists a continuous map $f_{n+1}(.) : S \to L^1(I, X)$ such that for any $s \in S$

$$f_{n+1}(s)(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) \quad a.e.\ (I),$$

$$|f_{n+1}(s)(t) - f_n(s)(t)| \le L(t)(p_{n+1}(s)(t) + \int_0^t L(u)p_{n+1}(s)(u)du) \quad a.e.\ (I).$$

From (16) and (d) we obtain

$$|x_{n+1}(s)(.) - x_n(s)(.)|_C \le M|f_{n+1}(s)(.) - f_n(s)(.)|_1 \le \frac{(M\alpha(m(T)))^n}{n!}(M|p(s)(.)|_1 + MTc(s) + d(s)). \tag{17}$$

Therefore $f_n(s)(.)$, $x_n(s)(.)$ are Cauchy sequences in the Banach space $L^1(I, X)$ and $C(I, X)$, respectively. Let $f(.) : S \to L^1(I, X)$, $x(.) : S \to C(I, X)$ be their limits. The function $s \to M|p(s)(.)|_1 + MTc(s) + d(s)$ is continuous, hence locally bounded. Therefore (17) implies that for every $s' \in S$ the sequence $f_n(s')(.)$ satisfies the Cauchy condition uniformly with respect to s' on some neighborhood of s. Hence, $s \to f(s)(.)$ is continuous from S into $L^1(I, X)$.

From (17), as before, $x_n(s)(.)$ is Cauchy in $C(I, X)$ locally uniformly with respect to s. So, $s \to x(s)(.)$ is continuous from S into $C(I, X)$. On the other hand, since $x_n(s)(.)$ converges uniformly to $x(s)(.)$ and

$$d(f_n(s)(t), F(t, x(s)(t), V(x(s))(t)) \le L(t)(|x_n(s)(t) - x(s)(t)| + \int_0^t L(u)|x_n(s)(u) - x(s)(u)|du) \quad a.e.\ (I),\ \forall s \in S$$

passing to the limit along a subsequence of $f_n(.)$ converging pointwise to $f(.)$ we obtain

$$f(s)(t) \in F(t, x(s)(t), V(x(s))(t)) \quad a.e.\ (I),\ \forall s \in S.$$

Passing to the limit in (d) we obtain

$$x(s)(t) = -\frac{\partial}{\partial s}\mathscr{U}(t, 0)a(s) + \mathscr{U}(t, 0)b(s) + \int_0^t \mathscr{U}(t, u)f(s)(u)du.$$

By adding inequalities (c) for all n and using the fact that $\sum_{i\ge 1} p_i(s)(t) \le \xi(s)(t)$ we obtain

$$|f_{n+1}(s)(t) - g(s)(t)| \le \sum_{l=0}^n |f_{l+1}(s)(u) - f_l(s)(u)| + |f_0(s)(t) - g(s)(t)| \le \sum_{l=0}^n L(t)p_{l+1}(s)(t) + p(s)(t) + \varepsilon_0(s) \le L(t)\xi(s)(t) + p(s)(t) + c(s). \tag{18}$$

Similarly, by adding (16) we get

$$|x_{n+1}(s)(t) - y(s)(t)| \le \sum_{l=0}^n p_l(s)(t) \le \xi(s)(t). \tag{19}$$

By passing to the limit in (18) and (19) we obtain the estimates in the statement of the theorem.

Theorem 2 allows to obtain the next corollary which is a general result concerning continuous selections of the solution set of problem (1).

Hypothesis H4. Hypothesis H1 is satisfied and there exists $p_0(.) \in L^1(I, \mathbf{R}_+)$ such that $d(0, F(t, 0, V(0)(t))) \leq p_0(t)$ a.e. (I).

Theorem 3 *Assume that Hypothesis H4 is satisfied.*
Then there exists a function $x(., .) : I \times X^2 \to X$ such that

(a) $x(., (\xi, \eta)) \in \mathscr{S}(\xi, \eta)$, $\forall (\xi, \eta) \in X^2$.
(b) $(\xi, \eta) \to x(., (\xi, \eta))$ is continuous from X^2 into $C(I, X)$.

Proof We take $S = X \times X$, $a(\xi, \eta) = \xi$, $b(\xi, \eta) = \eta$ $\forall (\xi, \eta) \in X \times X$, $c(.) : X \times X \to (0, \infty)$ an arbitrary continuous function, $g(.) = 0$, $y(.) = 0$, $p(\xi, \eta)(t) = p_0(t)$ $\forall (\xi, \eta) \in X \times X$, $t \in I$ and we apply Theorem 2 in order to obtain the conclusion of the theorem.

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Factorization Method and General Second Order Linear Difference Equation

Alina Dobrogowska and Mahouton Norbert Hounkonnou

Abstract This paper addresses an investigation on a factorization method for difference equations. It is proved that some classes of second order linear difference operators, acting in Hilbert spaces, can be factorized using a pair of mutually adjoint first order difference operators. These classes encompass equations of hypergeometric type describing classical orthogonal polynomials of a discrete variable.

Keywords Second order difference equations · Factorization method · Raising and lowering operators · Discrete polynomials

1 Introduction

The description of many problems in physics and mathematics, especially in probability, gives rise to difference equations. Difference equations relate to differential equations as discrete mathematics relates to continuous mathematics. The study of differential equations shows that even supposedly elementary examples can be hard to solve. By contrast, elementary difference equations are relatively easy to deal with. In general, the interest in difference equations can be justified for a number of reasons. Difference equations frequently arise when modelling real life situations. Since difference equations are readily handled by numerical methods, a standard approach to solving a nasty differential equation is to convert it to an approximately equivalent difference equation.

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A peculiar question in the field of differential or difference equations then remains to find appropriate analytical methods for their exact solvability. For differential or difference equations having polynomial solutions, it is well known that their solvability is closely related to the factorizability of their associated operators, (see [3] and references therein).

In the last few decades it was given a more prominent place in the discussion of operator factorization methods for solving second order differential or difference equations, the concept of which goes back to Darboux [6]. Later the method was rediscovered many times, in particular by the founders of quantum mechanics, (see Dirac [7], Schrödinger [25]), while solving the Schrödinger equation to study the angular momentum or the harmonic oscillator. In the work [17], which is now considered to be fundamental, Infeld and Hull summarized the quantum mechanical applications of the method. Later this technique was extended, see [14, 19, 20]. Some results were obtained also for q-difference and more general difference equations [1, 2, 4, 5, 10–12, 16, 22, 23]. In addition, special cases such as the factorization of Jacobi operators were also investigated [15]. If the operator in a second order linear ordinary differential or difference equation can be factorized, the problem of solving the equation is reduced to solving two first order linear equations; the latter can readily be solved.

Therefore, a nodal point in the application of this method consists in the existence of a pair of first order differential or difference operators, which the second order differential or difference operator decomposes into as their product, (see (9) in this work).

Using this method, we are here able to find the explicit solutions (18), via (17), to the eigenvalue problem (14) in a simple way. For additional readings, see monographs [13, 18, 21].

This work is an extension of a previous work [10]. Some results obtained in [1, 8, 9, 16] are used, and adapted to our context.

The paper is organized as follows. In Sect. 2, a detailed investigation of the factorization method applied to second order difference operators is given. In Sect. 3, our main results are described. Under given assumptions, the problem of operator factorization is solved.

2 Basic Tools

In this section, in the beginning, we introduce some notations and recall some basic facts about the factorization method. Let $\ell_k(\mathbb{Z}, \mathbb{R})$ and $\ell_k(\mathbb{Z}, \mathbb{C})$, $k \in \mathbb{N} \cup \{0\}$, be the sets of real-valued and complex-valued sequences $\{x(n)\}_{n\in\mathbb{Z}}$, respectively. We define the scalar product on $\ell_k(\mathbb{Z}, \mathbb{C})$ as follows:

$$\langle x|y\rangle_k := \sum_{n=a}^{b} \overline{x(n)}\, y(n)\rho_k(n), \tag{1}$$

where $a, b \in \mathbb{Z}$, $(a < b)$, and ρ_k is a weight function. We assume that the weight sequence satisfies the Pearson difference equation

$$\triangle \left(b_k(n)\rho_k(n)\right) = \left(c_k(n) - b_k(n)\right)\rho_k(n), \tag{2}$$

and the recursion relation

$$\rho_{k-1}(n) = c_k(n)\rho_k(n), \tag{3}$$

where $\{b_k\}$ and $\{c_k\}$ are some real-valued sequences. Moreover, the function ρ_k fulfills the boundary conditions

$$b_k(a)\rho_k(a) = b_k(b+1)\rho_k(b+1) = 0. \tag{4}$$

The forward and backward difference operators are defined by

$$\triangle\, x(n) := \left(\mathbf{S}^+ - \mathbf{1}\right) x(n) = x(n+1) - x(n), \tag{5}$$

$$\nabla\, x(n) := \left(\mathbf{1} - \mathbf{S}^-\right) x(n) = x(n) - x(n-1), \tag{6}$$

where the shift operators

$$\mathbf{S}^{\pm} x(n) := x(n \pm 1). \tag{7}$$

We want to apply the factorization method to the second order difference operators $\mathbf{H_k} : \ell_k(\mathbb{Z}, \mathbb{C}) \longrightarrow \ell_k(\mathbb{Z}, \mathbb{C})$ given by

$$\mathbf{H_k} := z_k(n)\mathbf{S}^+ + w_k(n)\mathbf{S}^- + v_k(n), \tag{8}$$

where $\{z_k\}$, $\{w_k\}$ and $\{v_k\}$ are real-valued sequences, $k \in \mathbb{N} \cup \{0\}$. Introducing the annihilation $\mathbf{A_k} : \ell_k(\mathbb{Z}, \mathbb{C}) \longrightarrow \ell_{k-1}(\mathbb{Z}, \mathbb{C})$, and creation operators $\mathbf{A_k^*} : \ell_{k-1}(\mathbb{Z}, \mathbb{C}) \longrightarrow \ell_k(\mathbb{Z}, \mathbb{C})$ (also called lowering and raising operators, respectively), we rewrite the above operators $\mathbf{H_k}$ in the form

$$\mathbf{H_k} := \mathbf{A_k^*}\mathbf{A_k} + \alpha_k = \mathbf{A_{k+1}}\mathbf{A_{k+1}^*} + \alpha_{k+1}, \tag{9}$$

where α_k are real constants. We construct the annihilation operator as

$$\mathbf{A_k} := \triangle + f_k(n) = \mathbf{S}^+ + f_k(n) - 1, \tag{10}$$

where $\{f_k\} \in \ell_k(\mathbb{Z}, \mathbb{R})$.

We seek the adjoint operator $\mathbf{A_k^*}$ of $\mathbf{A_k}$, obeying:

$$\langle \mathbf{A_k^*} x_{k-1} | y_k \rangle_k = \langle x_{k-1} | \mathbf{A_k} y_k \rangle_{k-1}. \tag{11}$$

A simple computation using (11) yields

$$\begin{aligned}\langle x_{k-1}|\mathbf{A_k}y_k\rangle_{k-1} &= \sum_{n=a}^{b}\overline{x_{k-1}(n)}y_k(n+1)\rho_{k-1}(n)\\ &+\sum_{n=a}^{b}\overline{x_{k-1}(n)}(f_k(n)-1)y_k(n)\rho_{k-1}(n)\\ &=\sum_{n=a+1}^{b+1}b_k(n)\overline{x_{k-1}(n-1)}y_k(n)\rho_k(n)\\ &+\sum_{n=a}^{b}(f_k(n)-1)c_k(n)\overline{x_{k-1}(n)}y_k(n)\rho_k(n)\\ &=\langle\big(b_k(n)\mathbf{S}^-+(f_k(n)-1)\,c_k(n)\big)\,x_{k-1}|y_k\rangle_k, \end{aligned} \tag{12}$$

where we applied the formulas (2)–(4). Finally, we obtain the explicit expression for the adjoint operator (also called creation operator)

$$\begin{aligned}\mathbf{A_k^*} &= -b_k(n)\bigtriangledown + b_k(n) + (f_k(n)-1)\,c_k(n)\\ &= b_k(n)\mathbf{S}^- + (f_k(n)-1)\,c_k(n). \end{aligned} \tag{13}$$

This type of factorization was presented in detail in papers [8, 11, 16] for $\tau-$, $q-$ and (q, h)–cases, respectively. Moreover, different cases, when the sequence b_k does not depend on parameter k, were considered in [10].

The operator $\mathbf{H_k} = \mathbf{A_k^*A_k} + \alpha_k$ is selfadjoint on $\ell_k(\mathbb{Z}, \mathbb{C})$. Its eigenvalue equation reads:

$$\mathbf{H_k}x_k^l(n) = \lambda_k^l x_k^l(n). \tag{14}$$

It is well known that the factorization gives us the eigenfunctions and corresponding eigenvalues. Indeed, the eigenvalue problem for the chain of operators (9) is equivalent to the two following equations:

$$\mathbf{A_k^*A_k}x_k^l(n) = \left(\lambda_k^l - \alpha_k\right)x_k^l(n), \tag{15}$$

$$\mathbf{A_{k+1}A_{k+1}^*}x_k^l(n) = \left(\lambda_k^l - \alpha_{k+1}\right)x_k^l(n). \tag{16}$$

Solving the first order homogeneous linear equation

$$\mathbf{A_k}x_k^0(n) = 0, \tag{17}$$

we observe that (15), (16) and (17) imply that the functions

$$x_k^{k-p}(n) = \mathbf{A_k^*A_{k-1}^*}\ldots\mathbf{A_{p+1}^*}x_p^0(n) \tag{18}$$

are solutions of the eigenvalue problem (14) for the eigenvalues $\lambda_k^{k-p} = \alpha_p$.

3 Factorization of Operators

In this section, we solve the factorization problem (9) under some assumptions. Finding a general solution remains a cumbersome task.

Comparing the coefficients of $\mathbf{S}^-$, $\mathbf{S}^+$ and $\mathbf{1}$ on both sides of the expression (9), we obtain the necessary and sufficient conditions for the existence of a factorizing pair of first order difference operators, $(\mathbf{A_k^*}, \mathbf{A_k})$, as follows:

$$f_{k+1}(n) - 1 = \frac{b_k(n)}{b_{k+1}(n)} \left(f_k(n-1) - 1\right), \tag{19}$$

$$c_{k+1}(n) = \frac{b_{k+1}(n)}{b_k(n)} c_k(n-1), \tag{20}$$

$$\begin{aligned} b_k(n) - b_{k+1}(n+1) = \alpha_{k+1} - \alpha_k + \frac{b_k(n)}{b_{k+1}(n)} (f_k(n-1) - 1)^2 c_k(n-1) \\ - (f_k(n) - 1)^2 c_k(n). \end{aligned} \tag{21}$$

The conditions (19) and (20) give us the transformation formulas for the sequences $\{f_k\}$ and $\{c_k\}$ as below:

$$f_k(n) = \prod_{i=1}^{k} \frac{b_{k-i}(n-i+1)}{b_{k-i+1}(n-i+1)} \Big(f_0(n-k) - 1\Big) + 1 \tag{22}$$

and

$$c_k(n) = \prod_{i=1}^{k} \frac{b_{k-i+1}(n-i+1)}{b_{k-i}(n-i+1)} c_0(n-k). \tag{23}$$

3.1 Example 1

We assume that $b_{k+1}(n) = b_k(n) =: b_0(n)$, i.e. the sequence $\{b_k\}$ does not depend on parameters k, see [10]. We show that, under this assumption, we can find a general solution to the factorization problem (9), i.e. we can solve the conditions (19)–(21).

We have:

$$\begin{cases} f_{k+1}(n) = f_k(n-1) \\ c_{k+1}(n) = c_k(n-1) \end{cases} \tag{24}$$

yielding

$$\begin{cases} f_k(n) = f_0(n-k) \\ c_k(n) = c_0(n-k). \end{cases} \tag{25}$$

Now, let us solve the third condition. The requirement (21), using the substitution $G_k(n) = (f_k(n) - 1)^2 \, c_k(n) - b_0(n+1)$, is equivalent to the equation

$$G_k(n) = G_k(n-1) + \alpha_{k+1} - \alpha_k. \tag{26}$$

By iterating we find

$$G_k(n) = G_k(0) + n\,(\alpha_{k+1} - \alpha_k)\,. \tag{27}$$

This gives us a formula for the sequence $\{b_0\}$:

$$b(n+1) = (f_0(n-k) - 1)^2\, c_0(n-k) - G_k(0) - n\,(\alpha_{k+1} - \alpha_k)\,. \tag{28}$$

But the left-hand side of the expression (28) does not depend on the parameter k. Then, we obtain the following sequence of conditions on the sequences $\{f_0\}$ and $\{c_0\}$:

$$(f_0(n) - 1)^2\, c_0(n) - G_0(0) - n\,(\alpha_1 - \alpha_0) = (f_0(n-k) - 1)^2\, c_0(n-k) \\ - G_k(0) - n\,(\alpha_{k+1} - \alpha_k)\,, \tag{29}$$

for all $k \in \mathbb{N}$. By introducing $F(n) = (f_0(n) - 1)^2\, c_0(n)$, we can write the above equation in the form:

$$F(n) = F(n-k) + G_0(0) - G_k(0) - n\,(\alpha_{k+1} - \alpha_k - \alpha_1 + \alpha_0)\,. \tag{30}$$

For $k = 1$, we get

$$F(n) = F(n-1) + G_0(0) - G_1(0) - n\,(\alpha_2 - 2\alpha_1 + \alpha_0)\,. \tag{31}$$

Next, by iteration we find

$$F(n) = F(0) + n\,(G_0(0) - G_1(0)) - \frac{n(n+1)}{2}\,(\alpha_2 - 2\alpha_1 + \alpha_0)\,. \tag{32}$$

We then arrive at a relationship between the sequences $\{f_0\}$ and $\{c_0\}$:

$$(f_0(n) - 1)^2\, c_0(n) = F(0) + n\,(G_0(0) - G_1(0)) - \frac{n(n+1)}{2}\,(\alpha_2 - 2\alpha_1 + \alpha_0)\,. \tag{33}$$

In addition, substituting (32) to (29) (because it is valid for all $k \in \mathbb{N}$), we find a recurrence relation on the constants α_k:

$$\alpha_{k+1} = \alpha_k + \alpha_1 - \alpha_0 + k\,(\alpha_2 - 2\alpha_1 + \alpha_0) \tag{34}$$

and the form of the constant

$$G_k(0) = G_0(0) + k\,(G_1(0) - G_0(0)) - \frac{k(k-1)}{2}\,(\alpha_2 - 2\alpha_1 + \alpha_0)\,. \tag{35}$$

A straightforward calculation affords

$$\alpha_k = \alpha_0 + k(\alpha_1 - \alpha_0) + \frac{k(k-1)}{2}(\alpha_2 - 2\alpha_1 + \alpha_0). \tag{36}$$

To sum up, the construction presented in (9) provides the chain of operators $\mathbf{H_k}$ parametrized by the freely chosen sequence $\{c_0\}$ and real parameters $\alpha_0, \alpha_1, \alpha_2, F(0)$, $G_0(0)$ and $G_1(0)$.

3.2 *Example 2*

Let us consider a case when $f_k(n) \equiv 0$. Then, the conditions (19)–(21) can be rewritten in the form:

$$b_{k+1}(n) = b_k(n) =: b_0(n), \tag{37}$$

$$c_k(n) = c_0(n-k), \tag{38}$$

$$b_0(n) - b_0(n+1) = \alpha_{k+1} - \alpha_k + c_0(n-k-1) - c_0(n-k). \tag{39}$$

This is a special case of Example 1. We find $b_0(n)$ by induction in the following form:

$$b_0(n) = b_0(0) - n(\alpha_{k+1} - \alpha_k) - c_0(-1-k) + c_0(n-1-k). \tag{40}$$

From (33) we obtain that the sequence $\{c_0\}$ is a polynomial of degree two:

$$c_0(n) = c_0(0) + n(G_0(0) - G_1(0)) - \frac{n(n+1)}{2}(\alpha_2 - 2\alpha_1 + \alpha_0). \tag{41}$$

Then, the relation $\mathbf{H_k} x_k^l(n) = \lambda_k^l x_k^l(n)$ is equivalent to

$$\Big(-b_0(n)\nabla + b_0(n) - c_0(n-k)\Big)\Delta x_k^l(n) = (\lambda_k^l - \alpha_k)x_k^l(n), \tag{42}$$

and the eigenvalue problem (14) is reduced to the difference equation of hypergeometric type:

$$-b_0(n)\bigtriangledown\bigtriangleup x_k^l(n) - (c_0(n-k) - b_0(n))\bigtriangleup x_k^l(n) + \left(\alpha_k - \lambda_k^l\right)x_k^l(n) = 0. \tag{43}$$

It is not difficult to see that b_0 is a second degree polynomial while the difference $c_0(n-k) - b_0(n)$ is a first degree polynomial. From Eq. (17), we find that the ground state is a constant sequence $\{x_k^0(n) \equiv 1\}$ (normalized to one) with the sequence of eigenvalues $\{\lambda_k^0 = \alpha_k = \alpha_0 + k(\alpha_1 - \alpha_0) + \frac{k(k-1)}{2}(\alpha_2 - 2\alpha_1 + \alpha_0)\}$. Expression (18) gives us a formula for polynomials

$$P_l(n) = x_k^l(n) = \prod_{i=0}^{l-1} \left(b_0(n)\mathbf{S}^- - c_0(n-k+i)\right) 1 \tag{44}$$

corresponding to eigenvalues $\lambda_k^l = \alpha_{k-l}$. Using the identity $\bigtriangledown\triangle = \triangle\bigtriangledown$ we transform the above Eq. (43) into the standard form

$$\sigma(n) \triangle \bigtriangledown x_k^l(n) + \tau(n) \triangle x_k^l(n) + \lambda x_k^l(n) = 0, \tag{45}$$

where

$$\sigma(n) = -b_0(n) = \frac{1}{2}(\alpha_2 - 2\alpha_1 + \alpha_0) n^2 + n \left(G_1(0) - G_0(0) + \alpha_1 - \alpha_0 - \frac{1}{2}(\alpha_2 - 2\alpha_1 + \alpha_0)\right) - b_0(0), \tag{46}$$

$$\tau(n) = b_0(n) - c_0(n-k) = (\alpha_0 - \alpha_1 + (1-k)(\alpha_2 - 2\alpha_1 + \alpha_0)) n + b_0(0) - c_0(0) + k(G_0(0) - G_1(0)) + \frac{k(k-1)}{2}(\alpha_2 - 2\alpha_1 + \alpha_0), \tag{47}$$

$$\lambda = \alpha_k - \lambda_k^l = \alpha_k - \alpha_{k-l} = l(\alpha_1 - \alpha_0) + \left(kl - \frac{l(l+1)}{2}\right)(\alpha_2 - 2\alpha_1 + \alpha_0) = -l\left(\tau'(n) + \frac{l-1}{2}\sigma''(n)\right). \tag{48}$$

It is well known that the above Eq. (43) describes classical orthogonal polynomials of a discrete variable such as the Charlier, Meixner, Kravchuk, Hahn polynomials. See [2, 24] for more details.

3.3 Example 3

We assume that $b_{k+1}(n) := \gamma_k b_k(n)$, where γ_k is some constant different from zero and one. Then, we get:

$$f_k(n) - 1 = \prod_{i=1}^{k} \gamma_{k-i}^{-1}(f_0(n-k) - 1), \tag{49}$$

$$c_k(n) = \prod_{i=1}^{k} \gamma_{k-i} c_0(n-k), \tag{50}$$

and

$$b_k(n) - \gamma_k b_k(n+1) = \alpha_{k+1} - \alpha_k + \gamma_k^{-1}(f_k(n-1) - 1)^2 c_k(n-1) - (f_k(n) - 1)^2 c_k(n). \tag{51}$$

Using previous results, (see Example 1), we solve by analogy the above difference equation. By introducing $R_k(n) = (f_k(n) - 1)^2 c_k(n) - \gamma_k b_k(n+1)$, we can write this equation in the form

$$R_k(n) = \gamma_k^{-1} R_k(n-1) + \alpha_{k+1} - \alpha_k. \tag{52}$$

By iterating we find

$$R_k(n) = \gamma_k^{-n} R_k(0) + \frac{1-\gamma_k^{-n}}{1-\gamma_k^{-1}} \left(\alpha_{k+1} - \alpha_k\right). \tag{53}$$

From here, expressing everything by the initial data we obtain

$$\begin{aligned} b_k(n+1) = {} & \gamma_k^{-1}\gamma_{k-1}^{-1}\ldots\gamma_0^{-1}\left(f_0(n-k)-1\right)^2 c_0(n-k) - \gamma_k^{-n-1} R_k(0) \\ & - \gamma_k^{-1} \frac{1-\gamma_k^{-n}}{1-\gamma_k^{-1}} \left(\alpha_{k+1} - \alpha_k\right). \end{aligned} \tag{54}$$

This must be consistent with the initial assumption, i.e. $b_{k+1}(n) := \gamma_k b_k(n)$. Then, we get the condition on the sequences $\{f_0\}$ and $\{c_0\}$:

$$\begin{aligned} & \gamma_{k+1}^{-1}\gamma_k^{-1}\ldots\gamma_0^{-1}\left(f_0(n-k-1)-1\right)^2 c_0(n-k-1) - \gamma_{k+1}^{-n-1} R_{k+1}(0) \\ & - \gamma_{k+1}^{-1}\frac{1-\gamma_{k+1}^{-n}}{1-\gamma_{k+1}^{-1}}\left(\alpha_{k+2}-\alpha_{k+1}\right) = \gamma_{k-1}^{-1}\ldots\gamma_0^{-1}\left(f_0(n-k)-1\right)^2 c_0(n-k) - \gamma_k^{-n} R_k(0) \\ & - \frac{1-\gamma_k^{-n}}{1-\gamma_k^{-1}}\left(\alpha_{k+1}-\alpha_k\right). \end{aligned} \tag{55}$$

Again, by entering the following auxiliary function:

$$S_k(n) = \gamma_{k-1}^{-1}\ldots\gamma_0^{-1}\left(f_0(n-k)-1\right)^2 c_0(n-k),$$

we get a recursion relation for S_k:

$$\begin{aligned} S_k(n) = {} & \gamma_{k+1}^{-1}\gamma_k^{-1} S_k(n-1) - \gamma_{k+1}^{-n-1} R_{k+1}(0) + \gamma_k^{-n} R_k(0) \\ & - \gamma_{k+1}^{-1}\frac{1-\gamma_{k+1}^{-n}}{1-\gamma_{k+1}^{-1}}\left(\alpha_{k+2}-\alpha_{k+1}\right) + \frac{1-\gamma_k^{-n}}{1-\gamma_k^{-1}}\left(\alpha_{k+1}-\alpha_k\right). \end{aligned} \tag{56}$$

By iteration,

$$
\begin{aligned}
S_k(n) &= \left(\gamma_{k+1}^{-1}\gamma_k^{-1}\right)^n S_k(0) - \gamma_{k+1}^{-n-1}\frac{1-\gamma_k^{-n}}{1-\gamma_k^{-1}}R_{k+1}(0) + \gamma_k^{-n}\frac{1-\gamma_{k+1}^{-n}}{1-\gamma_{k+1}^{-1}}R_k(0) \\
&- \gamma_{k+1}^{-1}\left(\alpha_{k+2}-\alpha_{k+1}\right)\sum_{i=0}^{n-1}\left(\gamma_{k+1}^{-1}\gamma_k^{-1}\right)^i \frac{1-\gamma_{k+1}^{-n+i}}{1-\gamma_{k+1}^{-1}} \\
&+ \left(\alpha_{k+1}-\alpha_k\right)\sum_{i=0}^{n-1}\left(\gamma_{k+1}^{-1}\gamma_k^{-1}\right)^i \frac{1-\gamma_k^{-n+i}}{1-\gamma_k^{-1}}. \qquad (57)
\end{aligned}
$$

4 Concluding Remarks

In this work, we have investigated a factorization method for difference equations, adapting and extending previous results known in the literature. We have showed that some classes of second order linear difference operators, acting in Hilbert spaces, are factorizable using a pair of mutually adjoint first order difference operators. These classes encompass equations of hypergeometric type describing classical orthogonal polynomials of a discrete variable. Other classes of difference equations are still under consideration, and will be in the core of our forthcoming papers.

An interesting outlook on which we are also working is the extension of this scheme to classes of higher order difference equations. It is in particular expected that this method for fourth order equations may allow to derive what one can call Krall–Laguerre–Hahn polynomials.

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Homogeneous Boundary Problem for the Compressible Viscous and Heat-Conducting Micropolar Fluid Model with Cylindrical Symmetry

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Abstract We consider nonstationary 3-D flow of a compressible viscous and heat-conducting micropolar fluid which is in the thermodynamical sense perfect and polytropic. We analyze the problem on the domain that is bounded by two coaxial cylinders which present solid thermo-insulated walls. Therefore we assume the cylindrical symmetry of the solution. In this work we present the existence and uniqueness results for corresponding problem with homogeneous boundary data for velocity, microrotation and heat flux, under the additional assumption that the initial density and initial temperature are strictly positive.

Keywords Micropolar fluids · Homogeneous boundary problem · Cylindrical symmetry

1 Introduction

The micropolar fluid model enables us to consider some physical phenomena that cannot be treated by the classical Navier–Stokes equations, with special emphasis to phenomena at the micro level. In this model microphenomena are modelled by new hydrodynamic variable which is called microrotation.

In this work we analyze the compressible flow of an isotropic, viscous and heat conducting micropolar fluid, which is in the thermodynamical sense perfect and polytropic. This kind of flow was introduced by Mujaković in [6], where she analysed the one dimensional model. Here we analyse the motion of the described fluid between two coaxial cylinders, which enables us to consider the cylindrically symmetric solution to the governing system, which is introduced in [3]. The motion of fluid between two concentric spheres has also been analysed and for details and recent progress in that model we refer to [1].

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The paper is organized as follows. In the next section we will describe the governing three-dimensional system and derive its cylindrically symmetric form in the Lagrangian description. Then we will give an overview of the current progress in mathematical analysis of this problem. We will introduce the generalized solution to the problem together with the existence and uniqueness theorem.

2 The Mathematical Model

The mathematical model of the described fluid is stated in the book of G. Lukaszewicz [5] and reads

$$\dot{\rho} = -\rho\nabla\cdot\mathbf{v}, \tag{1}$$

$$\rho\dot{\mathbf{v}} = \nabla\cdot\mathbf{T} + \rho\mathbf{f}, \tag{2}$$

$$\rho j_I\dot{\boldsymbol{\omega}} = \nabla\cdot\mathbf{C} + \mathbf{T}_x + \rho\mathbf{g}, \tag{3}$$

$$\rho\dot{E} = -\nabla\cdot\mathbf{q} + \mathbf{T} : \nabla\mathbf{v} + \mathbf{C} : \nabla\boldsymbol{\omega} - \mathbf{T}_x\cdot\boldsymbol{\omega}, \tag{4}$$

$$\mathbf{T}_{ij} = (-p + \lambda\mathbf{v}_{k,k})\delta_{ij} + \mu\left(\mathbf{v}_{i,j} + \mathbf{v}_{j,i}\right) + \mu_r\left(\mathbf{v}_{j,i} - \mathbf{v}_{i,j}\right) - 2\mu_r\varepsilon_{mij}\boldsymbol{\omega}_m, \tag{5}$$

$$\mathbf{C}_{ij} = c_0\boldsymbol{\omega}_{k,k}\delta_{ij} + c_d\left(\boldsymbol{\omega}_{i,j} + \boldsymbol{\omega}_{j,i}\right) + c_a\left(\boldsymbol{\omega}_{j,i} - \boldsymbol{\omega}_{i,j}\right), \tag{6}$$

$$\mathbf{q} = -k\nabla\theta, \tag{7}$$

$$p = R\rho\theta, \tag{8}$$

$$E = c_v\theta. \tag{9}$$

Equations (1)–(4) are, respectively, local forms of conservation laws for the mass, momentum, momentum moment and energy. Equations (5)–(6) are constitutive equations for the micropolar continuum. Equation (7) is the Fourier law and Eqs. (8)–(9) present the assumptions that our fluid is perfect and polytropic. We have the following notations:

- ρ - mass density,
- $\mathbf{v}$ - velocity,
- $\boldsymbol{\omega}$ - microrotation velocity,
- E - internal energy density,
- θ - absolute temperature,
- $\mathbf{T}$ - stress tensor,
- $\mathbf{C}$ - couple stress tensor,
- $\mathbf{q}$ - heat flux density vector,
- $\mathbf{f}$ - body force density,
- $\mathbf{g}$ - body couple density,
- p - pressure,
- j_I - microinertia density ($j_I > 0$),
- λ, μ - coefficients of viscosity,
- μ_r, c_0, c_d, c_a - coefficients of microviscosity,
- k - heat conduction coefficient ($k \geq 0$),
- R - specific gas constant,
- c_v - specific heat for a constant volume ($c_v > 0$).

Coefficients of viscosity and coefficients of microviscosity are related through the Clausius–Duhamel inequalities, as follows:

$$\mu \geq 0,\ 3\lambda + 2\mu \geq 0,\ \mu_r \geq 0,\ c_d \geq 0,\ 3c_0 + 2c_d \geq 0,\ |c_d - c_a| \leq c_d + c_a. \quad (10)$$

Vector $\mathbf{T}_x$ in Eqs. (3) and (4) is an axial vector with the Cartesian components $(\mathbf{T}_x)_i = \varepsilon_{ijk}\mathbf{T}_{jk}$, where ε_{ijk} is the Levi-Civita alternating tensor.[1] The differential (dot) operator in Eqs. (1)–(4) denotes the material derivative defined by

$$\dot{\mathbf{a}} = \frac{\partial \mathbf{a}}{\partial t} + (\nabla \mathbf{a}) \cdot \mathbf{v},$$

and the colon operator in Eq. (4) is the scalar product of tensors, i.e. $\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$.

We take the following homogeneous boundary conditions:

$$\mathbf{v}|_{\partial\Omega} = 0, \quad \boldsymbol{\omega}|_{\partial\Omega} = 0, \quad \left.\frac{\partial\theta}{\partial\boldsymbol{\nu}}\right|_{\partial\Omega} = 0, \quad (11)$$

where $\Omega \subset \mathbf{R}^3$ is the spatial domain of our problem and the vector $\boldsymbol{\nu}$ is the exterior unit normal vector. For simplicity reasons, we also assume that

$$\mathbf{f} = \mathbf{g} = 0. \quad (12)$$

To simplify the system (1)–(9) we will first substitute the (5)–(9) into (1)–(4) together with (2) and (12). We obtain:

$$\frac{\partial\rho}{\partial t} = -(\nabla\rho) \cdot \mathbf{v} - \rho\nabla \cdot \mathbf{v}, \quad (13)$$

$$\begin{aligned}\rho\frac{\partial \mathbf{v}}{\partial t} &= -\rho(\nabla\mathbf{v}) \cdot \mathbf{v} - R\nabla(\rho\theta) + (\lambda + \mu - \mu_r)\nabla(\nabla \cdot \mathbf{v}) \\ &\quad + (\mu + \mu_r)\Delta\mathbf{v} + 2\mu_r\nabla \times \boldsymbol{\omega},\end{aligned} \quad (14)$$

$$\begin{aligned}j_I\rho\frac{\partial\boldsymbol{\omega}}{\partial t} &= -\rho(\nabla\boldsymbol{\omega}) \cdot \mathbf{v} + 2\mu_r(\nabla \times \mathbf{v} - 2\boldsymbol{\omega}) \\ &\quad + (c_0 + c_d - c_a)\nabla(\nabla \cdot \boldsymbol{\omega}) + (c_d + c_a)\Delta\boldsymbol{\omega},\end{aligned} \quad (15)$$

$$\begin{aligned}c_v\rho\frac{\partial\theta}{\partial t} &= -c_v\rho(\nabla\theta) \cdot \mathbf{v} + k_\theta\Delta\theta - R\rho\theta(\nabla \cdot \mathbf{v}) + \lambda(\nabla \cdot \mathbf{v})^2 \\ &+ \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T) : (\nabla\mathbf{v} + (\nabla\mathbf{v})^T) + 4\mu_r\left(\frac{1}{2}\nabla \times \mathbf{v} - \boldsymbol{\omega}\right)^2 \\ &+ c_0(\nabla \cdot \boldsymbol{\omega})^2 + (c_d + c_a)\nabla\boldsymbol{\omega} : \nabla\boldsymbol{\omega} + (c_d - c_a)\nabla\boldsymbol{\omega} : (\nabla\boldsymbol{\omega})^T.\end{aligned} \quad (16)$$

[1] We assume the Einstein notation for summation.

Boundary conditions (11) mean that we analyze the flow of the fluid through a chamber with solid thermoinsulated walls. Here it is the flow between two coaxial cylinders and we have

$$\Omega = \{(x_1, x_2, x_3) \in \mathbf{R}^3, a < r < b,\ x_3 \in \mathbf{R}\},\ a > 0,\ r = \sqrt{x_1^2 + x_2^2}. \tag{17}$$

Because of the geometry of the spatial domain we introduce the cylindrical coordinate system $(O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by

$$\mathbf{e}_1 = \frac{1}{r}(x_1, x_2, 0), \quad \mathbf{e}_2 = \frac{1}{r}(-x_2, x_1, 0), \quad \mathbf{e}_3 = (0, 0, 1),$$

as well as cylindrically symmetric initial conditions

$$\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}) = \rho_0(r),\ \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) = v_0^r(r)\mathbf{e}_1 + v_0^\varphi(r)\mathbf{e}_2 + v_0^z(r)\mathbf{e}_3, \tag{18}$$

$$\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}) = \theta_0(r),\ \boldsymbol{\omega}(\mathbf{x}, 0) = \boldsymbol{\omega_0}(\mathbf{x}) = \omega_0^r(r)\mathbf{e}_1 + \omega_0^\varphi(r)\mathbf{e}_2 + \omega_0^z(r)\mathbf{e}_3 \tag{19}$$

where $\rho_0, v_0^r, v_0^\varphi, v_0^z, \theta_0, \omega_0^r, \omega_0^\varphi, \omega_0^z$ are given real functions of one variable on $]a, b[$.[2] Therefore we expect that the solution depends only on the radial variable r and the time variable t, so we take

$$\rho(\mathbf{x}, t) = \rho(r, t), \quad \theta(\mathbf{x}, t) = \theta(r, t), \tag{20}$$

$$\mathbf{v}(\mathbf{x}, t) = v^r(r, t)\mathbf{e}_1 + v^\varphi(r, t)\mathbf{e}_2 + v^z(r, t)\mathbf{e}_3, \tag{21}$$

$$\boldsymbol{\omega}(\mathbf{x}, t) = \omega^r(r, t)\mathbf{e}_1 + \omega^\varphi(r, t)\mathbf{e}_2 + \omega^z(r, t)\mathbf{e}_3, \tag{22}$$

for $(r, t) \in]a, b[\times]0, T[$. Using these assumptions the spatial domain (17) becomes a one-dimensional domain $]a, b[$. The governing system now takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial r} v^r + \rho \left(\frac{v^r}{r} + \frac{\partial v^r}{\partial r} \right) = 0, \tag{23}$$

$$\rho \left(\frac{\partial v^r}{\partial t} + \frac{\partial v^r}{\partial r} v^r \right) = -R \frac{\partial}{\partial r}(\rho\theta) + (\lambda + 2\mu) \frac{\partial}{\partial r} \left(\frac{\partial v^r}{\partial r} + \frac{v^r}{r} \right) + \rho \frac{(v^\varphi)^2}{r}, \tag{24}$$

$$\rho \left(\frac{\partial v^\varphi}{\partial t} + \frac{\partial v^\varphi}{\partial r} v^r \right) = (\mu + \mu_r) \frac{\partial}{\partial r} \left(\frac{\partial v^\varphi}{\partial r} + \frac{v^\varphi}{r} \right) - \rho \frac{v^r v^\varphi}{r} - 2\mu_r \frac{\partial \omega^z}{\partial r}, \tag{25}$$

$$\rho \left(\frac{\partial v^z}{\partial t} + \frac{\partial v^z}{\partial r} v^r \right) = (\mu + \mu_r) \left(\frac{\partial^2 v^z}{\partial r^2} + \frac{1}{r} \frac{\partial v^z}{\partial r} \right) + 2\mu_r \left(\frac{\partial \omega^\varphi}{\partial r} + \frac{\omega^\varphi}{r} \right), \tag{26}$$

[2] a and b are the radii of boundary cylinders from (17).

$$\rho j_I\left(\frac{\partial\omega^r}{\partial t}+\frac{\partial\omega^r}{\partial r}v^r\right)=(c_0+2c_d)\frac{\partial}{\partial r}\left(\frac{\partial\omega^r}{\partial r}+\frac{\omega^r}{r}\right)+\rho j_I\frac{\omega^\varphi v^\varphi}{r}-4\mu_r\omega^r, \quad (27)$$

$$\begin{aligned}\rho j_I\left(\frac{\partial\omega^\varphi}{\partial t}+\frac{\partial\omega^\varphi}{\partial r}v^r\right)&=(c_d+c_a)\frac{\partial}{\partial r}\left(\frac{\partial\omega^\varphi}{\partial r}+\frac{\omega^\varphi}{r}\right)\\&-\rho j_I\frac{\omega^r v^\varphi}{r}-2\mu_r\frac{\partial v^z}{\partial r}-4\mu_r\omega^\varphi,\end{aligned} \quad (28)$$

$$\begin{aligned}\rho j_I\left(\frac{\partial\omega^z}{\partial t}+\frac{\partial\omega^z}{\partial r}v^r\right)&=(c_d+c_a)\left(\frac{\partial^2\omega^z}{\partial r^2}+\frac{1}{r}\frac{\partial\omega^z}{\partial r}\right)\\&+2\mu_r\left(\frac{\partial v^\varphi}{\partial r}+\frac{v^\varphi}{r}\right)-4\mu_r\omega^z,\end{aligned} \quad (29)$$

$$\begin{aligned}c_v\rho\left(\frac{\partial\theta}{\partial t}+\frac{\partial\theta}{\partial r}v^r\right)&=k_\theta\left(\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}\right)-R\rho\theta\left(\frac{v^r}{r}+\frac{\partial v^r}{\partial r}\right)\\
+(\lambda+2\mu)\left(\frac{v^r}{r}+\frac{\partial v^r}{\partial r}\right)^2&-4\mu\frac{v^r}{r}\frac{\partial v^r}{\partial r}+(\mu+\mu_r)\left(\frac{v^\varphi}{r}+\frac{\partial v^\varphi}{\partial r}\right)^2\\
-4\mu\frac{v^\varphi}{r}\frac{\partial v^\varphi}{\partial r}&+(\mu+\mu_r)\left(\frac{\partial v^z}{\partial r}\right)^2+(c_0+2c_d)\left(\frac{\omega^r}{r}+\frac{\partial\omega^r}{\partial r}\right)^2\\
&-4c_d\frac{\omega^r}{r}\frac{\partial\omega^r}{\partial r}+(c_d+c_a)\left(\frac{\omega^\varphi}{r}+\frac{\partial\omega^\varphi}{\partial r}\right)^2\\
-4c_d\frac{\omega^\varphi}{r}\frac{\partial\omega^\varphi}{\partial r}&+(c_d+c_a)\left(\frac{\partial\omega^z}{\partial r}\right)^2+4\mu_r(\omega^r)^2+4\mu_r(\omega^\varphi)^2+4\mu_r(\omega^z)^2\\
&+4\mu_r\omega^\varphi\frac{\partial v^z}{\partial r}-4\mu_r\left(\frac{\partial v^\varphi}{\partial r}+\frac{v^\varphi}{r}\right)\omega^z\end{aligned} \quad (30)$$

$$\rho(r,0)=\rho_0(r),\quad \theta(r,0)=\theta_0(r), \quad (31)$$

$$v^r(r,0)=v_0^r(r),\ v^\phi(r,0)=v_0^\phi(r),\ v^z(r,0)=v_0^z(r), \quad (32)$$

$$\omega^r(r,0)=\omega_0^r(r),\ \omega^\phi(r,0)=\omega_0^\phi(r),\ \omega^z(r,0)=\omega_0^z(r), \quad (33)$$

$$v^r(a,t)=v^r(b,t)=0,\ v^\varphi(a,t)=v^\varphi(b,t)=0,\ v^z(a,t)=v^z(b,t)=0, \quad (34)$$

$$\omega^r(a,t)=\omega^r(b,t)=0,\ \omega^\varphi(a,t)=\omega^\varphi(b,t)=0,\ \omega^z(a,t)=\omega^z(b,t)=0, \quad (35)$$

$$\frac{\partial\theta}{\partial r}(a,t)=\frac{\partial\theta}{\partial r}(b,t)=0, \quad (36)$$

for $r\in]a,b[$ and $t\in]0,T[$.

In the mathematical analysis of compressible fluids it is convenient to use Lagrangian description. The Eulerian coordinates (r, t) are connected to the Lagrangian coordinates (ξ, t) by the relation

$$r(\xi, t) = r_0(\xi) + \int_0^t \tilde{v}^r(\xi, t)d\tau, \qquad r_0(\xi) = r(\xi, 0), \tag{37}$$

where $\tilde{v}^r(\xi, t)$ is defined by

$$\tilde{v}^r(\xi, t) = v^r\left(r(\xi, t), t\right). \tag{38}$$

We introduce the new function η by

$$\eta(r) = \int_a^r s\rho_0(s)ds, \quad r_0(\xi) = \eta^{-1}(\xi). \tag{39}$$

Therefore, we have $\xi \in]0, L[$, where

$$L = \int_a^b s\rho_0(s)ds. \tag{40}$$

Without danger of confusion, we write (x, t) instead of (ξ, t), omit $\sim$ notation and get the system in the Lagrangian form:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial}{\partial x}\left(rv^r\right) = 0. \tag{41}$$

$$\frac{\partial v^r}{\partial t} = -Rr\frac{\partial}{\partial x}(\rho\theta) + (\lambda + 2\mu)r\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(rv^r\right)\right) + \frac{(v^\varphi)^2}{r}, \tag{42}$$

$$\frac{\partial v^\varphi}{\partial t} = (\mu + \mu_r)r\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}(rv^\varphi)\right) - \frac{v^r v^\varphi}{r} - 2\mu_r r\frac{\partial \omega^z}{\partial x}, \tag{43}$$

$$\frac{\partial v^z}{\partial t} = (\mu + \mu_r)r\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(rv^z\right)\right) + (\mu + \mu_r)\frac{v^z}{\rho r^2} + 2\mu_r\frac{\partial}{\partial x}(r\omega^\varphi), \tag{44}$$

$$j_I\frac{\partial \omega^r}{\partial t} = (c_0 + 2c_d)r\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(r\omega^r\right)\right) + j_I\frac{\omega^\varphi v^\varphi}{r} - 4\mu_r\frac{\omega^r}{\rho}, \tag{45}$$

$$j_I\frac{\partial \omega^\varphi}{\partial t} = (c_d + c_a)r\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}(r\omega^\varphi)\right) - j_I\frac{\omega^r v^\varphi}{r} - 2\mu_r r\frac{\partial v^z}{\partial x} - 4\mu_r\frac{\omega^\varphi}{\rho}, \tag{46}$$

$$j_I \frac{\partial \omega^z}{\partial t} = (c_d + c_a) r \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r\omega^z\right)\right) + (c_d + c_a)\frac{\omega^z}{\rho r^2} + 2\mu_r \frac{\partial}{\partial x}(r v^\varphi) - 4\mu_r \frac{\omega^z}{\rho}, \tag{47}$$

$$\begin{aligned} c_v \frac{\partial \theta}{\partial t} &= k_\theta \frac{\partial}{\partial x}\left(r^2 \rho \frac{\partial \theta}{\partial x}\right) + \rho\left[(\lambda + 2\mu)\frac{\partial}{\partial x}\left(r v^r\right) - R\theta\right]\frac{\partial}{\partial x}\left(r v^r\right) + \\ &(\mu + \mu_r)\rho\left(\frac{\partial}{\partial x}(r v^\varphi)\right)^2 + (c_d + c_a)\rho\left(\frac{\partial}{\partial x}(r\omega^\varphi)\right)^2 \\ &+ (c_0 + 2c_d)\rho\left(\frac{\partial}{\partial x}\left(r\omega^r\right)\right)^2 + (\mu + \mu_r)\rho r^2\left(\frac{\partial v^z}{\partial x}\right)^2 \\ &+ (c_d + c_a)\rho r^2\left(\frac{\partial \omega^z}{\partial x}\right)^2 - 2c_d \frac{\partial}{\partial x}\left((\omega^r)^2 + (\omega^\varphi)^2\right) \\ &- 2\mu \frac{\partial}{\partial x}\left((v^r)^2 + (v^\varphi)^2\right) + 4\mu_r \frac{(\omega^r)^2}{\rho} + 4\mu_r \frac{(\omega^\varphi)^2}{\rho} + 4\mu_r \frac{(\omega^z)^2}{\rho} \\ &+ 4\mu_r r\omega^\varphi \frac{\partial v^z}{\partial x} - 4\mu_r \omega^z \frac{\partial}{\partial x}(r v^\varphi), \end{aligned} \tag{48}$$

$$\rho(x,0) = \rho_0(x), \quad v^r(x,0) = v_0^r(x),\ v^\varphi(x,0) = v_0^\varphi(x),\ v^z(x,0) = v_0^z(x), \tag{49}$$

$$\theta(x,0) = \theta_0(x), \quad \omega^r(x,0) = \omega_0^r(x),\ \omega^\varphi(x,0) = \omega_0^\varphi(x),\ \omega^z(x,0) = \omega_0^z(x), \tag{50}$$

$$v^r(0,t) = v^r(L,t) = 0,\ v^\varphi(0,t) = v^\varphi(L,t) = 0,\ v^z(0,t) = v^z(L,t) = 0, \tag{51}$$

$$\omega^r(0,t) = \omega^r(L,t) = 0,\ \omega^\varphi(0,t) = \omega^\varphi(L,t) = 0,\ \omega^z(0,t) = \omega^z(L,t) = 0, \tag{52}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(L,t) = 0 \tag{53}$$

considered on the domain $Q_T =]0, L[\times]0, T[$.

The function $r(x,t)$ is defined by

$$r(x,t) = r_0(x) + \int_0^t v^r(x,\tau)d\tau,\ (x,t) \in Q_T, \tag{54}$$

where

$$r_0(x) = \left(a^2 + 2\int_0^x \frac{1}{\rho_0(y)}\,dy\right)^{\frac{1}{2}}, \tag{55}$$

and $a > 0$ is a radius of smaller boundary cylinder.

3 Existence of the Solution

In this section we consider the existence and uniqueness of the so-called generalized solution to the problem (41)–(53).

Definition 1 A generalized solution to the problem (41)–(53) in the domain Q_T is a function

$$(x,t) \mapsto (\rho, v^r, v^\varphi, v^z, \omega^r, \omega^\varphi, \omega^z, \theta)(x,t), \quad (x,t) \in Q_T, \tag{56}$$

where

$$\rho \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(]0,L[)) \cap \mathrm{H}^1(Q_T)\ ,\ \inf_{Q_T} \rho > 0 \tag{57}$$

$$v^r, v^\varphi, v^z, \omega^r, \omega^\varphi, \omega^z, \theta \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(]0,L[)) \cap \mathrm{H}^1(Q_T) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(]0,L[)), \tag{58}$$

that satisfies the Eqs. (41)–(48) a.e. in Q_T and conditions (49)–(53) in the sense of traces.

Let us mention that by using the embedding and interpolation theorems one can conclude that our generalized solution could be treated as a strong solution. In fact, we have

$$\rho \in \mathrm{L}^\infty(0,T;\mathrm{C}([0,L])) \cap \mathrm{C}([0,T];\mathrm{L}^2(]0,L[))\ , \tag{59}$$

$$v^r, v^\varphi, v^z, \omega^r, \omega^\varphi, \omega^z, \theta \in \mathrm{L}^2(0,T;\mathrm{C}^1([0,L])) \cap \mathrm{C}([0,T];\mathrm{H}^1(]0,L[)), \tag{60}$$

$$v^r, v^\varphi, v^z, \omega^r, \omega^\varphi, \omega^z, \theta \in \mathrm{C}(\overline{Q}_T). \tag{61}$$

We first analyzed the existence of the generalized solution to the problem (41)–(51). Using the Faedo–Gelerikin method we proved in [4] the existence locally in time. After that we analyzed the uniqueness of the solution in [7], and finally based on extension principle, we proved in [2] the global existence theorem for the the problem (41)–(51). These results are summarized in the following theorem.

Theorem 1 *Let the functions* $\rho_0, \theta_0 \in \mathrm{H}^1(]0,L[)$, $v_0^r, v_0^\varphi, v_0^z, \omega_0^r, \omega_0^\varphi, \omega_0^z \in \mathrm{H}_0^1(]0, L[)$ *satisfy the conditions*

$$\rho_0(x) \geq m, \quad \theta_0(x) \geq m \quad \text{for} \ \ x \in]0,L[,$$

where $m \in \mathbf{R}^+$. *Then for any* $T \in \mathbf{R}^+$ *there exists unique generalized solution to the problem (41)–(51) on the domain* Q_T *having the property*

$$\theta > 0 \ \ \text{in} \ \overline{Q}_T. \tag{62}$$

Now we will briefly explain the proof of the Theorem 1.

In the first stage of the proof, based on Faedo–Galerkin method, we introduce the approximate solutions

$$(\rho^n, v^{r\,n}, v^{\varphi\,n}, v^{z\,n}, \omega^{r\,n}, \omega^{\varphi\,n}, \omega^{z\,n}, \theta^n)\ ,\ \ n \in \mathbf{N}. \tag{63}$$

The approximations $v^{r\,n}$ and r^n of the functions v^r and r were defined by

$$v^{r\,n}(x,t) = \sum_{i=1}^{n} v_i^{r\,n}(t) \sin\frac{\pi i x}{L}, \tag{64}$$

$$r^n(x,t) = r_0(x) + \int_0^t v^{r\,n}(x,\tau)d\tau, \tag{65}$$

where $r_0(x)$ is given by (55) and $v_i^{r\,n}$, $i = 1, 2, ..., n$ are unknown smooth functions defined on an interval $[0, T_n]$, $T_n \leq T$. Using the mass conservation law we obtain

$$\rho^n(x,t) = \frac{\rho_0(x)}{1 + \rho_0(x)\dfrac{\partial}{\partial x}\displaystyle\int_0^t r^n v^{r\,n}\, d\tau}. \tag{66}$$

We also define the approximations $v^{\varphi\,n}$, $v^{z\,n}$, $\omega^{r\,n}$, $\omega^{\varphi\,n}$, $\omega^{z\,n}$ and θ^n are the approximations of the functions v^φ, v^z, ω^r, ω^φ, ω^z and θ, respectively, by

$$v^{\varphi\,n}(x,t) = \sum_{i=1}^{n} v_i^{\varphi\,n}(t) \sin\frac{\pi i x}{L}, \tag{67}$$

$$v^{z\,n}(x,t) = \sum_{i=1}^{n} v_i^{z\,n}(t) \sin\frac{\pi i x}{L}, \quad \omega^{r\,n}(x,t) = \sum_{j=1}^{n} \omega_j^{r\,n}(t) \sin\frac{\pi j x}{L}, \tag{68}$$

$$\omega^{\varphi\,n}(x,t) = \sum_{j=1}^{n} \omega_j^{\varphi\,n}(t) \sin\frac{\pi j x}{L}, \quad \omega^{z\,n}(x,t) = \sum_{j=1}^{n} \omega_j^{z\,n}(t) \sin\frac{\pi j x}{L}, \tag{69}$$

$$\theta^n(x,t) = \sum_{k=0}^{n} \theta_k^n(t) \cos\frac{\pi k x}{L}, \tag{70}$$

where $v_i^{\varphi\,n}, v_i^{z\,n}$, $i = 1, 2, \ldots, n$, $\omega_j^{r\,n}, \omega_j^{\varphi\,n}, \omega_j^{z\,n}$, $j = 1, \ldots, n$ and θ_k^n, $k = 0, \ldots, n$ are unknown smooth functions defined on an interval $[0, T_n]$, $T_n \leq T$. Evidently, the boundary conditions

$$v^{r\,n}(0,t) = v^{r\,n}(L,t) = v^{\varphi\,n}(0,t) = v^{\varphi\,n}(L,t) = v^{z\,n}(0,t) = v^{z\,n}(L,t) = 0, \tag{71}$$

$$\omega^{r\,n}(0,t)=\omega^{r\,n}(L,t)=\omega^{\varphi\,n}(0,t)=\omega^{\varphi\,n}(L,t)=\omega^{z\,n}(0,t)=\omega^{z\,n}(L,t)=0, \tag{72}$$

$$\frac{\partial\theta^n}{\partial x}(0,t)=\frac{\partial\theta^n}{\partial x}(L,t)=0 \tag{73}$$

for $t\in]0,T_n[$ are satisfied.

According to the Faedo–Galerkin method, we take the following approximate conditions:

$$\int_0^L\left(\frac{\partial v^{r\,n}}{\partial t}+Rr^n\frac{\partial}{\partial x}\left(\rho^n\theta^n\right)-(\lambda+2\mu)r^n\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left(r^nv^{r\,n}\right)\right)-\frac{(v^{\varphi\,n})^2}{r^n}\right)\sin\frac{\pi i_1x}{L}dx=0, \tag{74}$$

$$\int_0^L\left(\frac{\partial v^{\varphi\,n}}{\partial t}-(\mu+\mu_r)r^n\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left(r^nv^{\varphi\,n}\right)\right)+\frac{v^{r\,n}v^{\varphi\,n}}{r^n}+2\mu_rr^n\frac{\partial\omega^{z\,n}}{\partial x}\right)\sin\frac{\pi i_2x}{L}dx=0, \tag{75}$$

$$\int_0^L\left(\frac{\partial v^{z\,n}}{\partial t}-(\mu+\mu_r)r^n\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left(r^nv^{z\,n}\right)\right)-(\mu+\mu_r)\frac{v^{z\,n}}{\rho^n(r^n)^2}-2\mu_r\frac{\partial}{\partial x}\left(r^n\omega^{\varphi\,n}\right)\right)\sin\frac{\pi i_3x}{L}dx=0, \tag{76}$$

$$\int_0^L\left(\frac{\partial\omega^{r\,n}}{\partial t}-\frac{c_0+2c_d}{j_I}r^n\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left(r^n\omega^{r\,n}\right)\right)-\frac{\omega^{\varphi\,n}v^{\varphi\,n}}{r^n}+4\frac{\mu_r}{j_I}\frac{\omega^{r\,n}}{\rho^n}\right)\sin\frac{\pi j_1x}{L}dx=0, \tag{77}$$

$$\int_0^L\left(\frac{\partial\omega^{\varphi\,n}}{\partial t}-\frac{c_d+c_a}{j_I}r^n\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left(r^n\omega^{\varphi\,n}\right)\right)+\frac{\omega^{r\,n}v^{\varphi\,n}}{r^n}+2\frac{\mu_r}{j_I}r^n\frac{\partial v^{z\,n}}{\partial x}+4\frac{\mu_r}{j_I}\frac{\omega^{\varphi\,n}}{\rho^n}\right)\sin\frac{\pi j_2x}{L}dx=0, \tag{78}$$

$$\int_0^L\left(\frac{\partial\omega^{z\,n}}{\partial t}-\frac{c_d+c_a}{j_I}r^n\frac{\partial}{\partial x}\left(\rho^n\frac{\partial}{\partial x}\left(r^n\omega^{z\,n}\right)\right)-\frac{c_d+c_a}{j_I}\frac{\omega^{z\,n}}{\rho^n(r^n)^2}-2\frac{\mu_r}{j_I}\frac{\partial}{\partial x}\left(r^nv^{\varphi\,n}\right)+4\frac{\mu_r}{j_I}\frac{\omega^{z\,n}}{\rho^n}\right)\sin\frac{\pi j_3x}{L}dx=0, \tag{79}$$

$$\int_0^L \left(\frac{\partial\theta^n}{\partial t} - \frac{k}{c_v}\frac{\partial}{\partial x}\left((r^n)^2\rho^n\frac{\partial\theta^n}{\partial x}\right) - \frac{\rho^n}{c_v}\left[(\lambda+2\mu)\frac{\partial}{\partial x}\left(r^n v^{r\,n}\right) - R\theta^n\right]\cdot\right.$$
$$\frac{\partial}{\partial x}\left(r^n v^{r\,n}\right) - \frac{\mu+\mu_r}{c_v}\rho^n\left(\frac{\partial}{\partial x}\left(r^n v^{\varphi\,n}\right)\right)^2 - \frac{c_d+c_a}{c_v}\rho^n\left(\frac{\partial}{\partial x}\left(r^n\omega^{\varphi\,n}\right)\right)^2$$
$$-\frac{c_0+2c_d}{c_v}\rho^n\left(\frac{\partial}{\partial x}\left(r^n\omega^{r\,n}\right)\right)^2 - \frac{\mu+\mu_r}{c_v}\rho^n(r^n)^2\left(\frac{\partial v^{z\,n}}{\partial x}\right)^2$$
$$-\frac{c_d+c_a}{c_v}\rho^n(r^n)^2\left(\frac{\partial\omega^{z\,n}}{\partial x}\right)^2 + 2\frac{c_d}{c_v}\frac{\partial}{\partial x}\left((\omega^{r\,n})^2+(\omega^{\varphi\,n})^2\right)+ \tag{80}$$
$$2\frac{\mu}{c_v}\frac{\partial}{\partial x}\left((v^{r\,n})^2+(v^{\varphi\,n})^2\right) - 4\frac{\mu_r}{c_v}\frac{(\omega^{r\,n})^2}{\rho^n} - 4\frac{\mu_r}{c_v}\frac{(\omega^{\varphi\,n})^2}{\rho^n} - 4\frac{\mu_r}{c_v}\frac{(\omega^{z\,n})^2}{\rho^n}$$
$$\left.-4\frac{\mu_r}{c_v}r^n\omega^{\varphi\,n}\frac{\partial v^{z\,n}}{\partial x} + 4\frac{\mu_r}{c_v}\omega^{z\,n}\frac{\partial}{\partial x}\left(r^n v^{\varphi\,n}\right)\right)\cos\frac{\pi k x}{L}dx = 0.$$

for $i_1, i_2, i_3, j_1, j_2, j_3 = 1, \ldots, n$, $k = 0, 1, \ldots, n$.

We take the initial conditions for $v^{r\,n}, v^{\varphi\,n}, v^{z\,n}, \omega^{r\,n}, \omega^{\varphi\,n}, \omega^{z\,n}$ and θ^n in the form:

$$v^{r\,n}(x,0) = v_0^{r\,n}(x), \quad v^{\varphi\,n}(x,0) = v_0^{\varphi\,n}(x), \quad v^{z\,n}(x,0) = v_0^{z\,n}(x), \tag{81}$$
$$\omega^{r\,n}(x,0) = \omega_0^{r\,n}(x), \quad \omega^{\varphi\,n}(x,0) = \omega_0^{\varphi\,n}(x), \quad \omega^{z\,n}(x,0) = \omega_0^{z\,n}(x), \tag{82}$$
$$\theta^n(x,0) = \theta_0^n(x), \quad x \in [0, L], \tag{83}$$

where $v_0^{r\,n}, v_0^{\varphi\,n}, v_0^{z\,n}, \omega_0^{r\,n}, \omega_0^{\varphi\,n}, \omega_0^{z\,n}$, and θ_0^n are defined by

$$v_0^{r\,n}(x) = \sum_{i=1}^n v_{0i}^r \sin\frac{\pi i x}{L}, \quad v_0^{\varphi\,n}(x) = \sum_{i=1}^n v_{0i}^\varphi \sin\frac{\pi i x}{L}, \tag{84}$$

$$v_0^{z\,n}(x) = \sum_{i=1}^n v_{0i}^z \sin\frac{\pi i x}{L}, \quad \omega_0^{r\,n}(x) = \sum_{i=1}^n \omega_{0i}^r \sin\frac{\pi j x}{L}, \tag{85}$$

$$\omega_0^{\varphi\,n}(x) = \sum_{i=1}^n \omega_{0i}^\varphi \sin\frac{\pi j x}{L}, \quad \omega_0^{z\,n}(x) = \sum_{i=1}^n \omega_{0i}^z \sin\frac{\pi j x}{L}, \tag{86}$$

$$\theta_0^n(x) = \sum_{k=0}^n \theta_{0k}\cos\frac{\pi k x}{L}, \tag{87}$$

and $v_{0i}^r, v_{0i}^\varphi, v_{0i}^z, \omega_{0j}^r, \omega_{0j}^\varphi, \omega_{0j}^z$, and θ_{0k} are the Fourier coefficients of the functions $v_0^r, v_0^\varphi, v_0^z, \omega_0^r, \omega_0^\varphi, \omega_0^z$, and θ_0, respectively.

Let z_m^n, λ_{pq}^n be:

$$z_m^n(t) = \int_0^t v_m^{r\,n}(\tau)d\tau, \quad m = 1, \ldots, n, \tag{88}$$

$$\lambda_{pq}^n(t) = \int_0^t z_p^n(\tau)v_q^{r\,n}(\tau)d\tau, \quad p, q = 1, \ldots, n \tag{89}$$

and we have

$$r^n(x,t) = r_0(x) + \sum_{i=1}^n z_i^n(t)\sin\frac{\pi i x}{L}, \tag{90}$$

$$\begin{aligned}\rho^n(x,t) = \rho_0(x)\Bigg(1 + \rho_0(x)\sum_{j=1}^n z_j^n(t)\frac{\partial}{\partial x}\left(r_0(x)\sin\frac{\pi j x}{L}\right) + \\ \rho_0(x)\sum_{i,j=1}^n \lambda_{ij}^n(t)\frac{\partial}{\partial x}\left(\sin\frac{\pi i x}{L}\sin\frac{\pi j x}{L}\right)\Bigg)^{-1},\end{aligned} \tag{91}$$

where $r_0(x)$ and $\rho_0(x)$ are known functions. Now we obtain the Cauchy problem for $v_{i_1}^{r\,n}, v_{i_2}^{\varphi\,n}, v_{i_3}^{z\,n}, \omega_{j_1}^{r\,n}, \omega_{j_2}^{\varphi\,n}, \omega_{j_3}^{z\,n}, \theta_k^n, z_m^n, \lambda_{pq}^n, i_1, i_2, i_3, j_1, j_2, j_3, m, p, q = 1, \ldots, n, k = 0, 1, \ldots, n$ which satisfies the conditions of the Cauchy-Picard theorem. Therefore, we can easily conclude that for each $n \in \mathbf{N}$ there exists T_n, $0 < T_n \leq T$ such that a set $Q_n =]0, L[\times]0, T_n[$ is a domain of the n-th approximate solution.

The next stage in the proof is to find such T_0, $0 < T_0 \leq T$, so that for each $n \in \mathbf{N}$ there exists a solution to the approximate problem defined on $[0, T_0]$. This is done by obtaining a series of uniform (in $n \in \mathbf{N}$) a priori estimates for the solutions (63). In the final stage of the proof the convergent subsequence of the sequence (63) was extracted and it has been shown that the limit of this subsequence is a solution to the analysed problem.

To prove the uniqueness of the generalized solution we derive the system for the functions $u = u_1 - u_2$, $u_i = \rho_i^{-1}$, $v^r = v_1^r - v_2^r$, $v^\varphi = v_1^\varphi - v_2^\varphi$, $v^z = v_1^z - v_2^z$, $\omega^r = \omega_1^r - \omega_2^r$, $\omega^\varphi = \omega_1^\varphi - \omega_2^\varphi$, $\omega^z = \omega_1^z - \omega_2^z$, $\theta = \theta_1 - \theta_2$ and $r = r_1 - r_2$, where ρ_i, v_i^r, v_i^φ, v_i^z, ω_i^r, ω_i^φ, ω_i^z, θ_i are two distinct generalized solution of the described problem. We obtain:

Lemma 3.1 *There exists the constant* $C > 0$ *such that for any* $t \in]0, T[$ *we have.*

$$\|u(t)\|^2 \leq C\int_0^t \left\|\frac{\partial v}{\partial x}(\tau)\right\|^2 d\tau, \tag{92}$$

$$\|v^r(t)\|^2 + \|v^\varphi(t)\|^2 + \|v^z(t)\|^2 + \|\omega^r(t)\|^2 + \|\omega^\varphi(t)\|^2 + \|\omega^z(t)\|^2 + \int_0^t \left(\left\| \frac{\partial v^r}{\partial x}(\tau) \right\|^2 + \left\| \frac{\partial v^\varphi}{\partial x}(\tau) \right\|^2 + \left\| \frac{\partial v^z}{\partial x}(\tau) \right\|^2 + \left\| \frac{\partial \omega^r}{\partial x}(\tau) \right\|^2 + \left\| \frac{\partial \omega^\varphi}{\partial x}(\tau) \right\|^2 + \left\| \frac{\partial \omega^z}{\partial x}(\tau) \right\|^2 \right) d\tau \le C \int_0^t \|\theta(\tau)\|^2 d\tau, \tag{93}$$

$$\|\theta(t)\|^2 + \int_0^t \left\| \frac{\partial \theta}{\partial x}(\tau) \right\|^2 d\tau \le C \int_0^t \|\theta(\tau)\|^2 d\tau. \tag{94}$$

Using this Lemma and Gronwall's inequality we immediately get that functions $u, v^r, v^\varphi, v^z, \omega^r, \omega^\varphi, \omega^z, \theta$ are equal to zero which ends the proof of the uniqueness.

The proof of the global existence is based on the a priori estimates and extension principle stated in the following proposition:

Proposition 3.1 *Let $T \in \mathbf{R}^+$ and let the function*

$$(x, t) \mapsto (\rho, v^r, v^\varphi, v^z, \omega^r, \omega^\varphi, \omega^z, \theta)(x, t), \quad (x, t) \in Q_{T'} \tag{95}$$

be the generalized solution to the problem (41)–(53) on the domain $Q_{T'}$, for any $T' < T$ with the property $\theta > 0$ in $\overline{Q}_{T'}$. Then (95) is the generalized solution of the same problem on the domain Q_T with the property $\theta > 0$ in $\overline{Q}_T$.

To be able to use the Proposition 3.1 it is crucial to find a set of global a priori estimates in which the constants can depend only on initial data and the constant T from the Proposition 3.1.

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Hermite–Fejer Polynomials as an Approximate Solution of Singular Integro-Differential Equations

Alexander Fedotov

Abstract For full singular integro-differential equations with Gilbert kernel, the collocation method is justified. The approximate solution is sought in the form of Hermite–Fejer polynomial. The convergence of the method is proved and the rate of convergence is estimated.

1 Introduction

Algebraic interpolation polynomials with multiple nodes, known as Hermite polynomials, are well-investigated and are successfully used to solve a wide range of application-oriented problems. Their trigonometric analogue is investigated much less and many questions concerning the existence, uniqueness, and approximate properties of such polynomials still remain open.

Early studies of trigonometric interpolation polynomials with multiple nodes apparently began toward the 30th years of the 20th century. S. M. Lozinsky [1] considered the approximation of the complex-variable functions regular in a single circle, and continuous on its boundary, by the trigonometric interpolation polynomials with multiple nodes located on a single circle's border. He was the first to call such polynomials Hermite–Fejer polynomials.

E. O. Zeel [2, 3], generalizing the results of the predecessors [4–7], proved the existence of the trigonometrical interpolation polynomials of the arbitrary multiplicity w.r.t. the system of the equidistant nodes for the real-valued 2π - periodic functions. Moreover, he showed the explicite form of the corresponding fundamental polynomials and established the conditions of uniform convergence of such polymials to the interpolated function depending on the parity of its multiplicity and the smoothness of the interpolated function.

B. G. Gabdulkhayev [8] obtained in a convenient form the best, in the sense of an order, estimates of the speed of convergence of trigonometrical interpolation

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polynomials of the first multiplicity to continuously differentiable functions. Also, in this work he investigated the properties of the quadrature formulas for Gilbert's kernel singular integrals based on such polynomials. Relying on the results of [3] and using B. G. Gabdulkhayev [8] technique Yu. Soliyev [9, 10] investigated systematically quadrature formulas based on the interpolation polynomials of different multiplicity for singular integrals with Cauchy and Gilbert kernels.

In this paper the calculation scheme of the collocation method based on trigonometric interpolation polynomials with the multiple nodes for the full singular integro-differential equation in periodic case is constructed and justified. Convergence of the method is proved, and the errors of the approximate solution are estimated.

2 Statement of the Problem

Consider the singular integro-differential equation

$$\sum_{\nu=0}^{1}(a_\nu(t)x^{(\nu)}(t) + b_\nu(t)(Jx^{(\nu)})(t) + (J_0 h_\nu x^{(\nu)})(t)) = y(t),\ t \in [0, 2\pi], \qquad (1)$$

where x is a required function, a_ν, b_ν, h_ν (by both variables), $\nu = 0, 1$, and y are known 2π-periodic functions, singular integrals

$$(Jx^{(\nu)})(t) = \frac{1}{2\pi}\int_0^{2\pi} x^{(\nu)}(\tau)\cot\frac{\tau - t}{2}d\tau,\ \nu = 0, 1,\ t \in [0, 2\pi],$$

are to be interpreted as the Cauchy–Lebesgues principal value, and

$$(J_0 h_\nu x^{(\nu)})(t) = \frac{1}{2\pi}\int_0^{2\pi} h_\nu(t, \tau)x^{(\nu)}(\tau)d\tau,\ \nu = 0, 1,\ t \in [0, 2\pi],$$

are regular integrals.

3 Calculation Scheme

Let's denote $\mathbb{N}$ the set of natural numbers, $\mathbb{N}_0$ the set of natural numbers with zero added, $\mathbb{R}$ the set of real numbers $\mathbb{C}$ the set of complex numbers.

Let's fix the natural number $n \in \mathbb{N}$. An approximate solution of the Eq. (1) we seek as a Hermite–Fejer polynomial

$$x_n(t) = \frac{1}{n^2} \sum_{k=0}^{n-1} (x_{2k} + x'_{2k} \sin(t - t_{2k})) \frac{\sin^2 \frac{n}{2}(t - t_{2k})}{\sin^2 \frac{t - t_{2k}}{2}}, \ t \in [0, 2\pi], \tag{2}$$

here t_{2k}, $k = 0, 1..., n-1$, are even numbered nodes of the mesh

$$t_k = \frac{\pi k}{n}, \quad k = 0, 1, ..., 2n - 1. \tag{3}$$

Unknown coefficients x_{2k}, x'_{2k}, $k = 0, 1..., n-1$, of the polynomial (2) we find out as a solution of the system of the algebraic equations

$$\sum_{\nu=0}^{1} (a_\nu(t_k) x_n^{(\nu)}(t_k) + b_\nu(t_k)(J x_n^{(\nu)})(t_k) + (J_0 P_{2n}^{\tau}(h_\nu x_n^{(\nu)}))(t_k)) = \tag{4}$$

$$= y(t_k), \quad k = 0, 1, ..., 2n - 1,$$

where

$$P_{2n}^{\tau}(h_\nu x_n^{(\nu)})(t, \tau) = \frac{1}{2n} \sum_{k=0}^{2n-1} h_\nu(t, t_k) x_n^{(\nu)}(t_k) \frac{\sin n(\tau - t_k) \cos \frac{\tau - t_k}{2}}{\sin \frac{\tau - t_k}{2}},$$

$$\nu = 0, 1, \ t, \tau \in [0, 2\pi],$$

is a Lagrange interpolation operator w.r.t. the nodes (3) applied by the variable τ to the functions $h_\nu x_n^{(\nu)}$, $\nu = 0, 1$, and

$$(J x_n)(t_k) = \frac{1}{n} \sum_{j=0}^{n-1} (\alpha^0_{0,k-2j} x_{2j} + \alpha^1_{0,k-2j} x'_{2j}), \quad k = 0, 1, ..., 2n - 1,$$

$$\alpha^0_{0,r} = \{-\cot \frac{r\pi}{2n} \quad \text{for} \quad r \neq 0, \quad 0 \quad \text{for} \quad r = 0\},$$

$$\alpha^1_{0,r} = \{-\frac{1}{n} \quad \text{for} \quad r \neq 0, \quad 2 - \frac{1}{n} \quad \text{for} \quad r = 0\};$$

$$(J x'_n)(t_{2k}) = \frac{1}{n} \sum_{j=0}^{n-1} (\alpha^0_{1,2k-2j} x_{2j} + \alpha^1_{1,2k-2j} x'_{2j}), \quad k = 0, 1, ..., n - 1,$$

$$(J x'_n)(t_{2k+1}) = \frac{1}{n} \sum_{j=0}^{n-1} \alpha^0_{1,2k-2j+1} x_{2j}, \quad k = 0, 1, ..., n - 1,$$

$$\alpha^0_{1,r} = \{\csc^2 \frac{r\pi}{2n} \quad \text{for} \quad r \neq 0, \quad -\frac{n^2-1}{3} \quad \text{for} \quad r = 0\},$$

$$\alpha^1_{1,r} = \{(-1)^r \csc \frac{r\pi}{2n} \quad \text{for} \quad r \neq 0, \quad 0 \quad \text{for} \quad r = 0\};$$

$$(J^0 P^\tau_{2n}(h_\nu x^{(\nu)}_n))(t_k) = \frac{1}{2n} \sum_{j=0}^{2n-1} h_\nu(t_k, t_j) x^{(\nu)}_n(t_j), \ \nu = 0, 1, \quad k = 0, 1, ..., 2n-1,$$

are the quadrature formulae.

4 Some Preliminaries

Let's denote C the space of continuous 2π-periodic functions with usual norm

$$\| f \|_{\mathrm{C}} = \sup_{t \in \mathbb{R}} | f(t) |, \ f \in \mathrm{C}.$$

For the fixed $m \in \mathbb{N}_0$ denote $\mathrm{C}^m \subset \mathrm{C}$ the set of the functions on $\mathbb{R}$ with continuous derivatives of order m ($\mathrm{C}^0 = \mathrm{C}$). The norm on C^m we define as follows:

$$\| f \|_{\mathrm{C}^m} = \max_{0 \leq \nu \leq m} \| f^{(\nu)} \|_{\mathrm{C}}, \ f \in \mathrm{C}^m.$$

Let's denote H_α the set of Hölder continuous functions of order $\alpha \in \mathbb{R}, 0 < \alpha \leq 1$. For the function f of this set let's denote

$$H(f; \alpha) = \sup_{\substack{t \neq \tau \\ t, \tau \in \mathbb{R}}} \frac{| f(t) - f(\tau) |}{| t - \tau |^\alpha},$$

the smallest constant of Hölder condition of the function f. With the help of this constant we can now define the norm on the set H_α, namely,

$$\| f \|_{\mathrm{H}_\alpha} = \max\{\| f \|_{\mathrm{C}}, H(f; \alpha)\}.$$

From the set C^m, for the fixed $\alpha \in \mathbb{R}, \ 0 < \alpha \leq 1$, we can select the set of the functions H^m_α with derivatives of order m satisfying Hölder condition

$$| f^{(m)}(t) - f^{(m)}(\tau) | \leq H(f^{(m)}; \alpha) \, | t - \tau |^\alpha, \ t, \tau \in \mathbb{R}.$$

The norm on the set H^m_α ($\mathrm{H}^0_\alpha = \mathrm{H}_\alpha$) we define as follows:

$$\| f \|_{\mathrm{H}^m_\alpha} = \max\{\| f \|_{\mathrm{C}^m}, H(f^{(m)}; \alpha)\}.$$

Denote $\mathscr{T}_n$ the set of all trigonometric polynomials of order not higher than n. For the follows we need 2 lemmas from the paper [11].

Lemma 1 *Let the numbers* $\alpha, \beta \in \mathbb{R}, 0 < \alpha \leq 1, 0 < \beta \leq 1, m, r \in \mathbb{N}_0, m \leq r$, *are such that* $m + \beta \leq r + \alpha$. *Then for any* $n \in \mathbb{N}$ *and any function* $x \in \mathrm{H}_\alpha^r$ *the following estimate is valid* [1]*:*

$$\|x - T_n\|_{\mathrm{H}_\beta^m} \leq cn^{m-r-\alpha+\beta} H(x^{(r)}; \alpha),$$

where $T_n \in \mathscr{T}_n$ *is a polynomial of the best approximation of the function* x.

Lemma 2 *For any* $n \in \mathbb{N}$, $\beta \in \mathbb{R}$, $0 < \beta \leq 1$ *and arbitrary trigonomentric polynomial* $T_n \in \mathscr{T}_n$ *the following estimate is valid:*

$$\|T_n\|_{\mathrm{H}_\beta} \leq (1 + 2^{1-\beta} n^\beta)\|T_n\|_{\mathrm{C}}.$$

An operator P_{2n} is exact for any polynomial of order $n - 1$ and, as it is shown in [12, 13], has the following properties:

$$\|P_{2n}\|_{\mathrm{H}_\beta^m \to \mathrm{H}_\beta^m} \leq c\|P_{2n}\|_{\mathrm{C} \to \mathrm{C}} \leq c \ln n \qquad (5)$$

for any $n \in \mathbb{N}$, $n \geq 2$, $\beta \in \mathbb{R}$, $0 < \beta \leq 1$, and arbitrary fixed number $m \in \mathbb{N}$.

5 Justification

Theorem 1 *Let the Eq. (1) and the calculation scheme (2)–(4) of the method satisfy the following conditions:*

A1 *functions* a_ν, b_ν, $\nu = 0, 1$, *and* y *belong to* H_α *for some* $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$*; functions* h_ν, $\nu = 0, 1$, *belong to* H_α *with the same* α *for each variable uniformly w.r.t. other variable,*

A2 $a_1^2(t) + b_1^2(t) \neq 0, \quad t \in [0, 2\pi]$,

A3 $\kappa = \mathrm{ind}(a_1 + ib_1) = 0$,

A4 *an Eq. (1) has a unique solution* $x^* \in \mathrm{H}_\beta^1$ *for each right-hand side* $y \in \mathrm{H}_\beta$, $0 < \beta < \alpha \leq 1$.

Then for n *large enough the system of equations (4) is uniquely solvable and approximate solutions* x_n^* *converge to the exact solution* x^* *of the Eq. (1) by the norm of the space* H_β^1

$$\|x^* - x_n^*\|_{\mathrm{H}_\beta^1} \leq cn^{-\alpha+\beta} \ln n, \quad 0 < \beta < \alpha \leq 1.$$

Proof Let's show first that the assumption **A4** of the Theorem 1 is not empty in the sense that there exist the equations of the class considered satisfying **A4**.

[1]Here and further c denotes generic real positive constants, independent from n.

In fact, consider an equation

$$a_1(t)(x'(t) + x(t)) + b_1(t)((Jx')(t) + (Jx)(t)) = y(t), \quad t \in [0, 2\pi]. \tag{6}$$

It is known [14], that the characteristic operator

$$Bx \equiv a_1(t)x(t) + b_1(t)(Jx)(t), \quad B : H_\beta \to H_\beta,$$

of the Eq. (6) is invertable, and an inverse operator $B^{-1} : H_\beta \to H_\beta$ could be written explicitly. Now apply the operator B^{-1} to both sides of the Eq. (6). Then we'll get an equivalent equation

$$x'(t) + x(t) = (B^{-1}y)(t), \quad t \in [0, 2\pi]. \tag{7}$$

In the couple of the spaces (H^1_β, H_β), an Eq. (7) is a Fredholm equation. Homogeneous equation

$$x'(t) + x(t) = 0, \quad t \in [0, 2\pi],$$

in the space of the real-valued functions has a solution $x(t) = ce^{-t}$, $t \in [0, 2\pi]$. However, this solution is not periodic for $c \neq 0$, so the only suitable value is $c = 0$. It means that in the space of the periodic functions H^1_β the homogeneous equation has the only zero solution $x(t) = 0$, $t \in [0, 2\pi]$, and it means that the Eq. (7), and thus the Eq. (6), are uniquely solvable for any right-hand side $y \in H_\beta$, $0 < \beta < \alpha \leq 1$.

For the following part of the proof of the Theorem 1 we'll use the method described in [15, 16].

Let's fix $\beta \in \mathbb{R}$, $0 < \beta < \alpha \leq 1$, and let $\mathrm{X} = \mathrm{H}^1_\beta$, $\mathrm{Y} = \mathrm{H}_\beta$. Then the Eq. (1) can be rewritten as an operator equation

$$Qx = y, \quad Q : \mathrm{X} \to \mathrm{Y}. \tag{8}$$

For each function $x \in \mathrm{X}$ we'll match the Cauchy integral

$$\Phi(z) = \Phi(x; z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{x(\tau)d\tau}{1 - z\exp(-i\tau)}, \ z \in \mathbb{C}.$$

Denote $x^+(t)$ $x^-(t)$ the limit values of the function $\Phi(z)$ as z trends to $\exp(it)$ by any ways inside and outside unit circle correspondently. For the functions x^+ and x^- the following Sokhotsky's formulae are valid means identical operator.

$$x^\pm(t) = \frac{1}{2}((\pm I - iJ)x)(t) + \frac{1}{2}J_0x, \ t \in \mathbb{R}. \tag{9}$$

Differentiating (9) and using known formulae

$$(x'(t))^{\pm} = (x^{\pm}(t))', \quad (Jx)'(t) = (Jx')(t),$$

we'll obtain

$$x'(t) = x'^{+}(t) - x'^{-}(t), \quad (Jx')(t) = i(x'^{+}(t) + x'^{-}(t)). \tag{10}$$

From the conditions **A2**, **A3**, according to [17] it follows

$$\frac{a_1 - ib_1}{a_1 + ib_1} = \frac{\psi^{+}}{\psi^{-}},$$

where

$$\psi(z) = e^{\theta(z)}, \quad \theta(z) = \Phi(u; z), \quad u = \ln \frac{a_1 - ib_1}{a_1 + ib_1}, \quad z \in \mathbb{C}.$$

Then, using (10), the characteristic operator of the Eq. (1) can be rewritten [14, 17] as

$$a_1(t)x'(t) + b_1(t)(Jx')(t) = \frac{(a_1(t) + ib_1(t))}{\psi^{-}(t)}(\psi^{-}(t)x'^{+}(t) - \psi^{+}(t)x'^{-}(t)).$$

The Eq. (1) or, in other notation, the Eq. (8) we rewrite as an equivalent operator equation

$$Kx \equiv Ux + Vx = f, \quad K : \mathrm{X} \to \mathrm{Y}, \tag{11}$$

where

$$Ux = \psi^{-}x'^{+} - \psi^{+}x'^{-}, \quad Vx = Ax + Bx + Wx,$$

$$Ax = v^{-1}a_0 x, \quad Bx = v^{-1}b_0 Jx, \quad Wx = v^{-1}\sum_{\nu=0}^{1} J^0 h_\nu x^{(\nu)},$$

$$f = v^{-1}y, \quad v = \frac{a_1 + ib_1}{\psi^{-}},$$

and according the condition **A2** of the Theorem 1, $v(t) \neq 0$, $t \in [0, 2\pi]$. An equivalence here means that the Eqs. (1) and (11) are both solvable or not solvable simultaneously and, if they are solvable, their solutions coincide.

Let $\mathrm{X}_n \subset \mathscr{T}_n$ be the set of trigonometrical polynomials of the form (2), and $\mathrm{Y}_n = P_{2n}\mathrm{Y} \subset \mathscr{T}_n$. Then the system of equations (4) is equivalent to the operator equation

$$K_n x_n \equiv U_n x_n + V_n x_n = f_n, \quad K_n : \mathrm{X}_n \to \mathrm{Y}_n, \tag{12}$$

where

$$U_n = P_{2n}U, \quad V_n x_n = P_{2n}Ax_n + P_{2n}Bx_n + W_n x_n,$$

$$W_n x_n = P_{2n}\sum_{\nu=0}^{1} J_0(P_{2n}^{\tau}(h_\nu x_n^{(\nu)})), \quad f_n = P_{2n}f.$$

Here an equivalence means that if the system of equations (4) has a solution x_{2k}^*, $x_{2k}'^*$, $k = 0, 1, ..., n-1$, then the Eq. (12) will also has a solution which coincide with the polynomial

$$x_n^*(t) = \frac{1}{n^2}\sum_{k=0}^{n-1}(x_{2k}^* + x_{2k}'^*\sin(t - t_{2k}))\frac{\sin^2\frac{n}{2}(t - t_{2k})}{\sin^2\frac{t - t_{2k}}{2}}, \ t \in \mathbb{R}.$$

Let's prove now that the operators K and K_n are close to each other on X_n.

For any $x_n \in \mathrm{X}_n$, using the polynomial of the best approximation $T_{n-1} \in \mathscr{T}_{n-1}$ for the function Ax_n, we'll have

$$\|Ax_n - P_{2n}Ax_n\|_{\mathrm{Y}} \le (1 + \|P_{2n}\|_{\mathrm{Y}\to\mathrm{Y}})\|Ax_n - T_{n-1}\|_{\mathrm{Y}}. \tag{13}$$

Now, taking into account the structural qualities of the function Ax_n, we can estimate

$$H(Ax_n; \alpha) \le c(\|x_n\|_C + \|x_n'\|_{\mathrm{C}}) \le c\|x_n\|_{\mathrm{X}}. \tag{14}$$

From (13), using Lemma 1, an estimation (5), and in view of (14) we have

$$\|Ax_n - P_{2n}Ax_n\|_{\mathrm{Y}} \le c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \tag{15}$$

In the same way, we obtain

$$\|Bx_n - P_{2n}Bx_n\|_{\mathrm{Y}} \le c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \tag{16}$$

Considering the trigonometrical degree of accuracy of the quadrature formulae for the regular integrals used in (4) we can write

$$\|Wx_n - W_n x_n\|_{\mathrm{Y}} \le \|\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - P_{2n}\sum_{\nu=0}^{1} J^0 P_{2n}^{\tau}(h_\nu x_n^{(\nu)})\|_{\mathrm{Y}} \le \tag{17}$$

$$\le \|\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - P_{2n}\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)})\|_{\mathrm{Y}} + \|P_{2n}\sum_{\nu=0}^{1} J^0(x_n^{(\nu)}(h_\nu - P_{2n}^{\tau}h_\nu))\|_{\mathrm{Y}}.$$

Now, using the polynomial of the best uniform approximation $T_{n-1} \in \mathscr{T}_{n-1}$ for the function $\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)}$, we get

$$\|\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)}) - P_{2n}\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)})\|_{\mathrm{Y}} \le (1 + \|P_{2n}\|_{\mathrm{Y}\to\mathrm{Y}})\|\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - T_{n-1}\|_{\mathrm{Y}}. \tag{18}$$

Considering the structural qualities of the function $h_\nu(t, \tau)$ by the variable t, it is easy to show that

$$H(\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)}); \alpha) \le c\sum_{\nu=0}^{1} \|x_n^{(\nu)}\|_{\mathrm{C}} \le c\|x_n\|_{\mathrm{X}}. \tag{19}$$

From (18) and (19), using Lemma 1 and an estimation (5), we get

$$\|\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - P_{2n}\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)}\|_{\mathrm{Y}} \le c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \tag{20}$$

Further, taking into account the structural qualities of the functions $h_\nu(t, \tau)$ by the variable τ, error estimations of the quadrature formulae, and Lemma 2, for the second summand of the right-hand side of the estimate (17) we get

$$\|P_{2n}\sum_{\nu=0}^{1} J^0(x_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_{\mathrm{Y}} \le \tag{21}$$

$$\le c(n^\beta \ln n)\|\sum_{\nu=0}^{1} J^0(x_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_{\mathrm{C}} \le c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}.$$

Finally, using the estimate (17), (20), and (21), we get

$$\|Wx_n - W_n x_n\|_{\mathrm{Y}} \le c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \tag{22}$$

Let's denote $\psi_{n-1}(t) \in \mathscr{T}_{n-1}$ the polynomial of the best uniform approximation of the function $\psi(t)$. Using an auxiliary operator

$$\bar{U}_n : \mathrm{X}_n \to \mathrm{Y}_n, \quad \bar{U}_n x_n = \psi_{n-1}^- x_n'^+ - \psi_{n-1}^+ x_n'^-,$$

we get

$$\|Ux_n - U_n x_n\|_{\mathrm{Y}} \le (1 + \|P_{2n}\|_{\mathrm{Y}\to\mathrm{Y}})\|Ux_n - \bar{U}_n x_n\|_{\mathrm{Y}}. \tag{23}$$

Futher, we have

$$\|Ux_n - \bar{U}_n x_n\|_{\mathrm{Y}} \le \|(\psi^- - \psi^-_{n-1})x'^{+}_n\|_{\mathrm{Y}} + \|(\psi^+ - \psi^+_{n-1})x'^{-}_n\|_{\mathrm{Y}}. \tag{24}$$

Each summand of the right-hand side of (24) we estimate, using Lemma 1 as follows:

$$\|(\psi^{\mp} - \psi^{\mp}_{n-1})x'^{\pm}_n\|_{\mathrm{Y}} \le \|\psi^{\mp} - \psi^{\mp}_{n-1}\|_{\mathrm{Y}}\|x'^{\pm}_n\|_{\mathrm{Y}} \le cn^{-\alpha+\beta}\|x_n\|_{\mathrm{X}}. \tag{25}$$

Now by using (24), (25), and (5) we can rewrite inequality (23) as

$$\|Ux_n - U_n x_n\|_{\mathrm{Y}} \le c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \tag{26}$$

And finally, using estimations (15), (16), (22), and (26), we get

$$\|K - K_n\|_{\mathrm{X}_n \to \mathrm{Y}} \le cn^{-\alpha+\beta}\ln n.$$

As the operators Q and K are both invertable and the inverse operator Q^{-1} is bounded, then

$$\|K^{-1}\|_{\mathrm{Y}\to\mathrm{X}} \le \|v\|_{\mathrm{Y}}\|Q^{-1}\|_{\mathrm{Y}\to\mathrm{X}} \le c. \tag{27}$$

So there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \ge n_0$,

$$\|K^{-1}\|_{\mathrm{Y}\to\mathrm{X}}\|K - K_n\|_{\mathrm{X}_n\to\mathrm{Y}} \le cn^{-\alpha+\beta}\ln n \le \frac{1}{2}.$$

For such n according to the Theorem 1.1 of the paper [16] there exist the operators $K_n^{-1} : \mathrm{Y}_n \to \mathrm{X}_n$, and they are bounded. Moreover, for the right-hand sides of the Eqs. (11), (12), using the condition **A1** of the Theorem 1, Lemma 1 and estimation (5), we have

$$\|y - y_n\|_{\mathrm{Y}} = \|y - P_{2n}y\|_{\mathrm{Y}} \le cn^{-\alpha+\beta}\ln n. \tag{28}$$

Now, using the corollary of the Theorem 1.2 [16], for the solutions x^* and x_n^* of the Eqs. (11), (12), taking into account (27), (28), we'll find

$$\|x^* - x_n^*\|_{\mathrm{X}} \le cn^{-\alpha+\beta}\ln n.$$

The Theorem 1 is proved. □

Corollary 1 *If, in the conditions of the Theorem 1, the functions a_ν, b_ν, h_ν (by both variables), $\nu = 0, 1$, and y belong to H^r_α, $r \in \mathbb{N}$. Then the approximate solutions x_n^* converge to the exact solution x^* of the Eq. (1) as $n \to \infty$ by the norm of the space H^1_β as follows:*

$$\|x^* - x_n^*\|_{\mathrm{H}^1_\beta} \le cn^{-r-\alpha+\beta}\ln n, \quad r + \alpha > \beta. \tag{29}$$

Proof Using the Theorem 6 from [15], we can write

$$\|x^* - x_n^*\|_X \le (1 + \|K_n^{-1} P_{2n} K\|)\|x^* - \bar{x}_n\|_X + \|K_n^{-1}\| \|K_n \bar{x}_n - P_{2n} K \bar{x}_n\|_Y, \tag{30}$$

where $\bar{x}_n$ is an arbitrary element of the space X_n. Under corollary 1 conditions the solution x^* of the Eq. (1) is so, that $x^{*\prime} \in H_\alpha^r$ for $0 < \alpha < 1$ and $x^{*(r+1)} \in Z$ for $\alpha = 1$ (Z means Zigmund class of the functions). Then, taking for the $\bar{x}_n \in \mathscr{T}_n$ the polynomial of the best uniform approximation for the function x^* and using Lemma 1, for the first summand of the right-hand side of (30) we'll obtain

$$(1 + \|K_n^{-1} P_{2n} K\|)\|x^* - \bar{x}_n\|_X \le c n^{-r-\alpha+\beta} \ln n. \tag{31}$$

Taking into account the structural qualities of the functions $h_\nu(t, \tau)$, $\nu = 0, 1$, by the variable τ, the error estimation of the quadrature formulae, using Lemma 2 and estimation (5) for the second summand of the right-hand side of the inequality (30), we get

$$\|K_n \bar{x}_n - P_{2n} K \bar{x}_n\|_Y = \|W_n \bar{x}_n - P_{2n} W \bar{x}_n\|_Y \le \tag{32}$$

$$\le \|P_{2n} \sum_{\nu=0}^{1} J_0(\bar{x}_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_Y \le$$

$$\le c(n^\beta \ln n)\| \sum_{\nu=0}^{1} J_0(\bar{x}_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_C \le c(n^{-r-\alpha+\beta}) \ln n \|\bar{x}_n\|_X.$$

Now, substituting estimations (31) and (32) in (30), and taking into account, that

$$\|\bar{x}_n\|_X \le \|x^*\|_X + \|x^* - \bar{x}_n\|_X \le \|x^*\|_X + c n^{-r-\alpha+\beta},$$

we get an estimation (29). Corollary 1 is proved. □

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On Nonexistence of Solutions to Some Nonlinear Functional Differential Inequalities

Evgeny Galakhov and Olga Salieva

Abstract We consider nonexistence of nontrivial solutions for several classes of nonlinear functional differential inequalities. In particular, we obtain sufficient conditions for nonexistence of such solutions for the following types of inequalities: semilinear elliptic inequalities with a transformed argument in the nonlinear term, including higher order ones; quasilinear elliptic inequalities with a transformed argument in the nonlinear term dependent on the absolute value of the gradient of the solution; elliptic inequalities with the principal part of the p-Laplacian type with similar transformations in the lower order terms; parabolic partial differential inequalities with a transformed temporal argument in the nonlinear term. In the case of the untransformed argument these results coincide with the well-known optimal results of Mitidieri and Pohozaev, but in the general case they depend on the character of the transformation of the argument. The results apply to different types of transformations of the argument, such as dilatations, rotations, contractions, and shifts.

Keywords Partial differential inequalities · Transformed argument · Nonexistence

1 Introduction

Sufficient conditions of nonexistence of solutions to nonlinear partial differential equations and inequalities are a popular field of studies in the recent years. This subject is not only interesting in itself, but has important mathematical and physical applications. In particular, some Liouville type theorems of nonexistence of nontrivial

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positive solutions to nonlinear equations can be used for obtaining a priori estimates of solutions to respective problems in bounded domains [1, 2].

In [3–5] (see also references therein) sufficient conditions for nonexistence of solutions were obtained for different classes of nonlinear partial differential inequalities using the test function method developed by S. Pohozaev [6]. But the respective functional differential inequalities with transformed argument was not covered by these results. Some special cases of such problems were treated in [7, 8] and in the papers of the second author [9, 10].

In this paper we obtain sufficient conditions for nonexistence of solutions to several classes of elliptic and parabolic functional differential inequalities and for systems of elliptic inequalities of this type.

The structure of the paper is as follows. In Sect. 2, we prove nonexistence theorems for semilinear elliptic inequalities of higher order; in Sect. 3, for quasilinear elliptic inequalities; and in Sect. 4, for nonlinear parabolic inequalities with a shifted time argument.

The letter c with different subscripts or without them denotes positive constants that may depend on the parameters of the inequalities and systems under consideration.

2 Semilinear Elliptic Inequalities

Let $k \in \mathbb{N}$. Consider a semilinear elliptic inequality

$$(-\Delta)^k u(x) \geq a(x)|u(g(x))|^q \qquad (x \in \mathbb{R}^n), \tag{1}$$

where $g \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is a mapping such that

- (g1) there exists a constant $c > 0$ and $\beta \in \mathbb{R}$ such that $|J_g^{-1}(x)| \geq c_1|x|^\beta > 0$ for all $x \in \mathbb{R}^n$;
- (g2) $|g(x)| \geq |x|$ for all $x \in \mathbb{R}^n$,

and

(a1) $a : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that there exist constants $c_2 > 0$ and $\alpha \in \mathbb{R}$ such that $a(x) \geq c_2|g(x)|^\alpha$ for all $x \in \mathbb{R}^n$.

Example 1 Transforms of the form $g(x) = \gamma(1 + |x|^\beta)x$, where $|\gamma^{-n}\beta| \geq 1$, satisfies assumptions (g1) with $c_1 = |\gamma^{-n}\beta|$ and (g2). In particular, the dilatation transform $g(x) = \gamma x$ with any $\gamma \in \mathbb{R}$ such that $|\gamma| > 1$ satisfies assumptions (g1) with $c = |\gamma|^{-n}$ and (g2).

Example 2 The rotation transform $g(x) = Ax$, where A is a $n \times n$ unitary matrix (and therefore $|g(x)| = |x|$ for all $x \in \mathbb{R}^n$), satisfies assumptions (g1) with $c = 1$ and (g2).

In some situations assumption (g2) can be replaced by a weaker one:

(g'2) there exist constants $c_0 > 0$ and $\rho > 0$ such that $|g(x)| \geq c_0|x|$ for all $x \in \mathbb{R}^n \setminus B_\rho(0)$.

Remark 1 We assume without loss of generality that $c_0 \leq 1$.

Example 3 The contraction transform $g(x) = \gamma x$ with $0 < |\gamma| \leq 1$ satisfies assumptions (g1) with $c = |\gamma|^{-n}$ and (g'2) with $c_0 = |\gamma|$ and any $\rho > 0$.

Example 4 So does the shift transform $g(x) = x - x_0$ for a fixed $x_0 \in \mathbb{R}^n$ with $c = 1$, $c_0 = 1/2$ and $\rho = 2|x_0|$.

Definition 1 A weak solution of inequality (1) is a function $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying the integral inequality

$$\int_{\mathbb{R}^n} u(x) \cdot (-\Delta)^k \varphi(x)\, dx \geq \int_{\mathbb{R}^n} a(x)|u(g(x))|^q \varphi(x)\, dx \tag{2}$$

for any nonnegative function $\varphi \in C_0^{2k}(\mathbb{R}^n)$.

For the proof of the following theorems we will need

Lemma 1 *There exists a non-increasing function $\varphi(s) \geq 0$ in $C^{2k}[0, \infty)$ satisfying conditions*

$$\varphi(s) = \begin{cases} 1 \ (0 \leq s \leq 1), \\ 0 \ (s \geq 2), \end{cases} \tag{3}$$

$$\int_1^2 |\varphi'(s)|^{q'} s^{-\frac{\alpha+\beta}{q-1}} \varphi^{1-q'}(s)\, ds < \infty, \tag{4}$$

$$\int_1^2 |\varphi'(s)|^{\frac{p(q+\lambda)}{q-p+1}} s^{-\frac{\alpha(\lambda+p-1)}{q-p+1}} \varphi^{1-\frac{p(q+\lambda)}{q-p+1}}(s)\, ds < \infty \tag{5}$$

for $\lambda < 0$ with sufficiently small absolute value, and

$$\int_1^2 |\Delta^k \varphi(s)|^{q'} s^{-\frac{\alpha+\beta}{q-1}} \varphi^{1-q'}(s)\, ds < \infty \tag{6}$$

where $q' = \frac{q}{q-1}$.

Proof Take $\varphi(s)$ equal to $(2 - s)^\lambda$ with a sufficiently large $\lambda > 0$ in a left neighborhood of 2 (see [3]).

Theorem 1 *Suppose that g satisfies assumptions (g1) and (g2), and a satisfies (a1). Let either $n \le 2k$ and $q > 1$, or $n > 2k$ and $1 < q \le \frac{n+\alpha+\beta}{n-2k}$. Then inequality (1) has no nontrivial solutions $u \in L^q_{loc}(\mathbb{R}^n)$.*

Proof Assume for contradiction that a nontrivial solutions of (1) does exist. Let $0 < R < \infty$ (in particular, the case $R = 1$ is possible). The function

$$\varphi_R(x) = \varphi\left(\frac{|x|}{R}\right),$$

where $\varphi(s)$ is from Lemma 1, will be used as a *test function* for inequality (1). Multiplying both sides of (1) by the test function φ_R and integrating by parts $2k$ times, we get

$$\int\limits_{\mathbb{R}^n} |u(x)| \cdot |\Delta^k \varphi_R(x)|\, dx \ge \int\limits_{\mathbb{R}^n} a(x)|u(g(x))|^q \varphi_R(x)\, dx. \tag{7}$$

Using (g1), (g2), and the monotonicity of φ_R, one can estimate the right-hand side of (7) from below as

$$\int\limits_{\mathbb{R}^n} a(x)|u(g(x))|^q \varphi_R(x)\, dx = \int\limits_{\mathbb{R}^n} a(g^{-1}(x))|u(x)|^q \varphi_R(g^{-1}(x))|J_g^{-1}(x)|\, dx \ge$$
$$\ge c \int\limits_{\mathbb{R}^n} |x|^{\alpha+\beta}|u(x)|^q \varphi_R(g^{-1}(x))\, dx \ge c \int\limits_{\mathbb{R}^n} |x|^{\alpha+\beta}|u(x)|^q \varphi_R(x)\, dx, \tag{8}$$

where $c = c_1 c_2 > 0$. On the other hand, applying the parametric Young inequality to the left-hand side of (7), we get

$$\int\limits_{\mathbb{R}^n} |u(x)| \cdot \left|\Delta^k \varphi_R(x)\right| dx \le$$
$$\le \frac{c}{q} \int\limits_{\mathbb{R}^n} |x|^{\alpha+\beta}|u(x)|^q \varphi_R(x)\, dx + \frac{c^{-q'}}{q'} \int\limits_{\mathbb{R}^n} \left|\Delta^k \varphi_R\right|^{q'} |x|^{-\frac{\alpha+\beta}{q-1}} \varphi_R^{1-q'}(x)\, dx \overset{x=Ry}{\le}$$
$$\overset{x=Ry}{\le} \frac{c}{q} \int\limits_{\mathbb{R}^n} |u(x)|^q |x|^{\alpha+\beta} \varphi_R(x)\, dx + \frac{c^{-q'}}{q'} R^{n-\frac{\alpha+\beta+2kq}{q-1}} \int\limits_{1\le|y|\le 2} \left|\Delta^k \varphi_1(y)\right|^{q'} |y|^{-\frac{\alpha+\beta}{q-1}} \varphi_1^{1-q'}(y)\, dy. \tag{9}$$

Combining (7)–(9), we have

$$\int\limits_{\mathbb{R}^n} |x|^{\alpha+\beta}|u(x)|^q \varphi_R(x)\, dx \le c^{-1-q'} A R^{n-\frac{\alpha+\beta+2kq}{q-1}},$$

where

$$A := \int\limits_{1\le|y|\le 2} \left|\Delta^k \varphi_1(y)\right|^{q'} |y|^{-\frac{\alpha+\beta}{q-1}} \varphi_1^{1-q'}(y)\, dy < \infty.$$

Restricting the integration domain in the left-hand side of the inequality, we obtain

$$\frac{c}{2}\int\limits_{B_R(0)} |x|^{\alpha+\beta}|u(x)|^q\,dx \le c^{-1-q'}AR^{n-\frac{\alpha+\beta+2kq}{q-1}}.$$

Taking $R\to\infty$, in all cases except the critical one (where the power exponent on the right hand is zero) we get a contradiction which proves the theorem.

In the critical case we get

$$\int\limits_{\mathbb{R}^n} |x|^{\alpha+\beta}|u(x)|^q\,dx < \infty$$

and hence

$$\int\limits_{\operatorname{supp}\Delta^k\varphi_R} |x|^{\alpha+\beta}|u(x)|^q\,dx \le \int\limits_{B_{2R}(0)\setminus B_R(0)} |u(x)|^q|x|^{\alpha+\beta}\,dx \to 0 \text{ as } R\to\infty.$$

But (7), (8), and the Hölder inequality imply

$$\begin{aligned} c\int\limits_{B_R(0)} |x|^{\alpha+\beta}|u(x)|^q\,dx &\le \left(\int\limits_{\operatorname{supp}\Delta^k\varphi_R} |x|^{\alpha+\beta}|u(x)|^q\,dx\right)^{\frac{1}{q}}\times \\ &\times\left(\int\limits_{\operatorname{supp}\Delta^k\varphi_R} \left|\Delta^k\varphi_R(x)\right|^{q'}\varphi_R^{1-q'}(x)\,dx\right)^{\frac{1}{q'}} \end{aligned} \tag{10}$$

and therefore

$$\int\limits_{B_R(0)} |u(x)|^q\,dx \le c\left(\int\limits_{\operatorname{supp}\Delta^k\varphi_R} |u(x)|^q\,dx\right)^{\frac{1}{q}} \to 0 \text{ as } R\to\infty$$

since the second factor in the right-hand side of (10) can be estimated from above by $c_2R^{n-\frac{\alpha+\beta+2kq}{q-1}}$ as before, where $n-\frac{\alpha+\beta+2kq}{q-1}=0$. Thus for a nontrivial u we obtain a contradiction in this case as well. This completes the proof.

Theorem 2 *Suppose that g satisfies assumptions (g1) and (g'2), and a satisfies (a1). Let either $n\le 2k$ and $q>1$, or $n>2k$ and $1<q\le\frac{n+\alpha+\beta}{n-2k}$. Then inequality (1) has no nontrivial solutions $u\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that for some $\rho\ge 0$*

$$\lim_{R\to\infty} \frac{\int\limits_{B_{2R}(0)} |x|^{\alpha+\beta}|u(x)|^q\,dx}{\int\limits_{B_{c_0R}(0)\setminus B_\rho(0)} |x|^{\alpha+\beta}|u(x)|^q\,dx} < \infty, \tag{11}$$

(in particular, $u \in L^q(\mathbb{R}^n)$ *in case* $\alpha+\beta=0$*).*

Proof Similarly to estimate (8), for $R > \rho$ we get

$$\int\limits_{\mathbb{R}^n} a(x)|u(g(x))|^q\varphi_R(x)\,dx = \int\limits_{\mathbb{R}^n} a(g^{-1}(x))|u(x)|^q\varphi_R(g^{-1}(x))|J_g^{-1}(x)|\,dx \geq$$
$$\geq c \int\limits_{\mathbb{R}^n\setminus B_\rho(0)} |x|^{\alpha+\beta}|u(x)|^q\varphi_R\left(\frac{x}{c_0}\right)dx \geq c \int\limits_{B_{c_0R}(0)\setminus B_\rho(0)} |x|^{\alpha+\beta}|u(x)|^q\,dx. \tag{12}$$

Then (7) and (9)–(12) imply

$$\int\limits_{B_{c_0R}(0)\setminus B_\rho(0)} |x|^{\alpha+\beta}|u(x)|^q\,dx \leq c_1 \int\limits_{B_{2R}(0)} |x|^{\alpha+\beta}|u(x)|^q\,dx + c_2R^{n-\frac{\alpha+\beta+2kq}{q-1}},$$

where $c_1, c_2 > 0$, and the constant c_1 can be chosen arbitrarily small. Hence by assumption (11) for $c_1 < \frac{1}{2l_\rho+1}$ and sufficiently large R we have

$$\int\limits_{B_{c_0R}(0)\setminus B_\rho(0)} |x|^{\alpha+\beta}|u(x)|^q\,dx \leq 2c_2R^{n-\frac{\alpha+\beta+2kq}{q-1}},$$

i.e., the conclusion of Theorem 1 remains valid in this case as well. The critical case can be treated similarly to the previous theorem.

3 Quasilinear Elliptic Inequalities

Consider the inequality

$$(-\Delta)^k u(x) \geq a(x)|Du(g(x))|^q \qquad (x \in \mathbb{R}^n). \tag{13}$$

Definition 2 A weak solution of inequality (13) is a function $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^n)$ satisfying the integral inequality

$$\int_{\mathbb{R}^n} u \cdot (-\Delta)^k \varphi \, dx \geq \int_{\mathbb{R}^n} a(x)|Du^q(g(x))|\varphi(x)\, dx \tag{14}$$

for any nonnegative function $\varphi \in C_0^1(\mathbb{R}^n)$.

Theorem 3 *Suppose that g satisfies assumptions (g1) and (g2), and a satisfies (a1). Let either $n \leq 2k-1$ and $q > 1$, or $n > 2k-1$ and $1 < q \leq \frac{n+\alpha+\beta}{n-2k+1}$. Then inequality (13) has no nontrivial solutions $u \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof Multiplying both sides of (13) by the test function φ_R and integrating by parts $2k-1$ times, we get

$$\int_{\mathbb{R}^n} (Du(x), D(\Delta^{k-1}\varphi_R(x)))\, dx \geq \int_{\mathbb{R}^n} a(x)|Du(g(x))|^q \varphi_R(x)\, dx,$$

which implies

$$\int_{\mathbb{R}^n} |Du(x)| \cdot |D(\Delta^{k-1}\varphi_R(x))|\, dx \geq \int_{\mathbb{R}^n} a(x)|Du(g(x))|^q \varphi_R(x)\, dx. \tag{15}$$

Using (g1) and (g2), we can estimate the right-hand side of (15) from below as

$$\int_{\mathbb{R}^n} a(x)|Du(g(x))|^q \varphi_R(x)\, dx = \int_{\mathbb{R}^n} a(g^{-1}(x))|Du(x)|^q \varphi_R(g^{-1}(x))|J_g^{-1}(x)|\, dx \geq$$
$$\geq c \int_{\mathbb{R}^n} |x|^{\alpha+\beta}|Du(x)|^q \varphi_R(x)\, dx. \tag{16}$$

On the other hand, applying the parametric Young inequality to the left-hand side of (15), we get

$$\int_{\mathbb{R}^n} |Du(x)| \cdot \left|D(\Delta^{k-1}\varphi_R(x))\right|\, dx \leq$$
$$\leq \frac{c}{2}\int_{\mathbb{R}^n} |x|^{\alpha+\beta}|Du(x)|^q \varphi_R(x)\, dx + c_1 \int_{\mathbb{R}^n} \left|D(\Delta^{k-1}\varphi_R(x))\right|^{q'} |x|^{-\frac{\alpha+\beta}{q-1}} \varphi_R^{1-q'}(x)\, dx \leq$$
$$\leq \frac{c}{2}\int_{\mathbb{R}^n} |x|^{\alpha+\beta}|Du(x)|^q \varphi_R(x)\, dx + c_2 R^{n-\frac{\alpha+\beta+(2k-1)q}{q-1}} \tag{17}$$

with some constants $c_1, c_2 > 0$. Combining (15)–(17), we have

$$\frac{c}{2}\int_{\mathbb{R}^n} |x|^{\alpha+\beta}|Du(x)|^q \varphi_R(x)\, dx \leq c_2 R^{n-\frac{\alpha+\beta+(2k-1)q}{q-1}}.$$

Restricting the integration domain in the left-hand side of the inequality, we obtain

$$\frac{c}{2}\int\limits_{B_R(0)} |x|^{\alpha+\beta}|Du(x)|^q\,dx \le c_2 R^{n-\frac{\alpha+\beta+(2k-1)q}{q-1}}.$$

Taking $R\to\infty$, we get a contradiction for $n-\dfrac{\alpha+\beta+(2k-1)q}{q-1}<0$. The critical case can be treated similarly to the previous theorems.

Theorem 4 *Suppose that g satisfies assumptions (g1) and (g'2), and a satisfies (a1). Let either* $n\le 2k-1$ *and* $q>1$, *or* $n>2k-1$ *and* $1<q\le\frac{n+\alpha+\beta}{n-2k+1}$. *Then inequality (13) has no nontrivial solutions* $u\in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ *such that*

$$m_\rho := \lim_{R\to\infty}\frac{\int\limits_{B_{2R}(0)} |x|^{\alpha+\beta}|Du(x)|^q\,dx}{\int\limits_{B_{c_0R}(0)\setminus B_\rho(0)} |x|^{\alpha+\beta}|Du(x)|^q\,dx}<\infty \tag{18}$$

(in particular, $u\in W^{1,q}(\mathbb{R}^n)$ *in case* $\alpha+\beta=0$*).*

Proof It is similar to that of Theorem 2.

Further consider the inequality

$$-\Delta_p u(x)\ge a(x)u^q(g(x))\qquad (x\in\mathbb{R}^n), \tag{19}$$

where $g:\mathbb{R}^n\to\mathbb{R}^n$ satisfies conditions (g1) with $0\le\alpha<n$, $\beta=0$ and (g'2), and the function $a(x)$ is as in the previous section. Without loss of generality, we can put $a(x)=c_2|g(x)|^\alpha$.

Definition 3 A weak solution of inequality (19) is a function $u\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)\cap L^q_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying the integral inequality

$$\int\limits_{\mathbb{R}^n}|Du|^{p-2}(Du,D\varphi)\,dx\ge\int\limits_{\mathbb{R}^n}a(x)u^q(g(x))\varphi(x)\,dx \tag{20}$$

for any nonnegative function $\varphi\in C^1_0(\mathbb{R}^n)$.

Theorem 5 *Let g satisfy conditions (g1) with* $0\le\alpha<n$, $\beta=0$ *and (g'2), and let*

$$p-1<q\le\frac{(n+\alpha)(p-1)}{n-p}.$$

Then inequality (19) has no nontrivial weak solutions $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^q_{\text{loc}}(\mathbb{R}^n)$ *satisfying condition (11) with* $\beta = 0$.

Remark 2 Condition (11) holds for all nonnegative solutions $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ of (19) (and even of the more general inequality $-\Delta_p u \geq 0$) due to the weak Harnack inequality

$$\begin{gathered} \forall s \in \left(0, \frac{n(p-1)}{n-p}\right) \exists C = C(n,p,s) > 0 : \\ \forall R > 0 \min_{x \in B_R(0)} u(x) \geq C R^{-\frac{n}{s}} \|u\|_{L^s(B_{2R}(0))} \end{gathered} \tag{21}$$

(see [11]).

Proof (Of Theorem 5.) We use test functions φ_R of the same structure as before. Choose λ so that $1 - p < \lambda < 0$. Multiplying both sides of (19) by $u^\lambda(x)\varphi_R(x)$, integrating by parts, and applying the parametric Young inequality with $\eta > 0$, we get

$$\begin{aligned} &\lambda \int_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p \varphi_R(x)\,dx + \int_{\mathbb{R}^n} u^\lambda(x)|Du(x)|^{p-1}|D\varphi_R(x)|\,dx \geq \\ &\geq \int_{\mathbb{R}^n} a(x)u^q(g(x))u^\lambda(x)\varphi_R(x)\,dx \geq \\ &\geq c_\eta \int_{\mathbb{R}^n} u^{q+\lambda}(g(x))\varphi_R(x)\,dx - \eta \int_{\mathbb{R}^n} a(x)u^{q+\lambda}(x)\varphi_R(x)\,dx. \end{aligned} \tag{22}$$

Further we note that inequalities $c_3|x| \geq |g(x)| \geq c_4|x|$ (see condition (g'2)) imply

$$c_2 c_3^\alpha |x|^\alpha \geq a(x) = c_2|g(x)|^\alpha \geq c_2 c_4^\alpha |x|^\alpha, \tag{23}$$

and one has

$$\begin{aligned} &\int_{\mathbb{R}^n} a(g^{-1}(x))u^{q+\lambda}(x)\varphi_R(g^{-1}(x))|J_g^{-1}(x)|\,dx \geq c_5 \int_{\mathbb{R}^n} |x|^\alpha u^{q+\lambda}(x)\varphi_R(g^{-1}(x))\,dx \geq \\ &\geq c_5 \int_{B_{c_0 R}(0)\setminus B_\rho(0)} |x|^\alpha u^{q+\lambda}(x)\,dx \geq c_6 \int_{B_{2R}(0)} |x|^\alpha u^{q+\lambda}(x)\,dx \end{aligned} \tag{24}$$

with some constants $c_5, c_6 > 0$.

Now, using (g1), (g'2) and (23)–(24), for a sufficiently small $\eta > 0$ (note that $c_\eta \to \infty$ as $\eta \to 0_+$) one can estimate the right-hand side of (24) from below as

$$
\begin{aligned}
&c_\eta \int\limits_{\mathbb{R}^n} a(x)u^{q+\lambda}(g(x))\varphi_R(x)\,dx - \eta \int\limits_{\mathbb{R}^n} a(x)u^{q+\lambda}(x)\varphi_R(x)\,dx = \\
&= c_\eta \int\limits_{\mathbb{R}^n} a(g^{-1}(x))u^{q+\lambda}(x)\varphi_R(g^{-1}(x))|J_g^{-1}(x)|\,dx - \eta \int\limits_{\mathbb{R}^n} a(x)u^{q+\lambda}(x)\varphi_R(x)\,dx \geq \\
&\geq c_\eta c_6 \int\limits_{B_{2R}(0)} |x|^\alpha u^{q+\lambda}(x)\,dx - c_2 c_3^\alpha \eta \int\limits_{\mathbb{R}^n} |x|^\alpha u^{q+\lambda}(x)\varphi_R(x)\,dx \geq \\
&\geq c_\eta c_6 \int\limits_{B_{2R}(0)} |x|^\alpha u^{q+\lambda}(x)\,dx - c_2 c_3^\alpha \eta \int\limits_{B_{2R}(0)} |x|^\alpha u^{q+\lambda}(x)\,dx = c_7 \int\limits_{\mathbb{R}^n} |x|^\alpha u^{q+\lambda}(x)\,dx
\end{aligned}
\tag{25}
$$

with a constant $c_7 = c_\eta c_6 - c_2 c_3^\alpha \eta > 0$.

On the other hand, applying the parametric Young inequality to the left-hand side of (22), similarly to the proof of Theorem 1 we get

$$
\begin{aligned}
&\lambda \int\limits_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p\varphi_R(x)\,dx + \int\limits_{\mathbb{R}^n} u^\lambda(x)|Du(x)|^{p-1}|D\varphi_R(x)|\,dx \leq \\
&\leq (\lambda+\varepsilon) \int\limits_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p\varphi_R(x)\,dx + c_\varepsilon \int\limits_{\mathbb{R}^n} u^{\lambda+p-1}(x)|D\varphi_R(x)|^p\varphi_R^{1-p}(x)\,dx \leq \\
&\leq (\lambda+\varepsilon) \int\limits_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p\varphi_R(x)\,dx + \frac{c_7}{2} \int\limits_{\mathbb{R}^n} |x|^\alpha u^{q+\lambda}(x)\varphi_R(x)\,dx + \\
&+ c_8 \int\limits_{\mathbb{R}^n} |D\varphi_R(x)|^{\frac{p(q+\lambda)}{q-p+1}} |x|^{-\frac{\alpha(\lambda+p-1)}{q-p+1}} \varphi_R^{1-\frac{p(q+\lambda)}{q-p+1}}(x)\,dx \leq (\lambda+\varepsilon) \int\limits_{\mathbb{R}^n} u^{\lambda-1}(x)|Du(x)|^p\varphi_R(x)\,dx + \\
&+ \frac{c_7}{2} \int\limits_{\mathbb{R}^n} |x|^\alpha u^{q+\lambda}(x)\varphi_R(x)\,dx + c_9 R^{n-\frac{\alpha(\lambda+p-1)+p(q+\lambda)}{q-p+1}}
\end{aligned}
\tag{26}
$$

with some constants $\varepsilon, c_\varepsilon, c_8, c_9 > 0$. Choosing $\varepsilon < |\lambda|$, from (24)–(26) we have

$$
\frac{c_7}{2} \int\limits_{\mathbb{R}^n} |x|^\alpha u^{q+\lambda}(x)\varphi(x)\,dx \leq c_9 R^{n-\frac{\alpha(\lambda+p-1)+p(q+\lambda)}{q-p+1}}.
$$

Choosing λ sufficiently close to 0 and taking $R \to \infty$, we obtain a contradiction for $n - \frac{\alpha(p-1)+pq}{q-p+1} < 0$, i.e., $p - 1 < q < \frac{(n+\alpha)(p-1)}{n-p}$. The critical case can be treated similarly to the previous theorems.

Further we consider the inequality

$$
-\Delta_p u(x) \geq a(x)|Du(g(x))|^q \qquad (x \in \mathbb{R}^n). \tag{27}
$$

Definition 4 A weak solution of inequality (27) is a function $u \in W_{\mathrm{loc}}^{1,\max(p,q)}(\mathbb{R}^n)$ satisfying the integral inequality

$$\int_{\mathbb{R}^n} u \cdot (-\Delta)^k \varphi \, dx \geq \int_{\mathbb{R}^n} a(x)|Du^q(g(x))|\varphi(x)\, dx \tag{28}$$

for any nonnegative function $\varphi \in C_0^1(\mathbb{R}^n)$.

Theorem 6 *Let $p-1 < q \leq \frac{n(p-1)}{n-1}$. Suppose that g satisfies assumptions (g1) and (g2). Then inequality (27) has no nontrivial nonnegative solutions $u \in W^{1,\max(p,q)}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof Inequality (28) implies

$$\int_{\mathbb{R}^n} |Du(x)|^{p-1} \cdot |D\varphi_R(x)|\, dx \geq \int_{\mathbb{R}^n} a(x)|Du(g(x))|^q \varphi_R(x)\, dx. \tag{29}$$

Using (g1), (g2), and the Hölder inequality, similarly to the previous arguments, we obtain

$$\int_{\mathbb{R}^n} |Du(x)|^q \varphi_R(x)\, dx \leq c_1 \int_{B_{2R}(0)} a^{-\frac{p-1}{q-p+1}}(x)|D\varphi_R(x)|^{\frac{q}{q-p+1}} \varphi_R^{1-\frac{q}{q-p+1}}(x)\, dx$$

and hence

$$\int_{B_R(0)} |Du(x)|^q \, dx \leq c_2 R^{n-\frac{q+\alpha}{q-p+1}}$$

with some constants $c_1, c_2 > 0$. Taking $R \to \infty$, we obtain a contradiction for $n - \frac{q+\alpha}{q-p+1} < 0$. The critical case can be treated similarly to the previous theorems.

Remark 3 If g satisfies (g'2) instead of (g2), a version of Theorem 4 can be proven for a class of solutions that satisfy (18) (in particular, $u \in W^{1,p}(\mathbb{R}^n) \cap W^{1,q}(\mathbb{R}^n)$) similarly to Theorems 2 and 4.

4 Nonlinear Parabolic Inequalities

Now let $\tau > 0$. Consider the semilinear parabolic inequality

$$\frac{\partial u(x,t)}{\partial t} + (-\Delta)^k u(x,t) \geq a(x,t)|u(x, g(t))|^q \qquad (x \in \mathbb{R}^n; t \in \mathbb{R}_+) \tag{30}$$

with initial condition

$$u(x,0) = u_0(x) \qquad (x \in \mathbb{R}^n), \tag{31}$$

where $u_0 \in C(\mathbb{R}^n)$ is a function that satisfies the condition

$$\int\limits_{\mathbb{R}^n} u_0(x)\,dx \geq 0, \tag{32}$$

$a \in C(\mathbb{R}^n \times \mathbb{R}_+)$ is a function that satisfies the condition

$$a(x,t) \geq ct^\alpha \text{ for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \tag{33}$$

with some constants $c > 0$ and $\alpha > -1$ independent of x and t, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that

(g3) $t \leq g(t)$ and $g'(t) \geq 1$ for any $t \geq 0$.

A solution of problem (30)–(31) will be defined in the distributional sense similarly to the previous sections.

Let $0 < R, T < \infty$. We will use as a test function the product of two functions

$$\Phi(x,t) = \varphi\left(\frac{|x|}{R}\right) \cdot \varphi\left(\frac{t}{T}\right),$$

where the function $\varphi(s)$ is the one from Lemma 1.

Theorem 7 *Problem (30)–(31) with u_0 that satisfies (32) and g that satisfies (g3) has no nontrivial solutions for $q > 1$ and $n - \frac{(2kq+\alpha)(1+\alpha)}{(q+\alpha)(q-1)} < 0$.*

Proof Multiplying both sides of (30) by the test function Φ and integrating by parts, we get

$$\begin{aligned} -\int\limits_{\mathbb{R}^n} u_0(x)\Phi(x,0)\,dx + \int\limits_0^\infty \int\limits_{\mathbb{R}^n} |u(x,t)| \cdot \left|\frac{\partial \Phi(x,t)}{\partial t}\right| dx\,dt + \\ + \int\limits_0^\infty \int\limits_{\mathbb{R}^n} |u(x,t)| \cdot \left|\Delta^k \Phi(x,t)\right| dx\,dt \geq \int\limits_0^\infty \int\limits_{\mathbb{R}^n} a(x,t)|u(x,g(t))|^q \Phi(x,t)\,dx. \end{aligned} \tag{34}$$

Since the function $\varphi(t/T)$ monotonically decreases, using (g3) and the monotonic decay of $\Phi(x,t)$ in t for each $x \in \mathbb{R}^n$, one can estimate the right-hand side of (34) from below as

$$\begin{aligned} \int\limits_0^\infty \int\limits_{\mathbb{R}^n} a(x,t)|u(x,g(t))|^q \Phi(x,t)\,dx\,dt = \int\limits_0^\infty \int\limits_{\mathbb{R}^n} a(x,g^{-1}(t))|u(x,t)|^q \Phi(x,g^{-1}(t))(g^{-1})'(t)\,dx\,dt \geq \\ \geq \int\limits_0^\infty \int\limits_{\mathbb{R}^n} a(x,t)|u(x,t)|^q \Phi(x,t)\,dx\,dt. \end{aligned} \tag{35}$$

On the other hand, applying the parametric Young inequality and Lemma 2.1 to the second and third terms of the left-hand side of (34), we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} |u(x,t)| \cdot \left| \frac{\partial \Phi(x,t)}{\partial t} \right| dx\, dt \le$$
$$\le \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^n} a(x,t)|u(x,t)|^q \Phi(x,t)\, dx\, dt + c_1 \int_0^\infty \int_{\mathbb{R}^n} a^{-\frac{q'}{q}}(x,t) \left| \frac{\partial \Phi(x,t)}{\partial t} \right|^{q'} \Phi^{1-q'}(x,t)\, dx\, dt \le$$
$$\le \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^n} a(x,t)|u(x,t)|^q \Phi(x,t)\, dx\, dt + c_2 R^n T^{-\frac{1+\alpha}{q-1}} \tag{36}$$

and

$$\int_0^\infty \int_{\mathbb{R}^n} |u(x,t)| \cdot \left| \Delta^k \Phi(x,t) \right| dx\, dt \le$$
$$\le \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^n} a(x,t)|u(x,t)|^q \Phi(x,t)\, dx\, dt + c_3 \int_0^\infty \int_{\mathbb{R}^n} a^{-\frac{q'}{q}}(x,t) \left| \Delta^k \Phi(x,t) \right|^{q'} \Phi^{1-q'}(x,t)\, dx\, dt \le$$
$$\le \frac{1}{4} \int_0^\infty \int_{\mathbb{R}^n} a(x,t)|u(x,t)|^q \Phi(x,t)\, dx\, dt + c_4 R^{n-\frac{2kq+\alpha}{q-1}} T \tag{37}$$

with some constants $c_1, \ldots, c_4 > 0$. Combining (34)–(37) and taking into account (32), we have

$$\frac{1}{2} \int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^q \Phi(x,t)\, dx\, dt \le c_2 R^n T^{-\frac{1+\alpha}{q-1}} + c_4 R^{n-\frac{2kq+\alpha}{q-1}} T.$$

Taking $T = R^{\frac{2kq+\alpha}{q+\alpha}}$ and $R \to \infty$, we obtain a contradiction for $n - \frac{2kq+\alpha}{q+\alpha} < 0$. The critical case can be considered similarly to the previous theorems.

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The Common Descent of Biological Shape Description and Special Functions

J. Gielis, D. Caratelli, C. Moreno de Jong van Coevorden and P. E. Ricci

Abstract Gielis transformations, with their origin in botany, are used to define square waves and trigonometric functions of higher order. They are rewritten in terms of Chebyshev polynomials. The origin of both, a uniform descriptor and the origin of orthogonal polynomials, can be traced back to a letter of Guido Grandi to Leibniz in 1713 on the mathematical description of the shape of flowers. In this way geometrical description and analytical tools are seamlessly combined.

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1 Gielis Transformations

Gielis transformations [1] are geometric transformations acting on planar functions $f(\vartheta)$ unifying a wide range of natural and abstract shapes (Eq. 1). Since its discovery two decades ago and the initial publications in 2001–2005 [1–3] they have been used in mathematics, biology and various fields of technology. Gielis transformations can morph a classic Euclidean circle or sphere, into an infinite number of shapes, including regular polygons, providing a designated unit circle or unit sphere. For example,

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it became possible to derive analytic solutions to a wide class of boundary value problems, using Fourier's classical methods. Heat distribution solved by Fourier on a circular plate, has now been extended Laplace, Helmholtz, wave and heat equations for 2D and 3D domains, including annuli and shells [4–8]. They can be extended in 3 or more dimensions [2, 9] also in relation to Generalized Möbius-Listing surfaces and bodies [10].

$$\kappa(\vartheta; a, b, m, n_1, n_2, n_3) = f(\vartheta)\left[\left|\frac{1}{A}\cos\left(\frac{m_1}{4}\vartheta\right)\right|^{n_2} \pm \left|\frac{1}{B}\sin\left(\frac{m_2}{4}\vartheta\right)\right|^{n_3}\right]^{-\frac{1}{n_1}} \quad (1)$$

Following a generalization of constant mean curvature surfaces for anisotropic energy functionals [11, 12], snowflakes and flowers can now be studied as minimal surfaces in the same way as soap bubbles and soap films are minimal surfaces for a given energy functional [13]. In general natural shapes can be described in a uniform way and studied via natural curvature conditions [2, 13, 14]. In biology, it has been used to drastically improve modelling of annual rings in trees [15], leaf shapes [16] and diatoms [17], to model human skin as dielectric materials [18, 19], or to model the backbone of RNA [20]. It was used to study the mechanical efficiency and stability of petioles [21, 22], or for biomechanical studies of knees and multi-dynamics mechanical systems in the body [23].

Vision algorithms developed with Gielis transformation allows for scanning objects or signals and for efficient compression algorithms, medical imaging and datamining [24, 25]. This has led to research in biomedical imaging of blood cells, heart, skulls and various organs [26, 27]. These algorithms are the first that can recognize self-intersecting curves or the symmetries of polygons and polygrams without prior learning algorithms encoded in the computer [24]. Intersecting curves are found in biomolecules, molecules, ropes and knots and a variety of other natural shapes.

In the field of nanotechnology alone Gielis transformations are used in at least 25 papers. For example, to compute the optimal shape of nanoparticles for applications in solar panels, cancer treatments with thermal methods, nano-antennas, in-body telemetry or to shrink sizes of chips in electronics [28–30]. In engineering they have been used to optimize the shapes of wind turbines [31], heat shields in manned space vehicles [32, 33], non-circular gears and gear tooth profiles [34, 35], or the shape of non-planar wings in aircraft [36]. In the field of telecommunications they has been used to design waveguides and antennas [37–41], whereby design can also be based on botanical shapes [42, 43], and to optimize lasers [44, 45]. All these developments originate in the study of plants [46].

In its original form it has six parameters, but to quantify bamboo leaves or tree rings, the optimization of nanoparticles or in the development of antennas, two or three parameters suffice for size and shape.

2 The Origin of Gielis Transformations: Botany

The origin of Gielis transformations is the study of square stems in plants using Lamé curves [1]. To extend this to other symmetries, inspiration was found in D'Arcy Thompson's *On Growth and Form* [47] showing the analogy between certain flowers, and Rhodonea curves, the oldest and most useful mathematical representation of flowers. Rhodonea curves were discovered by Guido Grandi and communicated to Leibniz in a letter [48, 49]. The observation that in Rhodonea or Grandi curves $\varrho(\vartheta) = \cos m\vartheta$ or $\varrho(\vartheta) = \sin m\vartheta$ the argument of the angle specified the frequency, was applied to Lamé superellipses $(\frac{x}{A})^n + (\frac{y}{B})^n = 1$ with n a positive integer [50], in particular to the polar representation of the Lamé curves or superellipses defined by $\varrho(\vartheta) = \frac{1}{\sqrt[n]{|\frac{1}{A}\cos\vartheta|^n + |\frac{1}{B}\sin\vartheta|^n}}$. The pivotal step to Eq. 1 was rewriting Lamé curves in polar coordinates and generalizing the symmetry from 4 to any real number.

Figure 1 displays the result of transformation on some simple function. In transforming the circle $f(\vartheta) = constant$, also regular polygons can result (Eq. 2), or self-intersecting shapes, for m a rational number (Fig. 2). Such shapes can be found in plant phyllotaxis [1, 46], the symmetry of DNA in planar view [53], or as separation zones in phase spaces with non-linear resonances [52].

$$\varrho(\vartheta) = \lim_{n_1\to\infty} \frac{1}{\left[|\cos(\frac{m}{4}\vartheta)|^{2(1-n_1\log_2\cos\frac{\pi}{m})} + |\sin(\frac{m}{4}\vartheta)|^{2(1-n_1\log_2\cos\frac{\pi}{m})}\right]^{\frac{1}{n_1}}} \tag{2}$$

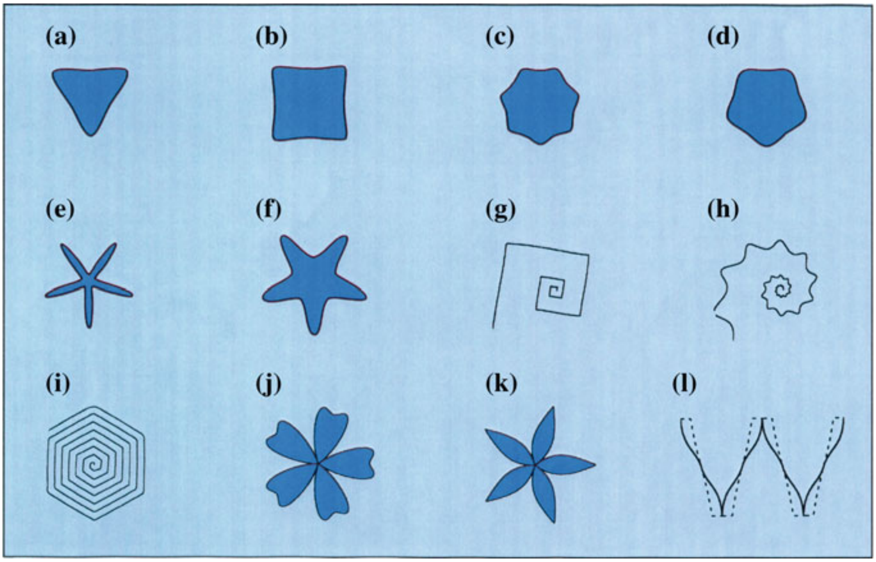


Fig. 1 **a**–**d** cross sections of plant stems; **e**–**f** starfish; **g**–**i** transformations of logarithmic (**g**–**h**) and Archimedean spirals (**i**); **j**–**l** transformations of cosines, as flowers or in wave view [1]

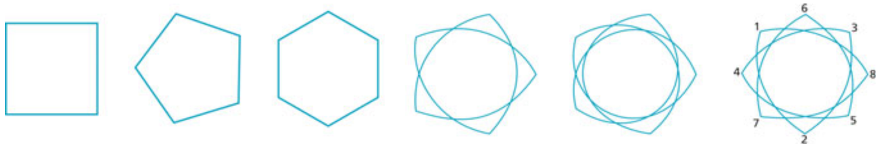
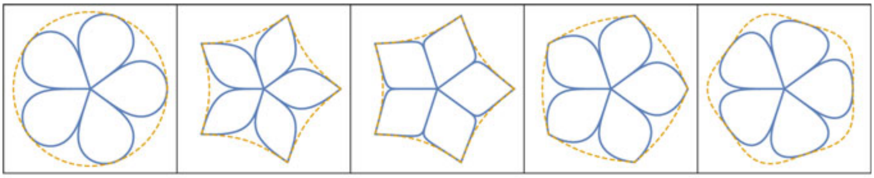


Fig. 2 Regular polygons for $m = 4, 5, 6$. Self-intersecting polygons for $m = \frac{5}{2}; \frac{5}{3}; \frac{8}{3}$



n_1	2	1	1	3	3
$n_{2,3}$	2	1	1	1	3
n_4	4	4	20	5	5

Fig. 3 Choripetalous five-petalled flowers with the corresponding constraining superpolygons and parameters

In Eq. 1 the function $f(\vartheta)$ can be regarded as the developing function *DF*, the function that want to grow or develop. The second part of Eq. 1, also known as Gielis formula, denotes the constraining function *CF* (κ), constraining the development of *DF*. When $f(\vartheta) = \delta(\vartheta; m, n_4) = |\cos(\frac{m}{2}\vartheta)|^{\frac{1}{n_4}}$, many natural flowers shapes result (Fig. 3; [53]).

3 Coordinate Functions of First and Higher Order, and Square Waves

The flower shapes and the wave-like shapes in Fig. 1, lower row, and the flowers in Fig. 3, defined by Eq. 1 are essentially the trigonometric functions associated with the shapes. Generalized trigonometric functions been defined beyond circular functions [54–58]. For Lamé curves with exponent *p* for example, the half perimeter is defined as π_p. For the Euclidean circle $\pi_{p=2} = \pi$. The corresponding trigonometric functions of Lamé curves are ${}_p\cos\vartheta$ and ${}_p\sin\vartheta$, with a Generalized Pythagorean Theorem $({}_p\cos\vartheta)^p + ({}_p\sin\vartheta)^p = 1$ [55, 56]. Likewise, the coordinate functions of shapes defined by Eq. 1, are cosine and sine moderated by Eq. 1.

With Eq. 1 we can modulate these or trigonometric functions. One example is the generation of square waves. A square wave may be generated in various ways, e.g. with reference to step functions, e.g. the Heaviside step function (Eq. 3). Note that the Dirac delta function is the derivative of the Heaviside function.

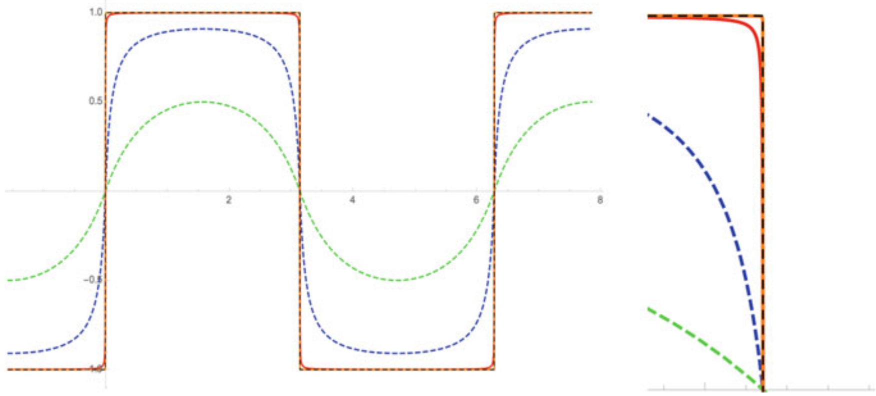


Fig. 4 Equation 5 Sines for varying $\epsilon = 10^{-\alpha}$ with $\alpha = 0$ green; $\alpha = 1$ blue; $\alpha = 3$ red solid; $\alpha = 5$ orange solid and $\epsilon = 0$ black dashed

$$H(x) = \begin{cases} 0, x < 0 \\ 1, x \geq 0 \end{cases} \tag{3}$$

An alternative method, is synthesis via Fourier series. One well-known disadvantage is the Fourier-Gibbs phenomenon, whereby oscillations occur in points of measure zero. These phenomena are an inherent feature of the method, but may be mediated in practice by using $sinc x = \frac{\sin \pi x}{\pi x}$ (Eq. 4):

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{m-1} sinc \frac{k}{m}\left(a_k \cos \frac{k\pi x}{L} + b_k \sin \frac{k\pi x}{L}\right) \tag{4}$$

Using Eq. 1 the ratio of the sine function and the absolute value of the sine can be taken. This is a special case of Eq. 1. In order to generate a square wave which is differentiable everywhere, all exponents in Eq. 1 are equal to 1, $m = 4$, and A very large, so that the cosine term becomes very small, ϵ (Eq. 5). In Fig. 4 the shape of the sine wave is given for various values of ϵ. As long as ϵ is finite and not zero, the function is differentiable everywhere. In this way Gibbs phenomena are avoided and differentiability can be ensured everywhere.

$$\frac{\sin \vartheta}{\epsilon + |\sin \vartheta|} \tag{5}$$

These curves can also be framed in a window, e.g. the interval [−1; 1] or in a Gaussian window $\frac{1}{\sigma\sqrt{2}} e^{-(\frac{(\vartheta-\mu)}{2\sigma})^n}$ (6) for various values of ϵ in (5) (Fig. 5). The Haar wavelet (Daubechies 1 wavelet) and various other step functions classically based on distributions, can be defined by Eq. 5, using the appropriate window and shifts.

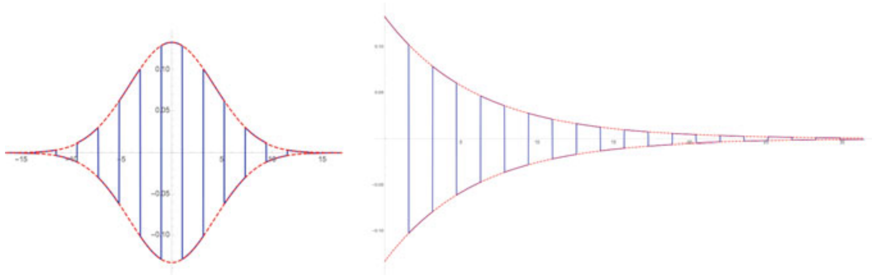


Fig. 5 Cosines for $\epsilon = 10^{-5}$ in Gaussian window with n = 2 in (6) (left). Decaying square wave with n = 1 in (6) (right)

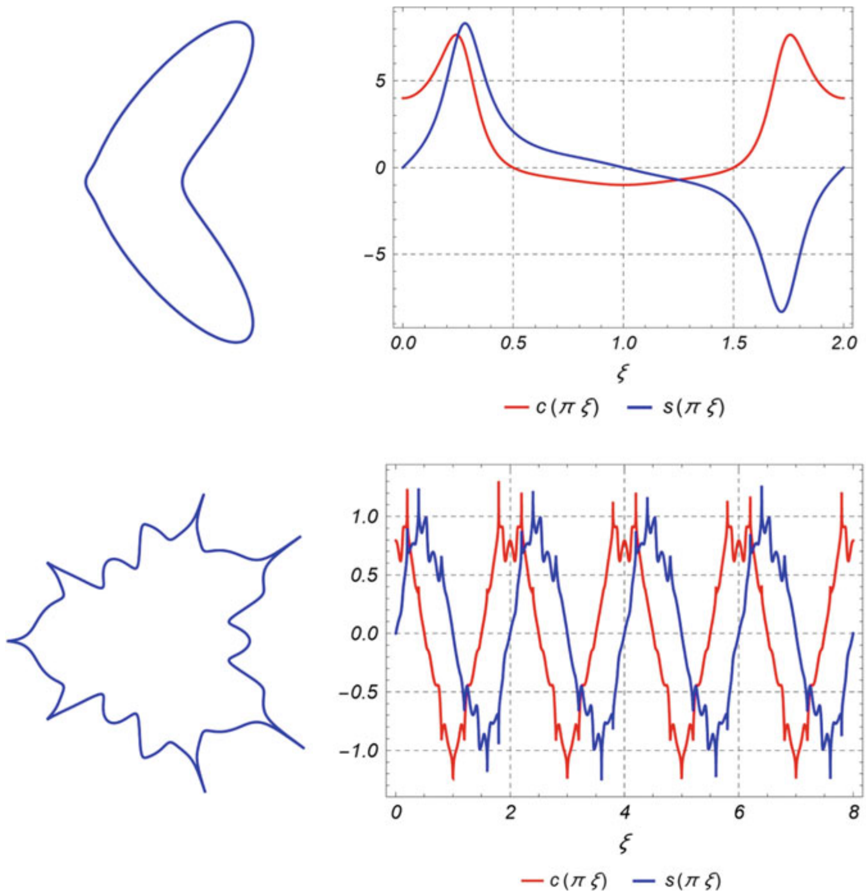


Fig. 6 First (upper row) and second order (lower row) supertrigonometric functions with associated polar graphs

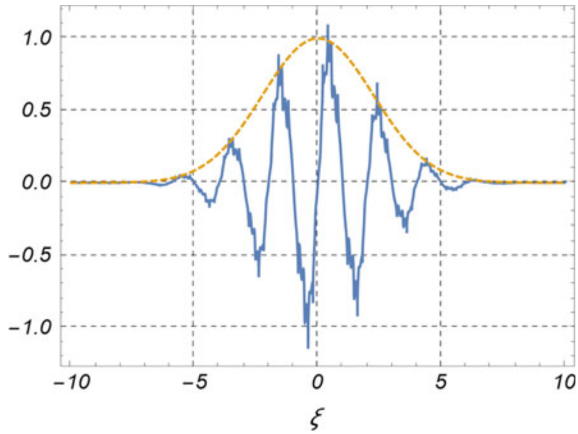


Fig. 7 The Gaussian version of Fig. 6 lower row

$$\psi(t) = \begin{cases} 1, \ 0 \le t \le \frac{1}{2} \\ -1, \ \frac{1}{2} \le t < 1 \\ 0, \ otherwise \end{cases} \tag{6}$$

Second and higher order trigonometric functions based on Eq. 1 can be generated. Given a shape $\gamma(\vartheta)$ defined by Eq. 1, the polar plot is generated by:

$$c(\vartheta) = \gamma(\vartheta)\cos\vartheta, \ s(\vartheta) = \gamma(\vartheta)\sin\vartheta \tag{7}$$

The functions $c(\vartheta)$ and $s(\vartheta)$ are displayed in Fig. 6 upper row (for $A = 2, B = 1; m_1 = 1.5; m_2 = 0.5; n_1 = 1; n_2 = 2; n_3 = 3$). They are used to define a second curve whereby $c(\vartheta)$ and $s(\vartheta)$ substitute for the original cosine and sine in Eq. 1 respectively. The second order curve and corresponding trigonometric curves are shown in Fig. 6 lower row (for $A = 2, B = 1; m_1 = 3; m_2 = 5; n_1 = 15; n_2 = 5; n_3 = 1$) and Fig. 7. This can be continued to any order and applying this to Eq. 1 would be similar to continued fractions.

4 From Transcendental to Algebraic Functions

The original Eq. 1 makes use of transcendental functions, defined on the shape (Fig. 8). What hitherto was missing is the reverse step, namely to express such shapes in Cartesian coordinates, to convert the transcendental functions into algebraic functions of one or more variables. This can be achieved using Chebyshev polynomials T, U, V and W. These originate from the work of the Russian mathematician Pafnuty Chebyshev in the mid 19th century, who laid the foundation for orthogonal polyno-

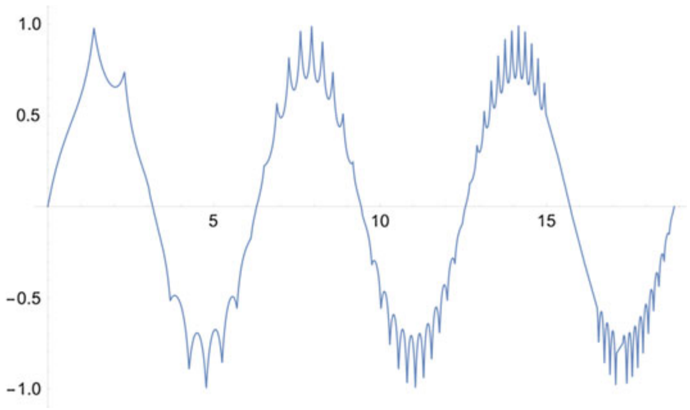


Fig. 8 Trigonometric functions defined according to [58] where $\frac{m_1}{4}\vartheta$ in is generalized to $f(\vartheta)$. In this case $f(\vartheta) = \frac{(\vartheta+\pi)}{4}$, with A, $B$and $n_{1,2,3} = 1$

mials [59, 60] with extremely wide applications in applied mathematics and physics. There is hardly any field in physics and technology where these special functions are not used.

Equation 1 can be rewritten as follows:

$$\varrho(x) = \frac{1}{\sqrt[n_1]{|\frac{1}{A}T_m(x)|^{n_2} + |\frac{1}{B}\sqrt{1-x^2}U_{m-1}(x)|^{n_3}}} \tag{8}$$

for $-1 \le x \le +1$. The substitution for Gielis curves is $x = cos\frac{\vartheta}{4}$. For Lamé curves $A = B = 1$, $n_1 = n_2 = n_3 = n$, and $m = 1$ so we have $T_m = T_1$ and $U_{m-1} = U_1$. This gives a rational polynomial function of transcendental arguments (although the argument could be any function). In a further step also the exponents can be internalized in the Chebyshev polynomials so that we have real polynomials, which can be calculated using the recurrence relations for Chebyshev polynomials. The roots of the polynomials correspond to the maxima and minima of the Superformula.

Each Chebyshev polynomial is composed of a finite number of terms. For example $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, etc. T_n contains terms in x of the powers n, $(n-2)$, $(n-4)$, ...ending in -1 or $+1$ for even powers x^n, and in $\pm nx$ for odd n. The expression in polar coordinates can then be studied in a single variable x. In this way it can be shown that for suitable choice of parameters (exponents n and symmetry parameter m are integers) Eq. 1 can be regarded as a algebraic function.

This inverse transformation is based on the obvious but novel observation that Grandi curves ($\varrho(\vartheta) = \cos m\vartheta$ or $\varrho(\vartheta) = \sin m\vartheta$) simply ARE Chebyshev polynomials of the first kind $T_m(\cos\vartheta) = \cos m\vartheta$ and of the third kind $\sin\vartheta U_{m-1}(x) = \sin m\vartheta$ for m integer. Three centuries after Grandi first discovered his curves, they led to the superformula in polar coordinates (Eq. 1) and the generalized Pythagorean Theorem, as well as the inverse transformation.

With further developments and generalizations (for example multivariate and multi-argument Chebyshev polynomials [60]), a wide variety of natural and abstract shapes can be considered and transformed as algebraic functions (for the appropriate choice of parameters) opening many possibilities in science and technology. For $\varrho(x) = \frac{1}{\sqrt[n]{|T_m(x)|^n + |\sqrt{1-x^2}U_{m-1}(x)|^n}}$ we immediately generalize Chebyshev polynomials and their multiple uses in the sense of Eq. 1, when $f(\vartheta)$ is a Chebyshev polynomial.

5 Fibonacci, Power Laws and Shape Description

Special functions and polynomials can thus become a standard tool in studies in botany and biology, since description (Eq. 1) and the analytical tools (Chebyshev polynomials) have the same origin with Grandi's observations. Chebyshev polynomials are finite and precise, and are known to give the best possible approximation to functions. Power laws, ubiquitous in the natural sciences, can be defined accurately in terms of Chebyshev polynomials. Another example in shape description related to rational approximations of shape descriptors in botany based on elliptic Fourier series: for even functions (for example the mirror symmetry of a leaf around the midrib) the Fourier series collapses to a Chebyshev polynomial series. Actually, one can rewrite Fourier series and consequently also the solutions for boundary value problems using the Fourier projection method [4–8, 46] in terms of Chebyshev polynomials. An open challenge is to combine Generalized Möbius-Listing bodies and their relation to knots and links [10, 61, 62] with the recent theorem that every knot is a Chebyshev knot [63].

There is the direct relation of Chebyshev polynomials to Lucas L_n and Fibonacci numbers F_n, widely used in describing plant phyllotaxy. They can all be considered as special cases of the homogeneous linear second order difference equation with constant coefficients $u_0; u_1; u_{n+1} = au_n + bu_{n-1}$, for $n \leq 1$. If a and b are polynomials in x, a sequence of polynomials is generated. In particular if $a = 2x$ and $b = -1$, we obtain Chebyshev polynomials. They are of the first kind $T_n(x)$ for $u_0 = 1; u_1 = x$, and of the second kind $U_n(x)$ for $u_0 = 1; u_1 = 2x$. Fibonacci numbers F_n arise for $a = b = 1; u_0 = 0; u_1 = 1$. For $a = b = 1;\ u_0 = 2; u_1 = 1$, we obtain Lucas numbers $L_{n.}$ Therefore, if in Chebyshev polynomials $i = \sqrt{-1}$ is used with $x = \frac{i}{2}$ the results are Lucas numbers L_n for Chebyshev polynomials of the first kind T_n, and Fibonacci numbers F_n for those of the second kind U_n. This also clarifies the direct relation between Fibonacci and Lucas numbers with the geometry of phyllotaxy given Eq. 1 with m a rational number ($m = \frac{p}{q} = 5/2, 8/3 \ldots = \frac{F_{n+2}}{F_n}$). This ratio gives the number of angles in the numerator $p(= F_{n+2})$ and the number of rotations needed to close $q(= F_n)$ as the inverse of the generally used $\frac{F_n}{F_{n+2}}$.

Obviously care must be taken when connecting Fibonacci to botany and Gielis transformations and Chebyshev polynomials are not restricted to these numbers.

Since the symmetry parameter m can be any real number, Eq. 1 can describe spiral monostichy or spiral distichy; it suffices for m to deviate slightly from 1 and 2 to accurately model this type of phyllotaxy as well [1, 46]. A precise description can help us understand the relation of shape to the nearly universal principle in the physical sciences is that the equilibrium configuration of a system can be found by minimizing its total energy among all admissible configurations, also for anisotropic shapes within the framework of Euclidean geometry. Once more, botany proves to be a fertile soil for science; Grandi and Gielis curves take the study of plants and natural organisms, living and non-living, further into the realms of the three pillars of mathematics, namely geometry, algebra and analysis.

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Variational Iteration Method for Solving Problems with Integral Boundary Conditions

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Abstract In this work, Variational Iteration Method is employed to solve parabolic partial differential equations subject to initial and nonlocal inhomogeneous boundary conditions of integral type. Since nonlocal boundary conditions considerably complicate the application of standard functional and numerical techniques, equations having such conditions are first transformed to local (classical) boundary conditions Then they are solved by Variational Iteration Method.

Keywords Parabolic differential equation · Variational iteration method Nonlocal conditions

2000 Mathematics Subject Classification Primary 05C38 · 15A15; Secondary 05A15 · 15A18

1 Introduction

Various problems arising in heat conduction [1–3], chemical engineering [4], thermo elasticity [5] and plasma physics [6] can be modeled by nonlocal initial boundary value problems with integral conditions. This class of boundary value problems has been investigated in [1, 3, 4, 7–16] for parabolic partial differential equations.

In recent years, Variational Iteration Method (VIM) has received more and more attention for solving this class of problems. This method was proposed by Ji-Huan He, in 1998, and has been applied to solve many different linear and non-linear functional equations, such as autonomous ordinary differential equations [17], wave equations

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[18], non-linear mixed Volterra–Fredholm integral equations [19], non-linear heat transfer equations [20], and many others.

Here, VIM is employed to solve parabolic partial differential equations subject to initial and nonlocal inhomogeneous boundary conditions of integral type. But, presence of nonlocal boundary conditions significantly complicate the application of standard functional and numerical techniques. So, we transform inhomogeneous linear parabolic equations with nonlocal boundary conditions to local (classical) boundary conditions. Then we apply the aforesaid method to find their solutions.

This paper is outlined as follows: In Sect. 2, It is explained how a given nonlocal initial boundary value problem for inhomogeneous linear parabolic equation subject to initial and nonlocal inhomogeneous boundary conditions of integral type can be transformed into a local Dirichlet initial boundary value problem. In Sect. 3, this transformation is applied to some numerical examples and then VIM is employed to solve the resulted equations. Finally, discussions and conclusions are presented in the last section.

2 Transforming Equations with Nonlocal Boundary Conditions into Local Boundary Conditions

2.1 *Linear Parabolic Equation with Purely Integral Conditions*

Consider the following inhomogeneous linear parabolic equation

$$\frac{\partial u(x,t)}{\partial x} - p(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t)\frac{\partial u(x,t)}{\partial x} + r(x,t)\,u(x,t) = f(x,t)\,,\ a \leq x \leq b,\ t \geq 0, \tag{1}$$

subject to initial conditions;

$$u(x,0) = \alpha(x)\,, \tag{2}$$

and the following nonlocal inhomogeneous boundary conditions of integral form.

$$\int_a^b \varphi_1(x)\,u(x,t)\,dx = \beta_1(x)\,,\ \int_a^b \varphi_2(x)\,u(x,t)\,dx = \beta_2(x)\,. \tag{3}$$

where $\varphi_i(x)$, $\beta_i(x)$, $i = 1, 2$, and $\alpha(x)$ are known continuous functions.

To transform nonlocal boundary conditions (1)–(3), into local boundary conditions, for a linear parabolic equation, let's proceed, by introducing a new function $v(x,t)$ as follows [5],

$$v(x,t) = \int_a^x \varphi(x)\, u(x,t)\, dx, \tag{4}$$

where

$\varphi(x) = \varphi_1(x) + \varphi_2(x)$.

Hence,

$$u(x,t) = \frac{1}{\varphi(x)} \frac{\partial v(x,t)}{\partial x}, \tag{5}$$

$$\frac{\partial u(x,t)}{\partial x} = \frac{1}{\varphi(x)} \frac{\partial^2 v(x,t)}{\partial x^2} + \left(\frac{1}{\varphi(x)}\right)' \frac{\partial v(x,t)}{\partial x}, \tag{6}$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{\varphi(x)} \frac{\partial^2 v(x,t)}{\partial t \partial x}, \tag{7}$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \left(\frac{1}{\varphi(x)}\right)'' \frac{\partial v(x,t)}{\partial x} + 2\left(\frac{1}{\varphi(x)}\right)' \frac{\partial^2 v(x,t)}{\partial x^2} + \frac{1}{\varphi(x)} \frac{\partial^3 v(x,t)}{\partial x^3}, \tag{8}$$

Substituting Eqs. (5)–(8) into Eq. (1), follows what we were looking for. This procedure states the following lemma.

Lemma 1 *Nonlocal initial-boundary value problem (1)–(3) can be always converted into a local initial-boundary value problem of the following form*

$$\begin{cases} \dfrac{\partial^2 v(x,t)}{\partial t \partial x} - h(x,t) \dfrac{\partial v(x,t)}{\partial x} + s(x,t) \dfrac{\partial^2 v(x,t)}{\partial x^2} - p(x,t) \dfrac{\partial^3 v(x,t)}{\partial x^3} = g(x,t), \\ \dfrac{\partial v(x,0)}{\partial x} = k_1(x), \\ v(a,t) = 0, \quad v(b,t) = \beta(t), \end{cases} \tag{9}$$

where

$$\begin{cases} h(x,t) = \left(-p(x,t)\left(\frac{1}{\varphi(x)}\right)'' + q(x,t)\left(\frac{1}{\varphi(x)}\right)'\right)\varphi(x) + r(x,t), \\ s(x,t) = -2p(x,t)\left(\frac{1}{\varphi(x)}\right)'\varphi(x) + q(x,t), \\ \beta(t) = \beta(t)_1 + \beta(t)_2, \\ k_1(x) = \varphi(x)\,\alpha(x), \\ g(x,t) = \varphi(x)\, f(x,t). \end{cases} \tag{10}$$

A solution of this problem will lead to a solution of the original problem, where $u(x,t)$ is given by (5). Once the function $u(x,t)$ is determined, we can return to the function $v(x,t)$ by Eq. (4).

3 Numerical Application

Example 1 Consider the following linear parabolic equation with nonlocal condition

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \dfrac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) = 0, \quad 0 \le x \le \pi,\ t \ge 0, \\ u(x,0) = \sin x, \\ \int_0^\pi xu(x,t)\,dx = \pi e^{-2t}, \\ \int_0^\pi (1-x)\,u(x,t)\,dx = (2-\pi)\,e^{-2t}. \end{cases} \tag{19}$$

Substituting these relations into (19) leads to the following local initial boundary value problem.

$$\begin{cases} \dfrac{\partial^2 v(x,t)}{\partial t \partial x} - \dfrac{\partial^3 v(x,t)}{\partial x^3} + \dfrac{\partial v(x,t)}{\partial x} = 0, \quad 0 \le x \le \pi,\ t \ge 0, \\ \dfrac{\partial v(x,0)}{\partial x} = \sin x, \\ v(0,t) = 0,\ v(\pi,t) = 2e^{-2t}. \end{cases} \tag{20}$$

where $\dfrac{\partial v(x,0)}{\partial x}$, $v(0,t)$, and $v(\pi,t)$ are determined according to the formulas stated in Lemma 1.

3.1 Solution by VIM

VIM well addressed in [3, 4, 16], is a known tool for solving functional equations. Let us apply this method for Example 1.

For simplicity let

$$w(x,t) = \frac{\partial v(x,t)}{\partial x}, \tag{25}$$

Then, Eq. (20) can be reduced to

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} - \dfrac{\partial^2 w(x,t)}{\partial x^2} + w(x,t) = 0, \quad 0 \le x \le \pi,\ t \ge 0, \\ w(x,0) = \sin x, \end{cases} \tag{26}$$

According to VIM, correction functional for Eq. (26) is constructed as follows.

$$w_{n+1} = w_n + \int_0^t \lambda(t,s)\left(\frac{\partial w_n}{\partial s} - \frac{\partial^2 \tilde{w}_n}{\partial x^2} + \tilde{w}_n\right) ds, \quad n = 0, 1, \ldots \tag{27}$$

where λ is a Lagrange multiplier, which can be identified optimally, via variational theory. w_n is the nth approximate solution, and $\tilde{w}_n$ denotes a restricted variation, that is, $\delta\tilde{w}_n = 0$. Considering stationary conditions, the Lagrange multiplier satisfies the following two conditions,

$$\begin{cases} \frac{\partial \lambda(t,s)}{\partial s} = 0, \\ 1 + \lambda(t,t) = 0, \end{cases}$$

Consequently, $\lambda(t,s) = -1$. So, Eq. (27) converts into the following equation.

$$w_{n+1} = w_n - \int_0^t \left(\frac{\partial w_n}{\partial s} - \frac{\partial^2 w_n}{\partial x^2} + w_n\right) ds, \quad n = 0, 1, \ldots \tag{28}$$

And it leads to the following results.

$w_0 = w(x,0) = \sin x,$
$w_1 = (1-2t)\sin x,$
$w_2 = \left(1 - 2t + 2t^2\right)\sin x,$
$w_3 = \left(1 - 2t + 2t^2 - \frac{4}{3}t^3\right)\sin x,$
$\vdots$

Hence the analytical solution to Eq. (26) can be obtained

$$w(x,t) = \lim_{n\to\infty} w_n(x,t) = \sin x \sum_{k=0}^{\infty} \frac{(-2t)^k}{k!} = (\sin x)\, e^{-2t},$$

Since $w(x,t) = \frac{\partial v(x,t)}{\partial x}$, $v(x,t)$ is derived as follows.

$v(x,t) = (-\cos x)\, e^{-2t} + c(t),$

Using both boundary conditions, $v(0,t)$ and $v(\pi,t)$, leads to $c(t) = e^{-2t}$ and so $v(x,t) = (1 - \cos x)\, e^{-2t}$, is an analytic solution of the Eq. (20). And finally, $u(x,t) = (\sin x)\, e^{-2t}$ will be a solution for Eq. (19).

For the second example the change of function (25), is applied for Eq. (21), this equation reduces to the following simple equation

$$\begin{cases} \frac{\partial^2 w(x,t)}{\partial t^2} - \frac{2}{(1+x)^2} w(x,t) - \frac{2}{1+x}\frac{\partial w(x,t)}{\partial x} - \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \quad 0 \le x \le 1,\ t \ge 0, \\ w(x,0) = x^2(x+1), \quad \frac{\partial w(x,0)}{\partial t} = 0, \end{cases} \tag{29}$$

Correction functional for Eq. (29) is constructed as follows.

$$w_{n+1} = w_n + \int_0^t \lambda(t,s)\left(\frac{\partial^2 w_n}{\partial s^2} - \frac{2}{(1+x)^2}\tilde{w}_n + \frac{2}{(1+x)}\frac{\partial \tilde{w}_n}{\partial x} - \frac{\partial^2 \tilde{w}_n}{\partial x^2}\right) ds, \quad n = 0, 1, \ldots \tag{30}$$

Considering the stationary conditions, the Lagrange multiplier satisfies the following system,

$$\begin{cases} \dfrac{\partial^2 \lambda(t,s)}{\partial s^2} = 0, \\ 1 - \left.\dfrac{\partial \lambda(t,s)}{\partial s}\right|_{s=t} = 0, \\ \lambda(t,t) = 0, \end{cases}$$

Which leads to $\lambda(t,s) = s - t$. So the iterative equation converts into the following equation.

$$w_{n+1} = w_n + \int_0^t \lambda(t,s) \left(\frac{\partial^2 w_n}{\partial s^2} - \frac{2}{(1+x)^2}\tilde{w}_n + \frac{2}{(1+x)}\frac{\partial w_n}{\partial x} - \frac{\partial^2 w_n}{\partial x^2} \right) ds, \quad n = 0, 1, \ldots \tag{31}$$

Using this iterative equation, leads to

$w_0 = x^3 + x^2$,
$w_1 = x^3 + x^2 + (1+x)t^2$,
$w_2 = x^3 + x^2 + (1+x)t^2$,
$w_3 = x^3 + x^2 + (1+x)t^2$,
$\vdots$

Thus an analytical solution to Eq. (29) is $w(x,t) = x^3 + x^2 + (1+x)t^2$. Hence $v(x,t) = \frac{x^4}{4} + \frac{x^3}{3} + \left(x + \frac{x^2}{2}\right)t^2 + c(t)$,

Using each boundary conditions, $v(0,t)$ and $v(1,t)$, leads to $c(0) = 0$, and finally an analytic solution to the Eq. (22), as follows.

$v(x,t) = \frac{x^4}{4} + \frac{x^3}{3} + \left(x + \frac{x^2}{2}\right)t^2$,

And $u(x,t) = x^2 + t^2$ is a solution to Eq. (21).

Now consider problem (23). Using (25) for it leads to

$$\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} - \frac{2}{x} w(x,t) + 2\dfrac{\partial w(x,t)}{\partial x} - x\dfrac{\partial^2 w(x,t)}{\partial x^2} = x - \frac{1}{x}(w(x,t))^2, \\ w(x,0) = x, \quad \dfrac{\partial w(x,0)}{\partial t} = 0, \quad 0 \le x \le 1, \ t \ge 0, \end{cases} \tag{32}$$

To solve by VIM, Correction functional for Eq. (31) is as follows.

$$w_{n+1} = w_n + \int_0^t \lambda(t,s) \left(\frac{\partial^2 w_n}{\partial s^2} - \frac{2}{x}\tilde{w}_n + 2\frac{\partial \tilde{w}_n}{\partial x} - x\frac{\partial^2 \tilde{w}_n}{\partial x^2} - x + \frac{1}{x}\tilde{w}_n^2 \right) ds, \quad n = 0, 1, \ldots \tag{33}$$

Similar to the last example, Lagrange multiplier is obtained as $\lambda(t,s) = s - t$. Substitution into Eq. (33), leads to the following iterative equation.

$$w_{n+1} = w_n + \int_0^t (s-t) \left(\frac{\partial^2 w_n}{\partial s^2} - \frac{2}{x}\tilde{w}_n + 2\frac{\partial w_n}{\partial x} - x\frac{\partial^2 w_n}{\partial x^2} - x + \frac{1}{x}w_n^2 \right) ds, \quad n = 0, 1, \ldots \tag{34}$$

So,

$w_0 = x$,
$w_1 = x$,

$w_2 = x,$

$\vdots$

Thus an analytical solution to Eq. (32) is $w(x,t) = x$. Hence

$v(x,t) = \frac{x^2}{2} + c(t),$

Using each boundary conditions, $v(0,t)$ and $v(1,t)$, leads to $c(t) = 0$ and finally an analytic solution to the Eq. (24), as follows.

$$v(x,t) = \frac{x^2}{2}$$

So, $u(x,t) = 1$ will be a solution to (23).

4 Discussion and Conclusion

In this paper, solving linear parabolic partial differential equations, subject to initial and nonlocal inhomogeneous boundary conditions of integral form is facilitated by introducing a change of function. Using this function nonlocal boundary conditions are transformed into local boundary ones, and these achieved equivalent equations can be solved by most of known approaches. Here, two well-known and analytical approaches; Variational Iteration is employed. Three illustrative examples are provided to verify the transformation and the aforementioned methods are utilized to solve them. Both of this method lead to the same exact solution which corroborates the computational efficacy of the transformation and the employed approaches.

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Kirchhoff-Type Boundary-Value Problems on the Real Line

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Abstract This paper deals with the existence and energy estimates of positive solutions for a class of Kirchhoff-type boundary-value problems on the real line, while the nonlinear part of the problem admits some hypotheses on the behavior at origin or perturbation property. In particular, for a precise localization of the parameter, applying a consequence of the local minimum theorem for differentiable functionals due to Bonanno the existence of a positive solution is established requiring the sublinearity of nonlinear part at origin and infinity. We also consider the existence of solutions for our problem under algebraic conditions with the classical Ambrosetti–Rabinowitz. In what follows, employing two consequences of the local minimum theorem for differentiable functionals due to Bonanno by combining two algebraic conditions on the nonlinear term which guarantees the existence of two positive solutions as well as applying the mountain pass theorem given by Pucci and Serrin, we establish the existence of the third positive solution for our problem. Moreover, concrete examples of applications are provided.

Keywords Boundary-value problems · Real line · Multiple solutions · Variational methods · Critical point theory

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1 Introduction

The goal of this paper is to study the existence and the qualitative properties of positive weak solutions for the Kirchhoff-type elliptic problem on the real line of the form

$$M(u) = \lambda\alpha(x)g(u(x)), \quad \text{for almost every } x \in R \tag{1}$$

where

$$M(u) := K\Big(\int_R (|u'(x)|^p + B|u(x)|^p)\mathrm{d}x\Big)\big(-(|u'(x)|^{p-2}u'(x))' + B|u(x)|^{p-2}u(x)\big)$$

in which $K : [0, +\infty) \to R$ is a nondecreasing continuous function such that there exist two positive constants κ_1 and κ_2 such that $\kappa_1 \le K(t) \le \kappa_2$ for all $t \ge 0$, λ is a real positive parameter, B is a real positive number, and $\alpha, g : R \to R$ are two functions such that $\alpha \in \mathrm{L}^1(R)$, $\alpha \not\equiv 0$ on any subset of positive measure in R and g is a non-negative continuous function.

The Kirchhoff equation refers back to Kirchhoff [23] in 1883 in the study on the oscillations of stretched strings and plates. It was suggested as an extended version of the classical D'Alembert's wave equation by taking into account the effects of the changes in the length of the string during the vibrations. Kirchhoff boundary value problems can be used for modeling several physical and biological systems where u describes a process which depend on the average of itself, such as the population density and dynamics [7]. Kirchhoff equation received great attention only after Lions [25] proposed an abstract framework for the problem. For some results on solvability of Kirchhoff type problems, we refer the reader to the papers [20, 21, 27].

Boundary value problems on infinite intervals arise in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena, such as the theory of drain flows and plasma physics, in the study of unsteady flow of a gas through a semi-infinite porous medium, discussion of electrostatic probe measurements in solid-propellant rocket exhausts, analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, investigation of the temperature distribution in the problem of phase change of solids with temperature dependent thermal conductivity, as well as numerous problems arising in the study of draining flows, circular membranes, nonlinear mechanics, and non-Newtonian fluid flows, see [1–5, 22, 28] and the references therein.

In recent years, boundary value problems in an infinite interval have been studied extensively and many results for the existence of solutions, positive solutions, multiple solutions have been obtained [6, 9, 10, 12, 14–16, 24, 26] and the references therein.

In this paper, we are interested in the existence results and energy estimates of solutions for problem (1). The main result of this paper ensures the existence of exact values of the parameter λ for which problem (1) admits at least one/two/three positive weak solutions. Several special cases of the main results and illustrating examples are also given. We also refer the reader to [13, 17–19] for some related results in this subject.

2 Preliminaries

In this section, we state some preliminary results. Let $(X, |\cdot|)$ be a real Banach space, X^* be the dual space of X and $\langle\cdot,\cdot\rangle$ be the duality pairing between X^* and X.

We denote by $|\cdot|$ and by $|\cdot|_t$ the usual norms on R and on $\mathrm{L}^t(R)$, for all $t \in [1, +\infty]$, while $W^{1,p}(R)$ indicates the closure of $\mathrm{C}_0^\infty(R)$ with respect to the norm $\|u\|_{1,p} = (|u'|_p^p + |u|_p^p)^{\frac{1}{p}}$. When $p = 2$ the norm is induced by the scalar product $(u, v) = (u', v')_{\mathrm{L}^2} + (u, v)_{\mathrm{L}^2}$. It is well known that $W^{1,p}(R) \equiv W_0^{1,p}(R)$ and $W^{1,p}(R)$ is embedded in $\mathrm{L}^t(R)$ for any $t \in [p, +\infty]$.

Remark 1 If $\{u_n\}_{n\in N}$ is a bounded sequence in $W^{1,p}(R)$, then it has a subsequence that point-wise converges to some $u \in W^{1,p}(R)$ and also weakly converges in $\mathrm{L}^\infty(R)$. Indeed, it can be inferred from the compact embedding $W^{1,p}(R) \hookrightarrow \mathrm{C}([-R, R])$, $R > 0$, and the continuity of $W^{1,p}(R) \to \mathrm{L}^\infty(R)$.

We consider $W^{1,p}(R)$ endowed by the norm $\|u\| = \left(\int_R (|u'(x)|^p + B|u(x)|^p)\mathrm{d}x\right)^{\frac{1}{p}}$, which is equivalent to the usual norm $\|.\|_1$, that is, when $B = 1$.

The following proposition corresponds to [14, Proposition 2.2].

Proposition 1 *One has* $|u|_\infty \leq C_B\|u\|$ *for all* $u \in W^{1,p}(R)$ *where* C_B *is a constant given by* $C_B := 2^{\frac{p-2}{p}}(\frac{p-1}{p})^{\frac{p-1}{p}}(\frac{1}{B})^{\frac{p-1}{p^2}}$.

Definition 1 We say that a function $u \in W^{1,p}(R)$ is a *weak solution* of problem (1) if for all $v \in W^{1,p}(R)$, $K(\|u\|^p)\left(\int_R (|u'(x)|^{p-2}u'(x)v'(x) + B|u(x)|^{p-2}u(x)v(x))\mathrm{d}x\right) - \lambda\int_R \alpha(x)g(u(x))v(x)\mathrm{d}x = 0$. Moreover, when α is, in addition, a continuous function on R, the (weak) solutions of (1) are actually classical, as standard computations show.

Put $G(t) = \int_0^t g(\xi)\mathrm{d}\xi$ for all $t \in R$ and $\widehat{K}(t) = \int_0^t K(\xi)\mathrm{d}\xi$ for all $t \geq 0$. Our hypotheses on g guarantee that $G \in \mathrm{C}^1(R)$ and $G'(t) = g(t) \geq 0$ for all $t \in R$, so G is non-decreasing.

Set

$$\rho(r) = \sup_{v\in\Phi^{-1}(r,+\infty)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{\Phi(v) - r}, \tag{2}$$

$$\beta(r_1, r_2) := \inf_{v\in\Phi^{-1}(r_1,r_2)} \frac{\sup_{u\in\Phi^{-1}(r_1,r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \tag{3}$$

and

$$\rho_2(r_1, r_2) := \sup_{v\in\Phi^{-1}(r_1,r_2)} \frac{\Psi(v) - \sup\limits_{u\in\Phi^{-1}(-\infty,r_1]} \Psi(u)}{\Phi(v) - r_1}. \tag{4}$$

In the proof of our main results, we will apply the following theorems.

Theorem 1 [11, Theorem 5.1] *Let X be a real Banach space and let $\Phi, \Psi : X \to R$ be two continuously Gâteaux differentiable functions. Assume that there exist r_1, $r_2 \in R$ with $r_1 < r_2$, such that $\beta(r_1, r_2) < \rho_2(r_1, r_2)$, where β and ρ_2 are given by (3) and (4), and for each $\lambda \in \left(\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right)$, the function $J_\lambda := \Phi - \lambda\Psi$ satisfies ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition (see [30]). Then for all $\lambda \in \left(\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right)$ there exists $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $J_\lambda(u_{0,\lambda}) \le J_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $J'_\lambda(u_{0,\lambda}) = 0$.*

Theorem 2 [11, Corollary 5.1] *Let X be a real Banach space and let $\Phi, \Psi : X \to R$ be two continuously Gâteaux differentiable functionals. Put*

$$\beta^* := \liminf_{r\to+\infty} \frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{r}$$

and assume that there is $\bar r \in R$ such that $\rho(\bar r) > \beta^*$ where ρ is given by (2). Moreover, assume that for each $\lambda \in (\frac{1}{\rho(\bar r)}, \frac{1}{\beta^*})$ the function $J_\lambda := \Phi - \lambda\Psi$ satisfies ${}^{[\bar r]}(\mathrm{PS})^{[r]}$-condition for all $r > \bar r$. Then there is $r_2 > \bar r$ such that for each $\lambda \in (\frac{1}{\rho(\bar r)}, \frac{1}{\beta^*})$, there is $u_{0,\lambda} \in \Phi^{-1}(\bar r, r_2)$ such that $J_\lambda(u_{0,\lambda}) \le J_\lambda(u)$ for all $u \in \Phi^{-1}(\bar r, r_2)$ and $J'_\lambda(u_{0,\lambda}) = 0$.

Proposition 2 *Let $J : W^{1,p}(R) \to W^{1,p}(R)^*$ be the operator defined by $J(u)(v) = K(\|u\|^p)\left(\int_R (|u'(x)|^{p-2}u'(x)v'(x) + B|u(x)|^{p-2}u(x)v(x))\mathrm{d}x\right)$ for every $u, v \in W^{1,p}(R)$. Then, J admits a continuous inverse on $W^{1,p}(R)^*$.*

Proof We have $J(u)(u) \ge \kappa_1\|u\|^p$, which means that J is coercive. Owing to our assumptions on the data, one has $\langle J(u) - J(v), u - v\rangle \ge C\|u - v\|^2 > 0$ for some $C > 0$ for every $u, v \in W^{1,p}(R)$, which means that J is strictly monotone. Moreover, since $W^{1,p}(R)$ is reflexive, for $u_n \to u$ strongly in $W^{1,p}(R)$ as $n \to +\infty$, one has $J(u_n) \to J(u)$ weakly in $W^{1,p}(R)^*$ as $n \to \infty$. Hence, J is demicontinuous, so by [31, Theorem 26.A(d)], the inverse operator J^{-1} of J exists and it is continuous. Indeed, let ρ_n be a sequence of $W^{1,p}(R)^*$ such that $\rho_n \to \rho$ strongly in $W^{1,p}(R)^*$ as $n \to \infty$. Let u_n and u in $W^{1,p}(R)$ such that $J^{-1}(\rho_n) = u_n$ and $J^{-1}(\rho) = u$. Since J is coercive, the sequence u_n is bounded in the reflexive space $W^{1,p}(R)$. For a suitable subsequence, we have $u_n \to \hat u$ weakly in $W^{1,p}(R)$ as $n \to \infty$, which concludes $\langle J(u_n) - J(u), u_n - \hat u\rangle = \langle \rho_n - \rho, u_n - \hat u\rangle = 0$. Note that if $u_n \to \hat u$ weakly in $W^{1,p}(R)$ as $n \to +\infty$ and $J(u_n) \to J(\hat u)$ strongly in $W^{1,p}(R)^*$ as $n \to +\infty$, one has $u_n \to \hat u$ strongly in $W^{1,p}(R)$ as $n \to +\infty$, and since J is continuous, we have $u_n \to \hat u$ weakly in $W^{1,p}(R)$ as $n \to +\infty$ and $J(u_n) \to J(\hat u) = J(u)$ strongly in $W^{1,p}(R)^*$ as $n \to +\infty$. Hence, taking into account that J is an injection, we have $u = \hat u$.

3 Main Results

In this section, we formulate our main results. Put $\alpha_0 := \int_{-1}^{1} \alpha(x)\mathrm{d}x$ and $\ell := C_B\left(2^{2p-1} + \frac{B}{2(p+1)} + 2B\right)^{\frac{1}{p}}$. Moreover, for every two nonnegative constants γ and σ with $\gamma \sqrt[p]{\kappa_1} \neq \sigma\ell \sqrt[p]{\kappa_2}$, set $b_\gamma(\sigma) = \frac{|\alpha|_1 G(\gamma) - \alpha_0 G(\sigma)}{\kappa_1\gamma^p - \kappa_2\sigma^p\ell^p}$. We denote by $\mathscr{G}$ the class of all continuous functions $g : R \to R$ satisfy in the following condition:

- there exist two non-negative constants a_1, a_2 such that

$$|g(t)| \leq a_1 + a_2|t|^{p-1} \quad \textit{for all } t \in R. \tag{5}$$

Theorem 3 *Assume that* $g \in \mathscr{G}$ *and there exist three real constants* γ_1, γ_2 *and* σ, *with* $0 < \gamma_1 < \sigma\ell < \sqrt[p]{\frac{\kappa_1}{\kappa_2}}\gamma_2$, *such that* $b_{\gamma_2}(\sigma) < b_{\gamma_1}(\sigma)$. *Then for each parameter* $\lambda \in \left(\frac{1}{pC_B^p b_{\gamma_1}(\sigma)}, \frac{1}{pC_B^p b_{\gamma_2}(\sigma)}\right)$, *problem (1) possesses at least one positive weak solution* $u_{0,\lambda} \in W^{1,p}(R)$, *such that* $\sqrt[p]{\frac{\kappa_1}{\kappa_2}}\frac{\gamma_1}{C_B} < \|u_{0,\lambda}\| < \frac{\gamma_2}{C_B}$.

Proof We will apply Theorem 1. Let $X := W^{1,p}(R)$ and consider the functionals $\Phi, \Psi : X \to R$ defined by $\Phi(u) := \frac{1}{p}\widehat{K}(\|u\|^p)$ and $\Psi(u) := \int_R \alpha(x)G(u(x))\mathrm{d}x$. Thus the functional $\Phi : X \to R$ is coercive. On the other hand, Φ and Ψ are continuously Gâteaux differentiable. More precisely, $\Psi'(u)(v) = \int_R \alpha(x)g(u(x))v(x)\mathrm{d}x$ and $\Phi'(u)(v) = K(\|u\|^p)\left(\int_R(|u'(x)|^{p-2}u'(x)v'(x) + B|u(x)|^{p-2}u(x)v(x))\mathrm{d}x\right)$ for every $u, v \in X$. Fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda\Psi$ is a function $u \in X$ such that $\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$ for every $v \in X$. Hence, the critical points of the functional J_λ are weak solutions of problem (1). At this point, let us observe that $\Phi(0_X) = \Psi(0_X) = 0$. Moreover, by choosing $r_1 = \frac{\kappa_1}{pC_B^p}{\gamma_1}^p$ and $r_2 = \frac{\kappa_1}{pC_B^p}{\gamma_2}^p$, from the definition of Φ and taking Proposition 1 into account, one has $\Phi^{-1}(-\infty, r_1) = \left\{u \in X;\ \frac{\widehat{K}(\|u\|^p)}{p} < r_1\right\} \subseteq \{u \in X;\ |u| \leq \gamma_1\}$ and $\Phi^{-1}(-\infty, r_2) = \left\{u \in X;\ \frac{\widehat{K}(\|u\|^p)}{p} < r_2\right\} \subseteq \left\{u \in X;\ |u| \leq \gamma_2\right\}$. Hence,

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u) \leq \int_R \alpha(x) \sup_{|\xi|\leq\gamma_1} G(\xi)\mathrm{d}x = |\alpha|_1 G(\gamma_1) \tag{6}$$

and

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u) \leq \int_R \alpha(x) \sup_{|\xi|\leq\gamma_2} G(\xi)\mathrm{d}x = |\alpha|_1 G(\gamma_2). \tag{7}$$

Now we define w_σ by

$$w_\sigma(x) = \begin{cases} 4\sigma(x+1)+\sigma, & x \in [-\frac{5}{4}, -1), \\ \sigma, & x \in [-1, 1], \\ 4\sigma(1-x)+\sigma, & x \in (1, \frac{5}{4}], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $w_\sigma \in X$. Simple computations show that $\|w_\sigma\|^p = \frac{(4\sigma)^p}{2} + B(\frac{1}{2(p+1)} + 2)$ $\sigma^p = \sigma^p(2^{2p-1} + \frac{B}{2(p+1)} + 2B) = \frac{\sigma^p \ell^p}{C_B^p}$. Thus, $\frac{\kappa_1 \sigma^p \ell^p}{pC_B^p} \leq \Phi(w_\sigma) \leq \frac{\kappa_2 \sigma^p \ell^p}{pC_B^p}$ and $\Psi(w_\sigma) = \int_{-\frac{5}{4}}^{\frac{5}{4}} \alpha(x)G(w_\sigma(x))\mathrm{d}x \geq \int_{-1}^{1} \alpha(x)G(w_\sigma(x))\mathrm{d}x = \alpha_0 G(\sigma)$. Taking $0 < \gamma_1 < \sigma\ell < \sqrt[p]{\frac{\kappa_1}{\kappa_2}}\gamma_2$ into account, by a direct computation, one has $r_1 < \Phi(w_\sigma) < r_2$. On the other hand, $\beta(r_1, r_2) \leq \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u) - \Psi(w_\sigma)}{r_2 - \Phi(w_\sigma)} \leq pC_B^p \frac{|\alpha|_1 G(\gamma_2) - \alpha_0 G(\sigma)}{\kappa_1 \gamma_2^p - \kappa_2 \sigma^p \ell^p}$ and $\rho_2(r_1, r_2) \geq \frac{\Psi(w_\sigma) - \sup_{u \in \Phi^{-1}(-\infty, r_1]} \Psi(u)}{\Phi(w_\sigma) - r_1} \geq pC_B^p \frac{|\alpha|_1 G(\gamma_1) - \alpha_0 G(\sigma)}{\kappa_1 \gamma_1^p - \kappa_2 \sigma^p \ell^p}$. Hence, by using the notation (3), from (6) and (7) together with $\frac{\kappa_1 \sigma^p \ell^p}{pC_B^p} \leq \Phi(w_\sigma) \leq \frac{\kappa_2 \sigma^p \ell^p}{pC_B^p}$, it follows that $\beta(r_1, r_2) \leq pC_B^p b_{\gamma_2}(\sigma)$ and $\rho_2(r_1, r_2) \geq pC_B^p b_{\gamma_1}(\sigma)$. The assumption $b_{\gamma_2}(\sigma) < b_{\gamma_1}(\sigma)$ yields $\beta(r_1, r_2) < \rho_2(r_1, r_2)$. Now, from above the functional Φ is continuously Gâteaux differentiable while by Proposition 2 admits a continuous inverse on X^*, the functional Φ is continuously Gâteaux differentiable whose Gâteaux derivative is compact and since $g \in \mathcal{G}$ the functional $\Phi - \Psi$ is coercive. Thus, from [11, Proposition 1], the functional J_λ satisfies the $^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition for all r_1 and r_2 with $r_1 < r_2 < +\infty$. Therefore, by Theorem 1, since g is nonnegative, for each $\lambda \in \left(\frac{1}{pC_B^p b_{\gamma_1}(\sigma)}, \frac{1}{pC_B^p b_{\gamma_2}(\sigma)}\right)$, the functional J_λ possesses at least one positive critical point $u_{0,\lambda}$ such that $r_1 < \Phi(u_{0,\lambda}) < r_2$, that is $\sqrt[p]{\frac{\kappa_1}{\kappa_2}} \frac{\gamma_1}{C_B} < \|u_{0,\lambda}\| < \frac{\gamma_2}{C_B}$. This completes the proof.

Remark 2 The result of Theorem 3 holds true if condition (5) is replaced by

- $\lim_{|t| \to \infty} \frac{g(t)}{|t|^{p-1}} = 0$, i.e., g is $p-1$-sublinear at infinity.

Now, we point out a particular case of Theorem 3.

Theorem 4 *Assume that $g \in \mathcal{G}$ and there exist two positive constants γ and σ with $0 < \sigma\ell < \sqrt[p]{\frac{\kappa_1}{\kappa_2}}\gamma$, such that $\frac{G(\gamma)}{\gamma^p} < \frac{\alpha_0 \kappa_1}{|\alpha|_1 \kappa_2 \ell^p} \frac{G(\sigma)}{\sigma^p}$. Then for each parameter*

$$\lambda \in \left(\frac{\kappa_2}{pC_B^p} \frac{\sigma^p \ell^p}{\alpha_0 G(\sigma)}, \frac{\kappa_1}{pC_B^p} \frac{\gamma^p}{|\alpha|_1 G(\gamma)}\right),$$

problem (1) possesses at least one positive weak solution $u_{0,\lambda} \in W^{1,p}(R)$ such that $\|u_{0,\lambda}\| < \frac{\gamma}{C_B}$.

Proof Taking $\gamma_1 = 0$ and $\gamma_2 = \gamma$ and bearing (3) in mind, we obtain $b_\gamma(\sigma) = \frac{|\alpha|_1 G(\gamma) - \alpha_0 G(\sigma)}{\kappa_1 \gamma^p - \kappa_2 \sigma^p \ell^p} \leq \frac{\alpha_0 G(\sigma)}{\kappa_2 \sigma^p \ell^p} = b_0(\sigma)$. Hence, Theorem 3 ensures the conclusion.

Now, we give an application of Theorem 2 which will be used later to obtain multiple solutions for the problem (1).

Theorem 5 *Assume that $g \in \mathscr{G}$ and there exist two constants $\bar{\gamma}$ and $\bar{\sigma}$ with $0 < \bar{\gamma} < \bar{\sigma}\ell$ such that $|\alpha|_1 G(\bar{\gamma}) < \alpha_0 G(\bar{\sigma})$. Then for each $\lambda > \tilde{\lambda}$, where*

$$\tilde{\lambda} := \frac{1}{pC_B^p} \frac{\kappa_1 \bar{\gamma}^p - \kappa_2 \bar{\sigma}^p \ell^p}{|\alpha|_1 G(\bar{\gamma}) - \alpha_0 G(\bar{\sigma})},$$

problem (1) possesses at least one positive weak solution $\bar{u}_{0,\lambda} \in W^{1,p}(R)$ such that $\|\bar{u}_{0,\lambda}\| > \sqrt[p]{\frac{\kappa_1}{\kappa_2}} \frac{\bar{\gamma}}{C_B}$.

Proof Take $X = W^{1,p}(R)$ and put $I_\lambda = \Phi - \lambda \Psi$, where Φ and Ψ are given as in the proof of Theorem 3. The functionals Φ and Ψ satisfy all assumptions requested in Theorem 2. Put $\bar{r} := \frac{\kappa_1}{pC_B^p} \bar{\gamma}^p$. From [11, Proposition 1], the functional J_λ satisfies ${}^{[\bar{r}]}(\mathrm{PS})^{[r]}$-condition for all r with $r > \bar{r}$. Arguing as in the proof of Theorem 3, we obtain that $\rho(\bar{r}) \geq \frac{\Psi(w_\sigma) - \sup_{u \in \Phi^{-1}(-\infty, \bar{r}]} \Psi(u)}{\Phi(w_\sigma) - \bar{r}} \geq pC_B^p \frac{|\alpha|_1 G(\bar{\gamma}) - \alpha_0 G(\bar{\sigma})}{\kappa_1 \bar{\gamma}^p - \kappa_2 \bar{\sigma}^p \ell^p}$. Hence, from our assumption it follows that $\rho(\bar{r}) > 0$. Therefore, It follows from Theorem 2 with $\beta^* = 0$, for each $\lambda > \tilde{\lambda}$, the functional J_λ admits at least one positive local minimum $\bar{u}_{0,\lambda} \in W^{1,p}(R)$ such that $\Phi(\bar{u}_{0,\lambda}) > \bar{r}$, which is just $\|\bar{u}_{0,\lambda}\| > \sqrt[p]{\frac{\kappa_1}{\kappa_2}} \frac{\bar{\gamma}}{C_B}$. Thus the conclusion is obtained.

The following result is a straight consequence of Theorem 4.

Theorem 6 *Assume that $g \in \mathscr{G}$ and*

$$\lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{p-1}} = +\infty. \tag{8}$$

Furthermore, let $\gamma > 0$ and set $\lambda_\gamma^\star := \frac{\kappa_1}{pC_B^p} \frac{\gamma^p}{|\alpha|_1 G(\gamma)}$. Then for every $\lambda \in (0, \lambda_\gamma^\star)$, problem (1) admits at least one positive weak solution $u_{0,\lambda} \in W^{1,p}(R)$ such that $\|u_{0,\lambda}\|_{a,p} < \frac{\gamma}{C_B}$.

Proof Fix $\lambda \in (0, \lambda_\gamma^\star)$. From (8) there exists a constant $\sigma > 0$ with $\sigma\ell < \sqrt[p]{\frac{\kappa_1}{\kappa_2}} \gamma$ such that $\frac{\kappa_2}{pC_B^p} \frac{\sigma^p \ell^p}{\alpha_0 G(\sigma)} < \lambda < \frac{\kappa_1}{pC_B^p} \frac{\gamma^p}{|\alpha|_1 G(\gamma)}$. Hence, by Theorem 4, problem (1) possesses at least one positive weak solution $u_{0,\lambda}$ such that $\|u_{0,\lambda}\| < \frac{\gamma}{C_B}$.

Example 1 Consider the problem

$$\begin{aligned} &\Big[1 + \tanh\Big(\int_R (|u'(x)|^4 + |u(x)|^4) \mathrm{d}x\Big)\Big]\big(-(|u'(x)|^2 u'(x))' + |u(x)|^2 u(x)\big) \\ &= \lambda \tfrac{g(u(x))}{1+x^2}, \qquad \text{for almost every } x \in R \end{aligned} \tag{9}$$

where

$$g(t)=\begin{cases} e^t, & t\in(-\infty,-1],\\ e^{\sin(\frac{\pi}{2}t)}, & t\in(-1,1),\\ e^{-\cos(\pi t)}, & t\in[1,\infty). \end{cases}$$

Direct calculations shows that $\kappa_1=1$, $\kappa_2=2$, $C_1=\frac{\sqrt[4]{27}}{2}$, $\alpha_0=\frac{\pi}{2}$, $|\alpha|_1=\pi$, $\lim_{\xi\to 0^+}\frac{g(\xi)}{\xi^{p-1}}=\lim_{\xi\to 0^+}\frac{e^{\sin(\frac{\pi}{2}\xi)}}{\xi^3}=+\infty$ and $\lim_{|\xi|\to+\infty}\frac{g(\xi)}{|\xi|^{p-1}}=\lim_{|\xi|\to+\infty}\frac{e^{-\cos(\pi\xi)}}{|\xi|^3}=0$. Now by choosing $\gamma=1$ we clearly see that all assumptions of Theorem 6 are satisfied. Hence, applying Theorem 6 and Remark 2 for every $\lambda\in\left(0,\frac{4}{27\pi e}\right)$, then problem (9) possesses at least one positive weak solution $u_{0,\lambda}\in W^{1,4}(R)$ such that $\|u_{0,\lambda}\|<\frac{2}{\sqrt[4]{27}}$.

Theorem 7 *Suppose that $g\in\mathscr{G}$. Then the mapping $\lambda\mapsto J_\lambda(u_{0,\lambda})$ is negative and strictly decreasing in $(0,\lambda^\star_\gamma)$.*

Proof The restriction of the functional J_λ to $\Phi^{-1}(0,r_2)$ where $r_2=\frac{\kappa_1}{pC_B^p}\gamma_2{}^p$, admits a global minimum, which is a critical point (local minimum) of J_λ in $W^{1,p}(R)$. Moreover, in view of $w_\sigma\in\Phi^{-1}(0,r_2)$ and $\frac{\Phi(w_\sigma)}{\Psi(w_\sigma)}\le\frac{\kappa_2\sigma^p\ell^p}{pC_B^p\alpha_0G(\sigma)}<\lambda$, we have $J_\lambda(u_{0,\lambda})\le J_\lambda(w_\sigma)<0$. Next, we see that $J_\lambda(u)=\lambda\left(\frac{\Phi(u)}{\lambda}-\Psi(u)\right)$ for every $u\in W^{1,p}(R)$ and fix $0<\lambda_1<\lambda_2<\lambda^\star_\gamma$. Put $m_{\lambda_1}:=\left(\frac{\Phi(u_{0,\lambda_1})}{\lambda_1}-\Psi(u_{0,\lambda_1})\right)=\inf_{u\in\Phi^{-1}(0,r_2)}\left(\frac{\Phi(u)}{\lambda_1}-\Psi(u)\right)$, $m_{\lambda_2}:=\left(\frac{\Phi(u_{0,\lambda_2})}{\lambda_2}-\Psi(u_{0\lambda_2})\right)=\inf_{u\in\Phi^{-1}(0,r_2)}\left(\frac{\Phi(u)}{\lambda_2}-\Psi(u)\right)$. Clearly, $m_{\lambda_i}<0$ (for $i=1,2$), and $m_{\lambda_2}\le m_{\lambda_1}$ thanks to $\lambda_1<\lambda_2$. Then the mapping $\lambda\mapsto J_\lambda(u_{0,\lambda})$ is strictly decreasing in $(0,\lambda^\star_\gamma)$ owing to $J_{\lambda_2}(u_{0,\lambda_2})=\lambda_2m_{\lambda_2}\le\lambda_2m_{\lambda_1}<\lambda_1m_{\lambda_1}=J_{\lambda_1}(u_{0,\lambda_1})$.

Remark 3 Generally, Theorem 6 ensures that if $g\in\mathscr{G}$ satisfies (8), then for every parameter λ belonging to $\Lambda_\Omega:=(0,\lambda^\star)$, where $\lambda^\star:=\frac{1}{pC_B^p}\sup_{\gamma>0}\frac{\kappa_1\gamma^p}{|\alpha|_1G(\gamma)}$, problem (1) possesses at least one positive weak solution $u_{0,\lambda}\in W^{1,p}(R)$.

Remark 4 We note that, in particular, if g is $(p-1)$-sublinear at infinity, Theorem 6 ensures that problem (1) admits at least one positive weak solution for every positive parameter λ. Moreover, in our case, the obtained solution is non-zero, while the classical direct method approach, that can be accept in this context, ensures the existence of at least one solution that may be zero.

Remark 5 A careful analysis of the proof of Theorem 6 ensures that the result still remains true if condition (8) is replaced by the more general assumption $\limsup_{\xi\to 0^+}\frac{G(\xi)}{\xi^p}=+\infty$. Moreover, the previous asymptotic condition at zero can be replaced by the following form

$$\limsup_{\xi\to 0^+}\frac{g(\xi)}{\xi^{p-1}}=+\infty. \tag{10}$$

Therefore, it is natural to obtain the following result.

Theorem 8 *Let* $\lim_{t\to 0^+}\frac{g(t)}{t^{p-1}} = +\infty$ *and* $\lim_{t\to+\infty}\frac{g(t)}{t^{p-1}} = 0$. *Then there exists* $\lambda^\star > 0$ *such that for every* $\lambda \in (0, \lambda^\star)$, *problem (1) possesses at least one positive weak solution* $u_{0,\lambda} \in W^{1,p}(R)$. *Moreover, we have* $\left(\int_R (|u'(x)|^p + B|u(x)|^p)\mathrm{d}x\right)^{\frac{1}{p}} \to 0$ *as* $\lambda \to 0^+$ *and the mapping* $\lambda \mapsto \frac{1}{p}\widehat{K}\left(\int_R (|u'(x)|^p + B|u(x)|^p)\mathrm{d}x\right) - \int_R \alpha(x) \left(\int_0^{u_{0,\lambda}} g(t)\mathrm{d}t\right)\mathrm{d}x$ *is negative and strictly decreasing in* $(0, \lambda^\star)$.

Below, we show how the former analysis can be used to pass from the existence of at least one positive solution to that of at least two nontrivial solutions. This objective will emerge by using the specific nature of the initially found solution, namely a local minimum. The information is then useful in guaranteeing the existence of a second solution as a critical point of mountain pass type. Accordingly, we start with the following theorem, where the celebrated Ambrosetti–Rabinowitz condition is necessary.

Theorem 9 *Let* g *be a nonnegative continuous function such that* $g(0) \neq 0$ *and the assumption (10) holds. Furthermore, assume that*

(AR) *there are constants* $\nu > p$ *and* $\rho > 0$ *such that, for all* $\xi \geq \rho$, *one has*

$$0 < \nu G(\xi) \leq \xi g(\xi). \tag{11}$$

Then for each $\lambda \in \Lambda_\Omega$, *problem (1) admits at least two positive weak solutions in the space* $W^{1,p}(R)$.

Proof Fix $\lambda \in \Lambda_\Omega$. Owing to the assumption (10), Theorem 6 ensures that problem (1) admits at least one weak positive solution u_1 in $W^{1,p}(R)$ which is a local minimum of the functional J_λ as defined in the proof of Theorem 3. Now, we prove the existence of the second local minimum distinct from the first one. To this goal, we verify the hypotheses of the mountain-pass theorem for the functional J_λ. Clearly, the functional J_λ is of class C^1 and $J_\lambda(0) = 0$. The first part of proof guarantees that $u_1 \in W^{1,p}(R)$ is a positive local minimum for J_λ in $W^{1,p}(R)$. We can assume that u_1 is a strict local minimum for J_λ in $W^{1,p}(R)$. Therefore, there is $\rho > 0$ such that $\inf_{\|u-u_1\|=\rho} I_\lambda(u) > I_\lambda(u_1)$, so condition [30, (I_1), Theorem 2.2] is verified. By integrating the condition (11), there exist constants $a_1, a_2 > 0$ such that $F(u) \geq a_1|u|^\nu - a_2$ for all $u \in W^{1,p}(R)$. Now, choosing any $u \in W^{1,p}(R)$, one has

$$\begin{aligned} J_\lambda(\tau u) = (\Phi - \lambda\Psi)(\tau u) &\leq \frac{1}{p}\widehat{K}(\|\tau u\|^p) - \lambda \int_R \alpha(x) G(\tau u(x))\mathrm{d}x \\ &\leq \frac{\kappa_2 \tau^p}{p}\|u\|^p - \lambda\tau^\nu a_1 \int_R \alpha(x)|u(x)|^\nu \mathrm{d}x + \lambda a_2|\alpha|_1 \to -\infty, \qquad \tau \to +\infty. \end{aligned}$$

Thus condition [30, (I_2), Theorem 2.2] is satisfied. Therefore the functional J_λ satisfies the geometry of mountain pass. Moreover, J_λ satisfies the (PS)-condition. Indeed, assume that $\{u_n\}_{n\in N} \subset X$ such that $\{J_\lambda(u_n)\}_{n\in N}$ is bounded and $J'_\lambda(u_n) \to 0$

as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|J_\lambda(u_n)| \le c_0$ and $|J'_\lambda(u_n)| \le c_0$ for all $n \in N$. Therefore, we infer to deduce from the definition of J'_λ and the assumption (AR) that

$$c_0 + c_1\|u_n\| \ge \nu J_\lambda(u_n) - J'_\lambda(u_n)(u_n) \ge \kappa_1\Big(\frac{\nu}{p} - 1\Big)\|u_n\|^p$$

$$- \lambda \int_\Omega \alpha(x)\,(\nu G(u_n(x)) - g(u_n(x))(u_n(x)))\,\mathrm{d}x \ge \kappa_1\Big(\frac{\nu}{p} - 1\Big)\|u_n\|^p,$$

for some $c_1 > 0$. Since $\nu > p$, this implies that (u_n) is bounded. Now, by simple computation we can prove $\{u_n\}$ converges strongly to u in $W^{1,p}(R)$. Consequently, J_λ satisfies (PS)-condition. Thus, by the classical theorem of Ambrosetti and Rabinowitz [8] we establish a nonnegative critical point u_2 of J_λ such that $J_\lambda(u_2) > J_\lambda(u_1)$. Since $g(0) \neq 0$, u_2 is positive. Hence, u_1 and u_2 are two distinct positive weak solutions of (1) and the proof is completed.

Remark 6 The non-triviality of the second weak solution ensured by Theorem 9 can be achieved also in the case $g(0) = 0$ requiring the extra conditions at zero

$$\limsup_{\xi\to 0^+} \frac{G(\xi)}{|\xi|^p} = +\infty \quad \text{and} \quad \liminf_{\xi\to 0^+} \frac{G(\xi)}{|\xi|^p} > -\infty. \tag{12}$$

Indeed, let $0 < \bar\lambda < \lambda^\star$ where $\lambda^\star = \frac{1}{pC_B^p}\sup_{\gamma>0}\frac{\kappa_1\gamma^p}{|\alpha|_1 G(\gamma)}$ and σ is a positive number with $\sigma\ell < \sqrt[p]{\frac{\kappa_1}{\kappa_2}}\gamma$. Then there exists $\bar\gamma > 0$ such that $\frac{\bar\lambda}{pC_B^p} < \frac{\kappa_1\bar\gamma^p}{|\alpha|_1 G(\bar\gamma)}$. Let Φ and Ψ be as given in the proof of Theorem 3. Due to Theorem 9, for every $\lambda \in (0, \bar\lambda)$ there exists a critical point of $J_\lambda = \Phi - \lambda\Psi$ such that $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$ where $r_\lambda := \frac{\kappa_1}{pC_B^p}\bar\gamma^p$. In particular, u_λ is a global minimum of the restriction of J_λ to $\Phi^{-1}(-\infty, r_\lambda)$. We will prove that the function u_λ cannot be trivial. Let us show that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{13}$$

Owing to the assumption (12), we can consider a sequence $\{\xi_n\} \subset R^+$ converging to zero and two constants ι, κ (with $\iota > 0$) such that $\lim_{n\to+\infty}\frac{G(\xi_n)}{|\xi_n|^p} = +\infty$ and $G(\xi) \ge \kappa|\xi|^{\alpha p}$ for every $\xi \in [0, \iota]$. We consider a set $\mathscr{F} \subset B$ of positive measure and a function $v \in W^{1,p}(R)$ such that

- $(k1)$ $v(x) \in [0, 1]$ for every $x \in R$;
- $(k2)$ $v(x) = 1$ for every $x \in \mathscr{F}$.

Hence, fix $N > 0$ and consider a real positive number η with

$$N < \frac{p\eta\int_{\mathscr{F}}\alpha(x)\mathrm{d}x + p\kappa\int_{R\setminus\mathscr{F}}|v(x)|^p\mathrm{d}x}{\kappa_2\|v\|^p}.$$

Then there is $n_0 \in N$ such that $\xi_n < \iota$ and $G(\xi_n) \geq \eta|\xi_n|^p$ for every $n > n_0$. Now, for every $n > n_0$, by the properties of the function v (that is $0 \leq \xi_n v(t) < \iota$ for n large enough), one has

$$\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = \frac{G(\xi_n)\int_{\mathscr{F}} \alpha(x)\mathrm{d}x + \int_{R\setminus\mathscr{F}} \alpha(x)G(\xi_n v(x))\mathrm{d}x}{\Phi(\xi_n v)}$$

$$> \frac{p\eta\int_{\mathscr{F}} \alpha(x)\mathrm{d}x + p\kappa\int_{R\setminus\mathscr{F}} |v(x)|^p\mathrm{d}x}{\kappa_2\|v\|^p} > N.$$

Since N could be arbitrarily large, we get $\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty$, from which (13) clearly follows. So, there exists a sequence $\{\omega_n\} \subset X$ strongly converging to zero such that, for n large enough, $\omega_n \in \Phi^{-1}(-\infty, r_\lambda)$ and $J_\lambda(\omega_n) = \Phi(\omega_n) - \lambda\Psi(\omega_n) < 0$. Since u_λ is a global minimum of the restriction of J_λ to $\Phi^{-1}(-\infty, r_\lambda)$, we obtain $J_\lambda(u_\lambda) < 0$, so that u_λ is not trivial.

Below, we present one application of Theorem 9 as follows.

Example 2 Let $K(t) = 2 + \tanh(t-1)$ for all $t \in [0, +\infty]$, $p = 3$, $B = 1$, $\alpha(x) = e^{-|x|}$ for all $x \in R$, $g(t) = 1 + t^6$ for all $t \in R$. Thus $\kappa_1 = 1$, $\kappa_2 = 3$, $|\alpha|_1 = 2$, $\alpha_0 = 2(1 - e^{-1})$, $C_B = \frac{2}{\sqrt[3]{9}}$ and $\ell = \sqrt[3]{\frac{91}{3}}$. Moreover, $g(0) = 1 \neq 0$, $\lim_{\xi\to 0^+} \frac{g(\xi)}{\xi^{p-1}} = \lim_{\xi\to 0^+} \frac{1+\xi^6}{\xi^2} = +\infty$ an taking into account that $\lim_{|\xi|\to+\infty} \frac{\xi g(\xi)}{G(\xi)} = \lim_{|\xi|\to+\infty} \frac{\xi+\xi^7}{\xi+\frac{1}{7}\xi^7} = 7 > 3 = p$, by choosing $\nu = 7 > 3 = p$, there exist $\rho > 1$ such that the assumption (AR) in Theorem 9 is fulfilled for all $\xi \geq \rho$. Hence, by applying Theorem 9 and Remark 2, for every $\lambda > 0$, problem (1), in this case possesses at least two positive weak solutions.

Finally, as a consequence of Theorems 4 and 5, we can obtain the following existence result of three solutions.

Theorem 10 *Assume that $g(0) \neq 0$ and there exist four positive constants γ, σ, $\bar{\gamma}$ and $\bar{\sigma}$ with $\sqrt[p]{\frac{\kappa_2}{\kappa_1}}\sigma\ell < \gamma \leq \bar{\gamma} < \bar{\sigma}\ell$ such that $\frac{G(\gamma)}{\gamma^p} < \frac{\alpha_0\kappa_1}{|\alpha|_1\kappa_2\ell^p}\frac{G(\sigma)}{\sigma^p}$ and $|\alpha|_1 G(\bar{\gamma}) < \alpha_0 G(\bar{\sigma})$ hold, and*

$$\frac{|\alpha|_1 G(\gamma)}{\kappa_1\gamma^p} < \frac{|\alpha|_1 G(\bar{\gamma}) - \alpha_0 G(\bar{\sigma})}{\kappa_1\bar{\gamma}^p - \kappa_2\bar{\sigma}^p\ell^p} \tag{14}$$

is satisfied. Then for each $\lambda \in \Lambda = \left(\max\left\{\tilde{\lambda}, \frac{1}{pC_B^p}\frac{\kappa_1\gamma^p - \kappa_2\sigma^p\ell^p}{|\alpha|_1 G(\gamma) - \alpha_0 G(\sigma)}\right\}, \frac{\kappa_1}{pC_B^p}\frac{\gamma^p}{|\alpha|_1 G(\gamma)}\right)$ with $\tilde{\lambda}$ given in Theorem 5, problem (1) possesses at least three positive weak solutions $u_{0,\lambda}$, $\bar{u}_{0,\lambda}$ and $\tilde{u}_{0,\lambda}$ such that $\|u_{0,\lambda}\| < \frac{\gamma}{C_B}$ and $\|\bar{u}_{0,\lambda}\| > \sqrt[p]{\frac{\kappa_1}{\kappa_2}}\frac{\bar{\gamma}}{C_B}$,

Proof First, in view of (14), we have $\Lambda \neq \emptyset$. Next, fix $\lambda \in \Lambda$. Employing Theorem 4, there is a positive weak solution $u_{0,\lambda}$ such that $\|u_{0,\lambda}\| < \frac{\gamma}{C_B^p}$, which is a local minimum for the associated functional J_λ, while Theorem 5 ensures a positive weak solution $\bar{u}_{0,\lambda}$ such that $\|\bar{u}_{0,\lambda}\| > \sqrt[p]{\frac{\kappa_1}{\kappa_2}}\frac{\bar{\gamma}}{C_B^p}$ which is a local minimum for J_λ. Arguing as in the proof of Theorem 3, we observe that the functional J_λ is coercive, then it satisfies the (PS)-condition. Hence, the conclusion follows from the mountain pass theorem as given by Pucci and Serrin (see [29]).

Remark 7 We note that, the results of this article extends the results obtained in [14].

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Comparison of Known Existence Results for One-Dimensional Beam Models of Suspension Bridges

Jakub Janoušek

Abstract The aim of this paper is to present our recent existence and uniqueness results for a one-dimensional damped model of a suspension bridge and compare them to previous results for either damped or non-damped one-dimensional beam models.

Keywords Suspension bridge · Beam equation · Weak solution

1 Introduction

Since the well-known collapse of Tacoma Narrows Bridge in 1940, efforts have been made to explain this disaster by mathematical modelling of suspension bridges. Starting with a very detailed report (see [2]) written by O. H. Amman, T. von Kármán and G. B. Woodruff, continuing through the rest of the twentieth century and still going on nowadays, the research has brought many approaches and important results. Some of them were obtained via the nonlinear approach, i.e., considering the bridge's cables to have no restoring force when being compressed. The simplest way how to describe the behaviour of such structure is to model the bridge as a one-dimensional bending beam with simply supported ends connected to an unmovable object by a set of nonlinear cables. These cables act as linear springs when being stretched, however, as already mentioned, when being compressed, they have no restoring force.

In this text, we provide a brief summary of so far known results for these simple one-dimensional beam models, all of them originating from a model presented by A. C. Lazer and P. J. McKenna in [14], which has the following form:

$$\begin{aligned} u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + ku^+ &= W(x) + \varepsilon f(x,t) \quad \text{in } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\tfrac{\pi}{2}, t\right) = u_{xx}\left(\pm\tfrac{\pi}{2}, t\right) &= 0, \quad u(x,t) = u(x, t+2\pi). \end{aligned} \tag{1}$$

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Here, the displacement $u(x, t)$ of the roadbed is measured as positive in the downward direction. Parameters α^2, β and k represent elastic forces inside the beam, viscous damping and the cables' stiffness, respectively. The term $W(x)$ stands for the weight per unit length of the roadbed, whereas $\varepsilon f(x, t)$ represents some external forces affecting the bridge. The nonlinear behaviour of the bridge's cables is described by the "positive part" function $(\cdot)^+$,

$$u^+(x, t) := \max\{u(x, t), 0\}.$$

Although it is true that such models are a major simplification of reality, they exhibit some phenomena closely connected to the behaviour of real structures such as the Golden Gate Bridge or even Tacoma Narrows itself (e.g., solutions of large amplitude or multiplicity of solutions, see [7, 9, 10, 12, 14, 15]). This suggests that even simple models yield relatively "enough" information about the suspension bridges' behaviour. These models can be simply improved by adding more "input data", such as additional terms, equations or corresponding boundary conditions. For more information, see, e.g., [9, 10] or [14].

Now let us return to simple one-dimensional models and start with the non-damped ones.

2 Models Without Damping

During the 1980s, A. C. Lazer, P. J. McKenna and W. Walter were studying multiplicity of solutions for various types of equations without damping, which were suggested as a possible tool for modelling suspension bridges (see, e.g. [13] or [15]). Their work was followed by Q. H. Choi and T. Jung (see [4]) and L. Humphreys, who also added some important numerical results in [12]. These results were extended later by P. Drábek and G. Holubová in [7] by employing global bifurcation theory. Generally, the results of all mentioned authors suggest that the more eigenvalues of the corresponding linear beam operator are crossed by the stiffness parameter k, the more solutions appear. Now, let us repeat all these results in more detail.

2.1 PDE Models

At first, in [13–15], models such as

$$\begin{aligned} u_{tt} + \alpha^2 u_{xxxx} + k u^+ &= W(x) + \varepsilon f(x, t) \quad \text{in} \quad \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\tfrac{\pi}{2}, t\right) = u_{xx}\left(\pm\tfrac{\pi}{2}, t\right) &= 0, \quad u(x, t) = u(x, t + 2\pi) \end{aligned} \tag{2}$$

were studied and the first results concerning multiplicity of solutions were obtained. For simplification, the right-hand side was considered in a more specific form by putting $W(x) = 1$. However, such a simplification is natural, since one expects that the weight per unit length is (more or less) constant for real structures (see e.g. [15]). After adding the symmetry conditions and normalizing by changing the variables, the authors of [4, 7, 12, 15] treated the following version of (2):

$$\begin{aligned} u_{tt} + u_{xxxx} + ku^{+} = 1 + \varepsilon f(x,t) \quad \text{in} \ \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\tfrac{\pi}{2}, t\right) = u_{xx}\left(\pm\tfrac{\pi}{2}, t\right) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x, \ t+\pi). \end{aligned} \tag{3}$$

First of all, according to the paper [15], if $k \in (3, 15)$ then at least two solutions of (3) exist. However, for $k \in (-1, 3)$, the problem (3) admits a unique solution.

Something more was proved in [4] by a variational reduction method. The authors brought additional information for $k \in (3, 15)$. By their result, under this assumption on k, the problem (3) has at least three solutions, two of them having large amplitude. The idea of "more solutions appear when k crosses more eigenvalues of the corresponding linear beam operator" was later numerically supported in [12]. The author also presented a large-amplitude numerical solution which was obtained by a mountain pass algorithm (cf. [5]).

Finally, the paper [7] comes with a different approach, looking at the problem (3) with ε sufficiently small, or with $\varepsilon = 0$, i.e.,

$$\begin{aligned} u_{tt} + u_{xxxx} + ku^{+} = 1 \quad \text{in} \ \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\tfrac{\pi}{2}, t\right) = u_{xx}\left(\pm\tfrac{\pi}{2}, t\right) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x, \ t+\pi) \end{aligned} \tag{4}$$

from the bifurcation theory point of view. This has brought also some other qualitative information about the solution set. Namely, the authors of [7] proved the following assertion.

> Each eigenvalue of the corresponding linear problem with an odd multiplicity is a point of global bifurcation and there exists a continuum of solutions, which is either unbounded, or reaches another eigenvalue. Moreover, for $k < -1$, the problem (4) has no solution, for $k \in (-1, 3)$ it has a unique, positive and stationary solution and for $k \in (3, 15)$ there exist at least two solutions, one of them being positive and stationary and the other one sign changing.

Still, some questions remained unanswered, e.g., whether for any $k > 3$ there exist multiple solutions of (4). However, for some even more simplified ODE models, one can obtain more precise results.

2.2 ODE Models

It is possible to consider the terms of the right-hand side in (3) being $W(x) = \cos x$ and $\varepsilon f(x,t) = \varepsilon f(t)\cos x$ and look for no-nodal solutions $u(x,t) = y(t)\cos x$ (see [7] or [14]). When we insert this additional information into (3), we get an ODE problem

$$\begin{aligned} y'' + y + ky^+ &= 1 + \varepsilon f(t), \\ y(t) = y(-t) &= y(t+\pi). \end{aligned} \tag{5}$$

Similar model has been dealt with in more detail in [14] and, again, the authors came to the conclusion that more crossed eigenvalues (by k) of the corresponding linear problem means more solutions of (5). Later, by taking ε sufficiently small or even $\varepsilon = 0$ in (5) and thus treating the model

$$\begin{aligned} y'' + y + ky^+ &= 1, \\ y(t) = y(-t) &= y(t+\pi), \end{aligned} \tag{6}$$

the authors of [7] obtained quite a strong result, which brings the following information about the set of solutions:

> There exists a sequence $\{k_m\}$ where $k_m = 4m^2 - 1,\ m \in \mathbb{N} \cup \{0\}$, such that (6) has exactly $2m+1$ solutions whenever $k \in (k_m, k_{m+1})$.

Using global bifurcation theorems, the authors also provided a detailed description of solution branches bifurcating from the points $k_m,\ m \geq 1$, which are the negatives of the corresponding linear operator's eigenvalues (see [7], Theorem 3.1).

Later, P. Drábek and P. Nečesal showed in [8] that the situation is not that simple, when one considers the solution of (6) not π-periodic, but generally T-periodic. Specifically, if $T \in (0,\pi)$, the solutions are all uniformly bounded, whereas if $T \geq \pi$ then there exist solutions with an arbitrarily large amplitude. Moreover, there are blow up points if $T > \pi$, that is, in such case, there exist nonstationary solutions with their amplitude approaching infinity. Further, the authors of [8] found out that this general T-periodic problem corresponds to the Fučík spectrum of

$$\begin{aligned} y'' + \alpha y^+ - \beta y^- &= 0, \\ y(t) = y(-t) &= y(t+T) \end{aligned} \tag{7}$$

in the following way:

> For a fixed T, the point k is a blow up point if and only if the couple $(k+1, 1)$ belongs to the Fučík spectrum of (7). Moreover, as T goes to infinity, the number of blow up points increases.

The authors also obtained similar results for the right-hand side of (6) being in the form $1 + \varepsilon f(t)$ for ε small enough (see [8] for details).

3 A Damped Model

Now, we turn our attention to a slightly more realistic approach, that is, using a one-dimensional model with viscous damping. Let us point out that the background of this model is also in the work of Lazer and McKenna, who introduced it in [14]. The model has the form of a boundary value problem

$$\begin{aligned} u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + ku^+ &= h(x,t), \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) &= 0, \\ u(x,t+2\pi) = u(x,t),\ -\infty < t < +\infty,\ x &\in (0,\pi), \end{aligned} \tag{8}$$

where $\alpha > 0$, $\beta > 0$ and $k \in \mathbb{R}$. The meaning of all parameters remains the same as in (1), however, the right-hand side $h(x,t)$ is considered in a more general form. This model has not been treated in so much detail as the above mentioned non-damped ones, but still some results have been obtained.

3.1 Previous Results

At the beginning of the 1990s, J. M. Alonso and R. Ortega studied the global asymptotic stability and uniqueness of a solution of a forced Newtonian system with dissipation (see [1]), i.e.,

$$u''(t) + cu'(t) + Au + \nabla G(u) = p(t), \tag{9}$$

where $u : \mathbb{R} \to \mathbb{R}^N$, $c > 0$, A is a symmetric positive semidefinite matrix, $G \in C^2(\mathbb{R}, \mathbb{R}^N)$ and the right-hand side $p \in C(\mathbb{R}, \mathbb{R}^N) \cap L^\infty(\mathbb{R}, \mathbb{R}^N)$. By considering the right-hand side of (8) $h(x,t)$ continuous and bounded, using the spatial discretization and the finite difference approach, the authors were able to interpret (8) in view of (9) and obtained a uniqueness result, which is in the form of a sufficient condition and gives the following information:

> If $k < \beta^2 + 2\alpha\beta$ then (8) has a unique bounded solution that is exponentially asymptotically stable.

This result partially coincides with one of the recent results obtained via a different technique, that can be seen further in this text (see Theorem 1) or with more details in [11].

Roughly at the same time, P. Drábek also studied the problem (8) (see [6]). He was able to prove the existence of at least one weak solution for a more general right-hand side than the one that was considered in [1]. Moreover, under additional assumptions, he showed that when the external forces are sufficiently small, there always exists a solution in some sense near to the equilibrium. The work of [6] was followed by G. Tajčová in [16]. By using Banach Contraction Theorem, she determined a sufficient condition of the existence and uniqueness of a weak solution from the space $L^2(\Omega)$, where $\Omega = (0, \pi) \times (0, 2\pi)$. By employing this abstract setting and considering the right-hand side to be an arbitrary function from $L^2(\Omega)$, the problem (8) can be transformed into an operator equation

$$Lu + ku^+ = h, \tag{10}$$

where the letter L stands for an $L^2(\Omega)$ abstract realization of the linear beam operator

$$u \mapsto u_{tt} + \alpha^2 u_{xxxx} + \beta u_t.$$

with the given periodic and boundary conditions. The spectrum $\sigma(L)$ of L consists only of the eigenvalues

$$\lambda_{mn} = \alpha^2 m^4 - n^2 + \mathrm{i}\beta n, \quad m \in \mathbb{N},\ n \in \mathbb{Z}. \tag{11}$$

Furthermore, for $\lambda \notin \sigma(L)$ the operator $L - \lambda I$ is invertible (let us denote the corresponding resolvent operator by L_λ^{-1}). This resolvent operator is linear, compact and bounded (see [3], [6] or [16]) with

$$|| L_\lambda^{-1} || \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(L))}. \tag{12}$$

Since zero is obviously not an eigenvalue of L, the operator equation (10) can be viewed as a fixed point problem

$$u = L_0^{-1}(-ku^+ + h), \tag{13}$$

which is finally suitable for employing the Banach Contraction Theorem. By doing this, G. Tajčová obtained the following result (see [16]):

> If $|k| < \operatorname{dist}(0, \sigma(L))$ then the problem (8) has a unique weak solution for an arbitrary right-hand side $h \in L^2(\Omega)$.

Unfortunately, such a result suggests, that the bridge is "safe enough" if its cables are not really stiff, which does not often correspond well to reality and therefore it gives an opportunity for possible improvement.

It is not necessary to change the whole approach. Instead of that, it is possible, with some minor updates, to get new sufficient conditions, some of them being less strict. These new conditions have been obtained in [11]. We used again the same abstract tools and settings (cf. [3, 16]), however, by proceeding with some new geometric arguments, we were able to extend the "uniqueness interval" for the stiffness parameter k.

3.2 Recent Results

The starting point for all upgrades in [11] is the fact that the eigenvalues (11) have their geometric interpretation in the complex plane. Actually, they can be viewed as intersections of parabolas

$$p_m = \left\{(x, y) : \; x = \alpha^2 m^4 - \frac{y^2}{\beta^2}\right\}, \quad m \in \mathbb{N},$$

with lines parallel to the real axis, i.e.,

$$l_n = \{(x, y) : \; y = \beta n, \}, \quad n \in \mathbb{Z}.$$

See Fig. 1 for illustration. With this interpretation, it is possible to work with the distance of parameter λ from the spectrum of L in a purely geometric way. So, the first possible improvement is to make the uniqueness condition from [16] more "readable" and easier to verify for specific input data. This can be done by determining for which type of relation between the parameters α and β the smallest real eigenvalue λ_{10} is the closest one to the origin, i.e., when

$$\mathrm{dist}(0, \sigma(L)) = |\lambda_{10}| = \alpha^2. \tag{14}$$

This equality means that the open disc $D_0 = \{z \in \mathbb{C} : \; |z| < \alpha^2\}$ does not contain any of the eigenvalues λ_{mn}. Since these eigenvalues can be identified as the above described intersections, it is enough to check when the first parabola p_1 is outside

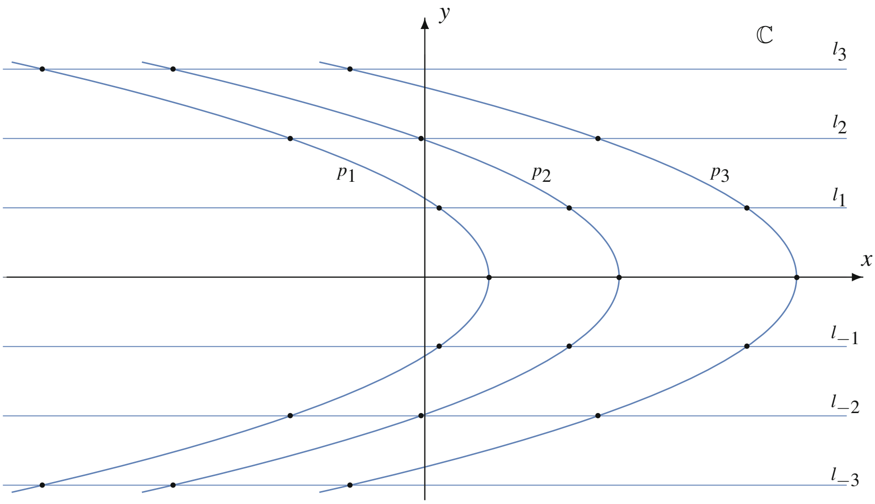

Fig. 1 Illustration of the eigenvalues λ_{mn} in the complex plane

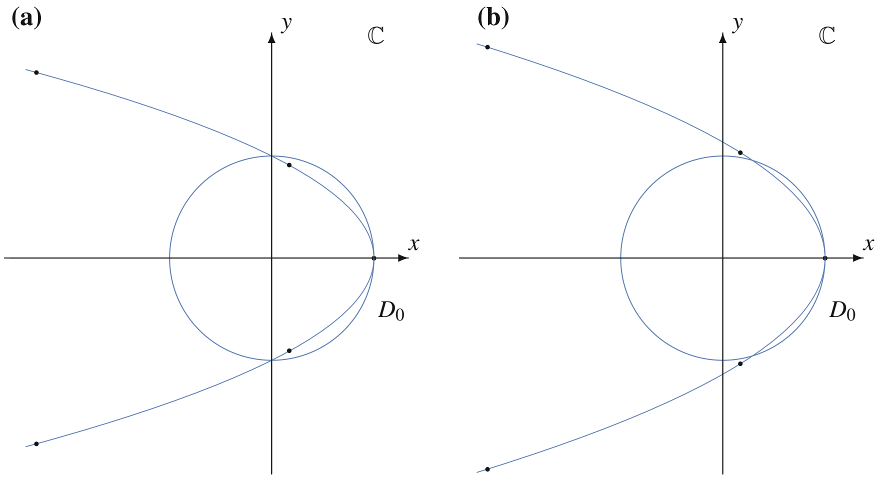

Fig. 2 Spectrum $\sigma(L)$ and disc D_0 for **a** $\alpha = 1.1$, $\beta = 1.1$, **b** $\alpha = 1.1$, $\beta = 1.25$. Here, $\sigma(L) \cap D_0 = \{\lambda_{1\pm1}\}$ in **a**, whereas $\sigma(L) \cap D_0 = \emptyset$ in **b**

D_0, or when the first pair of lines l_1, l_{-1} does not go through D_0. With this approach, we can observe the following behaviour (see Fig. 2 for illustration):

- If $\beta \geq \alpha^2$, then no horizontal line l_n, $n \in \mathbb{Z}$, intersects D_0.
- If $\beta \geq \sqrt{2}\alpha$, then no parabola p_m, $m \in \mathbb{N}$, intersects D_0.
- If $\alpha > 1$ and $\sqrt{2\alpha^2 - 1} \leq \beta < \sqrt{2}\alpha$, then the only parabola intersecting D_0 is p_1, but $\lambda_{1n} \notin D_0$ for all $n \in \mathbb{Z}$.

Hence, if one of these relations holds then (14) is true, we obtain (in view of the result from [16]) uniqueness of a weak solution for any $k \in \left(-\alpha^2, \alpha^2\right)$. Now, let us summarize all discussed facts.

Proposition 1 ([11]) *Let $\beta \geq \alpha^2$ for $\alpha < 1$ and $\beta \geq \sqrt{2\alpha^2 - 1}$ for $\alpha \geq 1$. Then the problem (8) has a unique weak solution $u \in H$ for an arbitrary right-hand side $h \in H$ whenever $k \in (-\alpha^2, \alpha^2)$.*

The reader should be aware of the fact, that although Proposition 1 brings a refining and better readability of the original condition, it is in fact weaker. Indeed, it makes the chain of implications, which lead to the existence and uniqueness of a weak solution, longer. If the assumptions of Proposition 1 are satisfied, then $|k| < \text{dist}(0, \sigma(L))$, then the operator $L_0^{-1}(-k(\cdot)^+ + h)$ is a contraction and then the existence and uniqueness result is obtained. Further improvement can be reached by modification of the operator equation (10) by an ε-shift, i.e.,

$$Lu - \varepsilon u + \varepsilon u + ku^+ = h,$$

thus getting an equation comprising an ε-shifted operator

$$(L - \varepsilon I)u = -(\varepsilon u + ku^+) + h. \tag{15}$$

The main reason of introducing this ε-shift can be seen in Fig. 2. Actually, checking the distance between the origin and $\sigma(L)$ is too restrictive and thanks to the shape of the parabolas p_m, one would rather check the distance between $\sigma(L)$ and some other point ε on the real axis, especially for $\varepsilon < 0$. However, some limited improvement is possible even for $\varepsilon > 0$. Thus, by considering ε not to be an eigenvalue of L and using the decomposition $\varepsilon u = \varepsilon u^+ - \varepsilon u^-$ on the right hand side of (15), we get a fixed point formulation

$$u = L_\varepsilon^{-1}\left(-(k+\varepsilon)u^+ + \varepsilon u^- + h\right). \tag{16}$$

Next, we again employ Banach Contraction Theorem together with the inequality

$$||\,(k+\varepsilon)(v^+ - u^+) - \varepsilon(v^- - u^-)|| \le \max\{|k+\varepsilon|,\ |\varepsilon|\}||\,v - u||$$

and obtain that if

$$\max\{|k+\varepsilon|,\ |\varepsilon|\} < \operatorname{dist}(\varepsilon, \sigma(L)), \tag{17}$$

then the operator $L_\varepsilon^{-1}\left(-(k+\varepsilon)(\cdot)^+ + \varepsilon(\cdot)^- + h\right)$ is contractive. Since $|k+\varepsilon|$ expresses the distance between ε and $-k$ and $|\varepsilon|$ the distance between ε and the origin, the inequality (17) reads

$$\operatorname{dist}(\varepsilon, 0) < \operatorname{dist}(\varepsilon, \sigma(L)) \quad \wedge \quad \operatorname{dist}(\varepsilon, -k) < \operatorname{dist}(\varepsilon, \sigma(L)).$$

Hence, it is optimal to consider $k = -2\varepsilon$, which implies $|k+\varepsilon| = |\varepsilon|$. Next, if we find the maximal positive values $\varepsilon_m, \varepsilon_M$ such that $\operatorname{dist}(\varepsilon, 0) < \operatorname{dist}(\varepsilon, \sigma(L))$ holds for any $\varepsilon \in (-\varepsilon_M, \varepsilon_m)$ then $\operatorname{dist}(\varepsilon, -k) < \operatorname{dist}(\varepsilon, \sigma(L))$ is satisfied for any $k \in (-2\varepsilon_m, 2\varepsilon_M)$. We thus get a larger "uniqueness interval" for values of the stiffness parameter k than the one obtained by the non-ε-shifted approach, however, finding the values $\varepsilon_m, \varepsilon_M$ is not necessarily simple. The first way how to deal with this problem is to find some "safe" estimates, as it can be seen in the following existence and uniqueness theorem, which brings a conclusion of the above sketched discussion and is available together with the proof in [11].

Theorem 1 ([11]) *Let $\varepsilon_M > 0$ and $\varepsilon_m > 0$ be the maximal real numbers for which*

$$\left\{z \in \mathbb{C}:\ (|z - \varepsilon_m| < \varepsilon_m) \vee (|z + \varepsilon_M| < \varepsilon_M)\right\} \cap \sigma(L) = \emptyset. \tag{18}$$

Then the problem (8) has a unique weak solution $u \in H$ for an arbitrary right-hand side $h \in H$ whenever $k \in (-2\varepsilon_m, 2\varepsilon_M)$. Moreover, the following estimates hold:

$$\varepsilon_M \geq \tilde{\varepsilon}_M = \begin{cases} \dfrac{2\alpha\beta + \beta^2}{2} & \textit{for } \beta \geq 2(1-\alpha), \\ \beta & \textit{for } \beta < 2(1-\alpha), \end{cases} \tag{19}$$

and

$$\varepsilon_m \geq \tilde{\varepsilon}_m = \begin{cases} \dfrac{\alpha^2}{2} & \textit{for} \quad \beta \geq \min\left\{\alpha,\ \frac{\alpha^2}{2}\right\}, \\ \dfrac{2\alpha\beta - \beta^2}{2} & \textit{for} \quad \beta \leq \min\{\alpha,\ 2(\alpha-1)\}, \\ \beta & \textit{for} \quad \alpha < 2 \ \textit{and} \ 2(\alpha-1) \leq \beta \leq \frac{\alpha^2}{2}. \end{cases} \tag{20}$$

Similarly as in the previous paragraph, with respect to the nature of the eigenvalues λ_{mn}, it is sufficient to check whether the first parabola or the first pair of lines go through the corresponding circles (see Figs. 3 and 4 for illustration). The estimates for $\varepsilon_m, \varepsilon_M$ based on this discussion are basic, since we do not question the position of other parabolas and pairs of lines, and can be improved. There is also another possibility, that is, to avoid any estimating and instead of that to compute the values $\varepsilon_m, \varepsilon_M$ directly via an algorithm working for specific given setting of α and β.

First, let us briefly mention the possibilities of improving the estimates $\tilde{\varepsilon}_m$ and $\tilde{\varepsilon}_M$. By a thorough checking of the parabolas p_1 and p_2, we find out that for $\alpha^2 < 1$ and $\beta < \frac{1-\alpha^2}{(4+\sqrt{15})\alpha}$ we may construct a disc touching the origin and p_2 with no eigenvalues

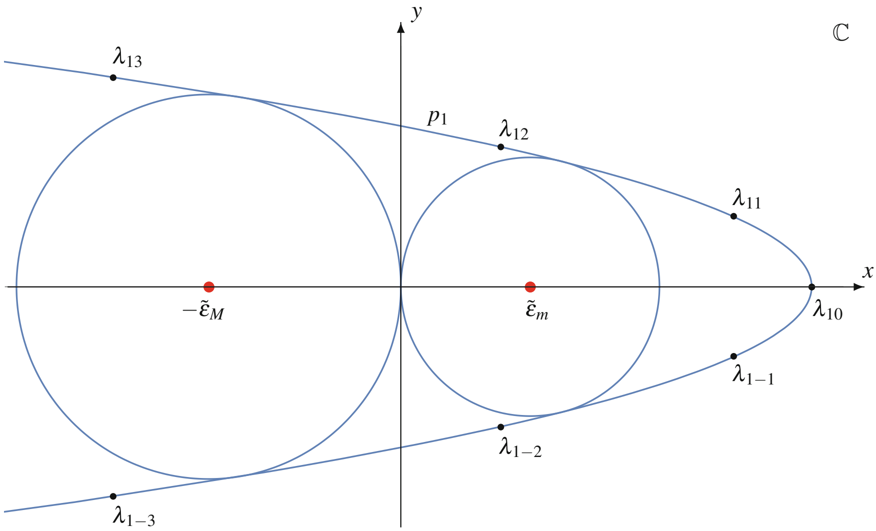

Fig. 3 The values $\tilde{\varepsilon}_m, \tilde{\varepsilon}_M$ for $2(1-\alpha) \leq \beta \leq \alpha$ and the corresponding "safe" discs, where none of the eigenvalues may appear

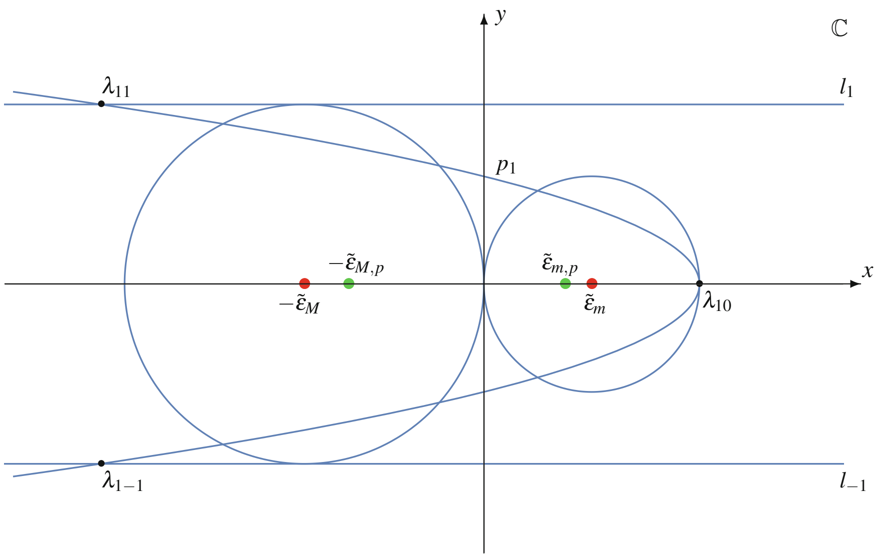

Fig. 4 The values $\tilde{\varepsilon}_m$, $\tilde{\varepsilon}_M$ and the corresponding "safe" discs, where none of the eigenvalues may appear, however, now for the case $\frac{\alpha^2}{2} \leq \beta < 2(1-\alpha)$, when the first pair of lines gains importance. Here, $\tilde{\varepsilon}_{M,p}$ and $\tilde{\varepsilon}_{m,p}$ stand for the more restrictive estimates, which would have been obtained by checking the position of the first parabola p_1

lying on p_1 in its interior. In this case, we can improve the estimate, which is then in the form $\varepsilon_M \geq \bar{\varepsilon}_M = \frac{8\alpha\beta+\beta^2}{2}$.

By trying to check the position of more parabolas and lines, we can obtain more precise estimates, but for the price of getting more and more complicated conditions on α and β. In order to avoid that, we may proceed via the above suggested algorithm and find the precise optimal values ε_m, ε_M. The algorithm can be described in four steps (cf. [11]):

1. Put $\lambda_{\text{opt}} = \lambda_{1n_0}$ with $n_0 = \lfloor \alpha + 1 \rfloor$. (Here, $\lfloor \cdot \rfloor$ denotes the integer part of a real number, and λ_{1n_0} is the closest eigenvalue to the imaginary axis with a negative real part on the parabola p_1.)
2. Find an open disc D with the center on the real axis, whose boundary is going through an eigenvalue λ_{opt} and the origin, i.e., $D = \{z \in \mathbb{C};\ |z + \varepsilon_D| < \varepsilon_D\}$ with

$$\varepsilon_D = \frac{|\lambda_{\text{opt}}|^2}{2\left|\text{Re}\,(\lambda_{\text{opt}})\right|}.$$

3. If there are no eigenvalues inside D, put $\varepsilon_M = \varepsilon_D$ and quit, in the other case find indexes $M = \max\{m : \lambda_{mn} \in D\}$ and $N = \min\{n : \lambda_{Mn} \in D\}$, i.e., find the indexes of such an eigenvalue with a negative real part inside D, which is the

closest one to the imaginary axis and lies on the parabola, whose “branches” are the furthest from the real axis.

4. Put $\lambda_{\text{opt}} = \lambda_{MN}$ and go back to Step 2.

Notice that in the case of ε_m we can proceed similarly. Now, let us show an example, which comprises the usage of all tools presented in this section.

Example 1 Let $s \in \mathbb{N}$ be arbitrary and put $\alpha = s$, $\beta = \frac{1}{s}$. Here, the refining of previous results, i.e., Proposition 1, gives no information about solvability of (8), since $\alpha^2 \geq 1$ and $\beta \leq 1 \leq \sqrt{2\alpha^2 - 1}$. However, since $s \in \mathbb{N}$, $\lambda_{1s} = 0 + \mathrm{i}$, and the open disc $D = \{z \in \mathbb{C};\ |z| < 1\}$ contains no other eigenvalue λ_{mn} (i.e., $\min_{m\in\mathbb{N}, n\in\mathbb{Z}} |\lambda_{mn}| = |\lambda_{1s}| = 1$), the original general result from [16] guarantees the existence and uniqueness of a weak solution of (8) for an arbitrary right-hand side $h \in H$ whenever $k \in (-1, 1)$.

By applying Theorem 1, the interval $(-1, 1)$ can be enlarged. It is easy to see, that for $s = 1$ the estimates (19), (20) yield $k \in \left(-\alpha^2,\ 2\alpha\beta + \beta^2\right) = (-1,\ 3)$, and for $s \geq 2$ we obtain $k \in \left(-2\alpha\beta + \beta^2,\ 2\alpha\beta + \beta^2\right) = \left(-2 + \frac{1}{s^2},\ 2 + \frac{1}{s^2}\right)$.

Note that these uniqueness intervals are twice as large as the original one. Moreover, e.g., for $s = 1$, the closest eigenvalue to the imaginary axis with a negative real part on p_1 is $\lambda_{12} = -3 + 2\mathrm{i}$ and the disc D, whose boundary is passing through it, contains no other eigenvalue in its interior. Hence, using our algorithm, we get $\varepsilon_M = \varepsilon_D = \frac{|\lambda_{12}|^2}{2|\mathrm{Re}\,(\lambda_{12})|} = \frac{13}{6}$ and the uniqueness result holds for any $k \in \left(-1,\ \frac{13}{3}\right)$.

Remark 1 Although it is questionable to compare results for non-damped and damped models, we may observe the following facts. Looking at Example 1 with using the estimates for $s = 1$ and at the result [7] for (3) and (4), we get the same uniqueness interval $(-1, 3)$. If we use our algorithm, the uniqueness interval is larger: $(-1, 13/3)$. This indeed suggests, that adding the damping term into the model may improve the “uniqueness behaviour” of it. For better illustration, we may consider $\alpha = 1$ and $\beta > 1$. By employing our estimates $\tilde{\varepsilon}_m$, $\tilde{\varepsilon}_M$, we obtain the uniqueness result for any $k \in \left(-1, 2\beta + \beta^2\right)$, where $2\beta + \beta^2 > 3$, i.e., for $3 < k < 15$ without damping, there are more solutions guaranteed, but possibly a unique solution with sufficient damping added.

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A Certain Class of Harmonic Mappings Related to Functions of Bounded Radius Rotation

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Abstract Let R_k be the class of functions with bounded radius rotation and let S_H be the class of sense-preserving harmonic mappings. In the present paper we investigate a certain class of harmonic mappings related to the function of bounded radius rotation.

Keywords Harmonic mapping · Bounded radius rotation · Distortion theorem · Growth theorem and radius of starlikeness

2010 Mathematics Subject Classification 30C45

1 Introduction

Let $\mathcal{A}$ be the class of functions in the open unit disc $\mathbb{D}$ that are normalized with $h(0) = 0, h'(0) = 1$, then a function $h(z) \in \mathcal{A}$ is called convex or starlike if it maps $\mathbb{D}$ onto a convex or starlike region, respectively. Corresponding classes are denoted by $\mathcal{C}$ and S^*. It is well known that $\mathcal{C} \subset S^*$, that both are subclasses of the univalent functions and have the following analytical representations.

$$h(z) \in \mathcal{C} \Longleftrightarrow Re\left(1 + z\frac{h''(z)}{h'(z)}\right) > 0, \quad z \in \mathbb{D} \tag{1}$$

and

$$h(z) \in S^* \Longleftrightarrow Re\left(z\frac{h'(z)}{h(z)}\right) > 0, \quad z \in \mathbb{D} \tag{2}$$

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More on these class can be found in [2]. Let $h(z)$ be an element of $\mathcal{A}$. If there is a function $s(z)$ in $\mathcal{C}$ such that

$$Re\left(\frac{h'(z)}{s'(z)}\right) > 0, \quad z \in \mathbb{D} \tag{3}$$

then $h(z)$ is called close-to-convex function in $\mathbb{D}$ and the class of such functions is denoted by $\mathcal{CC}$.

A function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation which is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit. Let V_k denote the class of functions $h(z) \in \mathcal{A}$ which maps $\mathbb{D}$ conformally onto an image domain of boundary rotation at most $k\pi$. The class of functions of bounded boundary rotation was introduced by Loewner [5] in 1917 and was developed by Paatero [6, 7] who systematically developed their properties and made an exhaustive study of the class V_k. Paatero has shown that $h(z) \in V_k$ if and only if

$$h'(z) = Exp\left[-\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right], \tag{4}$$

where $\mu(t)$ is real-valued function of bounded variation for which

$$\int_0^{2\pi} d\mu(t) = 2, \quad \int_0^{2\pi} |d\mu(t)| \leq k \tag{5}$$

for fixed $k \geq 2$ it can also be expressed as

$$\int_0^{2\pi} \left| Re\frac{(zh'(z))'}{h'(z)} \right| d\theta \leq 2k\pi, \quad z = re^{i\theta}. \tag{6}$$

Clearly, if $k_1 < k_2$ then $V_{k_1} \subset V_{k_2}$ that is the class V_k obviously expands on k increases. V_2 is the class of $\mathcal{C}$ of convex univalent functions. Paatero showed that $V_4 \subset \mathcal{S}$, where $\mathcal{S}$ is the class of normalized univalent functions [6, 7]. Later Pinchuk proved that V_k are close-to convex functions in $\mathbb{D}$ if $2 \leq k \leq 4$ [8].

Let R_k denote the class of analytic functions f of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ having the representation

$$f(z) = zExp\left[-\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right], \tag{7}$$

where $\mu(t)$ is given in (5). We note that the class R_k was introduced by Pinchuk and Pinchuk showed that Alexander type relation between the classes V_k and R_k exists,

$$h \in V_k \Leftrightarrow zh'(z) \in R_k$$

R_k consists of those function h which satisfy

$$\int_0^{2\pi} \left| Re(re^{i\theta} \frac{h'(re^{i\theta})}{h(re^{i\theta})}) \right| d\theta \leq k\pi, r < 1, z = re^{i\theta}. \tag{8}$$

Geometrically, the condition is that the total variation of angle between radius vector $f(re^{i\theta})$ makes with positive real axis is bounded $k\pi$. Thus, R_k is the class of functions of bounded radius rotation bounded by $k\pi$, therefore R_k generalizes the starlike functions.

P_k denote the class of functions $p(0) = 1$ analytic in $\mathbb{D}$ and having representation

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) \tag{9}$$

where $\mu(t)$ is given in (5). Clearly, $P_2 = P$ where P is the class of analytic functions with positive real part. For more details see [6].

From (9), one can easily find that $p(z) \in P_k$ can also written by

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad z \in \mathbb{D} \tag{10}$$

where $p_1(z), p_2(z) \in \mathcal{P}$. Pinchuk [8] has shown that the classes V_k and R_k can be defined by using the class P_k as gives below

$$h \in V_k \Leftrightarrow \frac{(zh'(z))'}{h'(z)} \in P_k \tag{11}$$

and

$$h \in R_k \Leftrightarrow \frac{zh'(z)}{h(z)} \in P_k \tag{12}$$

A planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function f which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping f has a canonical decomposition $f = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, 3, \ldots$. As usual we call $h(z)$ is analytic part of f and $g(z)$ is co-analytic part of f. An elegant and complete account of the theory of harmonic mappings are given in Duren's monograph [1]. Lewy [4] proved in 1936 that the harmonic mapping f is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f = |h'(z)|^2 - |g'(z)|^2$ is different from zero in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc $\mathbb{D}$ are either sense-preserving if

$|g'(z)| < |h'(z)|$ in $\mathbb{D}$, or sense-reversing if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$. Through this paper we will restrict ourselves to the study of sense- preserving harmonic mappings. We also note that $f = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in $\mathbb{D}$ and the second dilatation $w(z) = \left(\frac{g'(z)}{h'(z)}\right)$ has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. Therefore the class of all sense-preserving harmonic mappings in the open unit disc $\mathbb{D}$ with $a_0 = 0$, $b_0 = 0$, $a_1 = 1$ will be denoted by S_H. Thus S_H contains the standard class S of univalent analytic functions. The family of all mappings $f \in S_H$ with the additional property $g'(0) = 0$, i.e., $b_1 = 0$ is denoted by $\mathcal{S}_H^0$. Hence it is clear that $S \subset \mathcal{S}_H^0 \subset \mathcal{S}_H$ (see [2]).

Let $h, g \in \mathcal{A}$, then we say that h is subordinate to g, written as $h \prec g$ if there exists a Schwarz function $\phi \in \Omega$ $(z \in \mathbb{D})$ such that $h(z) = g(\phi(z))$, $z \in \mathbb{D}$. We also note that if g univalent in $\mathbb{D}$, then $h \prec g$ if and only if $h(0) = g(0)$, $h(\mathbb{D}) \subset g(\mathbb{D})$ implies $h(\mathbb{D}_r) \subset g(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z : |z| < r, 0 < r < 1\}$ (see [2]).

2 Main Results

Lemma 1 *Let $p(z)$ be an element of P_k, then*

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \leq \frac{k\pi}{1-r^2} \tag{13}$$

Proof Robertson [9] proved that if $h(z) \in V_k$, then

$$\left|z\frac{f''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{kr}{1-r^2}$$

Therefore the relation can be written in the following form,

$$\left|(1 + z\frac{f''(z)}{f'(z)}) - \frac{1+r^2}{1-r^2}\right| \leq \frac{k\pi}{1-r^2} \tag{14}$$

Using the definition of the class V_k, we obtain (13).

Corollary 1 *Let $p(z)$ be an element of P_k, then*

$$\frac{1-kr+r^2}{1-r^2} \leq |p(z)| \leq \frac{1+kr+r^2}{1-r^2} \tag{15}$$

$$\frac{1-kr+r^2}{1-r^2} \leq Rep(z) \leq \frac{1+kr+r^2}{1-r^2} \tag{16}$$

Theorem 1 *Let $h(z)$ be an element of R_k, then*

$$\frac{r}{(1-r)^{\frac{2-k}{2}}(1+r)^{\frac{2+k}{2}}} \leq |h(z)| \leq \frac{r}{(1-r)^{\frac{2+k}{2}}(1+r)^{\frac{2-k}{2}}} \tag{17}$$

Proof Let $h(z)$ be an element of R_k. Using the definition of R_k and P_k and Lemma 1, then we can write

$$\left| z\frac{h'(z)}{h(z)} - \frac{1+r^2}{1-r^2} \right| \leq \frac{kr}{1-r^2} \tag{18}$$

The inequality (18) can be written in the form

$$\frac{1-kr+r^2}{1-r^2} \leq Rez\frac{h'(z)}{h(z)} \leq \frac{1+kr+r^2}{1-r^2} \tag{19}$$

Since

$$Rez\frac{h'(z)}{h(z)} = r.\frac{\partial}{\partial r}log|h(z)|$$

Thus we have

$$\frac{1-kr+r^2}{r(1-r^2)} \leq \frac{\partial}{\partial r}log|h(z)| \leq \frac{1+kr+r^2}{r(1-r^2)} \tag{20}$$

Integrating both sides (20), we get (17).

Corollary 2 *For $k = 2$, we obtain*

$$\frac{r}{(1+r)^2} \leq |h(z)| \leq \frac{r}{(1-r)^2}$$

This is well known growth theorem for starlike functions [2].

Corollary 3 *Let $h(z)$ be an element of R_k, then*

$$\frac{1-kr+r^2}{(1-r)^{2-\frac{k}{2}}(1+r)^{2+\frac{k}{2}}} \leq |h'(z)| \leq \frac{1+kr+r^2}{(1-r)^{2+\frac{k}{2}}(1+r)^{2-\frac{k}{2}}} \tag{21}$$

Proof Since

$$\left| z\frac{h'(z)}{h(z)} - \frac{1+r^2}{1-r^2} \right| \leq \frac{kr}{1-r^2}$$

Then we have

$$\frac{1}{r}.\frac{1-kr+r^2}{1-r^2}|h(z)| \leq |h'(z)| \leq \frac{1}{r}.\frac{1+kr+r^2}{1-r^2}|h(z)| \tag{22}$$

Using Theorem 1 in the inequality (22), we get (21).

Corollary 4 *For $k = 2$, we obtain*

$$\frac{1-r}{(1+r)^3} \le |h'(z)| \le \frac{1+r}{(1-r)^3}$$

This is well known distortion theorem for starlike functions [2].

We note that all results are sharp because of extremal function is

$$h(z) = \frac{z(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}}.$$

3 Application to Harmonic Mapping

We now consider the subclass of harmonic mapping,

$$S_{HR_k}(A, B) = \left\{ f = h(z) + \overline{g(z)} : w(z) = \frac{g'(z)}{h'(z)} \prec b_1 \frac{1+Az}{1+Bz}, h(z) \in R_k \right\}$$

In this section we will investigate the subclass $S_{HR_k}(A, B)$.

Theorem 2 *Let $f = h(z) + \overline{g(z)}$ be an element of $S_{HR_k}(A, B)$, then*

$$\frac{g'(z)}{h'(z)} \prec b_1 \frac{1+Az}{1+Bz} \tag{23}$$

Proof We define the function ϕ by

$$\frac{g(z)}{h(z)} = b_1 \frac{1+A\phi(z)}{1+B\phi(z)} \tag{24}$$

then ϕ is analytic in $\mathbb{D}$ and

$$\left.\frac{g(z)}{h(z)}\right|_{z=0} = b_1 = b_1 \frac{1+A\phi(0)}{1+B\phi(0)} \Rightarrow \phi(0) = 0$$

On the other hand, if we take derivative from (24) and after simple calculations we get

$$\frac{g'(z)}{h'(z)} = b_1 \left(\frac{1+A\phi(z)}{1+B\phi(z)} + \frac{(A-B)z\phi'(z)}{(1+B\phi(z))^2} . \frac{h(z)}{zh'(z)} \right) \tag{25}$$

One can easily conclude that the subordination (23) is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Since $h(z) \in R_k$, then the boundary value of $z\frac{h'(z)}{h(z)}$ is

$$z\frac{h'(z)}{h(z)} = \frac{1 + kre^{i\theta} + r^2}{1 - r^2} \tag{26}$$

and Jack lemma says that "Let $\phi(z)$ be analytic in $\mathbb{D}$ with $\phi(0) = 0$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, then we have

$$z_0\phi'(z_0) = m\phi(z_0), m \geq 1.$$

Considering Jack lemma, (25) and (26), then we have

$$\frac{g'(z_0)}{h'(z_0)} = b_1\left(\frac{1 + A\phi(z_0)}{1 + B\phi(z_0)} + \frac{(A - B)z\phi'(z_0)}{(1 + B\phi(z_0))^2}.\frac{1 + kre^{i\theta} + r^2}{1 - r^2}\right).$$

This shows that

$$w(z_0) = \frac{g'(z_0)}{h'(z_0)} \notin w(\mathbb{D}_r)$$

where $\mathbb{D}_r = \{z : |z| < r < 1\}$. This is conradiction with (23). Then $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Corollary 5 *Let $f = h(z) + \overline{g(z)}$ be an element of $S_{HR_k}(A, B)$, then*

$$|b_1|\frac{1 - Ar}{1 - Br}\frac{1 - kr + r^2}{(1 - r)^{2-\frac{k}{2}}(1 + r)^{2+\frac{k}{2}}} \leq |g'(z)| \leq |b_1|\frac{1 + Ar}{1 + Br}\frac{1 + kr + r^2}{(1 - r)^{2+\frac{k}{2}}(1 + r)^{2-\frac{k}{2}}} \tag{27}$$

$$|b_1|\frac{1 - Ar}{1 - Br}\frac{r}{(1 - r)^{\frac{2-k}{2}}(1 + r)^{\frac{2+k}{2}}} \leq |g(z)| \leq |b_1|\frac{1 + Ar}{1 + Br}\frac{r}{(1 - r)^{\frac{2+k}{2}}(1 + r)^{\frac{2-k}{2}}} \tag{28}$$

Proof Since $\frac{g'(z)}{h'(z)} \prec b_1\frac{1+Az}{1+Bz}$ and $\frac{g(z)}{h(z)} \prec b_1\frac{1+Az}{1+Bz}$ then using subordination principle, we obtain

$$|b_1|\frac{1 - Ar}{1 - Br}|h'(z)| \leq |g'(z)| \leq |b_1|\frac{1 + Ar}{1 + Br}|h'(z)| \tag{29}$$

$$|b_1|\frac{1 - Ar}{1 - Br}|h(z)| \leq |g(z)| \leq |b_1|\frac{1 + Ar}{1 + Br}|h(z)| \tag{30}$$

Using Theorem 1 and Corollary 3 in (29) and (30), we get (27) and (28).

Theorem 3 *Let $f = h(z) + \overline{g(z)}$ be an element of $S_{HR_k}(A, B)$, then*

$$\frac{|b_1| - r}{1 - |b_1|r} \leq |w(z)| \leq \frac{|b_1| + r}{1 + |b_1|r} \tag{31}$$

This inequality is sharp because the extremal function is

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{z + b_1}{1 + \overline{b_1} z}$$

Proof Since

$$w(0) = \left.\frac{g'(z)}{h'(z)}\right|_{z=0} = b_1$$

then the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)} w(z)}$$

satisfies the conditions Schwarz lemma. Using the subordination

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{b_1 + \phi(z)}{1 + \overline{b_1}\phi(z)} \Leftrightarrow w(z) = \frac{g'(z)}{h'(z)} \prec \frac{b_1 + z}{1 + \overline{b_1} z}.$$

On the other hand the transformation $(\frac{b_1+z}{1+\overline{b_1}z})$ maps $|z| = r$ onto the disc with the centre $C(r) = \left(\frac{\alpha_1(1-r^2)}{1-|b_1|^2 r^2}, \frac{\alpha_2(1-r^2)}{1-|b_1|^2 r^2}\right)$ and the radius $\rho(r) = \frac{(1-|b_1|^2)r}{1-|b_1|^2 r^2}$, therefore we can write

$$\left| w(z) - \frac{b_1(1 - r^2)}{1 - |b_1|^2 r^2} \right| \leq \frac{(1 - |b_1|^2) r}{1 - |b_1|^2 r^2}$$

which gives (31).

Corollary 6 *Let $f = h(z) + \overline{g(z)}$ be an element of $S_{HR_k}(A, B)$, then*

$$\frac{(1 - r)(1 + |b_1|)}{1 - |b_1| r} \leq (1 + |w(z)|) \leq \frac{(1 + r)(1 + |b_1|)}{1 + |b_1| r} \tag{32}$$

$$\frac{(1 - r)(1 - |b_1|)}{1 + |b_1| r} \leq (1 - |w(z)|) \leq \frac{(1 + r)(1 - |b_1|)}{1 - |b_1| r} \tag{33}$$

This corollary is consequence of Theorem 3.

Corollary 7 *Let $f = h(z) + \overline{g(z)}$ be an element of $S_{HR_k}(A, B)$, then*

$$J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2 - |w(z) h'(z)|^2$$

$$J_{f(z)} = |h'(z)|^2 (1 - |w(z)|^2) \tag{34}$$

$$|h'(z)| - |g'(z)| = |h'(z)| - (w(z)|h'(z)|) = |h'(z)|(1 - |w(z)|) \tag{35}$$

$$|h'(z)| + |g'(z)| = |h'(z)| + (w(z)|h'(z)|) = |h'(z)|(1 + |w(z)|) \tag{36}$$

Using (34), (35) and (36) and related theorem and corrollaries, we obtain several inequlities of $f = h(z) + \overline{g(z)} \in S_{HR_k}(A, B)$. *At the same time if we give special values to A and B, we obtain new results of the subclasses of* $S_{HR_k}(A, B)$.

The special values of A and B are

$$(i) A = 1, B = -1, \quad (ii) A = (1 - 2\alpha), B = -1, 0 < \alpha < 1, \quad (iii) A = 1, B = 0$$

$$(iv) A = \alpha, B = 0, \quad (v) A = 1, B = -1 + \frac{1}{M}, M > \frac{1}{2}, \quad (vi) A = \alpha, B = -\alpha, 0 < \alpha < 1.$$

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Entropy of Nonautonomous Dynamical Systems

Christoph Kawan

Abstract Different notions of entropy play a fundamental role in the classical theory of dynamical systems. Unlike many other concepts used to analyze autonomous dynamics, both measure-theoretic and topological entropy can be extended quite naturally to discrete-time nonautonomous dynamical systems given in the process formulation. This paper provides an overview of the author's work on this subject. Also an example is presented that has not appeared before in the literature.

Keywords Nonautonomous dynamical system · Topological entropy · Measure-theoretic entropy · Variational principle

1 Introduction

In the 1950s, Kolmogorov and Sinai established the concept of measure-theoretic (or metric) entropy, based on Shannon entropy from information theory, as an invariant for measure-preserving maps on probability spaces. This invariant was used, e.g., by Ornstein [17] to classify Bernoulli shifts. Some years later, Adler, Konheim and McAndrew [1] defined in strict analogy a notion of entropy for continuous maps on compact spaces. They already conjectured that both entropy notions are related to each other in the sense of a variational principle, i.e., the topological entropy equals the supremum over all measure-theoretic entropies (supremizing over all invariant Borel probability measures). This was proved not much later by Goodman, Goodwyn and Dinaburg [5, 7, 9].

In the theory of dynamical systems, developed in the ensuing decades, both notions of entropy play a fundamental role as it turned out that they are related to many other dynamical characteristics such as Lyapunov exponents, dimensions of invariant measures and invariant sets and growth rates of periodic orbits, and also to the

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existence of horseshoes. Moreover, entropy has become a central concept in a branch of the topological theory of dynamical systems dedicated to the question of how well a dynamical system can be 'digitalized', i.e., modeled by a symbolic dynamical system [6].

Motivated by the study of triangular maps, Kolyada and Snoha [14] extended the notion of topological entropy to nonautonomous systems given by a sequence of continuous maps on a compact metric space. Together with Misiurewicz, they generalized this concept to sequences of maps between possibly different metric spaces in [15] and proved analogues of the Misiurewicz–Szlenk formula for the entropy of piecewise monotone interval maps. Further work on topological entropy of nonautonomous systems has been done in [18, 20–25] by several researchers with different motivations and partially independently of [14, 15]. An essential difference to the classical theory that should be mentioned is that the nonautonomous version of topological entropy is *not* a purely topological quantity. In fact, it depends on the sequence of metrics imposed on the time-varying state space.

Concepts of measure-theoretic entropy for sequences of maps were first introduced in the papers [4, 10, 25]. While [4, 25] require that all maps in the sequence preserve the same measure, a very restrictive condition, the approach in [10] is completely general. The invariant measure now becomes a sequence $(\mu_n)_{n\in\mathbb{Z}_+}$ of measures so that $(f_n)_*\mu_n = \mu_{n+1}$ for the given sequence of maps f_n. To introduce a reasonable notion of entropy in this general context, an additional structure (called an *admissible class*) needs to be imposed on the system, consisting in a family of sequences of measurable partitions. This family has to satisfy certain axioms in order to obtain structural results such as a power rule and invariance under a reasonably general class of transformations.

In the topological framework, a relation between the topological and the measure-theoretic entropy can be established through the definition of a suitable admissible class adapted to the metric space structure. We call this class the *Misiurewicz class*, since it allows for an easy adaptation of Misiurewicz's proof of the variational principle [19] to show that the measure-theoretic entropy is bounded above by the topological entropy. In the classical case of a single map, the entropy computed with respect to the Misiurewicz class reduces again to the Kolmogorov-Sinai measure-theoretic entropy.

It is still unclear whether a full variational principle holds in this context. One obstruction to a proof, amongst others, is that the Misiurewicz class might not contain elements of arbitrarily small diameter, in general. Some sufficient conditions for the existence of such sequences of small-diameter partitions have been identified in [13], but a general approach to this problem is still missing.

The paper is organized as follows. In Sect. 2, we motivate the entropy theory for nonautonomous dynamical systems by applications in networked control. Section 3 explains the entropy theory developed in [10, 11, 13–15], including the nonautonomous versions of topological and measure-theoretic entropy and their relation. Finally, an example for a system satisfying a full variational principle is presented in Sect. 4.

2 Motivation from Networked Control

The author's central motivation for the development of a nonautonomous entropy theory comes from problems arising in networked control. Networked control systems (NCS) are spatially distributed systems whose components (sensors, controllers and actuators) share a common digital communication network. Examples can be found in vehicle tracking, underwater communications for remotely controlled surveillance and rescue submarines, remote surgery, space exploration and aircraft design. Another large field of applications can be found in modern industrial systems, where industrial production is combined with information and communication technology ('Industry 4.0'). A fundamental problem in this field is to determine the minimal requirements on the communication network so that a specified control objective can be achieved.

The simplest model of an NCS consists of a single feedback loop containing a finite-capacity channel which transmits state information acquired by a sensor from a coder to the controller (see Fig. 1). The first task of the controller, before deciding on the control action, often consists in the computation of a state estimate. If the system is autonomous, it has been shown in [16] that the smallest channel capacity above which a state estimation of arbitrary precision can be achieved is given by the topological entropy of the system. If the problem setup is slightly changed, time-dependencies of many different sorts can appear. Here are some examples:

- Non-invariance of the region of relevant initial states leads to a time-dependent state space.
- The requirement of an exponential improvement of the estimate over time leads to a time-dependent metric on the state space.
- In a stochastic formulation of the problem, non-invariance of the distribution of x_0 (the initial state) leads to a time-dependent probability measure.
- Time-varying coding policies lead to time-dependent partitions of the state space (with respect to which entropy needs to be computed).

The entropy theory described in this paper is sufficiently general to handle all of these time-dependencies. A first application to a state estimation problem can be found in [12].

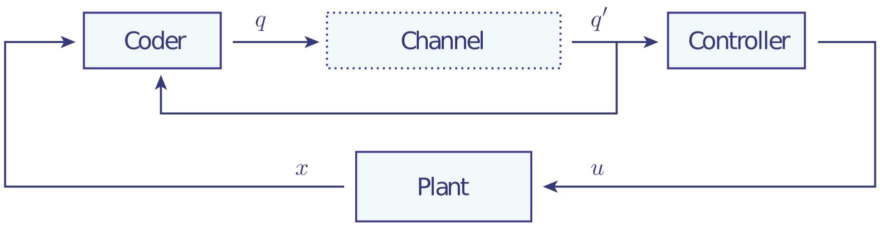


Fig. 1 The simplest model of an NCS

3 Entropy Theory for Nonautonomous Systems

Notation: We write $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. By δ_x we denote the Dirac measure concentrated at a point x. The cardinality of a finite set S is denoted by $\#S$. If A is a subset of a metric space (X, d), we write $\operatorname{diam} A = \sup\{d(x, y) : x, y \in A\}$. If $\mathscr{A}$ is a collection of sets $A \subset X$, we write $\operatorname{diam} \mathscr{A} = \sup\{\operatorname{diam} A : A \in \mathscr{A}\}$. All logarithms are taken to the base 2.

A nonautonomous dynamical system, or briefly an NDS, is a pair (X_∞, f_∞), where $X_\infty = (X_n)_{n\in\mathbb{Z}_+}$ is a sequence of sets and $f_\infty = (f_n)_{n\in\mathbb{Z}_+}$ a sequence of maps $f_n : X_n \to X_{n+1}$. For all $i \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, we define

$$f_i^0 := \mathrm{id}_{X_i}, \quad f_i^n := f_{i+n-1} \circ \dots \circ f_{i+1} \circ f_i, \quad f_i^{-n} := (f_i^n)^{-1}.$$

We do not assume that the maps f_i are invertible, so f_i^{-n} is only applied to sets. We speak of a *topological NDS* if each X_n is a compact metric space (X_n, d_n) and the sequence f_∞ is equicontinuous, i.e., for every $\varepsilon > 0$ there is $\delta > 0$ so that $d_n(x, y) < \delta$ implies $d_{n+1}(f_n(x), f_n(y)) < \varepsilon$ for any $n \in \mathbb{Z}_+$ and $x, y \in X_n$.

3.1 Topological Entropy

To define the topological entropy of a dynamical system, one needs to specify a *resolution* on the state space. Usually, this resolution is given by a finite $\varepsilon > 0$ or by an open cover. In the case of an NDS (X_∞, f_∞), we have to consider a sequence of open covers instead. Hence, let $\mathscr{U}_\infty = (\mathscr{U}_n)_{n\in\mathbb{Z}_+}$ be a sequence so that $\mathscr{U}_n$ is an open cover of X_n for every n. For all $i \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ define

$$\mathscr{U}_i^n := \bigvee_{j=0}^{n-1} f_i^{-j} \mathscr{U}_{i+j},$$

which is the common refinement of the open covers $f_i^{-j}\mathscr{U}_{i+j}$ of X_i, i.e., the open cover whose elements are of the form

$$U_{j_i} \cap f_i^{-1}(U_{j_{i+1}}) \cap \dots \cap f_i^{-n+1}(U_{j_{i+n-1}}), \quad U_{j_l} \in \mathscr{U}_l.$$

Then the entropy of f_∞ w.r.t. $\mathscr{U}_\infty$ is defined by

$$h(f_\infty; \mathscr{U}_\infty) := \limsup_{n\to\infty} \frac{1}{n} \log N(\mathscr{U}_0^n), \tag{1}$$

where $N(\cdot)$ denotes the minimal cardinality of a finite subcover. Here, unlike in the autonomous case, the lim sup in general is not a limit (see [14] for a counter-example).

To define a notion of topological entropy, independent of a given resolution, one usually takes the supremum over all resolutions. However, taking the supremum of $h(f_\infty;\mathscr{U}_\infty)$ over all sequences $\mathscr{U}_\infty$ would result in a quantity that is usually $+\infty$, because a sequence of open covers whose diameters exponentially converge to zero generates an increase of information that is not due to the dynamics of the system. Hence, such sequences have to be excluded. An elegant way how to do this, is to consider only sequences with Lebesgue numbers bounded away from zero. We thus let $\mathscr{L}(X_\infty)$ denote the family of all such sequences and define the *topological entropy of* (X_∞, f_∞) as

$$h_{\mathrm{top}}(f_\infty) := \sup_{\mathscr{U}_\infty \in \mathscr{L}(X_\infty)} h(f_\infty;\mathscr{U}_\infty).$$

This definition was first given in [15]. Some properties of h_{top} are the following:

- Alternative characterizations in terms of (n,ε)-spanning or (n,ε)-separated sets can be given. For instance, a set $E \subset X_0$ is $(n,\varepsilon;f_\infty)$-spanning if for every $x \in X_0$ there exists $y \in E$ such that $d_i(f_0^i(x), f_0^i(y)) < \varepsilon$ for $0 \le i < n$. Letting $r(n,\varepsilon;f_\infty)$ denote the minimal cardinality of an $(n,\varepsilon;f_\infty)$-spanning set,

$$h_{\mathrm{top}}(f_\infty) = \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\log r(n,\varepsilon;f_\infty). \tag{2}$$

- In the case where X_∞, d_∞ and f_∞ are constant, $h_{\mathrm{top}}(f_\infty)$ reduces to the usual notion of topological entropy for maps, which immediately follows from (2).
- The topological entropy $h_{\mathrm{top}}(f_\infty)$ also generalizes several other notions of entropy studied before, as for instance *topological sequence entropy* [8] and *topological entropy for uniformly continuous maps on non-compact metric spaces* [3].
- Fundamental properties of topological entropy for maps carry over to its nonautonomous generalization, as for instance the power rule, which can be formulated as follows. For $m \in \mathbb{N}$ define the mth power system $(X_\infty^{[m]}, f_\infty^{[m]})$ by $X_n^{[m]} := X_{nm}$ and $f_n^{[m]} := f_{nm}^m$. Then the following power rule holds:

$$h_{\mathrm{top}}(f_\infty^{[m]}) = m \cdot h_{\mathrm{top}}(f_\infty).$$

  Here the equicontinuity of f_∞ is essential, see [14] for a counter-example in the case when f_∞ is not equicontinuous.

3.2 Measure-Theoretic Entropy

To define measure-theoretic entropy, we consider systems given by measurable maps $f_n : X_n \to X_{n+1}$ between probability spaces $(X_n, \mathscr{F}_n, \mu_n)$, preserving the measures μ_n in the sense that $(f_n)_*\mu_n = \mu_{n+1}$ for all $n \in \mathbb{Z}_+$. In this case, we also call the sequence $\mu_\infty = (\mu_n)_{n\in\mathbb{Z}_+}$ an *invariant measure sequence*, or briefly an *IMS* for the given NDS (X_∞, f_∞), and we speak of a *measure-theoretic NDS*. Analogously to the topological framework, we define the entropy of f_∞ w.r.t. a sequence of finite measurable partitions $\mathscr{P}_n$ of X_n by

$$h(f_\infty;\mathscr{P}_\infty) = h_{\mu_\infty}(f_\infty;\mathscr{P}_\infty) := \limsup_{n\to\infty}\frac{1}{n}H_{\mu_0}(\mathscr{P}_0^n),$$

where $\mathscr{P}_0^n$ denotes the partition $\bigvee_{i=0}^{n-1} f_0^{-i}\mathscr{P}_i$ and $H_{\mu_0}(\cdot)$ is the Shannon entropy of a partition computed w.r.t. the measure μ_0.

To define measure-theoretic entropy independently of a sequence of partitions, we have to follow a similar strategy as in the topological case. However, the concept of Lebesgue numbers is not helpful here, and a similar construction of a family $\mathscr{L}(X_\infty)$, using the measures μ_n, does not lead to satisfying results. Looking at the topological theory, one sees that results for topological entropy such as the power rule rely on the equicontinuity of the sequence f_∞, and not on the mere continuity of each f_n. However, in the measure-theoretic framework considered here we do not require a similar property.

One way to overcome these obstructions is the study of the essential properties of the family $\mathscr{L}(X_\infty)$, defined in the topological framework, and enforcing these properties in the measure-theoretic framework by an axiomatic definition. As it turns out, the following definition leads to satisfying results.

Definition 1 A nonempty family $\mathscr{E}$ of sequences of finite measurable partitions for X_∞ is called an *admissible class* if it satisfies the following axioms:

(A) For each $\mathscr{P}_\infty = (\mathscr{P}_n)_{n\in\mathbb{Z}_+} \in \mathscr{E}$ there is a bound $N \in \mathbb{N}$ on the cardinality $\#\mathscr{P}_n$, i.e., $\#\mathscr{P}_n \leq N$ for all $n \in \mathbb{Z}_+$.
(B) If $\mathscr{P}_\infty = (\mathscr{P}_n)_{n\in\mathbb{Z}_+} \in \mathscr{E}$ and $\mathscr{Q}_\infty = (\mathscr{Q}_n)_{n\in\mathbb{Z}_+}$ is another sequence of finite measurable partitions for X_∞ such that each $\mathscr{Q}_n$ is coarser than $\mathscr{P}_n$, then $\mathscr{Q}_\infty \in \mathscr{E}$.
(C) If $\mathscr{P}_\infty = (\mathscr{P}_n)_{n\in\mathbb{Z}_+} \in \mathscr{E}$ and $m \in \mathbb{N}$, then also the sequence $\mathscr{P}_\infty^{\langle m\rangle}$, defined as follows, is an element of $\mathscr{E}$:

$$\mathscr{P}_n^{\langle m\rangle} := \bigvee_{i=0}^{m-1} f_n^{-i}\mathscr{P}_{i+n}, \quad n \in \mathbb{Z}_+.$$

Given an admissible class $\mathscr{E}$, we can define the measure-theoretic entropy of f_∞ w.r.t. this class as

$$h_{\mathscr{E}}(f_\infty) = h_{\mathscr{E}}(f_\infty;\mu_\infty) := \sup_{\mathscr{P}_\infty\in\mathscr{E}} h_{\mu_\infty}(f_\infty;\mathscr{P}_\infty).$$

Some elementary properties of admissible classes and their entropy are summarized in the following proposition, cf. [10].

Proposition 1 *Given a measure-theoretic NDS, the following statements hold:*

(i) *There exists a maximal admissible class $\mathscr{E}_{\max}$ defined as the family of all sequences $\mathscr{P}_\infty$ satisfying Axiom (A).*

(ii) *Unions and nonempty intersections of admissible classes are admissible classes.*

(iii) *For each $\emptyset \neq \mathscr{F} \subset \mathscr{E}_{\max}$ there exists a smallest admissible class $\mathscr{E}(\mathscr{F})$ containing $\mathscr{F}$, and its entropy satisfies*

$$h_{\mathscr{E}(\mathscr{F})}(f_\infty) = \sup_{\mathscr{P}_\infty \in \mathscr{F}} h(f_\infty; \mathscr{P}_\infty).$$

One might be tempted to regard the maximal admissible class $\mathscr{E}_{\max}$ as a canonical admissible class for the definition of entropy. However, this class is usually useless, because it contains two many elements. In [10, Example 18] it has been shown that $h_{\mathscr{E}_{\max}}(f_\infty) = \infty$ whenever the maps f_n are bi-measurable and the probability spaces X_n are non-atomic.

As in the classical theory, we can describe the dependence of $h(f_\infty; \mathscr{P}_\infty)$ on $\mathscr{P}_\infty \in \mathscr{E}_{\max}$, using a metric on $\mathscr{E}_{\max}$, defined as

$$D(\mathscr{P}_\infty, \mathscr{Q}_\infty) := \sup_{n \in \mathbb{Z}_+} \left(H_{\mu_n}(\mathscr{P}_n | \mathscr{Q}_n) + H_{\mu_n}(\mathscr{Q}_n | \mathscr{P}_n) \right)$$

with the conditional entropy $H(\cdot|\cdot)$. In the classical case, $D(\cdot, \cdot)$ reduces to the well-known *Rokhlin metric*. Just as in this case, the map $\mathscr{P}_\infty \mapsto h(f_\infty; \mathscr{P}_\infty)$ is Lipschitz continuous w.r.t. D with Lipschitz constant 1.

One particularly useful property of the measure-theoretic entropy w.r.t. an admissible class is the following power rule, cf. [10, Proposition 25].

Proposition 2 *Given a measure-theoretic NDS (X_∞, f_∞) and $m \in \mathbb{N}$, consider the mth power system $(X_\infty^{[m]}, f_\infty^{[m]})$. If $\mathscr{E}$ is an admissible class for (X_∞, f_∞), we denote by $\mathscr{E}^{[m]}$ the class of all sequences of partitions for $X_\infty^{[m]}$ which are defined by restricting the sequences in $\mathscr{E}$ to the spaces in $X_\infty^{[m]}$, i.e., $\mathscr{P}_\infty = \{\mathscr{P}_n\}_{n \in \mathbb{Z}_+} \in \mathscr{E}$ iff*

$$\mathscr{P}_\infty^{[m]} := \{\mathscr{P}_{nm}\}_{n \in \mathbb{Z}_+} \in \mathscr{E}^{[m]}.$$

Then $\mathscr{E}^{[m]}$ is an admissible class for $(X_\infty^{[m]}, f_\infty^{[m]})$ and

$$h_{\mathscr{E}^{[m]}}(f_\infty^{[m]}) = m \cdot h_{\mathscr{E}}(f_\infty).$$

3.3 Measure-Theoretic Entropy for Topological NDS

The concept of measure-theoretic entropy described in the preceding subsection appears to be too general and abstract for interesting applications. In this section, we explain how measure-theoretic and topological entropy interact through the definition of a specific admissible class adapted to the metric space structure of a topological NDS.

In the following, let (X_∞, f_∞) be a topological NDS and μ_∞ an associated IMS.

Definition 2 The *Misiurewicz class* $\mathscr{E}_{\mathrm{M}}$ associated with (X_∞, f_∞) and μ_∞ is defined as follows. A sequence $\mathscr{P}_\infty = (\mathscr{P}_n)_{n\in\mathbb{Z}_+}$ of finite Borel partitions, $\mathscr{P}_n = \{P_{n,1}, \dots, P_{n,k_n}\}$, belongs to $\mathscr{E}_{\mathrm{M}}$ if for every $\varepsilon > 0$ there are $\delta > 0$ and compact sets $K_{n,i} \subset P_{n,i}$ for $n \in \mathbb{Z}_+$, $1 \le i \le k_n$, such that the following holds for all $n \in \mathbb{Z}_+$:

(a) $\mu_n(P_{n,i}\backslash K_{n,i}) \le \varepsilon$ for $1 \le i \le k_n$.
(b) If $x \in K_{n,i}$, $y \in K_{n,j}$, $i \neq j$, then $d_n(x, y) \ge \delta$.

As it turns out, this definition in fact yields an admissible class that is well-adapted to the metric space structure, as expressed by the following theorem.

Theorem 1 *$\mathscr{E}_{\mathrm{M}}$ is an admissible class with the following properties:*

(i) $\mathscr{E}_{\mathrm{M}}$ and the associated entropy $h_{\mathscr{E}_{\mathrm{M}}}(f_\infty; \mu_\infty)$ are preserved by equi-conjugacies, i.e., equicontinuous changes of coordinates.
(ii) In the autonomous case, i.e., when $X_\infty, d_\infty, f_\infty$ and μ_∞ are constant, $h_{\mathscr{E}_{\mathrm{M}}}(f_\infty; \mu_\infty)$ reduces to the usual Kolmogorov-Sinai measure-theoretic entropy.
(iii) The inequality

$$h_{\mathscr{E}_{\mathrm{M}}}(f_\infty; \mu_\infty) \le h_{\mathrm{top}}(f_\infty)$$

holds (establishing one part of the variational principle).

The proofs of (i) and (iii) can be found in [10, Propositions 26 and 27, Theorem 28] and the proof of (ii) in [13, Corollary 3.1].

Since the definition of $\mathscr{E}_{\mathrm{M}}$ is tailored to the (first half of the) proof of the variational principle due to Misiurewicz [19], proving (ii) is an easy task. However, it is not as easy as it might seem to prove that $h_{\mathscr{E}_{\mathrm{M}}}$ generalizes the classical notion of measure-theoretic entropy, since even if $X_\infty, d_\infty, f_\infty$ and μ_∞ are assumed to be constant, we still have to deal with non-constant sequences of partitions. The proof is accomplished through the following result, cf. [13, Theorem 3.1].

Theorem 2 *Assume that there exists a sequence $(\mathscr{R}^k_\infty)_{k\in\mathbb{Z}_+}$ in $\mathscr{E}_{\mathrm{M}}$ with*

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}_+} \operatorname{diam} \mathscr{R}^k_n = 0.$$

Then the measure-theoretic entropy satisfies

$$h_{\mathscr{E}_{\mathrm{M}}}(f_\infty; \mu_\infty) = \lim_{k\to\infty} h(f_\infty; \mathscr{R}^k_\infty) = \sup_{k\in\mathbb{Z}_+} h(f_\infty; \mathscr{R}^k_\infty).$$

In the autonomous case, it is clear that every constant sequence of partitions is contained in $\mathscr{E}_{\mathrm{M}}$, hence any refining sequence of partitions defines a sequence $(\mathscr{R}^k_\infty)_{k\in\mathbb{Z}_+}$, as required in the theorem. Consequently, the theorem says that the entropy is already determined on the constant sequences of partitions, so the classical definition of Kolmogorov-Sinai entropy is retained.

In general, it is unclear whether the Misiurewicz class contains sequences as required in Theorem 2. The following result, proved in [13], yields several sufficient conditions in the case when the state space is time-invariant, cf. [13, Theorem 3.2].

Theorem 3 *Assume that $(X_n, d_n) \equiv (X, d)$ for some compact metric space (X, d). Then each of the following conditions guarantees that $\mathscr{E}_{\mathrm{M}}$ contains elements of arbitrarily (uniformly) small diameter:*

(i) *$\{\mu_n : n \in \mathbb{Z}_+\}$ is relatively compact in the strong topology on the space of measures.*
(ii) *For every $\alpha > 0$ there is a finite measurable partition $\mathscr{A}$ of X with $\mathrm{diam}\mathscr{A} < \alpha$ such that $\nu(\partial\mathscr{A}) = 0$ for all weak*-limits ν of μ_∞. (This holds, in particular, if there are only countably many non-equivalent weak*-limits.) In this case, $\mathscr{E}_{\mathrm{M}}$ contains all constant sequences of partitions of the form $(\mathscr{A}, \mathscr{A}, \mathscr{A}, \ldots)$.*
(iii) *$X = [0, 1]$ or $X = \mathrm{S}^1$ and there exists a dense set $D \subset X$ such that every $x \in D$ satisfies $\nu(\{x\}) = 0$ for all weak*-limits ν of μ_∞.*
(iv) *X has topological dimension zero.*

In each case, the sequences of partitions can in fact be chosen constant.

The following theorem provides an example, where both topological and measure-theoretic entropy can be computed, cf. [11, Theorem 5.4 and Theorem 5.5].

Theorem 4 *Let M be a compact Riemannian manifold and $f_\infty = (f_n)_{n\in\mathbb{Z}_+}$ a sequence of C^2-expanding maps $f_n : M \to M$ with expansion factors uniformly bounded away from one, and C^2-norms uniformly bounded. Then*

$$h_{\mathrm{top}}(f_\infty) = \limsup_{n\to\infty} \frac{1}{n} \log \int_M |\det \mathrm{D} f_0^n(x)| \mathrm{dvol},$$

and for any smooth initial measure μ_0, with $\mu_\infty = ((f_0^n)_*\mu_0)_{n\in\mathbb{Z}_+}$,

$$h_{\mathscr{E}_{\mathrm{M}}}(f_\infty; \mu_\infty) = \limsup_{n\to\infty} \frac{1}{n} \int_M \log |\det \mathrm{D} f_0^n(x)| \mathrm{dvol}.$$

The question under which conditions an NDS satisfies a full variational principle, i.e.,

$$h_{\mathrm{top}}(f_\infty) = \sup_{\mu_\infty} h_{\mathscr{E}_{\mathrm{M}}}(f_\infty; \mu_\infty)$$

is completely open. Only some examples are known which do not allow for a broad generalization.

4 An Example

In this section, we apply the theory explained above to an NDS which has been introduced in [2] by Balibrea and Oprocha. We will need the following proposition whose proof is completely analogous to the autonomous case, and hence is omitted.

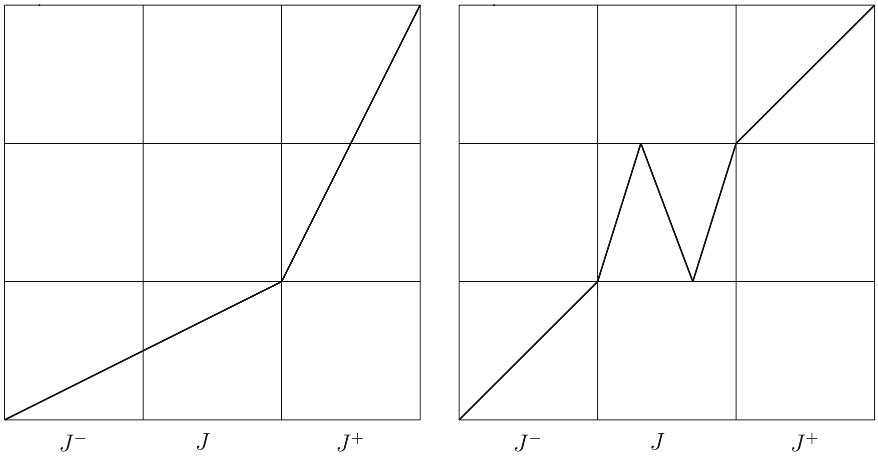
Proposition 3 *Let* (X_∞, f_∞) *be a topological NDS such that* f_n *is (globally) Lipschitz-continuous with Lipschitz constant* L_n *for each* n *and* X_0 *has finite upper capacitive dimension* $\overline{\dim}_C(X_0)$. *Then*

$$h_{\text{top}}(f_\infty) \leq \overline{\dim}_C(X_0) \cdot \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \max\{0, \log L_i\}.$$

Now consider the NDS from [2, Theorem 4], which is constructed from the two piecewise affine maps depicted in Fig. 2. More precisely, let $m_0 := 1$ and $m_n := 2^{n^2}$ for all $n \in \mathbb{N}$. Consider the maps $f, g : [0, 1] \to [0, 1]$ in Fig. 2, and the NDS $f_\infty = (f_n)_{n\in\mathbb{Z}_+}$ defined by

$$f_i := \begin{cases} f & \text{if } i = m_n \text{ for some } n \\ g & \text{otherwise} \end{cases}.$$

For the Lebesgue measure λ on $[0, 1]$ we have weak convergence $\mu_n = (f_0^n)_*\lambda \to \delta_0$, since every trajectory with initial value in $[0, 1)$ converges to zero. More precisely, this implies $\varphi \circ f_0^n(x) \to \varphi(0)$ for every $x \in [0, 1)$ and every continuous function $\varphi : [0, 1] \to \mathbb{R}$. Hence, $\int \varphi \mathrm{d}\mu_n = \int \varphi \circ f_0^n \mathrm{d}\lambda \to \int \varphi(0)\mathrm{d}\lambda$ by the theorem of dominated convergence. Consequently, by Theorem 3(ii), the admissible class $\mathscr{E}_M(\mu_\infty)$



(a) Graph of f (b) Graph of g

Fig. 2 The maps f and g

contains all constant sequences of partitions with δ_0-zero boundaries, in particular all constant sequences $\mathscr{P}_n \equiv \mathscr{P}$, where $\mathscr{P}$ consists of nontrivial subintervals of [0, 1].

Let $\mathscr{P}$ be a partition of [0, 1] into intervals of length $1/(3k)$ for some $k \in \mathbb{N}$. Then each interval in $\mathscr{P}$ is completely contained in $J^- := [0, 1/3]$, $J := [1/3, 2/3]$ or $J^+ := [2/3, 1]$ and

$$H_\lambda\left(\bigvee_{i=0}^{m_n} f_0^{-i}\mathscr{P}\right) = H_\lambda\left(\bigvee_{i=0}^{m_{n-1}} f_0^{-i}\mathscr{P} \vee \bigvee_{i=m_{n-1}+1}^{m_n} f_0^{-i}\mathscr{P}\right) \geq H_\lambda\left(\bigvee_{i=m_{n-1}+1}^{m_n} f_0^{-i}\mathscr{P}\right).$$

Note that for $m_{n-1}+1 \leq i \leq m_n$ we have

$$f_0^{-i} = \left(g^{i-m_{n-1}-1} \circ f_{m_{n-1}} \circ \cdots \circ f_1 \circ f_0\right)^{-1} = f_0^{-(m_{n-1}+1)} \circ g^{-(i-m_{n-1}-1)},$$

and hence, writing $l_n := m_n - m_{n-1} - 1$,

$$H_\lambda\left(\bigvee_{i=0}^{m_n} f_0^{-i}\mathscr{P}\right) \geq H_\lambda\left(f_0^{-(m_{n-1}+1)} \bigvee_{i=0}^{l_n} g^{-i}\mathscr{P}\right).$$

Now we look only at those members of $\bigvee_{i=0}^{l_n} g^{-i}\mathscr{P}$ that come from intervals $P \in \mathscr{P}$ with $P \subset J$. Let us write $\mathscr{P}^J$ for the set of all elements in $\mathscr{P}$ contained in J. Then the above can be estimated by

$$\begin{aligned} &\geq H_\lambda\left(f_0^{-(m_{n-1}+1)} \bigvee_{i=0}^{l_n} g^{-i}\mathscr{P}^J\right) \\ &= -\sum_{P \in \bigvee_{i=0}^{l_n} g^{-i}\mathscr{P}^J} \lambda(f_0^{-(m_{n-1}+1)}P) \log \lambda(f_0^{-(m_{n-1}+1)}P). \end{aligned}$$

Now we use that J is g-invariant and $f_0^{-(m_{n-1}+1)}(A) = f^{-n}(A)$ for any $A \subset J$ and $n \geq 1$. Moreover, we use that $f^{-1}(x) = (1/2)(x - (1/3)) + (2/3)$ on J. Together with the fact that g^{-1} is trivial on J^+, this gives

$$\begin{aligned} H_\lambda\left(\bigvee_{i=0}^{m_n} f_0^{-i}\mathscr{P}\right) &\geq -\left(\# \bigvee_{i=0}^{l_n} g^{-i}\mathscr{P}^J\right) \frac{1}{3^{l_n}2^n 3k} \log \frac{1}{3^{l_n}2^n 3k} \\ &= \log\left(3^{l_n}2^n 3k\right) = l_n \log(3) + n\log(2) + \log(3k). \end{aligned}$$

Dividing by m_n and sending n to infinity, gives log(3), since

$$\frac{m_n - m_{n-1} - 1}{m_n} = 1 - 2^{-2n-1} - \frac{1}{2^{n^2}} \to 1,$$

and $n/m_n \to 0$. Writing λ_∞ for the sequence $\lambda_n := (f_0^n)_*\lambda$, we obtain

$$h_{\mathscr{E}_M}(f_\infty; \lambda_\infty) \geq \log(3).$$

Since $L = 3$ is a Lipschitz constant for both f and g, Proposition 3 yields

$$\log(3) \leq h_{\mathscr{E}_M}(f_\infty; \lambda_\infty) \leq h_{\text{top}}(f_\infty) \leq \log(3),$$

implying that for f_∞ a full variational principle is satisfied with λ_∞ being an IMS of maximal entropy.

Remark 1 It is easy to see that every trajectory $\{f_0^n(x)\}_{n\in\mathbb{Z}_+}$ with $x \neq 1$ converges to 0. Hence, the example shows that both the measure-theoretic and the topological entropy can capture transient chaotic behavior, which is not seen in the asymptotic behavior of trajectories.

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A Proposal for an Application of a Max-Type Difference Equation to Epilepsy

David M. Chan, Candace M. Kent, Vlajko Kocić and Stevo Stević

Abstract We propose, *for the sake of dialogue*, that the nonautomomous reciprocal max-type difference equation,

$$x_{n+1} = \max\left\{\frac{A_n^{(0)}}{x_n}, \frac{A_n^{(1)}}{x_{n-1}}, \ldots, \frac{A_n^{(k)}}{x_{n-k}}\right\}, \quad n = 0, 1, \ldots,$$

where the parameters are positive periodic sequences and the initial conditions are positive, when $k = 1$ may serve as a *phenomenological model* of seizure activity as occurs in *mesial (or middle) temporal lobe epilepsy*.

Keywords Max-type difference equation · Heaviside function · Unbounded · Non-persistent · Temporal lobe epilepsy

1 Introduction

Difference equations with the maximum function, unlike differential equations with the maximum function, have up to now no known applications. The use of the maximum function with differential equations made its debut as early as the 1960s and

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such differential equations were of the form

$$x' = f\left(t, x(t), \max_{s\in S(t)} x(s)\right), \quad t \in [a, b),$$

where $x \in \mathbb{R}^n$, $a \geq 0$, $b \leq \infty$, $S(t) \in [\sigma(t), \tau(t)]$ with $\sigma, \tau : \mathbb{R} \to \mathbb{R}$. These differential equations were constructed with particular applications in mind such as automatic control, optimal control theory, and vision to name a few (see [3] and the references therein).

We propose, *for the sake of dialogue*, that the nonautomomous reciprocal max-type difference equation,

$$x_{n+1} = \max\left\{\frac{A_n^{(0)}}{x_n}, \frac{A_n^{(1)}}{x_{n-1}}, \dots, \frac{A_n^{(k)}}{x_{n-k}}\right\}, \quad n = 0, 1, \dots, \tag{1}$$

where the parameters $\{A_n^{(i)}\}_{n=0}^{\infty}$ are positive periodic sequences with periods $p_i \in \{1, 2, \dots\}$ and the initial conditions are positive, when $k = 1$ may serve as a *phenomenological model* of seizure activity as occurs in *mesial (or middle) temporal lobe epilepsy*.

When $k = 1$, Eq. (1) is written as

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-1}}\right\}, \quad n = 0, 1, \dots. \tag{2}$$

A phenomenological model is one that "does not follow directly from theory" and one whose variables do not directly represent measurable quantities, but rather is one which attempts to "describe the empirical relationship of phenomena" (–Wikipedia).

The *focus*, or starting point, of a seizure in mesial temporal lobe epilepsy exists in the middle structures of the temporal lobe, which include what is called the *hippocampus* (bearing resemblance to a seahorse). The hippocampus is involved in processing short- and long-term memory.

We ask the following questions.

Why use a *difference equation* at all to model particular physiological processes of the human brain?

Human brain function, from that which is physiological to that which is higher-level cognitive, can be viewed as *recursive* in nature. See, for example, the book by Corballis [7].

Why use the *max-type* difference equation, Eq. (1), to model normal functions of the human brain (e.g., the formation of memory) and abnormal function of the human brain (specifically, seizures)?

The behavior of Eq. (1) is such that either every positive solution is eventually periodic or every positive solution is unbounded (i.e., there exists a subsequence that diverges to $+\infty$) and does not persist (i.e. there exists a subsequence that converges to 0) (see [4, 9]). Afterall, under normal conditions, aggregates of neurons in the human brain undergo oscillatory electrochemical activity. Broadly speaking, epilepsy is characterized by seemingly spontaneously occurring and intermittent seizures, which, in turn, are characterized by a rapid spreading of hyperexcitability (i.e., being easily excited) of neurons and hypersynchronization (i.e., oscillations in synchrony) of activated neurons in certain regions of the brain, depending on the type of epilepsy.

Finally, why use Eq. (2) to model seizure activity in mesial temporal lobe epilepsy (where Eq. (2) is Eq. (1) for $k = 1$)?

According to the papers by Cranston and Kent [9] and Kent and Radin [17], the following can be said:

1. The smaller the delay k is in Eq. (1), the fewer the number of conditions on the parameters there are that need to be satisfied in order for there to be unboundedness and non-persistence of every solution of Eq. (1). Thus, the smaller the delay k is, the "easier" it is for Eq. (1) to have every solution unbounded and not persist.
2. The larger the delay k is in Eq. (1), the more sparsely distributed those periods of the parameters are that need to be avoided in order for there to be boundedness and persistence of every solution of Eq. (1). Thus, the larger the delay k is, the "easier" it is for Eq. (1) to have every solution bounded and persist and so be eventually periodic.

2 Epilepsy: A Brief Synopsis

Epilepsy is actually a constellation of syndromes that are chronic and have in common *spontaneous seizures* or *unprovoked recurrent seizures* (see [1, 5]). Lay persons often use the term *convulsion* instead of seizure; but a convulsion is a specific type of seizure, namely, one with a motor (i.e., movements) component.

Besides being characterized by recurrent seizures, epilepsy carries with it social and psychological problems, and is often co-morbid with other medical conditions (see, for example, [6]). Therefore, the most efficacious treatment is not simply with anti-seizure drugs, also known as *antiepileptic drugs* (AEDs) (see Chap. 1 in [10] and Chap. 121 in [11]), but drugs combined with a social support system and psychotherapy.

We present the highlights of a *1981 International League Against Epilepsy* (1981 ILAE) classification scheme of seizure types (see Chap. 44 in [10]). Note that this 1981 classification was modified in 2001.

1. **Partial Seizures**. (These are called *focal or local seizures* in the 2001 classification.) These seizures begin in one of the cerebral hemispheres of the brain, but may eventually spread to both hemispheres. When consciousness is not lost, a partial seizure is referred to as *simple*. When consciousness is lost, it is referred to as *complex*, and is more likely to involve eventually both cerebral hemispheres. Partial seizures include motor symptoms; somatosensory symptoms (e.g., auditory and visual hallucinations); autonomic symptoms (e.g., pallor, sweating); and psychic symptoms (e.g., impairment of higher cerebral function).

2. **Generalized Seizures**. These seizures begin in both cerebral hemispheres and are widespread. Consciousness is frequently lost. A typical example of generalized seizures is *petit mal or absence seizures*, in which the epileptic individual loses consciousness for a few seconds to half a minute, and appears as if he or she is day dreaming.
3. **Status Epilepticus**. These seizures last for a long period of time and are closely repetitive, with no recovery between seizures. Status epilepticus may be partial or generalized, and can be fatal without intervention.

The state of a seizure also provides another classification of seizures [1]: the *ictal state*, which is when the seizure is actually occurring; and the *interictal state*, the period between seizures, which includes the recovery time. There is also the *pre-ictal state* and the *post-ictal state*, whose details we omit.

Mesial temporal lobe epilepsy (MTLE), some aspects of which we will later propose as being successfully represented by a modified version of Eq. (2), is one of two forms of *temporal lobe epilepsy* (TLE), and involves the hippocampus, with or without sclerosis (i.e., scarring), which is located in the middle part of the temporal lobe [24]. The hippocampus is a part of the brain that resembles a seahorse and plays an important role in memory processing, as alluded to earlier. In particular, the hippocampus is important in synaptic plasticity (i.e., growth of neurons in making connections with other neurons), long-term memory formation, and maintenance of visuospatial working memory (i.e., the short-term maintenance and manipulation of visual sensory input), sometimes performing these functions in connection with such structures as the prefrontal cortex (i.e., the seat of impulse control and judgment, among other functions) (see [2, 8, 12, 14, 18, 19, 21]).

MTLE also involves other structures that are part of the limbic system (the limbic system has to do with human emotions, as well as other functions) [24]. Note that the other form of TLE, referred to as *neocortical temporal lobe epilepsy*, has its seizure focus located along the side or base of the temporal lobe [24]. TLE in general is the most common form of epilepsy characterized by partial seizures [24].

3 Equation (2): Boundedness and Persistence Results

We begin by covering some sufficient conditions on the unboundedness and non-persistence of Eq. (2).

Our first result is by Kent and Radin [17] and is on sufficient conditions on the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ for every positive solution $\{x_n\}_{n=0}^{\infty}$ of Eq. (2) to be unbounded and not to persist, with $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ *not necessarily periodic*.

Theorem 1 [17] (Unbounded and Non-Persistent Solution) *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq. (2). Suppose that there exists* $i \in \{0, 1, 2\}$ *such that*

$$\sup\{A_{3n+4+i} : n = 0, 1, \ldots\} < \inf\{B_{3n+3+i} : n = 0, 1, \ldots\},$$

and

$$\sup\{B_{3n+4+i} : n = 0, 1, \ldots\} < \inf\{A_{3n+2+i} : n = 0, 1, \ldots\}.$$

Then $\{x_n\}_{n=-1}^{\infty}$ *is unbounded and does not persist. In particular,* $\{x_n\}_{n=-1}^{\infty}$ *consists of the following three convergent and divergent subsequences:*

$$\lim_{n\to\infty} x_{3n+2+i} = 0,$$
$$\lim_{n\to\infty} x_{3n+3+i} = +\infty,$$
$$\lim_{n\to\infty} x_{3n+4+i} = +\infty,$$

with the subsequence $\{x_{3n+2+i}\}_{n=0}^{\infty}$ strictly decreasing to 0.

The next two results are also on sufficient conditions for every positive solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (2) to be unbounded and not to persist, only with these results $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are positive periodic sequences with period $p \in \{1, 2, \ldots\}$ and $q \in \{1, 2, \ldots\}$, respectively.

Corollary 1 [17] (Period of $\{A_n\}_{n=0}^{\infty}$ a Multiple of Three) *In Eq. (2), let* $\{A_n\}_{n=0}^{\infty}$ *be a positive periodic sequence with period* $p = 3k$, $k \in \{1, 2, \ldots\}$*, and let* $\{B_n\}_{n=0}^{\infty}$ *be a positive periodic sequence with period* $q \in \{1, 2, \ldots\}$ *such that for some* $i \in \{0, 1, 2\}$,

$$A_{3j+4+i} < B_0, B_1, \ldots, B_{q-1} < A_{3j+2+i},$$

for all $j = 0, 1, \ldots, k-1$. *Then every positive solution of Eq. (2) is unbounded and does not persist.*

Corollary 2 [17] (Period of $\{B_n\}_{n=0}^{\infty}$ a Multiple of Three) *In Eq. (2), let* $\{B_n\}_{n=0}^{\infty}$ *be a positive periodic sequence with period* $q = 3\ell$, $\ell \in \{1, 2, \ldots\}$*, and let* $\{A_n\}_{n=0}^{\infty}$ *be a positive periodic sequence with period* $p \in \{1, 2, \ldots\}$ *such that for some* $i \in \{0, 1, 2\}$,

$$B_{3j+4+i} < A_0, A_1, \ldots, A_{p-1} < B_{3j+3+i},$$

for all $j = 0, 1, \ldots, \ell - 1$. *Then every positive solution of Eq. (2) is unbounded and does not persist.*

Our last three results are on sufficient conditions for every positive solution of Eq. (1) or (2) to be bounded and persist (and therefore to be eventually periodic).

Theorem 2 [17] (Bounded and Persistent Solutions) *In Eq. (2), let* $\{A_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=0}^{\infty}$ *be positive periodic sequences with periods* $p \in \{1, 2, \ldots\}$ *and* $q \in \{1, 2, \ldots\}$*, respectively. Suppose that neither* p *nor* q *is a multiple of three. Then every positive solution of Eq. (2) is bounded and persists.*

We generalize.

Theorem 3 [9] (Boundedness and Persistence) *In Eq. (1), let* $\{A_n^{(i)}\}_{n=0}^{\infty}$ *be positive periodic sequences with periods* $p_i \in \{1, 2, \ldots\}$. *Suppose that there exists* $i \in \{1, 2, \ldots, k\}$ *such that the greatest common divisor* $\gcd(k+2, p_i p_{k-i}) = 1$. *Then every solution of Eq. (1) is bounded and persists.*

What will be specific to our model is the following.

Theorem 4 [16] *In Eq. (2), let* $\{A_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=0}^{\infty}$ *be positive periodic sequences with period* 2. *Then every solution of Eq. (2) is bounded and persists, and, in particular, is eventually periodic with period* 4.

4 A Proposal for a Model of Seizure Activity in MTLE

We propose, for the sake of dialogue, that a *modified version* of Eq. (2) serve as a phenomenological model of seizure activity in MTLE.

The modified version of Eq. (2) is the difference equation

$$x_{n+1} = \max\left\{\frac{A_n + D_n \cdot H(x_n)}{x_n}, \frac{B_n}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots, \tag{3}$$

where $H(x)$ is the *Heaviside function*,

$$H(x) = \begin{cases} 0, & 0 \le x < \varepsilon, \\ 1, & x \ge \varepsilon, \end{cases} \tag{4}$$

with ε sufficiently small (i.e., $\varepsilon \le 1$), and where the periodic parameters $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$, and $\{D_n\}_{n=0}^{\infty}$ will be specified below such that $\{A_n + D_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfy conditions for unboundedness and non-persistence and $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfy conditions for boundedness and persistence. Note that the role of the Heaviside function is to halt temporarily the ongoing unbounded and non-persistent behavior of solutions of Eq. (3). Motivated by numerical experiments, we call ε in Eq. (4) the *seizure threshold*. The state variable, x_n, represents the *density of activated neurons* in the middle of the temporal lobe, the region of the brain where seizures occur in MTLE. The magnitudes of the three parameters, $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$, and$\{D_n\}_{n=0}^{\infty}$, represent the *degree to which neurons are inherently hyperexcitable*.

An *activated neuron* is a neuron in a *depolarized state*, i.e., the voltage gradient across the membrane of the neuron is positive, compared to when the neuron is at rest and the voltage gradient across its membrane is negative. Activation is always followed immediately by *deactivation*, where the neuron ends up in a *hyperpolarized state*, i.e., the voltage gradient across the membrane is negative and the neuron is rendered temporarily refractory to further activation by incoming or afferent neurons.

The hallmarks of a seizure are *hyperexcitability* (easy activation of neurons) and *hypersynchronization* (synchronization of activation and deactivation of neurons, with an increase of synchronization over time). The model, Eq. (3), when its parameters satisfy the sufficient conditions for unboundedness and non-persistence, has these hallmarks: there are the three subsequences, $\{x_{3n+2+i}\}_{n=0}^{\infty}$, $\{x_{3n+3+i}\}_{n=0}^{\infty}$, and $\{x_{3n+4+i}\}_{n=0}^{\infty}$, of the solution $\{x_n\}_{n=0}^{\infty}$, where, as mentioned before in Theorem 1, two of which are tending to $+\infty$ (representing hyperexcitability and increasing hypersynchronization of activation), and one of which is strictly decreasing to 0 (representing increasing hypersynchronization of deactivation).

We state a definition which will play an important part in Sects. 5 and 6 on symbolic and numerical simulations.

Definition 1 *(Ictal State)* We define the *period of time of duration of a seizure generated by the model, Eq. (3)*, i.e., from onset to termination, as that period characterized by the following:

(i) The subsequence $\{x_{3n+2+i}\}_{n=0}^{\infty}$ is *strictly* decreasing to 0.
(ii) The terms of the three subsequences $\{x_{3n+2+i}\}_{n=0}^{\infty}$, $\{x_{3n+3+i}\}_{n=0}^{\infty}$, and $\{x_{3n+4+i}\}_{n=0}^{\infty}$ satisfy the *strict* inequality

$$x_{3n+2+i} < x_{3n+3+i}, x_{3n+4+i}.$$

Remark 1 Note that discontinuation of ongoing strict decrease to 0 with the subsequence $\{x_{3n+2+i}\}_{n=0}^{\infty}$ leads to the discontinuation of ongoing unbounded growth of the two subsequences $\{x_{3n+3+i}\}_{n=0}^{\infty}$ and $\{x_{3n+4+i}\}_{n=0}^{\infty}$.

We next list conditions that we place on the parameters of our model, which are sufficient for boundedness and persistence when $H(x_n) = 0$, and that are sufficient for unboundedness and non-persistence when $H(x_n) = 1$. (Note that, for convenience, we are going to let $i = 0$ in the three subsequences $\{x_{3n+2+i}\}_{n=0}^{\infty}$, $\{x_{3n+3+i}\}_{n=0}^{\infty}$, and $\{x_{3n+4+i}\}_{n=0}^{\infty}$.)

(C1) $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are both periodic with period 2. In this case, every positive solution of

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots$$

(i.e., when $H(x_n) = 0$ in Eq. (3)), is bounded and persists, and, in fact, is periodic with period 4. So, we expect that a seizure will terminate when $H(x_n) = 0$.

(C2) $\{D_n\}_{n=0}^{\infty}$ is periodic with period 3. Thus, $\{A_n + D_n\}_{n=0}^{\infty}$ (when $H(x_n) = 1$) is periodic with period 6, which is a multiple of 3. Note that for $m \geq 0$,

$$A_n + D_n = \begin{cases} A_0 + D_0, \; n = 6m, \\ A_1 + D_1, \; n = 6m+1, \\ A_0 + D_2, \; n = 6m+2, \\ A_1 + D_0, \; n = 6m+3, \\ A_0 + D_1, \; n = 6m+4, \\ A_1 + D_2, \; n = 6m+5. \end{cases}$$

This is one half of the sufficient conditions for unboundedness and non-persistence taken from Corollary 1.

(C3) The two sets of inequalities

$$\begin{aligned} A_0 + D_1 &< B_0, B_1 < A_0 + D_2, \\ A_1 + D_1 &< B_0, B_1 < A_1 + D_2 \end{aligned} \tag{5}$$

hold. This is the second half of the sufficient conditions for unboundedness and non-persistence taken from Corollary 1. Observe that these conditions imply further that

$$A_0, A_1, D_1 < B_0, B_1. \tag{6}$$

It should be reiterated that Conditions (C2) and (C3) generate the three subsequences $\{x_{3n+2}\}_{n=0}^{\infty}$, $\{x_{3n+3}\}_{n=0}^{\infty}$, and $\{x_{3n+4}\}_{n=0}^{\infty}$ such that

$$x_{3n+2} \downarrow 0, \quad x_{3n+3} \to +\infty, \quad x_{3n+4} \to +\infty$$

as $n \to \infty$ (as long as $H(x_n) = 1$).

5 Symbolic Simulation of Seizure Activity with Eq. (3)

We now exemplify how the model (Eq. (3)) works by symbolically simulating the occurrence of a seizure and its termination under a particular scenario of the following conditions (so that we can make certain assumptions in the computation of terms of the solution of Eq. (3)): we will assume that $\varepsilon \ll 1$, $A_0, A_1 > 1$, $B_0, B_1 > 1$, and $D_0, D_1, D_2 > 1$, and the inequalities in Eqs. (5) and (6) are close.

1. To begin the simulation of a seizure, we choose any initial values greater than ε, and the resulting solution of Eq. (3) will begin its unbounded growth and its non-persistent decay. However, because there is one subsequence, $\{x_{3n+2}\}_{n=0}^{\infty}$, that tends to 0, there must exist $n_0 \geq 0$ such that

$$x_{n_0+1} < \varepsilon,$$

where $x_{n_0+1} \in \{x_{3n+2}\}_{n=0}^{\infty}$ and where

$$x_{n_0+1} = \max\left\{\frac{A_{n_0} + D_{n_0}}{x_{n_0}}, \frac{B_{n_0}}{x_{n_0-1}}\right\}.$$

Observe that $H(x_{n_0}) = 1$, since x_{n_0+1} is extremely small (i.e., $x_{n_0+1} < \varepsilon \ll 1$) so that x_{n_0} (and x_{n_0-1}) must be extremely large (i.e., $x_{n_0}, x_{n_0-1} \gg 1 > \varepsilon$). Let $x_{n_0+1} \doteq \alpha > \varepsilon$.

2. We next have

$$x_{n_0+2} = \max\left\{\frac{A_{n_0+1}}{\alpha}, \frac{B_{n_0+1}}{x_{n_0}}\right\} = \frac{A_{n_0+1}}{\alpha} \quad \left(< \frac{A_{n_0+1} + D_{n_0+1}}{\alpha}\right),$$

since α is extremely small, x_{n_0} is extremely large, and A_0, A_1 are close in value to B_0, B_1. Note that $H(\alpha) = 0$, where $\alpha < \varepsilon$. Also notice that because

$$\frac{A_{n_0+1}}{\alpha} < \frac{A_{n_0+1} + D_{n_0+1}}{\alpha},$$

it appears as though ongoing unboundedness may be nearing a halt.

3. Then

$$x_{n_0+3} = \max\left\{\frac{A_{n_0} + D_{n_0+2}}{A_{n_0+1}/\alpha}, \frac{B_{n_0}}{\alpha}\right\} = \frac{B_{n_0}}{\alpha},$$

since $A_{n_0+1} > 1$ and $\alpha \ll 1$ so that we can assume that $\frac{A_{n_0+1}}{\alpha}$ is much larger than α and that this difference outweighs what difference there may be between $A_{n_0} + D_{n_0+2}$ and B_{n_0}. Note that $H\left(\frac{A_{n_0+1}}{\alpha}\right) = 1$, where $\frac{A_{n_0+1}}{\alpha} \gg 1 \gg \varepsilon$. The result here is something that we could have obtained without the Heaviside function incorporated in the model. In other words, here there is no indication that unboundedness is nearing a halt.

4. Finally,

$$x_{n_0+4} = \max\left\{\frac{A_{n_0+1} + D_{n_0}}{B_{n_0}/\alpha}, \frac{B_{n_0+1}}{A_{n_0+1}/\alpha}\right\} \geq \frac{B_{n_0+1}}{A_{n_0+1}/\alpha} > \alpha,$$

where the last inequality is justified by Eq. (6). Note that $H\left(\frac{B_{n_0}}{\alpha}\right) = 1$. Since x_{n_0+4} is a term in the subsequence $\{x_{3n+2}\}_{n=0}^{\infty}$, which should be *strictly* decreasing to 0 if non-persistence is occurring, we may say that ongoing non-persistence has come to a halt, where we have $x_{n_0+4} > \alpha$. Let $x_{n_0+4} \doteq \beta > \alpha$. We then have one of two situations:

Case 1. $x_{n_0+4} = \beta < \varepsilon$ and $H(\beta) = 0$ with the result that

$$x_{n_0+5} = \max\left\{\frac{A_{n_0}}{\beta}, \frac{B_{n_0}}{B_{n_0}/\alpha}\right\} = \max\left\{\frac{A_{n_0}}{\beta}, \alpha\right\} = \frac{A_{n_0}}{\beta} \quad \left(\ll \frac{A_{n_0} + D_{n_0+1}}{< \alpha}\right).$$

Therefore, with x_{n_0+4} and x_{n_0+5}, we no longer have ongoing non-persistence or ongoing unboundedness. So, by Definition 1, we can say that we are simulating being in an *interictal state* (i.e., the state between seizures) and may remain in such a state or eventually have another seizure.

Case 2. $x_{n_0+4} = \beta > \varepsilon$ and $H(\beta) = 1$, with the result that

$$x_{n_0+5} = \max\left\{\frac{A_{n_0} + D_{n_0+1}}{\beta}, \frac{B_{n_0}}{B_{n_0}/\alpha}\right\} = \max\left\{\frac{A_{n_0} + D_{n_0+1}}{\beta}, \alpha\right\} = \frac{A_{n_0} + D_{n_0+1}}{\beta} \quad \left(\ll \frac{A_{n_0} + D_{n_0+1}}{< \alpha}\right),$$

where we have that α is extremely small, β is still small, and $A_{n_0} + D_{n_0+1} > 1$, so that we can assume that $\frac{A_{n_0} + D_{n_0+1}}{\beta} > \alpha$. Therefore, again, with x_{n_0+4} and x_{n_0+5}, we can say that ongoing non-persistence and unboundedness stop, but only for a moment, for we also have $x_{n_0+4}, x_{n_0+5} > \varepsilon$. Hence, there is a short-lived interictal period. If the interictal period is almost nonexistent with a second seizure developing almost immediately, we can say that we probably are at the beginning of *status epilepticus* (i.e., the repetitive occurrence of seizures with no opportunity to recover fully between them).

6 Numerical Simulations of Seizure Activity

In the following simulations we study (1) the relative duration of seizures, and (2) the relative frequency of seizures as they correlate with the magnitudes of the parameters $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$, and $\{D_n\}_{n=0}^{\infty}$. See Figs. 1, 2, 3, 4 and 5 at the end of this section. Note the following, however: Figs. 1, 2, 3, 4 and 5 only roughly demonstrate the salient observations referred to below. All observations in this section are actually based on the multitude of data points which gave rise to Figs. 1, 2, 3, 4 and 5.

The numerical simulations are created using Eq. (3) with the Heaviside function defined by Eq. (4). In each figure we have the following:

1. The initial values are $x_{-1} = 0.5$ and $x_0 = 1.5$.
2. $\varepsilon = 1$.

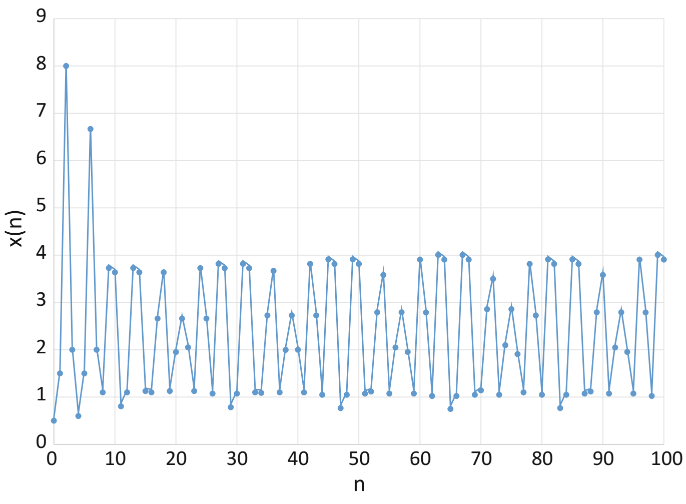

Fig. 1 Time series plot x_n versus n. $A_0 = 0.1$, $A_1 = 0.2$, $B_0 = 3$, $B_1 = 4$, $D_0 = 1$, $D_1 = 2$, and $D_2 = 4$

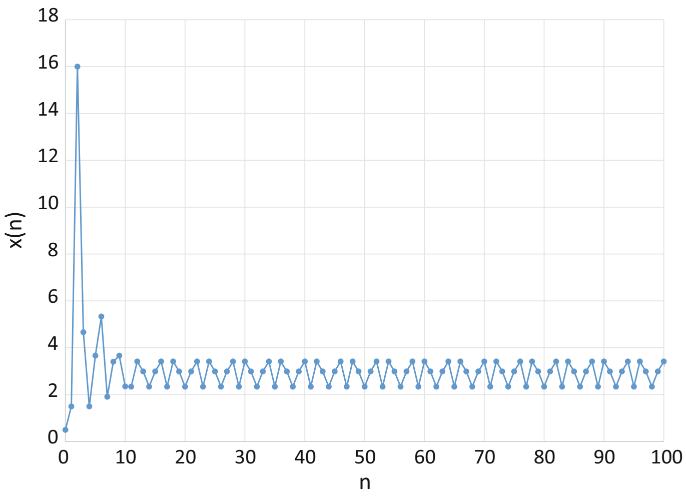

Fig. 2 Time series plot x_n versus n. $A_0 = 5$, $A_1 = 6$, $B_0 = 7$, $B_1 = 8$, $D_0 = 1$, $D_1 = 0.5$, and $D_2 = 4$

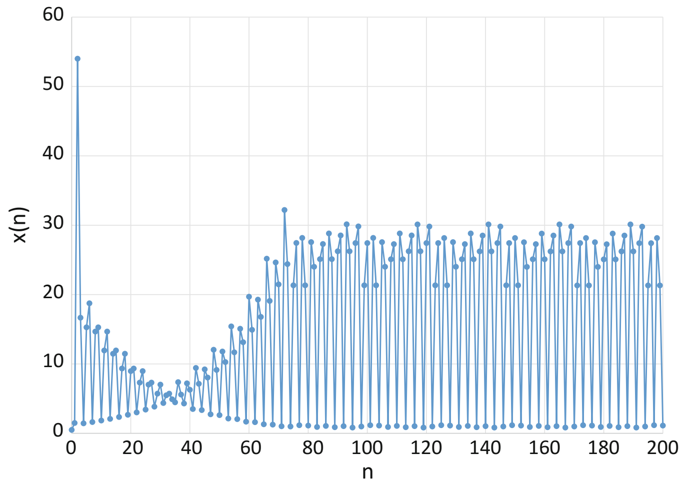

Fig. 3 Time series plot x_n versus n. $A_0 = 21$, $A_1 = 23$, $B_0 = 25$, $B_1 = 27$, $D_0 = 1$, $D_1 = 1$, and $D_2 = 10$

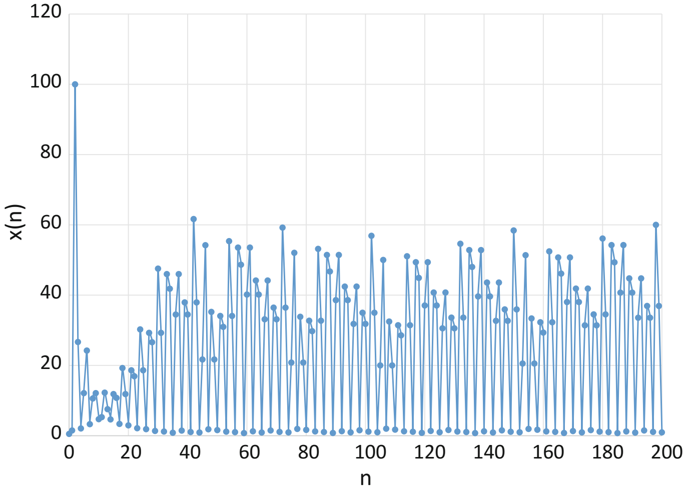

Fig. 4 Time series plot x_n versus n. $A_0 = 20$, $A_1 = 30$, $B_0 = 40$, $B_1 = 50$, $D_0 = 25$, $D_1 = 5$, and $D_2 = 35$

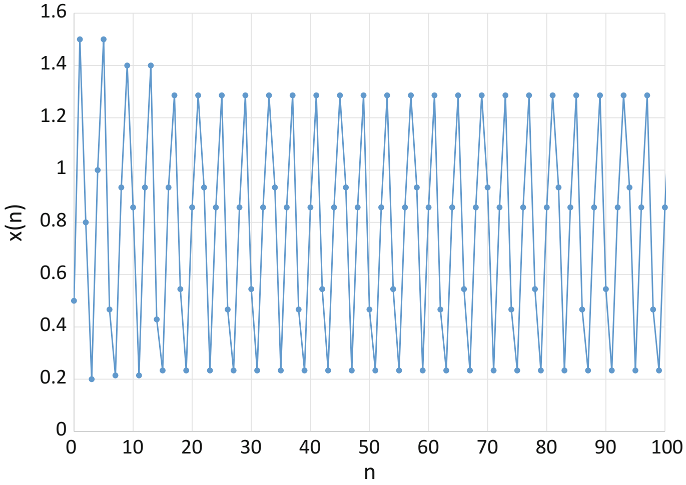

Fig. 5 Time series plot x_n versus n. $A_0 = 0.1$, $A_1 = 0.2$, $B_0 = 0.3$, $B_1 = 0.4$, $D_0 = 1$, $D_1 = 0.05$, and $D_2 = 0.5$

3. In each figure,
   (a) the subsequence $\{x_{3n+2}\}_{n=0}^{\infty}$ consists of the terms $x_2, x_5, x_8, \ldots$ of the solution $\{x_n\}_{n=0}^{\infty}$;
   (b) the subsequence $\{x_{3n+3}\}_{n=0}^{\infty}$ consists of the terms $x_3, x_6, x_9, \ldots$ of the solution $\{x_n\}_{n=0}^{\infty}$;
   (c) the subsequence $\{x_{3n+4}\}_{n=0}^{\infty}$ consists of the terms $x_4, x_7, x_{10}, \ldots$ of the solution $\{x_n\}_{n=0}^{\infty}$.

We do not make specific measurements of seizure durations in time and seizure frequencies of occurrence, but instead discuss relative seizure durations and frequencies as we *compare* Figs. 1, 2, 3, 4 and 5. We summarize our findings with each figure:

Figure 1. Here, $A_n \ll \varepsilon$ and $B_n, D_n > \varepsilon$. Seizure duration is relatively moderate, but seizure frequency is relatively low. Note that the solution of Eq. (3) is eventually periodic with period 36 according to the data points underlying Fig. 1. Additional symbolic hand computations indicate that, coincident with eventual periodicity of the solution, the first parameter $\{A_n + D_n \cdot H(x_n)\}_{n=0}^{\infty}$ in Eq. (3) is eventually periodic (see Theorem 5 below) (of course the second parameter $\{B_n\}_{n=0}^{\infty}$ in Eq. (3) is periodic with period 2 from the beginning of the computation of terms).

Figure 2. Here, $A_n, B_n > \varepsilon$ and $D_0 = \varepsilon$, $D_1 < \varepsilon$, and $D_2 > \varepsilon$. Seizure duration is relatively low and seizure frequency is relatively high. There is, however, no evidence of eventual periodicity of the solution of Eq. (3) from the data points underlying Fig. 2, although Fig. 2 seems to suggest an "almost eventual periodicity" of the solution.

Figure 3. Here, $A_n, B_n \gg \varepsilon$ and $D_n \geq \varepsilon$. Seizure duration and seizure frequency are both relatively high. Figure 3 itself appears bounded from above and bounded away from 0, and so we may say that the solution of Eq. (3) is perhaps bounded and persistent (see Lemma 1).

Figure 4. $A_n, B_n, D_n \ll \varepsilon$. Again, seizure duration and seizure frequency are both relatively high. Figure 4 itself appears bounded from above and bounded away from 0, and so we may say that the solution of Eq. (3) is perhaps bounded and persistent (see Lemma 1).

Figure 5. Here, $A_n, B_n \ll \varepsilon$ and $D_0 = \varepsilon$, $D_1 \ll \varepsilon$, and $D_2 < \varepsilon$, and, in this case, by Definition 1, there is no seizure activity according to the data points underlying Fig. 5. In fact, according to the data points, the solution is eventually periodic with period 4 and shows no ongoing unboundedness and ongoing non-persistence. Indeed, this behavior is characteristic of Eq. (2),

$$x_{n+1} = \max\left\{\frac{A_n + D_n \cdot 0}{x_n}, \frac{B_n}{x_{n-1}}\right\},$$

and is in accordance with Theorem 4.

The observations of Figs. 1, 2, 3, 4 and 5 along with their associated data points suggest that the greater the magnitudes of the two parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ compared to the seizure threshold $\varepsilon = 1$, the greater the seizure duration and seizure frequency. Also suggested is that if the three parameters $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$, and $\{D_n\}_{n=0}^{\infty}$ are less than the seizure threshold $\varepsilon = 1$, then there will be no seizure activity.

Figures 1, 2, 3, 4 and 5 and the data points associated with them additionally indicate that the solutions of Eq. (3) are bounded and persist, and that furthermore under certain circumstances the solutions are eventually periodic. Related to this latter tentative conclusion, we have the following lemma and theorem:

Lemma 1 *Consider Eq. (3) and suppose that the following hypotheses hold:*

(H1) $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ *are both positive periodic sequences with period* 2.
(H2) $\{D_n\}_{n=0}^{\infty}$ *is a positive periodic sequence with period* 3.
(H3) The inequalities in Eq. (5) hold.
(H4) $B_0, B_1 > \varepsilon$.
(H5) $\varepsilon \le 1$.

Then every positive solution of Eq. (3) is bounded and persists.

Proof We first prove a result that is a byproduct of the symbolic computations in Sect. 5; namely, if $x_{i-2} \in \{x_{3n+2}\}_{n=0}^{\infty}$, $x_{i-1} \in \{x_{3n+3}\}_{n=0}^{\infty}$, $x_i \in \{x_{3n+4}\}_{n=0}^{\infty}$, and $x_{i+1} \in \{x_{3n+2}\}_{n=0}^{\infty}$, where $x_{i-2}, x_{i-1}, x_i > \varepsilon$ and $x_{i+1} = \alpha < \varepsilon$, then $x_{i+4} \in \{x_{3n+2}\}_{n+0}^{\infty}$ and $x_{i+4} > x_{i+1} = \alpha$:

Specifically, let $x_{i-2} \in \{x_{3n+2}\}_{n=0}^{\infty}$ and $x_{i-2} > \varepsilon$ and let

$$x_{i+1} = \max\left\{\frac{A_i + D_i}{x_i}, \frac{B_i}{x_{i-1}}\right\},$$

where x_{i-1}, x_i are relatively large and greater than ε and x_{i+1} is relatively small and less than ε. Set $x_{i+1} = \alpha < \varepsilon$. Then

$$x_{i+2} = \max\left\{\frac{A_{i+1}}{\alpha}, \frac{B_{i+1}}{x_i}\right\} = \frac{A_{i+1}}{\alpha} \text{ or } \frac{B_{i+1}}{x_i},$$

where we may or may not have $\dfrac{A_{i+1}}{\alpha} > \varepsilon$ or $\dfrac{B_{i+1}}{x_i} > \varepsilon$. Next $x_{i+3} \in \{x_{3n+4}\}_{n=0}^{\infty}$ and $x_{i+3} > \varepsilon$. Finally, either

$$x_{i+4} = \max\left\{\frac{A_{i+1} + D_i}{x_{i+3}}, \frac{B_{i+1}}{A_{i+1}/\alpha}\right\} \ge \frac{B_{i+1}}{A_{i+1}/\alpha} > \alpha,$$

where $B_{i+1} > A_{i+1}$, or

$$x_{i+4} = \max\left\{\frac{A_{i+1} + D_i}{x_{i+3}}, \frac{B_{i+1}}{B_{i+1}/x_i}\right\} \ge \frac{B_{i+1}}{B_{i+1}/x_i} = x_i > \varepsilon > \alpha.$$

Thus, $x_{i+4} \in \{x_{3n+2}\}_{n=0}^{\infty}$ such that $x_{i+4} > x_{i+1}$.

Based upon the above finding, we look for the "smallest α" that can be attained such that every solution is bounded away from 0 by this "smallest α." Indeed, there exists $m > 0$ defined by

$$m = \frac{\min\{A_0 + D_1, A_1 + D_1, A_0 + D_0, A_1 + D_0\}}{\frac{1}{\varepsilon} \cdot \max\{A_0 + D_2, A_1 + D_2, A_0 + D_0, A_1 + D_0\}},$$

which can be observed to be a *lower bound* of every positive solution of Eq. (3). Observe that $m < \varepsilon$, where m is of the form

$$m = \frac{P}{Q/\varepsilon} = \frac{P}{Q} \cdot \varepsilon, \quad \text{with } P < Q.$$

Then there exists $M > 0$ defined by

$$M = \max\left\{\frac{A_0 + D_2}{m}, \frac{A_1 + D_2}{m}\right\},$$

which can be seen to be an *upper bound* of every positive solution of Eq. (3). Observe that $M > \varepsilon$, where we have

$$m < \varepsilon \leq 1 \text{ and } A_0 + D_2, A_1 + D_2 > B_0, B_1 > \varepsilon,$$

so that

$$\frac{A_0 + D_2}{m} > \varepsilon \text{ and } \frac{A_1 + D_2}{m} > \varepsilon.$$

□

We need the boundedness and persistence of every positive solution of Eq. (3) for the following theorem on eventual periodicity.

Theorem 5 (Eventual Periodicity) *Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (3), and suppose that Hypotheses (H1)-(H5) from Lemma 1 hold. If the parameter $\{A_n + D_n \cdot H(x_n)\}_{n=0}^{\infty}$ is eventually periodic, then $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic.*

Proof The results in the paper [4] by Bidwell and Franke hold in this case, with some minor inconsequential modifications. First of all, by Lemma 1, we know that our solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (3) is bounded and persists. We then take $\{x_n\}_{n=-1}^{\infty}$ and eliminate all terms up to the point when $\{A_n + D_n \cdot H(x_n)\}_{n=0}^{\infty}$ becomes periodic to obtain the new sequence $\{\bar{x}_n\}_{n=-1}^{\infty}$.

We make a transformation to a dynamically conjugate system that Bidwell and Franke refer to as the *log version*: Let

$$y_n = \ln(\bar{x}_n);$$

and starting with where $\{A_n + D_n \cdot H(x_n)\}_{n=0}^{\infty}$ becomes periodic, we define the parameters

$$a_n = \ln\left(A_n + D_n \cdot H(x_n)\right), \quad b_n = \ln(B_n).$$

The log version of Eq. (3) is then

$$y_{n+1} = \max\left\{a_n - y_n, b_n - y_{n-1}\right\},$$

where $\{a_n\}_{n=0}^{\infty}$ is periodic with period $p \in \{1, 2, \ldots\}$ and $\{b_n\}_{n=0}^{\infty}$ is periodic with period 2.

Then THEOREM 4 in [4] applies here and $\{y_n\}_{n=-1}^{\infty}$ is eventually periodic with period JP, where $J = \text{lcm}(p, 2)$ and P is some integer defined in [4]. □

Remark 2 The observation that if the parameter $\{A_n + D_n \cdot H(x_n)\}_{n=0}^{\infty}$ of Eq. (3) is eventually periodic, then the positive solutions of Eq. (3) are eventually periodic demonstrates that our model, Eq. (3), supports the occurrence of *unprovoked recurrent seizures*, a feature of MTLE.

7 Discussion

As was mentioned in Sect. 2, the hippocampus located in the middle region of the temporal lobe is significantly involved in certain aspects of memory processing and formation.

There are numerous reports that certain chemical receptors on hippocampal neurons (neurotransmitters like glutamate, dopamine, norepinephrine, and serotonin attach themselves to chemical receptors on neurons, which result in the activation of these neurons), called *Group I metabotropic glutamate receptors types 1 and 5* (mGluR1 and mGluR5), but particularly mGluR5, in conjunction with *N-methyl-*D-*aspartate receptors* (NMDARs), play an integral part in synaptic plasticity (the formation of new connections among neurons for the storage of memory), long-term memory processes such as long-term potentiation and depression, learning, and visuospatial working memory processing and maintenance (see [13, 19, 20, 22, 23]).

In MTLE, there can be subtle cognitive deficits in the form of memory impairment as measured by psychological tests. Coincidentally, dysregulation of mGluR5 plays a strong role in MTLE [15].

With our proposed model, Eq. (3), if we look at x_{n+1} as representing a future state, x_n as representing a present state, and x_{n-1} as representing a past state, we can say that the model does not go far back into the past, relatively speaking, where the delay $k = 1$ is as small as it can be, and so the model has "little memory."

Therefore, our proposed model could, in a *phenomenonological* manner, support the idea that dysregulation of the Group I mGluRs, especially dysregulation of mGluR5, plays a salient part in the etiology of seizure activity in MTLE. Following along these lines, we will make the further suggestion put forth in the paper [23] by Purgert *et al.* that intracellular (i.e., within the neuronal cell), as against extracellular (i.e., in the outside fluid bathing the neuronal cell), mGluR5 be targeted with antagonists (i.e., chemicals that block the attachment of glutamate to mGluR5) for an antiepileptic effect.

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On the Maximum Principle for Systems with Delays

A. V. Kim, V. M. Kormyshev and A. V. Ivanov

Abstract In this article we present the Pontryagin maximum principle of a time-optimal control problem for general form of functional-differential equations. The obtained results are the direct generalization of the case for ordinary differential equations: if the delay disappear then the results turn into the classic Pontryagin maximum principle for finite dimensional systems. In this work we apply the methodology and constructions of the i-Smooth analysis.

Keywords Functional-differential equations · Optimal control · Maximum principle · i-smooth analysis

1 Introduction

The delay phenomenon plays an important role in the study of processes arising in natural science, technology and society. First of all, this is due to the fact that the future development of many processes depends not only on their present state but is essentially influenced by their previous history. Such processes can be described mathematically using the functional-differential equations (hereinafter FDE). At present FDE theory is the well developed branch of the differential equations and offenly uses in description and modeling of automatic control processes with aftereffect, mechanics, technology, economics, medicine and other areas of human activity [6, 10].

This work is devoted to establishing the necessary optimality conditions in the form of Pontryagin's maximum principle for general FDEs. The discovery of the

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famous Pontryagin maximum principle [12] started the development of the mathematical theory of optimal processes. This classic fundamental book already included a variant of the maximum principle for systems with discrete delays. The origin of the development of the theory of delayed optimal processes goes back to [7], where an analog of the Pontryagin maximum principle was proved for optimal systems with constant delays in state coordinates. The maximum principle was later proved for some classes of systems with distributed delays ([1, 2, 5, 11, 13]). However, there is no principle maximum variant for general form FDE, that is systems without a priory specification of delay types. In this work we apply i-Smooth analysis [8, 9] to obtain the Pontryagin maximum principle for general form FDEs. i-Smooth analysis allows to obtain results by using methods and arguments similar to ordinary differential equations. In our article we apply an analog of the methodology developed in [3] for deriving the Pontryagin maximum principle for finite-dimensional systems.

This article is organized as follows. In the second section, we obtain special conditions of optimality in the form of the Bellman functional by applying the i-smooth analysis. In the third section we use these relations to obtain the maximum principle for general form of FDEs.

2 Problem Statement and Preliminaries

In the article we consider a control system with delays

$$\dot{x} = f(x(t), x(t+s), u(t)), \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in R^n$, $x(t+\cdot) = \{x(t+s), -\tau \le s < 0\}$, $f(x, y(\cdot), u) : R^n \times Q[-\tau, 0) \times P \to R^n$; $Q[-\tau, 0)$ is the space of piecewise continuous n-dimensional functions $x(\cdot)$ on $[-\tau, 0)$ (right continuous at points of discontinuity) with the norm $\|x(\cdot)\|_Q = \sup_{-\tau \le t < 0} \|x(t)\|$, $P \subseteq R^r$ is a control region; $h(x, y(\cdot)) \in H = R^n \times Q[-\tau, 0)$, $x_t = \{x(t), x(t+\cdot)\} \in H$.

The problem is to find a control which transfers the system (1) from a phase (functional) state (position) $h(x, y(\cdot)) \in H$ into a given point $x^* \in R^n$. Herewith as an initial position h we will consider various points of the phase space H.

We assume that further the following condition is valid

Assumption 1. For every position $h(x, y(\cdot)) \in H$ there is the time-optimal transition process from the position h into the point x^*.

We denote by $T[x, y(\cdot)]$ the optimal transition time from the position $h(x, y(\cdot)) \in H$ into a given point x^*. For the convenience we consider the functional

$$W[x, y(\cdot)] = -T[h], \tag{2}$$

which depends on $2n$ variables

$$W[x, y(\cdot)] = W[x^1, x^2, \dots, x^n, y^1(\cdot), y^2(\cdot), \dots, y^n(\cdot)].$$

We also assume that for the considered problem the following condition is also valid

Assumption 2. The functional $W[x, y(\cdot)]$ has the following partial and invariant derivatives

$$\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \ldots, \frac{\partial W}{\partial x_n}, \quad \partial W_{y^1}, \partial W_{y^2}, \ldots, \partial W_{y^n}.$$

which are invariantly continuous in domains.

Let $h(x_0, y_0(\cdot))$ be an arbitrary point of the phase space H, and $u_o \in P$ is an arbitrary point of the control region.

Consider a process which starts at a moment t_0 from the position h_0 under the constant control $u = u_0$. Therefore the phase trajectory of the process $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ satisfies the following functional differential equation

$$\dot{x} = f(x(t), x(t + \cdot), u_0), \quad \text{for} \quad t > t_0 \tag{3}$$

and the initial condition

$$x_{t_0} = h_0. \tag{4}$$

It takes time $t - t_0$ to move along this trajectory from the point x_0 to the point $x(t)$. Applying from the moment t an optimal control we move from x_t into the terminal point x^* during the time $T[x_t]$.

Such movement from the point x_0 into the terminal point x^* takes time $(t - t_0) + T[x_t]$. Taking into account that optimal (minimal) time from the position (point) $h_0(x^0)$ is equal to $T[h_0] = T[x_{t_0}]$ we obtain the following inequality

$$T[x_{t_0}] \leq (t - t_0) + T[x_t],$$

from which (see (2)) we have

$$-W[x_{t_0}] \leq (t - t_0) - W[x_t].$$

Therefore

$$W[x_t] - W[x_{t_0}] \leq t - t_0,$$

$$\frac{W[x_t] - W[x_{t_0}]}{t - t_0} \leq 1.$$

Proceeding in the last inequality to limit as $t \to t_0$ we obtain

$$\frac{d}{dt} W[x_t]|_{t=t_0} \leq 1. \tag{5}$$

The left-hand side of the inequality (5) can be expressed in terms of the partial and the invariant derivatives, then (5) can be presented in the form

$$\frac{\partial W[x_0, y_0(\cdot)]}{\partial x} \cdot f(x_0, y_0(\cdot), u_0) + \partial W[x_0, y_0(\cdot)] \leq 1.$$

$h = \{x_0, y_0\}$ and u_0 are arbitrary elements, therefore for any position $h = \{x_0, y(\cdot)\} \in H$ and every point $u \in P$ the following relation is valid

$$\frac{\partial W[x, y(\cdot)]}{\partial x} \cdot f(x, y(\cdot), u) + \partial W[x, y(\cdot)] \leq 1. \tag{6}$$

Let $\{x(\cdot), y(\cdot)\}$ be the time-optimal process of transferring the system from the position h_0 into the point x^*, and $[t_0, t_1]$ is the corresponding time interval, therefore: $x_{t_0} = h_0$, $x_{t_1} = x_1$ and $t_1 = t_1 + T[h_0]$.

The process satisfies the equation

$$\dot{x}(t) = f(x_t, u(t)), \quad t_0 \leq t \leq t_1. \tag{7}$$

Movement along the optimal trajectory from the position $h_0(x_0, y_0(\cdot))$ to a point $x(t)$ takes $t - t_0$, and from the point $x(t)$ to the terminal point x^* the system moves during $t_1 - t$, then $T[h_0] - (t - t_0)$ is the minimal time of transferring the system from the state x_t into the point x^*, that is

$$T[x_t] = T[h_0] - (t - t_0).$$

By virtue of $T[h] = -W[h]$ we obtain

$$W[x_t] = -W[h_0] + (t - t_0),$$

$$W[x(t), x(t + \cdot)] = -W[h_0] + (t - t_0).$$

Differentiating this equality by t we obtain

$$\sum_{i=1}^{n} \frac{\partial W[x_t]}{\partial x^i} \cdot \dot{x}^i(t) + \partial_{y^0} W[x(t)] = 1.$$

Taking into account (7) we have

$$\sum_{i=1}^{n} \frac{\partial W[x_t]}{\partial x^i} \cdot f^i(x_t, u(t)) + \partial_y W[x(t)] = 1, \quad t_0 \leq t \leq t_1. \tag{8}$$

Thus for every optimal process the equality (8) is valid during the process.

Consider the functional

$$B[x, y(\cdot), u] = \sum_{i=1}^{n} \frac{\partial W[h]}{\partial x^i} \cdot f^i(h, u) \tag{9}$$

then relations (6), (8) can be presented in the following form

$$B[h,u] \leq 1, \text{ for every } h \in H \text{ and } u \in P. \tag{10}$$

$$B[h,u] = 1, \text{ along any optimal process } (x(\cdot), y(\cdot)). \tag{11}$$

Thus the following theorem is proved

Theorem 1 *If the assumptions for the control system (1) and a fixed terminal point x^* are valid, then the relations (10) and (11) take place.*

This theorem presents the essence of the dynamic programming method for systems with delays. Its main mathematical relation can be expressed in other form.

From (11) with $t = t_0$ we have $B[h_0, u(t_0)] = 1$. Taking into account (10) we obtain relation

$$\max_{u \in P} B[h,u] = 1, \quad \forall h \in H,$$

or equivalently

$$\max_{u \in P} \sum_{i=1}^{n} \frac{\partial W[h]}{\partial x^i} \cdot f^i(x, y(\cdot), u) + \partial_y W[x(t)] = 1, \quad \forall h \in H. \tag{12}$$

3 Maximum Principle

Further along with the assumptions 1,2 we suppose that the following conditions are satisfied.

Assumption 1.

- The functional $W[x, y(\cdot)]$ has invariantly continuous derivatives with respect to x^i, $i = 1, \ldots, n$, up to the second order, that is functionals

$$\frac{\partial W[h]}{\partial x^i}, \frac{\partial^2 W[h]}{\partial x^i \partial x^j}, \quad i,j = 1, \ldots, n.$$

  are invariantly continuous.
- Functionals $f^i(x, y(\cdot), u)$, $i = 1, \ldots, n$ have invariantly continuous partial derivatives

$$\frac{\partial f^i(h,u)}{\partial x^j}, \quad i,j = 1, \ldots, n.$$

Let $(x(t), u(t))$, $t_0 \leq t \leq t_1$ be the time-optimal process transferring the system (1) from the position h_0 into the terminal point x^*.

Fix a moment $t \in [t_0, t_1)$ and consider the functional $B(x, y(\cdot), u(t))$ of variables $x, y(\cdot)$.

From the definition of the functional B (see. 9) and the hypothesis 3 it follows that the functional $B(x, y(\cdot), u(t))$ has the invariantly continuous derivatives with respect to variables $x^1, x^2, \ldots, x^n$:

$$\frac{\partial B(x, y(\cdot), u(t))}{\partial x^k} = \sum_{i=1}^{n} \frac{\partial^2 W[h]}{\partial x^i \partial x^k} \cdot f^i(h, u(t)) + \sum_{i=1}^{n} \frac{\partial W[h]}{\partial x^i} \cdot \frac{\partial f^i(x, y(\cdot), u(t))}{\partial x^k}, \quad k = 1, \ldots, n. \tag{13}$$

By virtue of (10), (11) we have

$$B[h, u(t)] \leq 1, \quad \forall h \in H;$$

$$B[h, u(t)] = 1, \quad \forall h = x_t.$$

These two relations mean that the functional achieves the maximum at the element $h = x_t$.

Therefore, if we fix $x(t + \cdot)$ and $u(t)$ in the functional $B[x, x(t + \cdot), u(t)]$, and consider it as the function of x, then this function has the maximum at the point $x = x(t)$. Hence its partial derivatives with respect to $x^1, x^2, ldots, x^n$ are equals to zero at this point:

$$\sum_{i=1}^{n} \frac{\partial^2 W[x_t]}{\partial x^i \partial x^k} \cdot f^i(h, u(t)) + \sum_{i=1}^{n} \frac{\partial W[x_t]}{\partial x^i} \cdot \frac{\partial f^i(x, y(\cdot), u(t))}{\partial x^k} = 0, \quad k = 1, \ldots, n. \tag{14}$$

(see (13)).

Differentiating the function $\frac{\partial W[x_t]}{\partial x^k}$ with respect to t and taking into account (7), we find

$$\frac{d}{dt}\left(\frac{\partial W[x_t]}{\partial x}\right) = \sum_{i=1}^{n} \frac{\partial W[x_t]}{\partial x^k \partial x^i} \dot{x}^i(t) = \sum_{i=1}^{n} \frac{\partial W[x_t]}{\partial x^k \partial x^i} f^i(x_t, u(t)), \quad k = 1, \ldots, n. \tag{15}$$

Then relation (15) can be presented in the following form:

$$\frac{d}{dt}\left(\frac{\partial W[x_t]}{\partial x}\right) = \sum_{i=1}^{n} \frac{\partial W[x_t]}{\partial x^k \partial x^i} \dot{x}^i(t) = \sum_{i=1}^{n} \frac{\partial W[x_t]}{\partial x^k \partial x^i} f^i(x_t, u(t)), \quad k = 1, \ldots, n. \tag{16}$$

(note, that $\frac{\partial^2 W}{\partial x^k \partial x^i} = \frac{\partial^2 W}{\partial x^i \partial x^k}$ due to continuity of the second derivatives).

Formulas (10)–(12), and (16) do not include the functional W, but only its partial derivatives with respect to $x^1, \ldots, x^n$: $\frac{\partial W}{\partial x^1}, \ldots, \frac{\partial W}{\partial x^n}$, so, for the convenience, we will use the following notation:

$$\frac{\partial W[x_t]}{\partial x^1} = \psi_1[t], \; \frac{\partial W[x_t]}{\partial x^2} = \psi_2[t], \ldots, \frac{\partial W[x_t]}{\partial x^n} = \psi_n[t]. \tag{17}$$

Then the functional B (see (9)) can be presented in the form:

$$B[x_t, y(\cdot), u(t)] = \sum_{i=1}^{n} \psi_i[t] \cdot f^i(x_t, u(t))$$

and the relation (11) becomes

$$\sum_{i=1}^{n} \psi_i[t] \cdot f^i(x_t, u(t)) \equiv 1 \text{ for optimal process } (x(t), u(t)),\, t_0 \le t \le t_1. \quad (18)$$

Besides, according to (10)

$$\sum_{i=1}^{n} \psi_i[t] \cdot f^i(x_t, u(t)) \le 1 \text{ for every point } u \in P \text{ and all } t_0 \le t \le t_1. \quad (19)$$

Finally, relations (15) can be presented in the following form:

$$\dot{\psi}_k[t] + \sum_{i=1}^{n} \psi_i[t] \cdot \frac{f^i(x_t, u(t))}{\partial x^k} = 0, \quad k = 1, \ldots, n. \quad (20)$$

In summary, if $(x(t), u(t))$, $t_0 \le t \le t_1$ is the optimal process, then there exist functionals $\psi_1[t], \psi_2[t], \ldots, \psi_n[t]$ (defined by (16)), such that the relations are valid.

The form of the left-hand sides of (17), (18) lead us to consideration of the functional

$$H[\psi, x, y(\cdot), u] = \sum_{i=1}^{n} \psi_i \cdot f^i(x, u) = \psi_1 \cdot f^1(x, u) + \cdots + \psi_n f^n(x, u), \quad (21)$$

depending on $2n + r$ variables $\psi_1, \ldots, \psi_n, x^1, \ldots, x^n, u^1, \ldots, u^r$. In terms of this functional relations (17), (18) can be presented in the form of two following relations:

$$H[\psi[t], x_t, y(\cdot), u(t)] \equiv 1 \text{ for optimal process } (x(t), u(t)),\, t_0 \le t \le t_1, \quad (22)$$

where $\psi[t] = (\psi_1[t], \ldots, \psi_n[t])$ is defined by (16).

$$H[\psi[t], x_t, y(\cdot), u(t)] \le 1 \text{ for every point } u \in P \text{ and all } t_0 \le t \le t_1. \quad (23)$$

Relations (22) and (23) can be unified in a compact form

$$\max_{u \in P} H[\psi[t], x(t), u(t)] = H[\psi[t], x_t, u(t)], \quad t_0 \le t \le t_1. \quad (24)$$

Additionally, the relation (19) can be presented in the form:

$$\dot{\psi}_k[t] = -\frac{\partial H[\psi[t], x_t, y(\cdot), u(t)]}{\partial x^k}, \quad k = 1, \ldots, n. \tag{25}$$

Thus, if $(x(t), u(t))$, $t_0 \leq t \leq t_1$ is the optimal process, then a function $\psi[t] = (\psi_1[t], \ldots, \psi_n[t])$ exists and the relations (22), (24), (25) are valid, in which the functional H is defined by (21).

Formulas (21), (22), (24), (25) do not contain explicitly the functional $W[x, y(\cdot)]$, so equalities (17), representing the functions $\psi_1[t], \ldots, \psi_n[t]$ by the functional W, do not give us additional information and will be out of our consideration. Relation (25) is the system of equations which satisfy these functions. Note that the functions $\psi_1[t], \ldots, \psi_n[t]$ are nontrivial solutions of this system (that is the functions do not equal to zero at the same time); indeed, if at some moment t we have $\psi_1[t] = \ldots = \psi_n[t] = 0$, then from (21) we obtain $H[\psi[t], x_t, u(t)] = 0$ that contradicts to equality (22). Thus we obtain the following theorem in the form of **the maximum principle**.

Theorem 2 *Let for the control system*

$$\dot{x}(t) = f(x(t), x(t+s), u(t)), \quad u \in P, \tag{26}$$

and a terminal point x^*, assumptions 1, 2 and 3 are valid, and let $(x(t), u(t))$, $t_0 \leq t \leq t_1$ be a process transferring the system from an initial state $h_0 \in H$ into the final point x_1. Consider a functional depending on variables $x^1, \ldots, x^n$, $u^1, \ldots, u^r$ and auxiliary variables $\psi_1, \ldots, \psi_n$ (cf. (21)):

$$H[\psi, x, y(\cdot), u] = \sum_{i=1}^{n} \psi_i f^i(x, y(\cdot), u). \tag{27}$$

Consider for the auxiliary variables the system of differential equations

$$\dot{\psi}_k[t] = -\frac{\partial H[\psi[t], x_t, u(t)]}{\partial x^k}, \quad k = 1, \ldots, n, \tag{28}$$

where $(x(t), u(t))$ is the process under consideration (cf. (25)). Then, if $(x(t), u(t))$, $t_0 \leq t \leq t_1$ is the time-optimal process, then there exists nontrivial solution $\psi_1[t], \ldots, \psi_n[t]$, $t_0 \leq t \leq t_1$ of the system (28) such that for every moment $t_0 \leq t \leq t_1$ the following maximum condition

$$H[\psi[t], x_t, u(t)] = \max_{u \in P} H[\psi[t], x(t), y(\cdot), u]. \tag{29}$$

(cf. (24)) and the equality (cf. (22)

$$H[\psi[t], x_t, u(t)] = 1$$

are valid.

The Theorem 2 presents necessary conditions for optimality of systems with delays in the form of the maximum principle.

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Hyperbolicity and Solvability for Linear Systems on Time Scales

Sergey Kryzhevich

Abstract We believe that the difference between time scale systems and ordinary differential equations is not as big as people use to think. We consider linear operators that correspond to linear dynamic systems on time scales. We study solvability of these operators in $\mathbb{L}^\infty$. For ordinary differential equations such solvability is equivalent to hyperbolicity of the considered linear system. Using this approach and transformations of the time variable, we spread the concept of hyperbolicity to time scale dynamics. We provide some analogs of well-known facts of Hyperbolic Systems Theory, e.g. the Lyapunov–Perron theorem on stable manifold.

Keywords Time scale · Hyperbolicity · Solvability · Stable manifolds · Exponential dichotomy

1 Introduction

Time scale systems play an important role in modern dynamics as they stand between discrete and continuous ones. For applications, they could be used for modelling strongly nonlinear phenomena e.g. impacts. There are hundreds of books and papers, devoted to time scale dynamics (see [1–7, 9, 11–13, 15–20, 23, 24, 28, 29] and references therein, the list is still incomplete). The main obstacle to study such systems is that they are in principle non-autonomous unless the time scale is periodic.

Here, we are mostly interested in stability of solutions of time scale systems. There were two principal approaches. One is related to Grobman–Bellman, Bihari and other similar estimates [6, 11–13, 16, 23, 24, 28, 29], see also [8] for the classical case of ordinary differential equations. Another powerful tool is the second Lyapunov

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method, related to constructing so-called Lyapunov functions ([2, 4–7, 12, 15–19, 23, 28], see also the classical book [21] for origins). However, in the ODE theory there is the third approach, the so-called first or direct Lyapunov method [10, 21]. Unlike implicit methods, listed above, this method allows to construct bounded solutions and even invariant manifolds as limits of successive approximations. The main aim of this paper is to generalise this approach, developed for non-autonomous ODEs, to the case of time scale dynamics. We study solvability of operators, corresponding to linear systems, we give analogs of classical result of hyperbolic theory: existence of bounded solutions for almost linear systems, Lyapunov–Perron theorem on invariant manifolds, etc.

A similar approach was developed in papers [12, 28], the principal difference of our approach is that we study equivalences between time scale equations and ODEs. This leads to different results. The key point is that many linear time scale systems can be represented as reductions of linear systems of ordinary differential equations and solvability of linear time scale operator follows from one of the differential operator.

In our paper, we always operate with the so-called Δ – derivatives, the case of ∇ – derivatives may be considered similarly. Studying the case of solvability of linear differential operators (and of the time scale ones), we always concentrate on results, related to hyperbolicity (exponential dichotomy) of the corresponding ODE systems. We could also consider the so-called regularity of linear systems or one of its generalisations instead (this would give solvability in the space of exponentially decaying solutions). However, we prefer to postpone this activity for the future. In this paper, we consider both systems on time scales and ordinary differential equations. We distinguish these two cases by the following formalism: solutions related to time scales are highlighted in bold. We use standard notions $B(\varepsilon, x)$ for ε – ball, centred in x and $|\cdot|$ for the Euclidean norm.

2 Dynamic Systems on Time Scales

Definition 2.1 Let the *time scale* be an unbounded closed subset of $[0, +\infty)$.

Let $\mathbb{T}$ be a time scale. Without loss of generality, we always assume that $0 \in \mathbb{T}$.

Definition 2.2 Given a $t_0 \in \mathbb{T}$, we denote $\sigma(t_0) := \inf\{t \in \mathbb{T} : t > t_0\}$, $\mu(t_0) := \sigma(t_0) - t_0$. Such $\mu(t_0)$ is called *graininess* function. We say that t_0 is *right-dense* if $\mu(t_0) = 0$ and *right-scattered* otherwise. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *rd-continuous* if it is continuous at all right-dense points and left continuous at all left-dense points.

Definition 2.3 The function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-*differentiable* at a point $t \in \mathbb{T}$ if there exists $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a neighborhood $\mathbf{W} \subset \mathbb{T}$ of t satisfying

$$|[f(\sigma(t)) - f(s)] - \gamma[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|$$

for all $s \in \mathbf{W}$. In this case, we write $f^{\Delta}(t) = \gamma$.

When $\mathbb{T} = \mathbb{R}$, $x^{\Delta}(t) = \dot{x}(t)$. When $\mathbb{T} = \mathbb{Z}$, $x^{\Delta}(n)$ is the standard forward difference operator $x(n+1) - x(n)$.

Definition 2.4 If $F^{\Delta}(t) = f(t)$, $t \in \mathbb{T}$, then F is a Δ-*antiderivative* of f, and the Cauchy Δ-*integral* is given by the formula

$$\int_{\tau}^{s} f(t)\Delta t = F(s) - F(\tau) \qquad \text{for all} \quad s, \tau \in \mathbb{T}.$$

Similarly, we may differentiate and integrate vector and matrix-valued functions.

Definition 2.5 A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$ and *positively regressive* if $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. The set of all positively regressive and rd-continuous function is denoted by $\mathcal{R}^{+}$.

Definition 2.6 A real non-degenerate matrix A is called *positive* if one of following three equivalent conditions is satisfied:

1. there is a real matrix B such that $A = \exp(B)$;
2. there is a real matrix C such that $A = C^2$;
3. for any negative value λ and for any $k \in \mathbb{N}$ the number of entries of the $k \times k$ block
$$B_{\lambda} = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 \\ 1 & \lambda & 0 & \dots & 0 \\ 0 & 1 & \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & \lambda \end{pmatrix}$$
in the Jordan normal form of the matrix A is even (that can be 0, of course).

Particularly, the positivity implies (but is not equivalent to) the fact that $\det A > 0$.

Now, we introduce a result from linear algebra. Let $M_{n,n}(\mathbb{R})$ (or $M_{n,n}(\mathbb{C})$) be the class of all real (or, respectively, complex) $n \times n$ matrices.

Proposition 2.7 *There exists a function* $\log$ *from the set of all non-degenerate* $n \times n$ *matrices such that the following holds.*

1. $B = \log A$ *implies* $A = \exp B$;
2. *this function is measurable and bounded on any set* $\{A : \max(|A|, |A^{-1}|) \leq R\}$, $R > 0$;
3. *if* A *is positive, then* $\log A$ *is real-valued.*

The construction of such logarithm is described in [14, Chapter VIII, Sect. 8].

Definition 2.8 A matrix-valued mapping $\mathbf{A} : \mathbb{T} \to M_{n,n}(\mathbb{R})$ is called *regressive* if for each $t \in \mathbb{T}$ the $n \times n$ matrix $E_n + \mu(t)\mathbf{A}(t)$ is invertible, and *uniformly regressive* if in addition the matrix-valued function $(E_n + \mu(t)\mathbf{A}(t))^{-1}$ is bounded. Here E_n is the unit matrix. We say that the matrix-valued function $\mathbf{A}$ is positively regressive if all matrices $E_n + \mu(t)\mathbf{A}(t)$ are positive.

Definition 2.9 We say that a time scale $\mathbb{T}$ is *syndetic* if $\sup\{\sigma(t) : t \in \mathbb{T}\} < +\infty$ or, in other words, gaps of the time scale are bounded.

We introduce a notion $[t]_{\mathbb{T}} = \max\{\tau \in \mathbb{T} : \tau \le t\}$. Clearly, $[t]_{\mathbb{T}} \le t$ and $[t]_{\mathbb{T}} = t$ if and only if $t \in \mathbb{T}$.

3 Solvability of Linear Non-homogenous Systems

Consider a time scale $\mathbb{T}$ and an rd-continuous matrix-valued function $A : \mathbb{T} \to \mathbb{R}^n$.

We study a linear system

$$x^{\Delta} = \mathbf{A}(t)x + \mathbf{f}(t) \tag{3.1}$$

and the corresponding homogeneous system

$$x^{\Delta} = \mathbf{A}(t)x. \tag{3.2}$$

Here $\mathbf{A}$ is a bounded uniformly regressive rd-continuous matrix-valued function, $\mathbf{f}$ is a bounded rd-continuous vector function. We are interested when systems (3.1) have bounded solutions for all admissible right-hand sides $\mathbf{f}$. We recall a notion from the theory of linear systems of ordinary differential equations. Given a linear system

$$\dot{x} = A(s)x \tag{3.3}$$

of ordinary differential equations, we consider the Cauchy matrix $\Phi_A(t,\tau) = \Phi_A(t)\Phi_A^{-1}(\tau)$.

Definition 3.1 A linear system (3.3) is called *hyperbolic* if for any $t \in [0,\infty)$ there exists linear spaces $U^+(t)$ and $U^-(t)$ called stable and unstable spaces respectively and positive values C and λ_0 such that

1. $U^+(t) \oplus U^-(t) = \mathbb{R}^n$;
2. $\Phi_A(t,\tau)U^{\pm}(\tau) = U^{\pm}(t)$;
3. $|\Phi_A(t,\tau)x_0| \le C\exp(-\lambda_0(t-\tau))|x_0|$ for all $t > \tau$, $x_0 \in U^+(\tau)$;
4. $|\Phi_A(t,\tau)x_0| \le C\exp(\lambda_0(t-\tau))|x_0|$ for all $t < \tau$, $x_0 \in U^-(\tau)$.

Many examples of hyperbolic systems, e.g., linear systems with constant matrices may be constructed, using approaches of the paper [28].

If a continuous function $f : [0,+\infty) \to \mathbb{R}^n$ is bounded, the system

$$\dot{x} = A(s)x + f(s) \tag{3.4}$$

has a bounded solution $\varphi := \mathcal{L}f$, where

$$\varphi(s) = \int_0^s \Phi_A(s,\tau)\Pi^+(\tau)f(\tau)\,d\tau - \int_t^{\infty} \Phi_A(t,\tau)\Pi^-(\tau)f(\tau)\,d\tau. \tag{3.5}$$

Here $\Pi^+(s)$ and $\Pi^-(s)$ are linear projector operators on the stable and the unstable spaces respectively such that $\Pi^+(s)x + \Pi^-(s)x \equiv x$. A similar fact is true for exponentially decaying right hand sides. There exists a $\lambda_1 > 0$ and $K > 0$ such that for any $\lambda \in [0, \lambda_1]$ if $|f(t)| \le C \exp(-\lambda t)$, then $|\mathcal{L} f(t)| \le K_\lambda C \exp(-\lambda t)$. Actually, we may take any $\lambda_1 \in (0, \lambda_0)$. The inverse statement is also true (see [22, 26] and also [30] for discrete case).

Theorem 3.2 ([22, 27]) (Pliss–Maizel Theorem) *If system (3.4), defined on* $[0, \infty)$ *has a bounded solution for any bounded function* f, *the corresponding system (3.3) is hyperbolic.*

4 Transformation of the Time Variable

Given a time scale $\mathbb{T}$, we define the function $s : \mathbb{R} \to \mathbb{R}$:

$$s(t) = \int_0^t \frac{\log(1 + \mu([t]_{\mathbb{T}}))}{\mu([t]_{\mathbb{T}})}\, d\tau.$$

Observe that $s(0) = 0$. The following statement is evident.

Lemma 4.1 *For any time scale* $\mathbb{T}$ *the function* $s(t)$ *is strictly increasing and unbounded;*

$$\limsup_{t\to+\infty} s(t)/t \le 1.$$

If the time scale is syndetic, we also have $\liminf_{t\to+\infty} s(t)/t > 0$.

Let $\Psi_{\mathbf{A}}(s, 0)$ be a fundamental matrix of the time scale system (3.2), such that $\Psi(0) = E_n$. We prove the following statement.

Lemma 4.2 *Given an rd-continuous uniformly regressive matrix* $\mathbf{A} : \mathbb{T} \to \mathbf{M}_{n,n}$ *there exists a piece-wise continuous complex matrix-valued function* $A : [0, +\infty) \to \mathbf{M}_{n,n}$ *such that for*

$$\Phi_A(s(t), 0) = \Psi_{\mathbf{A}}(t, 0) \tag{4.1}$$

for all $t \in \mathbb{T}$. *If* $\sup |\mathbf{A}(t)| < +\infty$ *and* $\sup |\mathbf{A}^{-1}(t)| < +\infty$, *then* $\sup |A(t)| < +\infty$. *If* $\mathbf{A}$ *is uniformly positively regressive, then the matrix* A *can be taken real.*

Proof We set $A(s(t)) = \mathbf{A}(t)$ for all $t \in \mathbb{T}$. For $t \notin \mathbb{T}$, we set

$$A(s(t)) = \frac{\log[E_n + \mu([t]_{\mathbb{T}})\mathbf{A}(([t]_{\mathbb{T}})]}{\log(1 + \mu([t]_{\mathbb{T}}))}.$$

By choice of the function $s(t)$ equality (4.1) is fulfilled. Evidently,

$$\frac{\log(E_n + \mu A)}{\log(1+\mu)} \to A$$

as $\mu \to 0$ uniformly on compact sets of matrices A. On the other hand,

$$\lim_{\mu\to+\infty} \frac{\log(E_n + \mu A)}{\log(1+\mu)} = \lim_{\mu\to+\infty} \frac{\log \mu E_n + \log(A + \mu^{-1} E_n)}{\log \mu} = E_n \tag{4.2}$$

for any non-degenerate matrix A and the limit is uniform on all compact subsets of $M_{n.n}$ that do not contain degenerate matrices. □

Definition 4.3 Consider a time-scale system (3.2) with a uniformly regressive matrix $\mathbf{A}(t)$ such that $\sup\{|\mathbf{A}(t)| + |\mathbf{A}^{-1}(t)| : t \in \mathbb{T}\} < +\infty$. We call it *hyperbolic* if the corresponding system of ordinary differential equations (3.3) is hyperbolic.

For hyperbolic time scale systems, we may take stable and unstable spaces $U^{\pm}(t)$ (same as for the corresponding systems of ordinary differential equations).

Proposition 4.4 *If (3.2) is hyperbolic, there exist constants $C, \lambda > 0$ such that*

1. $|\Psi_{\mathbf{A}}(t, t_0)x_0| \le C|x_0| \exp(-\lambda(s(t) - s(t_0)))$ *for all* $t, t_0 \in \mathbb{T}$, $t \ge t_0$, $x_0 \in U^+(t_0)$;
2. $|\Psi_{\mathbf{A}}(t, t_0)x_0| \le C|x_0| \exp(\lambda(s(t) - s(t_0)))$ *for all* $t, t_0 \in \mathbb{T}$, $t \le t_0$, $x_0 \in U^-(t_0)$.

Particularly, this statement implies that $\Psi_{\mathbf{A}}(t, t_0)x_0 \to 0$ as $t \to +\infty$ if $x_0 \in U^+(t_0)$ and $\Psi_{\mathbf{A}}(t, t_0)x_0 \to \infty$ as $t \to +\infty$ if $x_0 \in U^+(t_0) \setminus \{0\}$.

Remark 4.5 It follows from (4.2) that for any hyperbolic system (3.2) on a time scale $\mathbb{T}$ the following dichotomy takes place: either the time scale is syndetic or the system (3.2) is *unstable hyperbolic* i.e. $U^-(t) \equiv \mathbb{R}^n$.

5 Transformation of the Right Hand Side

Now, we consider a system (3.2) on a time scale $\mathbb{T}$. We fix the corresponding transformation $s(\cdot)$ of the time variable and the corresponding system (3.3) of ordinary differential equations. Suppose that the matrix $\mathbf{A}(t)$ is regressive and invertible for all t. Observe that on the time-scale $\mathbb{T}$ there exists the sigma–algebra, engendered from $\mathbb{R}$, so we can consider measurable functions on $\mathbb{T}$. Given a vector function $\mathbf{f} \in \mathbb{L}^\infty(\mathbb{T} \to \mathbb{R}^n)$, we construct a function $f \in \mathbb{L}^\infty(\mathbb{R} \to \mathbb{R}^n)$ such that

1. $f|_{\mathbb{T}} = \mathbf{f}$;
2. for any $x_0 \in \mathbb{R}^n$ and any $t \in \mathbb{T}$

$$\mathbf{x}(t, 0, x_0) = x(s(t), 0, x_0), \tag{5.1}$$

where $\mathbf{x}(t, 0, x_0)$ $(x(s(t), 0, x_0))$ is the solution of systems (3.2) (or, respectively (3.3)) with initial conditions $x(0) = x_0$;

3. $f|_{(t,\sigma(t))} = \text{const}$ for any $t \in \mathbb{T}$.

By (4.1) (Lemma 4.2) it suffices to check (5.1) for $x_0 = 0$ only. Then (5.1) is equivalent to

$$\mu(t_0)\mathbf{f}(t_0) = \int_{s_0}^{s_1} \Phi_A(s_1, \tau) f(\tau)\, d\tau. \tag{5.2}$$

Here $s_0 = s(t_0)$, $s_1 = s(\sigma(t_0))$. In our assumptions, setting $f_0 := f|_{(s_0,s_1)}$, we reformulate Eq. (5.2) as follows:

$$\mathbf{f}(t_0) = \frac{A^{-1}(\exp(A(s_0)(s_1 - s_0)) - E_n) f_0}{\mu(t_0)} = \log(1 + \mu(t_0))(\log[E_n + \mu(t_0)\mathbf{A}(t_0)])^{-1}\mathbf{A}(t_0) f_0$$

if $\mu(t_0) > 0$ or

$$f_0 = \frac{\log[E_n + \mu(t_0)\mathbf{A}(t_0)]}{\log(1 + \mu(t_0))}\mathbf{A}^{-1}(t_0)\mathbf{f}(t_0)$$

These formulae imply the following statement.

Theorem 5.1 *Let the matrix* $\mathbf{A}$ *be uniformly regressive with respect to the time scale* $\mathbb{T}$, *hyperbolic and uniformly bounded together with the inverse matrix* $\mathbf{A}^{-1}$. *Then, for any function* $\mathbf{f} \in \mathbb{L}^\infty(\mathbb{T} \to \mathbb{R}^n)$ *the corresponding system (3.1) has a bounded solution.*

In this case, there exists a continuous linear operator $\mathbf{L} : \mathbb{L}^\infty(\mathbb{T} \to \mathbb{R}^n) \to \mathbb{L}^\infty(\mathbb{T} \to \mathbb{R}^n)$ such that for any $\mathbf{f} \in \mathbb{L}^\infty(\mathbb{T} \to \mathbb{R}^n)$ the function $\mathbf{Lf}$ is a bounded solution of system (3.1). This operator $\mathbf{L}$ corresponds to the operator $\mathcal{L}$ that gives a bounded solution for Eq. (3.4) and is defined by formula (3.5). Let $\mathbf{K} = \|\mathbf{L}\|$.

An analog of Pliss–Maizel Theorem is also true for time scale systems.

Theorem 5.2 *Let the matrix* $\mathbf{A}$ *be uniformly regressive with respect to the time scale* $\mathbb{T}$. *Suppose that for any* $\mathbf{f} \in \mathbb{L}^\infty$ *the corresponding system (3.1) has a bounded solution and the time scale is syndetic. Then system (3.2) is hyperbolic on* $\mathbb{T}$.

Proof Suppose that system (3.3), constructed by system (3.2) as demonstrated is not hyperbolic. Then, there exists a bounded right hand side f such that the corresponding system (3.4) does not have any solutions, bounded on $[0, +\infty)$. Since system (3.4) is linear, all coefficients are bounded and the time scale is syndetic, all solutions of (3.4) are unbounded on $\mathbb{T}$. Consider the function $\mathbf{f} : \mathbb{T} \to \mathbb{R}^n$ such that $f(t) = \mathbf{f}(t)$ for all right-dense points t and Eq. (5.2) is satisfied for all right-scattered points. Then all solutions of Eq. (3.1) are unbounded. □

Similarly to what is done for ordinary differential equations, we can give estimates of the operator $\mathbf{L}$ in spaces of "exponentially small" functions.

Proposition 5.3 *Let the matrix* $\mathbf{A}$ *be hyperbolic on the time scale* $\mathbb{T}$. *Then there exist* $K > 0$ *and* $\lambda_1 > 0$ *such that for any* $\lambda \in [0, \lambda_0]$ *the inequality* $|\mathbf{f}(t)| \le C\exp(-\lambda s(t))\ \forall t \in \mathbb{T}$ *implies* $|\mathbf{L}f(t)| \le CK\exp(-\lambda s(t)) \quad \forall t \in \mathbb{T}$.

6 Conditional Stability by First Approximation

We can use the statement of Lemma 4.1 to prove some time scale analogs of famous statements from the theory of hyperbolic ODEs.

Theorem 6.1 *Let the matrix* $\mathbf{A}$ *satisfy conditions of Theorem 5.1. Let* $r_0 > 0$ *and the continuous function* $\mathbf{g} : \mathbb{T} \times B(0, r_0)$ *be such that*

1. $|\mathbf{g}(t, 0)| \le \varepsilon$ *for any* $t \in \mathbb{T}$;
2. $|\mathbf{g}(t, x_1) - \mathbf{g}(t, x_2)| \le l|x_1 - x_2|$ *for any* $t \in \mathbb{T}$, $x_{1,2} \in B(0, r_0)$.

Then given r_0 *there exist* $\varepsilon_0, l_0 > 0$ *such that if* $l < l_0$, $\varepsilon < \varepsilon_0$ *there exists a bounded solution* $\mathbf{X}(t)$ *of the system*

$$x^{\Delta} = \mathbf{A}(t)x + \mathbf{g}(t, x) \tag{6.1}$$

such that

$$|\mathbf{X}(t)| \le \frac{K\varepsilon}{1 - Kl}. \tag{6.2}$$

Let λ_0 be a constant of hyperbolicity of the matrix $\mathbf{A}$.

Theorem 6.2 (Lyapunov–Perron Theorem) *Let the matrix* $\mathbf{A}$ *satisfy conditions of Theorem 5.1. Let* $r_0 > 0$ *and the continuous function* $\mathbf{g} : \mathbb{T} \times B(0, r_0)$ *be such that*

1. $\mathbf{g}(t, 0) = 0$ *for any* $t \in \mathbb{T}$;
2. $|\mathbf{g}(t, x_1) - \mathbf{g}(t, x_2)| \le l|x_1 - x_2|$ *for any* $t \in \mathbb{T}$, $x_{1,2} \in B(0, r_0)$.

Then given r_0, $\lambda \in (0, \lambda_0)$, $t_0 \in \mathbb{T}$ *there exist* $D > 0$ $l_0 > 0$ *such that if* $l < l_0$, *there exists a map* $h : B(0, r_0) \bigcap U^+(t_0) \to U^-(t_0)$ *such that*

1. $h(0) = 0$;
2. $|h(x) - h(y)| \le Dl|x - y|$.
3. *If* x_0 *is such that* $x_0 = y_0 + h(y_0)$ *for some* y_0, *then* $\mathbf{x}(t, t_0, x_0)$ *tends to zero as* t *goes to infinity (in fact, it tends to zero). Here* $\mathbf{x}(t, t_0, x_0)$ *is the solution of system (6.1) with initial conditions* $\mathbf{x}(t_0) = x_0$.

This allows to construct the so-called local stable manifold as the image of the constructed map h. By Remark 4.5, this result is non-trivial only if the time scale is syndetic. Proofs of Theorems 6.1 and 6.2 are very close to ones of their classical analogs [8, 10, 21, 25–27].

Proof of Theorem 6.1 Consider the equation

$$\mathbf{x}(t) = \mathbf{L}[\mathbf{g}(\cdot, \mathbf{x})](t). \tag{6.3}$$

Any solution $\mathbf{X}(t)$ of (6.3) is a bounded solution of the equation $x^{\Delta} = \mathbf{A}(t)x + \mathbf{g}(t, X(t))$ and, hence, one of Eq. (6.1). Given r_0, we take ε_0 and l_0 so small that

$$Kl_0 \le \frac{1}{2}, \qquad \frac{K\varepsilon_0}{1 - Kl_0} \le \frac{r_0}{2}.$$

We set $\mathbf{x}^0(t) = 0$ for all t and define

$$\mathbf{x}^m(t) = \mathbf{L}[\mathbf{g}(\cdot, \mathbf{x}^{m-1})](t). \tag{6.4}$$

for all $m \in \mathbb{N}$.

Lemma 6.3 *All approximations* $\mathbf{x}^m(t)$ $(m \in \mathbb{N} \bigcup \{0\})$ *are*

1. *well-defined on* $\mathbb{T}$;
2. *such that* $|\mathbf{x}^m(t)| \leq r_0$ *for all* $t \in \mathbb{T}$, $m \in \mathbb{N}$;
3. *such that*

$$|\mathbf{x}^{m+1}(t) - \mathbf{x}^m(t)| \leq K\varepsilon(Kl)^m, \qquad t \in \mathbb{T}. \tag{6.5}$$

Observe that,

$$\|\mathbf{x}^1\| = \|\mathbf{x}^1 - \mathbf{x}^0\| \leq \|\mathbf{L}[\mathbf{g}(\cdot, 0)]\| \leq K\varepsilon \tag{6.6}$$

(all norms are considered in $\mathbb{L}^\infty(\mathbb{T})$). So, the statement of the lemma is true for $m = 0$.

Proceed by induction from the step $m - 1$ to m. If $\|\mathbf{x}^{m-1}\| \leq r_0$, the right hand side of Eq. (6.4) is well-defined and the solution $\mathbf{x}^m(t)$ can be found. Inequalities (6.5) considered for all previous steps and (6.6) imply that

$$\|\mathbf{x}^m\| \leq \frac{K\varepsilon(1 - (Kl)^{m+1})}{1 - Kl} \leq \frac{r_0}{2}. \tag{6.7}$$

Hence the iteration $\mathbf{x}^{m+1}$ is also well-defined and

$$\|\mathbf{x}^{m+1} - \mathbf{x}^m\| = \|\mathbf{L}[\mathbf{g}(\cdot, \mathbf{x}^m) - \mathbf{g}(\cdot, \mathbf{x}^{m-1})]\| \leq Kl\|\mathbf{x}^m - \mathbf{x}^{m-1}\|$$

that implies (6.5). □.

So, the iterations $\mathbf{x}^k$ converge uniformly and we may set $\mathbf{X} = \lim \mathbf{x}^m$. Since the function $\mathbf{g}$ is uniformly continuous w.r.t. x, we can proceed to limit in (6.4). So, $\mathbf{X}$ is a solution of (6.3). Proceeding to limit in Eq. (6.7), we get (6.2) that finishes the proof. □

Proof of Theorem 6.2 Without loss of generality, we suppose that $t_0 = 0$. Fix $y_0 \in U^+(0)$. Take $\mathbf{x}^0(t, y_0) = 0$, $\mathbf{x}^1(t, y_0) = \Psi_\mathbf{A}(t, 0)y_0$ and set $\mathbf{x}^{m+1}(t, y_0) = \Psi_\mathbf{A}(t, 0)y_0 + \mathbf{L}[\mathbf{g}(\cdot, \mathbf{x}^m)](t)$ for all $m \in \mathbb{N}$. By definition, we have $|\mathbf{x}^1(t, y_0)| \leq a|y_0| \exp(-\lambda s(t))$. We consider y_0 so small that $2a|y_0| < r_0$. We prove the following lemma, similar to Lemma 6.3.

Lemma 6.4 *All approximations* $\mathbf{x}^m(t)$ $(m \in \mathbb{N} \bigcup \{0\})$ *are*

1. *well-defined on* $\mathbb{T}$;
2. *such that*

$$|\mathbf{x}^m(t, y_0)| \leq 2a|y_0| \exp(-\lambda s(t)) \leq r_0, \qquad t \in \mathbb{T}, \quad m \in \mathbb{N}; \qquad \textit{and} \tag{6.8}$$

$$|\mathbf{x}^{m+1}(t, y_0) - \mathbf{x}^m(t, y_0)| \le a(Kl)^m |y_0| \exp(-\lambda s(t)), \qquad t \in \mathbb{T}. \tag{6.9}$$

Proof Inequalities (6.8) are evident for $m = 0$ and $m = 1$, inequality (6.9) is evident for $m = 0$. Now we are going to prove the lemma by induction.

If $|\mathbf{x}^m(t, y_0)| \le r_0$, $|\mathbf{x}^{m+1}(t, y_0)|$ is correctly defined. Moreover,

$$|\mathbf{x}^{m+1}(t, y_0) - \mathbf{x}^m(t, y_0)| = |\mathbf{L}[\mathbf{g}(\cdot, \mathbf{x}^m)](t, y_0) - \mathbf{L}[\mathbf{g}(\cdot, \mathbf{x}^{m-1})]$$
$$(t, y_0)| \le Kl|y_0|a(Kl)^m|y_0| \exp(-\lambda s(t))$$

which proves (6.9) for the given m. Taking sum of inequalities (6.9) for all previous values of m and taking into account the estimate for $\mathbf{x}^1$, we get

$$|\mathbf{x}^{m+1}(t, y_0)| \le \left(a + a\frac{Kl}{1 - Kl}\right) |y_0| \exp(-\lambda s(t)).$$

If $Kl < 1/2$, this implies (6.8) on the step $m + 1$. □

Now we prove that all iterations $\mathbf{x}^m$ are Lipschitz continuous. We set

$$\mathbf{x}^m(t, y_0) = \mathbf{x}^1(t, y_0) + \mathbf{z}^m(t, y_0) = \Psi_\mathbf{A}(t, 0)y_0 + \mathbf{z}^m(t, y_0).$$

Lemma 6.5 *All iterations* $\mathbf{x}^m(t, y_0)$ *and* $\mathbf{z}^m(t, y_0)$ *are Lipschitz continuous: for any* $t \in \mathbb{T}$, y_0, y_1 *such that* $|y_{0,1}| \le r_0$

$$\begin{aligned} |\mathbf{x}^m(t, y_0) - \mathbf{x}^m(t, y_1)| &\le 2a \exp(-\lambda s(t))|y_0 - y_1|; \\ |\mathbf{z}^m(t, y_0) - \mathbf{z}^m(t, y_1)| &\le 2Kal \exp(-\lambda s(t))|y_0 - y_1|. \end{aligned} \tag{6.10}$$

Proof For $m = 1$, (6.10) is evident:

$$|\mathbf{x}^1(t, y_0) - \mathbf{x}^1(t, y_1)| \le a \exp(-\lambda s(t))|y_0 - y_1|,$$

$\mathbf{z}^1(t, y_0) \equiv 0$. Then, we continue the proof by induction.

Let (6.10) be satisfied for a fixed value m. We write

$$|\mathbf{z}^{m+1}(t, y_0) - \mathbf{z}^{m+1}(t, y_1)| =$$
$$|\mathbf{L}[\mathbf{g}(\cdot, \mathbf{x}^{m+1}(\cdot, y_0)](t) - \mathbf{L}[\mathbf{g}(\cdot, \mathbf{x}^{m+1}(\cdot, y_0)](t)| \le 2aKl \exp(-\lambda s(t))|y_0 - y_1|;$$

$$|\mathbf{x}^{m+1}(t, y_0) - \mathbf{x}^{m+1}(t, y_1)| \le |\mathbf{x}^1(t, y_0) - \mathbf{x}^1(t, y_1)| + |\mathbf{z}^{m+1}(t, y_0) - \mathbf{z}^{m+1}(t, y_1)| \le$$
$$(a + 2Kal) \exp(-\lambda s(t))|y_0 - y_1| \le 2a \exp(-\lambda s(t))|y_0 - y_1|.$$

□

By Lemma 6.4, approximations $\mathbf{x}^k(t, y_0)$ converge to

$$\mathbf{x}^*(t, y_0) = \mathbf{x}^1(t, y_0) + \mathbf{z}^*(t, y_0)$$

that is a solution of the equation $\mathbf{x}(t) = \Psi_\mathbf{A}(t, 0)y_0 + \mathbf{L}[\mathbf{g}(\cdot, \mathbf{x}(\cdot))](t)$ with initial conditions $\mathbf{x}(0) = y_0 + \mathbf{z}^*(0, y_0) =: y_0 + h(y_0)$. Proceeding to limit in (6.8), we get

$$|\mathbf{x}^*(t, y_0)| \le 2a|y_0| \exp(-\lambda s(t)),$$

the second line of (6.10) implies

$$|h(y_0) - h(y_1)| \le 2aKl|y_0 - y_1|. \qquad \square$$

Many other analogs of classical results of hyperbolic systems of o.d.e.s may be proved for time scale systems. For example, following the lines of [27, Chap. 1], we can prove that all solutions that start in a small neighbourhood of zero out of the stable manifold, leave this small neighbourhood as time increases. Also, we can prove that for any $r \in \mathbb{N}$ the stable manifold is C^r - smooth provided the function $\mathbf{g}$ is C^r - smooth w.r.t. x.

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Oscillation of Third-Order Nonlinear Neutral Differential Equations

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Abstract In this paper, we study the oscillation and asymptotic properties of solutions of a certain nonlinear neutral third-order differential equation with either delay or advanced argument.

Keywords Neutral · Third-order · Almost oscillatory

1 Introduction

Consider third-order neutral differential equation of the form

$$\left(\frac{1}{p(t)}\left[\left(\frac{1}{r(t)}\big(u'(t)\big)^{\beta}\right)'\right]^{\alpha}\right)' + q(t)f\big(x\big(\delta(t)\big)\big) = 0, \tag{E}$$

where

$$u(t) = x(t) + a(t)x\big(\gamma(t)\big) \tag{1}$$

and $t \geq t_0$. We will always assume that

(i) $p(t),\ r(t),\ q(t),\ a(t),\ \gamma(t),\ \delta(t) \in C[t_0, \infty),\ p(t),\ r(t),\ q(t),\ \gamma(t),\ \delta(t)$ are positive for $t \geq t_0$,
(ii) α and β are ratios of odd positive integers,
(iii) $\int_{t_0}^{\infty} p^{\frac{1}{\alpha}}(t)\,\mathrm{d}t = \int_{t_0}^{\infty} r^{\frac{1}{\beta}}(t)\,\mathrm{d}t = \infty$,
(iv) $\gamma(t) \leq t$, $\lim_{t\to\infty} \gamma(t) = \infty$,
(v) $\lim_{t\to\infty} \delta(t) = \infty$
(vi) $0 \leq a(t) \leq a_0 < 1$ for $t \geq t_0$,
(vii) $f \in C(\mathbb{R}, \mathbb{R})$, f is odd, $f(v)v > 0$ for $v \neq 0$.

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Functional and neutral equations play an important role in many applications. Third-order differential equations arise in the study of entry-flow phenomenon, in mathematical theory of thyroid-pituitary interactions, in problems concerning nuclear reactor feedback etc.

In this paper we are going to enhance the results from [6, 7], where equation (E) was studied in the case $\alpha = \beta = 1$. There exist various papers studying equations similar to (E) in the case that $\beta = 1$, see e.g. [3, 8–11] or equations with $\beta = 1$ and additional middle term, see e.g. [2, 4] or [5]. Hence, it is natural to try to enhance our results and, if able, to relate them to the ones obtained by other authors.

If u is a function defined by (1), then functions

$$u^{[0]} = u, \qquad u^{[1]} = \frac{1}{r(t)}\left(u'\right)^{\beta}, \qquad u^{[2]} = \frac{1}{p(t)}\left[\left(\frac{1}{r(t)}\left(u'\right)^{\beta}\right)'\right]^{\alpha} = \frac{1}{p(t)}\left(\left(u^{[1]}\right)'\right)^{\alpha}$$

are called quasiderivatives of u. A solution x of (E) is said to be *proper* if it exists on the interval $[t_0, \infty)$ and satisfies the condition $\sup\{|x(s)| : t \le s < \infty\} > 0$ for any $t \ge t_0$. A proper solution is called *oscillatory* or *nonoscillatory* according to whether it does or does not have arbitrarily large zeros.

Definition 1 Equation (E) is said to have *property A* if any proper solution x of (E) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Some authors use a different terminology and instead of using property A they say that equation (E) is *almost oscillatory*.

2 Basic Properties of (E)

By a modification of the well-known result of Kiguradze we obtain lemma which characterizes behaviour of nonoscillatory solution of (E).

Lemma 1 *Let x be a nonoscillatory solution of (E) and let u be defined by* (1). *Then there are only two possible classes of solutions*

$$\mathcal{N}_0 = \left\{x \ \textit{solution}, \exists T_x : u(t)u^{[1]}(t) < 0,\ u(t)u^{[2]}(t) > 0 \ \textit{for}\ t \ge T_x\right\},$$
$$\mathcal{N}_2 = \left\{x \ \textit{solution}, \exists T_x : u(t)u^{[1]}(t) > 0,\ u(t)u^{[2]}(t) > 0 \ \textit{for}\ t \ge T_x\right\}.$$

It is clear that equation (E) has property A if and only if every nonoscillatory solution x belongs to class $\mathcal{N}_0$ and $\lim_{t\to\infty} x(t) = 0$.

One can simply prove following property of quasiderivatives.

Lemma 2 *Let x be a nonoscillatory solution of (E) and let u be defined by* (1). *Then* u, $u^{[1]}$ *and* $u^{[2]}$ *are monotone for large* t.

The basic properties of solutions in class $\mathcal{N}_2$ are described by the following lemma.

Lemma 3 *Assume that x is a solution of (E) from class $\mathscr{N}_2$. Then*

$$(1 - a_0)|u(t)| \leq |x(t)| \leq |u(t)| \tag{2}$$

for $t \geq T$ and

$$\lim_{t\to\infty} |u(t)| = \lim_{t\to\infty} |x(t)| = \infty. \tag{3}$$

Proof Let $x \in \mathscr{N}_2$. Without loss of generality we may assume that x is eventually positive, i.e. there exists $T_x \geq t_0$ such that $x(t) > 0$, $u(t) > 0$, $u^{[1]}(t) > 0$ and $u^{[2]}(t) > 0$ for $t \geq T_x$ and $T \geq T_x$ such that $\gamma(t) \geq T_x$ for $t \geq T$.

Since $\gamma(t) \leq t$ and u is an increasing function, we have $x\big(\gamma(t)\big) \leq u\big(\gamma(t)\big) \leq u(t)$ for $t \geq T$. Hence

$$x(t) = u(t) - a(t)x\big(\gamma(t)\big) \geq u(t) - a_0 x\big(\gamma(t)\big) \geq u(t) - a_0 u\big(\gamma(t)\big) \geq u(t)(1 - a_0).$$

To prove the second part we have that $u^{[1]}$ is positive and increasing function and therefore there exists $K > 0$ such that $u^{[1]}(t) \geq K$ for large t. Integrating this inequality from T to t we obtain

$$u(t) \geq u(T) + K^{\frac{1}{\beta}} \int_T^t r^{\frac{1}{\beta}}(s)\, \mathrm{d}s.$$

Letting $t \to \infty$ and using the fact that $\int_{t_0}^{\infty} r^{\frac{1}{\beta}}(t)\, \mathrm{d}t = \infty$, we obtain $u(t) \to \infty$. By the first part, $x(t) \geq (1 - a_0)u(t)$. From here it follows that $x(t) \to \infty$.

In our first theorem we will give the condition that ensures that every solution from class $\mathscr{N}_0$ has the desired asymptotic behaviour.

Theorem 1 *Assume that*

$$\int_{t_0}^{\infty} \left[r(t) \int_t^{\infty} \left[p(s) \int_s^{\infty} q(v)\, \mathrm{d}v \right]^{\frac{1}{\alpha}} \mathrm{d}s \right]^{\frac{1}{\beta}} \mathrm{d}t = \infty, \tag{4}$$

then every solution of (E) that belongs to class $\mathscr{N}_0$ satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Proof Without loss of generality we may assume that $x \in \mathscr{N}_0$ is an eventually positive solution. Then there exists $T_x \geq t_0$ such that $u(t) > 0$, $u\big(\gamma(t)\big) > 0$, $u^{[1]}(t) < 0$, $u^{[2]}(t) > 0$ for $t \geq T_x$. Since u is positive, there exists $\lim_{t\to\infty} u^{[i]}(t) = \ell_i$, $i = 0, 1, 2$. We will show that $\ell_i = 0$ for $i = 0, 1, 2$.

Assume that $\ell_1 < 0$, then

$$\lim_{t\to\infty} \frac{1}{r(t)}[u'(t)]^{\beta} = \ell_1 \Longrightarrow u'(t) \le \ell_1^{\frac{1}{\beta}} r^{\frac{1}{\beta}}(t)$$

and integrating the last inequality from T_x to t and letting $t \to \infty$ we get a contradiction with the positivity of u. In the similar manner we can see that $\ell_2 = 0$.

Assume by contradiction that $\ell_0 > 0$. Then for any $\varepsilon > 0$ we have that there exists $T_1 \ge T_x$ such that $\ell_0 + \varepsilon > u\big(\gamma(t)\big) > \ell_0$ and choosing $0 < \varepsilon < \frac{\ell_0(1-a_0)}{a_0}$ we obtain the lower estimate

$$x(t) = u(t) - a(t)x\big(\gamma(t)\big) > \ell_0 - a_0 u\big(\gamma(t)\big) > \ell_0 - a_0(\ell_0 + \varepsilon) = k(\ell_0 + \varepsilon) > k\ell_0, \quad (5)$$

where $k = \frac{\ell - a_0(l+\varepsilon)}{\ell+\varepsilon} > 0$. From (5) and in view of the fact that f is continuous, there exist K and $T_2 \ge T_1$ such that $f\Big(x\big(\delta(t)\big)\Big) \ge K$ for $t \ge T_2$. Hence from equation (E) it follows that $\big(u^{[2]}(t)\big)' \le -q(t)K$. Integrating this inequality two times from t to ∞ we obtain

$$-u^{[1]}(t) \ge K^{\frac{1}{\alpha}} \int_t^{\infty} \left[p(v) \int_v^{\infty} q(s)\,\mathrm{d}s\right]^{\frac{1}{\alpha}} \mathrm{d}v.$$

Integrating from T_2 to t we get

$$-u(t) + u(T_2) \ge K^{\frac{1}{\alpha\beta}} \int_{T_2}^{t} \left[r(w) \int_w^{\infty} \left[p(v) \int_v^{\infty} q(s)\,\mathrm{d}s\right]^{\frac{1}{\alpha}} \mathrm{d}v\right]^{\frac{1}{\beta}} \mathrm{d}w.$$

Letting $t \to \infty$ we obtain

$$\int_{T_2}^{\infty} \left[r(w) \int_w^{\infty} \left[p(v) \int_v^{\infty} q(s)\,\mathrm{d}s\right]^{\frac{1}{\alpha}} \mathrm{d}v\right]^{\frac{1}{\beta}} \mathrm{d}w < \infty,$$

which contradicts (4). Therefore $\ell = 0$ and the inequality $0 \le x(t) \le u(t)$ implies that $\lim_{t\to\infty} x(t) = 0$ as well.

3 Oscillation Theorems for Superlinear Case

In this section we treat the so-called superlinear case, i.e. we will assume the condition

$$\limsup_{v\to\infty} \frac{v}{f(v)} < \infty, \quad (6)$$

which is fulfilled for example for functions v^{λ}, where $\lambda > 1$.

The first two theorems deal with equation with delay, the third one is for the equation with advanced argument and the last one covers both cases.

Theorem 2 *Assume that* $\delta(t) \leq t$, $\alpha\beta > 1$ *and* (6). *If* (4) *holds and*

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{\delta(t)} r^{\frac{1}{\beta}}(s) \left[\int_{t_0}^{s} p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}} \mathrm{d}s\,\mathrm{d}t = \infty, \tag{7}$$

then equation (E) has property A.

Proof We rewrite equation (E) as a system

$$\begin{aligned} u'(t) &= r^{\frac{1}{\beta}}(t) y^{\frac{1}{\beta}}(t) \\ y'(t) &= p^{\frac{1}{\alpha}}(t) z^{\frac{1}{\alpha}}(t) \\ z'(t) &= -q(t) f\big(x\big(\delta(t)\big)\big) \end{aligned} \tag{8}$$

Without loss of generality assume that x is a positive solution from class $\mathscr{N}_2$. Then u, y and z are monotone and there exists T_x such that $x(t) > 0$, $u(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for $t \geq T_x$ and $T \geq T_x$ such that $\delta(t) \geq T_x$ for $t \geq T$.

Integrating the second equation of (8) and using the fact that z is monotone and decreasing we obtain

$$y(t) \geq z^{\frac{1}{\alpha}}(t) \int_{T}^{t} p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s$$

and therefore

$$y^{\frac{1}{\beta}}(t) \geq z^{\frac{1}{\alpha\beta}}(t) \left[\int_{T}^{t} p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}.$$

using this estimate we get from the first equation

$$u(t) \geq u(t) - u(T) = \int_{T}^{t} r^{\frac{1}{\beta}}(s) y^{\frac{1}{\beta}}(s)\,\mathrm{d}s \geq \int_{T}^{t} r^{\frac{1}{\beta}}(s) z^{\frac{1}{\alpha\beta}}(s) \left[\int_{T}^{s} p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}} \mathrm{d}s.$$

Using inequality (2) and the fact that z is monotone and decreasing yields

$$x(t) \geq (1 - a_0) z^{\frac{1}{\alpha\beta}}(t) \int_{T}^{t} r^{\frac{1}{\beta}}(s) \left[\int_{T}^{s} p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}} \mathrm{d}s. \tag{9}$$

Using (6) we get from the third equation of (8) for $t \geq T$

$$-z'(t) = q(t) f\big(x\big(\delta(t)\big)\big) \geq \inf_{s \geq t} \frac{f\big(x\big(\delta(s)\big)\big)}{x\big(\delta(s)\big)} q(t) x\big(\delta(t)\big),$$

i.e.

$$-z'(t)\sup_{s\geq t}\frac{x\big(\delta(s)\big)}{f\big(x\big(\delta(s)\big)\big)}\geq q(t)x\big(\delta(t)\big).$$

Substituting (9) into last inequality and using facts that $\delta(t)\leq t$ and z is decreasing we obtain

$$-z'(t)\sup_{s\geq t}\frac{x\big(\delta(s)\big)}{f\big(x\big(\delta(s)\big)\big)}\geq(1-a_0)q(t)z^{\frac{1}{\alpha\beta}}(t)\int_T^{\delta(t)}r^{\frac{1}{\beta}}(s)\left[\int_T^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}}\mathrm{d}s.$$

Dividing by $z^{\frac{1}{\alpha\beta}}(t)$ and integrating from T to t we get

$$\begin{aligned}\sup_{s\geq t}\frac{x\big(\delta(s)\big)}{f\big(x\big(\delta(s)\big)\big)}\int_T^t-\frac{z'(s)}{z^{\frac{1}{\alpha\beta}}(s)}\,\mathrm{d}s&\geq\\ \geq(1-a_0)\int_T^t q(s)\int_T^{\delta(s)}r^{\frac{1}{\beta}}(v)\left[\int_T^v p^{\frac{1}{\alpha}}(w)\,\mathrm{d}w\right]^{\frac{1}{\beta}}\mathrm{d}v\,\mathrm{d}s.\end{aligned}\qquad(10)$$

Quick computation shows that

$$\begin{aligned}\int_T^t-\frac{z'(s)}{z^{\frac{1}{\alpha\beta}}(s)}\,\mathrm{d}s&=\left(\frac{1}{\alpha\beta}-1\right)\int_T^t\left(\frac{1}{z^{\frac{1}{\alpha\beta}-1}(s)}\right)'\mathrm{d}s=\\&=\left(\frac{1}{\alpha\beta}-1\right)\left(\frac{1}{z^{\frac{1}{\alpha\beta}-1}(t)}-\frac{1}{z^{\frac{1}{\alpha\beta}-1}(T)}\right).\end{aligned}$$

Since $\alpha\beta>1$, we have that $\int_T^\infty-\frac{z'(s)}{z^{\frac{1}{\alpha\beta}}(s)}\,\mathrm{d}s<\infty$. Passing $t\to\infty$ in (10) we get the contradiction with condition (7), hence $\mathscr{N}_2=\emptyset$.

As condition (4) holds, by Theorem 1 we have that every solution x from class $\mathscr{N}_0$ satisfies $\lim_{t\to\infty}x(t)=0$, which completes the proof.

Adapting the useful trick from [1] we can extend previous theorem to the case $\alpha\beta=1$.

Theorem 3 *Assume that* $\delta(t)\leq t$, $\alpha\beta=1$ *and* (6).
If (4) *holds and*

$$\int_{t_0}^\infty q(t)\left[\int_{t_0}^{\delta(t)}r^{\frac{1}{\beta}}(s)\left[\int_{t_0}^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}}\mathrm{d}s\right]^{1-\varepsilon}\mathrm{d}t=\infty,\qquad(11)$$

where $0<\varepsilon<1$, *then equation (E) has property A.*

Proof Without loss of generality assume that x is a positive solution from class $\mathcal{N}_2$. Then u, y and z are monotone and there exist T_x such that $x(t) > 0$, $u(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for $t \geq T_x$ and $T \geq T_x$ such that $\delta(t) \geq T_x$ for $t \geq T$.

We proceed exactly as in the proof of the previous theorem and establish inequality (9). Raising this inequality to $(1-\varepsilon)$-th power and using $\alpha\beta = 1$ we obtain

$$x^{1-\varepsilon}(t) \geq (1-a_0)^{1-\varepsilon} z^{1-\varepsilon}(t) \left(\int_T^t r^{\frac{1}{\beta}}(s) \left[\int_T^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v \right]^{\frac{1}{\beta}} \mathrm{d}s \right)^{1-\varepsilon}. \tag{12}$$

Since x is nondecreasing, there exist $T_1 \geq T$ and constant $d > 0$ such that $x(t) \geq d$ for $t \geq T_1$. This implies that $x^{1-\varepsilon}(t) \leq \frac{1}{d^\varepsilon} x(t)$ for $t \geq T_1$. Combining this inequality with (12) we get

$$x(t) \geq d^\varepsilon (1-a_0)^{1-\varepsilon} z^{1-\varepsilon}(t) \left(\int_T^t r^{\frac{1}{\beta}}(s) \left[\int_T^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v \right]^{\frac{1}{\beta}} \mathrm{d}s \right)^{1-\varepsilon}.$$

Similarly to the proof of the preceding theorem, using the third equation, previous inequality and integrating from T_1 to t we obtain

$$\begin{aligned} \sup_{s\geq t} \frac{x\big(\delta(s)\big)}{f\big(x\big(\delta(s)\big)\big)} \int_{T_1}^t -\frac{z'(s)}{z^{1-\varepsilon}(s)}\,\mathrm{d}s \geq \\ \geq d^\varepsilon (1-a_0)^{1-\varepsilon} \int_{T_1}^t q(s) \left[\int_T^{\delta(s)} r^{\frac{1}{\beta}}(v) \left[\int_T^v p^{\frac{1}{\alpha}}(w)\,\mathrm{d}w \right]^{\frac{1}{\beta}} \mathrm{d}v \right]^{1-\varepsilon} \mathrm{d}s. \end{aligned} \tag{13}$$

Since $0 < 1-\varepsilon < 1$, we get by a direct computation that $\int_{T_1}^\infty -\frac{z'(s)}{z^{1-\varepsilon}(s)}\,\mathrm{d}s < \infty$. Hence, passing $t \to \infty$ in (13) gives a contradiction with (11), i.e. $\mathcal{N}_2 = \emptyset$.

Rest of the proof follows from Theorem 1.

Theorem 4 *Assume that* $\delta(t) \geq t$, $\alpha\beta < 1$ *and* (6).

If (4) *holds and*

$$\int_{t_0}^\infty r^{\frac{1}{\beta}}(s) \left[\int_{t_0}^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v \right]^{\frac{1}{\beta}} \left[\int_s^\infty q(v)\,\mathrm{d}v \right]^{\frac{1}{\alpha\beta}} \mathrm{d}s = \infty, \tag{14}$$

then equation (E) has property A.

Proof We rewrite equation (E) as a system (8). Without loss of generality assume that x is a positive solution from class $\mathcal{N}_2$. Then u, y and z are monotone and there exists T_x such that $x(t) > 0$, $u(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for $t \geq T_x$.

Integrating the third equation of the system (8) from t to ∞ ($t \geq T_x$) and using (in this order) that z is positive and decreasing, assumption (6), x is increasing and $\delta(t) \geq t$ and (2) we get

$$z(t) \geq \int_t^\infty q(s) f\big(x\big(\delta(s)\big)\big)\,\mathrm{d}s \geq \inf_{s\geq t} \frac{f\big(x\big(\delta(s)\big)\big)}{x\big(\delta(s)\big)} \int_t^\infty q(s) x\big(\delta(s)\big)\,\mathrm{d}s \geq$$

$$\geq \inf_{s\geq t} \frac{f\big(x\big(\delta(s)\big)\big)}{x\big(\delta(s)\big)} x(t) \int_t^\infty q(s)\,\mathrm{d}s \geq (1-a_0) \inf_{s\geq t} \frac{f\big(x\big(\delta(s)\big)\big)}{x\big(\delta(s)\big)} u(t) \int_t^\infty q(s)\,\mathrm{d}s.$$

We conclude from the second equation that

$$y^{\frac{1}{\beta}}(t) \geq z^{\frac{1}{\alpha\beta}}(t) \left[\int_T^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}.$$

Substituting into the first equation we obtain

$$u'(t) \geq r^{\frac{1}{\beta}}(t) \geq z^{\frac{1}{\alpha\beta}}(t) \left[\int_T^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}} \geq$$

$$\geq r^{\frac{1}{\beta}}(t)(1-a_0)^{\frac{1}{\alpha\beta}} \left(\inf_{s\geq t} \frac{f\big(x\big(\delta(s)\big)\big)}{x\big(\delta(s)\big)}\right)^{\frac{1}{\alpha\beta}} u^{\frac{1}{\alpha\beta}}(t) \left[\int_t^\infty q(s)\,\mathrm{d}s\right]^{\frac{1}{\alpha\beta}} \cdot \left[\int_T^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}.$$

Dividing by $u^{\frac{1}{\alpha\beta}}(t)$ and integrating from T to t we get

$$\int_T^t \frac{u'(s)}{u^{\frac{1}{\alpha\beta}}(s)}\,\mathrm{d}s \geq$$

$$\geq (1-a_0)^{\frac{1}{\alpha\beta}} \left(\inf_{s\geq t} \frac{f\big(x\big(\delta(s)\big)\big)}{x\big(\delta(s)\big)}\right)^{\frac{1}{\alpha\beta}} \int_T^t r^{\frac{1}{\beta}}(s) \left[\int_T^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}} \left[\int_s^\infty q(v)\,\mathrm{d}v\right]^{\frac{1}{\alpha\beta}} \mathrm{d}s. \tag{15}$$

Quick computation shows that

$$\int_T^t \frac{u'(s)}{u^{\frac{1}{\alpha\beta}}(s)}\,\mathrm{d}s = \left(1-\frac{1}{\alpha\beta}\right) \int_T^t \left(\frac{1}{u^{\frac{1}{\alpha\beta}-1}(s)}\right)' \mathrm{d}s =$$

$$= \left(1-\frac{1}{\alpha\beta}\right) \left(\frac{1}{u^{\frac{1}{\alpha\beta}-1}(t)} - \frac{1}{u^{\frac{1}{\alpha\beta}-1}(T)}\right).$$

Since $\alpha\beta < 1$ and u is positive and increasing, we have that $\int_T^\infty \frac{u'(s)}{u^{\frac{1}{\alpha\beta}}(s)}\,\mathrm{d}s < \infty$. Passing $t \to \infty$ in (15) we get the contradiction with condition (14), hence $\mathcal{N}_2 = \emptyset$.

Rest of the proof follows from Theorem 1.

The next theorem is less subtle than theorems before (no role of function p in the criteria), but it has two advantages. There are no additional assumptions on α and β and δ can be either delay or advanced argument.

Theorem 5 *Assume* (6). *If* (4) *and*

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{\delta(t)} r^{\frac{1}{\beta}}(s)\, \mathrm{d}s\, \mathrm{d}t = \infty, \tag{16}$$

then (E) has property A.

Proof Without loss of generality assume that x is a positive solution from class $\mathcal{N}_2$. Then u, y and z are monotone and there exists T_x such that $x(t) > 0$, $u(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for $t \geq T_x$ and $T \geq T_x$ such that $\delta(t) \geq T_x$ for $t \geq T$.

Since $u^{[1]}$ is an eventually positive increasing function, we have $u^{[1]}(t) > u^{[1]}(T)$ and by integrating from T to t we get

$$u(t) > \left(u^{[1]}(T)\right)^{\frac{1}{\beta}} \int_{T}^{t} r^{\frac{1}{\beta}}(s)\, \mathrm{d}s = K \int_{T}^{t} r^{\frac{1}{\beta}}(s)\, \mathrm{d}s.$$

Applying (2) yields

$$x\left(\delta(t)\right) \geq (1-a_0)u\left(\delta(t)\right) \geq (1-a_0)K \int_{T}^{\delta(t)} r^{\frac{1}{\beta}}(s)\, \mathrm{d}s. \tag{17}$$

Integrating equation (E) from T to ∞ we have

$$u^{[2]}(T) - u^{[2]}(\infty) = \int_{T}^{\infty} q(s) f\left(x\left(\delta(s)\right)\right) \mathrm{d}s.$$

Since $u^{[2]}$ is decreasing, $\int_{T}^{\infty} q(s) f\left(x\left(\delta(s)\right)\right) \mathrm{d}s < \infty$. Moreover, using (6) we obtain

$$\inf_{s \geq T} \frac{f\left(x\left(\delta(s)\right)\right)}{x\left(\delta(s)\right)} \int_{T}^{\infty} q(s) x\left(\delta(s)\right) \mathrm{d}s \leq \int_{T}^{\infty} q(s) f\left(x\left(\delta(s)\right)\right) \mathrm{d}s,$$

i.e.

$$\int_{T}^{\infty} q(s) x\left(\delta(s)\right) \mathrm{d}s < \infty.$$

Replacing $x\left(\delta(t)\right)$ by (17) we get

$$(1-a_0)K \int_{T}^{\infty} q(t) \int_{T}^{\delta(t)} r^{\frac{1}{\beta}}(s)\, \mathrm{d}s\, \mathrm{d}t < \infty,$$

which contradicts (16), i.e. $\mathcal{N}_2 = \emptyset$.

Rest of the proof follows from Theorem 1.

4 Oscillation Theorem for Sublinear Case

In this section we deal with the so-called sublinear case, i.e. we will assume that

$$\int_0^1 \frac{\mathrm{d}v}{f^{\frac{1}{\alpha\beta}}(v)} < \infty. \tag{18}$$

Observe that, if $\alpha = \beta = 1$ then condition (18) is fulfilled for example for $f(v) = v^\lambda$, where $0 < \lambda < 1$.

Theorem 6 *Assume that f is nondecreasing in $\mathbb{R}$ such that $f(uv) \geq f(u)f(v)$ for $u, v \in \mathbb{R}$, $\delta(t) \leq t$ and (18).*

If (4) and

$$\int_{t_0}^{\infty} q(t) f\left(\int_{t_0}^{\delta(t)} r^{\frac{1}{\beta}}(s)\left[\int_{t_0}^{s} p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}} \mathrm{d}s\right)\mathrm{d}t = \infty, \tag{19}$$

then equation (E) has property A.

Proof Without loss of generality assume that x is a positive solution from class $\mathcal{N}_2$. Then u, y and z are monotone and there exist T_x such that $x(t) > 0$, $u(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for $t \geq T_x$ and $T \geq T_x$ such that $\delta(t) \geq T_x$ for $t \geq T$.

Since $u^{[2]}$ is decreasing, we get by integrating from T to t

$$u^{[1]}(t) = u^{[1]}(T) + \int_T^t p^{\frac{1}{\alpha}}(s)\left(u^{[2]}(s)\right)^{\frac{1}{\alpha}} \mathrm{d}s \geq \left(u^{[2]}(t)\right)^{\frac{1}{\alpha}} \int_T^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s$$

and it follows that

$$u'(t) \geq \left(u^{[2]}(s)\right)^{\frac{1}{\alpha\beta}} r^{\frac{1}{\beta}}(t)\left[\int_T^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s\right]^{\frac{1}{\beta}}.$$

Using this inequality we obtain

$$u(t) \geq u(t) - u(T) = \int_T^t u'(s)\,\mathrm{d}s \geq \left(u^{[2]}(t)\right)^{\frac{1}{\alpha\beta}} \int_T^t r^{\frac{1}{\beta}}(s)\left[\int_T^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}} \mathrm{d}s.$$

Since $\delta(t) \leq t$, $u^{[2]}$ is decreasing and $f(uv) \geq f(u)f(v)$, we get

$$f\big(u\big(\delta(t)\big)\big) \geq f\left(\left(u^{[2]}(t)\right)^{\frac{1}{\alpha\beta}}\right) f\left(\int_T^{\delta(t)} r^{\frac{1}{\beta}}(s)\left[\int_T^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}} \mathrm{d}s\right) \geq$$
$$\geq f^{\frac{1}{\alpha\beta}}\left(u^{[2]}(t)\right) f\left(\int_T^{\delta(t)} r^{\frac{1}{\beta}}(s)\left[\int_T^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}} \mathrm{d}s\right). \tag{20}$$

Using (2) we get from (E) the following estimate

$$-\left(u^{[2]}(t)\right)' = q(t)f\left(x\left(\delta(t)\right)\right) \geq q(t)f(1-a_0)f\left(u\left(\delta(t)\right)\right).$$

Applying (20) and integrating from T to t yields

$$-\int_T^t \frac{\left(u^{[2]}(w)\right)'}{f^{\frac{1}{\alpha\beta}}\left(u^{[2]}(w)\right)}\mathrm{d}w \geq f(1-a_0)\int_T^t q(w)f\left(\int_T^{\delta(w)} r^{\frac{1}{\beta}}(s)\left[\int_T^s p^{\frac{1}{\alpha}}(v)\,\mathrm{d}v\right]^{\frac{1}{\beta}}\mathrm{d}s\right)\mathrm{d}w.$$

Passing $t \to \infty$ and using

$$-\int_T^\infty \frac{\left(u^{[2]}(w)\right)'}{f^{\frac{1}{\alpha\beta}}\left(u^{[2]}(w)\right)}\mathrm{d}w = \int_{u^{[2]}(\infty)}^{u^{[2]}(T)} \frac{\mathrm{d}s}{f^{\frac{1}{\alpha\beta}}(s)} < \infty$$

we get the contradiction, i.e. $\mathcal{N}_2 = \emptyset$.

Rest of the proof follows from Theorem 1.

5 Examples

Example 1 Consider the equation

$$\left(t\left[\left(\frac{1}{t}\left((x(t)+a(t)x(\gamma(t)))'\right)^{\frac{2}{3}}\right)'\right]^3\right)' + \frac{l}{t^4}x^\lambda\left(\frac{t}{k}\right) = 0, \tag{21}$$

where $\lambda > 1$, $k > 1$, $0 \leq a(t) \leq a_0 < 1$ and $t \geq 1$.

In this case we have $\alpha = 3$, $\beta = \frac{2}{3}$, $p(t) = \frac{1}{t}$, $r(t) = t$ and $q(t) = \frac{l}{t^4}$ and (6) holds. Condition (4), which reads as

$$\int_1^\infty \left[t\left[\int_t^\infty \frac{1}{s}\int_s^\infty \frac{l}{v^4}\,\mathrm{d}v\right]^{\frac{1}{3}}\mathrm{d}s\right]^{\frac{3}{2}}\mathrm{d}t = \infty,$$

is fulfilled for every $l > 0$. Similarly, condition (7) reads as

$$\int_1^\infty \frac{l}{t^4}\int_1^{\frac{t}{k}} s^{\frac{3}{2}}\left[\int_1^s \frac{1}{v^{\frac{1}{3}}}\,\mathrm{d}v\right]^{\frac{3}{2}}\mathrm{d}s\,\mathrm{d}t = \infty$$

and holds for every $l > 0$ as well. Hence, according Theorem 2 Eq. (21) has property A for every $l > 0$.

Note, that condition (16) is not fulfilled, so Theorem 5 cannot be applied.

Example 2 Consider the equation

$$\left(t\left[\left(x(t)+ax\left(\frac{t}{2}\right)\right)''\right]^{\alpha}\right)'+\frac{l}{t^2}x\,(t)=0,\qquad t\geq 1,\quad a\in[0,1),\quad l>0. \tag{22}$$

One can easily check that (16) is satisfied and (4) holds for every $\alpha \geq \frac{2}{3}$. Therefore Eq. (22) has property A for every $l > 0$ and $\alpha \geq \frac{2}{3}$ by Theorem 5.

This results generalizes and improves Corollary 3 in [3], where Eq. (22) has property A for $\alpha = 1$ and $l > \frac{2}{1-a}$.

Example 3 Consider the equation

$$\left(\left[\left(\left((x(t)+a(t)x\,(\gamma(t)))'\right)^2\right)'\right]^2\right)'+\frac{l}{t^2}x^{\frac{1}{2}}\left(\frac{t}{k}\right)=0, \tag{23}$$

where $k > 1$, $0 \leq a(t) \leq a_0 < 1$ and $t \geq 1$.

In this case we have $\alpha = \beta = 2$, $p(t) = r(t) = 1$, $q(t) = \frac{l}{t^2}$ and condition (6) does not hold. Since we have that

$$\int_0^1 \frac{\mathrm{d}v}{v^{\frac{1}{8}}} < \infty,$$

i.e. condition (18) holds, we can use Theorem 6.

It is obvious that condition (4) holds for every $l > 0$ and (19) reads as

$$\int_1^{\infty} \frac{l}{t^{\frac{3}{2}}}\left[\int_1^{\frac{t}{k}}\left[\int_1^{s}\mathrm{d}v\right]^{\frac{1}{2}}\mathrm{d}s\right]^{\frac{1}{2}}\mathrm{d}t=\infty$$

and holds for every $l > 0$ as well. Therefore, according Theorem 6, Eq. (23) has property A for every $l > 0$.

Conclusion

By taking more general form of equation (E), i.e. having $\alpha \neq 1$ and $\beta \neq 1$, we lost some precision in our criterias given in [6], because now we are not able to determine value of critical constant in borderline cases. On the other hand our technique of asymptotical integration gives new results in comparison to using Ricatti technique, which is used only in case when $\beta = 1$.

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Conjecture on Fučík Curve Asymptotes for a Particular Discrete Operator

Iveta Looseová

Abstract In this paper we study properties of the Neumann discrete problem. We investigate so called polar Pareto spectrum of a specific matrix which represents the Neumann discrete operator. There is a known relation between polar Pareto spectrum of any discrete operator and its Fučík spectrum. We also state a conjecture about asymptotes of Fučík curves with respect to the matrix and we illustrate a variety of polar Pareto eigenvectors corresponding to a fixed polar Pareto eigenvalue.

Keywords Fučík spectrum · Pareto spectrum · Discrete operator · Asymptotes of the Fučík curves

1 Introduction

First of all, let us consider sets of numbers $\mathbb{T} = \{0, \dots, n-1\}$ and $\hat{\mathbb{T}} = \{-1, \dots, n\}$, where $n \in \mathbb{N} \setminus \{1\}$. Let $u : \hat{\mathbb{T}} \to \mathbb{R}$ and let us denote a positive part of u by $u^+ : t \mapsto \max\{u(t), 0\}$ and negative part of u by $u^- : t \mapsto \max\{-u(t), 0\}$.

In this paper, we study the following discrete problem with Neumann boundary conditions

$$\begin{cases} -\Delta^2 u(t-1) = \alpha u^+(t) - \beta u^-(t),\ t \in \mathbb{T}, \\ \delta_c u(0) = \delta_c u(n-1) = 0, \end{cases} \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$, the difference operator is defined as

$$\Delta^2 u(t-1) = u(t-1) - 2u(t) + u(t+1)$$

and the boundary conditions are given by the central difference as

$$\delta_c u(0) = \frac{u(1) - u(-1)}{2}, \quad \delta_c u(n-1) = \frac{u(n) - u(n-2)}{2}.$$

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Problem (1) is equivalent to the matrix equation

$$\mathbf{A}^{\mathrm{N}}\mathbf{u} = \alpha\mathbf{u}^{+} - \beta\mathbf{u}^{-}, \tag{2}$$

where $\mathbf{u}^{\pm}$ is a vector $\mathbf{u}^{\pm} = \left[u^{\pm}(0), \ldots, u^{\pm}(n-1)\right]^{T}$ and matrix $\mathbf{A}^{\mathrm{N}}$ is tridiagonal non-symmetric square matrix of size $n \times n$ taking the form

$$\mathbf{A}^{\mathrm{N}} = \begin{bmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{bmatrix}. \tag{3}$$

We are interested in the properties of the discrete operator in (1), thus we investigate properties of matrix $\mathbf{A}^{\mathrm{N}}$ defined by (3).

The structure of the article is the following. In Sect. 2 we obtain results regarding the spectrum of matrix $\mathbf{A}^{\mathrm{N}}$ and the polar Pareto spectrum of $\mathbf{A}^{\mathrm{N}}$. In Sect. 3 we show the connection between this polar Pareto spectrum and corresponding Fučík spectrum. We also state a conjecture on asymptotes of the Fučík curves. A detailed investigation of Fučík spectrum problems for the second order difference operators is still an ongoing research topic.

A very little is known in general about the description of the Fučík spectrum for matrices. If we deal with a specific matrix, the situation is usually not any better. Thus any information about the Fučík spectrum or its structure (for example asymptotes of the Fučík curves) is appreciated and it might be helpful to other related research topics.

2 Polar Pareto Spectrum of $\mathbf{A}^{\mathrm{N}}$

First of all, we discuss the spectrum of matrix $\mathbf{A}^{\mathrm{N}}$.

Theorem 1 *Let* $\mathbf{A}^{\mathrm{N}}$ *be defined by* (3), $n \in \mathbb{N} \setminus \{1\}$. *Eigenvalues of* $\mathbf{A}^{\mathrm{N}}$ *are simple of the form*

$$\lambda_k = 4\sin^2 \frac{k\pi}{2(n-1)}, \quad k \in \{0, 1, \ldots, n-1\},$$

with the corresponding eigenvectors

$$u_k(t) = \cos \frac{k\pi t}{n-1}, \quad t \in \mathbb{T}.$$

Proof Eigenvalues of the matrix $\mathbf{A}^{\mathrm{N}}$ are real since $\mathbf{A}^{\mathrm{N}}$ is similar to the symmetric matrix with the change of basis matrix $\mathbf{P}$ such that $\mathbf{P}$ is diagonal with entries

$[1, \frac{\sqrt{2}}{2}, \ldots, \frac{\sqrt{2}}{2}, 1]^T$. Hence, the eigenvalues can be found using standard tools for retrieving eigenvalues of the second order difference operators. We can find this approach in the book [4] on the p. 280 in the Example 7.1.

In this part we define *Pareto spectrum* and *polar Pareto spectrum* of a matrix. More details about Pareto spectrum can be found in articles [1, 7, 8] and the concept of polar Pareto eigenvalues is introduced in article [3].

Let $\mathbb{M}_n$ be a set of all real square matrices of size $n \times n$. Let $\mathbf{x}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{x} = [\mathbf{x}(0), \ldots, \mathbf{x}(n-1)]^T$. Then $\mathbf{x} \geq \mathbf{0}$ is equivalent to $\mathbf{x}(0) \geq 0, \ldots, \mathbf{x}(n-1) \geq 0$. Scalar product of vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$. Let us define Pareto eigenvalues and polar Pareto eigenvalues.

Definition 1 A real number λ is called a *(polar) Pareto eigenvalue* of matrix $\mathbf{B} \in \mathbb{M}_n$ if there exists non-zero vector $\mathbf{x} \in \mathbb{R}^n$ satisfying

$$\mathbf{x} \geq \mathbf{0}, \quad \mathbf{Bx} - \lambda \mathbf{x} (\leq) \geq \mathbf{0}, \quad \langle \mathbf{x}, \mathbf{Bx} - \lambda \mathbf{x} \rangle = 0. \tag{4}$$

Moreover, $\mathbf{x}$ is a *(polar) Pareto eigenvector* of matrix $\mathbf{B}$ (corresponding to the (polar) Pareto eigenvalue λ). A set of all (polar) Pareto eigenvalues of matrix $\mathbf{B}$ is called *(polar) Pareto spectrum* of matrix $\mathbf{B}$ and it is denoted by $\sigma_{\text{pareto}}(\mathbf{B})$ or $\sigma^{\circ}_{\text{pareto}}(\mathbf{B})$ for polar Pareto spectrum respectively.

The difference between Pareto eigenvalues and polar Pareto eigenvalues lies only in the second inequality in (4). Let us point out, that there is a known relation between Pareto and polar Pareto eigenvalues. Let $\mathbf{B} \in \mathbb{M}_n$. Then $\lambda \in \sigma^{\circ}_{\text{pareto}}(\mathbf{B})$ if and only if $-\lambda \in \sigma_{\text{pareto}}(-\mathbf{B})$.

The estimation of the total number of Pareto eigenvalues δ_n with respect to a general matrix $B \in \mathbb{M}_n$ is (see article [7]) $1 \leq \delta_n \leq n2^{n-1} - (n-1)$. For obtaining the Pareto spectrum of some matrix we can use a numerical approach which is based on Semi-Smooth Newton method, details can be found in the article [1]. Advantage of this method is that some of the Pareto eigenvalues are obtained easily. On the other hand, it might not find all Pareto eigenvalues. In Fig. 1 there are polar Pareto eigenvectors corresponding to different polar Pareto eigenvalues of matrix $\mathbf{A}^{\mathrm{N}}$ for $n = 10$ obtained numerically by Semi-Smooth Newton method.

In the two following theorems we give a description of some numbers which are polar Pareto eigenvalues of matrix $\mathbf{A}^{\mathrm{N}}$ and its corresponding polar Pareto eigenvectors. Let the polar Pareto eigenvectors given by these theorems be referred to as *basic polar Pareto eigenvectors*.

Theorem 2 *Let* $\mathbf{A}^{\mathrm{N}}$ *be defined by* (3), $n \in \mathbb{N} \setminus \{1\}$. *Number* $\lambda_0 = 0$ *is polar Pareto eigenvalue of matrix* $\mathbf{A}^{\mathrm{N}}$ *and its corresponding polar Pareto eigenvector is* $\mathbf{u}_0 = [1, \ldots, 1]^T$.

Let us denote two sets of numbers E_n, O_n as

$$E_n = \{i : i = 2k, k \in \mathbb{N}, i \leq 2(n-1)\}, \tag{5}$$

$$O_n = \{i : i = 2k+1, k \in \mathbb{N}, i \leq n-1\}. \tag{6}$$

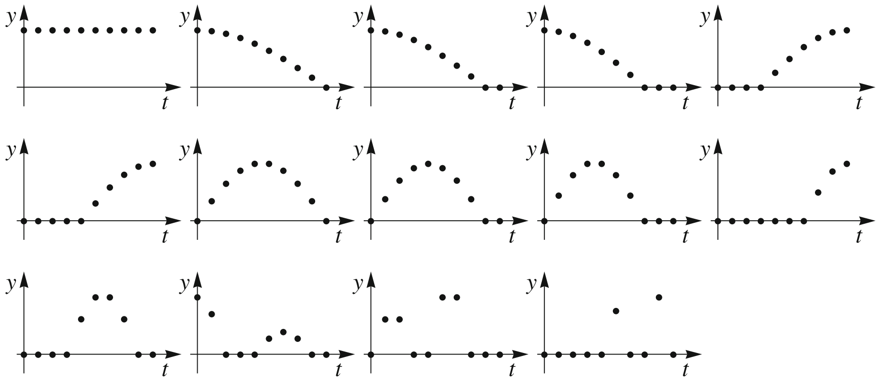

Fig. 1 Polar Pareto eigenvectors corresponding to different polar Pareto eigenvalues of matrix $\mathbf{A}^{\mathrm{N}}$ for $n = 10$ obtained numerically by Semi-Smooth Newton method

Theorem 3 *Let* $\mathbf{A}^{\mathrm{N}}$ *be defined by* (3)*,* $n \in \mathbb{N} \setminus \{1\}$*. Numbers*

$$\lambda_p = 2 - 2\cos\frac{\pi}{p}, \quad p \in E_n \cup O_n,$$

are polar Pareto eigenvalues of matrix $\mathbf{A}^{\mathrm{N}}$*. Corresponding polar Pareto eigenvector for polar Pareto eigenvalue* λ_p*, where* $p \in E_n$*, is*

$$\mathbf{u}_p(t) = \begin{cases} \cos\left(\frac{1}{p}\pi t\right) & \textit{for } t \in \{0, \ldots, \frac{p}{2}\}, \\ 0 & \textit{for } t \in \{\frac{p}{2} + 1, \ldots, n-1\}, \end{cases}$$

and for $p \in O_n$ *is*

$$\mathbf{u}_p(t) = \begin{cases} \sin\left(\frac{1}{p}\pi t\right) & \textit{for } t \in \{0, \ldots, p\}, \\ 0 & \textit{for } t \in \{p + 1, \ldots, n-1\}. \end{cases}$$

Proof (*Theorems* 2 and 3) Proof of assertions in Theorems 2 and 3 is technical and consists of verification of conditions in Definition 1.

Basic polar Pareto eigenvectors corresponding to different polar Pareto eigenvalues of matrix $\mathbf{A}^{\mathrm{N}}$ for $n = 10$ given by Theorems 2 and 3 are illustrated in Fig. 2. As a consequence we have that lower total number estimate of polar Pareto eigenvalues of matrix $\mathbf{A}^{\mathrm{N}}$ is $n + \lfloor \frac{n}{2} \rfloor - 1$.

Let us discuss in more detail polar Pareto eigenvectors. We have introduced basic polar Pareto eigenvectors. Using these we can generate other vectors which are also polar Pareto eigenvectors for the same polar Pareto eigenvalue. We want to illustrate (in Sects. 2.1 and 2.2) a variety of polar Pareto eigenvectors corresponding to a fixed polar Pareto eigenvalue.

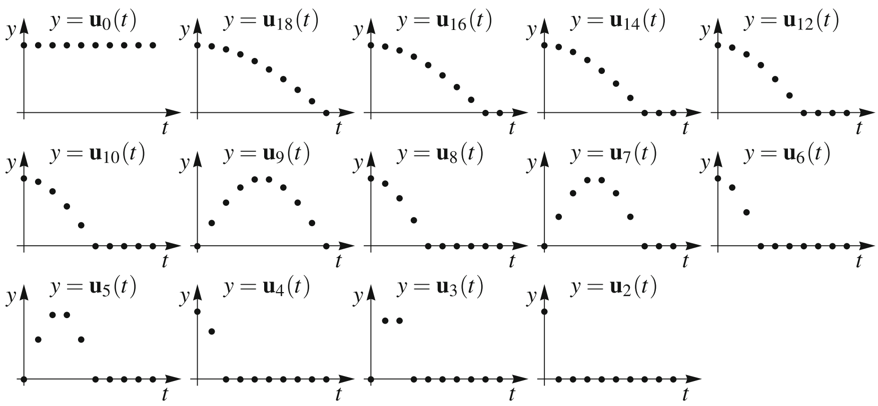

Fig. 2 Basic polar Pareto eigenvectors corresponding to different polar Pareto eigenvalues of matrix $\mathbf{A}^{\mathrm{N}}$ for $n = 10$

Let $\mathbf{u}_p = [\mathbf{u}_p(0), \ldots, \mathbf{u}_p(n-1)]^T$ be a basic polar Pareto eigenvector corresponding to the polar Pareto eigenvalue λ_p (see Theorem 3). Interval $[0, n-1]$ is split into $(n-1)$ equidistant subintervals d_i of length one

$$d_i = [i, i+1],\ i \in \{0, \ldots, n-2\},$$

i.e. $d_0 = [0, 1]$, $d_1 = [1, 2], \ldots, d_{n-2} = [n-2, n-1]$. Let us define set $J_{\mathbf{u}}$ as

$$J_{\mathbf{u}} = \{i \in \{0, \ldots, n-2\} : \mathbf{u}_p(i) \neq 0\ \vee\ \mathbf{u}_p(i+1) \neq 0\}. \tag{7}$$

Hence set $J_{\mathbf{u}}$ contains indices i of subintervals d_i for which the value of vector $\mathbf{u}_p$ is zero in at least one of the end points of subinterval d_i.

2.1 Polar Pareto Eigenvectors with Respect to the Set O_n

In this section we will only deal with polar Pareto eigenvalues λ_p, where p belongs to the number set O_n defined in (6).

Lemma 1 *Let $\mathbf{A}^{\mathrm{N}}$ be defined by (3), $n \in \mathbb{N} \setminus \{1\}$. Let $p \in O_n$. Vector $\mathbf{v}_p(t) = \mathbf{u}_p(t - n + 1)$ is polar Pareto eigenvector corresponding to the polar Pareto eigenvalue λ_p.*

Proof Showing that vector

$$\mathbf{v}_p(t) = \mathbf{u}_p(t-n+1) = \begin{cases} 0 & \text{for } t \in \{0, \ldots, n-2-\frac{p}{2}\}, \\ \cos\left(\frac{1}{p}\pi(t-n+1)\right) & \text{for } t \in \{n-1-\frac{p}{2}, \ldots, n-1\}, \end{cases}$$

satisfies conditions in Definition 1 (definition of polar Pareto eigenvectors) is straightforward.

Algorithms 1, 2 represent the way how to get other polar Pareto eigenvectors for a fixed polar Pareto eigenvalue using the basic polar Pareto eigenvector $\mathbf{u}_p$ (see Lemmas 2 and 3). In Fig. 3 we illustrate the upcoming algorithms and lemmas.

```
Data: n, u_p, p
Result: w_p
w_p := NULL;
w_p(0) := 0;
i := 1;
while i ≤ n − 1 do
    if i ≤ p/2 then
        w_p(i) := u_p(p/2 − i);
        w_p(p − i) := u_p(p/2 − i);
    end
    if i ≥ p then
        w_p(i) := 0;
    end
    i := i + 1;
end
```

Algorithm 1: Algorithm for retrieving polar Pareto eigenvector $\mathbf{w}_p$ (see Lemma 2).

Lemma 2 *Let* $\mathbf{A}^{\mathbb{N}}$ *be defined by* (3)*,* $n \in \mathbb{N} \setminus \{1\}$*. Let* $p \in O_n$*. If* $p \leq n-1$ *then vector* $\mathbf{w}_p$ *retrieved from Algorithm 1 is the polar Pareto eigenvector corresponding to the polar Pareto eigenvalue* λ_p*.*

Proof To show that vector

$$\mathbf{w}_p(t) = \begin{cases} \sin\left(\frac{1}{p}\pi t\right) & \text{for } t \in \{0, \ldots, p\}, \\ 0 & \text{for } t \in \{p+1, \ldots, n-1\}, \end{cases}$$

satisfies conditions in Definition 1 is straightforward.

```
Data: n, w_p, a, p
Result: x_p^a
x_p^a := NULL;
i := 0;
while i ≤ n − 1 do
    if i ≤ a − 1 then
        x_p^a(i) := 0;
    else
        x_p^a(i) := w_p(i − a);
    end
    i := i + 1;
end
```

Algorithm 2: Algorithm for retrieving polar Pareto eigenvectors $\mathbf{x}_p^a$ (see Lemma 3).

Lemma 3 *Let* $\mathbf{A}^{\mathrm{N}}$ *be defined by* (3), $n \in \mathbb{N} \setminus \{1\}$. *Let* $p \in O_n$ *and* $\mathbf{w}_p$ *vector retrieved from Algorithm 1. Then for* $a \in \{1, \ldots, n-1-p\}$ *vectors* $\mathbf{x}_p^a$ *retrieved from Algorithm 2 are polar Pareto eigenvectors corresponding to the polar Pareto eigenvalue* λ_p.

Proof Showing that for $a \in \{1, \ldots, n-1-p\}$ vectors

$$\mathbf{x}_p^a(t) = \begin{cases} \sin\left(\frac{1}{p}\pi(t-a)\right) & \text{for } t \in \{a, \ldots, p+a\}, \\ 0 & \text{for } t \in \{0, \ldots, n-1\} \setminus \{a, \ldots, p+a\}, \end{cases}$$

satisfy conditions in Definition 1 is straightforward.

Let us define set V_p^O (see Fig. 3) as set of all polar Pareto eigenvectors corresponding to the polar Pareto eigenvalue λ_p, $p \in O_n$ determined by Theorems 2, 3 and Lemmas 1–3, i. e:

$$V_p^O = \{\mathbf{u}_p, \mathbf{v}_p, \mathbf{w}_p, \mathbf{x}_p^1, \ldots, \mathbf{x}_p^{n-1-p}\}. \tag{8}$$

Next theorem shows us that any vector which is a certain combination of polar Pareto eigenvectors from the set V_p^O is (if one condition is satisfied) also a polar Pareto eigenvector to the same polar Pareto eigenvalue.

Theorem 4 *Let* $\mathbf{A}^{\mathrm{N}}$ *be defined by* (3), $n \in \mathbb{N} \setminus \{1\}$. *Let* $p \in O_n$. *Let* $\mathbf{a}, \mathbf{b} \in V_p^O$ *and let* $J_{\mathbf{a}}, J_{\mathbf{b}}$ *be corresponding sets defined in* (7). *If* $J_{\mathbf{a}} \cap J_{\mathbf{b}} = \emptyset$ *is satisfied, then*

$$\mathbf{z} = C_1\mathbf{a} + C_2\mathbf{b}, \quad C_1 \geq 0, \; C_2 \geq 0, \; C_1C_2 \neq 0,$$

is a polar Pareto eigenvector corresponding to the polar Pareto eigenvalue λ_p.

Proof Proof of the theorem is immediately obtained by substituting to Definition 1.

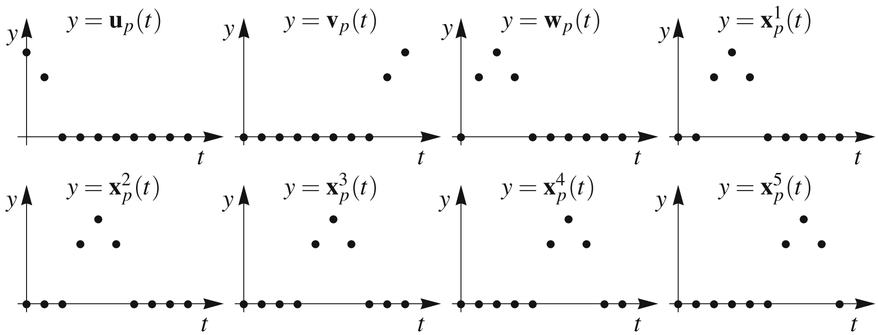

Fig. 3 Vectors belonging to the set V_p^O (defined in (8)) for $n = 10$ and $p = 4$

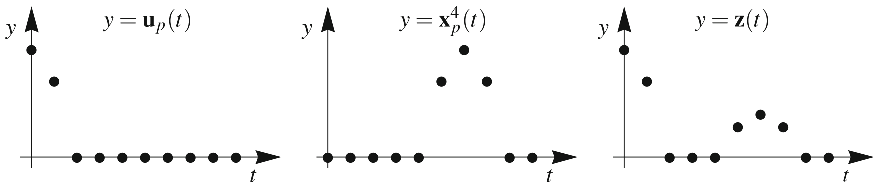

Fig. 4 Basic polar Pareto eigenvector $\mathbf{u}_p$, polar Pareto eigenvector $\mathbf{x}_p^4$ and polar Pareto eigenvector $\mathbf{z} = \mathbf{u}_p + 0.4\mathbf{x}_p^4$ for $n = 10$ and $p = 4$ (corresponding to polar Pareto eigenvalue λ_4 of matrix $\mathbf{A}^{\mathbb{N}}$)

In the Fig. 4 there is an example of results from Theorem 4. We have $\mathbf{a} = \mathbf{u}_p$, $\mathbf{b} = \mathbf{x}_p^4$, thus the final vector $\mathbf{z} = C_1\mathbf{a} + C_2\mathbf{b}$ (right) for $C_1 = 1$, $C_2 = 0.4$ is also polar Pareto eigenvector corresponding to the polar Pareto eigenvalue λ_p, where $p = 4$ and $n = 10$.

2.2 Polar Pareto Eigenvectors with Respect to the Set E_n

In this section we inspect polar Pareto eigenvalues λ_p, where p belongs to the number set E_n defined in (5). Situation is very similar as in the case $p \in O_n$.

Data: $n, \mathbf{u}_p, a, p$
Result: $\mathbf{y}_p^a$
$\mathbf{y}^a := NULL$;
$i := 0$;
while $i \leq n - 1$ **do**
  if $i \leq a - 1$ **then**
    $\mathbf{y}_p^a(i) := 0$;
  else
    $\mathbf{y}_p^a(i) := \mathbf{u}_p(i - a)$;
  end
  $i := i + 1$;
end

Algorithm 3: Algorithm for retrieving polar Pareto eigenvectors $\mathbf{y}_p^a$ (see Lemma 4).

Lemma 4 *Let* $\mathbf{A}^{\mathbb{N}}$ *be defined by* (3), $n \in \mathbb{N} \setminus \{1\}$. *Let* $p \in E_n$. *Then for* $a \in \{1, \ldots, n - 1 - p\}$ *vectors* $\mathbf{y}_p^a$ *retrieved from Algorithm 3 are polar Pareto eigenvectors corresponding to the polar Pareto eigenvalue* λ_p.

Proof Proof of this lemma is straightforward and it is obtained as in Lemma 3.

Notice that Algorithms 2 and 3 are almost the same with only difference in the input data (instead of $\mathbf{w}_p$ we use $\mathbf{u}_p$ in Algorithm 3).

Let us define set V_p^E as

$$V_p^E = \{\mathbf{u}_p, \mathbf{y}_p^1, \ldots, \mathbf{y}_p^{n-1-p}\}.$$

Theorem 5 *Let $\mathbf{A}^{\mathrm{N}}$ be defined by (3), $n \in \mathbb{N} \setminus \{1\}$. Let $p \in E_n$. Let $\mathbf{a}, \mathbf{b} \in V_p^E$ and let $J_{\mathbf{a}}, J_{\mathbf{b}}$ be corresponding sets defined in (7). If $J_{\mathbf{a}} \cap J_{\mathbf{b}} = \emptyset$ is satisfied, then*

$$\mathbf{z} = C_1\mathbf{a} + C_2\mathbf{b}, \quad C_1 \geq 0,\ C_2 \geq 0,\ C_1C_2 \neq 0,$$

is polar Pareto eigenvector corresponding to the polar Pareto eigenvalue λ_p.

Proof Again, we would approach the proof in the same way as for Theorem 4.

3 Fučík spectrum of $\mathbf{A}^{\mathrm{N}}$

In the previous section we studied specific properties of a discrete problem with Neumann boundary conditions (1) which can be written in the matrix form as $\mathbf{A}^{\mathrm{N}}\mathbf{u} = \alpha\mathbf{u}^+ - \beta\mathbf{u}^-$. We discussed the polar Pareto spectrum of matrix $\mathbf{A}^{\mathrm{N}}$. In this section we will explore properties of the Fučík spectrum of matrix $\mathbf{A}^{\mathrm{N}}$ and its connection with the corresponding polar Pareto spectrum. In Fig. 5, one can observe the Fučík spectrum of matrix $\mathbf{A}^{\mathrm{N}}$ for $n = 6$ and $n = 10$ obtained numerically. In general, a Fučík spectrum for a matrix is defined as follows.

Definition 2 The *Fučík spectrum* of matrix $\mathbf{B} \in \mathbb{M}_n$ is the set $\Sigma(\mathbf{B})$ of all pairs $(\alpha, \beta) \in \mathbb{R}^2$, for which there exists a non-trivial solution $\mathbf{u}$ of the problem $\mathbf{B}\mathbf{u} = \alpha\mathbf{u}^+ - \beta\mathbf{u}^-$. The pair $(\alpha, \beta) \in \Sigma(\mathbf{B})$ is called the Fučík eigenvalue.

Before we inspect a relation between asymptotes of Fučík curves and the polar Pareto spectrum, we introduce already known properties regarding Fučík spectrum with respect to discrete operators.

Authors Ma et al. [6] gave an expression of the Fučík spectrum regarding Dirichlet matrix (a similar discrete problem to (1), only Dirichlet boundary conditions are

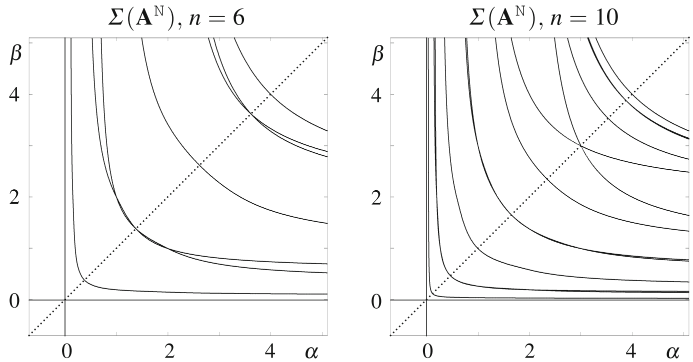

Fig. 5 Numerical reconstruction of the Fučík spectrum of matrix $\mathbf{A}^{\mathrm{N}}$ for $n = 6$ (left) a $n = 10$ (right)

considered instead of Neumann boundary conditions) via the matching-extension method. They described all points from $(0, 4) \times (0, 4)$ belonging to the Fučík spectrum. A part of this expression has to be calculated numerically before the matching-extension method can be used.

Stehlík [9] studied the first non-trivial Fučík curve of Dirichlet matrix in detail. In doing so, necessary conditions for points from $\mathbb{R}^2$ belonging to the first non-trivial Fučík curve were introduced. A conjecture that this curve has no elementary parametrization was stated in the article.

Authors Holubová and Nečesal [2] discussed similarities of structures in the Fučík spectrum for continuous and discrete operators. They also suggested an algorithm for numerical reconstruction of the Fučík spectrum for reasonably small matrices.

A connection between polar Pareto spectrum and Fučík spectrum was inspected in the article [3]. Authors discussed a more general relation between Fučík spectrum and so called K-spectrum.

Recently, a paper investigating Fučík spectrum of the Dirichlet matrix has appeared [5]. Moreover, all obtained results therein can be also applied for discrete problems with different local boundary conditions (for example problem (1) with Neumann boundary conditions). Authors provide the exact implicit description of all non-trivial Fučík curves. Furthermore, for each non-trivial Fučík curve, they give several different implicit descriptions, which differ in the level of depth of used nested functions. All presented descriptions of Fučík curves have the form of necessary and sufficient conditions.

Let $\mathbf{B} \in \mathbb{M}_n$. The Fučík spectrum of a matrix consists of finitely many algebraic curves, so called Fučík curves. Next theorem states the following. If the half-line $\alpha = \mu \in \mathbb{R},\ \beta \geq 0$, is a vertical asymptote of the Fučík curve (thus this Fučík curve has an asymptotic behaviour in the sense of Theorem 6) then μ is the polar Pareto eigenvalue of $\mathbf{B}$. But to the contrary, the opposite implication does not hold. Not every polar Pareto eigenvalue is vertical asymptote to some Fučík curve. Moreover, due to the fact, that Fučík spectrum is in general symmetric (with respect to the line $\alpha = \beta$ in the sense: $(\alpha, \beta) \in \Sigma(\mathbf{B}) \Leftrightarrow (\beta, \alpha) \in \Sigma(\mathbf{B})$), it is enough to investigate only vertical asymptotes of the Fučík curves.

Theorem 6 *Let $\mathbf{B} \in \mathbb{M}_n$ and let a sequence of Fučík eigenvalues $(\alpha_k, \beta_k) \in \Sigma(\mathbf{B})$ exist such that $\alpha_k \to \mu \in \mathbb{R}$ for $k \to +\infty$. If $\beta_k \to +\infty$ for $k \to +\infty$ then μ is a polar Pareto eigenvalue of $\mathbf{B}$.*

Proof Proof can be found in the article [3].

Our goal is to distinguish which of the polar Pareto eigenvalues of matrix $\mathbf{A}^{\mathrm{N}}$ actually make vertical asymptotes of some Fučík curve. The following conjecture concerning this issue has not been proved yet and therefore the distinction is still an open problem.

Conjecture 1 *Let $\mathbf{A}^{\mathrm{N}}$ be defined by (3), $n \in \mathbb{N} \setminus \{1\}$. Half-line $\alpha = \lambda_0$, $\beta \geq 0$, where λ_0 is polar Pareto eigenvalue of matrix $\mathbf{A}^{\mathrm{N}}$, is a vertical asymptote of the first Fučík curve of $\Sigma(\mathbf{A}^{\mathrm{N}})$.*

1. *Let* $p \in O_n$*. Then the half-line* $\alpha = \lambda_p, \beta \geq 0$*, where* λ_p *is polar Pareto eigenvalue of matrix* $\mathbf{A}^{\mathrm{N}}$*, is vertical asymptote of some Fučík curve if and only if there exists* $i \in \{0, 1, 2\}$*,* $j \in \left\{0, \ldots, \left\lceil \frac{n}{p} \right\rceil\right\}$ *such that* $i + j \neq 0$ *and exists* $k \in \{0, \ldots, i + j - 1\}$ *such that* $n = i\left\lceil \frac{p}{2} \right\rceil + jp + k + 1$.
2. *Let* $p \in E_n$*. Then half-line* $\alpha = \lambda_p, \beta \geq 0$*, where* λ_p *is polar Pareto eigenvalue of matrix* $\mathbf{A}^{\mathrm{N}}$*, is vertical asymptote of some Fučík curve if and only if there exists* $i \in \{0, 1\}$*,* $j \in \left\{1, \ldots, \left\lceil \frac{n}{p} \right\rceil\right\}$*,* $k \in \{0, \ldots, j - 1\}$ *such that* $n = i\left\lceil \frac{p}{2} \right\rceil + jp + k + 1$.

4 Conclusion

In this paper we investigated relation between the Fučík spectrum of matrix $\mathbf{A}^{\mathrm{N}}$ defined in (3) and polar Pareto spectrum of the matrix. We found numbers which belong to the polar Pareto spectrum of matrix $\mathbf{A}^{\mathrm{N}}$. We discussed in detail polar Pareto eigenvectors belonging to a single fixed polar Pareto eigenvalue and how to generate them from so called basic polar Pareto eigenvectors. Then we provided a conjecture on vertical asymptotes to the Fučík curves in the Fučík spectrum of matrix $\mathbf{A}^{\mathrm{N}}$.

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Interval Difference Methods for Solving the Poisson Equation

Andrzej Marciniak and Tomasz Hoffmann

Abstract In the paper we resemble interval difference method of second order designed by us earlier and present new, fourth order interval difference methods for solving the Poisson equation with Dirichlet boundary conditions. Interval solutions obtained contain all possible numerical errors. Numerical solutions presented confirm the fact that the exact solutions are within the resulting intervals.

Keywords Interval difference methods · Floating-point interval arithmetic · Poisson's equation · Boundary value problem in partial differential equations

1 Introduction

As it is well-known, there are two kinds of errors caused by floating-point arithmetic: representation errors and rounding errors. When we apply an approximate method to solve a problem on a computer we introduce the third kind of error—the error of method (usually called the truncation error). Using interval methods realized in interval floating-point arithmetic we can obtain solutions (in the form of intervals) which contain all these errors.

In this paper we resemble (see [6–8, 10]) second order interval difference methods for solving the Poisson equation with boundary conditions and present new, fourth order methods. The solutions obtained by these methods are in the form of intervals

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which contain all possible numerical errors. Moreover, it has been experimentally confirmed[1] that the exact solutions are placed inside the resulting intervals.

The paper is divided into seven sections. In Sect. 2 we present shortly the basis of interval arithmetic and their realization in floating-point computer arithmetic. In Sect. 3 we recall the well-known Poisson equation with Dirichlet boundary conditions which are of our interest. In the next section we resemble our second order interval difference methods. The main section of this paper is Sect. 5, where we present new, fourth order interval difference methods for using in proper and directed interval arithmetics. Numerical examples for second and fourth order interval methods developed are presented and compared in Sect. 6. These examples have been carried out in proper and directed floating-point interval arithmetics using our `IntervalArithmetic32and64` unit [12] written in the Delphi Pascal programming language. Finally, we shortly present conclusions and problems to consider in further research.

2 Interval Arithmetic

Verified numerical computing requires a mathematical tool to describe operations performed on computers. Such a mathematical tool, called interval arithmetic, has been developed by R. E. Moore in 1966 [15, 16] and extended by other researchers in the following years (see, e.g., [1, 3, 4]). As it is well-known, a real interval, or shortly an interval, is a closed and bounded subset of real numbers $\mathbb{R}$: $[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} : \underline{x} \leq x \leq \overline{x}\}$, where $\underline{x}$ and $\overline{x}$ denote the lower and upper bounds of the interval $[x]$, respectively. An interval is called a point interval if $\underline{x} = \overline{x}$. We can distinguish real (proper) and directed interval arithmetic. In real interval arithmetic it is excluded a division by an interval containing zero. This restriction may be removed in so called *extended* (*real*) *interval arithmetic*. Both of these interval arithmetics (real and extended real) are called *proper*, since for any interval $[x] = [\underline{x} \leq \overline{x}]$ we have $\underline{x} \leq \overline{x}$. It should be noted that the opposite and the inverse elements do not exist in proper interval arithmetic. Such elements exist in so called *directed interval arithmetic*, where for any interval $[x]$ we can have either $\underline{x} \leq \overline{x}$ or $\underline{x} \geq \overline{x}$.

The realization of proper interval arithmetic is based on simple rule, where left and right endpoints are calculated by using downward and upward roundings (see, e.g., [3]). In the case of directed interval arithmetic the rules of calculating endpoints are much more complicated. For each basic operation different rounding can be used for calculation of endpoints of the result interval. The accurate description of directed interval arithmetic is presented, among others, in [11, 18].

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[1] In our opinion, it is rather impossible to obtain a theoretical proof of this fact.

3 The Poisson Equation

Our problem is to find $u = u(x, y)$ satisfying the partial-differential equation (called the Poisson equation)

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = f(x, y), \quad 0 \le x \le \alpha, \quad 0 \le y \le \beta,$$

with the Dirichlet boundary conditions

$$u|_{\Gamma}(x, y) = \varphi(x, y) = \begin{cases} \varphi_1(y), & \text{for } x = 0, \\ \varphi_2(x), & \text{for } y = 0, \\ \varphi_3(y), & \text{for } x = \alpha, \\ \varphi_4(x), & \text{for } y = \beta, \end{cases}$$

where

$$\varphi_1(0) = \varphi_2(0), \quad \varphi_2(\alpha) = \varphi_3(0), \quad \varphi_3(\beta) = \varphi_4(\alpha), \quad \varphi_4(0) = \varphi_1(\beta),$$
$$\Gamma = \{(x, y) : \ x = 0, \alpha \text{ and } 0 \le y \le \beta \text{ or } 0 \le x \le \alpha \text{ and } y = 0, \beta\}.$$

4 Interval Difference Methods of Second Order

Interval difference methods of second order based on proper and directed interval arithmetic we developed in details in [6–8, 10]. Below we resemble some essential facts.

Partitioning the interval $[0, \alpha]$ into n equal parts of width h and the interval $[0, \beta]$ into m equal parts of width k provides a mean of placing a grid on the rectangle R with mesh points $(x_i, y_j) = (ih, jk)$, where $h = \alpha/n$, $k = \beta/m$, $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$. Assuming that the fourth order partial derivatives of u exist, for each mesh point in the interior of the grid we use the Taylor series in the variable x about x_i and in the variable y about y_j. This allows us to express the Poisson equation at the points (x_i, y_j) as

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i, y_j\right) - \frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}\left(x_i, \eta_j\right) = f_{ij}, \tag{1}$$

where

$$\delta_x^2 u_{ij} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \quad \delta_y^2 u_{ij} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2},$$

$u_{ij} = u(x_i, y_j), f_{ij} = f(x_i, y_j)$, and where $\xi_i \in (x_{i-1}, x_{i+1})$, $\eta_j \in (y_{j-1}, y_{j+1})$ are intermediate points, and the boundary conditions as

$$\begin{aligned} u(0, y_j) &= \varphi_1(y_j), && \text{for } j = 0, 1, \ldots, m, \\ u(x_i, 0) &= \varphi_2(x_i), && \text{for } i = 1, 2, \ldots, n-1, \\ u(\alpha, y_j) &= \varphi_3(y_j), && \text{for } j = 0, 1, \ldots, m, \\ u(x_i, \beta) &= \varphi_4(x_i), && \text{for } i = 1, 2, \ldots, n-1. \end{aligned} \tag{2}$$

Omitting in (1) the partial derivatives, we obtain a method, called the central-difference method, with local truncation error of order $O(h^2 + k^2)$ (see, e.g., [2, 9, 14]):

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} = f_{ij}. \tag{3}$$

Such formulas together with (2) present a system of linear equations (with respect to unknowns u_{ij}), which may be solved by any known exact or iterative method.

To construct an interval method, let us assume that there exists a constant M such that

$$\left| \frac{\partial^4 u}{\partial x^2 \partial y^2} \right| \le M \text{ for all } 0 \le x \le \alpha \text{ and } 0 \le y \le \beta,$$

and let

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}(x, y) = \frac{\partial^4 u}{\partial y^2 \partial x^2}(x, y).$$

Since from the Poisson equation (1) it follows that

$$\begin{aligned} \frac{\partial^4 u}{\partial x^4}(x, y) &= \frac{\partial^2 f}{\partial x^2} - \frac{\partial^4 u}{\partial x^2 \partial y^2}(x, y), \\ \frac{\partial^4 u}{\partial y^4}(x, y) &= \frac{\partial^2 f}{\partial y^2} - \frac{\partial^4 u}{\partial y^2 \partial x^2}(x, y), \end{aligned}$$

then it is obvious that we have

$$\begin{aligned} \frac{\partial^4 u}{\partial x^4}(\xi, y) &\in \Psi(X + [-h, h], Y) + [-M, M], \\ \frac{\partial^4 u}{\partial x^4}(x, \eta) &\in \Omega(X, Y + [-k, k]) + [-M, M], \end{aligned}$$

for any $\xi \in (x - h, x + h)$ and any $\eta \in (y - k, y + k)$, where X and Y denote interval extensions of x and y, respectively, and $\Psi(X, Y)$ and $\Omega(X, Y)$ are interval extensions of $\frac{\partial^2 f}{\partial x^2}(x, y)$ and $\frac{\partial^2 f}{\partial y^2}(x, y)$, respectively.

If we recall the Poisson equation at the mesh points (1) and write the partial derivatives at the right-hand side, it is easy to write an interval analogy to this equation. Assuming that all interval extensions are proper, we have

$$\begin{aligned}
&k^2U_{i-1,j} + h^2U_{i,j-1} - 2(h^2+k^2)U_{i,j} + k^2U_{i+1,j} + h^2U_{i,j+1} \\
&= h^2k^2\left(F_{i,j} + \frac{1}{12}\left(h^2\Psi(X_i+[-h,h],Y_j) + k^2\Omega(X_i,Y_j+[-k,k])\right.\right. \\
&\qquad\left.\left. +(h^2+k^2)[-M,M]\right)\right), \\
&\qquad i=1,2,\ldots,n-1,\quad j=1,2,\ldots,m-1,
\end{aligned} \tag{4}$$

where $F_{i,j} = F(X_i, Y_j)$, and where

$$\begin{aligned}
&U_{0,j} = \Phi_1(Y_j),\quad U_{i,0} = \Phi_2(X_i),\quad U_{n,j} = \Phi_3(Y_j),\quad U_{i,m} = \Phi_4(X_i) \\
&\qquad \text{for each } j = 0,1,\ldots,m \text{ and } i = 1,2,\ldots,n-1,
\end{aligned} \tag{5}$$

$\Phi_1(Y)$, $\Phi_2(X)$, $\Phi_3(Y)$ and $\Phi_4(X)$ denote interval extensions of the functions $\varphi_1(y)$, $\varphi_2(x)$, $\varphi_3(y)$ and $\varphi_4(x)$, respectively. The system of linear equations (4)–(5) can be solved in conventional (proper) floating-point interval arithmetic, because all intervals are proper. But we can consider another analogy of (1). Namely, we can write

$$\begin{aligned}
&k^2U_{i-1,j} + h^2U_{i,j-1} - 2(h^2+k^2)U_{i,j} + k^2U_{i+1,j} + h^2U_{i,j+1} \\
&\quad - \frac{h^2k^2}{12}\left(h^2\Psi(X_i+[-h,h],Y_j) + k^2\Omega(X_i,Y_j+[-k,k])\right. \\
&\qquad \left. +(h^2+k^2)[-M,M]\right) \\
&\qquad = h^2k^2F_{i,j}, \\
&\qquad i=1,2,\ldots,n-1,\quad j=1,2,\ldots,m-1.
\end{aligned}$$

Using directed interval arithmetic, we can add at both sides of this equation the opposites to some elements. We get

$$\begin{aligned}
&k^2U_{i-1,j} + h^2U_{i,j-1} - 2(h^2+k^2)U_{i,j} + k^2U_{i+1,j} + h^2U_{i,j+1} \\
&= h^2k^2\left(F_{i,j} + \frac{1}{12}\left(h^2\Psi(X_i+[-h,h],Y_j) + k^2\Omega(X_i,Y_j+[-k,k])\right.\right. \\
&\qquad\left.\left. +(h^2+k^2)[M,-M]\right)\right), \\
&\qquad i=1,2,\ldots,n-1,\quad j=1,2,\ldots,m-1.
\end{aligned} \tag{6}$$

The last Eq. (6) differs from the Eq. (4) only by the last term on the right-hand side, which is an improper interval. But using the directed interval floating-point arithmetic we can solve the system (6) (together with (5)). If the interval solutions of this system are in the form of improper intervals, to get the proper intervals we can use the so-called proper projection of intervals, i.e. transform each interval $[a, b]$, for which $b < a$, to the interval $[b, a]$.

We should also add a remark concerning the constant M. In general, when the exact solution is unknown and nothing can be concluded about M from physical or technical properties or characteristics of the problem considered, we propose to find this constant by the following procedure: It is obvious that

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}(x_i, y_j) = \lim_{h\to 0}\lim_{k\to 0}\left(\frac{u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}}{h^2k^2} + \frac{4u_{i,j} - 2(u_{i-1,j} + u_{i,j-1} + u_{i,j+1} + u_{i+1,j})}{h^2k^2}\right).$$

We can calculate

$$M_{n,m} = \frac{1}{h^2k^2}\max_{i,j}\left|u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + 4u_{i,j} - 2(u_{i-1,j} + u_{i,j-1} + u_{i,j+1} + u_{i+1,j})\right|$$

for $i = 1, 2, \ldots, n-1, j = 1, 2, ..., m-1$ and where u_{ij} are obtained by a conventional method for a variety of n and m, say $n = m = 10, 20, \ldots, N$, where N is sufficiently large. Then, we can plot $M_{n,m}$ against different $n = m$. The constant M can be easy determined from the obtained graph, since $\lim_{n\to\infty}\lim_{m\to\infty} M_{n,m} \le M$.

5 Interval Difference Methods of Fourth Order

Using the Taylor series of higher order, we can express the Poisson equation at the points (x_i, y_j) as

$$\begin{aligned}
&\delta_x^2 u_{ij} + \delta_y^2 u_{ij} + \frac{1}{12}(h^2 + k^2)\delta_x^2\delta_y^2 u_{ij} \\
&\quad - \frac{1}{240}\left(h^4\frac{\partial^6 u}{\partial x^4 \partial y^2}(\xi_i, y_j) + k^4\frac{\partial^6 u}{\partial x^2 \partial y^4}(x_i, \eta_j)\right) \\
&\quad - \frac{h^2k^2}{144}\left(\frac{\partial^6 u}{\partial x^4 \partial y^2}(\xi_i, \eta_j) + \frac{\partial^6 u}{\partial x^2 \partial y^4}(\xi_i, \eta_j)\right) \\
&= f_{ij} + \frac{1}{12}(h^2\delta_x^2 + k^2\delta_y^2)f_{ij} \\
&\quad - \frac{1}{240}\left(h^4\frac{\partial^4 f}{\partial x^4}(\xi_i, y_j) + k^4\frac{\partial^4 f}{\partial y^4}(x_i, \eta_j)\right).
\end{aligned} \tag{7}$$

If in (7) we omit the partial derivatives, we get the following conventional difference method of fourth order [5, 19]:

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} + \frac{1}{12}(h^2 + k^2)\delta_x^2\delta_y^2 u_{ij} = f_{ij} + \frac{1}{12}(h^2\delta_x^2 + k^2\delta_y^2)f_{ij}. \tag{8}$$

Let $\Theta(X, Y)$ and $\Xi(X, Y)$ denote interval extensions of $\frac{\partial^4 f}{\partial x^4}$ and $\frac{\partial^4 f}{\partial y^4}$, respectively, and let us assume that

$$\left|\frac{\partial^6 u}{\partial x^4 \partial y^2}\right| \le P \text{ and } \left|\frac{\partial^6 u}{\partial x^2 \partial y^4}\right| \le Q \text{ for all } 0 \le x \le \alpha \text{ and } 0 \le y \le \beta.$$

It is obvious that

$$\frac{\partial^4 f}{\partial x^4}(\xi, y) \in \Theta(X + [-h, h], Y), \quad \frac{\partial^4 f}{\partial y^4}(x, \eta) \in \Xi(X, Y + [-k, k]),$$
$$\frac{\partial^6 u}{\partial x^4 \partial y^2} \in [-P, P] \quad \frac{\partial^6}{\partial x^2 \partial y^4} \in [-Q, Q],$$

If in (7) we write all partial derivatives at the right-hand side, then it is easy to obtain an interval analogy to this equation. We have

$$\begin{aligned}
&(h^2 + k^2)(U_{i-1,j-1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i+1,j+1})\\
&\quad + 2(5k^2 - h^2)(U_{i-1,j} + U_{i+1,j}) + 2(5h^2 - k^2)(U_{i,j-1} + U_{i,j+1})\\
&\quad - 20(h^2 + k^2)U_{i,j}\\
&= h^2k^2\Big(F_{i-1,j} + F_{i+1,j} + 8F_{i,j} + F_{i,j-1} + F_{i,j+1}\\
&\quad - \frac{1}{20}\left(h^4\Theta(X_i + [-h, h], Y_j) + k^4\Xi(X_i, Y_j + [-k, k])\right)\\
&\quad + \frac{1}{20}\left(h^4[-P, P] + k^4[-Q, Q]\right) + \frac{h^2k^2}{12}[-P - Q, P + Q]\Big).
\end{aligned} \tag{9}$$

If in (7) we leave partial derivatives at the left-hand side, write an interval analogy to this equation, and then add adequate opposite interval elements (which exist in directed interval arithmetic), we get

$$\begin{aligned}
&(h^2 + k^2)(U_{i-1,j-1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i+1,j+1})\\
&\quad + 2(5k^2 - h^2)(U_{i-1,j} + U_{i+1,j}) + 2(5h^2 - k^2)(U_{i,j-1} + U_{i,j+1})\\
&\quad - 20(h^2 + k^2)U_{i,j}\\
&= h^2k^2\Big(F_{i-1,j} + F_{i+1,j} + 8F_{i,j} + F_{i,j-1} + F_{i,j+1}\\
&\quad - \frac{1}{20}\left(h^4\Theta(X_i + [-h, h], Y_j) + k^4\Xi(X_i, Y_j + [-k, k])\right)\\
&\quad + \frac{1}{20}\left(h^4[P, -P] + k^4[Q, -Q]\right) + \frac{h^2k^2}{12}[P + Q, -P - Q]\Big).
\end{aligned} \tag{10}$$

The difference between (9) and (10) occurs only in the last line. If from the problem considered there is no information about the constants P and Q, we can calculate

$$P_{n,m} = \frac{1}{h^4k^2} \max_{i,j} \big| u_{i-2,j-1} + u_{i-2,j+1} + u_{i+2,j-1} + u_{i+2,j+1} - 2(u_{i-2,j} + u_{i+2,j}) \\ - 4(u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}) \\ + 8(u_{i-1,j} + u_{i+1,j}) + 6(u_{i,j-1} + u_{i,j+1}) - 12u_{ij} \big| ,$$

$$Q_{n,m} = \frac{1}{h^2k^4} \max_{i,j} \big| u_{i-1,j-2} + u_{i+1,j-2} + u_{i-1,j+2} + u_{i+1,j+2} - 2(u_{i,j-2} + u_{i,j+2}) \\ - 4(u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1}) \\ + 8(u_{i,j-1} + u_{i,j+1}) + 6(u_{i-1,j} + u_{i+1,j}) - 12u_{ij} \big| ,$$

for $i = 2, 3, \ldots, n-2, \quad j = 2, 3, \ldots, m-2$, where u_{ij} are obtained by a conventional method for a variety of n and m. Then, the values of constants P and Q may be estimated from the fact that $\lim_{n\to\infty} \lim_{m\to\infty} P_{n,m} \le P$ and $\lim_{n\to\infty} \lim_{m\to\infty} Q_{n,m} \le Q$.

6 Numerical Examples

In the examples presented in this section we have used our own implementation of floating-point interval arithmetic written in Delphi Pascal. This implementation has been written in the form of a unit called `IntervalArithmetic32and64`, which current version one can find in [12]. All programs for the examples presented can be load from [13].

Example 1
Let us take into account the following boundary value problem:

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad 0 \le x \le 1, \quad 0 \le y \le 1,$$
$$u|_\Gamma(x,y) = \varphi(x,y) = \begin{cases} \varphi_1(y) = \cos(3y), & \text{for } x = 0, \\ \varphi_2(x) = \exp(3x), & \text{for } y = 0, \\ \varphi_3(y) = \exp(3)\cos(3y), & \text{for } x = 1, \\ \varphi_4(x) = \exp(3x)\cos(3), & \text{for } y = 1. \end{cases} \tag{11}$$

The exact solution is given by $u(x,y) = \exp(3x)\cos(3y)$. In Table 1 we present the results obtained by the second and fourth order methods in proper and directed arithmetics at the center of the region Γ.

In the second order methods we have assumed $M = 1627$. Of course, this estimation of can be calculated from the known exact solution, but a similar estimation one can obtain from the graph presented in Fig. 1.

Table 1 The interval solutions and the widths of intervals obtained in proper (U_p) and directed (U_d) interval arithmetics to the problem (11) at (0.5, 0.5) ($u_{exact}(0.5, 0.5) \approx 0.31702214358044366$)

$m = n$	$U_p(0.5, 0.5)$	$Width(U_p)$	$U_d(0.5, 0.5)$	$Width(U_d)$
20 (2nd order)	[0.26795781801796551, 0.36764778128690462]	0.099689963	[0.26795781801796628, 0.36764778128690385]	0.099689963
20 (4th order)	[0.31687231501883790, 0.31717197371330709]	0.000299659	[0.31687231501883870, 0.31717197371330630]	0.000299659
60 (2nd order)	[0.31156101681974879, 0.32265704145441798]	0.011096024	[0.31156101681975913, 0.32265704145440782]	0.011096024
60 (4th order)	[0.31702029383932179, 0.31702399332372090]	0.000003700	[0.31702029383933359, 0.31702399332370710]	0.000003699
100 (2nd order)	[0.31505586246198510, 0.31905099073825598]	0.003995128	[0.31505586246202793, 0.31905099073821316]	0.003995128
100 (4th order)	[0.31702190385388648, 0.31702238330710142]	0.000000048	[0.31702212784104150, 0.31702215931994640]	0.000000031

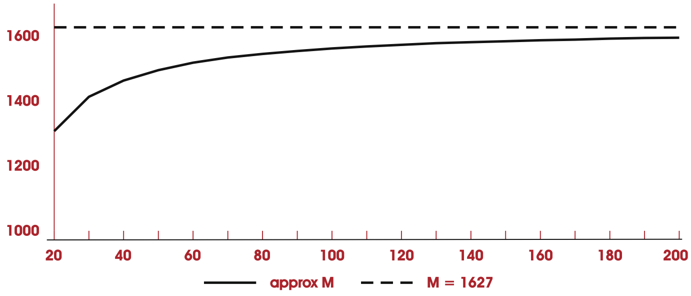

Fig. 1 Approximations to the constant M for the problem (11)

In the fourth order methods we have taken $P = Q = 14\,643$. In general, if the estimations of P and Q can not be obtained from any information about the problem considered, we can use similar technique as previously. Let us note that in both methods the exact solution belongs to the interval solutions obtained. ■

Example 2
As the second example let us consider the following problem:

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = -2\pi \sin(\pi x) \sin(\pi y), \quad 0 \le x \le 1, \quad 0 \le y \le 1, \qquad (12)$$
$$u|_\Gamma(x, y) = 0$$

with the exact solution $u(x, y) = \sin(\pi x) \sin(\pi y)$. The interval solutions obtained are presented in Table 2. To solve the problem (12) we have assumed $M = 97.5$ for the second order methods, and $P = Q = 961.4$ for the fourth order ones. As in

Table 2 The interval solutions and the widths of intervals obtained in proper (U_p) and directed (U_d) interval arithmetics to the problem (12) at (0.5, 0.5) ($u_{exact}(0.5, 0.5) = 1$)

$m = n$	$U_p(0.5, 0.5)$	$Width(U_p)$	$U_d(0.5, 0.5)$	$Width(U_d)$
20 (2nd order)	[0.9943031722943299, 1.0032966998827956]	0.008993528	[0.9972920186287353, 1.0003078535483902]	0.003015835
20 (4th order)	[0.9999825795708284, 1.0000059858600965]	0.000023406	[0.9999863114853876, 1.0000022539455373]	0.000015942
60 (2nd order)	[0.9993877757476903, 1.0001910675167757]	0.000803292	[0.9995227144656405, 1.0000561287988255]	0.000533414
60 (4th order)	[0.9999997879937084, 1.0000000498551452]	0.000000262	[0.9999998069620024, 1.0000000308868512]	0.000000217
100 (2nd order)	[0.9997852097215218, 1.0000562138028589]	0.000259718	[0.9998155730093303, 1.0000258505150504]	0.000210278
100 (4th order)	[0.9999999728030543, 1.0000000058410797]	0.000000033	[0.9999999743621285, 1.0000000042820054]	0.000000030

Table 3 The interval solutions and the widths of intervals obtained in proper (U_p) and directed (U_d) interval arithmetics to the problem (12) at (0.5, 0.5) for $M = 100$ and $P = Q = 1000$

$m = n$	$U_p(0.5, 0.5)$	$Width(U_p)$	$U_d(0.5, 0.5)$	$Width(U_d)$
20 (2nd order)	[0.9942265819722118, 1.0033732902049137]	0.009146708	[0.9972460644354644, 1.0003538077416611]	0.003107743
20 (4th order)	[0.9999821846099048, 1.0000063808210200]	0.000024196	[0.9999859165244640, 1.0000026489064608]	0.000016732
60 (2nd order)	[0.9993792508363238, 1.0001995924281422]	0.000820342	[0.9995175995188206, 1.0000612437456454]	0.000543644
60 (4th order)	[0.9999997831176551, 1.0000000547311985]	0.000000264	[0.9999998020859490, 1.0000000357629046]	0.000000234
100 (2nd order)	[0.9997821403236452, 1.0000592832007355]	0.000277143	[0.9998137313706043, 1.0000276921537746]	0.000213961
100 (4th order)	[0.9999999721711177, 1.0000000064730162]	0.000000034	[0.9999999737301920, 1.0000000049139419]	0.000000031

Example 1, the exact solution is within the interval solution obtained. Let us note that if we a little bit overestimate the constants M or P and Q (what can happen if we have no information about adequate partial derivatives) the interval results change insignificantly. In Table 3 we present the results obtained by our methods for the problem (12) with $M = 100$ and $P = Q = 1000$. ■

7 Conclusions and Further Studies

Interval methods for solving partial-differential equation problems in floating-point interval arithmetic give solutions in the form of intervals which contain all possible numerical errors, i.e. representation, rounding and truncation errors. The interval difference methods of fourth order are (of course) better than the methods of second order (give intervals with smaller widths). The interval difference methods realized in directed floating-point interval arithmetic are longer in time (approximately 15%) than by the methods realized in proper one, but yield interval solutions with a little bit smaller widths. Depending on the problem considered, the differences in widths may be decreasing or increasing in the number of mesh points, but in all cases the widths of intervals for directed interval arithmetic are a little bit smaller. To have more valuable approximations for constants used in our methods, in further studies we plan to use the Nakao interval estimations to partial derivatives (see, e.g., [17]). Moreover, according to a special form of the system of (interval) linear equations that have to be solved, some more effective methods should also be taken into account. We will also try to solve a generalized Poisson equation of the form

$$a(x, y)\frac{\partial^2 u}{\partial x^2}(x, y) + b(x, y)\frac{\partial^2 u}{\partial y^2}(x, y) = f(x, y), \tag{13}$$

where

$$a(x, y) \cdot b(x, y) > 0.$$

with some boundary conditions by interval difference methods of fourth order (interval difference methods of second order for solving (13) with the Dirichlet boundary conditions have been presented in [8]), and to use other interval difference methods.

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Gevrey Well Posedness of Goursat-Darboux Problems and Asymptotic Solutions

Jorge Marques and Jaime Carvalho e Silva

Abstract We consider the generalized Goursat-Darboux problem for a third order linear PDE with real constant coefficients. Our purpose is to find necessary conditions for the problem to be well-posed in the Gevrey classes. Since this problem can be reduced to the Cauchy problem using permutations of independent variables, we solve it for a ODE with complex coefficients and two unknown initial data. In order to prove our results, we first construct an explicit solution of a family of problems with initial data depending on a parameter $\eta > 0$ and then we obtain an asymptotic representation of a solution as η tends to infinity.

Keywords Goursat-Darboux problems · Gevrey classes · Asymptotic solutions

1 Introduction

The generalized Goursat-Darboux problem for a third order linear PDE with real constant coefficients in the space C^∞ was studied in [2, 3]. Given an open set $\Omega \subseteq \mathbf{R}^{3+m}$, neighborhood of the origin, the most general problem is defined on Ω by

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$$\begin{cases} \partial_t \partial_x \partial_y u(t,x,y,z) = \sum\limits_{\substack{l+k+j+|\xi|\leq 3 \\ l\neq 3, k\neq 3, j\neq 3}} a_{l,k,j,\xi} \partial_t^l \partial_x^k \partial_y^j \partial_z^\xi u(t,x,y,z) \\ u(0,x,y,z) = f_1(x,y,z) \\ u(t,0,y,z) = f_2(t,y,z) \\ u(t,x,0,z) = f_3(t,x,z) \end{cases} \tag{1}$$

where initial data satisfy the necessary compatibility conditions:

$$\begin{cases} f_1(0,y,z) = f_2(0,y,z) \\ f_1(x,0,z) = f_3(0,x,z) \\ f_2(t,0,z) = f_3(t,0,z) \\ f_1(0,0,z) = f_2(0,0,z) = f_3(0,0,z)\,. \end{cases} \tag{2}$$

It was showed in [3] that if the problem (1)–(2) is locally C^∞ well-posed in the neighborhood of origin then the coefficients $a_{0,0,0,\xi}$ with $|\xi| \leq 3$ are zero.

The necessary conditions for the problem to be C^∞ well-posed are very strong. Our goal is to investigate the local solvability of this problem in the classes of Gevrey functions [5].

Definition 1 (*Gevrey classes*)
Let $s > 1$ be a real number and Ω be an open subset of $\mathbf{R^n}$. The Gevrey class of index s on Ω, $\Gamma^s(\Omega)$, is the space of all the functions $f \in C^\infty(\Omega)$ such that for every compact $K \subset \Omega$ there exist constants $C > 0$ and $L > 0$ satisfying

$$\sup_{x\in K} |\, \partial^\alpha f(x)\, | \leq C L^{|\alpha|} \alpha!^s \tag{3}$$

for all multi-index α.

It is well known that there is a scale of Gevrey classes $\Gamma^s(\Omega)$ of index $s \geq 1$:

$$1 \leq s' < s \quad \Rightarrow \quad \Gamma^{s'}(\Omega) \subset \Gamma^s(\Omega)\,.$$

In fact these classes play an important role as spaces intermediate between the spaces of real analytic functions ($s = 1$) and $C^\infty(\Omega)$. In addition we have

$$\Gamma^1(\Omega) \subset \bigcap_{s>1} \Gamma^s(\Omega) \quad ; \quad \bigcup_{s>1} \Gamma^s(\Omega) \subset C^\infty(\Omega)\,.$$

We need to give a topology for $\Gamma^s(\Omega)$. Let L be a positive constant, we denote by $\Gamma^s_{L,K}$ the space of smooth functions $f \in \Omega$ such that for every compact $K \subset \Omega$,

$$\| f \|^s_{L,K} = \sup_\alpha [L^{-|\alpha|} \alpha!^{-s} \sup_{x\in K} |\, \partial^\alpha f(x)\, |] < \infty\,.$$

We also consider the space of functions in $\Gamma^s_{L,K}$ with compact support,

$$\Gamma^s_{L,K}(\Omega) = \{f \in C^\infty(\Omega) : \ \mathrm{supp} f \subset K \ , \ \|f\|^s_{L,K} < \infty\} \, ,$$

which is a Banach space endowed with the norm $\|f\|^s_{L,K}$. From a topological point of view, the Gevrey classes

$$\Gamma^s(\Omega) = \bigcup_{L>0, K\subset\Omega} \Gamma^s_{L,K}(\Omega)$$

are projective limits of inductive limits of Banach spaces [9].

2 Formulation of the Generalized Goursat-Darboux Problem

Let $m = 1$, without loss of generality, and let $\Omega \subseteq \mathbf{R^4}$ be an open set, neighborhood of the origin, defined by

$$\Omega = \{(t, x, y, z) : |t| < t_0 \quad \wedge \quad |x| < x_0 \quad \wedge \quad |y| < y_0 \quad \wedge \quad |z| < z_0\}.$$

We consider the simplest Goursat-Darboux problem on Ω for a third order linear PDE with real constant coefficients:

$$\begin{cases} \partial_t \partial_x \partial_y u(t, x, y, z) = \sum\limits_{0\le j\le 3} A_j \partial_z^j u(t, x, y, z) \\ u(0, x, y, z) = f_1(x, y, z) \\ u(t, 0, y, z) = f_2(t, y, z) \\ u(t, x, 0, z) = f_3(t, x, z) \end{cases} \tag{4}$$

where initial data satisfy compatibility conditions (2) on characteristic hyperplanes:

$$\Sigma_1 = \{(t, x, y, z) \in \mathbf{R^4} : t = 0\} \, , \ \Sigma_2 = \{(t, x, y, z) \in \mathbf{R^4} : x = 0\} \, , \tag{5}$$

$$\Sigma_3 = \{(t, x, y, z) \in \mathbf{R^4} : y = 0\} \, . \tag{6}$$

The problem (4)–(2) is a generalization of the problem studied by Hasegawa [7] for a second order linear PDE. It is called the Goursat problem of three faces.

Let us now introduce the definition of the well posed problem in the Gevrey classes in the sense of Hadamard [6].

Definition 2 (*Problem well-posed in the Gevrey classes*)
Let $s > 1$ be a real number and Ω be an open subset of $\mathbf{R^n}$, neighborhood of origin. We say that the problem (4)–(2) is $\Gamma^s(\Omega)$ well-posed on Ω if there exists a neighborhood $\mathscr{U} \subset \Omega$ such that

- For every $f_i \in \Gamma^s(\Omega \cap \Sigma_i)$, the problem (4)–(2) has a solution $u \in \Gamma^s(\mathscr{U})$;
- It is unique;
- It depends continuously on the data. This means that for every compact $K \subset \Omega$ and every constant $L > 0$ there exist compacts K_i and constants $L_i > 0$, $i = 1, 2, 3$, and $C > 0$ such that

$$\|u\|_{L,K}^s \le C \left(\|f_1\|_{L_1,K_1}^s + \|f_2\|_{L_2,K_2}^s + \|f_3\|_{L_3,K_3}^s \right) . \tag{7}$$

Our purpose is to find necessary conditions for the problem (4)–(2) to be well-posed in the Gevrey classes. We will try to find some critical index s_0 such that if the Goursat-Darboux problem is well posed in Γ^s for $s > s_0$ then the coefficients of the derivatives with respect to z are zero.

We begin by showing how the problem (4)–(2) can be reduced to a Cauchy problem following the ideas of Bronshtein [1]. It is easy to see that the differential operator

$$\partial_t \partial_x \partial_y - (A_3 \partial_{z^3}^3 + A_2 \partial_{z^2}^2 + A_1 \partial_z + A_0)$$

and the three characteristic hyperplanes Σ_i remain invariant under any permutation of the independent variables t, x and y. Let μ be the minimum value between t_0, x_0 and y_0 and

$$\Omega_\mu = \{(t, x, y, z) :| t |< \mu \wedge | x |< \mu \wedge | y |< \mu \wedge | z |< z_0 \}$$

be an open set, $\Omega_\mu \subset \Omega$. From now on we suppose that the problem (4)–(2) is Γ^s well-posed on Ω. By linearity, if $u(t, x, y, z)$ is a solution of the problem (4)–(2) on Ω then

$$v(t, x, y, z) = u(t, x, y, z) + u(x, y, t, z) + u(y, t, x, z) \tag{8}$$

is a solution of the corresponding problem on Ω_μ

$$\begin{cases} \partial_t \partial_x \partial_y v(t, x, y, z) = \sum_{j \le 3} A_j \partial_z^j v(t, x, y, z) \\ v(0, x, y, z) = f_1(x, y, z) + f_3(x, y, z) + f_2(y, x, z) \\ v(t, 0, y, z) = f_2(t, y, z) + f_1(y, t, z) + f_3(y, t, z) \\ v(t, x, 0, z) = f_3(t, x, z) + f_2(x, t, z) + f_1(t, x, z) . \end{cases} \tag{9}$$

We then reduce the number of the independent variables by setting $t = x = y$. We can define a function w by $w(r, z) = v(r, r, r, z)$ on

$$\tilde{\Omega} = \{(r, z) :| r |< \mu \wedge | z |< z_0\} \subseteq \mathbf{R}^2 .$$

Its partial derivatives with respect to r are given by

$$\partial_r w(r,z) = 3\partial_t v(r,r,r,z)\,,\ \partial^2_{r^2} w(r,z) = 9\partial_t\partial_x v(r,r,r,z)\,,$$

$$\partial^3_{r^3} w(r,z) = 27\partial_t\partial_x\partial_y v(r,r,r,z)\,.$$

For every parameter $\eta > 0$, taking

$$v(0,x,y,z) = v(t,0,y,z) = v(t,x,0,z) = e^{i\eta z}$$

we are looking for a unique solution depending continuously on the data. If v_η is solution of the problem on Ω_μ

$$\begin{cases} \partial_t\partial_x\partial_y v(t,x,y,z) = \sum_{j\leq 3} A_j \partial_z^j v(t,x,y,z) \\ v(0,x,y,z) = v(t,0,y,z) = v(t,x,0,z) = e^{i\eta z} \end{cases} \tag{10}$$

then $w_\eta(r,z) = v_\eta(r,r,r,z)$ is solution of the Cauchy problem on $\tilde{\Omega}$

$$\begin{cases} \partial^3_{r^3} w(r,z) = 27(A_3\partial^3_{z^3} + A_2\partial^2_{z^2} + A_1\partial_z + A_0) w(r,z) \\ \quad w(0,z) = e^{i\eta z}\,. \end{cases} \tag{11}$$

Notice that there are two arbitrary data $\partial_r w(0,z)$ and $\partial^2_{r^2} w(0,z)$.

3 Solving the Cauchy Problem

Applying the method of separation of variables we determine a unique solution of the Cauchy problem (11) in the form $w_\eta(r,z) = m_\eta(r)e^{i\eta z}$. Hence $m_\eta(r)$ is solution of the initial value problem

$$\begin{cases} m'''(r) = 27(-A_3 i\eta^3 - A_2\eta^2 + A_1 i\eta + A_0)m(r) \\ m(0) = 1 \\ m'(0) = \alpha \\ m''(0) = \beta \end{cases} \tag{12}$$

where α and β are unknown. In order to solve a third order linear ODE

$$m'''(r) = 27(-A_3 i\eta^3 - A_2\eta^2 + A_1 i\eta + A_0)m(r) \tag{13}$$

we use its characteristic equation

$$\lambda^3 - 27p(\eta) = 0 \tag{14}$$

where $p(\eta) = -A_3 i\eta^3 - A_2\eta^2 + A_1 i\eta + A_0$ is a polynomial with complex coefficients.

Lemma 1 *Let γ and $\overline{\gamma}$ be two conjugate complex roots of unity. If $A_\eta \neq 0$ is a solution of the Eq. (13) then the solution of the problem (12) is given by*

$$m_\eta(r) = \tfrac{1}{3}(1 + a_\eta + b_\eta)e^{A_\eta r} + \tfrac{1}{3}\left(1 + \overline{\gamma}a_\eta + \gamma b_\eta\right)e^{\gamma A_\eta r} + \\ + \tfrac{1}{3}\left(1 + \gamma a_\eta + \overline{\gamma} b_\eta\right)e^{\overline{\gamma} A_\eta r}. \tag{15}$$

where $a_\eta = \dfrac{\alpha}{A_\eta}$ *and* $b_\eta = \dfrac{\beta}{A_\eta^2}$.

Proof Let $A_\eta \neq 0$ be a solution of (13). If γ and $\overline{\gamma}$ are two conjugate complex roots of unity then by de Moivre's formula the general solution of the (13) is written in the form

$$m_\eta(r) = C_1 e^{A_\eta r} + C_2 e^{\gamma A_\eta r} + C_3 e^{\overline{\gamma} A_\eta r}$$

where $C_1, C_2, C_3 \in \mathbf{C}$ are arbitrary constants, which are determined from initial data of the problem (12) by solving a linear system.

If A_η is a real root of the (13) we simplify (15) by using the Euler's formula.

Corollary 1 (Characteristic equation with one real root) *If $A_\eta \in \mathbf{R} - \{0\}$ then*

$$m_\eta(r) = \tfrac{1}{3}(1 - c_\eta)e^{A_\eta r} + \tfrac{1}{3}\left(2 + c_\eta\right)\cos(\sqrt{3}A_\eta r/2)e^{-A_\eta r/2} + \\ + \tfrac{\sqrt{3}}{3}d_\eta \sin(\sqrt{3}A_\eta r/2)e^{-A_\eta r/2} \tag{16}$$

where $c_\eta = -a_\eta - b_\eta$ *and* $d_\eta = -i(a_\eta - b_\eta)$.

If A_η is a pure imaginary root of the (13), (15) can be written in a simpler expression.

Corollary 2 (Characteristic equation with a pure imaginary root) *If $A_\eta = -iB_\eta$ with $B_\eta \in \mathbf{R} - \{0\}$ then*

$$m_\eta(r) = \tfrac{1}{3}\left[(2 + c_\eta)\cosh(\sqrt{3}B_\eta r/2) + \sqrt{3}d_\eta \sinh(\sqrt{3}B_\eta r/2)\right]e^{iB_\eta r/2} + \\ + \tfrac{1}{3}(1 - c_\eta)e^{-iB_\eta r} \tag{17}$$

where $c_\eta = \dfrac{\beta}{B_\eta^2} - i\dfrac{\alpha}{B_\eta}$ *and* $d_\eta = \dfrac{\alpha}{B_\eta} - i\dfrac{\beta}{B_\eta^2}$.

4 Asymptotic Representation of Solutions

In previous works [2, 3, 7] an explicit solution of the generalized Goursat-Darboux problem involves a hypergeometric function of several variables. However some difficulties for obtaining asymptotic representations for these functions were pointed out in the paper [4].

In our work we have a linear combination of complex exponential functions as solution of the Cauchy problem. We provide asymptotic representations, as η tends to infinity, for the absolute value of complex functions m_η on a compact, which depends on η and s. Our approach is based on asymptotic analysis of the initial data in order to have only one exponential function as dominant term, that is, when one exponential function tends to infinity and the others tend to zero.

Here $\Re(p(\eta))$ and $\Im(p(\eta))$ denote the real part of $p(\eta)$ and the imaginary part of $p(\eta)$, respectively.

Proposition 1 *If* $\Im(p(\eta)) = 0$, $A_2 \neq 0$ *and* $s > 3/2$ *then there exist a constant* $c > 0$ *and a compact* K_η*, neighborhood of the origin, such that*

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim c e^{\sqrt[3]{|A_2|}\eta^{1/s}} \tag{18}$$

as η *tends to infinity.*

Proof By assumption $\Im(p(\eta)) = 0$ and $A_2 \neq 0$. The Eq. (13) has one real root $A_\eta = -3\sqrt[3]{A_2\eta^2 - A_0}$, then by Corollary 1 the solution of the problem (12), $m_\eta(r)$, is given by (16). Let's see three cases that may occur depending on complex values $c_\eta = -\dfrac{\alpha}{A_\eta} - \dfrac{\beta}{A_\eta^2}$ and $d_\eta = -i\left(\dfrac{\alpha}{A_\eta} - \dfrac{\beta}{A_\eta^2}\right)$. We first suppose that

$$\mid d_\eta \mid = O(\mid 1 - c_\eta \mid) \wedge \mid 2 + c_\eta \mid = O(\mid 1 - c_\eta \mid)\,.$$

We choose a compact K_η,

$$K_\eta = \{(r, z) : (r, z) = \pm\frac{1}{3}(\eta^{1/s - 2/3}, 0)\}\,,$$

in which

$$\sup_{r \in K_\eta} (A_\eta r) = \sqrt[3]{\mid A_2 \mid}\eta^{1/s}\,.$$

Notice that if $s > 3/2$ then K_η is a neighborhood of the origin on $\mathbf{R^2}$. Since

$$\mid (2 + c_\eta) \cos(2^{-1}\sqrt{3}\sqrt[3]{\mid A_2 \mid}\eta^{1/s}) \mid e^{-2^{-1}\sqrt[3]{|A_2|}\eta^{1/s}} = o(\mid 1 - c_\eta \mid e^{\sqrt[3]{|A_2|}\eta^{1/s}})$$

and

$$| d_\eta \sin(2^{-1}\sqrt{3}\sqrt[3]{| A_2 |}\eta^{1/s}) | e^{-2^{-1}\sqrt[3]{|A_2|}\eta^{1/s}} = o(| 1 - c_\eta | e^{\sqrt[3]{|A_2|}\eta^{1/s}})$$

we obtain

$$\sup_{r \in K_\eta} | m_\eta(r) | \sim \frac{1}{3} | 1 - c_\eta | e^{\sqrt[3]{|A_2|}\eta^{1/s}} \tag{19}$$

with $\frac{1}{3} | 1 - c_\eta | \geq c > 0$, as η tends to infinity. Then we suppose that

$$| d_\eta | = O(| 2 + c_\eta |) \quad \wedge \quad | 1 - c_\eta | = O(| 2 + c_\eta |) \,.$$

If $s > 3/2$ we choose a compact K_η, neighborhood of the origin on $\mathbf{R^2}$, in which

$$\sup_{r \in K_\eta} (-A_\eta r) = 2\sqrt[3]{| A_2 |}\eta^{1/s} \,.$$

Moreover, we choose a sequence of η values satisfying $\sup_{r \in K_\eta} \tan(\sqrt{3}A_\eta r/2) = 0$. Since

$$| 1 - c_\eta | e^{-2\sqrt[3]{|A_2|}\eta^{1/s}} = o(| 2 + c_\eta | e^{\sqrt[3]{|A_2|}\eta^{1/s}})$$

and

$$| d_\eta \sin(\sqrt{3}\sqrt[3]{| A_2 |}\eta^{1/s}) | = o(| (2 + c_\eta) \cos(\sqrt{3}\sqrt[3]{| A_2 |}\eta^{1/s}) |)$$

we obtain

$$\sup_{r \in K_\eta} | m_\eta(r) | \sim \frac{1}{3} | 2 + c_\eta | e^{\sqrt[3]{|A_2|}\eta^{1/s}} \tag{20}$$

with $\frac{1}{3} | 2 + c_\eta | \geq c > 0$, as η tends to infinity. Finally we suppose that

$$| 2 + c_\eta | = O(| d_\eta |) \wedge | 1 - c_\eta | = O(| d_\eta |) \,.$$

If $s > 3/2$ we choose a compact K_η, neighborhood of the origin on $\mathbf{R^2}$, in which

$$\sup_{r \in K_\eta} (-A_\eta r) = 2\sqrt[3]{| A_2 |}\eta^{1/s} \,,$$

and a sequence of η values satisfying $\sup_{r \in K_\eta} \cot(\sqrt{3}A_\eta r/2) = 0$. Since

$$| 1 - c_\eta | e^{-2\sqrt[3]{|A_2|}\eta^{1/s}} = o(| d_\eta | e^{\sqrt[3]{|A_2|}\eta^{1/s}})$$

and

$$| (2 + c_\eta) \cos(\sqrt{3}\sqrt[3]{| A_2 |}\eta^{1/s}) | = o(| d_\eta \sin(\sqrt{3}\sqrt[3]{| A_2 |}\eta^{1/s}) |)$$

we obtain

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim \frac{\sqrt{3}}{3} \mid d_\eta \mid e^{\sqrt[3]{|A_2|}\eta^{1/s}} \tag{21}$$

with $\frac{\sqrt{3}}{3} \mid d_\eta \mid \geq c > 0$, as η tends to infinity.

Proposition 2 *If* $\Re(p(\eta)) = 0,\ A_3 = 0,\ A_1 \neq 0$ *and* $s > 3$ *then there exist a constant* $c > 0$ *and a compact* K_η*, neighborhood of the origin, such that*

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim c e^{\sqrt[3]{|A_1|}\eta^{1/s}} \tag{22}$$

as η *tends to infinity.*

Proof By assumption we have $p(\eta) = A_1\eta i$ with $A_1 \neq 0$. The Eq. (13) has a pure imaginary root $A_\eta = -3i\sqrt[3]{A_1\eta}$. We consider $iA_\eta = 3\sqrt[3]{A_1\eta} = B_\eta$ with $B_\eta \in \mathbf{R}$, then by Corollary 2 the solution of the problem (12), $m_\eta(r)$, is given by (17), where $c_\eta = \dfrac{\beta}{B_\eta^2} - i\dfrac{\alpha}{B_\eta}$ and and $d_\eta = \dfrac{\alpha}{B_\eta} - i\dfrac{\beta}{B_\eta^2}$. We notice than $2 + c_\eta$ and d_η are not null simultaneously. If we suppose that

$$\mid 1 - c_\eta \mid = O(\mid 2 + c_\eta \mid) \wedge \mid d_\eta \mid = O(\mid 2 + c_\eta \mid)\,,$$

then for $s > 3$ we choose a compact K_η,

$$K_\eta = \left\{(r, z) : (r, z) = \pm\frac{2}{3\sqrt{3}}(\eta^{1/s-1/3}, 0)\right\}\,,$$

neighborhood of origin, in which

$$\frac{\sqrt{3}}{2}\sup_{r\in K_\eta}(B_\eta r) = \sqrt[3]{\mid A_1 \mid}\eta^{1/s}\,.$$

Since

$$\mid 1 - c_\eta \mid = o(\mid 2 + c_\eta \mid \cosh(\sqrt[3]{\mid A_1 \mid \eta^{1/s}}))$$

and

$$\mid d_\eta \mid \sinh(\sqrt[3]{\mid A_1 \mid \eta^{1/s}}) = O(\mid 2 + c_\eta \mid \cosh(\sqrt[3]{\mid A_1 \mid \eta^{1/s}}))$$

we obtain

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim \frac{1}{3} \mid 2 + c_\eta \mid e^{\sqrt[3]{|A_1|}\eta^{1/s}} \tag{23}$$

with $\frac{1}{3} \mid 2 + c_\eta \mid \geq c > 0$, as η tends to infinity. If we suppose that

$$| 1 - c_\eta | = O(| d_\eta |) \wedge | 2 + c_\eta | = O(| d_\eta |) \,,$$

then for $s > 3$ we can choose a compact K_η, neighborhood of the origin, in which

$$\frac{\sqrt{3}}{2} \sup_{r \in K_\eta} (B_\eta r) = \sqrt[3]{| A_1 |} \eta^{1/s} \,.$$

Since

$$| 1 - c_\eta | = o(| d_\eta | \sinh (\sqrt[3]{| A_1 | \eta^{1/s}}))$$

and

$$| 2 + c_\eta | \cosh (\sqrt[3]{| A_1 | \eta^{1/s}}) = O(| d_\eta | \sinh (\sqrt[3]{| A_1 | \eta^{1/s}}))$$

we obtain

$$\sup_{r \in K_\eta} | m_\eta(r) | \sim \frac{\sqrt{3}}{3} | d_\eta | e^{\sqrt[3]{|A_1|}\eta^{1/s}} \tag{24}$$

with $\frac{\sqrt{3}}{3} | d_\eta | \geq c > 0$, as η tends to infinity.

Lemma 2 *Let g_1, g_2, g_3 and h with $\Re(h(\eta)) > 0$ be complex functions of the real variable η. We consider m defined by*

$$m(\eta) = g_1(\eta) e^{h(\eta)} + g_2(\eta) e^{\gamma h(\eta)} + g_3(\eta) e^{\overline{\gamma} h(\eta)}$$

where $\gamma = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. If $|g_j(\eta)| = O(|g_1(\eta)|)$, $j \neq 1$, and $|\Im(h(\eta))| < \sqrt{3}\Re(h(\eta))$ then

$$|m(\eta)| \sim |g_1(\eta)| e^{\Re(h(\eta))}$$

as η tends to infinity.

Proof By assumption, we have $|g_j(\eta)| = O(|g_1(\eta)|)$, $j = 2, 3$, that is, there are constants $k_j > 0$, such that $|g_j(\eta)| \leq k_j |g_1(\eta)|$ for all $\eta \in]0, +\infty[$. From simple calculations we get

$$\Re((\gamma - 1)h(\eta)) = 3\Re(h(\eta) + \sqrt{3}\Im(h(\eta)) \,,\; \Re((\overline{\gamma} - 1)h(\eta)) = 3\Re(h(\eta) - \sqrt{3}\Im(h(\eta)) \,.$$

The condition $|\Im(h(\eta))| < \sqrt{3}\Re(h(\eta))$ it is equivalent to

$$\Re((\gamma - 1)h(\eta)) < 0 \quad \wedge \quad \Re((\overline{\gamma} - 1)h(\eta)) < 0 \,.$$

Since

$$|g_2(\eta)| e^{\Re(\gamma h(\eta))} = o(|g_1(\eta)| e^{\Re(h(\eta))}) \;\wedge\; |g_3(\eta)| e^{\Re(\overline{\gamma} h(\eta))} = o(|g_1(\eta)| e^{\Re(h(\eta))})$$

it implies

$$|g_2(\eta)|e^{\Re(\gamma h(\eta))} + |g_3(\eta)|e^{\Re(\overline{\gamma} h(\eta))} = o(|g_1(\eta)|e^{\Re(h(\eta))}),$$

by consequence

$$|m(\eta)| \sim |g_1(\eta)|e^{\Re(h(\eta))}$$

as η tends to infinity [8].

Proposition 3 *If* $\Re(p(\eta)) \neq 0$, $A_3 \neq 0$ *and* $s > 1$ *then there exist a constant* $c > 0$ *and a compact* K_η*, neighborhood of origin, such that*

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim c e^{\sqrt[3]{|A_3|}\eta^{1/s}} \tag{25}$$

as η *tends to infinity.*

Proof We have $\Re(p(\eta)) = -A_2\eta^2 + A_0 \neq 0$ and $\Im(p(\eta)) = -A_3\eta^3 + A_1\eta$ with $A_3 \neq 0$ by assumption. Let $A_\eta = 3\sqrt[3]{p(\eta)}$ be one of the three complex roots of the Eq. (13) whose principal argument is

$$\theta_1 \in \left(\left]-\frac{\pi}{3}, 0[\cup]0, \frac{\pi}{3}\right[\right) \ \vee \ \theta_2 = \theta_1 - \frac{2\pi}{3} \ \vee \ \theta_3 = \theta_1 + \frac{2\pi}{3}.$$

Then by Lemma 1 the solution of the problem (12), $m_\eta(r)$, is given by (15), where $a_\eta = \dfrac{\alpha}{A_\eta}$ and $b_\eta = \dfrac{\beta}{A_\eta^2}$. If we first suppose that

$$\mid 1 + \overline{\gamma} a_\eta + \gamma b_\eta \mid = O(\mid 1 + a_\eta + b_\eta \mid) \wedge \mid 1 + \gamma a_\eta + \overline{\gamma} b_\eta \mid = O(\mid 1 + a_\eta + b_\eta \mid)$$

we choose a compact K_η, neighborhood of origin for $s > 1$, in which

$$h(\eta) = \sup_{r \in K_\eta} A_\eta r = (1 + i \tan\theta_1)\sqrt[3]{\mid A_3 \mid}\eta^{1/s},$$

for some $\theta_1 \in (]-\frac{\pi}{3}, 0[\cup]0, \frac{\pi}{3}[)$. Since $|\Im(h(\eta))| < \sqrt{3}\Re(h(\eta))$ by Lemma 2 we obtain

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid \sim \frac{1}{3} \mid 1 + a_\eta + b_\eta \mid e^{\sqrt[3]{|A_3|}\eta^{1/s}} \tag{26}$$

with $\mid 1 + a_\eta + b_\eta \mid \geq c > 0$, as η tends to infinity. Then if we suppose that

$$\mid 1 + a_\eta + b_\eta \mid = O(\mid 1 + \overline{\gamma} a_\eta + \gamma b_\eta \mid) \wedge \mid 1 + \gamma a_\eta + \overline{\gamma} b_\eta \mid = O(\mid 1 + \overline{\gamma} a_\eta + \gamma b_\eta \mid)$$

we choose now a compact K_η, neighborhood of origin for $s > 1$, in which

$$h(\eta) = \sup_{r \in K_\eta} A_\eta r = \left(1 + i\frac{\tan\theta_2 - \sqrt{3}}{1 + \sqrt{3}\tan\theta_2}\right)\sqrt[3]{\mid A_3 \mid}\eta^{1/s},$$

for some $\theta_2 \in (]-\pi, -\frac{\pi}{2}[\cup]-\frac{\pi}{2}, -\frac{\pi}{3}[)$. Since $|\Im(h(\eta))| < \sqrt{3}\Re(h(\eta))$ by Lemma 2 we obtain

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim \frac{1}{3} \mid 1+\overline{\gamma}a_\eta + \gamma b_\eta \mid e^{\sqrt[3]{|A_3|}\eta^{1/s}} \tag{27}$$

with $\mid 1+\overline{\gamma}a_\eta + \gamma b_\eta \mid \geq c > 0$, as η tends to infinity. Finally if we suppose that

$$\mid 1+a_\eta+b_\eta \mid = O(\mid 1+\gamma a_\eta + \overline{\gamma} b_\eta \mid) \wedge \mid 1+\overline{\gamma}a_\eta+\gamma b_\eta \mid = O(\mid 1+\gamma a_\eta + \overline{\gamma} b_\eta \mid)$$

we take a compact K_η, neighborhood of origin for $s > 1$, in which

$$h(\eta) = \sup_{r\in K_\eta} A_\eta r = \left(1 + i\frac{\tan\theta_3 + \sqrt{3}}{1-\sqrt{3}\tan\theta_3}\right)\sqrt[3]{\mid A_3 \mid}\eta^{1/s},$$

for some $\theta_3 \in (]\frac{\pi}{3}, \frac{\pi}{2}[\cup]\frac{\pi}{2}, \pi[)$. Since $|\Im(h(\eta))| < \sqrt{3}\Re(h(\eta))$ by Lemma 2 we obtain

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim \frac{1}{3} \mid 1+\gamma a_\eta + \overline{\gamma} b_\eta \mid e^{\sqrt[3]{|A_3|}\eta^{1/s}} \tag{28}$$

with $\mid 1+\gamma a_\eta + \overline{\gamma} b_\eta \mid \geq c > 0$, as η tends to infinity.

Theorem 1 *If the problem (4)–(2) is Γ^s well-posed on Ω then*

(i)

$$s > 1 \quad \Rightarrow \quad A_3 = 0\,; \tag{29}$$

(ii)

$$s > \frac{3}{2} \quad \Rightarrow \quad A_2 = 0\,; \tag{30}$$

(iii)

$$s > 3 \quad \Rightarrow \quad A_1 = 0\,. \tag{31}$$

Proof We suppose that the problem (4)–(2) is Γ^s well-posed on Ω with $s > 1$. Then for every $\eta > 0$ the corresponding problem (10) has a unique solution v_η on Ω_μ.

On the one hand, we determine *a prior* an estimation for the Gevrey norm of v_η, an upper bound, from the initial data, $\|e^{i\eta z}\|^s_{L,K}$, for every compact $K \subset \Omega$ and every constant $L > 0$. The partial derivatives of $e^{i\eta z}$ with respect to multi-index (l, k, j, α), such that $l \neq 0$ or $k \neq 0$ or $j \neq 0$, are zero. Otherwise, it is clear that

$$\partial_z^\alpha(e^{i\eta z}) = (i\eta)^{|\alpha|}e^{i\eta z},$$

it follows that

$$\sup_{(t,x,y,z)\in K} \mid \partial^\alpha(e^{i\eta z}) \mid = \eta^{|\alpha|}$$

so that

$$\|e^{i\eta z}\|^s_{L,K} = \sup_\alpha \left(| \alpha |^{-s|\alpha|} L^{-|\alpha|}\eta^{|\alpha|}\right) .$$

Since the supremum is given by $e^{se^{-1}L^{-1/s}\eta^{1/s}}$ there exist constants $c_1 = se^{-1}L^{-1/s}$ and $C > 0$ such that

$$\|v_\eta\|^s_{L,K} \leq C\|e^{i\eta z}\|^s_{L,K} \leq Ce^{c_1\eta^{1/s}} \tag{32}$$

for every $\eta > 0$. This is a condition for stability of solution.

On the other hand, let's prove that if each coefficient of the equation is different from zero, $A_i \neq 0$, then there is some critical index s_0 such that if $s > s_0$ then (32) will be violated.

In (i) we suppose that $A_3 \neq 0$ and assume $A_2 = 0$, in (ii) we suppose that $A_2 \neq 0$ and assume $A_1 = 0$ and in (iii) we suppose that $A_1 \neq 0$ and assume $A_0 = 0$. We assume that some coefficient is null because we can do suitable dependent variable changes.

By using previous propositions we construct an asymptotic representation of a solution as η tends to infinity. For every neighborhood of the origin $\mathscr{O}$ there exist a compact K_η, $K_\eta \subset \mathscr{O}$, and constants $C > 0$ and $c_2 > 0$ such that

$$\sup_{r\in K_\eta} | v_\eta(r, r, r, z) |\sim Ce^{c_2\eta^{1/s}} .$$

Notice that $K_\eta \subset \mathscr{O}$ only if $s_0 = 1$ in (i), $s_0 = 3/2$ in (ii) and $s_0 = 3$ in (iii).
We have

$$\sup_{r\in K_\eta} | m_\eta(r) |= \sup_{r\in K_\eta} | w_\eta(r, z) |= \sup_{r\in K_\eta} | v_\eta(r, r, r, z) |$$

and as we know that

$$\|v_\eta\|^s_{L',K_\eta} > \sup_{r\in K_\eta} | v_\eta(r, r, r, z) |$$

we can choose a constant $L' > 0$ such that

$$\|v_\eta\|^s_{L',K_\eta} > \|v_\eta\|^s_{L,K}$$

as η tends to infinity. The condition (32) fails to hold since $\|v_\eta\|^s_{L',K_\eta}$ has exponential growth of higher order to $\eta^{1/s}$ as η tends to infinity.

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Oscillation Criteria for a Difference System with Two Delays

Pati Doi and Hideaki Matsunaga

Abstract The oscillation of all solutions of a linear autonomous difference system with two delays is studied. Explicit necessary and sufficient conditions in terms of the coefficient matrix and the delays are established, which are some extensions of the previous results. As an application, we can completely classify the oscillation and the asymptotic stability of a delay difference system.

Keywords Difference equations · Oscillation · Delay · Characteristic equation

1 Introduction

Delay difference equations have been actively investigated since Levin and May [6] studied them as discrete models corresponding to delay differential equations. Although delay difference equations are regarded as higher order difference equations, the special relations among the terms caused by delay are not reflected fully in the classical qualitative theory of higher order ones. But, the use of the method developed in the theory of delay differential equations enables us to analyze delay difference equations in detailed, and its new theory has been obtained.

In this paper we are concerned with a linear difference system

$$x(n+1) - ax(n-l) + Bx(n-k) = 0, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where a is a real number, B is a $d \times d$ real constant matrix, and k, l are nonnegative integer. By a solution of (1) we mean a sequence $\{x(n)\}$ of vectors in $\mathbf{R}^d$ is defined for $n \geq -\max\{k, l\}$ and satisfies (1) for $n \geq 0$.

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A sequence of real numbers $\{y(n)\}$ is said to be *oscillatory* if the terms $y(n)$ are not eventually positive or negative. Let $x(n)$ be a solution of (1) with $x(n) = \operatorname{col}\big(x_1(n), x_2(n), \ldots, x_d(n)\big)$ for $n \geq 0$. We say that the solution $x(n)$ is *oscillatory* if each component $x_j(n)$ is oscillatory. Otherwise it is called *nonoscillatory*.

For the simplest delay difference equation

$$x(n+1) - x(n) + bx(n-k) = 0, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where b is a real number and k is a positive integer, it is known [2–4] that *all nontrivial solutions of* (2) *are oscillatory if and only if*

$$b > \frac{k^k}{(k+1)^{k+1}}.$$

In the scalar case, system (1) is expressed as

$$x(n+1) - ax(n-l) + bx(n-k) = 0, \quad n = 0, 1, 2, \ldots, \tag{3}$$

where b is a real number. In [5], Ladas et al. gave the oscillation criterion for (3) as follows. Without loss of generality, one may assume that $k > l$.

Theorem A *Let k and l be positive integers with $k > l$. Then all nontrivial solutions of* (3) *are oscillatory if and only if*

$$a > 0, \;\; b^{l+1} > a^{k+1}\frac{(l+1)^{l+1}(k-l)^{k-l}}{(k+1)^{k+1}}$$

or

$$a \leq 0, \;\; b \geq 0.$$

In the case that $a = 1$ and $l = 0$, system (1) becomes

$$x(n+1) - x(n) + Bx(n-k) = 0, \quad n = 0, 1, 2, \ldots. \tag{4}$$

In [1], Chuanxi et al. obtained the following result:

Theorem B *Let k be a positive integer. Then all nontrivial solutions of* (4) *are oscillatory if and only if the matrix B has no real eigenvalues or*

$$\lambda_{\min}(B) > \frac{k^k}{(k+1)^{k+1}},$$

where $\lambda_{\min}(B)$ denotes the minimum of real eigenvalues of B.

The purpose of this paper is to establish explicit necessary and sufficient conditions for all nontrivial solutions of (1) to be oscillatory. For simplicity, we put

$$K = \frac{(l+1)^{l+1}(k-l)^{k-l}}{(k+1)^{k+1}}.$$

Our main result is stated in the following:

Theorem 1 *Let $k \geq l$. Then all nontrivial solutions of* (1) *are oscillatory if and only if any one of the following four conditions holds:*

(i) *The matrix B has no real eigenvalues;*

(ii) $k > l, \ a > 0, \ \lambda_{\min}(B) > a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}};$ (5)

(iii) $k > l, \ a \leq 0, \ \lambda_{\min}(B) \geq 0;$ (6)

(iv) $k = l, \ \lambda_{\min}(B) \geq a.$ (7)

Here $\lambda_{\min}(B)$ denotes the minimum of real eigenvalues of B.

Remark 1 In the case that $B = \text{diag}\,[b, \ldots, b]$ and $k > l$, Theorem 1 coincides with Theorem A. In the case that $a = 1$ and $l = 0$, Theorem 1 coincides with Theorem B. Hence, Theorem 1 is an extension of Theorems A and B.

Remark 2 In the case that $k < l$, the oscillation problem of (1) is unsolved and it is left for a future work.

2 Proof of Main Result

The characteristic equation of (1) is given by the form

$$\det(\lambda^{k+1} I - a\lambda^{k-l} I + B) = 0, \tag{8}$$

where I is the $d \times d$ identity matrix. Between the oscillation of solutions of (1) and the roots of the characteristic equation (8), the following proposition holds; see, e.g. [3, Chap. 7].

Proposition 1 *All nontrivial solutions of* (1) *are oscillatory if and only if the characteristic equation* (8) *has no positive roots.*

Remark 3 By virtue of Proposition 1, the following statements are equivalent:

(a) All nontrivial solutions of (1) are oscillatory componentwise;
(b) At least a component of all nontrivial solutions of (1) is oscillatory.

Proof of Theorem 1. By using Proposition 1, we will investigate explicit necessary and sufficient conditions for the characteristic equation (8) to have no positive roots. Equation (8) can also be written as

$$\det[(a\lambda^{k-l} - \lambda^{k+1})I - B] = 0.$$

We notice that λ_0 is a root of (8) if and only if $a\lambda_0^{k-l} - \lambda_0^{k+1}$ is an eigenvalue of B. Set

$$f(\lambda) = a\lambda^{k-l} - \lambda^{k+1} \quad \text{for } \lambda > 0.$$

Then

$$f'(\lambda) = a(k-l)\lambda^{k-l-1} - (k+1)\lambda^k = \lambda^{k-l-1}\{a(k-l) - (k+1)\lambda^{l+1}\}.$$

We will study necessary and sufficient conditions for the matrix B to have no real eigenvalues belonging to the range of $f(\lambda)$ for $\lambda > 0$. Our argument is divided into four cases.

Case (i): The matrix B has no real eigenvalues. In this case there are no real roots of (8).

Case (ii): $k > l$ and $a > 0$. One can easily find that the function $f(\lambda)$ attains the maximum value at $\tilde{\lambda} = \left\{\frac{a(k-l)}{k+1}\right\}^{\frac{1}{l+1}}$. It follows that

$$\begin{aligned}
f(\tilde{\lambda}) &= a\left\{\frac{a(k-l)}{k+1}\right\}^{\frac{k-l}{l+1}} - \left\{\frac{a(k-l)}{k+1}\right\}^{\frac{k+1}{l+1}} \\
&= \left(\frac{1}{k+1}\right)^{\frac{k+1}{l+1}} \left[a\{a(k-l)\}^{\frac{k-l}{l+1}}(k+1) - \{a(k-l)\}^{\frac{k+1}{l+1}}\right] \\
&= \left(\frac{1}{k+1}\right)^{\frac{k+1}{l+1}} \{a(k-l)\}^{\frac{k-l}{l+1}}\{a(k+1) - a(k-l)\} \\
&= \left(\frac{1}{k+1}\right)^{\frac{k+1}{l+1}} a^{\frac{k-l}{l+1}}(k-l)^{\frac{k-l}{l+1}} a(l+1)^{\frac{l+1}{l+1}} \\
&= a^{\frac{k+1}{l+1}} \left\{\frac{(l+1)^{l+1}(k-l)^{k-l}}{(k+1)^{k+1}}\right\}^{\frac{1}{l+1}} \\
&= a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}.
\end{aligned}$$

Since $\lim_{\lambda\to\infty} f(\lambda) = -\infty$, the range of $f(\lambda)$ is equal to $(-\infty, a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}]$. Therefore, Eq. (8) has no real roots if and only if no real eigenvalues of B belong to $(-\infty, a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}]$, that is,

$$\lambda_{\min}(B) > a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}.$$

Case (iii): $k > l$ and $a \leq 0$. Since $f(0) = 0$, $\lim_{\lambda\to\infty} f(\lambda) = -\infty$ and $f'(\lambda) < 0$ for $\lambda > 0$, the range of $f(\lambda)$ is equal to $(-\infty, 0)$. Hence, Eq. (8) has no real roots if and only if no real eigenvalues of B belong to $(-\infty, 0)$, that is,

$$\lambda_{\min}(B) \geq 0.$$

Case (iv): $k = l$. Since $f(0) = a$, $\lim_{\lambda\to\infty} f(\lambda) = -\infty$ and $f'(\lambda) < 0$ for $\lambda > 0$, the range of $f(\lambda)$ is equal to $(-\infty, a)$. Thus, Eq. (8) has no real roots if and only if no real eigenvalues of B belong to $(-\infty, a)$, that is,

$$\lambda_{\min}(B) \geq a.$$

This completes the proof. □

3 Two Dimensional Case

In this section we will consider the oscillation problem of (1) in the two dimensional case. The characteristic equation of the 2×2 matrix B becomes

$$\lambda^2 - (\mathrm{tr}\, B)\lambda + \det B = 0$$

and therefore, all the eigenvalues of B are explicitly given by

$$\lambda = \frac{\mathrm{tr}\, B \pm \sqrt{(\mathrm{tr}\, B)^2 - 4\det B}}{2}.$$

If $(\mathrm{tr}\, B)^2 - 4\det B < 0$, then the matrix B has no real eigenvalues, and thus, one can immediately obtain the following result from Theorem 1.

Theorem 2 *Let $d = 2$ and $k \geq l$. If $(\mathrm{tr}\, B)^2 - 4\det B < 0$, then all nontrivial solutions of (1) are oscillatory.*

On the other hand, if $(\mathrm{tr}\, B)^2 - 4\det B \geq 0$, then we have the explicit oscillation criterion for (1) from Theorem 1.

Theorem 3 *Let $d = 2$, $k \geq l$ and $(\mathrm{tr}\, B)^2 - 4\det B \geq 0$. Then all nontrivial solutions of (1) are oscillatory if and only if any one of the following three conditions holds:*

(i) $k > l,\ a > 0,\ \mathrm{tr}\, B > 2a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}},\ \det B > a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}\, \mathrm{tr}\, B - \left(a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}\right)^2;$

(ii) $k > l,\ a \leq 0,\ \mathrm{tr}\, B \geq 0,\ \det B \geq 0;$

(iii) $k = l,\ \mathrm{tr}\, B \geq 2a,\ \det B \geq a\, \mathrm{tr}\, B - a^2.$

Proof From $d = 2$ and $(\text{tr } B)^2 - 4 \det B \geq 0$, the matrix B has two real eigenvalues, and hence, we find

$$\lambda_{\min}(B) = \frac{\text{tr } B - \sqrt{(\text{tr } B)^2 - 4 \det B}}{2}.$$

Our argument is divided into three cases.

Case (i): $k > l$ and $a > 0$. Condition (5) in Theorem 1 asserts that all nontrivial solutions of (1) are oscillatory if and only if $\lambda_{\min}(B) > a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}$, or equivalently,

$$\begin{aligned}
&\text{tr } B - \sqrt{(\text{tr } B)^2 - 4 \det B} > 2a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}} \\
\iff\ & \text{tr } B - 2a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}} > \sqrt{(\text{tr } B)^2 - 4 \det B} \\
\iff\ & \begin{cases} \text{tr } B > 2a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}, \\ (\text{tr } B)^2 - 4a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}} \text{tr } B + 4\left(a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}\right)^2 > (\text{tr } B)^2 - 4 \det B \end{cases} \\
\iff\ & \text{tr } B > 2a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}, \quad \det B > a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}} \text{ tr } B - \left(a^{\frac{k+1}{l+1}} K^{\frac{1}{l+1}}\right)^2.
\end{aligned}$$

Case (ii): $k > l$ and $a \leq 0$. Condition (6) in Theorem 1 asserts that all nontrivial solutions of (1) are oscillatory if and only if $\lambda_{\min}(B) \geq 0$, or equivalently,

$$\begin{aligned}
&\text{tr } B - \sqrt{(\text{tr } B)^2 - 4 \det B} \geq 0 \\
\iff\ & \text{tr } B \geq \sqrt{(\text{tr } B)^2 - 4 \det B} \\
\iff\ & \text{tr } B \geq 0, \quad (\text{tr } B)^2 \geq (\text{tr } B)^2 - 4 \det B \\
\iff\ & \text{tr } B \geq 0, \quad \det B \geq 0.
\end{aligned}$$

Case (iii): $k = l$. Condition (7) in Theorem 1 asserts that all nontrivial solutions of (1) are oscillatory if and only if $\lambda_{\min}(B) \geq a$, or equivalently,

$$\begin{aligned}
&\text{tr } B - \sqrt{(\text{tr } B)^2 - 4 \det B} \geq 2a \\
\iff\ & \text{tr } B - 2a \geq \sqrt{(\text{tr } B)^2 - 4 \det B} \\
\iff\ & \text{tr } B \geq 2a, \quad (\text{tr } B)^2 - 4a \text{ tr } B + 4a^2 \geq (\text{tr } B)^2 - 4 \det B \\
\iff\ & \text{tr } B \geq 2a, \quad \det B \geq a \text{ tr } B - a^2.
\end{aligned}$$

This completes the proof. □

Finally, let us focus on the asymptotic behavior of solutions of (4) in the two dimensional case. By applying Theorems 2 and 3 to system (4), one can immediately obtain the following oscillation criterion for (4).

Corollary 1 *Let $d = 2$. Then all nontrivial solutions of (4) are oscillatory if and only if*

$$(\text{tr } B)^2 - 4 \det B < 0$$

or

$$(\mathrm{tr}\, B)^2 - 4 \det B \geq 0, \quad 2K_0 < \mathrm{tr} B < K_0 + \frac{\det B}{K_0},$$

where $K_0 = k^k/(k+1)^{k+1}$.

In [7], the second author has studied the asymptotic stability of (4). He presented the following result:

Theorem C *Let* $d = 2$. *Then the zero solution of* (4) *is asymptotically stable if and only if*

$$2\sqrt{\det B} \sin\left((2k+1)\sin^{-1}\left(\frac{\sqrt{\det B}}{2}\right)\right) < \mathrm{tr} B < \alpha + \frac{\det B}{\alpha}$$

and

$$0 < \det B < \alpha^2,$$

where $\alpha = 2\cos((k\pi)/(2k+1))$.

By combining Corollary 1 and Theorem C, we can completely classify the oscillation and the asymptotic stability of (4). The proof of the theorem will be omitted.

Theorem 4 *Let* $d = 2$. *Then all nontrivial solutions of* (4) *are oscillatory and tend to* 0 *as* $n \to \infty$ *if and only if*

$$2\sqrt{\det B} \sin\left((2k+1)\sin^{-1}\left(\frac{\sqrt{\det B}}{2}\right)\right) < \mathrm{tr} B$$

and

$$\mathrm{tr} B < \begin{cases} 2\sqrt{\det B} & (0 < \det B \leq {K_0}^2) \\ K_0 + \dfrac{\det B}{K_0} & ({K_0}^2 \leq \det B \leq \alpha K_0) \\ \alpha + \dfrac{\det B}{\alpha} & (\alpha K_0 < \det B < \alpha^2), \end{cases}$$

where $K_0 = k^k/(k+1)^{k+1}$ *and* $\alpha = 2\cos((k\pi)/(2k+1))$.

Remark 4 Theorem 4 is an extension of the following known result: *all nontrivial solutions of* (2) *are oscillatory and tend to* 0 *as* $n \to \infty$ *if and only if*

$$\frac{k^k}{(k+1)^{k+1}} < b < 2\cos\frac{k\pi}{2k+1}.$$

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$\log 0 = \log \infty = 0$ and Applications

Hiroshi Michiwaki, Tsutomu Matuura and Saburou Saitoh

Abstract In this paper, we will show that $\log 0 = \log \infty = 0$ by the division by zero $z/0 = 0$ and its fundamental applications. In particular, we will know that the division by zero is our elementary and fundamental mathematics.

Keywords Division by zero · $1/0 = 0/0 = 0$ · $\log 0 = 0$ · $\log \infty = 0$ · $0^0 = 1.0$ $e^0 = 1, 0$ · $\cos 0 = 1, 0$ · Y-field · Point at infinity · Infinity · Green function Robin constant · Capacity · Riemann mapping function · Laurent expansion

1 Introduction

By a **natural extension** of the fractions

$$\frac{b}{a} \tag{1}$$

for any complex numbers a and b, we found the simple and result, for any complex number b

$$\frac{b}{0} = 0, \tag{2}$$

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incidentally in [16] by the Tikhonov regularization for the Hadamard product inversions for matrices and we discussed their properties and gave several physical interpretations on the general fractions in [7] for the case of real numbers. The result is a very special case for general fractional functions in [5].

The division by zero has a long and mysterious story over the world (see, for example, H. G. Romig [15] and Google site with the division by zero) with its physical viewpoints since the document of zero in India on AD 628, however, Sin-Ei Takahasi [7] established a simple and decisive interpretation (2) by analyzing the extensions of fractions and by showing the complete characterization for the property (2).

We thus should consider, for any complex number b, as (2); that is, for the mapping

$$W = \frac{1}{z}, \tag{3}$$

the image of $z = 0$ is $W = 0$ (**should be defined**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere [1]. Therefore, the division by zero will give great impacts to complex analysis and to our ideas for the space and universe.

However, the division by zero (2) is now clear, indeed, for the introduction of (2), we have several independent approaches as in:

(1) by the generalization of the fractions by the Tikhonov regularization or by the Moore-Penrose generalized inverse,

(2) by the intuitive meaning of the fractions (division) by H. Michiwaki,

(3) by the unique extension of the fractions by S. Takahasi, as in the above,

(4) by the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ into $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$,

and

(5) by considering the values of functions with the mean values of functions.

Furthermore, in [9] we gave the results in order to show the reality of the division by zero in our world:

(A) a field structure containing the division by zero — the Yamada field $\mathbf{Y}$,

(B) by the gradient of the y axis on the (x, y) plane — $\tan \frac{\pi}{2} = 0$,

(C) by the reflection $W = 1/\overline{z}$ of $W = z$ with respect to the unit circle with center at the origin on the complex z plane — the reflection point of zero is zero,

and

(D) by considering rotation of a right circular cone having some very interesting phenomenon from some practical and physical problem.

See J. A. Bergstra, Y. Hirshfeld and J. V. Tucker [4] and J. A. Bergstra [3] for the relationship between fields and the division by zero, and the importance of the division by zero for computer science: It seems that the relationship of the division by zero and field structures are abstract in their papers.

Meanwhile, J. P. Barukčić and I. Barukčić [2] discussed the relation between the division 0/0 and special relative theory of Einstein. However, the result obtained contradicts with ours and their logic seems to be curious.

Furthermore, T.S. Reis and J.A.D.W. Anderson [13, 14] extend the system of the real numbers by introducing three infinities $1/0 = +\infty$, $-1/0 = -\infty$, $0/0 = \Phi$. Could we accept their theory as a natural one? They introduce a curious ideal number for the division 0/0.

Here, we recall Albert Einstein's words on mathematics: Black holes are where God divided by zero. I don't believe in mathematics. George Gamow (1904–1968) Russian-born American nuclear physicist and cosmologist remarked that "it is well known to students of high school algebra" that division by zero is not valid; and Einstein admitted it as **the biggest blunder of his life** (Gamow, G., My World Line (Viking, New York). p 44, 1970).

As the number system containing the division by zero, the Yamada structure is complete [9]. However, for applications of the division by zero to **functions**, we will need the concept of division by zero calculus for the sake of uniquely determinations of the results. See [10].

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i}, \tag{4}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)}, \tag{5}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i) \cdot \frac{1}{z+i}, \tag{6}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=i} \cdot \left[\frac{1}{z+i} \right|_{z=i} = 0 \cdot (-2i) = 0. \tag{7}$$

Therefore, we will introduce the division by zero calculus: For any formal Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n, \tag{8}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{9}$$

Note that here, there is no problem on any convergence of the expansion (8) at the point $z = a$. (Here, as convention, we consider as $0^0 = 1$.)

For the correspondence (9) for the function $f(z)$, we will call it **the division by zero calculus**. By considering the formal derivatives in (8), we can define any order derivatives of the function f at the singular point a.

However, for functions we see that the results by the division by zero calculus have not always practical senses and so, for the results by division by zero we should check the results, case by case; see many examples, [10].

2 Introduction of Formulas $\log 0 = \log \infty = 0$

For any fixed complex number a, we will consider the sector domain $\Delta_a(\alpha, \beta)$ defined by

$$0 \le \alpha < \arg(z-a) < \beta < 2\pi$$

on the complex z plane and we consider the conformal mapping of $\Delta_a(\alpha, \beta)$ into the complex W plane by the mapping

$$W = \log(z-a). \tag{10}$$

Then, the image domain is represented by

$$S(\alpha, \beta) = \{W; \alpha < \Im W < \beta\}.$$

Here, we will check the space structure by the division by zero.

We will be able to see the whole Euclidean plane by the stereographic projection into the Riemann sphere — *We think that in the Euclidean plane, there does not exist the point at infinity.*

However, we can consider it as a limit like ∞. Recall the definition of $z \to \infty$ by ε-δ logic; that is, $\lim_{z\to\infty} z = \infty$ if and only if for any large $M > 0$, there exists a number $L > 0$ such that for any z satisfying $L < |z|$, $M < |z|$. In this definition, the infinity ∞ does not appear. *The infinity is not a number, but it is an ideal space point* of the one point compactification of Aleksandrov.

The behavior of the space around the point at infinity may be considered (**may be defined**) by that around the origin by the linear transform $W = 1/z$ [1]. We thus see that

$$\lim_{z \to \infty} z = \infty, \tag{11}$$

however,

$$[z]_{z=\infty} = 0, \tag{12}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of (11) and (12) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x \to +\infty} x = +\infty, \quad \lim_{x \to -\infty} x = -\infty, \tag{13}$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0. \tag{14}$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions.

We were able to give also many evidences by analytic geometry in the Euclidean space for these properties [10]. In [8], we gave beautiful geometrical interpretations of determinants from the viewpoint of the division by zero.

Next, two lines $\{W; \Im W = \alpha\}$ and $\{W; \Im W = \beta\}$ usually were considered as having the common point at infinity, however, in the division by zero, the point is represented by zero (the point at infinity and zero point are coincident.).

Note that two parallel lines that are not the same have the common point the origin (0, 0) in the sense of the division by zero. Here, the common point is, of course, not in the usual sense. By writing the common point of the two lines that are not parallel, we obtain the common point (0, 0) applying the division by zero.

Indeed, we consider lines:

$$ax + by + c = 0, a'x + b'y + c' = 0. \tag{15}$$

The common point of the lines (15) is given by, if $ab' - a'b \neq 0$; that is, the lines are not parallel

$$\left(\frac{bc' - b'c}{ab' - a'b}, \frac{a'c - ac'}{ab' - a'b} \right). \tag{16}$$

By the division by zero, we can understand that if $ab' - a'b = 0$, then the common point is always given by

$$(0, 0), \tag{17}$$

even when the two lines are the same.

Meanwhile, we write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)}, \tag{18}$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle OP ($P = (r, \theta)$ on the line) and the positive x axis. Then, if $\theta - \alpha = \pi/2$: that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero; the point at infinity is represented by zero and we can consider the line passes the origin, however, it is in a discontinuous way.

That is, a line is, indeed, contains the origin; the true line should be considered as the sum of a usual line and the origin. We can say that it is a compactification of the line and the compacted point is the origin.

The similar property of a line passing the origin may be looked by using a Hesse representation of a line.

Therefore, $\log 0$ and $\log \infty$ **should be defined as zero**. Here, $\log \infty$ is precisely given in the sense of $[\log z]_{z=\infty}$. However, the properties of the logarithmic function should not be expected more, we should consider the value only. For example,

$$\log 0 = \log(2 \cdot 0) = \log 2 + \log 0$$

is not valid.

In particular, in many formulas in physics, in some expression, for some constants A, B

$$\log \frac{A}{B},$$

if we consider the case that A or B is zero, then we should consider it in the form

$$\log \frac{A}{B} = \log A - \log B, \tag{19}$$

and we should put zero in A or B. Then, in many formulas, we will be able to consider the case that A or B is zero. For the case that A or B is zero, the identity (19) is not valid, then the expression $\log A - \log B$ may be valid in many physical formulas. However, the results are case by case, and we should check the obtained results for applying the formula (19) for $A = 0$ or $B = 0$.

We can apply the result $\log 0 = 0$ for many cases as in the following way.

For example, we will consider the differential equation

$$xy' = xy^2 - a^2 x \log^{2k}(\beta x) + ak \log^{k-1}(\beta x). \tag{20}$$

For the solution $y = a \log^{2k}(\beta x)$ ([12], page 95, **5**), we can consider the solution $y = 0$ as $\beta = 0$.

In the famous function (Leminiscate)

$$x = a \log \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}, \quad a > 0, \tag{21}$$

we have

$$x = a \log \left[\frac{a + \sqrt{a^2 - y^2}}{y} \exp \left(-\frac{1}{a} \sqrt{a^2 - y^2} \right) \right]. \tag{22}$$

By the division by zero, at the point $y = 0$

$$\left[\frac{a + \sqrt{a^2 - y^2}}{y} \exp \left(-\frac{1}{a} \sqrt{a^2 - y^2} \right) \right] = 0. \tag{23}$$

Thus the curve passes also the origin (0.0).

In the differential equation

$$x^2 y''' + 4x^2 y'' - 2xy' - 4y = \log x, \tag{24}$$

we have the general solution

$$y = \frac{C_1}{x} + \frac{C_2}{x^2} + C_3 x^2 - \frac{1}{4} \log x + \frac{1}{4}, \tag{25}$$

satisfying that at the origin $x = 0$

$$y(0) = \frac{1}{4}, \, y'(0) = 0, \, y''(0) = 2C_3, \, y'''(0) = 0. \tag{26}$$

We can give the values C_1 and C_2. For the sake of the division by zero, we can, in general, consider differential equations even at analytic and isolated singular points.

In the formula ([6], p. 153), for $0 \le x, t \le \pi$

$$\sum_{n=1}^{\infty} \frac{\sin ns \sin nt}{n} = \frac{1}{2} \log |\sin((s+t)/2)/\sin((s-t)/2)|, \tag{27}$$

for $s = t = 0, \pi$, we can interpret that

$$0 = \frac{1}{2} \log \frac{0}{0} = \log 0. \tag{28}$$

In general, for $s = t$, we may consider that

$$\sum_{n=1}^{\infty} \frac{\sin^2 ns}{n} = \frac{1}{2} \log |\sin((s+s)/2)/0)| \tag{29}$$

$$= \frac{1}{2} \log |\sin ns/0| = \frac{1}{2} \log 0 = 0.$$

Note that this result is not a contradiction. Recall the case of the function $y = 1/x$ at the origin:

$$\lim_{x \to +0} \frac{1}{x} = +\infty, \tag{30}$$

in the monotonically increasing way, however,

$$\left[\frac{1}{x} \right]_{x=0} = 0. \tag{31}$$

Such a discontinuity property is important in the division by zero.

We will give a physical sense of $\log 0 = 0$. We shall consider a uniform line density μ on the $z-$ axis, then the force field $\mathbf{F}$ and the potential ϕ are given, for $\mathbf{p} = x\mathbf{i} + y\mathbf{j}$, $p = |\mathbf{p}|$,

$$\mathbf{F} = -\frac{2\mu}{p^2} \mathbf{p} \tag{32}$$

and

$$\phi = -2\mu \log \frac{1}{p}, \tag{33}$$

respectively. On the z- axis, we have, of course,

$$\mathbf{F} = \mathbf{0}, \phi = 0. \tag{34}$$

3 Robin Constant and Green's Functions

From the typical case, we will consider a fundamental application. Let $D(a, R) = \{|z| > R\}$ be the outer disc on the complex plane. Then, the Riemann mapping function that maps conformally onto the unit disc $\{|W| < 1\}$ and the point at infinity to the origin is given by

$$W = \frac{R}{z - a}. \tag{35}$$

Therefore, the Green function $G(z, \infty)$ of $D(a, R)$ is given by

$$G(z, \infty) = -\log \left\{ \frac{R}{|z - a|} \right\}. \tag{36}$$

Therefore, from the representation

$$G(z, \infty) = -\log R + \log |z| + \log \left(1 - \frac{a}{|z|}\right), \tag{37}$$

we have the identity

$$G(\infty, \infty) = -\log R, \tag{38}$$

that is the Robin constant of $D(a, R)$. This formula is valid in the general situation, because the Robin constant is defined by

$$\lim_{z \to b} \{G(z, b) + \log |z - b|\}, \tag{39}$$

for a general Green function with pole at b of some domain [1].

4 $e^0 = 1, 0$

By the introduction of the value $\log 0 = 0$, as the inversion function $y = e^x$ of the logarithmic function, we will consider that $y = e^0 = 0$. Indeed, we will show that this definition is very natural.

We will consider the conformal mapping $W = e^z$ of the strip

$$S(-\pi i, \pi i) = \{z; -\pi < \Im z < \pi\}$$

onto the whole W plane cut by the negative real line $(-\infty, 0]$. Of course, the origin 0 corresponds to 1. Meanwhile, we see that the negative line $(-\infty, 0]$ corresponds to the negative real line $(-\infty, 0]$. In particular, on the real line $\lim_{x \to -\infty} e^x = 0$. In our new space idea from the division by zero, the point at infinity is represented by zero and therefore, we should define as

$$e^0 = 0. \tag{40}$$

For the fundamental exponential function $W = \exp z$, at the origin, we should consider 2 valued function. The value 1 is the natural value as a regular point of analytic function, meanwhile the value 0 is given with a strong discontinuity; however, this value will appear in the universe as a natural way.

For the elementary functions $y = x^n$, $n = \pm 1, \pm 2, \ldots$, we have

$$y = e^{n \log x}. \tag{41}$$

Then, we wish to have

$$y(0) = e^{n \log 0} = e^0 = 0. \tag{42}$$

As a typical example, we will consider the simple differential equation

$$\frac{dx}{x} - \frac{2ydy}{1+y^2} = 0. \tag{43}$$

Then, by the usual method,

$$\log|x| - \log|1+y^2| = C; \tag{44}$$

that is,

$$\log\left|\frac{x}{1+y^2}\right| = \log e^C = \log K, \, K = e^C > 0 \tag{45}$$

and

$$\frac{x}{1+y^2} = \pm K. \tag{46}$$

However, the constant K may be taken zero, as we see directly $\log e^C = \log K = 0$.

In the differential equations

$$y' = -\lambda e^{\lambda x} y^2 + a e^{\mu x} y - a e^{(\mu-\lambda)x} \tag{47}$$

and

$$y' = -b e^{\mu x} y^2 + a\lambda e^{\lambda x} y - a^2 b e^{(\mu+2\lambda)x} \tag{48}$$

we have solutions

$$y = -e^{-\lambda x}, \tag{49}$$

$$y = a e^{\lambda x}, \tag{50}$$

respectively. For $\lambda = 0$, as $y = -1$, $y = a$ are solutions, respectively, however, the functions $y = 0$, $y = 0$ are not solutions, respectively. However, many and many cases, as the function $y = e^{0\cdot x} = 0$, we see that the function is solutions of differential equations, when $y = e^{\lambda \cdot x}$ is the solutions. See [12] for many concrete examples.

Meanwhile, we will consider the Fourier integral

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{-\alpha|t|} dt = \frac{2\alpha}{\alpha^2 + \omega^2}. \tag{51}$$

For the case $\alpha = 0$, if this formula valid, then we have to consider $e^0 = 0$.

Furthermore, by Poisson's formula, we have

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + (2\pi n)^2}. \tag{52}$$

If $e^0 = 0$, then the above identity is still valid, however, for $e^0 = 1$, the identity is not valid. We have many examples.

For the integral

$$\int_0^{\infty} \frac{x^3 \sin(ax)}{x^4 + 4} dx = \frac{\pi}{2} e^{-a} \cos a, \tag{53}$$

the formula is valid for $a = 0$.

For the integral

$$\int_0^{\infty} \frac{\xi \sin(x\xi)}{1 + a^2 \xi} d\xi = \frac{\pi}{2a^2} e^{-(x/a)}, \quad x > 0, \tag{54}$$

the formula is valid for $x = 0$.

5 $0^0 = 1, 0$

By the standard definition, we will consider

$$0^0 = \exp(0 \log 0) = \exp 0 = 1, 0. \tag{55}$$

The value 1 is famous which was derived by N. Abel, meanwhile, H. Michiwaki have directly derived it as 0 from the result of the division by zero. However, we now know that $0^0 = 1, 0$ is the natural result.

We will see its reality.

For $0^0 = 1$:

In general, for $z \neq 0$, from $z^0 = e^{0 \log z}$, $z^0 = 1$, and so, we will consider that $0^0 = 1$ in a natural way.

For example, in the elementary expansion

$$(1 + z)^n = \sum_{k=0}^{n} {}_nC_k z^k \tag{56}$$

the formula $0^0 = 1$ will be convenient for $k = 0$ and $z = 0$.

In the fundamental definition

$$\exp z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \tag{57}$$

in order to have a sense of the expansion at $z = 0$ and $k = 0$, we have to accept the formula $0^0 = 1$.

In the differential formula

$$\frac{d^n}{dx^n}x^n = nx^{n-1}, \tag{58}$$

in the case $n = 1$ and $x = 0$, the formula $0^0 = 1$ is convenient and natural.

For $0^0 = 0$:

For any positive integer n, since $z^n = 0$ for $z = 0$, we wish to consider that $0^0 = 0$ for $n = 0$.

6 $\cos 0 = 1, 0$

Since

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \tag{59}$$

we wish to consider also the value $\cos 0 = 0$.

The values $e^0 = 0$ and $\cos 0 = 0$ may be considered that the values at the point at infinity are reflected to the origin and other many functions will have the same property.

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Collocation Method to Solve Second Order Cauchy Integro-Differential Equations

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Abstract In this paper, we present a collocation method for solving the following second-order Cauchy integro-differential equation

$$x''(s) + \oint_{-1}^{1} \frac{\omega(t)x(t)}{s-t} dt = f(s), \quad -1 < s < 1,$$

$$x'(-1) = x(1) = 0,$$

in the space $\mathscr{X} := \mathscr{C}^0([-1, 1], \mathbb{C})$, with domain

$$\mathscr{D} := \left\{ x \in \mathscr{X} \ : \ x'' \in \mathscr{X}, \quad x'(-1) = x(1) = 0 \right\}.$$

The integral is a Cauchy principal value, and

$$\omega(s) := \sqrt{\frac{1+s}{1-s}}$$

is the weight function. We come up with a modified collocation method to build an approximate solution x_n using the airfoil polynomials of the first kind. Finally, we establish a numerical example to exhibit the theoretical results.

Keywords Cauchy kernel · Collocation method · Airfoil polynomials

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1 Introduction and Mathematical Background

Integro-differential equations appear in many applications in scientific fields such as biological, physical, and engineering problems. In [2], the authors have presented a high-order methods for the numerical solution of Volterra integro-differential equations. In [3], the authors have derived m-stage Runge-Kutta-Nystrm methods for the numerical solution of general second-order Volterra integro-differential equations. These implicit methods are based on collocation techniques in certain polynomial spline spaces. The modified trapezoidal method adapted for general second order initial value problems has been being given in [4]. In [6], the authors have presented a direct methods for a class of second order Volterra integro-differential equations which explicitly contain a first order derivative. In [8], the author has studied and presented a projection method for solving operator equations with bounded operator in Hilbert spaces. In [9], the author has introduced a projection method based on the Legendre polynomials, for solving integro-differential equations with Cauchy kernel. In [10], the author has studied a collocation method, for approximate solution of an integro-differential equations with logarithmic kernel, using airfoil polynomials. The goal of this study is to present a collocation method for solving second order integro-differential equations, using airfoil polynomials.

Let $L^2([-1, 1], \mathbb{C})$, be the space of complex-valued Lebesgue square integrable (classes of) functions on $[-1, 1]$.

We recall that the so-called airfoil polynomials are used as expansion functions to compute the pressure on an airfoil in steady or unsteady subsonic flow.

The airfoil polynomial t_n of the first kind is defined by

$$t_n(x) = \frac{\cos[(n+\frac{1}{2})\arccos x]}{\cos(\frac{1}{2}\arccos x)}.$$

The airfoil polynomial u_n of the second kind is defined by

$$u_n(x) = \frac{\sin[(n+\frac{1}{2})\arccos x]}{\sin(\frac{1}{2}\arccos x)}.$$

2 The Approximate Solution

Consider the following second order Fredholm integro-differential equation with Cauchy kernel:

$$\varphi''(s) + \oint_{-1}^{1} \frac{\omega(t)\varphi(t)}{t-s}dt = f(s), \quad -1 < s < 1. \tag{1}$$

$$x'(-1) = x(1) = 0,$$

with the domain

$$\mathscr{D} := \left\{ x \in \mathscr{X} \ : \ x'' \in \mathscr{X}, \quad x'(-1) = x(1) = 0 \right\}.$$

The following two formulas (cf. [5])

$$(1+s)t_i'(s) = (i + \frac{1}{2})u_i(s) - \frac{1}{2}t_i(s),$$

$$(1-s^2)t_i''(s) + (1-2s)t_i'(s) + n(n-1)t_i(s) = 0$$

give

$$t_i''(s) = \frac{(2s-1)(n+1/2)}{(1-s^2)(1+s)}u_i(s) - \frac{(2s-1)+2n(n-1)}{2(1-s^2)}. \tag{2}$$

We recall that (cf. [5]),

$$\oint_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{t_i(t)}{t-s} dt = \pi u_i(s). \tag{3}$$

Our goal is to approximate the solution of Eq. (1) via the airfoil polynomials of the first kind t_n as

$$\varphi_n(s) = \sum_{i=0}^{n} a_i t_i(s).$$

Consider the set of $n+1$ collocation points s_j, which are the zeros of u_{n+1}:

$$s_j = -\cos \frac{2j-1}{2n+3}\pi, \quad j = 0, 1, \ldots, n.$$

Letting

$$(V_1 y)(s) = \int_a^s y(t)dt,$$

$$(V_2 \psi)(s) = \int_s^b \psi(t)dt.$$

We recall that $V_1, V_2 : \mathscr{H} \to \mathscr{D}$ are compact.
Moreover,
For all $\varphi \in \mathscr{D}$,

$$(V_2 V_1 S)\varphi = -\varphi.$$

Consider the space $C^{0,\lambda}[-1,1]$ of all functions φ defined on $[-1,1]$ satisfying the following Hölder condition: $\exists M \geq 0$ such that

$$\forall s_1, s_2 \in [-1,1], \quad |\varphi(s_1) - \varphi(s_2)| \leq M \, |s_1 - s_2|^{\lambda},$$

where $0 < \lambda \leq 1$.

Let

$$\mathscr{H} := \left\{\varphi \in L^2[-1,1] \, : \, \varphi'' \in L^2([-1,1]), \quad \varphi'(-1) = \varphi(1) = 0\right\}.$$

Note that the the operator T is bounded from $L^2[-1,1]$ into itself and also from $C^{0,\lambda}[-1,1]$ into itself.

Consider hat functions $e_0, e_1, e_2, \ldots, e_n$ in $C^0[-1,1]$ such that

$$e_j(x_k) = \delta_{j,k}.$$

Define the projection operators π_n from $C^0[-1,1]$ into the space of continuous functions by

$$\pi_n g(x) := \sum_{j=0}^{n} g(x_j) e_j(x).$$

Let us define the operators

$$V_n := V_2 V_1 \pi_n T, \quad V := V_2 V_1 T.$$

Consider the following approximate equation in the unknown φ_n:

$$-\varphi_n + V_n \varphi_n = V_2 V_1 f.$$

Theorem 1 *Assume that* $f \in C^0[-1,1]$. *There exists a positive constant* α*, such that*

$$\|\varphi - \varphi_n\|_{\infty} \leq \alpha \| \left(V_n - V\right) \varphi\|_{\infty}$$

for n *large enough.*

Proof It is well-known that $\|\pi_n x - x\|_{\infty} \to 0$, for all $x \in C^0[-1,1]$. Since $V_2 V_1$ is compact, it is clear that V is compact. In (cf. [1, 7]) it is shown that the inverse operator $(-I + V_n)^{-1}$ exists and is uniformly bounded for n large enough. On the other hand,

$$\varphi_n - \varphi = [V_n \varphi_n - V_2 V_1 f] - [V\varphi - V_2 V_1 f],$$

hence

$$\varphi_n - \varphi = [V_n\varphi_n - V\varphi].$$

This leads to

$$\varphi_n - \varphi = [(V_n - V)\varphi + V_n(\varphi_n - \varphi)].$$

Thus

$$(-I + V_n)(\varphi - \varphi_n) = (V_n - V)\varphi.$$

Consequently

$$\varphi - \varphi_n = (-I + V_n)^{-1}[(V_n - V)\varphi],$$
$$\|\varphi_n - \varphi\|_\infty \le \alpha\|(V_n - V)\varphi\|_\infty,$$

where

$$\alpha := \sup_{n\ge N}\left\|(-I + V_n)^{-1}\right\|,$$

which is finite.

Thus, we obtain the following system:

$$S\varphi_n(s_j) + T\varphi_n(s_j) = f(s_j), \quad j = 0, 1, \ldots, n.$$

By (2) and (3),

$$\sum_{i=0}^{n} a_i\left\{\left[\frac{(2s_j - 1)(i + 1/2)}{(1 - s_j^2)(1 + s_j)} - \pi\right]u_i(s_j) - \frac{(2s_j - 1) + 2i(i - 1)}{2(1 - s_j^2)}t_i(s_j)\right\} = f(s_j), \quad j = 0, 1, \ldots, n.$$

3 Numerical Results

Let us consider the integro-differential equation (1), with the following exact solution

$$\varphi(x) = x^3 - 3x + 2.$$

Table 1 gives the numerical results for Example 1.

Table 1 Example 1

x	n = 6	n = 22	n = 120
−0.8	0.133e-1	0.154e-2	0.142e-3
−0.6	0.172e-1	0.179e-2	0.147e-3
−0.4	0.124e-1	0.163e-2	0.241e-4
−0.2	0.321e-1	0.165e-2	0.134e-3
0.0	0.156e-1	0.187e-2	0.201e-4
0.2	0.179e-1	0.177e-2	0.443e-4
0.4	0.195e-1	0.165e-2	0.781e-4
0.6	0.541e-1	0.167e-2	0.795e-4
0.8	0.325e-1	0.167e-2	0.807e-4

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Approximative Solutions to Autonomous Difference Equations of Neutral Type

Janusz Migda

Abstract Asymptotic properties of solutions to difference equations of the form

$$\Delta^m(x_n - u_n x_{n-k}) = a_n f(x_{\sigma(n)}) + b_n$$

Using a new version of the Krasnoselski fixed point theorem and the iterated remainder operator, we establish sufficient conditions under which a given solution of the equation

$$\Delta^m(x_n - u_n x_{n-k}) = b_n$$

is an approximative solution to the above equation. Our approach, based on the iterated remainder operator, allows us to control the degree of approximation. We use $\mathrm{o}(n^s)$, for a given nonpositive real s, as a measure of approximation.

Keywords Difference equation · Neutral equation · Prescribed asymptotic behavior · Asymptotically polynomial solution · Convergent solution

1 Introduction

Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{N}$ denote the set of all integers, the set of all real numbers, and the set of nonnegative integers respectively. Fix a positive integer m. We consider the difference equation of the form

$$\Delta^m(x_n - c_n x_{n-k}) = a_n f(x_{\sigma(n)}) + b_n, \tag{E}$$

$$a_n, b_n, c_n \in \mathbb{R}, \quad k \in \mathbb{Z}, \quad f : \mathbb{R} \to \mathbb{R}, \quad \sigma : \mathbb{N} \to \mathbb{N}, \quad \sigma(n) \to \infty.$$

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By a *solution* of (E) we mean a sequence $x : \mathbb{N} \to \mathbb{R}$ satisfying (E) for all large n. If $y : \mathbb{N} \to \mathbb{R}$ is a given sequence and x is solution of (E) such that $y_n - x_n = \mathrm{o}(1)$, then we say that y is an *approximative solution* and x is a solution with *prescribed asymptotic behavior*. In our investigations we replace o(1) by $\mathrm{o}(n^s)$ for a given nonpositive real s.

Asymptotic properties of solutions to neutral equations were investigated in many papers. See, for example, [1–9, 13, 14]. The existence of solutions with prescribed asymptotic behavior to difference equations of neutral type is the theme of many papers. See, for example, [3, 5–9, 12] or [16].

This paper completes the paper [12]. In Theorem 1 we generalize [12, Theorem 4.1], which is the main result of [12]. Next, in Theorems 2 and 3, we present some additional results.

2 Notation and Terminology

We will use the following notation

$$\mathbb{R}^* = \mathbb{R} \setminus \{0\}, \quad \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}, \quad \mathbb{N}^* = \mathbb{N} \setminus \{0\}.$$

The space of all sequences $x : \mathbb{N} \to \mathbb{R}$ we denote by SQ. If $x, y \in \mathrm{SQ}$, then

$$xy \quad \text{and} \quad |x|$$

denotes the sequences defined by $xy(n) = x_n y_n$ and $|x|(n) = |x_n|$ respectively. If there exists a positive constant c such that $x_n - y_n \geq c$ for any n, then we write

$$x \gg y.$$

Let $a \in \mathrm{SQ}$, $t \in [1, \infty)$. We will use the following notations

$$\mathrm{Fin} = \{x \in \mathrm{SQ} : x_n = 0 \text{ for all large } n\},$$

$$\mathrm{o}(1) = \{x \in \mathrm{SQ} : \ x \text{ is convergent to zero}\}, \quad \mathrm{O}(1) = \{x \in \mathrm{SQ} : \ x \text{ is bounded}\},$$

$$\mathrm{o}(a) = \{ax : \ x \in \mathrm{o}(1)\} + \mathrm{Fin}, \quad \mathrm{O}(a) = \{ax : \ x \in \mathrm{O}(1)\} + \mathrm{Fin},$$

$$\mathrm{A}(t) := \{a \in \mathrm{SQ} : \sum_{n=1}^{\infty} n^{t-1}|a_n| < \infty\}, \qquad \mathrm{Pol}(m-1) = \mathrm{Ker}\, \Delta^m.$$

Note that $\mathrm{Pol}(m-1)$ is the space of all polynomial sequences of degree less than m. Let $f : \mathbb{R} \to \mathbb{R}$. We say that a sequence $x : \mathbb{N} \to \mathbb{R}$ is *uniformly f-bounded* if there exists a positive ε such that f is bounded on the set

$$\bigcup_{n=0}^{\infty}[x_n - \varepsilon, x_n + \varepsilon].$$

For a sequence $x \in \mathrm{SQ}$ we define

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n| \in [0, \infty].$$

We say that a subset X of SQ is *ordinary* if $\|x - y\| < \infty$ for all $x, y \in X$. We regard any ordinary subset X of SQ as a metric space with metric defined by

$$d(x, y) = \|x - y\|.$$

Remainder operator. Let

$$\mathrm{S}(m) = \left\{ a \in \mathrm{SQ} : \text{series} \quad \sum_{i_1=0}^{\infty}\sum_{i_2=i_1}^{\infty}\cdots\sum_{i_m=i_{m-1}}^{\infty} a_{i_m} \quad \text{is convergent} \right\}.$$

For any $a \in \mathrm{S}(m)$ we define the sequence $r^m(a)$ by

$$r^m(a)(n) = \sum_{i_1=n}^{\infty}\sum_{i_2=i_1}^{\infty}\cdots\sum_{i_m=i_{m-1}}^{\infty} a_{i_m}.$$

Then $\mathrm{S}(m)$ is a linear subspace of $\mathrm{o}(1)$, $r^m(a) \in \mathrm{o}(1)$ for any $a \in \mathrm{S}(m)$ and

$$r^m : \mathrm{S}(m) \to \mathrm{o}(1)$$

is a linear operator which we call the *remainder operator* of order m.

Lemma 1 *Assume* $a \in \mathrm{A}(m)$*,* $x \in \mathrm{SQ}$*,* $u \in \mathrm{O}(1)$*, and* $p \in \mathbb{N}$*. Then*

(a) $x \in \mathrm{A}(m) \Leftrightarrow |x| \in \mathrm{S}(m) \Leftrightarrow \mathrm{O}(x) \subset \mathrm{S}(m)$, $\mathrm{O}(a) \subset \mathrm{A}(m) \subset \mathrm{o}(n^{1-m})$,

$$r^m(a)(p) = \sum_{k=0}^{\infty}\binom{m-1+k}{m-1}a_{n+p}, \quad |r_p^m a| \le r_p^m|a| \le \sum_{n=p}^{\infty} n^{m-1}|a_n|,$$

(b) $\Delta^m r^m a = (-1)^m a$, $|r^m(ua)| \le \|u\| r^m|a|$, $\Delta r^m|a| \le 0$,
(c) *if* $x, y \in \mathrm{S}(m)$ *and* $x_n \le y_n$ *for* $n \ge p$*, then* $r_n^m x \le r_n^m y$ *for* $n \ge p$*,*
(d) *if* $s \in (-\infty, 0]$ *and* $x \in \mathrm{A}(m - s)$*, then* $r^m x \in \mathrm{o}(n^s)$.

Proof This lemma is a consequence of [10, Lemma 3.1] and [11, Lemma 4,2].

Fundamental equation of neutral type. Let $m \in \mathbb{N}^*$, $k \in \mathbb{Z}$, $\lambda \in \mathbb{R}$. We consider the equations

$$\Delta^m(x_n - \lambda x_{n-k}) = 0 \tag{F}$$

$$x_n - \lambda x_{n-k} = 0 \tag{G}$$

which we call a *fundamental equation of neutral type* and a *geometric equation* respectively. By a solution of (F) we mean a real sequence x such that (F) is satisfied for all $n \geq \max(0, k)$. Analogously we define solutions of (G). We denote by

$$\mathrm{PG}(m, \lambda, k), \qquad \mathrm{Geo}(\lambda, k)$$

the set of all solutions of (F) and (G) respectively. Let $k \in \mathbb{Z}^*$, $x, y \in \mathrm{SQ}$. If

$$x_{n+|k|} = x_n, \qquad y_{n+|k|} = -y_n$$

for any $n \in \mathbb{N}$, then we say that x is *k-periodic* and y is *k-alternating*. We denote by

$$\mathrm{Per}(k), \qquad \mathrm{Alt}(k)$$

the set of all k-periodic sequences and the set of all k-alternating sequences respectively. Note that $\mathrm{Per}(k)$, and $\mathrm{Alt}(k)$ are linear subspaces of SQ and

$$\dim \mathrm{Per}(k) = |k| = \dim \mathrm{Alt}(k), \qquad \mathrm{Alt}(k) \subset \mathrm{Per}(2k).$$

Let $n \in \mathbb{N}$, $k \in \mathbb{Z}^*$, and $\lambda \in \mathbb{R}^*$. We define

$$n \operatorname{div} k := (\operatorname{sgn} k) \max\{j \in \mathbb{Z} : \ j|k| \leq n\}, \quad n \bmod k := n - |k|\,(n \operatorname{div} |k|),$$

$$\mathrm{geo}(\lambda, k), \mathrm{alt}(k) : \mathbb{N} \to \mathbb{R}, \quad \mathrm{geo}(\lambda, k)(n) = \lambda^{n \operatorname{div} k}, \quad \mathrm{alt}(k) = \mathrm{geo}(-1, k).$$

Note that

$$n \operatorname{div}(-k) = -(n \operatorname{div} k), \quad n \bmod(-k) = n \bmod k, \quad \mathrm{geo}(\lambda, -k) = \mathrm{geo}(\lambda^{-1}, k).$$

Moreover, $\mathrm{geo}(\lambda, k)$ is an "expanded" geometric sequence. Note also, that for a fixed k, the sequence $(n \bmod k)$ is k-periodic.

Example 1

$$(n \operatorname{div} 3) = (0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, \dots),$$

$$(n \bmod 3) = (0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, \dots),$$

$$\mathrm{geo}(\lambda, 1) = (\lambda^{n \operatorname{div} 1}) = (\lambda^n) = (1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \dots),$$

$$\mathrm{geo}(\lambda, 3) = (\lambda^{n \operatorname{div} 3}) = (1, 1, 1, \lambda, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2, \lambda^3, \lambda^3, \lambda^3, \dots),$$

$$\operatorname{alt}(3) = (1, 1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1, \dots).$$

Lemma 2 (Solutions of geometric equation) *If $k \in \mathbb{Z}^*$ and $\lambda \in \mathbb{R}^*$, then a sequence $x : \mathbb{N} \to \mathbb{R}$ is a solution of the geometric equation (G) if and only if for any $n \in \mathbb{N}$ we have*

$$x_n = \lambda^{n \operatorname{div} k} x_{n \bmod k}.$$

Proof See [12, Lemma 3.2].

Lemma 3 (Solutions of fundamental equation) *If $k \in \mathbb{Z}^*$, $\lambda \in \mathbb{R}^*$, then*

$$\operatorname{PG}(m, \lambda, k) = \operatorname{Pol}(m-1) \oplus \operatorname{Geo}(\lambda, k).$$

Moreover, if $\rho = \sqrt[k]{|\lambda|}$, then $k(|\lambda| - 1)(\rho - 1) \geq 0$ and

$$\operatorname{Geo}(\lambda, k) = \operatorname{geo}(\lambda, k)\operatorname{Per}(k) = \begin{cases} (\rho^n)\operatorname{Per}(k) & \text{if } \lambda > 0 \\ (\rho^n)\operatorname{Alt}(k) & \text{if } \lambda < 0 \end{cases}, \quad \operatorname{Alt}(k) = \operatorname{alt}(k)\operatorname{Per}(k).$$

Proof See [12, Theorem 3.1].

Hence any solution $y \in \operatorname{PG}(m, \lambda, k)$ of the fundamental equation

$$\Delta^m (y_n - \lambda y_{n-k}) = 0$$

is of the form

$$y_n = \varphi(n) + \omega_n \rho^n = \varphi(n) + \mathrm{O}(\rho^n),$$

where $\varphi \in \operatorname{Pol}(m-1)$, $\rho = \sqrt[k]{|\lambda|}$, and ω is $2k$-periodic.

Moreover, if $k(|\lambda| - 1) < 0$, then $\rho < 1$ and the polynomial part φ of y is dominating. On the other hand, if $k(|\lambda| - 1) > 0$, then $\rho > 1$ and the geometric part $\omega_n \rho^n$ is dominating.

3 Approximative Solutions

In this section, in Theorems 1, 2, and 3, we present our main results. First we need some lemmas.

Lemma 4 *Assume $k \in \mathbb{N}(0)$, $x, z, c \in \mathrm{SQ}$, $\alpha \in (0, 1)$, $|c| \leq \alpha$, $s \in \mathbb{R}$,*

$$z_n = x_n - c_n x_{n-k}$$

for $n \geq k$ and $z = \mathrm{o}(n^s)$. Then $x = \mathrm{o}(n^s)$.

Proof See [12, Lemma 4.1].

Lemma 5 (Krasnosielski fixed point lemma) *Assume X is an ordinary compact and convex subset of* SQ, $A, B : X \to \mathrm{SQ}$, $AX + BX \subset X$, $\alpha \in (0, 1)$, *A is continuous and B is an α-contraction. Then there exists a point $x \in X$ such that $Ax + Bx = x$.*

Proof This lemma is a consequence of [12, Theorem 2.2].

Lemma 6 *If $y \in \mathrm{SQ}$, $\rho \in \mathrm{o}(1)$, then the set $S = \{x \in \mathrm{SQ} : |x - y| \le |\rho|\}$ is ordinary, convex and compact.*

Proof See [12, Lemma 2.2].

Lemma 7 *Assume $X \subset \mathrm{SQ}$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\sigma : \mathbb{N} \to \mathbb{N}$, $\lim_{n\to\infty} \sigma(n) = \infty$, $a \in \mathrm{A}(m)$, $L \in \mathbb{R}$, and $\|f \circ x \circ \sigma\| \le L$ for any $x \in X$. Then the map $R : X \to \mathrm{SQ}$ defined by $R(x) = r^m(a(f \circ x \circ \sigma))$ is continuous.*

Proof Let $x \in X$ and $\varepsilon > 0$. There exist $p \in \mathbb{N}$ and $\alpha > 0$ such that

$$2L \sum_{n=p}^{\infty} n^{m-1}|a_n| < \varepsilon \qquad \text{and} \qquad \alpha \sum_{n=1}^{p} n^{m-1}|a_n| < \varepsilon.$$

There exists an index q such that $\sigma(\{0, 1, \dots, p\}) \subset \{0, 1, \dots, q\}$. Let

$$W = [x_0 - 1, x_0 + 1] \cup [x_1 - 1, x_1 + 1] \cup \cdots \cup [x_q - 1, x_q + 1].$$

Then W is compact and f is uniformly continuous on W. Choose a $\delta \in (0, 1)$ such that for $s, t \in W$ the condition $|s - t| < \delta$ implies $|f(s) - f(t)| < \alpha$. Assume $z \in X$, $\|x - z\| < \delta$. Then

$$\|Rx - Rz\| = \sup_{n\ge 0} |R(x)(n) - R(z)(n)| = \sup_{n\ge 0} |r_n^m(a(f \circ x \circ \sigma)) - r_n^m(a(f \circ z \circ \sigma))|$$

$$= \sup_{n\ge 0} |r_n^m(a(f \circ x \circ \sigma - f \circ z \circ \sigma))| \le \sup_{n\ge 0} r_n^m(|a||f \circ x \circ \sigma - f \circ z \circ \sigma|)$$

$$= r_0^m(|a||f \circ x \circ \sigma - f \circ z \circ \sigma|) \le \sum_{n=0}^{\infty} n^{m-1}|a_n||f(x_{\sigma(n)}) - f(z_{\sigma(n)})|$$

$$\le \sum_{n=0}^{p} n^{m-1}|a_n||f(x_{\sigma(n)}) - f(z_{\sigma(n)})| + \sum_{n=p}^{\infty} n^{m-1}|a_n||f(x_{\sigma(n)}) - f(z_{\sigma(n)})|$$

$$\le \alpha \sum_{n=0}^{p} n^{m-1}|a_n| + 2L \sum_{n=p}^{\infty} n^{m-1}|a_n| < \varepsilon + \varepsilon.$$

Theorem 1 *Assume* $\lambda \in \mathbb{R}^*$, $k \in \mathbb{Z}^*$, $s \in (-\infty, 0]$, $a, b \in \mathrm{A}(m-s)$, f *is continuous, and one of the following conditions is satisfied*

$$(a)\quad k(|\lambda|-1) < 0 \text{ and } u_n = \lambda + \mathrm{o}(n^{s-m+1}), \qquad (b)\quad k(|\lambda|-1) > 0 \text{ and } u_n = \lambda.$$

Then for any uniformly f*-bounded sequence*

$$y \in \mathrm{PG}(m, \lambda, k)$$

there exists a solution x *of (E) such that* $x_n = y_n + \mathrm{o}(n^s)$.

Proof Modify the proof of [12, Theorem 4.1], using Lemma 7.

Corollary 1 (Polynomial approximative solutions) *Assume* $\lambda \in \mathbb{R}^*$, $k \in \mathbb{Z}^*$, $s \in (-\infty, 0]$, $a, b \in \mathrm{A}(m-s)$, f *is continuous, and one of the following conditions is satisfied*

$$(a)\quad k(|\lambda|-1) < 0 \text{ and } u_n = \lambda + \mathrm{o}(n^{s-m+1}), \quad (b)\quad k(|\lambda|-1) > 0 \text{ and } u_n = \lambda.$$

Then for any uniformly f*-bounded polynomial sequence* $\varphi \in \mathrm{Pol}(m-1)$ *there exists a solution* x *of (E) such that* $x_n = \varphi(n) + \mathrm{o}(n^s)$.

Proof Since $\mathrm{Pol}(m-1) \subset \mathrm{PG}(m, \lambda, k)$, the assertion is a consequence of Theorem 1.

Corollary 2 (Convergent solutions) *Assume* $\lambda \in \mathbb{R}^*$, $k \in \mathbb{Z}^*$, $s \in (-\infty, 0]$, $a, b \in \mathrm{A}(m-s)$, f *is continuous, and one of the following conditions is satisfied*

$$(a)\quad k(|\lambda|-1) < 0 \text{ and } u_n = \lambda + \mathrm{o}(n^{s-m+1}), \quad (b)\quad k(|\lambda|-1) > 0 \text{ and } u_n = \lambda.$$

Then for any constant $d \in \mathbb{R}$ *there exists a solution* x *of (E) such that* $x_n = d + \mathrm{o}(n^s)$.

Proof Note that if f is continuous, then any constant sequence is uniformly f-bounded. Hence the assertion is a consequence of Corollary 1.

Example 2 Assume $f(t) = e^t$, $m = 3$, $s = -1$, $k(|\lambda|-1) < 0$, $a, b \in \mathrm{A}(4)$, $u_n = \lambda + \mathrm{o}(n^{-3})$. Then, by Theorem 1, for any $c_2 < 0$ and all $c_1, c_0 \in \mathbb{R}$ there exists a solution x of the equation

$$\Delta^3(x_n - u_n x_{n-k}) = a_n e^{x_n} + b_n$$

such that $x_n = c_2 n^2 + c_1 n + c_0 + \mathrm{o}(n^{-1})$.

Example 3 Assume $s \in (-\infty, 0]$, $a, b \in \mathrm{A}(m-s)$, and $f(t) = e^{-t}$. Then, by Theorem 1, for any polynomial sequence $\varphi \in \mathrm{Pol}(m-1)$ and any positive 3-periodic sequence ω there exists a solution x of the equation

$$\Delta^m(x_n - 27x_{n-3}) = a_n e^{-x_n} + b_n$$

such that $x_n = \varphi(n) + \omega_n 3^n + \mathrm{o}(n^s)$.

Theorem 2 *Assume $s \in (-\infty, 0]$, $a \in \mathrm{A}(m-s)$, $|c| \ll 1$, and f is continuous. Then for any uniformly f-bounded solution y of the equation*

$$\Delta^m(y_n - c_n y_{n-k}) = b_n$$

there exists a solution x of (E) such that $x = y + \mathrm{o}(n^s)$.

Proof Choose a real number α such that $|c| \leq \alpha < 1$. Let $\beta = (1-\alpha)^{-1}$. Assume y is a uniformly f-bounded solution of the equation

$$\Delta^m(y_n - c_n y_{n-k}) = b_n.$$

There exists a positive number μ such that f is bounded on the set

$$Z = \bigcup_{n=0}^{\infty}[y_n - \mu, y_n + \mu].$$

Choose a constant L such that $|f(t)| \leq L$ for any $t \in Z$. Let

$$Y = \{x \in \mathrm{SQ} : \|x - y\| \leq \mu\}.$$

Note that $\|f \circ x \circ \sigma\| \leq L$ for any $x \in Y$. Hence

$$a(f \circ x \circ \sigma) \in \mathrm{A}(m-s) \subset \mathrm{A}(m)$$

for any $x \in Y$. Let

$$\rho, \gamma^* \in \mathrm{SQ}, \qquad \rho = Lr^m|a|, \qquad \gamma_n^* = \begin{cases} \beta\rho_n & \text{if } k \leq 0 \text{ and } n \geq 0 \\ \rho_n & \text{if } k > 0 \text{ and } n < k \\ \rho_n + \alpha\gamma_{n-k}^* & \text{if } k > 0 \text{ and } n \geq k \end{cases}.$$

By Lemma 1 we have $\rho = \mathrm{o}(n^s)$. If $k > 0$ and $n \geq k$, then

$$\rho_n = \gamma_n^* - \alpha\gamma_{n-k}^*$$

By Lemma 4 we have $\gamma^* \in \mathrm{o}(n^s) \subset \mathrm{o}(1)$. Note also, that $\gamma^* \geq 0$. Choose an index $p \geq \max(k, 0)$ such that

$$\gamma_n^* \leq \mu \quad \text{for any} n \geq p.$$

Let

$$\gamma \in \mathrm{SQ}, \qquad \gamma_n = \begin{cases} 0 & \text{if n<p} \\ \gamma_n^* & \text{if } n \ge 0 \end{cases}, \qquad X = \{x \in \mathrm{SQ} : |x - y| \le \gamma\}.$$

By Lemma 6, X is an ordinary compact and convex subset of SQ. Moreover $X \subset Y$. Let

$$B : \mathrm{SQ} \to \mathrm{SQ}, \qquad B(x)(n) = \begin{cases} 0 & \text{if} \quad n < p \\ c_n(x_{n-k} - y_{n-k}) & \text{if} \quad n \ge p \end{cases},$$

$$R, A : Y \to \mathrm{SQ}, \quad R(x)(n) = \begin{cases} 0 & \text{if } n < p \\ (-1)^m r_n^m(a(f \circ x \circ \sigma)) & \text{if } n \ge p \end{cases}, \quad Ax = y + Rx.$$

If $n \ge p$, then

$$|R(x)(n)| = |(-1)^m r_n^m(a(f \circ x \circ \sigma))| = |r_n^m(a(f \circ x \circ \sigma))| \le r_n^m(|a(f \circ x \circ \sigma)|)$$

$$\le r_n^m(L|a|) = Lr_n^m|a| = \rho_n.$$

Let $x, z \in X$. For $n < p$ we have

$$|Ax + Bz - y|(n) = 0 = \gamma_n.$$

Assume $n \ge p$. If $k \ge 0$, then

$$|Ax + Bz - y|(n) = |Rx + Bz|(n) \le \rho_n + |c_n(z_{n-k} - y_{n-k})| \le \rho_n + \alpha\gamma_{n-k} = \gamma_n.$$

If $k \le 0$, then

$$|Ax + Bz - y|(n) \le \rho_n + |c_n(z_{n-k} - y_{n-k})| \le \rho_n + \alpha\gamma_{n-k} =$$

$$\rho_n + \alpha\beta\rho_{n-k} \le \rho_n + \alpha\beta\rho_n = (1 + \alpha\beta)\rho_n = \beta\rho_n = \gamma_n.$$

Therefore $AX + BX \subset X$. Using Lemma 7, it is easy to see that the map A is continuous. Obviously B is an α-contraction. By Lemma 5 there exists a point $x \in X$ such that $x = Ax + Bx$. Then for $n \ge p$ we have

$$x_n = R(x)(n) + y_n + B(x)(n) = (-1)^m r_n^m(a(f \circ x \circ \sigma)) + y_n + c_n x_{n-k} - c_n y_{n-k},$$

$$x_n - c_n x_{n-k} = y_n - c_n y_{n-k} + (-1)^m r_n^m(a(f \circ x \circ \sigma)).$$

Since $\Delta^m(y_n - c_n y_{n-k}) = b_n$ and

$$\Delta^m((-1)^m r^m(a(f \circ x \circ \sigma))) = a(f \circ x \circ \sigma)$$

we obtain

$$\Delta^m(x_n - c_n x_{n-k}) = a_n f(x_{\sigma(n)}) + b_n$$

for $n \geq p$. Since $x \in X$ we have $x = y + \mathrm{o}(n^s)$.

Theorem 3 *Assume* $s \in (-\infty, 0]$, $a \in \mathrm{A}(m-s)$, $|c| \gg 1$, *and* f *is continuous. Then for any uniformly* f*-bounded solution* y *of the equation*

$$\Delta^m(y_n - c_n y_{n-k}) = b_n$$

there exists a solution x *of (E) such that* $x = y + \mathrm{o}(n^s)$.

Proof Choose a real number β such that $|c| \geq \beta > 1$. Let

$$\alpha = \beta^{-1} \quad \text{and} \quad \lambda = (1-\alpha)^{-1}.$$

Assume y is a uniformly f-bounded solution of the equation

$$\Delta^m(y_n - c_n y_{n-k}) = b_n.$$

There exists a positive number μ such that f is bounded on the set

$$Z = \bigcup_{n=0}^{\infty} [y_n - \mu, y_n + \mu].$$

Choose a constant L such that $|f(t)| \leq L$ for any $t \in Z$. Let

$$Y = \{x \in \mathrm{SQ} : \|x - y\| \leq \mu\}.$$

Since $\|f \circ x \circ \sigma\| \leq L$ for any $x \in Y$, we have

$$a(f \circ x \circ \sigma) \in \mathrm{A}(m-s) \subset \mathrm{A}(m)$$

for any $x \in Y$. Let

$$\rho \in \mathrm{SQ}, \qquad \rho_n = \begin{cases} 0 & \text{if } n < \max(0, -k) \\ L\alpha r^m_{n+k}|a| & \text{if } n \geq \max(0, -k) \end{cases}.$$

$$\gamma^* \in \mathrm{SQ}, \qquad \gamma^*_n = \begin{cases} \lambda\rho_n & \text{if } k \geq 0 \text{ and } n \geq 0 \\ \rho_n & \text{if } k < 0 \text{ and } n < -k \\ \rho_n + \alpha\gamma^*_{n+k} & \text{if } k < 0 \text{ and } n \geq -k \end{cases}.$$

By Lemma 1 we have $\rho = \mathrm{o}(n^s)$. If $k < 0$ and $n \geq -k$, then

$$\rho_n = \gamma_n^* - \alpha\gamma_{n+k}^*.$$

By Lemma 4 we have $\gamma^* \in \mathrm{o}(n^s) \subset \mathrm{o}(1)$. Note also, that $\gamma^* \geq 0$. Choose an index $p \geq \max(0, -k)$ such that

$$\gamma_n^* \leq \mu \quad \text{for any } n \geq p.$$

Let

$$\gamma \in \mathrm{SQ}, \qquad \gamma_n = \begin{cases} 0 & \text{if } n < p \\ \gamma_n^* & \text{if } n \geq 0 \end{cases}, \qquad X = \{x \in \mathrm{SQ} : |x - y| \leq \gamma\}.$$

Then X is an ordinary compact and convex subset of SQ. Moreover $X \subset Y$. Let

$$d \in \mathrm{SQ}, \qquad d_n = \begin{cases} 0 & \text{if} \quad n < p \\ c_{n+k}^{-1} & \text{if} \quad n \geq p \end{cases},$$

$$B : \mathrm{SQ} \to \mathrm{SQ}, \qquad B(x)(n) = \begin{cases} 0 & \text{if} \quad n < p \\ d_n(x_{n+k} - y_{n+k}) & \text{if} \quad n \geq p \end{cases},$$

$$R : Y \to \mathrm{SQ}, \quad R(x)(n) = \begin{cases} 0 & \text{if } n < p \\ (-d_n)(-1)^m r_{n+k}^m (a(f \circ x \circ \sigma)) & \text{if } n \geq p \end{cases},$$

$$A : Y \to \mathrm{SQ}, \qquad Ax = y + Rx.$$

If $n \geq p$, then

$$|R(x)(n)| = |(-d_n)(-1)^m r_{n+k}^m (a(f \circ x \circ \sigma))| = |d_n||r_{n+k}^m (a(f \circ x \circ \sigma))|$$

$$\leq \alpha r_{n+k}^m (|a(f \circ x \circ \sigma)|) \leq \alpha r_{n+k}^m (L|a|) = L\alpha r_{n+k}^m |a| = \rho_n.$$

Let $x, z \in X$. For $n < p$ we have

$$|Ax + Bz - y|(n) = 0 = \gamma_n.$$

Assume $n \geq p$. If $k < 0$, then

$$|Ax + Bz - y|(n) = |Rx + Bz|(n) \leq \rho_n + |d_n(z_{n+k} - y_{n+k})| \leq \rho_n + \alpha\gamma_{n+k} = \gamma_n.$$

If $k \geq 0$, then

$$|Ax + Bz - y|(n) \leq \rho_n + |c_n(z_{n+k} - y_{n+k})| \leq \rho_n + \alpha\gamma_{n+k}$$

$$= \rho_n + \alpha\lambda\rho_{n+k} \leq \rho_n + \alpha\lambda\rho_n = (1 + \alpha\lambda)\rho_n = \lambda\rho_n = \gamma_n.$$

Therefore $AX + BX \subset X$. Using Lemma 7, it is easy to see that the map A is continuous. Obviously B is an α-contraction. By Lemma 5 there exists a point $x \in X$ such that $x = Ax + Bx$. Then for $n \geq p$ we have

$$x_n = R(x)(n) + y_n + B(x)(n) = (-d_n)(-1)^m r^m_{n+k}(a(f \circ x \circ \sigma)) + y_n + d_n(x_{n+k} - y_{n+k}),$$

$$x_n - d_n x_{n+k} = y_n - d_n y_{n+k} - d_n(-1)^m r^m_{n+k}(a(f \circ x \circ \sigma)).$$

Multiplying by $-c_{n+k}$ we have

$$-c_{n+k}x_n + x_{n+k} = -c_{n+k}y_n + y_{n+k} + (-1)^m r^m_{n+k}(a(f \circ x \circ \sigma)).$$

Replacing n by $n - k$ we obtain

$$x_n - c_n x_{n-k} = y_n - c_n y_{n-k} + (-1)^m r^m_n(a(f \circ x \circ \sigma)).$$

Since $\Delta^m(y_n - c_n y_{n-k}) = b_n$ and

$$\Delta^m((-1)^m r^m(a(f \circ x \circ \sigma))) = a(f \circ x \circ \sigma)$$

we obtain

$$\Delta^m(x_n - c_n x_{n-k}) = a_n f(x_{\sigma(n)}) + b_n$$

for $n \geq p$. Since $x \in X$ we have $x = y + \mathrm{o}(n^s)$.

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Asymptotic Properties of Nonoscillatory Solutions of Third-Order Delay Difference Equations

Alina Gleska and Małgorzata Migda

Abstract We study a third-order delay trinomial difference equation. We transform this equation to a binomial third-order difference equation with quasidifferences. Using comparison theorems with a certain first order delay difference equation we establish results on asymptotic properties of nonoscillatory solutions of the studied equation. We give an easily verifiable criterium which ensures that all nonoscillatory solutions tend to zero.

Keywords Third-order difference equation · Asymptotic behavior
Non-oscillation · Oscillation

1 Introduction

In this paper we consider the linear third-order difference equation of the form

$$\Delta^3 x_n + p_n \Delta x_{n+1} + q_n x_{n-\tau} = 0, \tag{E}$$

where $n \in \mathbb{N}(n_0)$, $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$, n_0 is fixed in $\mathbb{N} = \{1, 2, \dots\}$, τ is a positive integer, (p_n) is a sequence of nonnegative real numbers, (q_n) is a sequence of positive real numbers. Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$.

By a solution of equation (E) we mean a real sequence (x_n) defined for $n \geq n_0 - \tau$ which satisfies (E) for all $n \geq n_0$. A solution is said to be *nonoscillatory* if it is eventually positive or eventually negative. Otherwise, the solution is said to be

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oscillatory. Equation (E) is called *oscillatory* if all its solutions are oscillatory. For $k \in \mathbb{N}$ we use the usual factorial notation

$$n^{\underline{k}} = n(n-1)\dots(n-k+1) \quad \text{with} \quad n^{\underline{0}} = 1.$$

The third-order difference equations often arise in the study of many problems of economics, mathematical biology, ecology and engineering. So, in recent years, there has been an increasing interest in the study of qualitative properties of solutions of such equations. For example, the third-order binomial difference equation related to Eq. (E), i.e.,

$$\Delta^3 x_n + p_n x_{n+1} = 0$$

and its various generalization including quasidifferences have been studied, e.g., in [2, 3, 8, 12, 16, 18–21]. For some background details we refer to [20].

Equation (E) may be considered as a discrete analogue of the delay differential equation

$$y'''(t) + p(t)y'(t) + q(t)y(\tau(t)) = 0.$$

For some recent results on oscillation and nonoscillation of this equation, see, for example, [4, 5, 10].

In this paper we study asymptotic properties of nonoscillatory solutions of equation (E) by transforming this equation to a third-order binomial difference equation. Using comparison theorems with a certain first order delay difference equation we establish results on asymptotic properties of solutions of equation (E). We also present sufficient conditions which ensure that all nonoscillatory solutions of equation (E) tend to zero. The presented criteria are easily applicable.

2 Preparatory Results

The main results of this paper are based on the connection between the properties of Eq. (E) and positive solutions of the auxiliary second-order difference equation

$$\Delta^2 u_n + p_n u_{n+1} = 0. \tag{1}$$

Lemma 1 *Let (u_n) be a positive solution of (1). Then Eq. (E) can be written in the form*

$$\Delta\left(u_{n+1}u_n\Delta\left(\frac{1}{u_n}\Delta x_n\right)\right) + u_{n+1}q_n x_{n-\tau} = 0. \tag{E'}$$

Proof Let (u_n) be a positive solution of (1). It is easy to check that

$$\frac{1}{u_{n+1}}\Delta\left(u_{n+1}u_n\left(\Delta\left(\frac{1}{u_n}\Delta x_n\right)\right)\right) = \frac{1}{u_{n+1}}\left(\Delta\left(u_n\Delta^2 x_n - \Delta x_n\Delta u_n\right)\right)$$

$$= \frac{1}{u_{n+1}} \left(u_{n+1} \Delta^3 x_n - \Delta x_{n+1} \Delta^2 u_n \right) = \Delta^3 x_n + p_n \Delta x_{n+1}.$$

Hence we see that Eq. (E) takes the form of Eq. (E'). □

In the sequel, we first investigate the existence and the properties of positive solutions of (1) and then, instead of studying the properties of the trinomial equation (E), we study the properties of its binomial representation (Eq. (E')).

The following result, (see e.g. [1, Theorem 1.14.2]), enables us to determine if there exist positive solutions of (1).

Theorem 1 ([1]) *Suppose that* $p_n \geq 0$*,*

$$\sum_{n=n_0}^{\infty} p_n < \infty, \quad \limsup_{n \longrightarrow \infty} n \sum_{s=n}^{\infty} p_s < \frac{1}{4}. \tag{2}$$

Then (1) *possesses a positive solution.*

It is well-known, see [1] or [14], that if the second order linear equation (1) is nonoscillatory, then there exists a nontrivial positive solution (u_n) uniquely determined up to a constant factor, such that

$$\lim \frac{u_n}{v_n} = 0,$$

where (v_n) denotes an arbitrary nontrivial solution of (1), linearly independent of (u_n). Solution u is called a *recessive solution*. Any solution v linearly independent of u is called a *dominant solution*. Recessive solutions u and dominant solutions v have the following useful properties

$$\sum_{n=n_0}^{\infty} \frac{1}{u_n u_{n+1}} = \infty, \tag{3}$$

$$\sum_{n=n_0}^{\infty} \frac{1}{v_n v_{n+1}} < \infty. \tag{4}$$

Following Trench [22], we say that a linear difference operator

$$L x_n = \Delta \left(r_n^{m-1} \left(\ldots \Delta \left(r_n^1 \Delta x_n \right) \ldots \right) \right)$$

is in a canonical form if

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n^j} = \infty, \quad j = 1, 2, \ldots, m-1.$$

For our purpose it will be convenient to have Eq. (E') in a canonical form. So, in the sequel we require that the condition

$$\sum_{n=n_0}^{\infty} u_n = \infty \tag{5}$$

and the condition (3) hold. The next result is obvious.

Lemma 2 *Let (u_n) be a positive solution of (1). Then $\Delta u_n > 0$, $\Delta^2 u_n < 0$, and (5) holds.*

Therefore, every recessive solution (u_n) of Eq. (1) satisfies the conditions (3) and (5). Hence, combining Lemmas 1 and 2 and Theorem 1, we get the following result.

Corollary 1 *Assume that condition (2) is satisfied and that (u_n) is a recessive solution of equation (1). Then Eq. (E) can be written in a binomial form (E'), which is in a canonical form.*

Now, denote for Eq. (E') the quasidifferences by

$$\begin{aligned} &L_0 x_n = x_n, \quad L_1 x_n = \frac{1}{u_n} \Delta \left(L_0 x_n \right), \\ &L_2 x_n = u_n u_{n+1} \Delta \left(L_1 x_n \right), \quad L_3 x_n = \Delta \left(L_2 x_n \right). \end{aligned} \tag{6}$$

Observe, that if x is a solution of equation (E'), then $-x$ is also a solution of equation (E'). Thus, during studying the nonoscillation of Eq. (E'), for the sake of simplicity, we restrict our attention to eventually positive solutions.

Using the generalized Kiguradze's Lemma (see [17]), we get the following structure of nonoscillatory solutions of equation (E') (as well as Eq. (E)).

Lemma 3 *Let condition (2) holds. Assume that (u_n) is such a positive solution of (1) for which (3) and (5) are satisfied. Then every positive solution (x_n) of Eq. (E') is either of degree 0,*

$$L_0 x_n > 0, \quad L_1 x_n < 0, \quad L_2 x_n > 0, \quad L_3 x_n < 0 \tag{7}$$

or of degree 2

$$L_0 x_n > 0, \quad L_1 x_n > 0, \quad L_2 x_n > 0, \quad L_3 x_n < 0. \tag{8}$$

We say that (E') has *property (A)* if its every nonoscillatory solution (x_n) is of degree 0, that it satisfies (7). We say that (E) has *property (P)* if all its nonoscillatory solutions (x_n) satisfy the condition

$$x_n \Delta x_n < 0. \tag{9}$$

To the end of this section we present a useful comparison result (see [13], Corollary 7.6.1).

Lemma 4 ([13]) *Assume* (a_n) *is a sequence of nonnegative real numbers and* τ *is a positive integer such that* $\sum_{k=n-\tau}^{n-1} a_k > 0$ *for large* n. *Then the difference inequality*

$$\Delta x_n + a_n x_{n-k} \leq 0$$

has an eventually positive solution if and only if the equation

$$\Delta x_n + a_n x_{n-k} = 0$$

has an eventually positive solution.

3 Main Results

First we show that property (P) of Eq. (E) and property (A) of Eq. (E') are equivalent.

Theorem 2 *Let condition* (2) *hold. Assume that* (u_n) *is a recessive solution of equation* (1). *Then Eq. (E') has property (A) if and only if Eq. (E) has property (P).*

Proof Let (x_n) be a positive solution of equation (E'). Suppose that Eq. (E') has property (A). Then it is easy to check that $\Delta x_n < 0$. Hence $x_n \Delta x_n < 0$ what means that Eq. (E) has property (P).

Now suppose that Eq. (E) has property (P). Then

$$L_0 x_n = x_n > 0, \quad L_1 x_n = \frac{1}{u_n} \Delta x_n < 0.$$

Let us observe that also $L_3 x_n < 0$ as the first part of Eq. (E'). It means that $(L_2 x_n)$ is decreasing. We have two possibilities: $(L_2 x_n)$ is positive for $n \geq n_0$ or $(L_2 x_n)$ is eventually negative, say for $n \geq n_1 \geq n_0$. The second case means that $(L_1 x_n)$ is decreasing. Then for $n \geq n_1$ the condition

$$L_1 x_n < L_1 x_{n_1}$$

implies that

$$\Delta L_0 x_n < u_n L_1 x_{n_1}.$$

Summing now from n_1 to $n-1$ we get

$$L_0 x_n < L_0 x_{n_1} + L_1 x_{n_1} \sum_{i=n_1}^{n-1} u_i.$$

From (5) the right-hand side of the above inequality tends to $-\infty$ which contradicts that (x_n) is positive. Thus $(L_2 x_n)$ is positive and (7) holds. □

Now, we provide criteria that enable us to deduce property (P) of Eq. (E) from the oscillation of certain first-order difference equation.

Theorem 3 *Let condition* (2) *hold. Assume that* (u_n) *is a recessive solution of equation* (1)*. Let us denote*

$$a_n = u_{n+1} q_n \left[\sum_{i=n_1}^{n-\tau-1} u_i \sum_{j=n_1}^{i} \frac{1}{u_j u_{j+1}} \right]$$

for some $n_1 \geq n_0$*. If the first-order difference equation*

$$\Delta z_n + a_n z_{n-\tau} = 0 \tag{10}$$

is oscillatory, then Eq. (E) *has property* (P)*.*

Proof Assume that Eq. (E) has a positive solution (x_n). Then from Lemma 3 it follows that (x_n) is either of degree 2 or of degree 0. If (x_n) is of degree 2, then using the fact that $z_n = u_n u_{n+1} \Delta \left(\frac{1}{u_n} \Delta x_n \right)$ is decreasing there exists $n_1 \geq n_0$ such that for $n \geq n_1$ we obtain

$$\frac{1}{u_n} \Delta x_n \geq \sum_{j=n_1}^{n-1} \Delta \left(\frac{1}{u_j} \Delta x_j \right) \geq z_n \sum_{j=n_1}^{n-1} \frac{1}{u_j u_{j+1}}.$$

Summing now from n_1 to $n-1$, we get

$$x_n \geq \sum_{i=n_1}^{n-1} \left(z_i u_i \sum_{j=n_1}^{i} \frac{1}{u_j u_{j+1}} \right) \geq z_n \sum_{i=n_1}^{n-1} \left(u_i \sum_{j=n_1}^{i} \frac{1}{u_j u_{j+1}} \right).$$

Then also

$$x_{n-\tau} \geq z_{n-\tau} \sum_{i=n_1}^{n-\tau-1} \left(u_i \sum_{j=n_1}^{i} \frac{1}{u_j u_{j+1}} \right). \tag{11}$$

Combining (11) with Eq. (E'), we see that

$$-\Delta z_n = u_{n+1} q_n x_{n-\tau} \geq u_{n+1} q_n z_{n-\tau} \sum_{i=n_1}^{n-\tau-1} \left(u_i \sum_{j=n_1}^{i} \frac{1}{u_j u_{j+1}} \right).$$

In other words, (z_n) is a positive solution of the difference inequality

$$\Delta z_n + a_n z_{n-\tau} \leq 0,$$

where

$$a_n = u_{n+1} q_n \sum_{i=n_1}^{n-\tau-1} \left(u_i \sum_{j=n_1}^{i} \frac{1}{u_j u_{j+1}} \right).$$

Hence, by Lemma 4, we conclude that the corresponding difference equation (10) has also a positive solution which contradicts the oscillation of (10). Therefore (x_n) is of degree 0 what means that Eq. (E) has property (P). □

Using known oscillation criteria to the first order difference equation with delay of the form (10) (see, for example, [6, 7, 11, 15]), we immediately get from Theorem 3 various criteria for property (P) of Eq. (E).

Corollary 2 *Assume condition* (2) *holds and* (u_n) *is a recessive solution of equation* (1). *If one of the following conditions holds*

$$\liminf_{n\to\infty} \sum_{k=n-\tau}^{n-1} u_{k+1} q_k \left[\sum_{i=n_1}^{k-\tau-1} u_i \sum_{j=n_1}^{i} \frac{1}{u_j u_{j+1}} \right] > \left(\frac{\tau}{\tau+1} \right)^{\tau+1} \tag{12}$$

or

$$\limsup_{n\to\infty} \sum_{k=n-\tau}^{n} u_{k+1} q_k \left[\sum_{i=n_1}^{k-\tau-1} u_i \sum_{j=n_1}^{i} \frac{1}{u_j u_{j+1}} \right] > 1, \tag{13}$$

then Eq. (E) *has property* (P).

Example 1 Consider the difference equation

$$\Delta^3 x_n + \frac{1}{2(2^{n+1}-1)} \Delta x_{n+1} + \frac{1}{8(2-2^{-n})} x_{n-1} = 0, \quad n \geq 0. \tag{14}$$

It is easy to check that for this equation conditions (2) and (13) are satisfied. Hence, by Corollary 2, every nonoscillatory solution of equation (14) satisfies the condition $x_n \Delta x_n < 0$. Indeed, the sequence $x_n = \frac{1}{2^n}$ is one of such solutions.

Our comparison method is based on the canonical representation (E') of Eq. (E). The condition (2) guarantees the existence of the positive solution (u_n) of (1); then the canonical representation (E') is possible. Here arises a natural question, what to do when we are not able to find (u_n). So next we give criteria in which instead of (u_n) its asymptotic representation is used. The assumption on the coefficient (p_n) is stronger, but the presented results are easier to use.

The following theorem will be used.

Theorem 4 ([9]) *Suppose that* $p_n \geq 0$ *and*

$$\sum_{n=n_0}^{\infty} np_n < \infty. \tag{15}$$

Then for every $d \in \mathbb{R}$ *there exists a solution* (u_n) *of the Eq.* (1) *such that* $\lim_{n\longrightarrow\infty} u_n = d$.

Combining Theorem 4 with Corollary 2, we get the following result.

Theorem 5 *Suppose that* (15) *holds. If for some* $c \in (0, 1)$ *one of the following conditions is satisfied*

$$\liminf_{n\to\infty} \sum_{k=n-\tau}^{n-1} \frac{c^2\,(k-\tau-n_0)^2}{2}\, q_k > \left(\frac{\tau}{\tau+1}\right)^{\tau+1} \tag{16}$$

or

$$\limsup_{n\to\infty} \sum_{k=n-\tau}^{n} \frac{c^2\,(k-\tau-n_0)^2}{2}\, q_k > 1, \tag{17}$$

then Eq. (E) *has property* (P).

Proof Let (u_n) be a positive solution of (1). It follows from Theorem 4 and Lemma 2 that for any $c \in (0, 1)$ we have

$$c < u_n < 1,$$

for sufficiently large n, say $n \geq n_1$. Hence, by (16), we get

$$\begin{aligned}
&\liminf_{n\to\infty} \sum_{k=n-\tau}^{n-1} u_{k+1} q_k \left(\sum_{i=n_1}^{k-\tau-1} u_i \sum_{j=n_1}^{i} \frac{1}{u_j u_{j+1}} \right) \\
&> \liminf_{n\to\infty} \sum_{k=n-\tau}^{n-1} u_{k+1} q_k \left(\sum_{i=n_1}^{k-\tau-1} (i-n_1+1) u_i \right) \\
&> \liminf_{n\to\infty} \sum_{k=n-\tau}^{n-1} \frac{c(k-\tau-n_1)^2}{2}\, u_{k+1} q_k \\
&> \liminf_{n\to\infty} \sum_{k=n-\tau}^{n-1} \frac{c^2(k-\tau-n_0)^2}{2}\, q_k.
\end{aligned}$$

Thus, by Corollary 2, Eq. (E) has property (P). □

Example 2 Consider the difference equation

$$\Delta^3 x_n + \frac{a}{n^3}\Delta x_{n+1} + \frac{4b}{n^{\beta}} x_{n-\tau} = 0, \quad n \geq 1, \tag{18}$$

where $a, b, \beta > 0$, $\tau \geq 1$. It is easy to see that (15) holds. Let us take $c = \frac{1}{2}$. For $\beta \leq 2$ and $b > 1$ we have

$$\limsup_{n\to\infty}\sum_{k=n-\tau}^{n}\frac{(k-\tau-1)^2}{8}\frac{4b}{k^\beta}\geq b>1.$$

Hence, condition (17) is satisfied and by Theorem 5, Eq. (18) has property (P).

In the next theorem we derive conditions guaranteeing that every nonoscillatory solution of equation (E) tends to zero as $n\to\infty$.

Theorem 6 *Assume condition* (2) *holds and* (u_n) *is a recessive solution of equation* (1). *Assume that Eq.* (E) *has property* (P). *If*

$$\sum_{k=n_0}^{\infty}u_k\sum_{j=k}^{\infty}\frac{1}{u_ju_{j+1}}\sum_{i=j}^{\infty}u_{i+1}q_i=\infty, \tag{19}$$

then every nonoscillatory solution of (E) *has the property* $\lim_{n\to\infty}x_n=0$.

Proof Let (x_n) be a nonoscillatory solution of equation (E) satisfying (P). Without loss of generality, we may assume that (x_n) is eventually positive. Therefore there exists $n_1\geq n_0$ such that $x_n>0$ and $x_{n-\tau}>0$ for $n\geq n_1$. Then $\Delta x_n<0$ for all $n\geq n_2$ for some $n_2\geq n_1$. Hence, there exists a limit $\lim_{n\longrightarrow\infty}x_n=l\geq 0$. Assume for contradiction that $l>0$. By Theorem 2, (x_n) is also a solution of degree 0 of Eq. (E'). Since $x_{n-\tau}>l$ for $n\geq n_2$, from Eq. (E') we get

$$-\Delta\left(u_{n+1}u_n\Delta\left(\frac{1}{u_n}\Delta x_n\right)\right)\geq l u_{n+1}q_n,\quad n\geq n_2. \tag{20}$$

From (6) it follows that (L_1x_n) and (L_2x_n) tends to zero. Hence, summing (20) from n to ∞, we obtain

$$u_{n+1}u_n\left(\Delta\frac{1}{u_n}\Delta x_n\right)\geq l\sum_{i=n}^{\infty}u_{i+1}q_i.$$

Summing again we get

$$-\frac{1}{u_n}\Delta x_n\geq l\sum_{j=n}^{\infty}\frac{1}{u_ju_{j+1}}\sum_{i=j}^{\infty}u_{i+1}q_i.$$

Finally, summing the above equation from n_2 to $n-1$, we have

$$x_n\leq x_{n_2}-l\sum_{k=n_2}^{n-1}u_k\sum_{j=k}^{\infty}\frac{1}{u_ju_{j+1}}\sum_{i=j}^{\infty}u_{i+1}q_i.$$

Letting $n\to\infty$ and using (19) we get $\lim_{n\to\infty}x_n=-\infty$ which contradicts the fact that (x_n) is eventually positive. Hence $l=0$. The proof is complete. □

From Theorems 4 and 6, we get the following result. The proof is similar to the proof of Theorem 5, so it is omitted.

Theorem 7 *Let condition* (15) *be satisfied. Assume that Eq. (E) has property (P). If*

$$\sum_{n=n_0}^{\infty} n^2 q_n = \infty, \tag{21}$$

then every nonoscillatory solution of equation (E) has the property $\lim_{n\to\infty} x_n = 0$.

Combining Theorem 7 with Theorem 5 we get following result.

Corollary 3 *Let condition* (15) *be satisfied. If condition* (16) *or* (17) *is satisfied, then every nonoscillatory solution of equation (E) has the property* $\lim_{n\to\infty} x_n = 0$.

Proof Observe that condition (16) or (17) implies condition (21). Thus we get the result. □

Example 3 Consider the difference equations (14) and (18). It is easy to see that

$$\sum_{n=0}^{\infty} \frac{n}{2(2^{n+1}-1)} < \infty.$$

Moreover, by Example 1, Eq. (14) has property (P). Hence, by Corollary 3, every nonoscillatory solution of this equation tends to zero as $n \to \infty$. Similarly, for Eq. (18) by Example 2, assumptions of Corollary 3 hold. Hence every nonoscillatory solution of equation (18) tends to zero as $n \to \infty$, too.

4 Summary

In this paper we have studied asymptotic properties of nonoscillatory solutions of the third-order delay trinomial equation (E) by transforming this equation to a binomial third-order difference equation with quasidifferences. We have deduced property (P) of Eq. (E) from the oscillation of a certain first-order difference equation. Finally, in Corollary 3, we have presented an easily verifiable criterium which ensures that all nonoscillatory solutions of equation (E) tend to zero. We point out that the assumptions of this criterium depend only on the coefficients (p_n), (q_n) and the delay τ.

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On Copson's Theorem and Its Generalizations

A. Linero Bas and D. Nieves Roldán

Abstract E.T. Copson generalized the well-known result about the convergence of bounded and monotonic sequences of real numbers. Over the years, generalizations of this result have been made concerning linear and nonlinear inequalities that gave us a wide range of criteria for the convergence of sequences in relationship to the characteristic polynomial, monotonicity of the variables, etc. In this paper, we present a survey about these generalizations of Copson's result, focusing in the state-of-art of the problem, and bring up some open questions that could lead us to future research.

Keywords Sequences · Convergence · Copson's Theorem · Characteristic polynomial · Monotonicity · Monotone convergence theorem

1 Introduction

It is a widely known result of real mathematical analysis that a bounded monotonic sequence of real numbers converges. In 1969, Professor E.T. Copson, inspired by two suggestions of J. M. Whittaker and J. B. Tatchell, generalized this result by changing the monotonicity of the sequence by a convex inequality that involves r consecutive members of the sequence as follows:

Theorem 1 ([3]) *If* (a_n) *is a bounded sequence of real numbers which satisfies the inequality*

$$a_{n+r} \leq \sum_{s=1}^{r} k_s a_{n+r-s}, \tag{1}$$

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where the coefficients k_s *are strictly positive and* $k_1 + \cdots + k_r = 1$, *with* $r \geq 1$, *then* (a_n) *is a convergent sequence. But if* (a_n) *is unbounded, it diverges to* $-\infty$.

It is worth mentioning that, apart from its own proof, Copson included in his article another one due to R. A. Rankin. On one hand, Copson's proof underlines the relevance of the characteristic polynomial associated to inequality (1), given by $P(\lambda) = \lambda^r - k_1\lambda^{r-1} - k_2\lambda^{r-2} - \ldots - k_{r-1}\lambda - k_r$. The main point of the proof is to bound the sequence as follows: $a_{n+r} \leq \sum_{s=1}^{r} A_s(l)a_{n-l+r-s}$, where the coefficients $A_s(l)$ are given by $A_s(l+1) = k_s A_1(l) + A_{s+1}(l)$. After that, using an auxiliary result, it can be assured the convergence of $A_s(l)$, and then, by applying some properties of the inferior and superior limit the proof ends. On the other hand, the technique developed by Rankin, based on defining an auxiliary sequence $A_n = \max\{a_{n-1}, \ldots, a_{n-r}\}$ which is monotonic under the conditions of the theorem, allows us to extend this result to other mathematical objects like double sequences or sequences of functions.

It should be highlighted that the direction of inequality (1) is clearly immaterial, since we may replace a_n by $-a_n$ to reach the same conclusion. Also, it should be emphasized that the conclusion does not necessarily follow if some of the coefficients k_s are zero, as the following example shows.

Example 1 If (a_n) is bounded and

$$a_{n+4} \leq \frac{1}{2}(a_{n+2} + a_n),$$

then the sequences (a_{2n}) and (a_{2n+1}) are convergent, but (a_n) is not necessarily convergent. To show this, just consider the sequence

$$(a_n) = (1, -1, 1, -1, 1, -1, \ldots).$$

□

In addition, the coefficients k_s does not have to be all positive. In fact, as we will see later, if (a_n) is bounded and satisfies

$$a_{n+3} \leq -\frac{1}{2}a_{n+2} + \frac{3}{4}a_{n+1} + \frac{3}{4}a_n,$$

then it is convergent.

These remarks let us conclude that the condition of Copson's Theorem is sufficient but not necessary. This gave rise to numerous generalizations in the literature. In the present paper we are going to survey some of these generalizations (see Sects. 2 and 3) and we will finish extending the result to other mathematical objects, including a generalization of the monotone convergence theorem of Lebesgue (see Sect. 4).

2 Characterization of Convergence in Terms of the Characteristic Polynomial

As we have previously commented, Copson's proof suggests that the characteristic polynomial associated to inequality (1) is the key to develop new results in order to assure or characterize the convergence of real bounded sequences. Firstly, J. D. Kečkić justified that all the coefficients k_s of the inequality need not be positive [6]. In this sense, he presented a sufficient condition of convergence that depends on the roots of the characteristic polynomial. Under some conditions, if all the roots are distinct and lie in the open unit disk, then we have the convergence of the sequence. Namely, he proved the following:

Theorem 2 ([6]) *Let (a_n) be a bounded sequence of real numbers, which satisfies inequality (1), with $\sum_{s=1}^{r} k_s = 1$. If $l_s = 1 - k_1 - \cdots - k_s$ $(s = 1, 2, \ldots, r-1)$ and if all roots of the equation*

$$\lambda^{r-1} + l_1\lambda^{r-2} + \cdots + l_{r-1} = 0, \tag{2}$$

are distinct and lie in the disk $|\lambda| < 1$, then (a_n) is a convergent sequence.

Sometimes, it is not necessary to look for the roots of the equation, for instance by a direct application of Rouché's theorem (see [9]), which says that two holomorphic functions f and g have the same number of zeros in a region Ω (the unit disk in our case) if it is verified that $|f(z) - g(z)| < |f(z)|$ for all $z \in \gamma$, being γ a closed path in Ω; we can see that all the roots of equation (2) lie in the open unit disk whenever $|l_1| + |l_2| + \cdots + |l_{r-1}| < 1$. Indeed, it suffices to consider the polynomials $g(\lambda) = \lambda^{r-1} + \sum_{j=1}^{r-1} l_j \lambda^{r-1-j}$ and $f(\lambda) = \lambda^{r-1}$ in the disk $|z| \leq 1$.

Example 2 ([6]) If (a_n) is bounded and verifies

$$a_{n+5} \leq \frac{13}{12}a_{n+4} - \frac{1}{2}a_{n+3} + \frac{1}{3}a_{n+2} + \frac{1}{4}a_{n+1} - \frac{1}{6}a_n,$$

then it is convergent, since

$$\left|-\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6}\right| + \left|\frac{1}{3} + \frac{1}{4} - \frac{1}{6}\right| + \left|\frac{1}{4} - \frac{1}{6}\right| + \left|-\frac{1}{6}\right| = \frac{3}{4} < 1.$$

□

Afterwards, S. Stević characterized in [11] the convergence of real bounded sequences by giving a sufficient and necessary condition related to the characteristic polynomial too. In this case, the roots need not lie in the unit disk.

Theorem 3 ([11]) *Let k_s be real, $\sum_{s=1}^{r} k_s = 1$, $P_r(z) = z^r - k_1 z^{r-1} - \cdots - k_{r-1}z - k_r$ and let the real sequence (a_n) satisfy the inequality*

$$a_{n+r} \leq k_1 a_{n+r-1} + \cdots + k_r a_n.$$

Then the boundedness of (a_n) *implies its convergence if and only if zeros of polynomial* $P_r(z)$ *belong to the set* $\mathbb{C} \setminus \{z : |z| = 1, z \neq 1\}$.

The main point of the proof is to reduce the order of the corresponding difference inequality by a suitable substitution $b_n = a_{n+1} + \alpha a_n$.

We should remark that before Stević presented his result, D.C. Russell published an analogous result [10] that also characterized the convergence, but he appealed to summability theory giving a tougher proof. In that sense, Stević's result is given in a more natural form, since he expresses it in terms of the characteristic polynomial of the corresponding difference equation.

3 Sufficient Conditions of Monotonicity

In this section we will handle with different generalizations related to monotonicity criteria. In this way, we will present sufficient conditions of convergence by studying the monotonicity properties of the functions, not necessarily linear, that define the Copson-type inequalities verified by the bounded sequences.

Firstly, we will start defining two notions involved in Bibby's theorem [1].

Definition 1 A function $f : \mathbb{R}^r \to \mathbb{R}$ is said to be an averaging function if it is continuous, strictly increasing in each argument and satisfies

$$x = f(x, \ldots, x),$$

for all $x \in \mathbb{R}$.

Definition 2 A sequence (a_n) is said to be g-decreasing if there exists an averaging function f such that

$$a_n \leq f(a_{n-1}, a_{n-2}, \ldots, a_{n-r}), \tag{3}$$

for all $n > r$.

If inequality (3) is reversed, we say the sequence is g-increasing. A sequence is g-monotonic if it is either g-decreasing or g-increasing.

We can now state the following theorem due to J. Bibby [1].

Theorem 4 ([1]) *If a real sequence is bounded and g-monotonic then it is convergent.*

Example 3 The weighted arithmetic mean (with positive weights $\alpha_1, \ldots, \alpha_r$) is an example of an averaging function

$$f(x_1, \ldots, x_r) = \frac{\sum_{i=1}^r \alpha_i x_i}{\sum_{i=1}^r \alpha_i},$$

which is continuous, strictly increasing and $f(x, \ldots, x) = x$.

Then if (a_n) is a bounded sequence satisfying

$$a_n \leq \frac{\sum_{i=1}^r \alpha_i a_{n-i}}{\sum_{i=1}^r \alpha_i},$$

it is convergent.

Another averaging function is given by

$$f(x_1, \ldots, x_r) = \sqrt[r]{x_1 \cdot \ldots \cdot x_r},$$

with r an odd positive integer. Therefore, if the bounded sequence (a_n) holds

$$a_n \leq \sqrt[r]{a_{n-1} \cdot \ldots \cdot a_{n-r}},$$

we can ensure its convergence. □

Now we will change the convex combination that appears in Copson's inequality (1) to a continuous real-valued function. In this way we characterize the convergence of the considered sequence (a_n) establishing conditions of monotonicity in the function.

Theorem 5 ([12]) *Assume that f is a continuous real-valued function defined on $\mathbb{R}^r$ satisfying the following conditions:*

1. *f is nondecreasing in each of its arguments;*
2. *$f(x_1, \ldots, x_r)$ is strictly increasing in x_1;*
3. *for every $x \in \mathbb{R}$ it holds*
$$f(x, x, \ldots, x) \leq x.$$

Then every bounded solution of the difference inequality

$$x_n \leq f(x_{n-1}, \ldots, x_{n-r})$$

converges.

The condition of the existence of a variable such that $f(x_1, \ldots, x_r)$ is increasing in it, is necessary. Indeed, let $f(x_1, \ldots, x_r) = \max\{x_1, \ldots, x_r\}$. It is easy to check that it is a continuous real function, nondecreasing in each variable and it verifies $f(x, \ldots, x) \leq x$ for every $x \in \mathbb{R}$. Now, considering the sequence

$$(a_n) = (1, 2, \ldots, r, 1, 2, \ldots, r, 1, 2, \ldots, r, \ldots),$$

it satisfies the inequality

$$a_{n+r} \leq \max\{a_{n+r-1}, a_{n+r-2}, \ldots, a_n\},$$

and it is not convergent.

Now, let us consider an autonomous difference equation of order $r \in \mathbb{N}$

$$x_n = f(x_{n-1}, \ldots, x_{n-r}), \tag{4}$$

in order to generate sequences of real numbers by applying (4) from the initial values $x_{-r}, \ldots, x_{-1}$.

Theorem 6 ([5]) *Let $f(x_1, \ldots, x_r)$ be a continuous function from I^r into I, where I is bounded or unbounded interval of $\mathbb{R}$, which satisfies the following conditions:*

1.

$$f(x_1, x_2, \ldots, x_r) \geq f(x_2, \ldots, x_r, x_1),$$

if $x_1 \geq \max\{x_2, \ldots, x_r\}$;

2.

$$f(x_1, x_2, \ldots, x_r) \leq f(x_2, \ldots, x_r, x_1),$$

if $x_1 \leq \min\{x_2, \ldots, x_r\}$;

3. f is nondecreasing in the last variable x_r.

Then every bounded solution of (4) with initial values $x_{-k}, \ldots, x_{-1} \in I$ converges, and every unbounded solution of (4) tends either to $+\infty$ or to $-\infty$.

Example 4 ([5]) Let us consider the function f and suppose that it is a linear combination of any monotonous nondecreasing function g, not necessarily linear, with respect to variables z_i, $i = 1, 2, \ldots, r$, that is

$$f(z_1, \ldots, z_r) = a_1 g(z_1) + \cdots + a_r g(z_r), \; a_1 \geq a_2 \geq \cdots \geq a_r \geq 0.$$

The case when all constants a_i, $i = 1, 2, \ldots, r$, are the same, obviously satisfies all conditions. So, suppose that $a_1 > a_r$. Now, assume that $z_1 \geq z_i$ for all $i = 1, 2, \ldots, r$. It is well known that for any monotonous nondecreasing function g (not necessarily differentiable), inequality $g(z_1) \geq g(z_i)$ holds for all $i = 1, 2, \ldots, r$. After the multiplication of all these inequalities by $(a_{i-1} - a_i)/(a_1 - a_r)$, respectively, and the summation from $i = 2$ to $i = r$, for some $\eta \leq z_1$ we get (notice that $\sum_{i=2}^{r}((a_{i-1} - a_i)/(a_1 - a_r)) = 1$)

$$g(z_1) \geq g(\eta) = \frac{1}{a_1 - a_r} \left(\sum_{i=2}^{r} a_{i-1} g(z_i) - \sum_{i=2}^{r} a_i g(z_i) \right),$$

that is

$$\sum_{i=1}^{r} a_i g(z_i) \geq \sum_{i=2}^{r} a_{i-1} g(z_i) + a_r g(z_1),$$

which constitutes the first condition of the theorem. Similar algebraic manipulations lead us to verify the other two conditions.

Therefore, we can apply Theorem 6 to conclude that every bounded solution of

$$x_n = f(x_{n-1}, \ldots, x_{n-r}) = \sum_{j=1}^{r} a_j g(x_{n-j}),$$

is convergent. □

We will finish this section by considering inequalities involving two different sequences. It would be interesting to study the generalization to an arbitrary number of sequences and to look for possible applications.

Theorem 7 ([13]) *Let $f(x_1, x_2, \ldots, x_r)$ be a continuous real function on $\mathbb{R}^r$ which satisfies the following conditions:*

1. *f is nondecreasing in each variable and increasing in the first one;*
2. *$f(x, x, \ldots, x) \leq x$, for every $x \in \mathbb{R}$.*

If (a_n) is a sequence bounded from below and satisfies the inequality

$$a_{n+r} \leq f(a_{n+r-1}, a_{n+r-2}, \ldots, a_n) + b_n,$$

where (b_n) is a sequence of real numbers such that $\sum_{n=0}^{\infty} |b_n| < \infty$, then it converges.

The stated results have a lot of applications to other fields, since the difference inequality

$$a_{n+r} \leq f(a_{n+r-1}, a_{n+r-2}, \ldots, a_n),$$

includes the difference equation

$$a_{n+r} = f(a_{n+r-1}, a_{n+r-2}, \ldots, a_n),$$

which appears in a large class of mathematical biology models. As an example we can consider the following difference equations that describe population models:

$$x_{n+1} = \left(a x_n + b x_{n-1} e^{x_{n-1}}\right) e^{x_n},$$

$$y_{n+1} = (\alpha y_n + \beta y_{n-1})\, e^{-y_n}.$$

In particular, the first one describes the growth of a mosquito population. If the reader is interested in these or other population models, see [4].

4 Extensions of Copson's Theorem

Apart from the numerous generalizations that have been made from Copson's theorem, the result can be extended to other fields or mathematical objects.

Firstly, D. Borwein extended Theorem 1 to the complex plane [2]. Let us introduce some preliminary notation in order to understand the statement of Borwein's result.

Let (K_n) be a sequence of complex numbers, let

$$K(z) = \sum_{n=0}^{\infty} K_n z^n, \quad z \in \mathbb{C},$$

and let $k_0 = K_0,\ k_n = K_n - K_{n-1}$.

Let D be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, let $\bar{D}$ be its closure and let $\partial D = \bar{D} - D$ be its border.

Theorem 8 ([2]) *Under the above notation, if*

$$\sum_{n=0}^{\infty} |K_n| < \infty,$$

$$K(z) \neq 0 \text{ on } \partial D,$$

and if (a_n) is a bounded sequence such that, for some positive integer N,

$$\sum_{s=0}^{n} k_s a_{n-s} \geq 0 \quad (n = N, N+1, \ldots),$$

then (a_n) is convergent.

Example 5 Every bounded sequence (a_n) satisfying

$$a_n - \frac{i}{1+i} a_{n-1} - \frac{i}{(1+i)^2} a_{n-2} - \ldots - \frac{i}{(1+i)^n} a_0 \geq 0,$$

is convergent.

To prove it, consider $K_n = \left(\frac{1}{1+i}\right)^n$ verifying

$$\sum_{n=0}^{\infty} \left|\frac{1}{1+i}\right|^n = \sum_{n=0}^{\infty} \frac{1}{2^{\frac{n}{2}}} = \frac{1}{\sqrt{2}-1} < \infty,$$

and $K(z) \neq 0$ in ∂D since $K(z) = \sum_{n\geq 0} \left(\frac{1}{1+i}\right)^n z^n = \frac{1}{1-\frac{z}{1+i}}$ is the sum of a geometric progression well defined for $|z| < |1+i| = \sqrt{2}$. Finally, notice that $k_s = K_s - K_{s-1} = -i\frac{1}{(1+i)^s}$. □

Realize that if $k_0 = -1 = K_0$, then it is easily seen that $K_1 = -1 + k_1$, $K_2 = -1 + k_1 + k_2$, and, in general, $K_j = -1 + k_1 + k_2 + \ldots + k_j$ for $j \geq 2$. This shows us that, in essence, we can deduce Copson's Theorem if in Theorem 8 we replace the conditions $\sum_{n=0}^{\infty} |K_n| < \infty$, $K(z) \neq 0$ on ∂D, by the single condition

$$-1 = K_0 < K_1 < \ldots < K_{r-1} < K_r = K_{r+1} = K_{r+2} = \ldots = 0,$$

since in this case $K(z)$ reduces to the polynomial of degree $r-1$, $K(z) = -1 + (-1 + k_1)z + \ldots + (-1 + k_1 + \ldots + k_{r-1})z^{r-1}$ and, under the hypothesis of Theorem 1, we have that $K(1) < 0$ and $\mathrm{Re}\,((1-z)K(z)) = -\sum_{s=0}^{r} k_s(1 - \cos(r\theta)) < 0$ for $z = e^{i\theta}$, $0 < \theta < 2\pi$.

Secondly, G.G. Vrănceanu extended in [14] the result to other mathematical objects getting convergence criteria based on Copson-type inequalities. For example, we can extend it to double sequences.

Theorem 9 ([14]) *If (a_m^n) is a bounded double sequence which satisfies the inequality*

$$a_{m+r}^{n+l} \leq \sum_{s,p=1}^{r,l} k_{s,p} a_{m+r-s}^{n+l-p},$$

where the coefficients $k_{s,p}$ are strictly positive and $\sum k_{s,p} = 1$, then (a_m^n) is convergent.

Also, we can extend it to real continuous functions in a way that generalize Dini's theorem (see [8]) about monotonic sequences of continuous functions on a compact space.

Theorem 10 ([14]) *Let X be a compact space and (f_n) be a bounded sequence of real continuous functions such that:*

1. *$f_n \to f$ simple and f continuous.*
2. *There exists strictly positive constants k_s with $\sum_{s=1}^{r} k_s = 1$ and*

$$f_{n+r}(t) \leq \sum_{s=1}^{r} k_s f_{n+r-s}(t).$$

Then (f_n) converges uniformly to f.

It is easy to see that the properties established about double sequences and sequences of continuous functions can be proved if we reverse the inequality, following the same strategy we remarked with Copson's Theorem.

Moreover, he presented an extension to hermitian operators.

Theorem 11 ([14]) *If (T_n) is a bounded sequence of hermitian operators and for each n we have*

$$T_{n+r} = \sum_{s=1}^{r} k_s T_{n+r-s},$$

where k_s are strictly positive numbers and $\sum_{s=1}^{r} k_s = 1$, then (T_n) converges strongly to a hermitian operator.

Finally, we announce a proper result, whose proof is not published yet. By applying the ideas and techniques developed by R.A. Rankin in his proof of Copson's theorem, we are able to generalize the monotone convergence theorem of Lebesgue (for consulting the statement of this well-known result from Measure Theory, the reader is referred to [8], where he/she can also find the associate notions of measurable functions and Lebesgue integral).

Theorem 12 *Let (f_n) be a sequence of measurable and positive functions over X. Assuming*

$$f_{n+k}(x) \geq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x),$$

for all $x \in X$, where $\sum_{j=1}^{k} \alpha_j = 1,\ 0 < \alpha_j < 1,\ j = 1, \ldots, k$, and

$$f_n(x) \to f(x),$$

for all $x \in X$ when $n \to \infty$. Then f is measurable and $\int_X f_n d\mu \to \int_X f d\mu$ when $n \to \infty$.

We emphasize that our proof uses both the ideas of Copson's Theorem and elementary techniques from Measure Theory. We hope that we will publish it in a forthcoming publication [7].

Another future line of research deals with the analysis of the convergence of sequences satisfying a Copson-type inequality involving two or more sequences, in the line of Theorem 7. As a first step in this direction, we could think about new sufficient conditions on the auxiliary sequence (b_n) in that theorem.

We conclude our survey by emphasizing that the evolution of a simple and elegant result has been the origin of a wide range of results about its generalizations and extensions, enriching the literature in relation with the topic of the convergence of sequences. In our opinion, there are still some interesting questions related with it, in particular in the setting of difference equations, and applications to mathematical models will be welcome.

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Global Asymptotic Stability of a Non-linear Population Model of Diabetes Mellitus

Silvia Rodrigues de Oliveira, Soumyendu Raha and Debnath Pal

Abstract A preliminary mathematical model of diabetes has been proposed in [4], in which the evolution of the size of a population of diabetes mellitus patients and the number of patients with complications, has been modeled by second order system of nonlinear differential equations. The model, has already been analyzed for the linear local stability of the equilibria of the system. However, the global behavior of the flow of the nonlinear system has not been studied. The present article analyzes the global behavior of the trajectories of the population growth using Lyapunov stability analysis. Toward this, we construct a suitable Lyapunov function corresponding to an interior equilibrium point and show that it is asymptotically stable within the entire open first quadrant of the planar state space which is the region of interest. Further, transient or incremental stability in the phase plane has been studied via Lyapunov exponent analysis. The stability analysis has also been verified through numerical simulations, under various parameters. A physical interpretation of the parametric dependence of the flows of the nonlinear system is provided from the point of view of diabetic population dynamics.

Keywords Diabetes mellitus · Stability theory · Mathematical model

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1 Introduction

Diabetes is a group of diseases characterized by high levels of blood glucose resulting from defect in insulin production, insulin action or both. It is now commonly admitted that diabetes is an important public health problem, worrisome and serious worldwide. According to the World Health Organization [3], the diabetes is on the rise. No longer its a disease of predominantly rich nations; the prevalence and incidence of the disease are steadily increasing everywhere. Is it estimated that globally, 422 million adults were living with diabetes in 2014, compared to 108 million in 1980 [3].

Diabetes can lead to complications in many parts of the body, reduced quality of life and premature death. In addition to placing a large financial burden on individuals and their families due to the cost of insulin and other essential medicines, diabetes also has a substantial economic impact on countries and national health systems [1].

Based on the cost estimates from a recent systematic review, the World Health Organization has estimated the direct annual cost of diabetes to the world to be more than US$ 827 billion [3]. Thus, the growing trend of diabetes requires urgent measures and effective strategies for the prevention and management of diabetes and its complications.

Currently, there is a growing interest in the study and development of population level models and in the behavior of non-communicable disease. The combination of theoretical methods with mathematical methods has played an essential role in the development of this area. Specifically, the role and sensitivity of various parameters of these models to the dynamics of diabetic population growth is significant since estimation of the parameters couple the multiscale models.

Abundant mathematical models have been developed to understand diabetes and it is mainly devoted to simulate and analyze the dynamics of glucose and insulin. The literature shows different models using differential equations, delayed differential equations, integro-differential equations, stochastic differential equations, optimal control and others methods for glycaemic control, blood glucose monitoring and devices devoted to diabetes prevention [2]. In this way, few authors have proposed epidemiological models for diabetes for understanding the populations dynamics.

A mathematical model of diabetes by Boutayeb et al., for the evolution of the size of a population of Diabetes Mellitus patients and the number of patients with complications, has been studied for stability with respect to various parameters.The model, a second order system of nonlinear differential equations, has already been analysed for the linear local stabilityof the equilibria of the system. However, the global behavior of the flow of the nonlinear system has not been studied as it requires further phase plane analysis, which in turn gives parametric insight into the model.

The aim of this paper is to extend the analysis of the stability of the system proposed by [4]. Thus, we study the global behavior of the trajectories of the population growth using Lyapunov stability analysis. Toward this, we construct a suitable Lyapunov function corresponding to an interior equilibrium point and show that it is asymptotically stable within the entire open first quadrant of the planar state space which is the region of interest.

2 The Mathematical Model

A. Boutayed [4] proposed the following two-dimensional population model of Diabetes Mellitus,

$$\frac{dD}{dt} = I - (\lambda + \mu)D(t) + \lambda C(t) \tag{1}$$

$$\frac{dC}{dt} = \lambda D(t) - (\gamma + \mu + v + \delta)C(t) \tag{2}$$

It will now be assumed that the probability of developing a complication λ is given by:

$$\lambda = \lambda(t) = \frac{\beta C(t)}{N(t)}$$

which $N(t) = C(t) + D(t)$ give rise to the initial-value problem (IVP)

$$\frac{dC}{dt} = -(\lambda + \theta)C(t) + \lambda N(t), t > 0; C(0) = C_0 \tag{3}$$

$$\frac{dN}{dt} = I - (v + \gamma)C(t) - \mu N(t), t > 0; N(0) = N_0 \tag{4}$$

where $\theta = \gamma + \mu + v + \delta$.

$$\frac{dC}{dt} = (\beta - \theta)C(t) - \beta\frac{C^2(t)}{N(t)}, t > 0; C(0) = C_0 \tag{5}$$

$$\frac{dN}{dt} = I - (v + \delta)C(t) - \mu N(t), t > 0; N(0) = N_0 \tag{6}$$

The possible steady states of the dynamical system equation (5) and (6) are: $S_0 = (0, \frac{I}{\mu})$, $S_1 = (C^*, N^*)$ where

$$(\beta - \theta)N^* - \beta C^* = 0 \tag{7}$$

$$(v + \delta)C^* + \mu N^* = I \tag{8}$$

The Eqs. (7) and (8) may be written in matrix-vector form as:

$$\begin{bmatrix} -\beta & (\beta-\theta) \\ (v+\delta) & \mu \end{bmatrix} \begin{bmatrix} C^* \\ N^* \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$-\beta\mu - (v+\delta)(\beta-\theta) \neq 0 \text{ uniquesolution}$$

$$\begin{pmatrix} C^* \\ N^* \end{pmatrix} = \frac{-1}{\mu\beta + (v+\delta)(\beta-\theta)} \begin{pmatrix} \mu & -(\beta-\theta) \\ -(v+\delta) & -\beta \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix}$$

hence, $C^* = \frac{(\beta-\theta)I}{\mu\beta+(v+\delta)(\beta-\theta)}$ and $N^* = \frac{-\beta I}{\mu\beta+(v+\delta)(\beta-\theta)}$
(the non-trivial fixed point)
The Jacobian matrix of model system 5 and 6 at the trivial critical point, is given by:

$$J(s_0) = \begin{pmatrix} \beta-\theta & 0 \\ -(v+\delta) & -\mu \end{pmatrix} \tag{9}$$

The eigenvalues of $J(s_0)$ are the roots χ_1 and χ_2 of the characteristic equation:

$$\chi^2 + (\mu+\theta-\beta)\chi + (\theta-\beta)\mu = 0,$$

so that

$$\chi_1 = (\beta-\theta) > 0 \text{ and } \chi_2 = -\mu < 0$$

thus, the trivial critical point is saddle point.
Similarly, the Jacobian matrix associated with the non-trivial critical point is given by:

$$J(s_1) = \begin{pmatrix} -(\beta-\theta) & \frac{(\beta-\theta)^2}{\beta} \\ -(v+\delta) & -\mu \end{pmatrix} \tag{10}$$

The eigenvalues of $J(s_1)$ are the roots χ_1 and χ_2 of the characteristic equation:

$$\chi^2 + (\beta-\theta+\mu)\chi + (\beta-\theta)\mu + \frac{(v+\delta)(\beta-\theta)^2}{\beta} = 0,$$

which is of the form:

$$\chi^2 + b\chi + c = 0; b > 0, c > 0.$$

Hence, χ_1 and χ_2 may be:

- Case 1: both real and negative, so that the non-trivial critical point given is a stable node.

- Case 2: complex conjugates with negative real parts, so that the non-trivial critical point is a stable spiral and the solution of the ODE system in (3), (4) spirals into the non-trivial critical point.

3 Global Stability

We consider the global stability of the system of equation (5) and (6) by constructing a suitable Lyapunov function:

$$V(C,N) = \left[\left(C - C^*\right) - C^* ln\left(\frac{C}{C^*}\right)\right] + W\left[\left(N - N^*\right) - N^* ln\left(\frac{N}{N^*}\right)\right] \quad (11)$$

where w is a suitable constant to be determined in the subsequent steps. It can be easily verified that V is zero at the equilibrium point (C^*, N^*) and positive for all other positive value of C, N. The time derivative of V along the trajectories of equation (5) and (6),

$$\frac{dV}{dt} = \frac{(C - C^*)}{C}\frac{dC}{dt} + W\left(1 - \frac{N^*}{N}\right)\frac{dN}{dt} \quad (12)$$

$$= \left(C - C^*\right)\left[(\beta - \theta) - \beta\frac{C}{N}\right] + W\left(1 - \frac{N^*}{N}\right)\left[I - (v + \delta)\,C - \mu N\right]$$

Also we have the set of equilibrium equations:

$$(\beta - \theta)\,N^* - \beta C^* = 0 \text{ and}$$
$$(v + \delta)\,C^* + \mu N^* = I$$

corresponding the steady state $S_1 = (C^*, N^*)$
We can write the Eq. (12) together with the above two equations in the form:

$$= (C - C^*)\left[(\beta - \theta) - \frac{\beta}{N}(C - C^*) - \frac{\beta}{N}C^*\right] +$$
$$W\frac{(N - N^*)}{N}[I - (v + \delta)(C - C^*) - \mu(N - N^*) - (v + \delta)C^* - \mu N^*]$$
$$= (C - C^*)\left[\frac{(\beta - \theta)}{N}(N - N^*) - \frac{\beta}{N}(C - C^*)\right] +$$
$$W\frac{(N - N^*)}{N}[-(v + \delta)(C - C^*) - \mu(N - N^*)]$$
$$= (C - C^*)^2\left(\frac{-\beta}{N}\right) + \frac{(\beta - \theta)}{N}(C - C^*)(N - N^*) +$$

$$W(C-C^*)(N-N^*)\left(\frac{-(v+\delta)}{N}\right)-W\frac{\mu}{N}(N-N^*)^2$$
$$=-\Big[(C-C^*)^2\left(\frac{\beta}{N}\right)+(C-C^*)(N-N^*)\left[W\frac{(v+\delta)}{N}-\frac{(\beta-\theta)}{N}\right]+W\frac{\mu}{N}(N-N^*)^2\Big]$$
$$=-\left[(C-C^*)^2\left(\frac{\beta}{N}\right)+W\frac{\mu}{N}(N-N^*)^2\right]$$

We choose W = $\frac{(\beta-\theta)}{(v-\delta)}$ here.
Now since dV/dt is negative semidefinite in some neighbourhood of (C^*, N^*), the interior equilibrium point (C^*, N^*) is globally asymptotically stable.

4 Numerical Results

In this section, we consider some numerical examples (Table 1). Parameters values are taken as follows:

Table 1 Parameters values used in numerical experiments

Parameters	Values
I	60 000
β	1
v	0.05
δ	0.05
μ	0.02
γ	0.08
θ	0.2

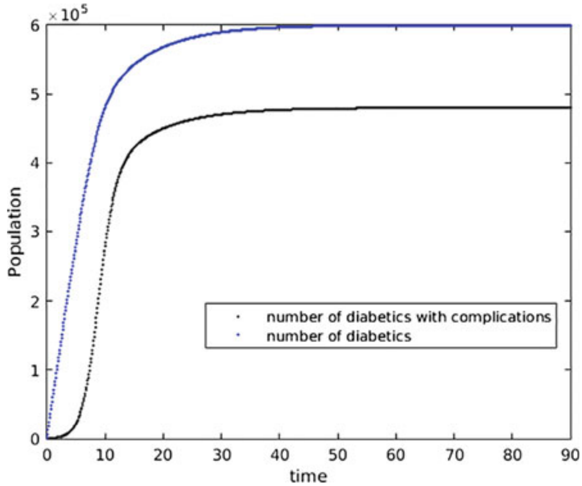

Fig. 1 Profiles of $C(t)$ and $N(t)$

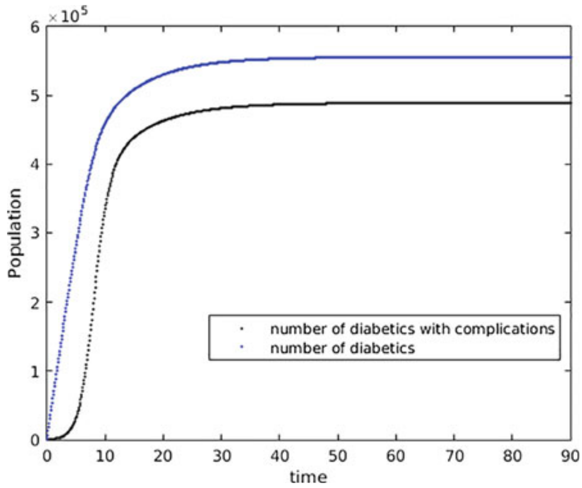

Fig. 2 Profiles of $C(t)$ and $N(t)$

Figure 1 shows the behavior of C and N and Fig. 2 shows also the behavior of the populations using the previous values of parameters and the rate of recovery from complications $\gamma = 0$.

The behavior of system is exhibited similar of behaviour as with $\gamma = 0.08$.

5 Summary

A mathematical model is considered to investigate the number of diabetics and those with complications. By utilizing a Lyapunov function, global stability of one equilibria is studied.The stability analysis has also been also verified through numerical simulations.

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On a Nonlocal Boundary Value Problem for First Order Nonlinear Functional Differential Equations

Zdeněk Opluštil

Abstract A nonlocal boundary value problem for nonlinear functional equations is studied. New effective conditions are found for solvability a unique solvability of considered problem. Obtained results are concretized for differential equation with deviating argument.

Keywords Functional differential equation · Solvability · Unique solvability
Equations with deviating arguments

1 Introduction

On the interval $[a, b]$, we consider the functional differential equation

$$u'(t) = F(u)(t), \tag{1}$$

where $F : C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ is a continuous (in general) nonlinear operator. As usual, by a solution of this equation we understand an absolutely continuous function $u : [a, b] \to \mathbb{R}$ satisfying the equality (1) almost everywhere on $[a, b]$. Along with the Eq. (1), we consider the nonlocal boundary condition

$$h(u) = \varphi(u), \tag{2}$$

where $\varphi : C([a, b]; \mathbb{R}) \to \mathbb{R}$ is a continuous (in general) nonlinear functional and $h : C([a, b]; \mathbb{R}) \to \mathbb{R}$ is a (non-zero) linear bounded functional.

The following notation is used in the sequel.

$\mathbb{R}$ is the set of all real numbers. $\mathbb{R}_+ = [0, +\infty[$.

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$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v:\ [a,b]\to\mathbb{R}$ with the norm $\|v\|_C=\max\{|v(t)|:\ t\in[a,b]\}$.
$C([a,b];\mathbb{R}_+)=\{v\in C([a,b];\mathbb{R}):v(t)\ge 0 \text{ for } t\in[a,b]\}$.
$AC([a,b];\mathbb{R})$ is the set of absolutely continuous functions $v:\ [a,b]\to\mathbb{R}$.
$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p:\ [a,b]\to\mathbb{R}$ with the norm $\|p\|_L=\int_a^b|p(s)|\,\mathrm{d}s$.
$L([a,b];\mathbb{R}_+)=\big\{p\in L([a,b];\mathbb{R}):p(t)\ge 0 \text{ for almost all } t\in[a,b]\big\}$.
$\mathscr{L}_{ab}$ is the set of linear operators $\ell:\ C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ for which there is a function $\eta\in L([a,b];\mathbb{R}_+)$ such that

$$|\ell(v)(t)|\le\eta(t)\|v\|_C\quad\text{for a.e. } t\in[a,b] \text{ and all } v\in C([a,b];\mathbb{R}).$$

P_{ab} is the set of so-called *positive* operators $\ell\in\mathscr{L}_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.
F_{ab} is the set of linear bounded functionals $h:\ C([a,b];\mathbb{R})\to\mathbb{R}$.
PF_{ab} is the set of so-called *positive* functionals $h\in F_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.
$\mathscr{B}_{hc}=\{u\in C([a,b];\mathbb{R}):\ h(u)\operatorname{sgn}u(a)\le c\}$, where $h\in F_{ab}$, $c\in\mathbb{R}$.
$K([a,b]\times A;B)$, where $A,B\subseteq\mathbb{R}$, is the set of function $f:[a,b]\times A\to B$ satisfying the Carathéodory conditions, i.e., $f(\cdot,x):[a,b]\to B$ is a measurable function for all $x\in A$, $f(t,\cdot):A\to B$ is a continuous function for almost every $t\in[a,b]$, and for every $r>0$ there exists $q_r\in L([a,b];\mathbb{R}_+)$ such that

$$|f(t,x)|\le q_r(t)\quad\text{for a.e. } t\in[a,b] \text{ and all } x\in A,\ |x|\le r.$$

As it is usual, we suppose following assumptions on a nonlinear operator F and a functional φ throughout the paper:

(H_1) $F:C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ is a continuous operator such that the relation

$$\sup\big\{|F(v)(\cdot)|:v\in C([a,b];\mathbb{R}),\ \|v\|_C\le r\big\}\in L([a,b];\mathbb{R}_+)$$

is satisfied for every $r>0$.

(H_2) $\varphi:C([a,b];\mathbb{R})\to\mathbb{R}$ is a continuous functional such that the condition

$$\sup\big\{|\varphi(v)|:v\in C([a,b];\mathbb{R}),\ \|v\|_C\le r\big\}<+\infty$$

holds for every $r>0$.

The question on the solvability and unique solvability of various types of boundary value problems for functional differential equations and their systems is a classical topic in the theory of differential equations (see, e.g., [1–6, 8] and references therein). There is a lot of interesting general results but only a few efficient conditions is known, namely, in the case where the boundary condition considered is nonlocal.

One can find (e.g. see [3]) that it is very useful to consider the boundary condition (2) as a nonlocal perturbation of the two-point condition

$$u(a) + \lambda u(b) = \varphi(u), \tag{3}$$

where $\lambda \in \mathbb{R}_+$. Consequently, it is natural to consider, in what follows, that the linear functional h in (2) is defined by the formula

$$h(v) = v(a) + \lambda v(b) - h_0(v) + h_1(v) \quad \text{for } v \in C([a, b]; \mathbb{R}),$$

where $\lambda \in \mathbb{R}_+$ and $h_0, h_1 \in PF_{ab}$. We should mention that there is no loss of generality to assume h like this, because an arbitrary linear functional h can be represented in this form.

One can see that a particular case of the Eq. (1) is, for example, a differential equation with deviating arguments

$$u'(t) = p(t)u(\tau(t)) - g(t)u(\sigma(t)) + f(t, u(t), u(\mu(t))), \tag{4}$$

where $p, g \in L([a, b]; \mathbb{R}_+)$, $\tau, \sigma, \mu : [a, b] \to [a, b]$ are measurable functions, and $f \in K([a, b] \times \mathbb{R}^2; \mathbb{R})$.

On the other hand, particular case of boundary condition (2) are, for example, Cauchy problem, anti-periodic problem (if $\varphi \equiv c$ and h_0, h_1 are trivial functionals), and some integral condition of the form $\int_a^b u(s)ds = c$.

In this paper, we extended results presented in [7] to nonlinear case, as well as, some results stated in [3] concerning the problem (1), (3). New efficient conditions are found for the solvability and unique solvability of the problem (1), (2). Moreover, below presented statements are concretized for the differential equation with deviating argument (4).

2 Main Results

Firstly we formulate statements, which guarantee solvability of considered boundary value problem.

Theorem 1 *Let $c \in \mathbb{R}_+$, $\lambda > 0$ and $h_0 \in PF_{ab}$ is such that*

$$h_0(1) < 1. \tag{5}$$

Let, moreover, the condition

$$\varphi(v)\operatorname{sgn} v(a) \le c \quad \textit{for } v \in C([a, b]; \mathbb{R}) \tag{6}$$

be fulfilled and there exist

$$\ell_0, \ell_1 \in P_{ab} \tag{7}$$

such that, on the set $\mathscr{B}_{hc}([a, b]; \mathbb{R})$*, the inequality*

$$\big(F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big)\operatorname{sgn} v(t) \le q(t, \|v\|_C) \quad \text{for a.e. } t \in [a, b] \tag{8}$$

holds, where the function $q \in K([a, b] \times \mathbb{R}_+; \mathbb{R}_+)$ *satisfies*

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s, x)\,\mathrm{d}s = 0. \tag{9}$$

If, in addition,

$$1 - h_0(1) - (\lambda + h_1(1))^2 \le \|\ell_0(1)\|_L < 1 - h_0(1), \tag{10}$$

$$\|\ell_0(1)\|_L + (\lambda + h_1(1))\,\|\ell_1(1)\|_L < 1 - h_0(1), \tag{11}$$

then the problem (1), (2) has at least one solution.

The next theorem can be regarded as complement of previous one in the sense, we consider the condition $\|\ell_0(1)\|_L < 1 - h_0(1) - (\lambda + h_1(1))^2$ holds instead of (10).

Theorem 2 *Let* $c \in \mathbb{R}_+$, $\lambda > 0$ *and* $h_0, h_1 \in PF_{ab}$ *be such that*

$$(\lambda + h_1(1))^2 < 1 - h_0(1) \tag{12}$$

holds. Let, moreover, the condition (6) be fulfilled and there exist $\ell_0, \ell_1 \in P_{ab}$ *such that, on the set* $\mathscr{B}_{hc}([a, b]; \mathbb{R})$*, the inequality (8) hold, where the function* $q \in K([a, b] \times \mathbb{R}_+; \mathbb{R}_+)$ *satisfies (9). If, in addition,*

$$\|\ell_0(1)\|_L < 1 - h_0(1) - (\lambda + h_1(1))^2, \tag{13}$$

$$\|\ell_1(1)\|_L < 2\sqrt{1 - h_0(1) - \|\ell_0(1)\|_L} - \lambda - h_1(1), \tag{14}$$

then the problem (1), (2) has at least one solution.

The following assertion immediately follows from previous theorems.

Corollary 1 *Let* $c \in \mathbb{R}_+$, $\lambda > 0$ *and* $h_0, h_1 \in PF_{ab}$ *be such that (12) holds. Let, moreover, (6) and*

$$f(t, x, y)\operatorname{sgn} x \le q(t) \quad \text{for a.e. } t \in [a, b] \text{ and all } x, y \in \mathbb{R}$$

be satisfied, where $q \in L([a, b]; \mathbb{R}_+)$*. If, in addition,*

$$\int_a^b p(s)\,\mathrm{d}s < 1 - h_0(1), \quad \int_a^b g(s)\,\mathrm{d}s < \omega(\lambda, h_0(1), h_1(1)), \tag{15}$$

where

$$\omega(\lambda, h_0(1), h_1(1)) = \begin{cases} \frac{1-h_0(1)-\int_a^b p(s)\,\mathrm{d}s}{\lambda+h_1(1)} & \text{if } \int_a^b p(s)\,\mathrm{d}s \geq 1-h_0(1)-(\lambda+h_1(1))^2, \\ 2\sqrt{1-h_0(1)-\int_a^b p(s)\,\mathrm{d}s}-\lambda-h_1(1) & \text{if } \int_a^b p(s)\,\mathrm{d}s < 1-h_0(1)-(\lambda+h_1(1))^2, \end{cases}$$

then problem (4), (2) has at least one solution.

Now we formulate statements concerning the unique solvability of the considered problems.

Theorem 3 *Let* $\lambda > 0$ *and the condition*

$$\big(\varphi(v)-\varphi(w)\big)\operatorname{sgn}\big(v(a)-w(a)\big) \leq 0 \tag{16}$$

hold for every $v, w \in C([a,b];\mathbb{R})$ *and there exist* $\ell_0, \ell_1 \in P_{ab}$ *such that, on the set* $\mathscr{B}_{hc}([a,b];\mathbb{R})$ *with* $c = |\varphi(0)|$*, the inequality*

$$\big(F(v)(t)-F(w)(t)-\ell_0(v-w)(t)+\ell_1(v-w)(t)\big)\operatorname{sgn}\big(v(t)-w(t)\big) \leq 0 \tag{17}$$

is fulfilled for a.e. $t \in [a,b]$*. If, in addition, either conditions (5), (10), (11) or conditions (12)–(14) are satisfied, then the problem (1), (2) is uniquely solvable.*

Finally we established assertion for the unique solvability of problem (4), (2), which immediately follows from the previous theorem.

Corollary 2 *Let* $\lambda > 0$ *and* $h_0, h_1 \in PF_{ab}$ *satisfy the relation (12). Let, moreover, conditions (15) and*

$$\big[f(t,x_1,x_2)-f(t,y_1,y_2)\big]\operatorname{sgn}(x_1-x_2) \leq 0 \text{ for a.e. } t \in [a,b], \text{ and all } x_1,x_2,y_1,y_2 \in \mathbb{R}$$

hold. If, in addition, condition (16) is fulfilled for every $v, w \in C([a,b];\mathbb{R})$*, then the problem (4), (2) is uniquely solvable.*

3 Auxiliary Propositions

The main results are proved using the lemma on a priory estimate stated in [6] by Kiguradze and Půža. This lemma can be formulated as follows.

Lemma 1 ([6, Corollary 2]) *Let there exist a positive number* ρ *and an operator* $\ell \in \mathscr{L}_{ab}$ *such that the homogeneous problem*

$$u'(t) = \ell(u)(t), \qquad h(u) = 0 \tag{18}$$

has only the trivial solution, and, for every $\delta \in]0,1[$*, an arbitrary function* $u \in AC([a,b];\mathbb{R})$ *satisfying the relations*

$$u'(t) = \ell(u)(t) + \delta[F(u)(t) - \ell(u)(t)] \quad \text{for a.e. } t \in [a,b], \quad h(u) = \delta\varphi(u) \tag{19}$$

admits the estimate

$$\|u\|_C \le \rho. \tag{20}$$

Then the problem (1), (2) has at least one solution.

Definition 1 Let $h \in F_{ab}$. We say that an operator $\ell \in \mathscr{L}_{ab}$ belongs to the set $\mathscr{U}(h)$, if there exists $r > 0$ such that for arbitrary $q^* \in L([a,b];\mathbb{R}_+)$ and $c \in \mathbb{R}_+$, every function $u \in AC([a,b];\mathbb{R})$ satisfying the inequalities

$$h(u)\operatorname{sgn} u(a) \le c, \tag{21}$$

$$\big(u'(t) - \ell(u)(t)\big)\operatorname{sgn} u(t) \le q^*(t) \quad \text{for a.e. } t \in [a,b] \tag{22}$$

admits the estimate

$$\|u\|_C \le r(c + \|q^*\|_L). \tag{23}$$

Lemma 2 *Let* $c \in \mathbb{R}_+$ *and (6) hold. Let, moreover, there exist* $\ell \in \mathscr{U}(h)$ *such that, on the set* $\mathscr{B}_{hc}([a,b];\mathbb{R})$*, the inequality*

$$\big(F(v)(t) - \ell(v)(t)\big)\operatorname{sgn} v(t) \le q(t, \|v\|_C) \quad \text{for a.e. } t \in [a,b] \tag{24}$$

is fulfilled, where the function $q \in K([a,b] \times \mathbb{R}_+;\mathbb{R}_+)$ *satisfies (9). Then the problem (1), (2) has at least one solution.*

Proof Firstly, we note that, due to the condition $\ell \in \mathscr{U}(h)$, the homogeneous problem (18) has only the trivial solution.

Let r be the number appearing in Definition 1. According to (9), there exists $\rho > 2rc$ such that

$$\frac{1}{x}\int_a^b q(s,x)\,ds < \frac{1}{2r} \quad \text{for } x > \rho.$$

Assume that a function $u \in AC([a,b];\mathbb{R})$ satisfies (19) for some $\delta \in]0,1[$. Then, according to (6), u satisfies inequality (21), i.e., $u \in \mathscr{B}_{hc}([a,b];\mathbb{R})$. By (24), we obtain that inequality (22) is fulfilled with $q^* \equiv q(\cdot\,, \|u\|_C)$. Hence, by virtue of the condition $\ell \in \mathscr{U}(\lambda)$ and the definition of the number ρ, we get the estimate (20).

Since ρ depends neither on u nor on δ, it follows from Lemma 1 that the problem (1), (2) has at least one solution.

4 Proofs of Main Results

Proof of Theorem 1 Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$ are such that the conditions (10) and (11) hold. We show that ℓ belongs to the set $\mathscr{U}(h)$.

Let $c \in \mathbb{R}_+$, $q^* \in L([a,b];\mathbb{R}_+)$, and $u \in AC([a,b];\mathbb{R})$ satisfy (21) and (22). We prove that the estimate (23) hold, where r depends only on $\|\ell_0(1)\|_L$, $\|\ell_1(1)\|_L$, λ, $h_0(1)$, and $h_1(1)$.

It is clear that

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \widetilde{q}(t) \quad \text{for a.e. } t \in [a,b], \tag{25}$$

where

$$\widetilde{q}(t) = u'(t) - \ell(u)(t) \quad \text{for a.e. } t \in [a,b].$$

From (21) and (22), we get

$$\big(u(a) + \lambda u(b) - h_0(u) + h_1(u)\big)\mathrm{sgn}u(a) \le c \tag{26}$$

and

$$\widetilde{q}(t)\mathrm{sgn}u(t) \le q^*(t) \quad \text{for a.e. } t \in [a,b]. \tag{27}$$

First suppose that the function u does not change its sign. We set

$$\bar{M} = \max\{|u(t)| : t \in [a,b]\}, \qquad \bar{m} = \min\{|u(t)| : t \in [a,b]\} \tag{28}$$

and choose $t_{\bar{M}}, t_{\bar{m}} \in [a,b]$ such that $t_{\bar{M}} \ne t_{\bar{m}}$ and

$$|u(t_{\bar{M}})| = \bar{M}, \qquad |u(t_{\bar{m}})| = \bar{m}. \tag{29}$$

It is clear that $\bar{M} \ge 0$, $\bar{m} \ge 0$, and either

$$t_{\bar{M}} < t_{\bar{m}} \tag{30}$$

or

$$t_{\bar{M}} > t_{\bar{m}}. \tag{31}$$

Moreover, according to (7), (27), and (28), from (25) we obtain

$$|u(t)|' \le \bar{M}\,\ell_0(1)(t) + q^*(t) \quad \text{for a.e. } t \in [a,b]. \tag{32}$$

If $u(a) = 0$ then $\bar{m} = 0$ and integrating inequality (32) from a to $t_{\bar{M}}$ and taking into account (29), one gets

$$\bar{M} \leq \bar{M} \int_a^{t_{\bar{M}}} \ell_0(1)(s)\,\mathrm{d}s + \int_a^{t_{\bar{M}}} q^*(s)\,\mathrm{d}s.$$

It follows from the last inequality and (7) that

$$\bar{M} \leq \bar{M}\|\ell_0(1)\|_L + \|q^*\|_L + c.$$

Hence, in view of (10), we have

$$||u||_C \leq \left(\|q^*\|_L + c\right)(1 - \|\ell_0(1)\|_L)^{-1}.$$

Consequently, the estimate (23) holds with $r = (1 - \|\ell_0(1)\|_L)^{-1}$.

If $u(a) \neq 0$ then, according to (26), we obtain

$$|u(a)| + \lambda|u(b)| \leq h_0(|u|) - h_1(|u|) + c. \tag{33}$$

Let first (30) hold. Then integrating of inequality (32) from a to $t_{\bar{M}}$ one gets

$$\bar{M} - |u(a)| \leq \bar{M} \int_a^{t_{\bar{M}}} \ell_0(1)(s)\,\mathrm{d}s + \int_a^{t_{\bar{M}}} q^*(s)\,\mathrm{d}s$$

Latter two inequalities give

$$\bar{M} + \lambda|u(b)| - h_0(|u|) + h_1(|u|) - c \leq \bar{M} \int_a^{t_{\bar{M}}} \ell_0(1)(s)\,\mathrm{d}s + \int_a^{t_{\bar{M}}} q^*(s)\,\mathrm{d}s.$$

Therefore, in view of (7), (29), (33) and $h_0,\ h_1 \in PF_{ab}$, we obtain

$$\bar{M} - \bar{M}h_0(1) - c \leq \bar{M}\|\ell_0(1)\|_L + ||q^*||_L. \tag{34}$$

Now suppose that (31) is fulfilled. Then the integration of (32) from $t_{\bar{m}}$ to $t_{\bar{M}}$, on account of (7) and (28), yields

$$\bar{M} - \bar{m} \leq \bar{M}\|\ell_0(1)\|_L + \|q^*\|_L.$$

On the other hand, by virtue of (28) and $h_0,\ h_1 \in PF_{ab}$, inequality (33) implies

$$\bar{m} + \lambda\bar{m} - \bar{M}h_0(1) + \bar{m}h_1(1) \leq c.$$

Summing the last inequalities, we get

$$\bar{M} + \lambda\bar{m} - \bar{M}h_0(1) + mh_1(1) \leq \bar{M}\|\ell_0(1)\|_L + \|q^*\|_L + c.$$

Hence, the inequality (34) holds in both cases (30) and (31). Consequently, in view of the assumption (10) we obtain

$$\bar{M} \leq (1 - h_0(1) - \|\ell_0(1)\|_L)^{-1}(\|q^*\|_L + c).$$

Thus, estimate (23) hold, where $r = \big(1 - h_0(1) - \|\ell_0(1)\|_L\big)^{-1}$.

Let now u change its sign. We choose $t_m, t_M \in [a, b]$ such that

$$u(t_m) = \min\{u(t) : t \in [a, b]\}, \quad u(t_M) = \max\{u(t) : t \in [a, b]\} \tag{35}$$

and we denote

$$-m = u(t_m), \quad M = u(t_M). \tag{36}$$

It is clear that $m > 0$ and $M > 0$ and either

$$t_m < t_M \tag{37}$$

or

$$t_m > t_M. \tag{38}$$

Suppose that (37) holds firstly. Then there exists $a_2 \in]t_m, t_M[$ such that

$$u(a_2) = 0, \quad u(t) > 0 \quad \text{for} \quad a_2 < t \leq t_M. \tag{39}$$

On the other hand, we put

$$a_1 = \inf\{t \in [a, t_m] : u(s) < 0 \text{ for } t \leq s \leq t_m\}.$$

Obviously,

$$u(t) < 0 \quad \text{for} \quad a_1 < t \leq t_m \quad \text{and if} \quad a_1 > a \quad \text{then} \quad u(a_1) = 0. \tag{40}$$

Hence, by virtue of conditions $\lambda > 0$ and (26), we have

$$u(a_1) \geq -\lambda M - mh_0(1) - Mh_1(1) - c. \tag{41}$$

Integration equality (25) from a_1 to t_m and from a_2 to t_M and taking into account (7), (27), (35), (36), (39)–(41), one gets

$$m - \lambda M - m h_0(1) - M h_1(1) - c \le M \int_{a_1}^{t_m} \ell_1(1)(s)\,\mathrm{d}s + m \int_{a_1}^{t_m} \ell_0(1)(s)\,\mathrm{d}s + \int_{a_1}^{t_m} q^*(s)\,\mathrm{d}s$$

$$M \le M \int_{a_2}^{t_M} \ell_0(1)(s)\,\mathrm{d}s + m \int_{a_2}^{t_M} \ell_1(1)(s)\,\mathrm{d}s + \int_{a_2}^{t_M} q^*(s)\,\mathrm{d}s.$$

From the last two inequalities we obtain

$$\begin{aligned} m\,(1 - C - h_0(1)) &\le M\,(\lambda + h_1(1) + A) + ||q^*||_L, \\ M\,(1 - D) &\le mB + ||q^*||_L, \end{aligned} \tag{42}$$

where

$$\begin{aligned} A &= \int_{a_1}^{t_m} \ell_1(1)(s)\,\mathrm{d}s, \quad B = \int_{a_2}^{t_M} \ell_1(1)(s)\,\mathrm{d}s, \\ C &= \int_{a_1}^{t_m} \ell_0(1)(s)\,\mathrm{d}s, \quad D = \int_{a_2}^{t_M} \ell_0(1)(s)\,\mathrm{d}s. \end{aligned} \tag{43}$$

It follows from (10) that $1 > C + h_0(1)$ and $1 > D$. Consequently, inequalities (42) yield

$$\begin{aligned} 0 &< m\,(1 - C - h_0(1))\,(1 - D) \le \\ &\qquad mB\,(\lambda + h_1(1) + A) + \big(||q^*||_L + c\big)\,(1 + \lambda + h_1(1) + \|\ell_1(1)\|_L)\,, \\ 0 &< M\,(1 - C - h_0(1))\,(1 - D) \le \\ &\qquad MB\,(\lambda + h_1(1) + A) + \big(||q^*||_L + c\big)\,(1 + \|\ell_1(1)\|_L)\,. \end{aligned} \tag{44}$$

It is clear that

$$(1 - C - h_0(1))\,(1 - D) \ge 1 - (C + D) - h_0(1) \ge 1 - \|\ell_0(1)\|_L - h_0(1). \tag{45}$$

On the other hand, we get from the first inequality in (10) and (11) that

$$B \le \|\ell_1(1)\|_L < \lambda + h_1(1)$$

and therefore

$$(A + \lambda + h_1(1))\,B = AB + (\lambda + h_1(1))\,B \le (\lambda + h_1(1))\,\|\ell_1(1)\|_L. \tag{46}$$

By using (11), (45), and the last inequality, we obtain from (44) that

$$\begin{aligned} m \le r_0 \left(1 + \lambda + h_1(1) + \|\ell_1(1)\|_L\right) \left(c + ||q^*||_L\right) \\ M \le r_0 \left(1 + \|\ell_1(1)\|_L\right) \left(c + ||q^*||_L\right), \end{aligned} \tag{47}$$

where

$$r_0 = \left(1 - h_0(1) - \|\ell_0(1)\|_L - (\lambda + h_1(1)) \|\ell_1(1)\|_L\right)^{-1}. \tag{48}$$

Consequently, the estimate (23) is fulfilled with $r = r_0 \left(1 + \lambda + h_1(1) + \|\ell_1(1)\|_L\right)$.

Let now (38) hold. Then there exists $a_4 \in \,]t_M, t_m[\,$ such that

$$u(a_4) = 0, \quad u(t) < 0 \quad \text{for} \quad a_4 < t \le t_m. \tag{49}$$

We put

$$a_3 = \inf\{t \in [a, t_M] : u(s) > 0 \text{ for } t \le s \le t_M\}$$

It is clear that

$$u(t) > 0 \quad \text{for} \quad a_3 < t \le t_M \quad \text{and if} \quad a_3 > a \quad \text{then} \quad u(a_3) = 0. \tag{50}$$

Consequently, from $\lambda > 0$, (26) and (50), we obtain

$$u(a_3) \le \lambda m + M h_0(1) + m h_1(1) + c. \tag{51}$$

Integration (25) from a_3 to t_M and from a_4 to t_m and taking into account (7), (27), (35), (36), (49)–(51), one gets

$$M - \lambda m - M h_0(1) - m h_1(1) - c \le M \int_{a_3}^{t_M} \ell_0(1)(s)\,\mathrm{d}s + m \int_{a_3}^{t_M} \ell_1(1)(s)\,\mathrm{d}s + \int_{a_3}^{t_M} q^*(s)\,\mathrm{d}s,$$

$$m \le M \int_{a_4}^{t_m} \ell_1(1)(s)\,\mathrm{d}s + m \int_{a_4}^{t_m} \ell_0(1)(s)\,\mathrm{d}s + \int_{a_4}^{t_m} q^*(s)\,\mathrm{d}s.$$

The last two inequalities yield

$$\begin{aligned} M\left(1 - \widetilde{C} - h_0(1)\right) \le m\left(\lambda + h_1(1) + \widetilde{A}\right) + c + ||q^*||_L, \\ m\left(1 - \widetilde{D}\right) \le M\widetilde{B} + ||q^*||_L, \end{aligned} \tag{52}$$

where

$$\widetilde{A} = \int_{a_3}^{t_M} \ell_1(1)(s)\,\mathrm{d}s, \quad \widetilde{B} = \int_{a_4}^{t_m} \ell_1(1)(s)\,\mathrm{d}s,$$
$$\widetilde{C} = \int_{a_3}^{t_M} \ell_0(1)(s)\,\mathrm{d}s, \quad \widetilde{D} = \int_{a_4}^{t_m} \ell_0(1)(s)\,\mathrm{d}s. \tag{53}$$

It follows from (10) that $\widetilde{C} < 1 - h_0(1)$ and $\widetilde{D} < 1$. Consequently, inequalities (52) imply

$$0 < M\big(1 - \widetilde{C} - h_0(1)\big)\big(1 - \widetilde{D}\big) \leq$$
$$M\widetilde{B}\big(\lambda + h_1(1) + \widetilde{A}\big) + \big(||q^*||_L + c\big)\big(1 + \lambda + h_1(1) + \|\ell_1(1)\|_L\big),$$
$$0 < m\big(1 - \widetilde{C} - h_0(1)\big)\big(1 - \widetilde{D}\big) \leq$$
$$m\widetilde{B}\big(\lambda + h_1(1) + \widetilde{A}\big) + \big(||q^*||_L + c\big)\left(1 + \|\ell_1(1)\|_L\right).$$

From latter inequalities, analogously as in the case (37), we can show that relations (47) hold, i.e. the estimate (23) is fulfilled with $r = r_0$, where r_0 is introduces in (48).

Consequently, function $u \in AC([a,b];\mathbb{R})$ satisfies estimates (23) in all cases and therefore, operator $\ell = \ell_0 - \ell_1$ belongs to the set $\mathscr{U}(h)$. The assertion of Theorem 1 follows from the Lemma 2.

Proof of Theorem 2 Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$ are such that the conditions (13) and (14) hold. We show that ℓ belongs to the set $\mathscr{U}(h)$.

Let $c \in \mathbb{R}_+$, $q^* \in L([a,b];\mathbb{R}_+)$, and $u \in AC([a,b];\mathbb{R})$ satisfy (21) and (22). We prove that the estimate (23) hold, where r depends only on $\|\ell_0(1)\|_L$, $\|\ell_1(1)\|_L$, λ, $h_0(1)$, and $h_1(1)$.

Analogously to the proof of Theorem 1, if the function u does not change its sign, one can prove that estimate (23) hold.

Let now the function u change its sign. Then either (37) or (38) is fulfilled, where t_m, t_M are introduced in relations (35) and (36).

Similarly as in the proof of Theorem 1, we can show that inequalities (47) are satisfied in both cases (37) and (38) with

$$r_0 = \left[1 - h_0(1) - \|\ell_0(1)\|_L - \frac{1}{4}\Big(\|\ell_1(1)\|_L + \lambda + h_1(1)\Big)^2\right]^{-1}.$$

We only use conditions (13), (14) instead of (10), (11) and relation

$$(\mathscr{A} + \lambda + h_1(1))\,\mathscr{B} \leq \frac{1}{4}\Big(\mathscr{A} + \mathscr{B} + \lambda + h_1(1)\Big)^2 \leq \frac{1}{4}\Big(\|\ell_1(1)\|_L + \lambda + h_1(1)\Big)^2$$

instead of (46). We put $\mathscr{A} := A$, $\mathscr{B} = B$ in the case (37) and $\mathscr{A} := \widetilde{A}$, $\mathscr{B} = \widetilde{B}$ in the case (38), where A, B are introduced by (43) and $\widetilde{A}$, $\widetilde{B}$ by (53).

Hence, function $u \in AC([a,b];\mathbb{R})$ satisfies estimates (23) in all cases and therefore operator $\ell = \ell_0 - \ell_1$ belongs to the set $\mathscr{U}(h)$. Now the assertion of Theorem 2 follows from the Lemma 2.

Proof of Theorem 3 It follows from the condition (16) that the inequality (6) is fulfilled, where $c = |\varphi(0)|$. Moreover, from (17) we get that the inequality (8) holds on the set $\mathscr{B}_{hc}([a,b];\mathbb{R})$, where $q \equiv |F(0)|$. Hence, if conditions (5), (10), (11) hold, then all the assumptions of Theorem 1 are fulfilled. On the other hand, if conditions (12)–(14) hold, then all the assumptions of Theorem 2 are satisfied. Consequently, in both cases the problem (1), (2) has at least one solution and, moreover, it follows from the proofs of Theorems 1 and 2 that operator $\ell = \ell_0 - \ell_1$ belongs to the set $\mathscr{U}(h)$.

It remains to show that problem (1), (2) has at most one solution. Let u_1, u_2 be arbitrary solutions of the problem (1), (2). Put $u(t) = u_1(t) - u_2(t)$ for $t \in [a,b]$. Then, by virtue of (16) and (17), we get $u_1, u_2 \in \mathscr{B}_{hc}([a,b];\mathbb{R})$ and

$$h(u)\operatorname{sgn} u(a) \le 0,$$
$$\big(u'(t) - \ell(u)(t)\big)\operatorname{sgn} u(t) \le 0 \quad \text{for a.e. } t \in [a,b].$$

The last relations, together with $\ell \in \mathscr{U}(h)$, result in $u \equiv 0$. Consequently, $u_1 \equiv u_2$.

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Existence Results for Fuzzy Differential Equations via Truncation Operators Between an Upper and a Lower Solution and Fixed Point Results

Rosana Rodríguez-López

Abstract In this work, we analyze the existence of solution to a fuzzy differential equation of first order in the fuzzy functional interval determined by an upper and a lower solution. The approach followed consists in the study of an auxiliary problem that is defined through a proper 'truncation operator' based on the choice of well ordered upper and lower solutions to the problem of interest. To our purpose, we justify that the truncation operator is well defined and satisfies some monotonicity properties. Finally, using the lattice structure of some subsets of the space of continuous fuzzy-valued functions and imposing some restrictions on the nonlinearity, we conclude the existence of solution to the equation on the interval $[0, +\infty)$ by the application of Tarski's fixed point theorem.

Keywords Fuzzy differential equations · Upper and lower solutions
Truncation operator · Fixed point theory

1 Introduction

The use of fuzzy mathematics can be an adequate tool to model processes which are subject to imprecise factors such as inexact physical measurements or uncertain information. In particular, fuzzy differential equations [3, 4, 13] have relevant applications in different scientific and social fields.

When considering a fuzzy differential equation, we often have to specify the type of fuzzy derivative chosen for fuzzy-valued functions [10]. We illustrate the method followed by using Hukuhara differentiability, which has several drawbacks, but the procedure can be extended to equations under more general types of fuzzy derivatives [1, 14].

In what follows, we present the problem of interest, the fixed point result we apply, and some notation and basic results on the properties of fuzzy sets (Sect. 2).

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In the main section (Sect. 3), we define the truncation operators and give their main properties, and we obtain some existence results by applying fixed point theory. For some other results concerning the application of fixed point results to the solvability of fuzzy differential equations, see, for instance, [6].

2 Notation and Preliminaries

We consider the space E^1 of normal, upper semicontinuous, fuzzy-convex and compact-supported mappings $u : \mathbb{R} \longrightarrow [0, 1]$. In this space E^1 (of fuzzy intervals), we define the distance

$$d_\infty(x, y) = \sup_{a\in[0,1]} d_H([x]^a, [y]^a), \quad x,\ y \in E^1,$$

where d_H denotes the Hausdorff distance in $\mathscr{K}_C$ (the set of nonempty compact and convex subsets of $\mathbb{R}$).

We consider the following nonlinear first-order fuzzy differential equation

$$x'(t) = f(t, x(t)),\ t \in [t_0, +\infty),$$

where $t_0 \in \mathbb{R}$ and $f : [t_0, +\infty) \times E^1 \longrightarrow E^1$, although, by simplicity, we will study the case $t_0 = 0$. The previous equation is written in integral form by

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds, \qquad t \geq t_0.$$

Here, the level sets of $x \in E^1$ are represented by $[x]^a = [x_{al}, x_{ar}]$, for $a \in [0, 1]$. We also use the functions $x_L : [0, 1] \longrightarrow \mathbb{R}$ and $x_R : [0, 1] \longrightarrow \mathbb{R}$ defined, respectively, by $x_L(a) = x_{al}$, and $x_R(a) = x_{ar}$, for all $a \in [0, 1]$.

Here, we consider the problem of the existence of solution between an upper and a lower solution for one-dimensional fuzzy differential equations by passing to a modified 'truncated' problem through an operator obtained by using the upper and lower solutions. To prove the existence of solution, we use Tarski's Fixed Point Theorem.

Theorem 1 ([15]) *Suppose that X is a complete lattice and $F : X \longrightarrow X$ is a nondecreasing function, that is, $F(x) \leq F(y)$ whenever $x \leq y$. Moreover, suppose that there exists $x_0 \in X$ such that $F(x_0) \geq x_0$. Then F has at least one fixed point in X.*

In this context, the maximal fixed point for F is obtained as the supremum of the set $Y = \{x \in X\ :\ F(x) \geq x\}$. On the other hand, if there exists $x_1 \in X$ such that $F(x_1) \leq x_1$, then the minimal fixed point can be obtained as the infimum of the set $Z = \{x \in X\ :\ F(x) \leq x\}$. If there exist simultaneously x_0 and x_1 such that $F(x_0) \geq x_0$ and $F(x_1) \leq x_1$, then $z = \sup Y = \sup\{x \in X\ :\ F(x) \geq x\}$ is the maximal fixed point of F in X and $\hat{z} = \inf Z = \inf\{x \in X \ : \ F(x) \leq x\}$ is the minimal fixed point of F in X.

In the sequel, we use the following two order relations in E^1.

Definition 1 ([9]) Let $x, \ y \in E^1$. We say that $x \leq y$ if and only if

$$x_{al} \leq y_{al} \text{ and } x_{ar} \leq y_{ar}, \text{ for every } a \in [0, 1].$$

On the other hand, we say that $x \preceq y$ if and only if

$$x_{al} \geq y_{al} \text{ and } x_{ar} \leq y_{ar}, \text{ for every } a \in [0, 1],$$

that is, $[x]^a \subseteq [y]^a, \forall\, a \in [0, 1]$.

Remark 1 ([9]) Note that:

- $x \leq y$ is equivalent to $x_L \leq y_L$ and $x_R \leq y_R$ on $[0, 1]$.
- $x \preceq y$ is equivalent to $y_L \leq x_L$ and $x_R \leq y_R$ on $[0, 1]$.

Lemma 1 ([6]) *Consider the above-introduced partial orderings $\leq$ and $\preceq$ in E^1.*

(i) Given $x, \ y \in E^1$, the following properties are valid:

$$x = y \text{ if and only if } x \leq y \text{ and } y \leq x,$$

$$x = y \text{ if and only if } x \preceq y \text{ and } y \preceq x.$$

(ii) If $x, \ y, \ z \in E^1$ are such that $x \leq y$, then $x + z \leq y + z$.

(iii) If $x, \ y, \ z \in E^1$ are such that $x \preceq y$, then $x + z \preceq y + z$.

In the following result, the product of fuzzy intervals is defined by the Zadeh's Extension Principle.

Lemma 2 *If $x \leq y$ and $z \geq \chi_{\{0\}}$, then $xz \leq yz$. Similarly, if $x \preceq y$ and $z \succeq \chi_{\{0\}}$, then $xz \preceq yz$.*

Proof If $x \leq y$, then $x_L \leq y_L$ and $x_R \leq y_R$ on $[0, 1]$, and using that $z_L(a), \ z_R(a) \geq 0$ for $a \in [0, 1]$, we obtain, for every $a \in [0, 1]$, that

$$(xz)_L(a) = x_L(a)z_L(a) \leq y_L(a)z_L(a) = (yz)_L(a),$$

$$(xz)_R(a) = x_R(a)z_R(a) \leq y_R(a)z_R(a) = (yz)_R(a).$$

If $x \preceq y$, then $y_L \leq x_L, x_R \leq y_R$ on $[0, 1]$, thus, for every $a \in [0, 1]$, using the properties of z, we have

$$(yz)_L(a) = y_L(a)z_L(a) \leq x_L(a)z_L(a) = (xz)_L(a), \qquad (xz)_R(a) \leq (yz)_R(a). \ \square$$

Definition 2 For $f, g : I \to E^1$, with I a real interval, we say that $f \leq g$ if $f(t) \leq g(t)$, for every $t \in I$. Analogously, for the order relation $\preceq$.

Lemma 3 ([6]) *Consider* $t \in [0, +\infty)$ *arbitrarily fixed. Then:*

(i) *If* $x(s) \leq y(s),\ \forall s \in [0, t]$, *then* $\int_0^t x(s)\,ds \leq \int_0^t y(s)\,ds$.
(ii) *If* $x(s) \preceq y(s),\ \forall s \in [0, t]$, *then* $\int_0^t x(s)\,ds \preceq \int_0^t y(s)\,ds$.

Proof Indeed, for the partial ordering $\leq$ and $a \in [0, 1]$,

$$\left[\int_0^t x(s)\,ds\right]_{al} = \int_0^t x(s)_{al}\,ds \leq \int_0^t y(s)_{al}\,ds = \left[\int_0^t y(s)\,ds\right]_{al},$$

$$\left[\int_0^t x(s)\,ds\right]_{ar} = \int_0^t x(s)_{ar}\,ds \leq \int_0^t y(s)_{ar}\,ds = \left[\int_0^t y(s)\,ds\right]_{ar}.$$

Similarly, for $\preceq$ and $a \in [0, 1]$,

$$\left[\int_0^t x(s)\,ds\right]^a = \int_0^t [x(s)]^a\,ds \subseteq \int_0^t [y(s)]^a\,ds = \left[\int_0^t y(s)\,ds\right]^a. \qquad \square$$

3 Main Results

We consider the equation

$$u'(t) = f(t, u(t)),\ t \in [0, +\infty), \tag{1}$$

where $f : [0, +\infty) \times E^1 \longrightarrow E^1$.

Our approach is based on the use of upper and lower solutions in order to obtain a modified auxiliary problem, and the main interest is to prove the existence of solution to (1) between an upper and a lower solution to this equation.

We consider $C^1([0, +\infty), E^1)$ as the set of continuous functions $x : [0, +\infty) \to E^1$ with continuous derivative under a certain kind of fuzzy derivative. The type of derivative considered in the rest of the paper will be Hukuhara derivative for fuzzy-valued functions (see [2]).

Definition 3 The function $\alpha \in C^1([0, +\infty), E^1)$ is a $\leq$-lower solution to equation (1) if $\alpha'(t) \leq f(t, \alpha(t)),\ t \in [0, +\infty)$. Similarly, $\beta \in C^1([0, +\infty), E^1)$ is $\leq$-upper solution to (1) if the reversed inequality is satisfied.

We can define the corresponding concepts for the partial ordering $\preceq$.

3.1 Definition and Properties of the Truncation Operators

Suppose that $\alpha,\ \beta$ are, respectively, lower and upper solutions for equation (1) such that

$$\alpha \le \beta \text{ on } [0, +\infty) \text{ (resp. } \alpha \preceq \beta \text{ on } [0, +\infty)),$$

and define the truncation operator

$$\begin{array}{rcl} q : [0, +\infty) \times E^1 & \longrightarrow & E^1 \\ (t, x) & \longrightarrow & q(t, x), \end{array}$$

respectively,

$$\begin{array}{rcl} \tilde{q} : [0, +\infty) \times E^1 & \longrightarrow & E^1 \\ (t, x) & \longrightarrow & \tilde{q}(t, x), \end{array}$$

as we explain in the sequel.

First, we consider $\alpha \le \beta$, then $\alpha(t) \le \beta(t)$, $\forall t \in [0, +\infty)$, that is,

$$\alpha(t)_{al} \le \beta(t)_{al} \text{ and } \alpha(t)_{ar} \le \beta(t)_{ar}, \text{ for every } a \in [0, 1],$$

and q is defined in such a way that, for every $a \in (0, 1]$,

$$[q(t, x)]^a := [\max\{\alpha(t)_{al}, \min\{x_{al}, \beta(t)_{al}\}\}, \max\{\alpha(t)_{ar}, \min\{x_{ar}, \beta(t)_{ar}\}\}]$$

and $[q(t, x)]^0 = \overline{\bigcup_{a \in (0,1]} [q(t, x)]^a}$. We have to prove that the sets given previously define a fuzzy interval. To this purpose, we use the following characterization of Theorem 1.5.1 [5] in terms of the functions x_L and x_R (branches) associated to a fuzzy interval x.

Theorem 2 *If $u \in E^1$, then the following conditions hold:*

$$u_L(a) \le u_R(a), \text{ for every } a \in [0, 1], \tag{2}$$

$$u_L \text{ is nondecreasing and } u_R \text{ is nonincreasing on } [0, 1], \tag{3}$$

$$u_L,\ u_R \text{ are left-continuous on } (0, 1]. \tag{4}$$

Conversely, if $f_L : [0, 1] \longrightarrow \mathbb{R}$ and $f_R : [0, 1] \longrightarrow \mathbb{R}$ are two functions satisfying conditions (2)–(4), then there exists $u \in E^1$ such that

$$u_L(a) = f_L(a),\ u_R(a) = f_R(a),\ \forall a \in (0, 1]$$

and

$$u_L(0) = \inf_{a>0} f_L(a) \geq f_L(0),\ u_R(0) = \sup_{a>0} f_R(a) \leq f_R(0). \tag{5}$$

Proof The proof is obvious considering that conditions (2)–(4) are equivalent to hypotheses (1.5.1)–(1.5.3) in Theorem 1.5.1 [5]. Indeed, for $x : \mathbb{R} \to [0, 1]$,

$$[x_L(a), x_R(a)] \in \mathscr{K}_C,\ \text{ for every } a \in [0, 1]$$

if and only if

$$x_L(a) \leq x_R(a),\ \text{ and } x_L, x_R \text{ bounded on } [0, 1].$$

On the other hand,

$$\begin{aligned}&[x_L(a_2), x_R(a_2)] \subseteq [x_L(a_1), x_R(a_1)],\ \text{ for } 0 \leq a_1 \leq a_2 \leq 1\\ &\qquad\iff x_L(a_1) \leq x_L(a_2),\ x_R(a_1) \geq x_R(a_2),\ \text{ for } 0 \leq a_1 \leq a_2 \leq 1,\end{aligned}$$

and for $(a_k)_{k\in\mathbb{N}}$ a nondecreasing sequence converging to $a > 0$ ($(a_k) \to a^-$), taking into account that $\{[x_L(a_k), x_R(a_k)] : k \in \mathbb{N}\}$ is a nested sequence and the characterization of the Hausdorff distance in terms of the functions x_L and x_R, we obtain that

$$\begin{aligned}[x_L(a), x_R(a)] &= \cap_{k\geq 1}[x_L(a_k), x_R(a_k)]\\ &\iff x_L(a_k) \to x_L(a),\ \ x_R(a_k) \to x_R(a).\end{aligned}$$

In consequence, the conclusion follows from Theorem 1.5.1 [5]. □

Remark 2 In Theorem 2, if $u \in E^1$, then u_L and u_R are right-continuous at 0. Conversely, if we add the right-continuity of u_L and u_R at 0 in condition (4), then an identity is obtained in (5).

Lemma 4 *Suppose that* $\alpha \leq \beta$ *on* $[0, +\infty)$*, then* $q(t, x) \in E^1$ *for every* $t \in [0, +\infty)$ *and* $x \in E^1$.

Proof Let $t \in [0, +\infty)$ and $x \in E^1$ fixed. We take the functions

$$f_L = \max\{\alpha(t)_L, \min\{x_L, \beta(t)_L\}\}, \quad \text{and} \quad f_R = \max\{\alpha(t)_R, \min\{x_R, \beta(t)_R\}\},$$

which are defined on [0, 1].

Since $\alpha(t)$, x and $\beta(t)$ are fuzzy intervals, then conditions (2)–(4) of Theorem 2 hold and, therefore, they are also satisfied for f_L and f_R. Hence, $q(t, x)$ is a fuzzy interval, and

$$[q(t, x)]^a = [q(t, x)_L(a), q(t, x)_R(a)],\ \forall a \in [0, 1],$$

where

$$q(t,x)_L(a) = \max\{\alpha(t)_L(a), \min\{x_L(a), \beta(t)_L(a)\}\},\ \forall a \in (0,1],$$

$$q(t,x)_L(0) = \inf_{a>0} q(t,x)_L(a),$$

and

$$q(t,x)_R(a) = \max\{\alpha(t)_R(a), \min\{x_R(a), \beta(t)_R(a)\}\}, \forall a \in (0,1],$$

$$q(t,x)_R(0) = \sup_{a>0} q(t,x)_R(a),$$

which means that the endpoints of the level sets of $q(t,x)$ are given by the truncation of x_L and x_R in $[\alpha(t)_L, \beta(t)_L]$ and $[\alpha(t)_R, \beta(t)_R]$, respectively. □

Now, for the partial ordering $\preceq$, if $\alpha \preceq \beta$, then, for every $t \in [0,+\infty)$,

$$\beta(t)_{al} \le \alpha(t)_{al} \text{ and } \alpha(t)_{ar} \le \beta(t)_{ar}, \text{ for every } a \in [0,1],$$

and we define $\tilde{q}(t,x)$ such that, for every $a \in (0,1]$,

$$[\tilde{q}(t,x)]^a = [\max\{\beta(t)_{al}, \min\{x_{al}, \alpha(t)_{al}\}\}, \max\{\alpha(t)_{ar}, \min\{x_{ar}, \beta(t)_{ar}\}\}],$$

and $[\tilde{q}(t,x)]^0 = \overline{\bigcup_{a\in(0,1]} [\tilde{q}(t,x)]^a}$.

Lemma 5 *If $\alpha \preceq \beta$ on $[0,+\infty)$, then $\tilde{q}(t,x)$ is a fuzzy interval for every $t \in [0,+\infty)$ and $x \in E^1$.*

Proof For $t \in [0,+\infty)$ and $x \in E^1$ fixed, we apply Theorem 2 to functions

$$g_L = \max\{\beta(t)_L, \min\{x_L, \alpha(t)_L\}\}, \quad \text{and} \quad g_R = \max\{\alpha(t)_R, \min\{x_R, \beta(t)_R\}\},$$

defined on $[0,1]$. Using that $\alpha(t)$, x and $\beta(t)$ are fuzzy intervals, then conditions (2)–(4) of Theorem 2 hold and, in consequence, functions g_L and g_R are under the assumptions of Theorem 2. This proves that $\tilde{q}(t,x) \in E^1$ and

$$[\tilde{q}(t,x)]^a = [\tilde{q}(t,x)_L(a), \tilde{q}(t,x)_R(a)],\ a \in [0,1],$$

where

$$\tilde{q}(t,x)_L(a) = \max\{\beta(t)_L(a), \min\{x_L(a), \alpha(t)_L(a)\}\},\ \forall a \in (0,1],$$

$$\tilde{q}(t,x)_L(0) = \inf_{a>0} \tilde{q}(t,x)_L(a),$$

and

$$\tilde{q}(t,x)_R(a) = \max\{\alpha(t)_R(a), \min\{x_R(a), \beta(t)_R(a)\}\},\ \forall a \in (0,1],$$

$$\tilde{q}(t,x)_R(0) = \sup_{a>0} \tilde{q}(t,x)_R(a).$$

Note that the right-continuity at $a = 0$ also holds. In this case, the endpoints of the level sets of $\tilde{q}(t,x)$ are given by the truncation of x_L and x_R in the intervals $[\beta(t)_L, \alpha(t)_L]$ and $[\alpha(t)_R, \beta(t)_R]$, respectively. □

Next, we present some properties of the truncation operators q and $\tilde{q}$ that will be useful later in our procedure.

Lemma 6 *The following assertions are valid for the operators q and $\tilde{q}$:*

(i) If $\alpha \le \beta$ on $[0,+\infty)$, then $\alpha(t) \le q(t,x) \le \beta(t),\ \forall t \in [0,+\infty),\ \forall x \in E^1$.
(ii) If $\alpha \preceq \beta$ on $[0,+\infty)$, then $\alpha(t) \preceq \tilde{q}(t,x) \preceq \beta(t),\ \forall t \in [0,+\infty),\ \forall x \in E^1$.
(iii) Taking $t \in [0,+\infty)$ fixed, then $q(t,x) = x$ if and only if $\alpha(t) \le x \le \beta(t)$.
(iv) Taking $t \in [0,+\infty)$ fixed, then $\tilde{q}(t,x) = x$ if and only if $\alpha(t) \preceq x \preceq \beta(t)$.

Proof Properties (i) and (ii) are obtained from Remark 1, since, given $t \in [0,+\infty)$, and $x \in E^1$ fixed,

$$\alpha(t)_L \le q(t,x)_L = \max\{\alpha(t)_L, \min\{x_L, \beta(t)_L\}\} \le \beta(t)_L \quad \text{on } [0,1],$$

$$\beta(t)_L \le \tilde{q}(t,x)_L = \max\{\beta(t)_L, \min\{x_L, \alpha(t)_L\}\} \le \alpha(t)_L \quad \text{on } [0,1],$$

and

$$\alpha(t)_R \le q(t,x)_R = \tilde{q}(t,x)_R = \max\{\alpha(t)_R, \min\{x_R, \beta(t)_R\}\} \le \beta(t)_R \quad \text{on } [0,1].$$

To prove (iii), consider $t \in [0,+\infty)$ fixed, and take into account that $q(t,x) = x$ means that

$$q(t,x)_L = \max\{\alpha(t)_L, \min\{x_L, \beta(t)_L\}\} = x_L,$$

$$q(t,x)_R = \max\{\alpha(t)_R, \min\{x_R, \beta(t)_R\}\} = x_R,$$

or, equivalently, $\alpha(t)_L \le x_L \le \beta(t)_L,\ \alpha(t)_R \le x_R \le \beta(t)_R$. Analogous conclusion is obtained for $\tilde{q}$. □

Moreover, functions q and $\tilde{q}$ are nondecreasing in the second variable.

Lemma 7 *If $x \le y$, then $q(t,x) \le q(t,y)$, for every $t \in [0,+\infty)$. On the other hand, if $x \preceq y$, then $\tilde{q}(t,x) \preceq \tilde{q}(t,y)$, for every $t \in [0,+\infty)$.*

Proof If $x \le y$, then $x_L \le y_L$ and $x_R \le y_R$ on $[0,1]$ so that, for $t \in [0,+\infty)$, we obtain

$$q(t,x)_L = \max\{\alpha(t)_L, \min\{x_L, \beta(t)_L\}\} \le \max\{\alpha(t)_L, \min\{y_L, \beta(t)_L\}\} = q(t,y)_L,$$

and

$$q(t,x)_R = \max\{\alpha(t)_R, \min\{x_R, \beta(t)_R\}\} \le \max\{\alpha(t)_R, \min\{y_R, \beta(t)_R\}\} = q(t,y)_R.$$

Similarly, if $x \preceq y$, then $y_L \le x_L$ and $x_R \le y_R$ on $[0,1]$, so that, for every $t \in [0,+\infty)$,

$$\tilde{q}(t,x)_L = \max\{\beta(t)_L, \min\{x_L, \alpha(t)_L\}\} \ge \max\{\beta(t)_L, \min\{y_L, \alpha(t)_L\}\} = \tilde{q}(t,y)_L,$$

and $\tilde{q}(t,x)_R \le \tilde{q}(t,y)_R$. □

To apply the fixed point theorem we use in the main existence results, it is important to determine the lattice structure of certain sets of fuzzy intervals and sets of fuzzy-valued functions.

Lemma 8 (Proposition 2.1 [7]) *The set* $(E^1, \le)$ *is a lattice.*

Proof Given $x,\ y \in E^1$, it is easy to prove that there exist lower and upper bounds (namely, $\min\{x,y\}$ and $\max\{x,y\}$) of x and y in E^1. Indeed (see [7]), for $a \in [0,1]$,

$$\begin{aligned}
[max\{x,y\}]^a &= [\max\{x_{al}, y_{al}\}, \max\{x_{ar}, y_{ar}\}]\\
&= [\max\{x_L(a), y_L(a)\}, \max\{x_R(a), y_R(a)\}],
\end{aligned}$$

$$\begin{aligned}
[min\{x,y\}]^a &= [\min\{x_{al}, y_{al}\}, \min\{x_{ar}, y_{ar}\}]\\
&= [\min\{x_L(a), y_L(a)\}, \min\{x_R(a), y_R(a)\}].
\end{aligned}$$

It is easy to check that each pair of functions

$$(f_1)_L = \max\{x_L, y_L\},\ (f_1)_R = \max\{x_R, y_R\},$$

$$(f_2)_L = \min\{x_L, y_L\},\ (f_2)_R = \min\{x_R, y_R\},$$

satisfy the conditions in Theorem 2, then the fuzzy intervals $max\{x,y\}$ and $min\{x,y\}$ are well-defined and

$$max\{x,y\}_L = (f_1)_L,\ max\{x,y\}_R = (f_1)_R,\ min\{x,y\}_L = (f_2)_L,\ min\{x,y\}_R = (f_2)_R,$$

From these relations and Remark 1, it can be easily proved that $min\{x,y\} \le x,\ y \le max\{x,y\}$. See [7] for further details. □

Although $(E^1, \preceq)$ is not a lattice, we can work on some of their subsets which are lattices.

Lemma 9 (Proposition 3.2 [7]) *Let* $p \in \mathbb{R}$ *and* $S \subseteq E^1$ *be such that* $\chi_{\{p\}} \preceq x$, *for all* $x \in S$. *Then* $(S, \preceq)$ *is a lattice.*

Proof In this case, for $x, y \in S$, we can take (see [7]),

$$\left[\widetilde{max}\{x,y\}\right]^a = [\min\{x_{al}, y_{al}\}, \max\{x_{ar}, y_{ar}\}] \\ = [\min\{x_L(a), y_L(a)\}, \max\{x_R(a), y_R(a)\}],$$

and

$$\left[\widetilde{min}\{x,y\}\right]^a = [\max\{x_{al}, y_{al}\}, \min\{x_{ar}, y_{ar}\}] \\ = [\max\{x_L(a), y_L(a)\}, \min\{x_R(a), y_R(a)\}].$$

Again, each pair of functions

$$(f_3)_L = \min\{x_L, y_L\},\ (f_3)_R = \max\{x_R, y_R\},$$

$$(f_4)_L = \max\{x_L, y_L\},\ (f_4)_R = \min\{x_R, y_R\},$$

satisfy conditions in Theorem 2, then the fuzzy intervals $\widetilde{max}\{x, y\}$ and $\widetilde{min}\{x, y\}$ are well-defined and

$$\widetilde{max}\{x,y\}_L = (f_3)_L,\ \widetilde{max}\{x,y\}_R = (f_3)_R,\ \widetilde{min}\{x,y\}_L = (f_4)_L,\ \widetilde{min}\{x,y\}_R = (f_4)_R.$$

Hence $\widetilde{min}\{x, y\} \preceq x,\ y \preceq \widetilde{max}\{x, y\}$. See [7] for further details. □

Lemma 10 ([7]) *The following properties are valid:*

- *If* $\mu, \nu \in E^1$ *are such that* $\mu \le \nu$, *then* $([\mu, \nu], \le)$ *is a complete lattice, where*

$$[\mu,\nu] := \{x \in E^1 \ :\ \mu \le x \le \nu\}.$$

- *If* $\mu, \nu \in E^1$ *are such that* $\mu \preceq \nu$, *then* $([\mu, \nu], \preceq)$ *is a complete lattice, where*

$$[\mu,\nu] := \{x \in E^1 \ :\ \mu \preceq x \preceq \nu\}.$$

Corollary 1 ([7]) *Consider I a real interval. If* $\alpha, \beta \in C(I, E^1)$ *are such that* $\alpha \le \beta$ *on* I, *then*

$$[\alpha,\beta]_{\le} := \{x \in C(I,E^1) \ :\ \alpha \le x \le \beta \text{ on } I\}$$

is a complete lattice. On the other hand, if $\alpha, \beta \in C(I, E^1)$ *are such that* $\alpha \preceq \beta$ *on* I, *then*

$$[\alpha,\beta]_{\preceq} := \{x \in C(I,E^1) \ :\ \alpha \preceq x \preceq \beta \text{ on } I\}$$

is a complete lattice.

Proof This property is clear taking into account that, for $x, y \in C(I, E^1)$, we can define, respectively, lower and upper bounds for x and y for the ordering $\le$ as

$$\begin{array}{rcl} \min\{x,y\}: I & \longrightarrow & E^1 \\ t & \longrightarrow & \min\{x(t),y(t)\}, \end{array} \qquad \begin{array}{rcl} \max\{x,y\}: I & \longrightarrow & E^1 \\ t & \longrightarrow & \max\{x(t),y(t)\}' \end{array}$$

and, for the partial ordering $\preceq$, as

$$\begin{array}{rcl} \widetilde{min}\{x,y\}: I & \longrightarrow & E^1 \\ t & \longrightarrow & \widetilde{min}\{x(t),y(t)\}, \end{array} \qquad \begin{array}{rcl} \widetilde{max}\{x,y\}: I & \longrightarrow & E^1 \\ t & \longrightarrow & \widetilde{max}\{x(t),y(t)\}. \end{array} \quad \square$$

3.2 Study of the Nonlinear Fuzzy Differential Equation

To find a solution to equation (1), given α, β, respectively, $\leq$-lower and $\leq$-upper solutions for (1) with $\alpha \leq \beta$ on $[0, +\infty)$, we consider the following auxiliary fuzzy differential equation

$$u'(t) = f(t, q(t, u(t))), \;\; t \in [0, +\infty). \tag{6}$$

Clearly, by the properties stated in Lemma 6, if $u : [0, +\infty) \longrightarrow E^1$ is a solution to (6) such that $\alpha \leq u \leq \beta$ on $[0, +\infty)$, then u is a solution to (1), and this solution belongs to the functional interval $[\alpha, \beta]_{\leq} := \{x \in C([0, +\infty), E^1) \; : \; \alpha \leq x \leq \beta \text{ on } [0, +\infty)\}$. We show that the Eq. (6) has a solution in the functional interval $[\alpha, \beta]_{\leq}$ under suitable hypotheses. To this aim, we write the mentioned Eq. (6) in integral form and try to apply some appropriate fixed point results. We proceed similarly for the case where $\alpha \preceq \beta$ on $[0, +\infty)$, taking the equation

$$u'(t) = f(t, \tilde{q}(t, u(t))), \;\; t \in [0, +\infty). \tag{7}$$

In this case, if u is a solution to (7) such that $\alpha \preceq u \preceq \beta$ on $[0, +\infty)$, then u is a solution to (1). In the following, we denote $I := [0, +\infty)$.

Theorem 3 *If α, β are, respectively, $\leq$-lower and $\leq$-upper solutions for (1) such that $\alpha \leq \beta$ on I, and $f(t, x)$ continuous is $\leq$-nondecreasing in the second variable x for $t \in I$ fixed and x in the interval $[\alpha(t), \beta(t)] := \{x \in E^1 \; : \; \alpha(t) \leq x \leq \beta(t)\}$, then (1) has at least one solution in $[\alpha, \beta]_{\leq} := \{u \in C(I, E^1) \; : \; \alpha \leq u \leq \beta \text{ on } I\}$.*

Proof We consider the auxiliary problem (6) and define

$$\begin{array}{rcl} F : C(I, E^1) & \longrightarrow & C(I, E^1) \\ x & \longrightarrow & Fx, \end{array}$$

given by $[Fx](t) = x(0) + \int_0^t f(s, q(s, x(s)))\, ds, \;\; t \in I$. Note that the space where the mapping F is defined is not a complete metric space, since the interval $I = [0, +\infty)$ is not compact. However, $C(I, E^1)$ is a partially ordered set, with the induced ordering relation given in Definition 2. According to this order relation, function F is nondecreasing. Indeed, for $x \leq y$, then $x(s) \leq y(s)$, for every $s \in I$. Using Lemma 7,

$$q(s, x(s)) \leq q(s, y(s)), \ \forall s \in I,$$

and, by using the $\leq$-nondecreasing character of $f(t, x)$ in the second variable x relative to the interval $[\alpha(t), \beta(t)]$, we obtain, applying Lemma 6, that

$$f(s, q(s, x(s))) \leq f(s, q(s, y(s))), \ \forall s \in I.$$

Using Lemmas 1 and 3, we get

$$[Fx](t) = x(0) + \int_0^t f(s, q(s, x(s)))\, ds \leq y(0) + \int_0^t f(s, q(s, y(s)))\, ds = [Fy](t), \ t \in I.$$

Moreover, using Lemma 6 and the definitions of lower and upper solutions for equation (1), we obtain

$$\begin{aligned} \alpha(t) &= \alpha(0) + \int_0^t \alpha'(s)\, ds \leq \alpha(0) + \int_0^t f(s, \alpha(s))\, ds \\ &= \alpha(0) + \int_0^t f(s, q(s, \alpha(s)))\, ds = [F\alpha](t), \ t \in I, \\ [F\beta](t) &= \beta(0) + \int_0^t f(s, q(s, \beta(s)))\, ds = \beta(0) + \int_0^t f(s, \beta(s))\, ds \\ &\leq \beta(0) + \int_0^t \beta'(s)\, ds = \beta(t), \ t \in I, \end{aligned}$$

so that $\alpha \leq F\alpha$ and $F\beta \leq \beta$. The proof is completed using Tarski's Fixed Point Theorem (see [15]) from the considerations in [7] (see Corollary 1). □

Theorem 4 *If α, β are, respectively, $\preceq$-lower and $\preceq$-upper solutions for (1) such that $\alpha \preceq \beta$ on I, and $f(t, x)$ continuous is $\preceq$-nondecreasing in the second variable x for $t \in I$ fixed and x in the interval $[\alpha(t), \beta(t)] := \{x \in E^1 \ : \ \alpha(t) \preceq x \preceq \beta(t)\}$, then (1) has at least one solution in $[\alpha, \beta]_{\preceq} := \{u \in C(I, E^1) \ : \ \alpha \preceq u \preceq \beta \text{ on } I\}$.*

The use of comparison results can be also useful to study some problems for fuzzy differential equations (see [11, 12]), allowing to replace the condition of monotonicity of f in the second variable by a similar property of a modified function related to f. However, in some cases, additional conditions have to be imposed in order to guarantee the existence of the appropriate Hukuhara differences. See [8] for details on the existence of solution to some boundary value problems and also [6] for other approaches. The use of other more general types of derivatives can also lead to weaker restrictions on the nonlinearity f.

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On Systems of Nonlinear ODE Arising in Gas Dynamics: Application to Vortical Motion

Olga S. Rozanova and Marko K. Turzynski

Abstract We show that with the multidimensional system of gas dynamics with a special forcing one can associate a quadratically nonlinear ODE system which describes a special class of motion. The system can be obtained by two different ways. In particular, we study the influence of Coriolis and frictional terms. We review the result about the non-frictional case and study the influence of constant dry friction.

Keywords Gas dynamics · Vortex motion · Nonlinear stability · Linear profile of velocity · Coriolis force · Dry friction

Introduction

We study the model of gas dynamics in the uniformly rotating reference frame. This model is important due to applications in geophysics. Namely, in the middle scale approximation, the motion of air can be considered on the l-plane (i.e. on the plane tangent to the Earth surface at a fixed point). In this approximation, the Coriolis parameter l is a constant. Moreover, the horizontal dimension of the atmosphere is much more large than the vertical one, therefore the atmosphere often is modeled in the two-dimensional setting. Many important systems of equations arising from physics (so called systems of hydrodynamical type) possess a special class of solutions. Such solutions can be obtained by reducing to a nonlinear system of ODEs. The reducing is possible if we make an assumption on the structure of velocity: it has to be linear with respect to the space coordinates. The existence of this class of motions implies that under some special conditions the liquid or gas behaves like a rigid body. First, this class of solution was applied to the incompressible fluid, then many other models, including geophysical ones, were analyzed from this point of view. This helped to study many important properties of complicated physical models within a subclass of motions. The motion with a linear profile of velocity is

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meaningful by itself since any large atmospherical vortex near its center has such structure [11]. Also, this class of motions can help to study trajectories of tropical cyclones [8, 9].

An extensive review of the state of art can be found, for example, in [1, 4].

Considering geophysical models it is natural to take into account interaction of air with the underlying surface. The dry friction, when the friction force is proportional to velocity, is the simplest way to describe this interaction.

In this paper, we are going to show that under the assumption of linear structure of velocity the model of two-dimensional rotating gas influenced by the dry friction can be reduced to a system of nonlinear ODEs, having independent mathematical interest.

1 Model of Two Dimensional Gas Dynamics

We consider the system of non-isentropic polytropic gas dynamics equations in a uniformly rotating reference frame for unknown functions $\rho \ge 0$, $p \ge 0$, $U = (U_1, U_2, U_3)$, S (density, pressure, velocity, and entropy), in the presence of the horizontal dry friction (e.g. [7]), namely

$$\rho(\partial_t U + (U, \nabla)U + l e_3 \times U + \mu U + g e_3) = -\nabla p, \tag{1}$$

$$\partial_t \rho + \operatorname{div}(\rho U) = 0, \tag{2}$$

$$\partial_t S + (U, \nabla S) = 0. \tag{3}$$

The functions depend on time t and on point $x \in \mathbb{R}^3$, $e_3 = (0, 0, 1)$ is the “upward” unit vector, l is the Coriolis parameter, μ is the friction coefficient, g is the acceleration due to gravity (in $-e_3$ direction), $\mu \ge 0$ is the friction coefficient. The state equation is

$$p = \rho^\gamma e^S, \tag{4}$$

where $\gamma \in (1, 2)$ is the adiabatic exponent. We assume $U = (U_H, 0) = (U_1, U_2, 0)$, therefore the model becomes two-dimensional.

For $\mu = 0$, the system implies the conservation of mass $\mathscr{M} = \int\limits_{\Omega(t)} \rho\, dx$, momentum $P = \int\limits_{\Omega(t)} \rho U\, dx$ and energy

$$\mathscr{E} = \mathscr{E}_k(t) + \mathscr{E}_p(t) = \int\limits_{\Omega(t)} \left(\frac{\rho |U|^2}{2} + \frac{1}{\gamma - 1} p \right) dx,$$

inside a material volume $\Omega(t)$, if we assume the hydrostatic balance

$$\partial_{x_3} p = -g\rho. \tag{5}$$

To prove these conservation laws we apply the formula for the derivative with respect to time of integral taken over a material volume [3], namely,

$$\frac{d}{dt}\int\limits_{\Omega(t)} f(t,x)dx = \int\limits_{\Omega(t)} \big(\partial_t f(t,x) + \operatorname{div}(f(t,x)\,v)\big)\,dx. \tag{6}$$

Let us introduce the following functionals:

$$G(t) = \frac{1}{2}\int\limits_{\Omega(t)} \rho|X_1|^2\,dx_1dx_2, \quad F_i(t) = \int\limits_{\Omega(t)} (U, X_i)\rho\,dx_1dx_2,$$

$$G_{x_1}(t) = \frac{1}{2}\int\limits_{\Omega(t)} \rho x_1^2 dx_1dx_2, \quad G_{x_2}(t) = \frac{1}{2}\int\limits_{\Omega(t)} \rho x_2^2 dx_1dx_2,$$

$$G_{x_1x_2}(t) = \frac{1}{2}\int\limits_{\Omega(t)} \rho x_1x_2\,dx_1dx_2,$$

where $X_1 = (x_1, x_2),\ X_2 = (x_2, -x_1),\ i = 1, 2$. We note that $G(t) > 0$ and $\Delta(t) = G_{x_1}G_{x_2} - G^2_{x_1x_2} > 0$ for nontrivial solutions to (1)–(5).

We assume that l and μ are positive constants.

Lemma 1 *For the classical solutions to* (1)–(5) *the following relations hold:*

$$\dot G = F_1, \qquad \dot F_2 = lF_1 - \mu F_2,$$

$$\dot F_1 = 2(\gamma-1)E_p + 2E_k - lF_2 - \mu F_1,$$

$$\dot E = -2\mu E_k, \tag{7}$$

where $E_k(t) = \int\limits_{\Omega(t)} \frac{\rho|U_H|^2}{2}\,dx_1dx_2,\ E_k(t) = \int\limits_{\Omega(t)} \frac{1}{\gamma-1} p\,dx_1dx_2.$

Proof To prove the identities it is enough to apply formula (6) with respect to the variables x_2 and x_2. For example, taking into account (1), we get

$$\frac{dG}{dt} = \frac{1}{2}\int\limits_{\Omega(t)} \partial_t\rho|X_1|^2\,dx_1dx_2 =$$

$$\int\limits_{\Omega(t)} \Big(-\frac{1}{2}\operatorname{div}(\rho\,U)\,|X_1|^2 + \frac{1}{2}\operatorname{div}\big(\rho\,U_H\,|X_1|^2)\big)\Big)\,dx_1dx_2 =$$

$$= \int_{\Omega(t)} (X_1, U)\, \rho\, dx_1 dx_2 = F_1.$$

The proof of other identities are analogous. It is convenient to take into account that (2)–(4) imply

$$\partial_t p + (U, \nabla p) + \gamma\, p\, \mathrm{div} U = 0. \tag{8}$$

□

System (7) is not closed. Nevertheless, we can use a special assumption on the velocity structure inside $\Omega(t)$ to close it.

Namely, we set

$$U_H = Q\mathbf{x}, \quad Q = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad V_3 = 0. \tag{9}$$

Lemma 2 *For the velocity* (9) *we have*

$$\dot{G}_{x_1} = 2aG_{x_1} + 2bG_{x_1x_2}, \quad \dot{G}_{x_2} = 2dG_{x_2} + 2cG_{x_1x_2},$$

$$\dot{G}_{x_1x_2} = (a+d)G_{x_1x_2} + bG_{x_2} + cG_{x_1},$$

$$\dot{E}_p = -(\gamma - 1)(a+d)E_p, \quad \dot{\Delta} = 2(a+d)\Delta.$$

The potential energy E_p is connected with Δ as

$$E_p(t,x) = E_p(0,x)\Delta^{(\gamma-1)/2}(0)\Delta^{(-\gamma+1)/2}(t,x).$$

Proof The proof is a direct computation with taking into account formula (6). The expression for potential energy can be obtained my means of (8).

Let us introduce new functions

$$G_1 = G_{x_1}\Delta^{-(\gamma+1)/2}, \quad G_2 = G_{x_2}\Delta^{-(\gamma+1)/2}, \quad G_3 = G_{x_1x_2}\Delta^{-(\gamma+1)/2}.$$

Lemmas 1 and 2 imply that for the elements of the matrix Q and G_1, G_2, G_3 the following closed system of equations can be obtained:

$$\begin{aligned}
\dot{G}_1 &= ((1-\gamma)a - (1+\gamma)d)G_1 + 2bG_3, \\
\dot{G}_2 &= ((1-\gamma)d - (1+\gamma)a)G_2 + 2cG_3, \\
\dot{G}_3 &= cG_1 + bG_2 - \gamma(a+d)G_3, \\
\dot{a} &= -a^2 - bc + lc - \mu a - \mathscr{K}G_2, \\
\dot{b} &= -b(a+d) + ld - \mu b_H + \mathscr{K}G_3, \\
\dot{c} &= -c(a+d) - la - \mu c_H + \mathscr{K}G_3,
\end{aligned} \tag{10}$$

$$\dot{d} = -d^2 - bc - lb - \mu d - \mathscr{K} G_1,$$

with $\mathscr{K} = -\frac{\gamma-1}{2} E_p \Delta^{(\gamma-1)/2}|_{t=0}$.

This system for the components of the matrices Q and $R = \begin{pmatrix} A(t) & \frac{1}{2}B(t) \\ \frac{1}{2}B(t) & C(t), \end{pmatrix}$ can be written as:

$$\begin{aligned} \dot{R} + RQ + Q^T R + (\gamma - 1)\mathbf{tr} QR = 0, \\ \dot{Q} + Q^2 + lLQ + \mu Q + 2c_0 R = 0, \end{aligned} \tag{11}$$

where $A = G_2$, $B = -2G_3$, $C = G_1$,

The same system of ODEs can be obtained in the barotropic model. Indeed, if $P = C\rho^\gamma$, $C = \text{const}$ and $\pi = P^{\frac{\gamma-1}{\gamma}}$, the system under consideration can be reduced to two equations

$$\begin{aligned} \partial_t U + (U \cdot \nabla) U + (l\, L + \mu\, I) U + c_0\, \nabla \pi &= 0, \\ \partial_t \pi + (\nabla \pi \cdot U) + (\gamma - 1)\, \pi \operatorname{div} U &= 0, \end{aligned}$$

with $c_0 = \frac{\gamma}{\gamma-1} C^{\frac{1}{\gamma}}$. Here $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, I is the identity matrix.

We consider a simple class of exact solutions which correspond to the first terms of expansion of the local field π at a critical point and look for the solution of form (9),

$$\pi(t, \mathbf{x}) = A(t)x_1^2 + B(t)x_1x_2 + C(t)x_2^2 + \Pi(t).$$

On this way we get (11) with $\mathscr{K} = 2c_0$. The system of matrix equations consists of 7 nonlinear ODEs. Π does not appear there, this component can be found from a separate linear equation.

1.1 A Friction-Free Vortex ($\mu = 0$)

Let us make some review of known results about the non-frictional case [8, 10].

1.1.1 Axisymmetric Case

It is easy to see that (11) has a closed submanifold of solutions having additional properties $a = d, c = -b, A = C, B = 0$. These solutions corresponds to the axisymmetric motion. Note that it is the most interesting case related to the vortex in atmosphere. Here we get a system of 3 ODEs:

$$\begin{aligned} \dot{A} + 2\gamma a A &= 0, \\ \dot{a} + a^2 - b^2 + lb + 2c_0 A &= 0, \\ \dot{b} + 2ab - la &= 0. \end{aligned} \tag{12}$$

The functions a, b, A correspond to one half of divergence, one half of vorticity and the fall of pressure in the center of vortex, respectively. The only nontrivial equilibrium point that relates to a vortex motion is

$$a = 0, \; b = -c = b^*, \; A = A* = \frac{b^*(b^* - l)}{2c_0}. \tag{13}$$

Further, there exists one first integral

$$b = \frac{l}{2} + C|A|^{\frac{1}{\gamma}}, \tag{14}$$

where C is a constant. Thus, (12) can be reduced to the following system:

$$\begin{aligned} \dot{A} &= -2\gamma a A, \\ \dot{a} &= -a^2 - \frac{l^2}{4} + C^2 A^{\frac{2}{\gamma}} - 2c_0 A. \end{aligned}$$

For $A > 0$ the system has the unique equilibrium, it is stable in the Lyapunov sense.

1.1.2 General Case

Theorem 1 *If*

$$b^* < \frac{1 - \sqrt{2}}{2} l \quad or \quad b^* > \frac{1 + \sqrt{2}}{2} l > l,$$

then the equilibrium of system (11) is unstable.

Proof The point (13) is the only equilibrium of the full system (11). It is the same point of equilibrium as in the axisymmetric case (12). Nevertheless, in the symmetric case this equilibrium is always stable in the Lyapunov sense, whereas in the general case the situation is different. Indeed, the eigenvalues of matrix corresponding to the linearization at the equilibrium point are the following:

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{-(2(2-\gamma)b^*(b^* - l) + l^2)},$$

$$\lambda_{4,5,6,7} = \pm\sqrt{2}\sqrt{-l\left(b^* + \frac{l}{4}\right) \pm \sqrt{\left(b^* + \frac{l}{2}\right)^2 \left(\frac{l^2}{4} + b^* l - (b^*)^2\right)}}.$$

Since $(2-\gamma)b^*(b^*-l)+l^2>0$ for $\gamma\in(1,2)$, then $\Re(\lambda_{2,3})=0$. Eigenvalues λ_i, $i=4,5,6,7$ have zero real part if and only if b^* satisfies the following inequalities simultaneously: $l(b^*+\frac{l}{4})\ge 0$, $\frac{l^2}{4}+b^*l-(b^*)^2>0$, $l^2\left(b^*+\frac{l}{4}\right)^2>\left(b^*+\frac{l}{2}\right)^2\left(\frac{l^2}{4}+b^*l-(b^*)^2\right)$, that is $b^*\in\left[\frac{1-\sqrt{2}}{2}l,\ \frac{1+\sqrt{2}}{2}l\right]$. For others values of b^* the eigenvalues $\lambda_{4,5,6,7}=\pm\alpha\pm i\beta$, $\alpha\ne 0$, $\beta\ne 0$, therefore there exist an eigenvalue with a positive real part. Thus, the Lyapunov theorem implies instability of the equilibrium for $b^*<\frac{1-\sqrt{2}}{2}l$ and $b^*>\frac{1+\sqrt{2}}{2}l>l$. □

Let us recall the following properties of solution to (11).

Theorem 2 *System* (11) *has three first integrals:*

$$(b-c-l)\mathscr{D}^{-\frac{1}{2\gamma}}=I_1, \tag{15}$$

$$((d-a)B+2bA-2cC-l(A+C))\mathscr{D}^{-\frac{\gamma+1}{2\gamma}}=I_2, \tag{16}$$

$$((a^2+c^2)C+(b^2+d^2)A+(ac+bd)B-\frac{4c_0}{\gamma-1}\mathscr{D})\mathscr{D}^{-\frac{\gamma+1}{2\gamma}}=I_3, \tag{17}$$

where $\mathscr{D}=AC-B^2/4$.

Theorem 3 *The equilibrium* (13) *is nonlinearly stable in the Lyapunov sense for* $0<b^*<l$.

To proof it is enough to construct the Lyapunov function. The function $\Lambda(a,b,c,d,A,B,C)=b^*I_2-I_3-\Lambda_0$, where I_2 and I_3 are given by equalities (16) and (17), the constant Λ_0 is the value of Λ at the equilibrium point (13), satisfies all necessary properties.

First integral (15) reduces the full system (11) to the system of 6 equations. If $b^*\in\Sigma$, $\Sigma=\left(\frac{1-\sqrt{2}}{2}l,0\right)\cup\left(l,\frac{1+\sqrt{2}}{2}l\right)$, then the matrix, corresponding to the system, linearized at the equilibrium, has 3 pairs of pure imaginary complex conjugate roots λ_i, $i=2,\dots,7$ (for the range of parameters under consideration the roots are simple). It can be proved that in the general case of rationally independent frequencies almost all trajectories in ϵ-neighborhood of the equilibrium are quasi-periodic. This means that the equilibrium is "practically" stable in the Lyapunov sense. The cases of possible resonant frequencies correspond to several values of b^*, very close to the boundaries of Σ, see [10] for details.

Instability of the equilibrium $(0,b_*,A_*)$ in the cases $b_*=0$ and $b_*=l$ can be proven directly, since here $A_*=0$.

1.2 Influence of the Friction on an Axisymmetric Vortex

The system of equations, describing a vortex with a rotational symmetry is the following:

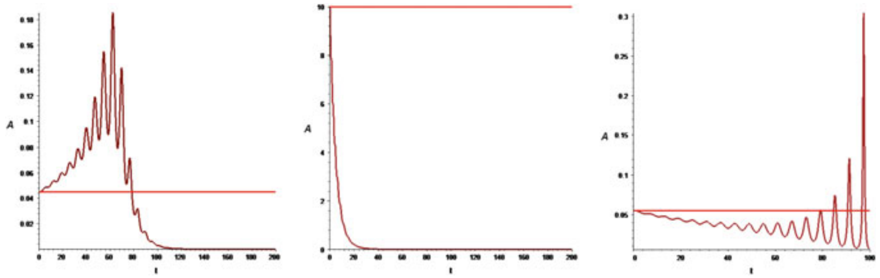

Fig. 1 Influence of friction on the steady solution (the graph is a straight line) on example of component A. Here $l = 1$, $\mu = 0.05$, $\gamma = 9/7$. Given b^* and c_0, the value of A^*, corresponding to the equilibrium, is found from (13). Left: $b^* = 0.1$, $c_0 = -1$. Center: $b^* = 5$, $c_0 = 1$. Right: $b^* = -0.1$, $c_0 = 1$

$$\begin{aligned} \dot{A} + 2\gamma a A &= 0, \\ \dot{a} + a^2 - b^2 + lb + 2c_0 A &= -\mu a, \\ \dot{b} + 2ab - la &= -\mu b. \end{aligned} \tag{18}$$

The solution to the equation has a complicated oscillating behavior. Influenced by a small friction, the vortex, which is stable at $\mu = 0$, can quickly decay or develop increasing oscillations. The oscillations in its turn can decay or not (a typical behavior one can see in Fig. 1). Nevertheless it is possible to study it analytically to a certain extent.

Theorem 4 *System (18) has two equilibriums* $(a_1^*, b_1^*, A_1^*) = (0, 0, 0)$ *and* $(a_2^*, b_2^*, A_2^*) = (-\mu, l, 0)$*, both are unstable.*

Proof Indeed, the matrix of the system linearized at the point (A_0, a_0, b_0) is

$$\mathscr{L}(A_0, a_0, b_0) = \begin{pmatrix} -2\gamma a_0 & -2\gamma A_0 & 0 \\ -2c_0 & -2a_0 - \mu & 2b_0 - l \\ 0 & -2b_0 + l & -2a_0 - \mu \end{pmatrix}.$$

The eigenvalues of $\mathscr{L}(0, -\mu, l)$ solve the equation

$$(2\gamma\mu - k)((\mu - k)^2 + l^2) = 0.$$

The polynomial has a positive root, this means instability of equilibrium $(-\mu, l, 0)$. The eigenvalues of $\mathscr{L}(0, 0, 0)$ are $(0, -\mu \pm il)$, therefore the linearized theory does not give an answer on the stability or instability of zero equilibrium. However in the critical case we can use the theory of [5], Sect. 4. Namely, we consider expansions into series $a(A) = a_1 A + O(A^2)$, $b(A) = b_1 A + O(A^2)$ as $A \to 0$, substitute the expansions into (18) and get $a_1 = -\frac{2\mu c_0}{\mu^2 + l^2}$, $b_1 = -\frac{2lc_0}{\mu^2 + l^2}$, therefore $\dot{A} = \frac{4\gamma\mu c_0}{\mu^2 + l^2} A^2 + O(A^2)$. This implies instability of zero equilibrium. □

Remark 1 Theorem 4 implies that the the zero equilibrium of the full system (11) is also unstable.

Theorem 5 *Let $c_0 \neq 0$ and $A^* > 0$. Then solutions to system (18) has no finite time blow up points at $t > 0$. Moreover, the following estimates hold:*

- *for $c_0 > 0$*

$$A(t) \leq K_+ e^{\frac{2\mu\gamma}{2-\gamma}t}, \tag{19}$$

- *for $c_0 < 0$*

$$A(t) \leq K_-, \tag{20}$$

with positive constants K_+ and K_-, depending only on initial data.

Proof 1. $c_0 > 0$. Let us denote $\Lambda = \frac{a^2+b^2}{2}$. The first equation and two latter equations of (18) imply

$$\dot{A}^{-1/\gamma} - 2aA^{-1/\gamma} = 0 \tag{21}$$

and

$$\dot{\Lambda} + 2a\Lambda + 2c_0 aA + 2\mu\Lambda = 0, \tag{22}$$

respectively. Equations (21) and (22) result

$$\frac{d}{dt}\left(\Lambda A^{-\frac{1}{\gamma}} - \frac{c_0}{\gamma - 1}A^{\frac{\gamma-1}{\gamma}}\right) = -2\mu\Lambda A^{-\frac{1}{\gamma}} \leq 0. \tag{23}$$

From (23) we obtain

$$\Lambda \leq \frac{c_0}{\gamma - 1}A + k_0 A^{\frac{1}{\gamma}} e^{-2\mu t}, \quad k_0 > 0. \tag{24}$$

Inequality (24) implies that there exists a constant $\bar{A}$, depending on initial data such that for $A > \bar{A}$ we have

$$\Lambda \leq k_1 A \tag{25}$$

with a positive constant k_1. Further, (25) and the first equation of (18) imply $\dot{A} \leq k_2 A^{3/2}$, for some $k_2 > 0,\ t < t_* = \frac{1}{k_3}$, and

$$A \leq \frac{A(0)}{(1 - k_3 t)^2},\ k_3 > 0. \tag{26}$$

Thus, we get a rough upper bound for a possible growth rate for A. To refine the upper estimate we introduce a new variable $H(t) = (b - \frac{l}{2})A^{-\frac{1}{\gamma}}e^{\mu t}$. It is easy to check that

$$\dot{H} = -\frac{l\mu}{2}A^{-\frac{1}{\gamma}}e^{\mu t} \leq 0. \tag{27}$$

As follows from (27), $H(t) \le H(0)$. The second equation of (18) takes the form

$$\dot{a} = -a^2 - \mu a + H^2 A^{2/\gamma} e^{-2\gamma\mu t} - 2c_0 A - \frac{l^2}{4}. \tag{28}$$

First, we consider the cases $l\mu = 0$, where $H(t) = H(0)$, and $b(0) < \frac{l}{2}$ (or $H(0) < 0$), for $l\mu \neq 0$. Then (28) and (25) imply that for sufficiently large A we have

$$\dot{a} \ge H^2(0) A^{2/\gamma} e^{-2\gamma\mu t} - k_4 A, \quad k_4 > 0.$$

Thus, if there exits an interval of t such that the inequality

$$A^{\frac{2-\gamma}{\gamma}} > k_5 e^{2\mu t} \tag{29}$$

holds with a positive constant k_5, depending only on initial data, then for these t the function $a(t)$ increases and, as follows from the first equation of (18), $A(t)$ decreases. Thus $A(t)$ can increase if and only if inequality (19), opposite to (29), holds.

The last case is $l\mu \neq 0$, $b(0) \ge \frac{l}{2}$ or $H(0) \ge 0$. Here we use the rough upper estimate for $A(t)$, (26), to obtain from (27)

$$\dot{H} \le -\frac{l\mu}{2} A^{-\frac{1}{\gamma}} \le -k_5(1 - k_3 t)^{\frac{2}{\gamma}}, \quad k_5 > 0,\ t < t_*.$$

Integrating this inequality we can see that $H(t_*) \le H(0) - \frac{\gamma k_5}{k_3(\gamma+2)} < H(0)$. Let $t = t_*$ be the initial moment of time. If $H(t_*) < 0$ we get estimates (19) and (24) as before. If $H(t_*) > 0$, we apply the estimate (26) to inequality (27) again. One can see that at a finite step n, we get $H(nt_*) < 0$.

2. $c_0 < 0$. In this case we are in the frame of the model described by (10) and the balance of energy $E'(t) = -\mu E_k$, obtained for a smooth solution in a moving volume (see Lemma 1) prevents an unbounded growth of A. Indeed, if the velocity field has the form (9), $a = d$, $c = -b$, then $E_k = (a^2 + b^2)G \ge 0$, $E_p = \delta_1 G^{1-\gamma} \ge 0$, $E_k + E_p \le \delta_2$, where δ_1 and δ_2 are positive constants, depending on initial data. This implies $G \ge \left(\frac{\delta_1}{\delta_2}\right)^{\frac{1}{\gamma-1}} := \delta_3 > 0$. However, as follows from the results of Sect. 1, $A = G_1 = G^{-\gamma} \le \delta_3^{-\gamma}$. Thus, (20) is proved. □

Remark 2 For $\mu = 0$ inequalities (19) and (24) imply that the solution to system (12) for $c_0 > 0$ is bounded for all $t > 0$ by a constant depending on initial data.

Remark 3 As follows from (27), the value of $H(t)$ is constant for $l\mu = 0$. From the conservation of H for $\mu = 0$ we get integral (14).

Remark 4 If $c_0 = 0$, then (18) splits into two part, one of them is (30), solved in Sect. 1.4. Thus, as follows from the explicit form of solution, a can blow up within a finite time. First equation of (18) implies that in his case A blows up, too.

1.3 Small μ Expansion

For small μ solution to the system (18) can be expanding into a convergent Taylor series with respect to parameter μ. The proof of this fact is standard [6]. Let us take as a zero approximation a steady state solution to (11) and find the first term of the expansion. Thus, $A(t) = A_0(t) + \mu A_1(t) + o(\mu)$, $a(t) = a_0(t) + \mu a_1(t) + o(\mu)$, $b(t) = b_0(t) + \mu b_1(t) + o(\mu)$. Functions $A_1(t), a_1(t), b_1(t)$ satisfy the following linear system of ODEs:

$$\dot{A}_1 = -2\gamma a_1 A^*,$$
$$\dot{a}_1 = (2b^* - l)b_1(t) - 2c_0 A_1,$$
$$\dot{b}_1 = (l - 2b^*)a_1 - b^*,$$

subject to initial conditions $A_1(0) = 0$, $a_1(0) = 0$, $b_1(0) = 0, b^*$ and A^* correspond to the stationary point (13). The solution is $a_1(t) = \frac{b^* d}{\beta}(1 - \cos\sqrt{\beta}t)$, $A_1(t) = \frac{2\gamma A^* b^* d}{\beta}\left(\frac{\sin\sqrt{\beta}t}{\sqrt{\beta}} - t\right)$, $b_1(t) = b^*\left(\frac{d^2}{\beta} - 1\right)t - \frac{b^* d^2}{\beta\sqrt{\beta}}\sin\sqrt{\beta}t$, where $d = l - 2b^* = -2C|A^*|^{\frac{1}{\gamma}}$, $\beta = d^2 - 4c_0\gamma A^*$. The constant β is positive, since $\beta = 4(C^2 |A^*|^{\frac{2}{\gamma}} - c_0\gamma A^*) > 4(C^2|A^*|^{\frac{2}{\gamma}} - \frac{l^2}{4} - 2c_0 A^*) = 0$.

1.4 Special Class of Solutions to (18) for $A(t) \equiv 0$

For $A(t) \equiv 0$ the system (18) can be explicitly solved. Indeed, it takes the form

$$\dot{a} = -a^2 + b^2 - lb - \mu a,$$
$$\dot{b} = -2a\,b + la - \mu b. \tag{30}$$

If we introduce the new complex variable $z = a + ib$, we can rewrite (30) as $\dot{z} = -z^2 + Kz$, where $K = il - \mu$. Thus, $z = \dfrac{K}{1 + CKe^{-Kt}}$, where $C = C_1 + iC_2$, C_1, C_2 are real constants. Taking real and imaginary part of z we get

$$a = \frac{1}{D}\big((\mu^2 + l^2)(C_1 \cos lt + C_2 \sin lt)\, e^{\mu t} - \mu\big),$$

$$b(t) = \frac{1}{D}\big((\mu^2 + l^2)(C_1 \sin lt - C_2 \cos lt)e^{\mu t} + l\big),$$

$$D = 2((C_1 l - C_2\mu)\sin lt - (C_1\mu + C_2 l)\cos lt)\, e^{\mu t} + (\mu^2 + l^2)(C_1^2 + C_2^2)e^{2\mu t} + 1.$$

Proposition 1 *System (30) has two equilibria:* $(a_1^*, b_1^*) = (0, 0)$ *(stable focus) and* $(a_2^*, b_2^*) = (-\mu, l)$ *(unstable focus).*

The equilibria can be found from equation $-z^2 + ilz - \mu z = 0$, i.e. $z_1^* = 0$ and $z_2^* = \mu - il$. Thus, in variables a, b we get $(a_1^*, b_1^*) = (0, 0)$ and $(a_2^*, b_2^*) = (-\mu, l)$. Matrix of linearization at some point (a_0, b_0) is

$$\mathscr{L}(a_0, b_0) = \begin{pmatrix} -2a_0 - \mu & 2b_0 - l \\ -2b_0 + l & -2a_0 - \mu \end{pmatrix}.$$

Eigenvalues of $\mathscr{L}(0, 0)$ and $\mathscr{L}(-\mu, l)$ are $-\mu \pm il$ and $\mu \pm il$, respectively. □

Conclusion

We considered a special class of solutions of the gas dynamics equations in the rotating reference frame. It is characterized by the linear profile of velocity. We studied the influence of small friction on the stationary vortex from the above class of solutions. In particular, we showed that the presence of friction in this system does not necessarily lead to depletion of the vortex, as intuition suggests. Sometimes the vortex demonstrates the appearance of strong oscillations. They can decay to zero after some period or continue to rise. A similar influence of friction is known in the nonlinear solid mechanics [2]. Nevertheless, the solution will never blow up within a finite time.

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Division by Zero Calculus and Differential Equations

Sandra Pinelas and Saburou Saitoh

Abstract In this paper, we will show and give applications of the division by zero $z/0 = 1/0 = 0/0 = 0$ in calculus and differential equations. In particular, we will know that the division by zero is our elementary and fundamental mathematics.

Keywords Division by zero calculus · Singularity · Derivative
Differential equation · Division by zero · $0/0 = 1/0 = z/0 = 0$
Point at infinity · Infinity · Gradient · Laurent expansion

1 Introduction

By a **natural extension** of the fractions

$$\frac{b}{a} \tag{1}$$

for any complex numbers a and b, we found the simple result, for any complex number b

$$\frac{b}{0} = 0, \tag{2}$$

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incidentally in [20] by the Tikhonov regularization for the Hadamard product inversions for matrices and we discussed their properties and gave several physical interpretations on the general fractions in [10] for the case of real numbers. The result is a very special case for general fractional functions in [5].

The division by zero has a long and mysterious story over the world (see, for example, H.G. Romig [18] and Google site with the division by zero) with its physical viewpoints since the document of zero in India on AD 628. In particular, note that Brahmagupta (598–668?) established the four arithmetic operations by **introducing** 0 **and at the same time he defined as** $0/0 = 0$ in Brāhmasphuṭasiddhānta. Our world history, however, stated that his definition $0/0 = 0$ is wrong over 1300 years. We will see that his definition is right and suitable.

Indeed, we will show typical examples:

The conditional probability $P(A|B)$ for the probability of A under the condition that B happens is given by the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$, then, of course, $P(A \cap B) = 0$ and $P(A|B) = 0$ and so, $0/0 = 0$.

For the differential equation

$$\frac{dy}{dx} = \frac{2y}{x},$$

we have the general solution with constant C

$$y = Cx^2.$$

At the origin $(0, 0)$ we have

$$y'(0) = \frac{0}{0} = 0.$$

We have many and many concrete examples.

However, we do not know the reason and motivation of the definition of $0/0 = 0$, furthermore, for **the important case** $1/0$ we do not know any result there.

Meanwhile, Sin-Ei Takahasi [10] established a simple and decisive interpretation (2) by analyzing the extensions of fractions and by showing the complete characterization for the property (2):

Proposition 1 *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ satisfying the product property*

$$F(b, a)F(c, d) = F(bc, ad)$$

for all

$$a, b, c, d \in \mathbf{C}$$

and

$$F(b,a) = \frac{b}{a}, \quad a, b \in \mathbf{C}, a \neq 0.$$

Then, we obtain, for any $b \in \mathbf{C}$

$$F(b,0) = 0.$$

Note that the complete proof of this proposition is simply given by 2 or 3 lines in [10].

We thus should consider, for any complex number b, as (2); that is, for the mapping

$$W = \frac{1}{z}, \tag{3}$$

the image of $z = 0$ is $W = 0$ (**should be defined**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere [1]. As the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a **strong discontinuity at the point of infinity** on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [6–8]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see [13, 14] for example. Therefore, the division by zero will give great impacts to complex analysis and to our basic ideas for the space and universe.

However, the division by zero (2) is now clear, indeed, for the introduction of (2), we have several independent approaches as in:

(1) by the generalization of the fractions by the Tikhonov regularization or by the Moore-Penrose generalized inverse,
(2) by the intuitive meaning of the fractions (division) by H. Michiwaki,
(3) by the unique extension of the fractions by S. Takahasi, as in the above,
(4) by the extension of the fundamental function $W = 1/z$ from $\mathbf{C}\setminus\{0\}$ into $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C}\setminus\{0\}$ onto $\mathbf{C}\setminus\{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$,
and
(5) by considering the values of functions with the mean values of functions.

Furthermore, in [12] we gave the results in order to show the reality of the division by zero in our world:

(A) a field structure containing the division by zero — the Yamada field **Y**,
(B) by the gradient of the y axis on the (x, y) plane — $\tan \frac{\pi}{2} = 0$,
(C) by the reflection $W = 1/\overline{z}$ of $W = z$ with respect to the unit circle with center at the origin on the complex z plane — the reflection point of zero is zero, and
(D) by considering rotation of a right circular cone having some very interesting phenomenon from some practical and physical problem.

Furthermore, in [13, 14, 19, 20, 22], we discussed many division by zero properties in the Euclidean plane. In [11], we gave geometrical interpretations of determinants from the viewpoint of the division by zero.

See also J.A. Bergstra, Y. Hirshfeld and J.V. Tucker [4] and J.A. Bergstra [3] for the relationship between fields and the division by zero, and the importance of the division by zero for computer science. It seems that the relationship of the division by zero and field structures are abstract in their paper.

Meanwhile, J.P. Barukcic and I. Barukcic [2] discussed the relation between the division $0/0$ and special relative theory of Einstein. However it seems that their results are curious with their logics. Their results contradict with ours.

Furthermore, T.S. Reis and J.A.D.W. Anderson [16, 17] extend the system of the real numbers by defining division by zero with three infinities $+\infty, -\infty. \Phi$. Could we accept their theory as a natural one? They introduce a curious ideal number for the division $0/0 = \Phi$.

Here, we recall Albert Einstein's words on mathematics: **Blackholes are where God divided by zero.** I don't believe in mathematics. George Gamow (1904–1968) Russian-born American nuclear physicist and cosmologist remarked that "it is well known to students of high school algebra" that division by zero is not valid; and Einstein admitted it as **the biggest blunder of his life** (Gamow, G., My World Line (Viking, New York). p 44, 1970).

In this paper, we will discuss the division by zero in calculus and differential equations, and we will be able to see that the division by zero is our elementary and fundamental mathematics.

In particular, we would like to express our deep thanks Dr. Masako Takagi who initially considered the applications of the division by zero to differential equations.

2 Calculation by Division by Zero

As the number system containing the division by zero, the Yamada structure is complete, however for applications of the division by zero to functions, we will need the concept of division by zero calculus for the sake of uniquely determinations of the results. See [13] for examples:

For any formal Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n, \tag{4}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{5}$$

Note that here, there is no problem on any convergence of the expansion (4) at the point $z = a$. (Here, as convention, we consider $0^0 = 1$.) For the correspondence (5) for the function $f(z)$, we will call it **the division by zero calculus**. By considering the formal derivatives in (4), we can define any order derivatives of the function f at the singular point a.

In order to avoid any logical confusion in the division by zero, we would like to refer to the logical essence:

For the elementary function $W = f(z) = 1/z$, we define $f(0) = 0$ and we will write it by $1/0$ following the form, apart from the intuitive sense of fraction. With only this new definition, we can develop our mathematics, through the division by zero calculus.

We will give typical and various examples.

For the typical function $\sin x/x$, we have

$$\frac{\sin x}{x}(0) = \frac{\sin 0}{0} = \frac{0}{0} = 0,$$

however, by the division by zero calculus, we have, for the function $(\sin x)/x$

$$\frac{\sin x}{x}(0) = 1,$$

that is more reasonable in analysis.

However, for functions we see that the results by the division by zero calculus have not always practical senses and so, for the results by division by zero we should check the results, case by case, see many examples, [13].

For the function

$$f(x) = x \sin \frac{1}{x},$$

if $f(0) = 0$, then the function is continuous at $x = 0$, however, it is not differentiable at the origin. By the division by zero calculus, we have, automatically

$$f(0) = 1.$$

We will show division by zero calculus examples:

1.
$$\left[\frac{(ax+b)^{n+1}}{a\,(n+1)}\right]_{n=-1} = \int (ax+b)^{-1}\,dx = \frac{\ln|ax+b|}{a},\ a \neq 0;$$

and
$$\left[\frac{(ax+b)^{n+1}}{a\,(n+1)}\right]_{a=0} = \int b^n dx = b^n x.$$

2.
$$\left[\frac{\arctan(x/a)}{a}\right]_{a=0} = \int \frac{1}{x^2}dx = -\frac{1}{x}.$$

3.
$$\left[\frac{a^x}{\log a}\right]_{a=1} = \int dx = x.$$

4. For the integral
$$\int x(x^2+1)^a dx = \frac{(x^2+1)^{a+1}}{2(a+1)} \quad (a \neq -1), \tag{6}$$
we obtain, by the division by zero,
$$\int x(x^2+1)^{-1}dx = \frac{\log(x^2+1)}{2}. \tag{7}$$

5. For the integral
$$\int \sin ax \cos x dx = \frac{\sin ax \sin x + a \cos ax \cos x}{1-a^2} \quad (a^2 \neq 1), \tag{8}$$
we obtain, by the division by zero, for the case $a = 1$
$$\int \sin x \cos x dx = \frac{\sin^2 x}{2} - \frac{1}{4}. \tag{9}$$

6. For the integral
$$\int \sin^{\alpha-1} x \cos(\alpha+1)x dx = \frac{1}{\alpha}\sin^{\alpha} x \cos \alpha x, \tag{10}$$
we obtain, by the division by zero, for the case $\alpha = 0$
$$\int \sin^{-1} x \cos x dx = \log \sin x. \tag{11}$$

7. Meanwhile, for many generating functions we can obtain some interesting identities. For example, we will consider the mapping

$$\zeta \in C \setminus \{0\} \to F(z,\zeta) := \exp \frac{z}{2}\left(\zeta - \frac{1}{\zeta}\right).$$

Then, from

$$F(z,\zeta) = \sum_{n=-\infty}^{+\infty} J_n(z)\zeta^n, \tag{12}$$

we obtain:

$$F(z,0) = J_0(z).$$

8. **Difficulty in Maple for specialization problems**
For the Fourier coefficients a_k of a function:

$$\frac{a_k \pi k^3}{4}$$

$$= \sin(\pi k)\cos(\pi k) + 2k^2\pi^2 \sin(\pi k)\cos(\pi k) + 2\pi(\cos(\pi k))^2 - \pi k, \tag{13}$$

for $k = 0$, we obtain, by the division by zero calculus, immediately

$$a_0 = \frac{8}{3}\pi^2 \tag{14}$$

(see [9], (3.4)).

9. **Reproducing kernels**
The function

$$K_{a,b}(x,y) = \frac{1}{2ab}\exp\left(-\frac{b}{a}|x-y|\right)$$

is the reproducing kernel for the space $H_{K_{a,b}}$ equipped with the norm

$$\|f\|^2_{H_{K_{a,b}}} = \int (a^2 f'(x)^2 + b^2 f(x)^2)dx$$

([21], pp. 15–16). If $b = 0$, then

$$K_{a,0}(x,y) = -\frac{1}{2a^2}|x-y|$$

is the reproducing kernel for the space $H_{K_{a,0}}$ equipped with the norm

$$\|f\|^2_{H_{K_{a,b}}} = a^2 \int (f'(x)^2 dx.$$

Meanwhile, if $a = 0$, $K_{0,b}(x, y) = 0$, then it is the trivial reproducing kernel for the zero function space.

3 Derivatives of a Function

On derivatives, we obtain new concepts, from the division by zero.

From the viewpoint of the division by zero, when there exists the limit, at x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \infty \tag{15}$$

or

$$f'(x) = -\infty, \tag{16}$$

both cases, we can write them as follows:

$$f'(x) = 0. \tag{17}$$

This property was derived from the fact that the gradient of the y axis is zero; that is,

$$\tan \frac{\pi}{2} = 0, \tag{18}$$

that was derived from many geometric properties in [13], and also from the formal way from the result $1/0 = 0$.

We will look this fundamental result by elementary functions. For the function

$$y = \sqrt{1 - x^2}, \tag{19}$$

$$y' = \frac{-x}{\sqrt{1 - x^2}}, \tag{20}$$

and so, by the division by zero calculus,

$$[y']_{x=1} = 0, \quad [y']_{x=-1} = 0. \tag{21}$$

Of course, depending on the context, we should refer to the derivatives of a function at a point from the right hand direction and the left hand direction.

Here, note that, for $x = \cos\theta$, $y = \sin\theta$,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1} = -\cot\theta.$$

Note also that from the expansion

$$\cot z = \frac{1}{z} + \sum_{\nu=-\infty, \nu\neq 0}^{+\infty} \left(\frac{1}{z - \nu\pi} + \frac{1}{\nu\pi} \right) \tag{22}$$

or the Laurent expansion

$$\cot z = \sum_{n=-\infty}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

we have, by the division by zero calculus,

$$\cot 0 = 0.$$

Note that in (22), since

$$\left(\frac{1}{z - \nu\pi} + \frac{1}{\nu\pi} \right)_{\nu=0} = \frac{1}{z}, \tag{23}$$

we can write it simply

$$\cot z = \sum_{\nu=-\infty}^{+\infty} \left(\frac{1}{z - \nu\pi} + \frac{1}{\nu\pi} \right). \tag{24}$$

The differential equation

$$y' = -\frac{x}{y} \tag{25}$$

with a general solution

$$x^2 + y^2 = a^2 \tag{26}$$

is satisfied for all the points of the solutions by the division by zero, however, the differential equations

$$x + yy' = 0, \quad y' \cdot \frac{y}{x} = -1 \tag{27}$$

are not satisfied for all points of the solutions.

4 Continuation of Solution

We will consider the differential equation

$$\frac{dx}{dt} = x^2 \cos t. \tag{28}$$

Then, as the general solution, we obtain, for a constant C

$$x = \frac{1}{C - \sin t}. \tag{29}$$

For $x_0 \neq 0$, for any given initial value (t_0, x_0) we obtain the solution satisfying the initial condition,

$$x = \frac{1}{\sin t_0 + \frac{1}{x_0} - \sin t}. \tag{30}$$

If

$$\left| \sin t_0 + \frac{1}{x_0} \right| < 1, \tag{31}$$

then the solution has many poles and L.S. Pontrjagin stated in his book that the solution is disconnected by the poles and so, the solution may be considered as infinitely many solutions.

However, by the viewpoint of the division by zero, the solution takes the value zero at the singular points and the derivatives at the singular points are all zero; that is, the solution (30) may be understood as one solution.

Furthermore, by the division by zero, the solution (30) has its sense for even the case $x_0 = 0$ and it is the solution of (28) satisfying the initial condition $(t_0, 0)$.

We will consider the differential equation

$$y' = y^2. \tag{32}$$

For $a > 0$, the solution satisfying $y(0) = a$ is given by

$$y = \frac{1}{\frac{1}{a} - x}. \tag{33}$$

Note that the solution satisfies on the whole space $(-\infty, +\infty)$ even at the singular point $x = \frac{1}{a}$, in the sense of the division by zero, as

$$y'\left(\frac{1}{a}\right) = y\left(\frac{1}{a}\right) = 0. \tag{34}$$

5 Singular Solutions

We will consider the differential equation

$$(1 - y^2)dx = y(1 - x)dy. \tag{35}$$

By the standard method, we obtain the general solution, for a constant C ($C \neq 0$)

$$\frac{(x-1)^2}{C} + y^2 = 1. \tag{36}$$

By the division by zero, for $C = 0$, we obtain the singular solution

$$y = \pm 1,$$

like the singular solution $x = 1$.

For the simple Clairaut differential equation

$$y = px + \frac{1}{p}, \quad p = \frac{dy}{dx}, \tag{37}$$

we have the general solution

$$y = cx + \frac{1}{c}, \tag{38}$$

with a general constant c and the singular solution

$$y^2 = 4x. \tag{39}$$

Note that we have also the solution $y = 0$ from the general solution, by the division by zero $1/0 = 0$ from $c = 0$ in (38).

6 Solutions with Singularities

(1) We will consider the differential equation

$$y' = \frac{y^2}{2x^2}. \tag{40}$$

We will consider the solution with an isolated singularity at a point a with taking the value $-2a$ in the sence of division by zero.

First, by the standard method, we have the general solution, with a constant C

$$y = \frac{2x}{1 + 2Cx}. \tag{41}$$

From the singularity, we have, $C = -1/2a$ and we obtain the desired solution

$$y = \frac{2ax}{a - x}. \tag{42}$$

Indeed, from the expansion

$$\frac{2ax}{a - x} = -2a - \frac{2a^2}{x - a}, \tag{43}$$

we see that it takes $-2a$ at the point a in the sense of the division by zero calculus. This function was appeared in [12].

(2) For any fixed $y > 0$, we will consider the differential equation

$$E(x, y)\frac{\partial E(x, y)}{\partial x} = \frac{y^2 d^2}{(y - x)^3} \tag{44}$$

for $0 \leq x \leq y$. Then, note that the function

$$E(x, y) = \frac{y}{y - x}\sqrt{d^2 + (y - x)^2} \tag{45}$$

satisfies the differential equation (44) satisfying the condition

$$[E(x, y)]_{x=y} = 0, \tag{46}$$

in the sense of the division by zero. This function was appeared in showing a strong discontinuity of the curvature center (the inversion of EM diameter) of the circle movement of the rotation of two circles with radii x and y in [12].

(3) We will consider the singular differential equation

$$\frac{d^2 y}{dx^2} + \frac{3}{x}\frac{dy}{dx} - \frac{3}{x^2}y = 0. \tag{47}$$

By the series expansion, we obtain the general solution, for any constants a, b

$$y = \frac{a}{x^3} + bx. \tag{48}$$

We see that by the division by zero

$$y(0) = 0, y'(0) = b, y''(0) = 0. \tag{49}$$

The solution (48) has its sense and the Eq. (47) is satisfied even at the origin. The value $y'(0) = b$ may be given arbitrary, however, in order to determine the value a, we have to give some value for the regular point $x \neq 0$. Of course, we can give the information at the singular point with the Laurent coefficient a, that may be interpreted with the value at the singular point zero, with the division by zero. Indeed, the value a may be considered at the value

$$[y(x)x^3]_{x=0} = a. \tag{50}$$

(4) Next, we will consider the Euler differential equation

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0. \tag{51}$$

We obtain the general solution, for any constants a, b

$$y = \frac{a}{x} + \frac{b}{x^2}. \tag{52}$$

The solution (52) is satisfied even at the origin, by the division by zero and furthermore, all the derivatives of the solution of any order are all zero at the origin.

7 Solutions with an Analytic Parameter

For example, in the ordinary differential equation

$$y'' + 4y' + 3y = 5e^{-3x}, \tag{53}$$

in order to look for a special solution, by setting $y = Ae^{kx}$ we have, from

$$y'' + 4y' + 3y = 5e^{kx}, \tag{54}$$

$$y = \frac{5e^{kx}}{k^2 + 4k + 3}. \tag{55}$$

For $k = -3$, by the division by zero calculus, we obtain

$$y = e^{-3x}\left(-\frac{5}{2}x - \frac{5}{4}\right), \tag{56}$$

and so, we can obtain the special solution

$$y = -\frac{5}{2}xe^{-3x}. \tag{57}$$

For example, for the differential equation

$$y'' + a^2 y = b \cos \lambda x, \tag{58}$$

we have a special solution

$$y = \frac{b}{a^2 - \lambda^2} \cos \lambda x. \tag{59}$$

Then, when for $\lambda = a$, by the division by zero, we obtain the special solution

$$y = \frac{bx \sin(ax)}{2a} + \frac{b \cos(ax)}{4a^2}. \tag{60}$$

We can find many examples.

8 Special Reductions by Division by Zero of Solutions

We will consider the differential equation, for a constant R

$$y' = Ry.$$

Then, we have the solution

$$y(t) = y(0)e^{Rt}.$$

For the differential equation, for constants R, K

$$y' = Ry\left(1 - \frac{y}{K}\right),$$

we have the solution

$$y(t) = \frac{y(0)e^{Rt}}{1 + \frac{y(0)(e^{Rt}-1)}{K}}.$$

If $K = 0$, then, by the division by zero, we obtain the previous result, immediately.

We will consider the fundamental ordinary differential equations

$$x''(t) = g - kx'(t) \tag{61}$$

with the initial conditions

$$x(0) = -h, x'(0) = 0. \tag{62}$$

Then we have the solution

$$x(t) = \frac{g}{k}t + \frac{g(e^{-kt} - 1)}{k^2} - h. \tag{63}$$

Then, for $k = 0$, we obtain, immediately, by the division by zero calculus

$$x(t) = \frac{1}{2}gt^2 - h. \tag{64}$$

For the differential equation

$$x''(t) = g - k(x'(t))^2 \tag{65}$$

satisfying the same condition with (62), we obtain the solution

$$x(t) = \frac{1}{2k} \log \frac{\left(e^{2t\sqrt{kg}} + 1\right)^2}{4e^{2t\sqrt{kg}}} - h. \tag{66}$$

Then, for $k = 0$, we obtain

$$x(t) = \frac{1}{2} g t^2 - h. \tag{67}$$

immediately, by the division by zero calculus.

For the differential equation

$$x''(t) = -g + k(x'(t))^2 \tag{68}$$

satisfying the initial conditions

$$x(0) = 0, x'(0) = V, \tag{69}$$

we have

$$x'(t) = -\sqrt{\frac{g}{k}} \tan(\sqrt{kg} t - \alpha), \tag{70}$$

with

$$\alpha = \tan^{-1} \sqrt{\frac{k}{g}} V \tag{71}$$

and the solution

$$x(t) = \frac{1}{k} \log \frac{\cos\left(\sqrt{kg} t - \alpha\right)}{\cos \alpha}. \tag{72}$$

Then we obtain for $k = 0$, by the division by zero calculus

$$x'(t) = -gt + V \tag{73}$$

and

$$x(t) = -\frac{1}{2} g t^2 + V t. \tag{74}$$

We can find many and many such examples. However, note that the following fact.

For the differential equation

$$y''' + a^2 y' = 0, \tag{75}$$

we obtain the general solution, for $a \neq 0$

$$y = A \sin ax + B \cos ax + C. \tag{76}$$

For $a = 0$, from this general solution, how can we obtain the correspondent solution

$$y = Ax^2 + Bx + C, \tag{77}$$

naturally?

For the differential equation

$$y' = ae^{\lambda x} y^2 + afe^{\lambda x} y + \lambda f, \tag{78}$$

we obtain a special solution, for $a \neq 0$

$$y = -\frac{\lambda}{a} e^{-\lambda x}. \tag{79}$$

For $a = 0$, from this solution, how can we obtain the correspondent solution

$$y = \lambda f x + C, \tag{80}$$

naturally?

9 Partial Differential Equations

For the partial differential equation

$$\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + bx \frac{\partial w}{\partial x} + (cx + d)w, \tag{81}$$

we have a special solution

$$w(x, t) = \exp\left[-\frac{c}{b} x + \left(d + \frac{ac^2}{b^2} \right) t \right]. \tag{82}$$

For $b = 0$, how will be the correspondent solution? If $b = 0$, then $c = 0$ and

$$\frac{c}{b} = \frac{0}{0} = 0, \tag{83}$$

and we obtain the correspondent solution.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (be^{\beta t} + c)w, \tag{84}$$

we have special solutions

$$w(x,t) = (Ax + B)\exp\left[\frac{b}{\beta}e^{\beta t} + ct\right], \tag{85}$$

$$w(x,t) = A(x^2 + 2at)\exp\left[\frac{b}{\beta}e^{\beta t} + ct\right], \tag{86}$$

and

$$w(x,t) = A\exp\left[\lambda x + a\lambda^2 t + \frac{b}{\beta}e^{\beta t} + ct\right]. \tag{87}$$

Then, we see that for $\beta = 0$, by the interpretation

$$\left[\frac{1}{\beta}e^{\beta t}\right]_{\beta=0} = t, \tag{88}$$

we can obtain the correspondent solutions.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bxe^{\beta x} + c)w, \tag{89}$$

we have a special solution

$$w(x,t) = A\exp\left[\frac{b}{\beta}xe^{\beta t} + \frac{ab^2}{2\beta^3}e^{2\beta t} + ct\right]. \tag{90}$$

Then, for $\beta = 0$, by the interpretation

$$\left[\frac{1}{\beta^j}e^{\beta t}\right]_{\beta=0} = \frac{1}{j!}t^j, \tag{91}$$

we can obtain the correspondent solution.

However, the above properties will be, in general, complicated.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw, \tag{92}$$

we have the fundamental solution

$$w(x,t) = \frac{1}{2\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at} + bt\right). \tag{93}$$

For $a = 0$, we have the correspondent solution

$$w(x,t) = \exp bt. \tag{94}$$

For the factor

$$\frac{1}{2\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at}\right) \tag{95}$$

we have, for letting $a \to 0$,

$$\delta(x), \tag{96}$$

meanwhile, at $a = 0$, by the division by zero, we have 0. So, the reduction problem is a delicate open problem.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (-bx^2 + ct + d)w, \tag{97}$$

we have a special solution

$$w(x,t) = \exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}x^2 + \frac{1}{2}ct^2 + (\sqrt{ab} + d)t\right]. \tag{98}$$

For $a = 0$, how will be the correspondent solution? Since we have the solution

$$w(x,t) = \exp\left[-bx^2 t + \frac{1}{2}ct^2 + dt\right], \tag{99}$$

for the factor

$$\frac{1}{2}\sqrt{\frac{b}{a}}x^2 \tag{100}$$

we have to have

$$-bx^2 t. \tag{101}$$

We can see many and many interesting examples in [15].

10 Conclusion

The division by zero is uniquely and reasonably determined as

$$1/0 = 0/0 = z/0 = 0$$

in the natural extensions of fractions.

We have to change our basic ideas for our space and world.

We have to change our textbooks and scientific books on the division by zero.

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Optimality Conditions for Multidimensional Variational Problems Involving the Caputo-Type Fractional Derivative

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Abstract We study multidimensional variational problems, where the Lagrange function depends on the partial Caputo–Katugampola fractional derivatives, generalizing the Caputo and the Caputo–Hadamard fractional derivatives. We present sufficient and necessary conditions which determine the extremizers of a functional.

Keywords Fractional calculus
Multidimensional variational calculus · Caputo-type fractional derivative

AMS Subject classification 26A33 · 34A08 · 34K28

1 Introduction

Fractional variational calculus studies problems of extremizing (minimizing or maximizing) functionals with integrands depending on fractional derivatives (derivatives of real or complex order). In the first works on this subject, published by Fred Riewe in 1996–1997, it was noted that fractional derivatives can describe non-conservative systems in mechanics [18, 19]. It is an important issue because frictional and non-conservative forces are under macroscopic variational treatment and, therefore, they are studied using the most advanced methods of classical mechanics [10]. So far

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several remarkable results concerning fractional calculus of variations were obtained including necessary optimality conditions for fundamental and isoperimetric problems, transversality conditions or Noether's theorem [3, 12, 14, 15, 17]. In the simplest case, one thinks of one-dimensional problems. However, results were also generalized to the multi-dimensional case [4, 14, 16]. For the comprehensive study on the fractional variational calculus we refer the reader to the recent books [9, 13].

In the theory of fractional calculus one can find several types of differential operators and, depending on the considered system, one should choose the most appropriate one [8, 9]. In order to unify the theory, interesting approach was introduced in the works [6, 7], where author defines new derivatives and integrals which in particular cases reduce to the Riemann–Liouville and Hadamard operators. Some properties of these operators were studied in [11]. Moreover, in the works [1, 5], extension of the Caputo and the Caputo–Hadamard operators was proposed.

In this work, in contrary to [1] where the one-dimensional fractional variational problems were studied, our goal is to develop non-integer order calculus of variations by considering multidimensional problems with Lagrangians depending on the new partial Caputo-type operators (generalizing the Caputo and the Caputo–Hadamard partial derivatives). First we prove generalized integration by parts formula and next we apply this result to obtain necessary optimality conditions of Euler–Lagrange type to the fundamental and isoperimetric problems. Notice that, a formula of integration by parts for arbitrary $\alpha > 0$ in the one dimensional case was proven in [2].

The text is organized as follows. In Sect. 2 we present definitions of the new Caputo-type partial fractional derivatives and prove generalized integration by parts formula. Section 3 is devoted to the fundamental problem- we prove necessary and sufficient conditions for extremizers. Finally, in Sect. 4, we derive necessary optimality conditions for isoperimetric problem.

2 Preliminaries

In this section we introduce notions of the Caputo–Katugampola fractional derivatives in a multidimensional finite domain and obtain the generalized integration by parts formula. Along the work, for $i = 1, \ldots, n$, let $0 < a_i < b_i < \infty, a_i, b_i \in \mathbb{R}$ and $t = (t_1, \ldots, t_n)$ be a point in Ω_n, where $\Omega_n = (a_1, b_1) \times \cdots \times (a_n, b_n)$ is a subset in $\mathbb{R}^n$. Moreover, we denote by $dt = dt_1 \ldots dt_n$.

Definition 1 Let $t \in \Omega_n$ and $\rho > 0$. The left and the right partial Caputo–Katugampola fractional derivatives of order $\alpha \in (0, 1)$ of a function $x \in C^1(\bar{\Omega}_n; \mathbb{R})$, with respect to the ith variable t_i, are defined by

$$^{C}D^{\alpha,\rho}_{a_i+,t_i} x(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_{a_i}^{t_i} \frac{1}{(t_i^\rho - \tau^\rho)^\alpha} \frac{\partial}{\partial \tau} x(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau$$

and

$$ {}^C D^{\alpha,\rho}_{b_i-,t_i} x(t) = -\frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_{t_i}^{b_i} \frac{1}{(\tau^\rho - t_i^\rho)^\alpha} \frac{\partial}{\partial \tau} x(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau, $$

respectively.

Remark 1 Note that, for $\rho = 1$, operators ${}^C D^{\alpha,\rho}_{a_i+,t_i}$ and ${}^C D^{\alpha,\rho}_{b_i-,t_i}$ become standard partial left and right Caputo fractional derivatives, while for $\rho \to 0^+$, they recover classical partial left and right Caputo–Hadamard fractional derivatives (see e.g., [5–7, 13]).

Following the idea from [1], in order to obtain integration by parts formula for Caputo–Katugampola partial derivatives, we introduce two fractional operators acting on functions of several variables. Let $t \in \Omega_n$, then the fractional integral type operator is defined by

$$ I^{\alpha,\rho}_{b_i-,t_i} x(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_i}^{b_i} \frac{1}{(\tau^\rho - t_i{}^\rho)^{1-\alpha}} x(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau, $$

while the fractional differential type operator is given by

$$ D^{\alpha,\rho}_{b_i-,t_i} x(t) = -\frac{\rho^\alpha}{\Gamma(1-\alpha)} \frac{\partial}{\partial t_i} \int_{t_i}^{b_i} \frac{1}{(\tau^\rho - t_i{}^\rho)^\alpha} x(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau. $$

Theorem 1 *If $x \in C(\bar{\Omega}_n; \mathbb{R})$ and $y \in C^1(\bar{\Omega}_n; \mathbb{R})$, then*

$$ \int_{\Omega_n} x(t)^C D^{\alpha,\rho}_{a_i+,t_i} y(t) dt = -\int_{\Omega_n} D^{\alpha,\rho}_{b_i-,t_i} x(t) y(t) dt + \int_{\partial\Omega_n} y(t) I^{1-\alpha,\rho}_{b_i-,t_i} x(t) \nu^i d(\partial\Omega_n), $$

where ν^i is the outward pointing unit normal to $\partial\Omega_n$.

Proof Using Fubini's theorem we change the order of integration in the iterated integrals

$$ \begin{aligned} &\int_{\Omega_n} x(t)^C D^{\alpha,\rho}_{a_i+,t_i} y(t) dt \\ &= \int_{\Omega_n} \int_{a_i}^{t_i} x(t) \frac{\rho^\alpha}{\Gamma(1-\alpha)} \frac{1}{(t_i^\rho - \tau^\rho)^\alpha} \frac{\partial y}{\partial \tau}(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau dt \\ &= \int_{\Omega_n} \frac{\partial y}{\partial \tau} \left(\frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_\tau^{b_i} x(t) \frac{1}{(\tau^\rho - t_i{}^\rho)^\alpha} dt_i \right) dt_n \ldots dt_{i-1} d\tau dt_{i+1} \ldots dt_1 \\ &= \int_{\Omega_n} \frac{\partial y}{\partial \tau} I^{1-\alpha,\rho}_{b_i-,\tau} x(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) dt_n \ldots dt_{i-1} d\tau dt_{i+1} \ldots dt_1, \end{aligned} $$

where $\frac{\partial y}{\partial \tau}$ means $\frac{\partial y}{\partial \tau}(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)$.

Integrating by parts, we obtain

$$\int_{\Omega_n} I^{1-\alpha,\rho}_{b_i-,t_i} x(t) \frac{\partial}{\partial t_i} y(t) dt = -\int_{\Omega_n} \frac{\partial}{\partial t_i} I^{1-\alpha,\rho}_{b_i-,t_i} x(t) y(t) dt + \int_{\partial\Omega_n} y(t) I^{1-\alpha,\rho}_{b_i-,t_i} x(t) \nu^i d(\partial\Omega_n)$$
$$= -\int_{\Omega_n} D^{\alpha,\rho}_{b_i-,t_i} x(t) y(t) dt + \int_{\partial\Omega_n} y(t) I^{1-\alpha,\rho}_{b_i-,t_i} x(t) \nu^i d(\partial\Omega_n).$$

3 The Fundamental Problem

In the space $C^1(\bar{\Omega}_n; \mathbb{R})$ consider the norm $\|\cdot\|$ given by

$$\|x\| = \max_{t \in \bar{\Omega}_n} |x(t)| + \sum_{i=1}^{n} \max_{t_i \in [a_i, b_i]} \left| {}^C D^{\alpha,\rho}_{a_i+,t_i} x(t) \right|.$$

Let $\mathscr{A}$ be a nonempty subset of $C^1(\bar{\Omega}_n; \mathbb{R})$ and $\mathscr{I}$ be a functional defined on $\mathscr{A}$. We say that $\bar{x}$ is a local minimizer of $\mathscr{I}$ in the set $\mathscr{A}$ if there exists a neighborhood $\mathscr{N}_\delta(\bar{x})$ of $\bar{x}$ such that for all $x \in \mathscr{N}_\delta(\bar{x}) \cap \mathscr{A}$, we have

$$\mathscr{I}(x) \leq \mathscr{I}(\bar{x}).$$

Note that any function $x \in \mathscr{N}_\delta(\bar{x}) \cap \mathscr{A}$ can be represented in the form $x = \bar{x} + \varepsilon h$, where $|\varepsilon| \ll 1$ and h is such that $\bar{x} + \varepsilon h \in \mathscr{A}$.

Let $\zeta : \partial\Omega_n \to \mathbb{R}$ be a given function and $n \in \mathbb{N}$. We consider the following functional

$$\begin{aligned} \mathscr{I} : \mathscr{A}(\zeta) &\longrightarrow \mathbb{R} \\ x &\longmapsto \int_{\Omega_n} F(t, x(t), \nabla^{\alpha,\rho} x(t))\, dt, \end{aligned} \tag{1}$$

where

$$\mathscr{A}(\zeta) := \left\{ x \in C^1(\bar{\Omega}_n, \mathbb{R}) : x|_{\partial\Omega_n} = \zeta \right\},$$

$$\nabla^{\alpha,\rho} x(t) = \sum_{i=1}^{n} {}^C D^{\alpha,\rho}_{a_i+,t_i} x(t) \cdot e_i = \left({}^C D^{\alpha,\rho}_{a_1+,t_1} x(t), \ldots, {}^C D^{\alpha,\rho}_{a_n+,t_n} x(t) \right).$$

Moreover, function $F : \bar{\Omega}_n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and satisfies the following conditions

- $\partial_{2+i} F(t, x(t), \nabla^{\alpha,\rho} x(t)) \in C^1(\bar{\Omega}_n; \mathbb{R})$ for all $i = 1, \ldots, n$
- $\partial_2 F(t, x(t), \nabla^{\alpha,\rho} x(t)) \in C(\bar{\Omega}_n; \mathbb{R})$
- ${}_i D^{\alpha,\rho}_{b_i-} \partial_{2+i} F(t, x(t), \nabla^{\alpha,\rho} x(t)) \in C(\bar{\Omega}_n; \mathbb{R})$ for all $i = 1, \ldots, n$.

Here and along the work $\partial_i F$ denotes the partial derivative of function F with respect to its ith argument.

Theorem 2 *Suppose that* $\bar{x} \in \mathscr{A}(\zeta)$ *is a minimizer of the functional* (1)*. Then,* $\bar{x}$ *satisfies the following generalized Euler–Lagrange equation*

$$\partial_2 F(*_x)(t) - \sum_{i=1}^{n} {}^C D^{\alpha,\rho}_{a_i+,t_i} \partial_{2+i} F(*_x)(t) = 0, \; t \in \Omega_n, \tag{2}$$

where $(*_x)(t) = (t, x(t), \nabla^{\alpha,\rho} x(t))$.

Proof Let $\bar{x} \in \mathscr{A}(\zeta)$ be a minimizer of (1). Then, for any $|\varepsilon| \ll 1$ and every $h \in \mathscr{A}(0)$, it satisfies

$$\mathscr{I}(\bar{x}) \leq \mathscr{I}(\bar{x} + \varepsilon h).$$

Now, let us define the function $j : [-1, 1] \to \mathbb{R}$ as follows

$$j(\varepsilon) = \mathscr{I}(\bar{x} + \varepsilon h) = \int_{\Omega_n} F(t, \bar{x}(t) + \varepsilon h(t), \nabla^{\alpha,\rho} (\bar{x}(t) + \varepsilon h(t)))dt.$$

Since $\bar{x}$ is a minimizer of (1), $\varepsilon = 0$ is minimizer of j and so $j'(0) = 0$. Computing $j'(0)$ and using Theorem 1, we obtain

$$\begin{aligned} j'(0) = & \int_{\Omega_n} \partial_2 F(*_{\bar{x}})(t) \cdot h(t)dt + \sum_{i=1}^{n} \int_{\Omega_n} \partial_{2+i} F(*_{\bar{x}})(t) \cdot^C D^{\alpha,\rho}_{a_i+,t_i} h(t)dt \\ = & \int_{\Omega_n} \partial_2 F(*_{\bar{x}})(t) \cdot h(t)dt - \sum_{i=1}^{n} \int_{\Omega_n} h(t) \cdot D^{\alpha,\rho}_{b_i-,t_i} \partial_{2+i} F(*_{\bar{x}})(t)dt \\ & + \int_{\partial\Omega_n} h(t) \cdot I^{1-\alpha,\rho}_{b_i-,t_i} \partial_{2+i} F(*_{\bar{x}})(t)\nu^i d(\partial\Omega_n). \end{aligned}$$

Since $h \in \mathscr{A}(0)$ and h is arbitrary elsewhere, we conclude

$$\partial_2 F(*_{\bar{x}})(t) - \sum_{i=1}^{n} D^{\alpha,\rho}_{b_i-,t_i} \partial_{2+i} F(*_{\bar{x}})(t) = 0, \; t \in \Omega_n$$

by the fundamental lemma of the calculus of variations.

We remark that Eq. (2) gives only a necessary condition. To deduce a sufficient condition, we recall the notion of convex function. Let $f : \mathbb{R}^m \to \mathbb{R}$ and $f \in C^1(\mathbb{R}^m; \mathbb{R})$. The function f is convex if and only if

$$f(x) \geq f(y) + \langle \nabla f(y); x - y \rangle, \; x, y \in R^m,$$

where $\langle\cdot;\cdot\rangle$ denotes the scalar product in $\mathbb{R}^m$.

Theorem 3 *Suppose that $\bar{x} \in \mathscr{A}(\zeta)$ satisfies (2) and function $(u, v) \mapsto F(t, u, v)$ is convex for every $t \in \bar{\Omega}_n$. Then, $\bar{x}$ is a minimizer of functional (1).*

Proof If $\bar{x} \in \mathscr{A}(\zeta)$ satisfies (2) and function $(u, v) \mapsto F(t, u, v)$ is convex for every $t \in \bar{\Omega}_n$, then

$$\mathscr{I}(x) \geq \mathscr{I}(\bar{x}) + \int_{\Omega_n} \left(\partial_2 F(*_{\bar{x}})(t) \cdot (x - \bar{x}) + \sum_{i=1}^{n} \partial_{2+i} F(*_{\bar{x}})(t) \cdot {}^C D^{\alpha,\rho}_{a_i+,t_i}[x - \bar{x}] \right) dt,$$

for every $x \in \mathscr{A}(\zeta)$. By Theorem 1 and the fact that $x - \bar{x}|_{\partial\Omega_n} = 0$, we get

$$\mathscr{I}(x) \geq \mathscr{I}(\bar{x}) + \int_{\Omega_n} \left(\partial_2 F(*_{\bar{x}})(t) - \sum_{i=1}^{n} D^{\alpha,\rho}_{b_i-,t_i} \partial_{2+i} F(*_{\bar{x}})(t) \right) (x - \bar{x}) dt.$$

Finally, applying Eq. (2), we have $\mathscr{I}(x) \geq \mathscr{I}(\bar{x})$ for any $x \in \mathscr{A}(\zeta)$.

4 Isoperimetric Problem

Let us define the functional $\mathscr{J} : \mathscr{A}(\zeta) \to \mathbb{R}$ by

$$\mathscr{J}(x) = \int_{\Omega_n} G(t, x(t), \nabla^{\alpha,\rho} x(t)) dt, \tag{3}$$

where operator $\nabla^{\alpha,\rho}$ and function G are of the same class as in the case of functional (1). In the next theorem we give a necessary optimality condition for a function to be a minimizer of (1) subject to the isoperimetric constraint $\mathscr{J}(x) = \xi$.

Theorem 4 *Suppose that $\bar{x}$ is a minimizer of functional (1) on the set*

$$\mathscr{A}_\xi(\zeta) := \left\{ x \in \mathscr{A}(\zeta) : \ \mathscr{J}(x) = \xi \right\}$$

and that the following condition is satisfied

$$\partial_2 G(*_{\bar{x}})(t) - \sum_{i=1}^{n} D^{\alpha,\rho}_{b_i-,t_i} \partial_{2+i} G(*_{\bar{x}})(t) \neq 0. \tag{4}$$

Then, there exists a real constant λ_0 such that, for $H = F + \lambda_0 G$, equation

$$\partial_2 H(*_{\bar{x}})(t) - \sum_{i=1}^{n} D^{\alpha,\rho}_{b_i-,t_i} \partial_{2+i} H(*_{\bar{x}})(t) = 0, \tag{5}$$

holds.

Proof By the fundamental lemma of the calculus of variations and hypothesis (4), there exists $h_2 \in \mathscr{A}(0)$ so that

$$\int_{\Omega_n} \left(\partial_2 G(*_{\bar{x}})(t) - \sum_{i=1}^{n} D^{\alpha,\rho}_{b_i-,t_i} \partial_{2+i} G(*_{\bar{x}})(t) \right) h_2(t)dt = 1.$$

Now, with function h_2 and an arbitrary $h_1 \in \mathscr{A}(0)$, let us define two functions $\varphi,\ \psi : [-1, 1] \times [-1, 1] \to \mathbb{R}$ such that

$$\begin{aligned} \varphi(\varepsilon_1, \varepsilon_2) &:= \mathscr{I}(\bar{x} + \varepsilon_1 h_1 + \varepsilon_2 h_2) \\ \psi(\varepsilon_1, \varepsilon_2) &:= \mathscr{J}(\bar{x} + \varepsilon_1 h_1 + \varepsilon_2 h_2) - \xi. \end{aligned}$$

Note that, $\psi(0, 0) = 0$ and that

$$\left.\frac{\partial \psi}{\partial \varepsilon_2}\right|_{(0,0)} = \int_{\Omega_n} \left(\partial_2 G(*_{\bar{x}})(t) - \sum_{i=1}^{n} D^{\alpha,\rho}_{b_i-,t_i} \partial_{2+i} G(*_{\bar{x}})(t) \right) h_2(t)dt = 1.$$

The implicit function theorem implies, that there are $\delta_0 > 0$ and a function $s \in C^1([-\delta_0, \delta_0]; \mathbb{R})$ with $s(0) = 0$ such that

$$\psi(\varepsilon_1, s(\varepsilon_1)) = 0, \qquad |\varepsilon_1| \le \delta_0,$$

and then $\bar{x} + \varepsilon_1 h_1 + s(\varepsilon_1) h_2 \in \mathscr{A}_\xi(\zeta)$. Moreover,

$$\frac{\partial \psi}{\partial \varepsilon_1} + \frac{\partial \psi}{\partial \varepsilon_2} \cdot s'(\varepsilon_1) = 0, \qquad |\varepsilon_1| \le \delta_0,$$

and then

$$s'(0) = -\left.\frac{\partial \psi}{\partial \varepsilon_1}\right|_{(0,0)}.$$

Because $\bar{x} \in \mathscr{A}(\zeta)$ is a minimizer of $\mathscr{I}$ we have

$$\varphi(0, 0) \le \varphi(\varepsilon_1, s(\varepsilon_1)), \qquad |\varepsilon_1| \le \delta_0,$$

and hence

$$\left.\frac{\partial \varphi}{\partial \varepsilon_1}\right|_{(0,0)} + \left.\frac{\partial \varphi}{\partial \varepsilon_2}\right|_{(0,0)} \cdot s'(0) = 0.$$

Letting $\lambda_0 = -\left.\frac{\partial \varphi}{\partial \varepsilon_2}\right|_{(0,0)}$ be the Lagrange multiplier we find

$$\left.\frac{\partial \varphi}{\partial \varepsilon_1}\right|_{(0,0)} + \lambda_0 \left.\frac{\partial \psi}{\partial \varepsilon_1}\right|_{(0,0)} = 0$$

or, in other words

$$\int_{\Omega_n} \left(\partial_2 F(*_{\bar{x}})(t) + \sum_{i=1}^{n} \partial_{2+i} F(*_{\bar{x}})(t) D^{\alpha,\rho}_{b_i-,t_i} h_2(t) \right) dt$$
$$+\lambda_0 \left[\int_{\Omega_n} \left(\partial_2 G(*_{\bar{x}})(t) + \sum_{i=1}^{n} \partial_{2+i} G(*_{\bar{x}})(t) D^{\alpha,\rho}_{b_i-,t_i} h_2(t) \right) dt \right] = 0$$

Finally, applying Theorem 1 and fundamental lemma of the calculus of variations we obtain (5).

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Maximum Principle for a Kind of Elliptic Systems with Morrey Data

Lubomira G. Softova

Abstract We consider nonlinear elliptic systems satisfying componentwise coercivity condition. The nonlinear terms have controlled growths with respect to the solution and its gradient, while the behaviour in the independent variable x is governed by functions in Morrey spaces. We obtain maximum principle for such kind of systems.

Keywords Nonlinear elliptic systems · Morrey spaces · Maximum principle

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain. We are interested in boundedness of the weak solutions to nonlinear elliptic systems of the type

$$\operatorname{div}\mathbf{A}(x, \mathbf{u}, D\mathbf{u}) = \mathbf{b}(x, \mathbf{u}, D\mathbf{u}), \qquad x \in \Omega \tag{1}$$

where the nonlinear terms

$$\begin{aligned} \mathbf{A}(x, \mathbf{u}, \mathbf{z}) &: \Omega \times \mathbb{R}^N \times \mathbb{M}^{N\times n} \to \mathbb{R}^{N\times n}, \\ \mathbf{b}(x, \mathbf{u}, \mathbf{z}) &: \Omega \times \mathbb{R}^N \times \mathbb{M}^{N\times n} \to \mathbb{R}^N \end{aligned} \tag{2}$$

are Carathéodory maps. That is, they are measurable in $x \in \Omega$ for each $(\mathbf{u}, \mathbf{z}) \in \mathbb{R}^N \times \mathbb{M}^{N\times n}$ and continuous in $(\mathbf{u}, \mathbf{z})$ for almost all $x \in \Omega$.

The celebrated result of De Giorgi [6] and Nash [14] implies that any weak solution $u \in W_0^{1,2}(\Omega)$ of the linear elliptic equation $\operatorname{div}(\mathbf{A}(x)Du + \mathbf{g}(x)) = f(x)$ is locally

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Hölder continuous when $\mathbf{g} \in L^p(\Omega, \mathbb{R}^n)$ with $p > n$ and $f \in L^q(\Omega)$ with $q > n/2$, even if the coefficients are only L^∞. Unfortunately the De Giorgi–Nash result does not hold anymore if we consider a system of uniformly elliptic equations because of the lack of *Maximum principle*, as it was shown by De Giorgi himself almost ten years later, constructing a counterexample [7].

Moreover, the result of De Giorgi–Nash cannot be extended to quasilinear systems even if the coefficients are analytic functions, as it was shown by Giusti and Miranda in [9]. In order to get a maximum principle for elliptic systems we need to impose some quite restrictive structural conditions. The simplest one requires the system to be in diagonal form, or *decoupled.*

Example 1 Consider the operator $\operatorname{div}(\mathbf{A}(x, D\mathbf{u})) = 0$ in Ω with coefficients

$$A_i^\alpha(x, D\mathbf{u}) = \sum_{j=1}^{n} \sum_{\beta=1}^{N} \delta_{\alpha\beta} A_{ij}^{\alpha\beta}(x) D_j u^\beta$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Then u^α solves a single elliptic equation and $\sup_\Omega u^\alpha \le \sup_{\partial\Omega} u^\alpha$, for each $\alpha = 1, \dots, N$.

One more example was given by Nečas and Stará in [15].

Example 2 Consider the system $\operatorname{div}\mathbf{A}(x, \mathbf{u}, D\mathbf{u}) = 0$ in Ω that is diagonal for large values of u^α, that is,

$$0 < \theta^\alpha \le u^\alpha \implies A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) = \sum_{j=1}^{n} \sum_{\beta=1}^{N} \delta_{\alpha\beta} A_{ij}^{\alpha\beta}(x, \mathbf{u}) D_j u^\beta \tag{3}$$

with bounded and elliptic $A_{ij}^{\alpha\beta}$. It turns out that

$$\sup_\Omega u^\alpha \le \max\left\{\theta^\alpha; \sup_{\partial\Omega} u^\alpha\right\}.$$

The situation becomes more complicated if we consider *general nonlinear systems.* Along with the Carathéodory conditions on the maps $\mathbf{A}(x, \mathbf{u}, \mathbf{z})$ and $\mathbf{b}(x, \mathbf{u}, \mathbf{z})$ we need to control also the growths of $\mathbf{A}$ and $\mathbf{b}$ with respect to $\mathbf{u}$ and $\mathbf{z}$. These additional *controlled growth conditions* ensure the convergence of the integrals in the definition of *weak solution* to (1).

In [13] Leonetti and Petricca assume *componentwise coercivity* of $\mathbf{A}$ and positivity of $\mathbf{b}$ for large values of u^α, that is, for each $\alpha = 1, \dots, N$, there exist positive constants θ^α such that

$$\theta^\alpha \le u^\alpha \implies \begin{cases} \nu|\mathbf{z}^\alpha|^p - M_\alpha \le \sum_{i=1}^{n} A_i^\alpha(x, \mathbf{u}, \mathbf{z}) z_i^\alpha \\ 0 \le b^\alpha(x, \mathbf{u}, \mathbf{z}). \end{cases} \tag{4}$$

Combining the *Sobolev inequality* with the *Stampacchia Lemma* [23] they get a componentwise bound of the solution, covering this way also the systems studied in [15], since (3) is a special case of (4). Let us note that getting essential boundedness of the weak solution to (1) is a starting point for a further study of its regularity in various function spaces. In [8, 16, 18] the authors obtain better integrability and Hölder regularity of the bounded solutions to quasilinear elliptic equations ($N = 1$) under controlled growth conditions on the nonlinear terms. Further this result has been extended in [20] to semilinear uniformly elliptic systems of the form

$$\operatorname{div}(\mathbf{A}(x)D\mathbf{u} + \mathbf{a}(x, \mathbf{u})) = \mathbf{b}(x, \mathbf{u}, D\mathbf{u}) \qquad \text{in } \Omega \tag{5}$$

with minimal regular assumptions on the coefficients and the underlying domain. Precisely, it is shown that if the nonlinear terms satisfy the controlled growth conditions with $\varphi \in L^p(\Omega)$, $p > 2$ and $\psi \in L^q(\Omega)$, $q > \frac{2n}{n+2}$ then any bounded weak solution to (5) belongs to $W_0^{1,r}(\Omega; \mathbb{R}^N)$ with $r = \min\{p, q^*\}$.

The natural question that arises is what kind of regularity of the solution to (1) we can expect if the given functions φ and ψ belong to some Morrey space. In the case of a single equation we count with the results of Byun and Palagachev [2, 4]. Combining the Gehring–Giaquinta–Modica lemma, the Adams trace inequality and the Hartmann–Stampacchia maximum principle they obtain L^∞ estimate of the solution. Further, the Morrey-type estimate of the gradient permits the authors to show also Hölder regularity of the solution.

Our goal is to obtain a componentwise maximum principle for any component of the solution $\mathbf{u}$ of (1) supposing that the operators $\mathbf{A}$ and $\mathbf{b}$ satisfy structural conditions expressed in terms of Morrey functions.

As a consequence we obtain also Morrey regularity of the gradient of each component of the solution, extending such a way the regularity results obtained in [2–4, 8, 10, 17, 20, 21] for linear and quasilinear equations and systems with Morrey data to nonlinear systems with Morrey data.

Recall that a real valued function $f \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ with $p \in [1, \infty)$, $\lambda \in (0, n)$, if

$$\|f\|_{p,\lambda;\Omega} = \left(\sup_{\mathscr{B}_r(x)} \frac{1}{r^\lambda} \int_{\Omega \cap \mathscr{B}_r(x)} |f(y)|^p \, dy \right)^{1/p} < \infty \tag{6}$$

where the supremum is taken over all balls $\mathscr{B}_r(x)$, $r \in (0, \operatorname{diam} \Omega]$ and $x \in \overline{\Omega}$. Working in the framework of the Morrey spaces we note that the Sobolev trace inequality is not enough anymore. For this goal we will use the following result due to Adams.

Lemma 1 (Adams Trace Inequality, [1, 5, 19]) *Let $d\varsigma$ be a positive Radon measure defined in Ω and such that for each ball $\mathscr{B}_\rho$ it holds*

$$\varsigma(\mathscr{B}_\rho) \le K\rho^{\tau_0}, \quad \tau_0 = \frac{s}{r}(n - r), \quad 1 < r < s < \infty, \quad r < n \tag{7}$$

with an absolute constant $K > 0$. *Then*

$$\left(\int_{\Omega} |v(x)|^s \, d\varsigma\right)^{\frac{1}{s}} \leq C(n,s,r) K^{\frac{1}{s}} \left(\int_{\Omega} |Dv(x)|^r \, dx\right)^{\frac{1}{r}} \tag{8}$$

for each function $v \in W_0^{1,r}(\Omega)$.

In what follows we suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain satisfying the (A)-condition, that is, there exists a constant $A_\Omega > 0$ such that

$$|\Omega_r(x)| \geq A_\Omega \, r^n \qquad \forall \, x \in \overline{\Omega}, \; r \in (0, \operatorname{diam} \Omega] \tag{A}$$

where $\Omega_r(x) = \Omega \cap \mathscr{B}_r(x)$. It is worth noting that the (A)-condition excludes interior cusps at each point of the boundary and guarantees the validity of the Sobolev embedding theorem in $W^{1,p}(\Omega)$. This geometric property is surely satisfied when $\partial\Omega$ has the uniform interior cone property (e.g. C^1-smooth or Lipschitz continuous boundaries), but it holds also for the Reifenberg falt domains (cf. [18]).

Throughout the text the standard summation convention on the repeated indexes is adopted. The letter $C > 0$ is used for various constants and may change from one occurrence to another.

2 Maximum Principle

Consider in Ω the system (1) under the basic assumptions (2). In [22] we have studied the properties of the weak solutions $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^N)$ to such a system. Our goal is to extend this study to solutions with higher integrability. Such a way we cover the case of m-Laplacian systems. For this goal we impose the following *controlled growth conditions*. Suppose that for each $(x, \mathbf{u}, \mathbf{z}) \in \Omega \times \mathbb{R}^N \times \mathbb{M}^{N \times n}$ and $m \geq 2$ holds

$$\begin{cases} |\mathbf{A}(x, \mathbf{u}, \mathbf{z})| \leq \Lambda\big(\varphi(x) + |\mathbf{u}|^{\frac{m^*(m-1)}{m}} + |\mathbf{z}|^{m-1}\big) \\ |\mathbf{b}(x, \mathbf{u}, \mathbf{z})| \leq \Lambda\big(\psi(x) + |\mathbf{u}|^{m^*-1} + |\mathbf{z}|^{m\frac{m^*-1}{m^*}}\big) \end{cases} \tag{9}$$

as $|\mathbf{u}|, |\mathbf{z}| \to \infty$, with some positive constant Λ (cf. [12]). Here m^* is the Sobolev conjugate of m, that is,

$$m^* = \begin{cases} \dfrac{mn}{n-m} & \text{if } n > m \\ \text{any large number} & \text{if } n \leq m, \end{cases}$$

and the given functions φ and ψ satisfy

$$\begin{cases} \varphi \in L^{p,\lambda}(\Omega), \quad p > \frac{m}{m-1}, \ \lambda \in (0,n), \ (m-1)p + \lambda > n \\ \psi \in L^{q,\mu}(\Omega), \quad q > \frac{m^*}{m^*-1}, \ \mu \in (0,n), \ mq + \mu > n. \end{cases} \tag{10}$$

A *weak solution* of (1) is a function $\mathbf{u} \in W^{1,m}(\Omega;\mathbb{R}^N) \cap L^{m^*}(\Omega;\mathbb{R}^N)$, satisfying

$$\sum_{i=1}^{n} \int_\Omega A_i^\alpha(x,\mathbf{u},D\mathbf{u}) D_i\phi^\alpha(x)\,dx + \int_\Omega b^\alpha(x,\mathbf{u}(x),D\mathbf{u}(x))\phi^\alpha(x)\,dx = 0 \tag{11}$$

for all $\phi = (\phi^1,\ldots,\phi^N) \in W_0^{1,m}(\Omega;\mathbb{R}^N)$. The conditions (9) and (10) are the natural assumptions that ensure the convergence of the integrals in (11). Moreover, they are optimal as it is shown in [12] in case of single equation.

Generally we cannot expect boundedness of the solutions to (1) unless we add some restrictions on the structure of the operator (see for example [11, 13]). For this goal we impose componentwise coercivity on A_i^α and a sign condition on b^α.

For every $\alpha \in \{1,\ldots,N\}$ there exist positive constants θ^α, γ and a function φ such that for each $u^\alpha \geq \theta^\alpha$ we have

$$\begin{cases} \gamma|\mathbf{z}^\alpha|^m - \Lambda\varphi(x)^{\frac{m}{m-1}} \leq \sum_{i=1}^{n} A_i^\alpha(x,\mathbf{u},\mathbf{z}) z_i^\alpha \\ \varphi \in L^{p,\lambda}(\Omega), \ p > \dfrac{m}{m-1}, \ \lambda \in (0,n), \ (m-1)p + \lambda > n \\ 0 \leq b^\alpha(x,\mathbf{u},\mathbf{z}) \quad \text{for a.a. } x \in \Omega, \ \forall\, \mathbf{z} \in \mathbb{M}^{N\times n}. \end{cases} \tag{12}$$

Theorem 1 (Maximum principle) *Let Ω be (A)-type and $\mathbf{u} \in W^{1,m}(\Omega;\mathbb{R}^N) \cap L^{m^*}(\Omega;\mathbb{R}^N)$ be a weak solution to* (1) *under the conditions* (9)–(12) *and such that* $\sup_{\partial\Omega} u^\alpha < \infty$*. Then*

$$\sup_\Omega u^\alpha \leq \max\left\{\theta^\alpha, \sup_{\partial\Omega} u^\alpha\right\} + M_\alpha \qquad \alpha \in \{1,\ldots,N\}$$

where M_α depends on $n, m, p, \lambda, \Lambda, \gamma, \|\varphi\|_{p,\lambda;\Omega}$, and $|\Omega|$.

Proof We choose a constant $L > 0$ such that $L \geq \max\{\theta^\alpha; \sup_{\partial\Omega} u^\alpha\}$ and define the set $\mathscr{A}_L^\alpha = \{x \in \Omega : \ u^\alpha(x) - L > 0\}$. Then we take a vector function $\mathbf{v}$ as follows

$$v^\beta = \begin{cases} \max\{u^\alpha - L; 0\} & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases}, \qquad Dv^\beta = \begin{cases} Du^\alpha \chi_{\mathscr{A}_L^\alpha} & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases}.$$

It is clear that $\mathbf{v} \in W_0^{1,m}(\Omega;\mathbb{R}^N)$ and hence $\mathbf{v} \in L^{m^*}(\Omega;\mathbb{R}^N)$ by the Sobolev embedding. Choosing $\phi^\alpha = v^\alpha$ as a test function we obtain

$$\sum_{i=1}^{n}\int_{\mathscr{A}_L^\alpha} A_i^\alpha(x,\mathbf{u},D\mathbf{u})D_i u^\alpha(x)\,dx \;+\int_{\mathscr{A}_L^\alpha} b^\alpha(x,\mathbf{u},D\mathbf{u})(u^\alpha(x)-L)\,dx=0\,.$$

We start with the case $n>m$ when $m^*=mn/(n-m)$. Define a positive measure $d\varsigma$ supported in Ω by

$$d\varsigma := (\chi_\Omega(x)+\varphi(x)^{\frac{m}{m-1}})\,dx,$$

where χ_Ω is the characteristic function of Ω. Then by (12) we get the estimate

$$\int_{\mathscr{A}_L^\alpha}|Du^\alpha(x)|^m\,dx \le \frac{\Lambda}{\gamma}\int_{\mathscr{A}_L^\alpha}\varphi(x)^{\frac{m}{m-1}}\,dx \le \frac{\Lambda}{\gamma}\int_{\mathscr{A}_L^\alpha}(\chi_\Omega(x)+\varphi(x)^{\frac{m}{m-1}})\,dx$$
$$\le C(\Lambda,\gamma)\varsigma(\mathscr{A}_L^\alpha)\,. \tag{13}$$

We extend the solution u^α and φ as zero out of Ω. Direct calculations give that

$$\varsigma(\mathscr{B}_\rho) = \int_{\mathscr{B}_\rho}\left(\chi_\Omega(x)+\varphi(x)^{\frac{m}{m-1}}\right)dx$$
$$\le C(n,\operatorname{diam}\Omega)\rho^{n-(n-\lambda)\frac{m}{p(m-1)}}\|\varphi\|_{L^{p,\lambda}(\Omega)}^{\frac{m}{m-1}} = K\rho^{\tau_0} \tag{14}$$

with $K=K(n,p,\lambda,m,\operatorname{diam}\Omega,\|\varphi\|_{p,\lambda;\Omega})$ and

$$\tau_0 = n-\frac{n-\lambda}{p}\frac{m}{m-1} > n-m.$$

Applying (8) with $r=m<n$ and calculating s from (7) we get

$$\int_{\mathscr{A}_L^\alpha}(u^\alpha(x)-L)\,d\varsigma \le \left(\int_{\mathscr{A}_L^\alpha}|u^\alpha(x)-L|^s\,d\varsigma\right)^{\frac{1}{s}}\varsigma(\mathscr{A}_L^\alpha)^{1-\frac{1}{s}}$$
$$\le CK^{\frac{1}{s}}\left(\int_{\mathscr{A}_L^\alpha}|Du^\alpha(x)|^m\,dx\right)^{\frac{1}{m}}\varsigma(\mathscr{A}_L^\alpha)^{1-\frac{1}{s}} \tag{15}$$
$$\le C(n,p,K,\gamma,\Lambda)\varsigma(\mathscr{A}_L^\alpha)^{1+\frac{1}{m}-\frac{1}{s}}$$

with $s=\frac{m}{n-m}\left[n-\frac{(n-\lambda)m}{p(m-1)}\right]>m.$

A similar bound holds also in the case $n=m$. In fact, for any ball $\mathscr{B}_\rho\subset\mathbb{R}^m$ we have

$$\varsigma(\mathscr{B}_\rho)\le C\rho^{m-\frac{(m-\lambda)m}{p(m-1)}}\|\varphi\|_{p,\lambda;\Omega}^{\frac{m}{m-1}}\le K\rho^{\tau_0}\,,$$

with $\tau_0=m-\frac{m-\lambda}{p}\frac{m}{m-1}>0$. Choosing $s=m$ we calculate r from (7)

$$r = \frac{mp(m-1)}{2p(m-1)-(m-\lambda)} \in (1, m).$$

Then by the Hölder inequality, the Adams trace inequality and (13) we obtain

$$\begin{aligned} \int_{\mathscr{A}_L^\alpha} (u^\alpha(x) - L)\, d\varsigma &\le \left(\int_{\mathscr{A}_L^\alpha} (u^\alpha(x) - L)^m\, d\varsigma \right)^{\frac{1}{m}} \varsigma(\mathscr{A}_L^\alpha)^{1-\frac{1}{m}} \\ &\le C K^{\frac{1}{2}} \left(\int_{\mathscr{A}_L^\alpha} |Du^\alpha(x)|^r\, dx \right)^{\frac{1}{r}} \varsigma(\mathscr{A}_L^\alpha)^{1-\frac{1}{m}} \\ &\le C K^{\frac{1}{m}} \left(\int_{\mathscr{A}_L^\alpha} |Du^\alpha(x)|^m\, dx \right)^{\frac{1}{m}} \left(\int_{\mathscr{A}_L^\alpha} \chi_\Omega(x)\, dx \right)^{\frac{1}{r}-\frac{1}{m}} \varsigma(\mathscr{A}_L^\alpha)^{1-\frac{1}{m}} \\ &= C K^{\frac{1}{m}} \left(\int_{\mathscr{A}_L^\alpha} |Du^\alpha(x)|^m\, dx \right)^{\frac{1}{m}} \varsigma(\mathscr{A}_L^\alpha)^{1+\frac{1}{r}-\frac{2}{m}} \le C \varsigma(\mathscr{A}_L^\alpha)^{1+\frac{1}{r}-\frac{1}{m}} \end{aligned} \tag{16}$$

with $C = C(n, p, \lambda, K, \gamma, \Lambda)$. Unifying the estimates (15) and (16) we obtain

$$\int_{\mathscr{A}_L^\alpha} (u^\alpha(x) - L)\, dm \le C m(\mathscr{A}_L^\alpha)^{1+\sigma_0} \tag{17}$$

where

$$\sigma_0 = \begin{cases} \dfrac{1}{m} - \dfrac{1}{s} = \dfrac{p(m-1)+\lambda-n}{np(m-1)-(n-\lambda)m} & \text{if } n > m \\ \dfrac{1}{r} - \dfrac{1}{m} = \dfrac{p(m-1)+\lambda-m}{mp(m-1)} & \text{if } n = m\,. \end{cases}$$

Suppose now that $\varsigma(\mathscr{A}_L^\alpha) > 0$, otherwise $\sup_\Omega u^\alpha(x) \le L$. For any $L_1 > L$ we have $\mathscr{A}_{L_1}^\alpha \subset \mathscr{A}_L^\alpha$ and therefore (17) yields

$$\begin{aligned} (L_1 - L)\varsigma(\mathscr{A}_{L_1}^\alpha) &\le \int_{\mathscr{A}_{L_1}^\alpha} (u^\alpha(x) - L)\, d\varsigma \\ &\le \int_{\mathscr{A}_L^\alpha} (u^\alpha(x) - L)\, d\varsigma \le C\varsigma(\mathscr{A}_L^\alpha)^{1+\sigma_0}\,. \end{aligned}$$

Hence

$$\varsigma(\mathscr{A}_{L_1}^\alpha) \le \frac{C}{L_1 - L} \varsigma(\mathscr{A}_L^\alpha)^{1+\sigma_0}\,.$$

In order to estimate the measure of the set $\mathscr{A}_L^\alpha$ we apply the following *Lemma of Stampacchia* [23, Lemma 4.1].

Lemma 2 *Let $\Theta : [L_0, \infty) \to [0, \infty)$ be a decreasing function. Assume that there exist $c, a \in (0, \infty)$ and $b \in (1, \infty)$ such that*

$$L_1 > L \geq L_0 \implies \Theta(L_1) \leq \frac{c}{(L_1 - L)^\alpha}(\Theta(L))^b.$$

Then

$$\Theta(L_0 + d) = 0 \quad \textit{where} \quad d = \left[c\Theta(L_0)^{b-1}2^{\frac{ab}{b-1}}\right]^{\frac{1}{a}}.$$

The application of the Lemma 2 to the function $\Theta(L) = \varsigma(\mathscr{A}_L^\alpha)$ with $a = 1$, $b = 1 + \sigma_0$ and $L_0 = \max\{\theta^\alpha, \sup_{\partial\Omega} u^\alpha\}$ yields

$$\varsigma(\mathscr{A}^\alpha_{L_0+d_\alpha}) = 0 \qquad \text{where} \qquad d_\alpha \leq C\varsigma(\Omega)^{\sigma_0}2^{1+\frac{1}{\sigma_0}}. \tag{18}$$

The last assertion means that for each $\alpha = 1, \ldots, N$ there exists a constant M_α depending on $n,\ p,\ \lambda,\ \gamma,\ \Lambda,\ |\Omega|,\ \|\varphi\|_{p,\lambda;\Omega}$ such that

$$\sup_\Omega u^\alpha < \max\left\{\theta^\alpha; \sup_{\partial\Omega} u^\alpha\right\} + M_\alpha \tag{19}$$

and this completes the proof of the theorem.

Corollary 1 *Let $\mathbf{u} \in W_0^{1,m}(\Omega; \mathbb{R}^N)$ be a solution of (1) under the assumptions (A), (2), (9), and (10). Suppose that instead of (12) holds*

$$\begin{cases} \gamma|\mathbf{z}^\alpha|^m - \Lambda\varphi(x)^{\frac{m}{m-1}} \leq \sum_{i=1}^n A_i^\alpha(x, \mathbf{u}, \mathbf{z})z_i^\alpha \\ \varphi \in L^{p,\lambda}(\Omega),\ p > \dfrac{m}{m-1},\ \lambda \in (0, n),\ (m-1)p + \lambda > n \\ 0 \leq b^\alpha(x, \mathbf{u}, \mathbf{z})\operatorname{sign} u^\alpha(x) \end{cases} \tag{20}$$

for $|u^\alpha| \geq \theta^\alpha > 0$, $\alpha = 1, \ldots, N$. Then there exists a constant M depending on known quantities such that

$$\|\mathbf{u}\|_{\infty,\Omega} \leq M.$$

Proof Take a positive constant L such that $L \geq \theta^\alpha$ and consider the set $\bar{\mathscr{A}}_L^\alpha = \{x \in \Omega : u^\alpha(x) + L < 0\}$. Then the Theorem 1 applied to $-u^\alpha$ gives

$$\inf_\Omega u^\alpha > -\theta^\alpha - M_\alpha. \tag{21}$$

Unifying (19) and (21) we get boundedness of $\|u^\alpha\|_{\infty;\Omega}$ for each $\alpha = 1, \ldots, N$. Then

$$\|\mathbf{u}\|_{\infty;\Omega} = \max_{1\leq\alpha\leq N} \|u^\alpha\|_{\infty;\Omega} =: M < \infty.$$

Theorem 2 (Morrey Regularity of the Gradient) *Let Ω be a bounded (A)-type domain in $\mathbb{R}^n$, $n > m$, and $\mathbf{u} \in W_0^{1,m}(\Omega, \mathbb{R}^N)$ be a weak solution to (1) under the assumptions (A), (2), (9), (10), and (20). Then $Du^\alpha \in L^{m,n-m}(\Omega)$ and*

$$\int_{\Omega_\rho(x_0)} |Du^\alpha(x)|^m \, dx \le C\rho^{n-m} \qquad \forall x_0 \in \Omega,\ \rho \in (0, \operatorname{diam} \Omega] \tag{22}$$

and constant depending on known quantities.

Proof Fix $x_0 \in \Omega$ and $\rho > 0$ be such that $\mathscr{B}_\rho(x_0) \subset \mathscr{B}_{2\rho}(x_0) \Subset \Omega$, $\rho > 1$. Define a cut-off function $\zeta(x) \in C^1(\mathbb{R}^n)$

$$\zeta(x) = \begin{cases} 1 & x \in \mathscr{B}_\rho(x_0), \\ 0 & x \notin \mathscr{B}_{2\rho}(x_0), \end{cases} \qquad |D\zeta| \le \frac{C}{\rho}.$$

For any fixed α take $\phi^\alpha(x) = e^{u^\alpha(x)}\zeta(x)^m$ as a test function in (11) to get

$$\begin{aligned} \sum_{i=1}^n \int_\Omega A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{u^\alpha(x)} D_i u^\alpha(x) \zeta(x)^m \, dx \\ = -\sum_{i=1}^n \int_\Omega A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{u^\alpha(x)} m\zeta(x)^{m-1} D_i\zeta(x) \, dx \\ - \int_\Omega b^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{u^\alpha(x)} \zeta(x)^m \, dx\,. \end{aligned}$$

The left-hand side can be estimated by (20) while for the right-hand side we use (9) and (10)

$$\begin{aligned} e^{-M} \int_\Omega \big(\gamma |Du^\alpha(x)|^m - \Lambda \varphi(x)^{\frac{m}{m-1}}\big) \zeta(x)^m \, dx \\ \le mn\Lambda e^M \int_\Omega \big(\varphi(x) + |\mathbf{u}|^{\frac{n(m-1)}{n-m}} + |D\mathbf{u}|^{m-1}\big) \zeta(x)^{m-1} |D\zeta(x)| \, dx \\ + \Lambda e^M \int_\Omega \big(\psi(x) + |\mathbf{u}|^{\frac{mn-n+m}{n-m}} + |D\mathbf{u}|^{\frac{mn-n+m}{n}}\big) \zeta(x)^m \, dx. \end{aligned}$$

To proceed further, we use the Young inequality $ab \le \varepsilon a^p + \frac{b^{p/(p-1)}}{\varepsilon^{1/(p-1)}}$, whence

$$\begin{aligned} &\int_\Omega \varphi(x)\zeta(x)^{m-1} |D\zeta(x)| \, dx \le \frac{1}{2} \int_\Omega \varphi(x)^{\frac{m}{m-1}} \zeta(x)^m \, dx + 2^m \int_\Omega |D\zeta(x)|^m \, dx \\ &\int_\Omega |\mathbf{u}|^{\frac{n(m-1)}{n-m}} \zeta(x)^{m-1} |D\zeta(x)| \, dx \le M^{\frac{n(m-1)}{n-m}} \left(\frac{1}{2} \int_\Omega \zeta(x)^m \, dx + 2^{m-1} \int_\Omega |D\zeta(x)|^m \, dx \right) \\ &\int_\Omega |D\mathbf{u}|^{m-1} \zeta(x)^{m-1} |D\zeta| \, dx \le \varepsilon \int_\Omega |D\mathbf{u}|^m \zeta(x)^m \, dx + \frac{1}{\varepsilon^{m-1}} \int_\Omega |D\zeta(x)|^m \, dx \end{aligned}$$

$$\int_{\Omega} |D\mathbf{u}|^{\frac{mn-n+m}{n}} \zeta(x)^m\, dx \le \varepsilon \int_{\Omega} |D\mathbf{u}|^m \zeta(x)^m\, dx + \varepsilon^{-\frac{mn-n+m}{n-m}} \int_{\Omega} \zeta(x)^m\, dx\,.$$

Unifying the above estimates we get

$$\begin{aligned} &\int_{\Omega} |Du^\alpha(x)|^m \zeta(x)^m\, dx \\ &\quad \le C \int_{\Omega} (1+\psi(x)+\varphi(x)^{\frac{m}{m-1}})\zeta(x)^m\, dx \\ &\quad + C\int_{\Omega} |D\zeta(x)|^m\, dx + \varepsilon C \int_{\Omega} |D\mathbf{u}(x)|^m \zeta(x)^m\, dx \end{aligned} \tag{23}$$

with constants depending on n, Λ, γ, M, and ε. Then we sum up (23) over α and fix ε small enough, such that to have the estimate

$$\int_{\mathscr{B}_\rho} |D\mathbf{u}|^m\, dx \le C \int_{\mathscr{B}_{2\rho}} (1+\psi(x)+\varphi(x)^{\frac{m}{m-1}})\, dx + C\int_{\mathscr{B}_{2\rho}} |D\zeta(x)|^m\, dx\,. \tag{24}$$

Then, by (10) we have

$$\begin{aligned} &\int_{\mathscr{B}_{2\rho}} \big(1+\psi(x)+\varphi(x)^{\frac{m}{m-1}}\big)\, dx \le C\big[\rho^n + \rho^{n-\frac{n-\mu}{q}} \|\psi\|_{q,\mu;\Omega} \\ &\qquad + \rho^{n-\frac{n-\lambda}{p}\frac{m}{m-1}} \|\varphi\|^{\frac{m}{m-1}}_{p,\lambda;\Omega}\big] \\ &\int_{\mathscr{B}_{2\rho}} |D\zeta(x)|^m\, dx \le C\rho^{n-m}\,. \end{aligned}$$

Hence

$$\int_{\mathscr{B}_\rho} |Du^\alpha|^m\, dx \le C\rho^{\lambda_0} \tag{25}$$

with $\lambda_0 = \min\left\{n - \frac{(n-\lambda)}{p}\frac{m}{m-1}, n-\frac{n-\mu}{q}, n-m\right\} = n-m$ and the constant depends on known quantities.

Let $\mathscr{B}_\rho(x_0)\cap\partial\Omega \ne \emptyset$. Then we extend u^α and the given functions φ and ψ as zero in Ω^c and consider the test functions

$$\phi^\alpha(x) = (e^{|u^\alpha(x)|}-1)\zeta(x)^m \operatorname{sign} u^\alpha(x)\,.$$

Then the estimate follows the same line of the proof as before.

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A Survey on the Oscillation of Delay Equations with A Monotone or Non-monotone Argument

G. M. Moremedi and I. P. Stavroulakis

Abstract Consider the first-order linear differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0,$$

where the functions $p, \tau \in C([t_{0,}\infty), \mathbb{R}^+)$, (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$. A survey on the oscillation of all solutions to this equation is presented in the case of monotone and non-monotone argument and especially in the critical case where $\liminf_{t\to\infty} p(t) = 1/e\tau$ and also when the known oscillation conditions $\limsup\limits_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1$ and $\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}$ are not satisfied. Examples illustrating the results are given.

Keywords Oscillation · Retarded · Differential equations · Non-monotone arguments

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1 Introduction

Consider the differential equation with a retarded argument of the form

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$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where the functions $p, \tau \in C([t_{0,}\infty), \mathbb{R}^+)$, (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

By a solution of the Eq. (1.1) we understand a continuously differentiable function defined on $[\tau(T_0), +\infty)$ for some $T_0 \geq t$ and such that (1.1) is satisfied for $t \geq T_{0.}$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

It is noteworthy to observe that a first-order linear differential equation of the form (1.1) without delay ($\tau(t) \equiv t$) *does not possess ocillatory solutions*. Therefore the investigation of oscillatory solutions is of interest for equations of the form (1.1). Furthermore, the mathematical modelling of several real-world problems leads to differential equations that depend on the past history (like equations of the form (1.1)) rather than only the current state. For the general theory of this equation the reader is referred to [15, 18–20].

In this paper we present a survey on the oscillation of all solutions to this equation in the case of a monotone or non-monotone argument and especially in the critical case where $\liminf_{t\to\infty} p(t) = \frac{1}{e\tau}$ and also when the well-known oscillation conditions

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1 \quad \text{and} \quad \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}.$$

are not satisfied.

2 Oscillation Criteria for Eq. (1.1)

In this section we study the delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where the functions $p, \tau \in C([t_{0,}\infty), \mathbb{R}^+)$, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

The problem of establishing sufficient conditions for the oscillation of all solutions to the delay differential equation (1.1) has been the subject of many investigations. See, for example, [1–40] and the references cited therein.

The first systematic study for the oscillation of all solutions to Eq. (1.1) was made by Myshkis. In 1950 [32] he proved that every solution of Eq. (1.1) oscillates if

$$\limsup_{t\to\infty}[t - \tau(t)] < \infty \quad \text{and} \quad \liminf_{t\to\infty}[t - \tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{e}.$$

In 1972, Ladas, Lakshmikantham and Papadakis [28] proved that the same conclusion holds if

$$\tau \text{ is a non-decreasing function and } A := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1. \tag{C_1}$$

In 1979, Ladas [27] established integral conditions for the oscillation of equation (1.1) with constant delay, while in 1982, Koplatadze and Canturija [24] established the following result. If

$$\mathfrak{a} := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{C_2}$$

then all solutions of Eq. (1.1) oscillate; If

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < \frac{1}{e}, \tag{N_1}$$

then Eq. (1.1) has a non-oscillatory solution.

Set

$$\mathcal{P} = \limsup_{t\to\infty} p(t)$$

and

$$\mathfrak{p} = \liminf_{t\to\infty} p(t).$$

Observe that in the case of the equation

$$x'(t) + p(t)x(t-\tau) = 0, \quad t \ge t_0, \tag{1.1$'$}$$

the results by Myshkis [32] reduce to the following conditions: If

$$\mathfrak{p}\tau > \frac{1}{e}, \tag{c_2}$$

then all solutions of Eq. (1.1)$'$ oscillate, while

$$\mathcal{P}\tau < \frac{1}{e} \tag{$(N_1)'$}$$

implies the existence of a non-oscillatory solution of (1.1)$'$. Thus, for the oscillation of all solutions to (1.1)$'$ a necessary condition is the following

$$\mathcal{P}\tau \ge \frac{1}{e}. \tag{2.1}$$

At this point it should be pointed out that in the special case of the delay equation with a constant positive coefficient p and a constant positive delay τ, that is in the case of the equation

$$x'(t) + px(t-\tau) = 0, \quad t \geq t_0, \tag{1.1$''$}$$

$$p\tau > \frac{1}{e} \tag{$(c_2)'$}$$

is a *necessary and sufficient condition* [29] for all solutions to $(1.1)''$ to oscillate.

In 2017, Pituk [34] studied the delay equation $(1.1)'$ in the case where the function $p \in C([t_{0,}\infty), \mathbb{R}^+)$ is *slowly varying at infinity,* that is, for every $s \in \mathbb{R}$,

$$p(t+s) - p(t) \to 0 \quad as \quad t \to \infty,$$

and proved the following theorem.

Theorem 2.1 ([34]) *Suppose that the function* p *is slowly varying at infinity and* $\mathfrak{p} > 0$. *Then*

$$\mathcal{P}\tau > \frac{1}{e}, \tag{$(C_2)'$}$$

implies that all solutions of Eq. $(1.1)'$ *oscillate.*

Remark 2.1 ([34]) It is easy to see that

$$\mathfrak{p}\tau \leq \mathfrak{a} \leq A \leq \mathcal{P}\tau.$$

Thus the above oscillation results by Ladas [27] and Koplatadze and Chanturija [24] imply the results by Myshkis [32]. As it is shown in [34], when the function p is slowly varying at infinity, then

$$\mathfrak{p}\tau = \mathfrak{a} \quad \text{and} \quad \mathcal{P}\tau = A. \tag{2.2}$$

Therefore in that case both results are equivalent. Moreover, condition (C_1) together with (2.2) implies that if p is slowly varying at infinity, then the condition

$$\mathcal{P}\tau > 1, \tag{$(C_1)'$}$$

guarantees the oscillation of all solutions to Eq. $(1.1)'$ Consequently, if instead of $(C_2)'$ the stronger condition $(C_1)'$ is assumed, then the uniform positivity condition $\mathfrak{p} > 0$ can be omitted.

Note the analogy of the conditions $(C_1)'$, (C_1) also $(C_2)'$, $(c_2)'$, (c_2), (C_2) and $(N_1)'$, (N_1)

Remark 2.2 ([34]) The conclusion of Theorem 2.1 does not hold if $(C_2)'$ is replaced by (2.1). Indeed, if $p(t) = \frac{1}{\tau e}$ identically for $t \geq t_0$, then the function p is slowly varying at infinity with $\mathfrak{p} = \mathcal{P} = \frac{1}{\tau e}$ so that $\mathcal{P}\tau = \frac{1}{e}$. Observe that in this case Eq. (1.1)′ admits a non-oscillatory solution given by $x(t) = e^{-t/\tau}$ for $t \geq t_0$. Furthermore in the case that $\mathfrak{p} = \mathcal{P} = \frac{1}{\tau e}$ so that $\mathcal{P}\tau = \frac{1}{e}$ and

$$p(t) \to \frac{1}{\tau e} \text{ as } t \to \infty,$$

although p is slowly varying at infinity, Theorem 2.1 does not apply because in this case the oscillation of all solutions depends on the rate of convergence of $p(t)$ to the limit $\frac{1}{\tau e}$ as $t \to \infty$ as it is explained below.

In 1995 Elbert and Stavroulakis [13] established sufficient conditions under which all solutions to Eq. (1.1) oscillate in the critical case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}.$$

In 1996 Domshlak [6, 7] investigated Eq. (1.1)′ in the *critical case* where $\mathfrak{p} = \frac{1}{\tau e}$ and sufficient conditions for the oscillation of all solutions were established in spite of the fact that the corresponding "limiting" equation

$$x'(t) + \frac{1}{\tau e} x(t-\tau) = 0, \quad t \geq t_0,$$

admits a non-oscillatory solution $x(t) = e^{-t/\tau}$. Indeed, in [6, 7] it was proved that if

$$\liminf_{t\to\infty} p(t) = \frac{1}{\tau e} \quad \text{and} \quad \liminf_{t\to\infty} \left[\left(p(t) - \frac{1}{\tau e} \right) t^2 \right] > \frac{\tau}{8e} \tag{2.3}$$

then all solutions of (1.1)′ oscillate.

Also in 1996 this result was improved by Domshlak and Stavroulakis [8] as follows.

Theorem 2.2 ([8]) *Assume that*

$$\liminf_{t\to\infty} p(t) = \frac{1}{\tau e}, \liminf_{t\to\infty} \left[\left(p(t) - \frac{1}{\tau e} \right) t^2 \right] = \frac{\tau}{8e}$$

and

$$C := \liminf_{t\to\infty} \left\{ \left[\left(p(t) - \frac{1}{\tau e} \right) t^2 - \frac{\tau}{8e} \right] \ln^2 t \right\} > \frac{\tau}{8e}. \tag{2.4}$$

Then all solutions of Eq. (1.1)′ oscillate.

Example 2.1 ([8]) Consider the equation (cf. Theorem 3 in [13])

$$x'(t) + p(t)x(t-1) = 0, \quad t \geq 1,$$

where

$$p(t) = \frac{(2t-1)\ln t - 1}{2e\sqrt{t(t-1)\ln t \ln(t-1)}}.$$

It is easy to see that $x(t) = e^{-t}\sqrt{t \ln t}$ is a non-oscillatory solution. In this case one can check that

$$\liminf_{t\to\infty} \left\{ \left[\left(p(t) - \frac{1}{\tau e} \right) t^2 - \frac{\tau}{8e} \right] \ln^2 t \right\} = \frac{1}{8e},$$

that is, condition (2.4) is not satisfied (as expected). Thus the inequality $C > \frac{\tau}{8e}$ can not be replaced by the corresponding equality.

Later in 1998 and 2000 the above results were extended by Diblik [9–11] using the *iterated* logarithm as follows. Call the expression $\ln_k t, k \geq 1$, defined by the formula

$$\ln_k t = \underbrace{\ln \ln \ldots \ln}_{k} t, \;\; k \geq 1$$

the kth *iterated* logarithm if $t > \exp_{k-2} 1$ where

$$\exp_k t \equiv (\underbrace{\exp(\exp(\ldots \exp}_{k} t))), \;\; k \geq 1,$$

$\exp_0 t \equiv t$ and $\exp_{-1} t \equiv 0$. Moreover, let us define $\ln_0 t \equiv t$ and also instead of expressions $\ln_0 t$, $\ln_1 t$, we will write only t and $\ln t$. Then the following results were established.

Theorem 2.3 ([9–11]) *If for some integer $k \geq 0$*

$$p(t) \leq \frac{1}{\mathrm{e}\tau} + \frac{\tau}{8\mathrm{e}t^2} + \frac{\tau}{8\mathrm{e}(t\ln t)^2} + \frac{\tau}{8\mathrm{e}(t \ln t \ln_2 t)^2} + \cdots + \frac{\tau}{8\mathrm{e}(t \ln t \ln_2 t \ldots \ln_k t)^2} \text{ as } t \to \infty,$$

then there exists a positive solution $x = x(t)$ of Eq. (1.1)$'$ and moreover,

$$x(t) < \mathrm{e}^{-t/\tau}\sqrt{t \ln t \ln_2 t \ldots \ln_k t} \text{ as } t \to \infty,$$

while if for a constant $\theta > 1$,

$$p(t) \geq \frac{1}{\mathrm{e}\tau} + \frac{\tau}{8\mathrm{e}t^2} + \frac{\tau}{8\mathrm{e}(t\ln t)^2} + \cdots + \frac{\tau}{8\mathrm{e}(t \ln t \ln_2 t \ldots \ln_{k-1} t)^2} + \frac{\theta\tau}{8\mathrm{e}(t \ln t \ln_2 t \ldots \ln_k t)^2} \tag{2.5}$$

as $t \to \infty$, then all solutions of Eq. (1.1)$'$ oscillate.

It is obvious that there is a gap between the conditions (C_1) and (C_2) when the limit $\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$ does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [16] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible non-oscillatory solutions $x(t)$ of Eq. (1.1). Their result says that all the solutions of Eq. (1.1) are oscillatory, if $0 < \mathfrak{a} \le \frac{1}{e}$ and

$$A > 1 - \frac{\mathfrak{a}^2}{4}. \qquad (C_3)$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$.

In 1991, Jian [22] derived the condition

$$A > 1 - \frac{\mathfrak{a}^2}{2(1-\mathfrak{a})}, \qquad (C_4)$$

while in 1992, Yu, Wang, Zhang and Qian [38] obtained the condition

$$A > 1 - \frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}. \qquad (C_5)$$

In 1990, Elbert and Stavroulakis [12] and in 1991 Kwong [26], using different techniques, improved (C_3), in the case where $0 < \mathfrak{a} \le \frac{1}{e}$, to the conditions

$$A > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2 \qquad (C_6)$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \qquad (C_7)$$

respectively, where $\lambda_1$is the smaller real root of the equation $\lambda = e^{\mathfrak{a}\lambda}$.

In 1998, Philos and Sficas [33] and in 1999, Zhou and Yu [40] and Jaroš and Stavroulakis [21] improved further the above conditions in the case where $0 < \mathfrak{a} \le \frac{1}{e}$ as follows

$$A > 1 - \frac{\mathfrak{a}^2}{2(1-\mathfrak{a})} - \frac{\mathfrak{a}^2}{2}\lambda_1, \qquad (C_8)$$

$$A > 1 - \frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2} - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2, \qquad (C_9)$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}, \qquad (C_{10})$$

respectively.

Consider equation (1.1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p(\tau(t))\tau'(t) \geq \theta p(t)$ eventually for all t. Under this additional assumption, in 2000, Kon, Sficas and Stavroulakis [23] and in 2003, Sficas and Stavroulakis [35] established the conditions

$$A > 2\mathfrak{a} + \frac{2}{\lambda_1} - 1, \qquad (C_{11})$$

and

$$A > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\mathfrak{a}\lambda_1}}{\lambda_1}, \qquad (C_{12})$$

respectively. In the case where $\mathfrak{a} = \frac{1}{e}$, then $\lambda_1 = e$, and (C_{12}) leads to

$$A > \frac{\sqrt{7 - 2e}}{e} \approx 0.459987065.$$

It is to be noted that for small values of $\mathfrak{a}$ ($\mathfrak{a} \to 0$), all the previous conditions $(C_3) - (C_{11})$ reduce to the condition (C_1), i.e. $A > 1$. However, the condition (C_{12}) leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is an essential improvement. Moreover (C_{12}) improves all the above conditions for all values of $\mathfrak{a} \in (0, \frac{1}{e}]$. Note that the value of the lower bound on A can not be less than $\frac{1}{e} \approx 0.367879441$. Thus, the aim is to establish a condition which leads to a value *as close as possible to* $\frac{1}{e}$.

For illustrative purpose, we give the values of the lower bound on A under these conditions when (i) $\mathfrak{a} = 1/1000$ and (ii) $\mathfrak{a} = 1/e$.

	(i)	(ii)
(C_3) :	0.999999750	0.966166179
(C_4) :	0.999999499	0.892951367
(C_5) :	0.999999499	0.863457014
(C_6) :	0.999999749	0.845181878
(C_7) :	0.999999499	0.735758882
(C_8) :	0.999998998	0.709011646
(C_9) :	0.999999249	0.708638892
(C_{10}) :	0.999998998	0.599215896
(C_{11}) :	0.999999004	0.471517764
(C_{12}) :	0.733050517	0.459987065

We see that the condition (C_{12}) essentially improves all the known results in the literature.

Moreover, it should be pointed out that in 1994, Koplatadze and Kvinikadze [25] improved (C_5) as follows: Assume

$$\sigma(t) := \sup_{s \le t} \tau(s), \quad t \ge 0. \tag{2.6}$$

Clearly $\sigma(t)$ is non-decreasing and $\tau(t) \le \sigma(t)$ for all $t \ge 0$. Define

$$\psi_1(t) = 0, \psi_i(t) = \exp\left\{\int_{\tau(t)}^{t} p(\xi)\psi_{i-1}(\xi)d\xi\right\}, i = 2, 3, \dots \text{ for } t \in \mathbb{R}^+. \tag{2.7}$$

Then the following theorem was established in [25].

Theorem 2.4 ([25]) *Let $k \in \{1, 2, \dots\}$ exist such that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{\int_{\sigma(s)}^{\sigma(t)} p(\xi)\psi_k(\xi)d\xi\right\} ds > 1 - c(\mathfrak{a}), \tag{2.8}$$

where $\sigma, \psi_k, \mathfrak{a}$ are defined by (2.6), (2.7),(C_2) respectively, and

$$c(\mathfrak{a}) = \begin{cases} 0 & \text{if } \mathfrak{a} > \frac{1}{e}, \\ \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}\right) & \text{if } 0 < \mathfrak{a} \le \frac{1}{e}. \end{cases} \tag{2.9}$$

Then all solutions of Eq. (1.1) oscillate.

Concerning the constants 1 and $\frac{1}{e}$ which appear in the conditions (C_1), (C_2) and (N_1), in 2011, Berezansky and Braverman [1] established the following:

Theorem 2.5 ([1]) *For any $\alpha \in (1/e, 1)$ there exists a non-oscillatory equation*

$$x'(t) + p(t)x(t - \tau) = 0, \quad \tau > 0$$

with $p(t) \ge 0$ such that

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds = \alpha.$$

Also in 2011, Braverman and Karpuz [2] investigated equation (1.1) in the case of a general argument (τ is not assumed monotone) and proved that:

Theorem 2.6 ([2]) *There is no constant $K > 0$ such that*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > K \tag{2.10}$$

implies oscillation of equation (1.1) for arbitrary (not necessarily non-decreasing) argument $\tau(t) \le t$.

Remark 2.3 Observe that, because of the condition (N_1), the constant K in the above inequality makes sense for $K > 1/e$.

Moreover in [2] the following result was established.

Theorem 2.7 ([2]) *Assume that*

$$B := \limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{\int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi\right\} ds > 1, \tag{2.11}$$

where $\sigma(t)$ *is defined by (2.6). Then all solutions of Eq. (1.1) oscillate.*

Observe that condition (2.11) improves (C_1).

Using the upper bound of the ratio $\frac{x(\tau(t))}{x(t)}$ for possible non-oscillatory solutions $x(t)$ of Eq. (1.1), presented in [12, 21, 23, 35], the above result was recently essentially improved in [36].

Theorem 2.8 ([36]) *Assume that* $0 < \mathfrak{a} \le \frac{1}{e}$ *and*

$$B := \limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{\int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi\right\} ds > 1 - \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}\right) \tag{2.12}$$

where $\sigma(t)$ *is defined by (2.6). Then all solutions of Eq. (1.1) oscillate.*

Remark 2.4 ([36]) Observe that as $\mathfrak{a} \to 0$, then condition (2.12) reduces to (2.11). However the improvement is clear as $\mathfrak{a} \to \frac{1}{e}$. Actually, when $\mathfrak{a} = \frac{1}{e}$, the value of the lower bound on B is equal to $\approx$0.863457014. That is, (2.12) essentially improves (2.11).

Remark 2.5 ([36]) Note that, under the additional assumption that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p(\tau(t))\tau'(t) \ge \theta p(t)$ eventually for all t, (see [23, 35]) the condition (2.12) of Theorem 2.8 reduces to

$$B > 1 - \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{(1-\mathfrak{a})^2 - 4M}\right), \tag{2.12$'$}$$

where M is given by

$$M = \frac{e^{\lambda_1\theta\mathfrak{a}} - \lambda_1\theta\mathfrak{a} - 1}{(\lambda_1\theta)^2}$$

and λ_1 is the smaller root of the equation $\lambda = e^{\lambda\mathfrak{a}}$. When $\theta = 1$, then from [35] it follows that

$$\frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{(1-\mathfrak{a})^2 - 4M}\right) = 1 - \mathfrak{a} - \frac{1}{\lambda_1}$$

and in the case that $\mathfrak{a} = \frac{1}{e}$, then $\lambda_1 = e$ and $(2.12)'$ leads to

$$B > 1 - \left(1 - \frac{2}{e}\right) = \frac{2}{e} \approx 0.735758882.$$

That is, condition (2.12)′ essentially improves (2.12) but of course under the additional (stronger) assumptions on $\tau(t)$ and $p(t)$.

The following example illustrates the significance of Theorem 2.8.

Example 2.2 (cf. [2, 36]) Consider the equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq 0, \tag{2.13}$$

where $p(t) = \frac{0.93}{e}$, and

$$\tau(t) := \begin{cases} t - 1, & t \in [3n, 3n+1] \\ -3t + (12n+3), & t \in [3n+1, 3n+2] \\ 5t - (12n+13), & t \in [3n+2, 3n+3]. \end{cases}$$

We see that

$$\sigma(t) = \begin{cases} t - 1, & t \in [3n, 3n+1] \\ 3n, & t \in [3n+1, 3n+2.6] \\ 5t - (12n+13), & t \in [3n+2.6, 3n+3]. \end{cases}$$

Observe that

$$\mathfrak{a} = \liminf_{t\to\infty} \int_{\tau(t)}^{t} \frac{0.93}{e} ds = \frac{0.93}{e} \approx 0.34212788 < \frac{1}{e},$$

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \frac{0.93}{e} ds = 2.6\frac{0.93}{e} = 0.889532488 < 1.$$

Moreover, for $n \geq 0$ we have

$$\begin{aligned} \int_{\sigma(3n+3)}^{3n+3} \frac{0.93}{e} \exp\left\{\int_{\tau(s)}^{\sigma(3n+3)} \frac{0.93}{e} d\xi\right\} ds &= \int_{3n+2}^{3n+3} \frac{0.93}{e} \exp\left\{\int_{5s-(12n+13)}^{3n+2} \frac{0.93}{e} d\xi\right\} ds \\ &= \int_{3n+2}^{3n+3} \frac{0.93}{e} \exp\left\{\frac{4.65}{e}[3n+3-s]\right\} ds \\ &= \frac{1}{5}\left[\exp\left\{\frac{4.65}{e}\right\} - 1\right] \approx 0.906499566 < 1. \end{aligned}$$

That is, the conditions (C_1), (C_2), and (2.11) are not satisfied. Observe, however, that for $\mathfrak{a} \approx 0.34212788$

$$1-\frac{1}{2}\left(1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}\right)\approx 0.893938766$$

and we see that

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}p(s)\exp\left\{\int_{\tau(s)}^{\sigma(t)}p(\xi)d\xi\right\}ds>0.90>1-\frac{1}{2}\left(1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}\right)\approx 0.89,$$

that is, the conditions of Theorem 2.8 are satisfied and therefore all solutions of the Eq. (2.13) are oscillatory.

In 2016, El- Morshedy and Attia [14] proved that, if

$$\limsup_{t\to\infty}\left[\int_{g(t)}^{t}p_n(s)ds+c(\mathfrak{a})\exp\left(\int_{g(t)}^{t}\sum_{j=0}^{n-1}p_j(s)ds\right)\right]>1, \tag{2.14}$$

where

$$p_n(t)=p_{n-1}(t)\int_{g(t)}^{t}p_{n-1}(s)\exp\left(\int_{g(s)}^{t}p_{n-1}(u)du\right)ds,\ n\geq 1,\text{ with } p_0(t)=p(t) \tag{2.15}$$

and $c(\mathfrak{a})$ is given by (2.9), then all solutions of (1.1) oscillate. Here, $g(t)$ is a non-decreasing continuous function such that $\tau(t)\leq g(t)\leq t$, $t\geq t_1$ for some $t_1\geq t_0$. Clearly, $g(t)$ is more general than $\sigma(t)$ given by (2.6).

Recently, Chatzarakis [3, 4], proved that if for some $j\in\mathbb{N}$

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)}p_j(u)du\right)ds>1 \tag{2.16}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)}p_j(u)du\right)ds>1-\frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}, \tag{2.17}$$

where

$$p_j(t)=p(t)\left[1+\int_{\tau(t)}^{t}p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)}p_{j-1}(u)du\right)ds\right],\text{ with } p_0(t)=p(t), \tag{2.18}$$

and $0<\mathfrak{a}\leq\frac{1}{e}$, then all solutions of (1.1) oscillate.

Very recently, Chatzarakis, Purnaras and Stavroulakis [5] improved the above conditions as follows.

Theorem 2.9 ([5]) *Assume that for some* $j\in\mathbb{N}$

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)}P_j(u)du\right)ds>1, \tag{2.19}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_j(u)du\right)ds > 1-\frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}, \tag{2.20}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} P_j(u)du\right)ds > \frac{2}{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}, \tag{2.21}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)} P_j(u)du\right)ds > \frac{1+\ln\lambda_1}{\lambda_1}-\frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}, \tag{2.22}$$

where

$$P_j(t) = p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_{j-1}(\xi)d\xi\right)du\right)ds\right], \tag{2.23}$$

with $P_0(t) = p(t)$. $0 < \mathfrak{a} \le \frac{1}{e}$, *and* λ_1 *is the smaller root of the transcendental equation* $\lambda = e^{\mathfrak{a}\lambda}$. *Then all solutions of (1.1) oscillate.*

Theorem 2.10 ([5]) *Assume that for some* $j \in \mathbb{N}$

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)} P_j(u)du\right)ds > \frac{1}{e}, \tag{2.24}$$

where P_j *is defined by (2.23). Then all solutions of (1.1) oscillate.*

Before closing this section we note that one can easily see that the conditions (2.19), (2.20), (2.22), and (2.24) substantially improve the conditions (C_1), (2.11), (2.16), (2.12), (C_{10}) and (C_2). That can immediately be observed, if we compare the corresponding parts on the left-hand side of these conditions.

3 Examples

The examples below illustrate that the oscillation conditions presented in Theorems 2.9 and 2.10 essentially improve known results in the literature yet indicate a type of independence among some of them. The calculations were made by the use of MATLAB software.

Example 3.1 ([5]) Consider the retarded differential equation

$$x'(t) + \frac{1}{8}x(\tau(t)) = 0, \qquad t \ge 0, \tag{3.1}$$

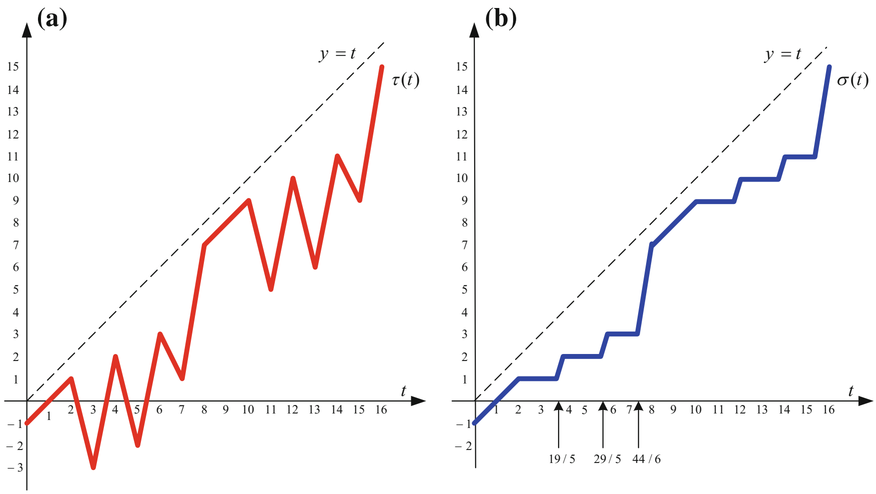


Fig. 1 The graphs of $\tau(t)$ and $\sigma(t)$

with (see Fig. 1a)

$$\tau(t) = \begin{cases} t-1, & \text{if } t \in [8k, 8k+2] \\ -4t+40k+9, & \text{if } t \in [8k+2, 8k+3] \\ 5t-32k-18, & \text{if } t \in [8k+3, 8k+4] \\ -4t+40k+18, & \text{if } t \in [8k+4, 8k+5] \\ 5t-32k-27, & \text{if } t \in [8k+5, 8k+6] \\ -2t+24k+15, & \text{if } t \in [8k+6, 8k+7] \\ 6t-40k-41, & \text{if } t \in [8k+7, 8k+8] \end{cases}, \quad k \in \mathbb{N}_0,$$

where $\mathbb{N}_0$ is the set of non-negative integers.

By (2.6), we see (Fig. 1b) that

$$\sigma(t) = \begin{cases} t-1, & \text{if } t \in [8k, 8k+2] \\ 8k+1, & \text{if } t \in [8k+2, 8k+19/5] \\ 5t-32k-18, & \text{if } t \in [8k+19/5, 8k+4] \\ 8k+2, & \text{if } t \in [8k+4, 8k+29/5] \\ 5t-32k-27, & \text{if } t \in [8k+29/5, 8k+6] \\ 8k+3, & \text{if } t \in [8k+6, 8k+44/6] \\ 6t-40k-41, & \text{if } t \in [8k+44/6, 8k+8] \end{cases}, \quad k \in \mathbb{N}_0.$$

Let the function $F_j : \mathbb{R}_0 \to \mathbb{R}_+$ ($j \in \mathbb{N}$) be defined by

$$F_j(t) = \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(t)} P_j(u)du \right) ds, \tag{3.2}$$

with P_j given by (2.23). Noting that F_j attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, for every $j \in \mathbb{N}$, and using an algorithm on MATLAB software, we obtain

$$\limsup_{t\to\infty} F_1(t) = \limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)du\right) ds \simeq 1.0097 > 1.$$

That is, condition (2.19) of Theorem 2.9 is satisfied for $j = 1$, and therefore all solutions of (3.1) oscillate.

Observe, however, that

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s)ds = \limsup_{k\to\infty} \int_{8k+3}^{8k+44/6} \frac{1}{8} ds = 0.5417 < 1,$$

$$\mathfrak{a} = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \liminf_{k\to\infty} \int_{8k+1}^{8k+2} \frac{1}{8} ds = 0.125 < \frac{1}{e},$$

$$0.5417 < \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2} \simeq 0.9815,$$

where $\lambda_1 = 1.15537$ is the smaller solution of $e^{0.125\lambda} = \lambda$.

Noting that the function Φ_j defined by

$$\Phi_j(t) = \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\sigma(s)}^{\sigma(t)} p(u)\psi_j(u)du\right) ds, \ (j \geq 2), \tag{3.3}$$

(with ψ_j defined by (2.7)) attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$ for every $j \geq 2$. Specifically, we find

$$\limsup_{t\to\infty} \Phi_2(t) \simeq 0.6450 < 1 - \frac{1-\mathfrak{a}-\sqrt{1-\mathfrak{a}-\mathfrak{a}^2}}{2} \simeq 0.99098.$$

Also

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} p(u)du\right) ds \simeq 0.74354 < 1$$

and

$$0.74354 < 1 - \frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2} \simeq 0.99098.$$

As each one of the functions G_j ($j \in \mathbb{N}$) defined by

$$G_j(t) = \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} p_j(u)du\right) ds, \qquad (j \in \mathbb{N}) \tag{3.4}$$

attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, for every $j \in \mathbb{N}$ we find

$$\limsup_{t\to\infty} G_1(t) = \limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(t)} p_1(u)du\right) ds \simeq 0.8626 < 1$$

and

$$0.8626 < 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} \simeq 0.99098.$$

That is, none of the conditions (C_1), (C_2) (C_{10}), (2.8) (for $j = 2$), (2.11), (2.12) and (2.16) (for $j = 1$), is satisfied. In addition, observe that conditions (2.8) and (2.16) do not lead to oscillation at the first iteration. On the contrary, condition (2.19) is satisfied from the first iteration, which means that it is much faster than (2.8) and (2.16).

In addition,

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} P_1(u)du\right) ds \simeq 4.8243 < \frac{2}{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}} \simeq 110.85,$$

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(s)} P_1(u)du\right) ds \simeq 0.7983$$
$$< \frac{1 + \ln \lambda_1}{\lambda_1} - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} \simeq 0.9815,$$

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(s)} P_1(u)du\right) ds = 0.125 < \frac{1}{e},$$

that is, none of the conditions (2.21) (for $j = 1$), (2.22) (for $j = 1$) and (2.24) (for $j = 1$), is satisfied.

The next example concerns the condition (2.20) of Theorem 2.9. It will be apparent that it may imply oscillation when other known criteria cited in the paper (including condition (2.19)) fail.

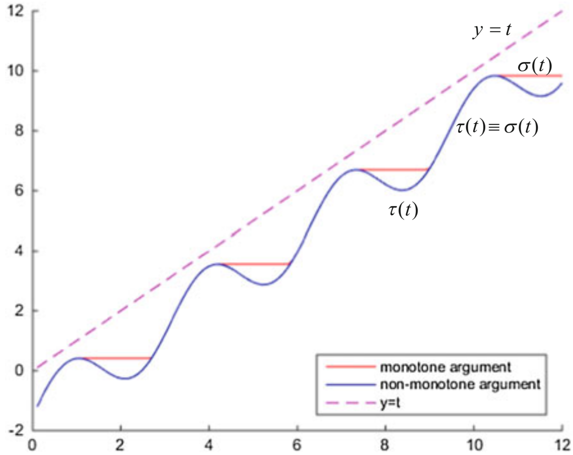
Example 3.2 ([5]) Consider the retarded differential equation

$$x'(t) + \frac{25}{27e} x(\tau(t)) = 0, \quad t \geq 0, \tag{3.5}$$

with (see Fig. 2, blue line)

$$\tau(t) = t - 1.5 + \sin(2t), \; t \geq 0.$$

Fig. 2 The graphs of $\tau(t)$ and $\sigma(t)$



By (2.6), we see (Fig. 2, red line) that

$$\sigma(t) = \begin{cases} t - 1.5 + \sin(2t), & \text{if } t \in [0, \pi/3] \cup \bigcup_{k=0}^{\infty} [2.6938 + k\pi, (k+1)\pi + \pi/3] \\ \frac{2\pi - 9 + 3\sqrt{3}}{6} + k\pi & \text{if } t \in \bigcup_{k=0}^{\infty} [k\pi + \pi/3, 2.6938 + k\pi] \end{cases}.$$

It is easy to see that

$$\mathfrak{a} = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \liminf_{k\to\infty} \int_{\pi/4+k\pi-0.5}^{\pi/4+k\pi} \frac{25}{27e} ds \simeq 0.170314556 < \frac{1}{e}.$$

Observe that the function F_j defined by (3.2) in Example 3.1, attains its maximum at $t = 2.6938 + k\pi$, $k \in \mathbb{N}_0$, for every $j \in \mathbb{N}$. By using an algorithm on MATLAB software, we obtain

$$\limsup_{t\to\infty} F_1(t) \simeq 0.9836 > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} \simeq 0.9629.$$

That is, condition (2.20) of Theorem 2.9 is satisfied for $j = 1$, and therefore all solutions of (3.5) oscillate.

However,

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s)ds = \limsup_{k\to\infty} \int_{\frac{2\pi-9+3\sqrt{3}}{6}+k\pi}^{2.6938+k\pi} \frac{25}{27e} ds \simeq 0.7768 < 1,$$

and the value of the constant $\mathfrak{a}$ is found to be

$$\mathfrak{a} \simeq 0.170314556 < \frac{1}{e}.$$

Consequently, the smaller root of the equation $e^{\mathfrak{a}\lambda} = \lambda$ is approximately $\lambda_1 = 1.23386$, so

$$0.7768 < \frac{1 + \ln \lambda_1}{\lambda_1} - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} \simeq 0.9629,$$

indicating that condition (C_{10}) does not hold.

Observe that the function Φ_2 defined by (3.3) in Example 3.1 attains its maximum at $t = 2.6938 + k\pi$, $k \in \mathbb{N}_0$. Specifically, we find

$$\limsup_{t\to\infty} \Phi_2(t) \simeq 0.7971 < 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} \simeq 0.9821,$$

and

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(t)} p(u)du\right) ds \simeq 0.8776 < 1,$$

$$0.8776 < 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} \simeq 0.9821.$$

Also, specifically for the function $G_1 : \mathbb{R}_0 \to \mathbb{R}_+$ defined by (3.4) in Example 3.1, we find

$$\limsup_{t\to\infty} G_1(t) = \limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(t)} p_1(u)du\right) ds \simeq 0.9555 < 1,$$

so we see that

$$0.9555 < 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} \simeq 0.9821.$$

That is, none of conditions (2.19) (for $j = 1$), (C_1), (C_2) (C_{10}), (2.8) (for $j = 2$), (2.11), (2.12), (2.16) (for $j = 1$) and (2.17) (for $j = 1$), is satisfied. In addition, observe that conditions (2.19), (2.8), (2.16) and (2.17) do not lead to oscillation at the first iteration. On the contrary, condition (2.20) is satisfied from the first iteration, which means that it is much faster than (2.19), (2.17), (2.16) and (2.8).

In addition,

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} P_1(u)du\right) ds \simeq 3.87 < \frac{2}{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}} \simeq 55.974,$$

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)} P_1(u)du\right)ds \simeq 0.170314556 < \frac{1}{e}.$$

That is, none of the conditions (2.21) (for $j = 1$) and (2.24) (for $j = 1$) is satisfied.

The last example deals with the condition (2.21) of Theorem 2.9.

Example 3.3 ([5]) Consider the retarded differential equation

$$x'(t) + \frac{97}{625}x(\tau(t)) = 0, \quad t \geq 0, \tag{3.6}$$

where $\tau(t)$ is defined as in Example 3.1.

It is easy to see that

$$\mathfrak{a} = \liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds = \liminf_{k\to\infty}\int_{7k+1}^{7k+2} p(s)ds = 0.1552 < \frac{1}{e}.$$

As before, we may see that the function $\widehat{F}_j$ $(j \in \mathbb{N})$ defined by

$$\widehat{F}_j(t) = \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} P_j(u)du\right)ds, \qquad (j \in \mathbb{N}),$$

attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, for every $j \in \mathbb{N}$. An algorithm on MATLAB software gives

$$\limsup_{t\to\infty}\widehat{F}_1(t) \simeq 69.8327 > \frac{2}{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}} \simeq 68.9412,$$

that is, condition (2.21) of Theorem 2.9 is satisfied for $j = 1$, and therefore all solutions of (3.6) oscillate.

However, we find

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)ds = \limsup_{k\to\infty}\int_{8k+3}^{8k+44/6}\frac{97}{625}ds \simeq 0.6725 < 1,$$

and since $\mathfrak{a} = 0.1552 < \frac{1}{e}$,

$$0.6725 < \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2} \simeq 0.97,$$

where $\lambda_1 = 1.2058$ is the smaller solution of $e^{0.1552\lambda} = \lambda$.

Recalling that the function Φ_j defined as in Example 3.1, attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, for every $j \geq 2$. Specifically, we find

$$\limsup_{t\to\infty} \Phi_2(t) \simeq 0.84 < 1 - \frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2} \simeq 0.9855,$$

that is, none of conditions (C_1), (C_2), (C_{10}) and (2.8) (for $j = 1$) is satisfied.

In addition,

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(s)} P_1(u)du\right) ds = 0.1552 < \frac{1}{e},$$

that is, condition (2.24) (for $j = 1$) is not satisfied.

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Discrete Versions of Some Dirac Type Equations and Plane Wave Solutions

Volodymyr Sushch

Abstract A discrete version of the plane wave solution to some discrete Dirac type equations in the spacetime algebra is established. The conditions under which a discrete analogue of the plane wave solution satisfies the discrete Hestenes equation are briefly discussed.

Keywords Dirac–Kähler equation · Hestenes equation · Clifford product
Spacetime algebra · Plane wave solution · Discrete models

MSC 81Q05 · 39A12 · 39A70

1 Introduction

This work is a direct continuation of that described in my previous papers [11, 12]. In [11], a discrete analogue of the Dirac equation for a free electron in the Hestenes form was constructed based on the discretization scheme [10]. In [12], a relationship between the discrete Dirac–Kähler equation and discrete analogues of some Dirac type equations in the spacetime algebra was discussed. In this paper, we establish a discrete version of plane wave solutions to discrete Dirac type equations.

We first briefly review some definitions and basic facts on the Dirac–Kähler equation [7, 8] and the Dirac equation in the spacetime algebra [4, 5]. Let $M = \mathbb{R}^{1,3}$ be Minkowski space with metric signature $(+, -, -, -)$. Denote by $\Lambda^r(M)$ the vector space of smooth differential r-forms, $r = 0, 1, 2, 3, 4$. We consider $\Lambda^r(M)$ over $\mathbb{C}$. Let $d : \Lambda^r(M) \to \Lambda^{r+1}(M)$ be the exterior differential and let $\delta : \Lambda^r(M) \to \Lambda^{r-1}(M)$ be the formal adjoint of d with respect to the natural inner product in $\Lambda^r(M)$. We have $\delta = *d*$, where $*$ is the Hodge star operator $* : \Lambda^r(M) \to \Lambda^{4-r}(M)$ with respect to the Lorentz metric. Denote by $\Lambda(M)$ the set of all differential forms on M. We have

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$$\Lambda(M) = \Lambda^{ev}(M) \oplus \Lambda^{od}(M),$$

where $\Lambda^{ev}(M) = \Lambda^0(M) \oplus \Lambda^2(M) \oplus \Lambda^4(M)$ and $\Lambda^{od}(M) = \Lambda^1(M) \oplus \Lambda^3(M)$.

Let $\Omega \in \Lambda(M)$ be an inhomogeneous differential form, i.e. $\Omega = \sum_{r=0}^{4} \overset{r}{\omega}$, where $\overset{r}{\omega} \in \Lambda^r(M)$. The Dirac–Kähler equation is given by

$$i(d+\delta)\Omega = m\Omega, \tag{1}$$

where i is the usual complex unit and m is a mass parameter. It is easy to show that Eq. (1) is equivalent to the set of equations

$$\begin{aligned} i\delta\overset{1}{\omega} &= m\overset{0}{\omega}, \\ i(d\overset{0}{\omega} + \delta\overset{2}{\omega}) &= m\overset{1}{\omega}, \\ i(d\overset{1}{\omega} + \delta\overset{3}{\omega}) &= m\overset{2}{\omega}, \\ i(d\overset{2}{\omega} + \delta\overset{4}{\omega}) &= m\overset{3}{\omega}, \\ id\overset{3}{\omega} &= m\overset{4}{\omega}. \end{aligned}$$

The operator $d+\delta$ is an analogue of the gradient operator $\nabla = \sum_{\mu=0}^{3} \gamma_\mu \partial^\mu$ in Minkowski spacetime, where γ_μ is the Dirac gamma matrix and ∂^μ is a partial derivative. Think of $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ as a vector basis in spacetime. Then the gamma matrices γ_μ can be considered as generators of the Clifford algebra $C\ell(1,3)$ [1, 2]. Hestenes [5] calls this algebra the spacetime algebra. Denote by $C\ell_{\mathbb{R}}(1,3)$ ($C\ell_{\mathbb{C}}(1,3)$) the real (complex) Clifford algebra. It is known that an inhomogeneous form Ω can be represented as element of $C\ell_{\mathbb{C}}(1,3)$. Then the Dirac–Kähler equation can be written as the algebraic equation

$$i\nabla\Omega = m\Omega, \quad \Omega \in C\ell_{\mathbb{C}}(1,3). \tag{2}$$

Equation (2) is equivalent to the four Dirac equations (traditional column-spinor equations) for a free electron. Let $C\ell^{ev}(1,3)$ be the even subalgebra of the algebra $C\ell(1,3)$. The equation

$$-\nabla\Omega^{ev}\gamma_1\gamma_2 = m\Omega^{ev}\gamma_0, \quad \Omega^{ev} \in C\ell^{ev}_{\mathbb{R}}(1,3), \tag{3}$$

is called the Hestenes form of the Dirac equation [4, 5]. Consider also the equation

$$i\nabla\Omega^{ev} = m\Omega^{ev}\gamma_0, \quad \Omega^{ev} \in C\ell^{ev}_{\mathbb{C}}(1,3). \tag{4}$$

In [6], this equation is called the “generalized bivector Dirac equation”. Following Baylis [2] we call Eq. (4) the Joyce equation. This equation admits the plane wave solution of the form

$$\Psi = A \exp\left(i \sum_{\mu=0}^{3} p^{\mu} x_{\mu}\right), \tag{5}$$

where $A \in C\ell^{ev}_{\mathbb{C}}(1,3)$ is a constant element and $\{p^0, p^1, p^2, p^3\}$ is a four-momentum. Suppose that for exterior forms (elements of $\Lambda(M)$) the Clifford multiplication is defined. It should be noted that the graded algebra $\Lambda(M)$ endowed with the Clifford multiplication is an example of the Clifford algebra. In this case the basis covectors $e^{\mu} = dx^{\mu}$, $\mu = 0, 1, 2, 3$, of spacetime are considered as generators of the Clifford algebra. Let $\Lambda_{\mathbb{R}}(M)$ denote the set of real-valued differential forms. Then Eqs. (3) and (4) can be rewritten in terms of inhomogeneous forms as

$$-(d+\delta)\Omega^{ev} e^1 e^2 = m\Omega^{ev} e^0, \quad \Omega^{ev} \in \Lambda^{ev}_{\mathbb{R}}(M), \tag{6}$$

and

$$i(d+\delta)\Omega^{ev} = m\Omega^{ev} e^0, \quad \Omega^{ev} \in \Lambda^{ev}(M). \tag{7}$$

The aim of the present paper is to construct a discrete version of the plane wave solution (5). In much the same way as in the continuum case [2, 6] we show that the discrete Joyce equation admits eight linearly independent plane wave solutions in the discrete formulation. We briefly discuss the conditions under which the obtained plane wave solutions satisfy the discrete Hestenes equation.

2 Discrete Dirac–Kähler equation

In this section, we start off with a discretization scheme. The scheme is based on the language of differential forms and is described in [10]. This approach was originated by Dezin in [3]. Due to space limitations, we skip the relevant material from [10]. For the convenience of the reader, we fix only some notation and recall some facts concerning discrete analogues of the differential operators d and δ. All details can be found in [9, 10].

Let $K(4) = K \otimes K \otimes K \otimes K$ be a cochain complex with complex coefficients, where K is the 1-dimensional complex generated by 0- and 1-dimensional basis elements x^{κ} and e^{κ}, $\kappa \in \mathbb{Z}$, respectively. Then an arbitrary r-dimensional basis element of $K(4)$ can be written as $s^k_{(r)} = s^{k_0} \otimes s^{k_1} \otimes s^{k_2} \otimes s^{k_3}$, where s^{k_μ} is either x^{k_μ} or e^{k_μ}, $k = (k_0, k_1, k_2, k_3)$ and $k_\mu \in \mathbb{Z}$. Let

$$x^k = x^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes x^{k_3}, \qquad e^k = e^{k_0} \otimes e^{k_1} \otimes e^{k_2} \otimes e^{k_3}$$

denote the 0- and 4-dimensional basis elements of $K(4)$. The dimension r of a basis element $s^k_{(r)}$ is given by the number of factors e^{k_μ} that appear in it. For example, the 1-dimensional basis element $e^k_\mu \in K(4)$ can be written as

$$e_0^k = e^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes x^{k_3}, \qquad e_1^k = x^{k_0} \otimes e^{k_1} \otimes x^{k_2} \otimes x^{k_3},$$
$$e_2^k = x^{k_0} \otimes x^{k_1} \otimes e^{k_2} \otimes x^{k_3}, \qquad e_3^k = x^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes e^{k_3},$$

where the subscript $\mu = 0, 1, 2, 3$ indicates a place of e^{k_μ} in e_μ^k. Similarly, $e_{\mu\nu}^k$, $\mu < \nu$, and $e_{\iota\mu\nu}^k$, $\iota < \mu < \nu$, denote the 2- and 3-dimensional basic elements of $K(4)$. The complex $K(4)$ is a discrete analogue of $\Lambda(M)$ and cochains play the role of differential forms. Let us call them forms or discrete forms to emphasize their relationship with differential forms. Denote by $K^r(4)$ the set of all r-forms. Then we have

$$K(4) = K^{ev}(4) \oplus K^{od}(4),$$

where $K^{ev}(4) = K^0(4) \oplus K^2(4) \oplus K^4(4)$ and $K^{od}(4) = K^1(4) \oplus K^3(4)$. Any r-form $\overset{r}{\omega} \in K^r(4)$ can be expressed as

$$\overset{0}{\omega} = \sum_k \overset{0}{\omega}_k x^k, \qquad \overset{2}{\omega} = \sum_k \sum_{\mu<\nu} \omega_k^{\mu\nu} e_{\mu\nu}^k, \qquad \overset{4}{\omega} = \sum_k \overset{4}{\omega}_k e^k, \tag{8}$$

$$\overset{1}{\omega} = \sum_k \sum_{\mu=0}^{3} \omega_k^{\mu} e_{\mu}^k, \qquad \overset{3}{\omega} = \sum_k \sum_{\iota<\mu<\nu} \omega_k^{\iota\mu\nu} e_{\iota\mu\nu}^k, \tag{9}$$

where $\overset{0}{\omega}_k$, $\omega_k^{\mu\nu}$, $\overset{4}{\omega}_k$, ω_k^{μ} and $\omega_k^{\iota\mu\nu}$ are complex numbers. A discrete inhomogeneous form $\Omega \in K(4)$ is defined to be

$$\Omega = \sum_{r=0}^{4} \overset{r}{\omega}. \tag{10}$$

Let $d^c : K^r(4) \to K^{r+1}(4)$ be a discrete analogue of the exterior derivative d and let $\delta^c : K^r(4) \to K^{r-1}(4)$ be a discrete analogue of the codifferential δ. For more precise definitions of these operators we refer the reader to [10]. In this paper we give only the difference expressions for d^c and δ^c. Let the difference operator Δ_μ be defined by

$$\Delta_\mu \omega_k^{(r)} = \omega_{\tau_\mu k}^{(r)} - \omega_k^{(r)}, \tag{11}$$

where $\omega_k^{(r)} \in \mathbb{C}$ is a component of $\overset{r}{\omega} \in K^r(4)$ and τ_μ is the shift operator which acts as $\tau_\mu k = (k_0, \dots k_\mu + 1, \dots k_3)$, $\mu = 0, 1, 2, 3$. For forms (8), (9) we have

$$d^c \overset{0}{\omega} = \sum_k \sum_{\mu=0}^{3} (\Delta_\mu \overset{0}{\omega}_k) e_\mu^k, \qquad d^c \overset{1}{\omega} = \sum_k \sum_{\mu<\nu} (\Delta_\mu \omega_k^{\nu} - \Delta_\nu \omega_k^{\mu}) e_{\mu\nu}^k, \tag{12}$$

$$d^c\overset{2}{\omega} = \sum_k \big[(\Delta_0\omega_k^{12} - \Delta_1\omega_k^{02} + \Delta_2\omega_k^{01})e_{012}^k + (\Delta_0\omega_k^{13} - \Delta_1\omega_k^{03} + \Delta_3\omega_k^{01})e_{013}^k$$
$$+ \ (\Delta_0\omega_k^{23} - \Delta_2\omega_k^{03} + \Delta_3\omega_k^{02})e_{023}^k + (\Delta_1\omega_k^{23} - \Delta_2\omega_k^{13} + \Delta_3\omega_k^{12})e_{123}^k\big], \quad (13)$$

$$d^c\overset{3}{\omega} = \sum_k(\Delta_0\omega_k^{123} - \Delta_1\omega_k^{023} + \Delta_2\omega_k^{013} - \Delta_3\omega_k^{012})e^k, \qquad d^c\overset{4}{\omega} = 0, \quad (14)$$

$$\delta^c\overset{0}{\omega} = 0, \qquad \delta^c\overset{1}{\omega} = \sum_k(\Delta_0\omega_k^{0} - \Delta_1\omega_k^{1} - \Delta_2\omega_k^{2} - \Delta_3\omega_k^{3})x^k, \quad (15)$$

$$\delta^c\overset{2}{\omega} = \sum_k \big[(\Delta_1\omega_k^{01} + \Delta_2\omega_k^{02} + \Delta_3\omega_k^{03})e_0^k + (\Delta_0\omega_k^{01} + \Delta_2\omega_k^{12} + \Delta_3\omega_k^{13})e_1^k$$
$$+ (\Delta_0\omega_k^{02} - \Delta_1\omega_k^{12} + \Delta_3\omega_k^{23})e_2^k + (\Delta_0\omega_k^{03} - \Delta_1\omega_k^{13} - \Delta_2\omega_k^{23})e_3^k\big], \quad (16)$$

$$\delta^c\overset{3}{\omega} = \sum_k \big[(-\Delta_2\omega_k^{012} - \Delta_3\omega_k^{013})e_{01}^k + (\Delta_1\omega_k^{012} - \Delta_3\omega_k^{023})e_{02}^k$$
$$+ (\Delta_1\omega_k^{013} + \Delta_2\omega_k^{023})e_{03}^k + (\Delta_0\omega_k^{012} - \Delta_3\omega_k^{123})e_{12}^k$$
$$+ (\Delta_0\omega_k^{013} + \Delta_2\omega_k^{123})e_{13}^k + (\Delta_0\omega_k^{023} - \Delta_1\omega_k^{123})e_{23}^k\big], \quad (17)$$

$$\delta^c\overset{4}{\omega} = \sum_k \big[(\Delta_3\overset{4}{\omega}_k)e_{012}^k - (\Delta_2\overset{4}{\omega}_k)e_{013}^k + (\Delta_1\overset{4}{\omega}_k)e_{023}^k + (\Delta_0\overset{4}{\omega}_k)e_{123}^k\big]. \quad (18)$$

Let $\Omega \in K(4)$ be given by (10). A discrete analogue of the Dirac–Kähler equation (1) can be defined as

$$i(d^c + \delta^c)\Omega = m\Omega. \quad (19)$$

We can write this equation more explicitly by separating its homogeneous components as

$$i\delta^c\overset{1}{\omega} = m\overset{0}{\omega}, \quad i(d^c\overset{1}{\omega} + \delta^c\overset{3}{\omega}) = m\overset{2}{\omega}, \quad id^c\overset{3}{\omega} = m\overset{4}{\omega},$$
$$i(d^c\overset{0}{\omega} + \delta^c\overset{2}{\omega}) = m\overset{1}{\omega}, \qquad i(d^c\overset{2}{\omega} + \delta^c\overset{4}{\omega}) = m\overset{3}{\omega}. \quad (20)$$

Substituting (12)–(18) into (20) one obtains the set of 16 difference equations [10].

3 Discrete Hestenes and Joyce Equations

As in [11], we define the Clifford multiplication of the basis elements x^k and e_μ^k, $\mu = 0, 1, 2, 3$, by the following rules:

(a) $x^k x^k = x^k, \quad x^k e^k_\mu = e^k_\mu x^k = e^k_\mu$;
(b) $e^k_\mu e^k_\nu + e^k_\nu e^k_\mu = 2g_{\mu\nu} x^k$;
(c) $e^k_{\mu_1} \cdots e^k_{\mu_s} = e^k_{\mu_1 \cdots \mu_s}$ for $0 \leq \mu_1 < \cdots < \mu_s \leq 3$.

Here $g_{\mu\nu} = diag(1, -1, -1, -1)$ is the metric tensor. Note that the multiplication is defined for the basis elements of $K(4)$ with the same multi-index $k = (k_0, k_1, k_2, k_3)$ supposing the product to be zero in all other cases. The operation is linearly extended to arbitrary discrete forms.

Consider the following unit forms

$$x = \sum_k x^k, \qquad e = \sum_k e^k, \qquad e_\mu = \sum_k e^k_\mu, \qquad e_{\mu\nu} = \sum_k e^k_{\mu\nu}, \tag{21}$$

where $\mu, \nu = 0, 1, 2, 3$. The unit 0-form x plays a role of the unit element in $K(4)$, i.e. for any r-form $\overset{r}{\omega}$ we have $x\overset{r}{\omega} = \overset{r}{\omega} x = \overset{r}{\omega}$.

Proposition 1 *The following holds:*

$$e_\mu e_\nu + e_\nu e_\mu = 2g_{\mu\nu} x, \qquad \mu, \nu = 0, 1, 2, 3. \tag{22}$$

Proof By the rule (b) it is obvious.

Proposition 2 *For any inhomogeneous form $\Omega \in K(4)$ we have*

$$(d^c + \delta^c)\Omega = \sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega, \tag{23}$$

where Δ_μ is the difference operator which acts on each component of Ω by the rule (11).

Proof See Proposition 1 in [12].

Thus the discrete Dirac–Kähler equation can be rewritten in the form

$$i \sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega = m\Omega.$$

Let $\Omega^{ev} \in K^{ev}(4)$ be a real-valued even inhomogeneous form, i.e. $\Omega^{ev} = \overset{0}{\omega} + \overset{2}{\omega} + \overset{4}{\omega}$. A discrete analogue of the Hestenes equation (6) is defined by

$$-(d^c + \delta^c)\Omega^{ev} e_1 e_2 = m\Omega^{ev} e_0, \tag{24}$$

or equivalently,

$$-\sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega^{ev} e_1 e_2 = m\Omega^{ev} e_0,$$

where e_1, e_2 and e_0 are given by (21). A discrete analogue of the Joyce equation (7) is given by

$$i(d^c + \delta^c)\Omega^{ev} = m\Omega^{ev} e_0, \tag{25}$$

where $\Omega^{ev} \in K^{ev}(4)$ is a complex-valued even inhomogeneous form. Clearly, Eq. (25) can be rewritten in the form

$$i\sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega^{ev} = m\Omega^{ev} e_0.$$

Applying (12)–(18) Eqs. (24) and (25) can be expressed also in terms of difference equations (see [12]).

Consider the following constant forms

$$P_{\pm 0} = \frac{1}{2}(x \pm e_0), \qquad P_{\pm 12} = \frac{1}{2}(x \pm ie_1e_2). \tag{26}$$

Since

$$(P_{\pm 0})^2 = P_{\pm 0}P_{\pm 0} = P_{\pm 0}, \qquad (P_{\pm 12})^2 = P_{\pm 12}P_{\pm 12} = P_{\pm 12},$$

it follows that $P_{\pm 0}$ and $P_{\pm 12}$ are projectors. The projectors $P_{\pm 0}$ and $P_{\pm 12}$ have the following properties:

$$P_{\pm 0}P_{\pm 12} = P_{\pm 12}P_{\pm 0}, \quad e_0P_{\pm 0} = P_{\pm 0}e_0, \quad e_1e_2P_{\pm 12} = P_{\pm 12}e_1e_2, \tag{27}$$

$$P_{\pm 0} = \pm P_{\pm 0}e_0, \qquad P_{\pm 12} = \pm iP_{\pm 12}e_1e_2. \tag{28}$$

See [11] for more details.

Recall that the Hestenes equation is defined on real-valued even forms. First suppose that the discrete Hestenes equation (24) acts in $K(4)$, i.e. acts in the same space as the discrete Dirac–Kähler equation.

Proposition 3 *Let $\Omega^{ev} \in K^{ev}(4)$ be a solution of the discrete Joyce equation, then*

$$\Omega^{ev} = \Omega^{ev}P_{+0} + \Omega^{ev}P_{-0},$$

where $\Omega^{ev}P_{+0}$ satisfies the discrete Dirac–Kähler equation while $\Omega^{ev}P_{-0}$ satisfies the same equation but the sign of the right-hand side changed to its opposite.

Proposition 4 *Let $\Omega^{ev} \in K^{ev}(4)$ be a solution of the discrete Joyce equation, then*

$$\Omega^{ev} = \Omega^{ev} P_{+12} + \Omega^{ev} P_{-12},$$

where $\Omega^{ev} P_{+12}$ satisfies the discrete Hestenes equation while $\Omega^{ev} P_{-12}$ satisfies the same equation but the sign of the right-hand side changed to its opposite.

By (27) and (28), the proof is straightforward.

In the case of the real-defined discrete Hestenes equation, we have the following results. It is proven in [12, Proposition 5] that by a solution of the discrete Dirac–Kähler equation four independent solutions of the discrete Hestenes equation (24) are constructed. Every solution of the discrete Joyce equation can be represented in the form in which each term of the real and imaginary parts is a real even solution of the discrete Hestenes equation with the correct or reversed sign on the right-hand side [11, Proposition 6]. These are discrete versions of well-known results for corresponding continuum equations.

4 Plane Wave Solutions

Let us consider the following 0-form

$$\psi = \sum_k \psi_k x^k, \tag{29}$$

where

$$\psi_k = (ip_0 + 1)^{k_0}(ip_1 + 1)^{k_1}(ip_2 + 1)^{k_2}(ip_3 + 1)^{k_3}, \qquad p_\mu \in \mathbb{R}. \tag{30}$$

We wish to find a solution of the discrete Joyce equation of the form

$$\Omega = A\psi, \tag{31}$$

where $A \in K^{ev}(4)$ is an inhomogeneous constant form. A constant form means that its components do not depend on k. More explicitly, let

$$A = \overset{0}{\alpha} + \overset{2}{\alpha} + \overset{4}{\alpha},$$

where

$$\overset{0}{\alpha} = \alpha^0 x, \qquad \overset{2}{\alpha} = \sum_{\mu<\nu} \alpha^{\mu\nu} e_{\mu\nu}, \qquad \overset{4}{\alpha} = \alpha^4 e,$$

and $\alpha^0, \alpha^{\mu\nu}, \alpha^4 \in \mathbb{C}$. Recall that x, $e_{\mu\nu}$ and e are the unit forms given by (21). The form (31), where the components of ψ are given by (30), is a discrete version of the plane wave solution (5).

It is easy to check that

$$\Delta_\mu \psi_k = i p_\mu \psi_k, \quad \mu = 0, 1, 2, 3.$$

Consequently, the factor ip_μ is an eigenvalue of the difference operator Δ_μ. This clearly forces

$$d^c \psi = \sum_k \sum_{\mu=0}^{3} (i p_\mu \psi_k) e_\mu^k.$$

According to (23) we have

$$(d^c + \delta^c)\Omega = (d^c + \delta^c)A\psi = \sum_{\mu=0}^{3} e_\mu \Delta_\mu (A\psi) = \sum_{\mu=0}^{3} e_\mu A(\Delta_\mu \psi)$$

$$= \sum_{\mu=0}^{3} e_\mu A \left(\sum_k (\Delta_\mu \psi_k) x^k \right) = i \sum_{\mu=0}^{3} e_\mu p_\mu A \left(\sum_k \psi_k x^k \right) = i \left(\sum_{\mu=0}^{3} e_\mu p_\mu \right) A\psi.$$

Substituting into equation (25) we obtain

$$-\left(\sum_{\mu=0}^{3} e_\mu p_\mu \right) A\psi = mA\psi e_0.$$

Since $\psi e_0 = e_0 \psi$, this equation reduces to

$$-\left(\sum_{\mu=0}^{3} e_\mu p_\mu \right) A = mAe_0. \tag{32}$$

From (22) it follows that $e_0 e_0 = x$. Then Eq. (32) can be written as

$$-\left(p_0 x + \sum_{\mu=1}^{3} p_\mu e_0 e_\mu \right) A = me_0 A e_0. \tag{33}$$

By (22) and by trivial computation, we have

$$\left(p_0 x - \sum_{\mu=1}^{3} p_\mu e_0 e_\mu \right) \left(p_0 x + \sum_{\mu=1}^{3} p_\mu e_0 e_\mu \right) = \left(p_0^2 - \sum_{\mu=1}^{3} p_\mu^2 \right) x.$$

Therefore, multiplying both sides of (33) by the same factor $-\big(p_0 x - \sum_{\mu=1}^{3} p_\mu e_0 e_\mu\big)$ gives

$$\left(p_0^2 - \sum_{\mu=1}^{3} p_\mu^2\right) xA = -m \left(p_0 x - \sum_{\mu=1}^{3} p_\mu e_0 e_\mu\right) e_0 A e_0.$$

This implies

$$\left(p_0^2 - \sum_{\mu=1}^{3} p_\mu^2\right) A = -m \left(\sum_{\mu=0}^{3} p_\mu e_\mu\right) A e_0.$$

Applying (32) again we obtain

$$\left(p_0^2 - \sum_{\mu=1}^{3} p_\mu^2\right) A = m^2 A e_0 e_0,$$

or equivalently,

$$\left(p_0^2 - \sum_{\mu=1}^{3} p_\mu^2\right) A = m^2 A.$$

Thus we have the following assertion.

Proposition 5 *The form (31) is a non-trivial solution of Eq. (25) if and only if*

$$p_0^2 = \sum_{\mu=1}^{3} p_\mu^2 + m^2,$$

or equivalently,

$$p_0 = \pm\sqrt{m^2 + p_1^2 + p_2^2 + p_3^2}. \tag{34}$$

The condition (34) is the same as in the case of continuum counterpart [2].

Let us represent the even form A as

$$A = A_+ + A_-,$$

where

$$A_+ = \alpha^0 x + \alpha^{12} e_{12} + \alpha^{13} e_{13} + \alpha^{23} e_{23}, \tag{35}$$

$$A_- = \alpha^{01} e_{01} + \alpha^{02} e_{02} + \alpha^{03} e_{03} + \alpha^4 e. \tag{36}$$

It is easy to check that A_+ commutes with e_0 and A_- anticommutes with it, i.e.

$$e_0 A_\pm = \pm A_\pm e_0. \tag{37}$$

Lemma 1 *The form $e_{0\mu}A_-$ commutes with e_0 and has the view (35), while $e_{0\mu}A_+$ anticommutes with e_0 and has the view (36) for any $\mu = 1, 2, 3$.*

Proof For $\mu = 1$ we have

$$\begin{aligned} e_{01}A_- &= e_{01}(\alpha^{01}e_{01} + \alpha^{02}e_{02} + \alpha^{03}e_{03} + \alpha^4 e) \\ &= \alpha^{01}x - \alpha^{02}e_{12} - \alpha^{03}e_{13} + \alpha^4 e_{23}. \end{aligned}$$

The same proof remains valid for all other cases.

Theorem 1 *The form $\Omega = A\psi$ is a non-trivial solution of the discrete Joyce equation if and only if the condition*

$$A_- = \frac{p_1e_0e_1 + p_2e_0e_2 + p_3e_0e_3}{m - p_0}A_+ \tag{38}$$

holds, or equivalently,

$$A_+ = -\frac{p_1e_0e_1 + p_2e_0e_2 + p_3e_0e_3}{m + p_0}A_-. \tag{39}$$

Proof Let $\Omega = A\psi$ satisfy (25). Then we have

$$-\left(\sum_{\mu=0}^{3} e_\mu p_\mu\right)(A_+ + A_-) = m(A_+ + A_-)e_0,$$

or

$$-\left(\sum_{\mu=1}^{3} e_0e_\mu p_\mu\right)(A_+ + A_-) = p_0(A_+ + A_-) + me_0(A_+ + A_-)e_0.$$

Applying (37) we can rewrite the above relationship as

$$-\left(\sum_{\mu=1}^{3} e_0e_\mu p_\mu\right)(A_+ + A_-) = (p_0 + m)A_+ + (p_0 - m)A_-.$$

By Lemma 1, collecting like terms gives

$$-(e_0e_1p_1 + e_0e_2p_2 + e_0e_3p_3)A_+ = (p_0 - m)A_-,$$

$$-(e_0e_1p_1 + e_0e_2p_2 + e_0e_3p_3)A_- = (p_0 + m)A_+.$$

Conversely, substituting (38) into (39) yields the condition (34). It follows that $A\psi$ is a non-trivial solution of (25).

Note that the condition (38) can be rewritten as the following system of equations

$$
\begin{aligned}
(m-p_0)\alpha^{01} - p_1\alpha^0 - p_2\alpha^{12} - p_3\alpha^{13} &= 0,\\
(m-p_0)\alpha^{02} - p_2\alpha^0 + p_1\alpha^{12} - p_3\alpha^{23} &= 0,\\
(m-p_0)\alpha^{03} - p_3\alpha^0 + p_1\alpha^{13} + p_2\alpha^{23} &= 0,\\
(m-p_0)\alpha^{4} - p_1\alpha^{23} + p_1\alpha^{13} - p_3\alpha^{12} &= 0.
\end{aligned}
$$

Similarly, the condition (39) gives the following equivalent system of equations

$$
\begin{aligned}
(m+p_0)\alpha^{0} + p_1\alpha^{01} + p_2\alpha^{02} + p_3\alpha^{03} &= 0,\\
(m+p_0)\alpha^{12} - p_1\alpha^{02} + p_2\alpha^{01} + p_3\alpha^{4} &= 0,\\
(m+p_0)\alpha^{13} - p_1\alpha^{03} - p_2\alpha^{4} + p_3\alpha^{01} &= 0,\\
(m+p_0)\alpha^{23} + p_1\alpha^{4} - p_2\alpha^{03} + p_3\alpha^{02} &= 0.
\end{aligned}
$$

According to (34), p_0 can be positive or negative. Hence for given p_μ, $\mu = 1, 2, 3$, there are four linearly independent solutions of the form (31) for positive p_0 and four for negative p_0.

It should be noted that in the continuum case there are also eight linearly independent plane-wave solutions of the Joyce equation for a given momentum vector [6]. Moreover, the conditions (38) and (39) are the same in both the continuum and discrete cases [2].

Proposition 6 *Let the form (31) be a solution of the discrete Joyce equation. If*

$$\alpha^0 = -i\alpha^{12}, \qquad \alpha^{13} = i\alpha^{23}, \tag{40}$$

then $A\psi$ satisfies the discrete Hestenes equation.

Proof By Proposition 4, we have

$$A\psi = A\psi P_{+12} + A\psi P_{-12},$$

where $A\psi P_{+12}$ satisfies the discrete Hestenes equation.

Let us compute A_+P_{-12}. We have

$$
\begin{aligned}
A_+P_{-12} &= \frac{1}{2}(A_+ - iA_+e_{12}) = \frac{1}{2}(A_+ - i\alpha^0 e_{12} + i\alpha^{12}x + i\alpha^{13}e_{23} - i\alpha^{23}e_{13})\\
&= \frac{1}{2}\big((\alpha^0 + i\alpha^{12})x + (\alpha^{12} - i\alpha^0)e_{12} + (\alpha^{13} - i\alpha^{23})e_{13} + (\alpha^{23} + i\alpha^{13})e_{23}\big).
\end{aligned}
$$

Applying (40) gives $A_+P_{-12} = 0$. From (38) it follows that $A_-P_{-12} = 0$ also. This gives $AP_{-12} = 0$. Since the 0-form ψ (29) commutes with any form, we thus get

$$A\psi = A\psi P_{+12}.$$

Corollary 1 *The conditions (40) are equivalent to the following conditions*

$$\alpha^{01} = i\alpha^{02}, \qquad \alpha^{03} = -i\alpha^4.$$

Corollary 2 *If*

$$\alpha^0 = i\alpha^{12}, \qquad \alpha^{13} = -i\alpha^{23},$$

or equivalently,

$$\alpha^{01} = -i\alpha^{02}, \qquad \alpha^{03} = i\alpha^4,$$

then $A\psi$ satisfies the discrete Hestenes equation with a reversed mass sign.

Proof Under these conditions, the part $A\psi P_{+12}$ vanishes and $A\psi = A\psi P_{-12}$ satisfies the discrete Hestenes equation with a reversed mass sign.

As a final remark, it is worth pointing out that in the case of a discrete version of the real plane wave solution to the Hestenes equation the components of (29) should be given in the form

$$\psi_k = (x - p_0 e_{12})^{k_0}(x - p_1 e_{12})^{k_1}(x - p_2 e_{12})^{k_2}(x - p_3 e_{12})^{k_3},$$

where x and e_{12} are given by (21) and $p_\mu \in \mathbb{R}$. This is the subject of current work in progress.

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Analytic Representation of Generalized Möbius-Listing's Bodies and Classification of Links Appearing After Their Cut

Sandra Pinelas and Ilia Tavkhelidze

Abstract For more than almost 200 years the Möbius strip and its "mysterious" property attracts the attention of mathematicians. After a "complete cut" of this surface, one object appears, but already with a fourfold twist. The generalization of this phenomenon to figures of a more complex configuration led to an "unexpected" result: after the cut of the generalized Möbius-Listing body, more than two geometric shapes may appear. In this paper, we consider all possible cases of a complete cut of the generalized Möbius-Listing body with a regular hexagon as radial section. In early works, together with different colleagues, on the basis of importance, they separately examined the case of Möbius-Listing's bodies with a radial section of regular 3, 4 and 5 angular figures. Also, cases of similar bodies with a radial section of convex regular two and three angular figures were considered separately. One possible application of these results is assumed in the description of the properties of the middle surfaces in the theory of elastic shells [14] (Vekua, Shell Theory: General Methods of Construction. Pitman Advanced Publishing Program, Boston, p. 287, 1985).

Keywords Analytic representation · Möbius strip · Möbius-Listing's surfaces
Knots · Link-2

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Notations

In this article we use notations whose are similar to the notations in previous articles [9, 11], but for clarity, we repeat some basic definitions and notation:

• X; Y; Z denote, as usual, the Cartesian coordinates, τ, ψ, θ - are space values (local coordinates or parameters in "parallelogram"):

1. $\tau \in [\tau_*, \tau^*]$, where $\tau_* \leq \tau^*$ usually are non-negative constants;
2. $\Psi \in [0, 2\pi]$;
3. $\theta \in [0, 2\pi h]$, where $h \in R(Real)$;

• $P \equiv A_0 A_1 \dots A_{m-1}$ - denotes **"Plane figure with m-symmetry"**, in particular P_m is a "regular polygon" and m is the number of its angles or vertices. In the general case the edges of "regular polygons" are not always straight lines ($A_i A_{i+1}$ may be, for example: edge of epicycloid, or edge of hypocycloid, or part of lemniscate of Bernoulli, and so on). A wide class of P_m may be represented by Gielis super formula (1);

$$p(\tau, \psi) = \left[\left| \frac{\cos\left(\frac{m_1\psi}{4}\right)}{a} \right|^{n_2} + \left\| \frac{\sin\left(\frac{m_2\psi}{4}\right)}{b} \right\|^{n_3} \right]^{-\frac{1}{n_1}}, \qquad (1)$$

• $PR_m \equiv A_0 A_1 \dots A_{m-1} A'_0 A'_1 \dots A'_{m-1}$ denotes an orthogonal prism, whose ends $A_0 A_1 \dots A_{m-1}$ and $A'_0 A'_1 \dots A'_{m=1}$ are "Plane m-symmetric figures P_m;

– $PR_2 \equiv A_0 A_1 A'_0 A'_1$ is a rectangle, if $P_2 \equiv A_0 A_1$ is a segment of straight line; but also PR_2 may be a cylinder with cross section P_2 (ellipse, or lemniscate of Bernoulli and so on);
– PR_∞ - is an orthogonal cylinder, whose cross section is a P_∞-circle.

• The OO' - axis of symmetry (middle line) of the prism PR_m is transformed into a **"Basic line"** of the GML_m^n (in generally of the GTR_m^n) body;

In the article, just in case, we give a verbal definition of the **Generalized Möbius Listing's body** and, without loss of generality, we shall use one of the forms of its analytic representation given in [5, 9–13].

Definition 1 (*Generalized Möbius Listing's body*) - shortly GML_m^n - is obtained by identifying the opposite ends of the prism PR_m in such a way that:

(**A**) For any integer $n \in Z$ and $i = 1, \dots, m$ each vertex A_i coincides with $A'_{i+n} \equiv A'_{mod_m(i+n)}$, and each edge $A_i A_{i+1}$ coincides with the edge

$$A'_{i+n} A'_{i+n+1} \equiv A'_{mod_m(i+n)} A'_{mod_m(i+n+1)}$$

correspondingly;

(**B**) The integer $n \in Z$ denotes the number of rotations of the end of the prism with respect to the axis OO' before the identification. If $n > 0$, the rotations are counter-clockwise, and if $n < 0$ then rotations are clockwise. Some particular examples of

GML_m^n and its graphical realizations can be found in [3, 5, 9–11, 13], but in this article, without loss of generality we use following analytic representation of these bodies:

$$\begin{aligned} X(\tau,\theta) &= \left[R + r_1 \cos\left(\frac{n_1\theta}{m_1}\right) + p(\tau,\psi)\cos\left(\psi + \frac{n\theta}{6}\right)\right]\cos(\theta) \\ Y(\tau,\theta) &= \left[R + r_1 \cos\left(\frac{n_1\theta}{m_1}\right) + p(\tau,\psi)\cos\left(\psi + \frac{n\theta}{6}\right)\right]\sin(\theta) \\ Z(\tau,\theta) &= r_1 \sin\left(\frac{n_1\theta}{m_1}\right) + p(\tau,\psi))\sin\left(\psi + \frac{n\theta}{6}\right), \end{aligned} \tag{2}$$

According to the above notation in [9]:

More precise information about the analytic representation of these bodies can be found in [1, 4, 7, 9, 10, 13].

I. Generalized Möbius-Listing's Bodies GML_m^n and the corresponding sets of bulky knots and bulky links

Based on analytical representation (2), and on the definition of operation of cutting defined earlier by the second author, some basic questions to be answered appear, for example:

1. How many objects appear after cutting of the GML_m^n surfaces or bodies?
2. What type of the $GML_?^?\{?\}$ surfaces or bodies appear after cutting of the $GML_m^n\{0\}$? (this question for Möbius strip $GML_2^1\{0\}$ was first formulated by Sosinsky see e.g. [8])?
3. What is a link-structure of the surfaces or bodies, which appear after cutting?
4. What are the shapes of radial cross sections of the bodies which appear after cutting of GML_m^n surfaces or bodies?
5. How many different combinations of shapes of the bodies appear after cutting for specific number m?
6. What are differential geometric characteristics of GML_m^n surfaces or bodies?

At this stage, we unfortunately do not have answers to all these questions raised in the case of arbitrary values of $\boldsymbol{m}$, however, some particular cases have been studied in previous papers (see e.g. [3, 11, 12]) by the second author and his colleagues.

A tabulation of knots and links of small complexity (thread structure without interior geometry) is well known (see e.g. [2, 6] or [15]). In this part,we use the analytic representation (2) to study bulky knots and links which appear after a cutting process of the Generalized Möbius - Listing's bodies along "parallel" surfaces of their "Ribs".

Definition 2 Basic line of the GML_6^n body is a continuous, closed, in the general case, the spatial, line on which transforms the axis of symmetry OO' of the prism, after identifying the ends of the figure. This line is represented by (2) when the arguments $\tau = 0$ and $\psi = 0$.

• In this article, without loss of generality, before cutting of GML body will be always $GML_6^n \equiv GML_6^n\{0\}$, i.e. means body with plane basic line–circle;

• $GML_6^n\{\frac{n}{m}\}$ is a body whose basic line, is the spatial toroidal line with characteristic n/m; (precise definitions see in [9, 11]).

Definition 3 **Rib** of the GML_6^n is a continuous closed line, in which only the vertices of the radial cross sections (plane figures) of this body are situated.

Definition 4 **Side** of the GML_6^n - is a continuous closed surface, in which only the sides of the radial cross section (plane figures) of this body are situated.

Definition 5 We call **Slit-surface** or s-**surface** of the GML_m^n body a surface $GML_2^k\{\frac{n}{m}\}$ (GML body whose basic line, the spatial toroidal line with characterictic n/m; see [9, 11]) such that:

1. Its basic line is strictly contained within the GML_m^n body and it is "parallel" to the basic line and ribs of this body;
2. Its radial cross section is a straight line;
3. The line of intersection of the $GML_2^k\{\frac{n}{m}\}$ with the GML_m^n body, which is situated on the side of this body, is "parallel" to the rib line of the GML_m^n body; This restriction defines the number of rotation k (of surface) which strictly depends on the number of rotation n - of the body;

Definition 6 For s-surfaces, without loss of generality, we will use the following notations (examples are in Table 1):

1. $S_{1,j}$ - **surface** of the GML_6^n body is a slit-surface $GML_2^k\{\frac{n}{6}\}$ such that the ends of the straight line (radial cross section) are situated on the sides with the numbers 1 (or A_0A_1) and j (or $A_{j-1}A_j$) where $j = 2, 3, 4$; correspondingly of the plane figures (6 symmetric polygon, hexagon) of the radial cross section of the GML_6^n body;
2. SB - **surface** of the GML_6^n body is such $S_{1,4}$ slit-surface $GML_2^k\{\frac{n}{6}\}$, whose radial cross section (straight line) contains the center of symmetry and does not contain vertices of the radial cross section of the GML_6^n body;
3. $VS_{0,j}$ - **surface** of the body is a slit-surface GML_6^n, whose radial cross section (straight line) is situated on the edges with the numbers j (where $j = 2, 3$) and contains vertex number 0 of the radial cross section of the GML_6^n body;
4. $V_{0,2}$ - **surface** of the GML_6^n body is a slit-surface $GML_2^k\{\frac{n}{6}\}$, whose radial cross sections (straight line) contain correspondingly vertexes numbers 0 and 2 of the radial cross section of hexagon.
5. $VB_{0,3}$ - **surface** of the GML_6^n body is a slit-surface $GML_2^k\{\frac{n}{6}\}$, whose radial cross sections (straight line) contains correspondingly vertices numbers 0 and 3 and the center of symmetry of the radial cross section of hexagon.

Remark 1 According to the regularity of hexagon it is clear that previous designations are sufficient and do not limit the generality; In the future we will have to consider several cases separately and therefore it is necessary to introduce the following notation:

1. $A \equiv |A_0A_1|$ - full length of the side of the regular polygon (radial cross section of GML_6^n body);

2. $b_j^1 \equiv |C_j^1 A_1|$ and $b_j^2 \equiv |C_j^2 A_1| b_2 \equiv |C_2^2 A_1|$, where $j = 2, 3$ or 4 and points $C_j^1 \in A_0 A_1$ and $C_j^2 \in A_{j-1} A_j$ are correspondingly ends of straight line $C_j^1 C_j^2$ (line is a radial cross section of the corresponding slit-surface $GML_2^k\{\frac{n}{6}\}$;

Definition 7 A domain, part of the $GML_6^n\{\mu\}$ body (having similar structure to the $GML_?^k$? body, usually radial cross section is not symmetric figures), whose two opposite parallel to the side-surfaces (see Definition 2) are slit-surfaces, is called a "Slit zone" or shortly an "*s*-zone".

• "Thickness" of the slit-zone is the distance between two opposite parallel slit-surfaces (distance between two opposite parallel straight line in the radial cross section of the slit-zone);

• If the thickness of the slit-zone is zero, then it coincides with a slit-surface. Without loss of generality, in this article we assume that the "-thickness" of the slit-zone is very small with respect to the size of the body.

Definition 8 The "process of cutting" or shortly the "cutting" of a GML_6^n body is always realized along some *s*-surface and produces the vanishing (i.e. elimination) of the corresponding *s*-zone (which possibly reduces to a slit-surface).

• If a GML_6^n body is cut along an $S_{1,j}$-surface ($\xrightarrow{S_{1,j}}$), where $j = 2, 3$, or 4 then the corresponding vanishing zone is called an $S_{1,j}$-**slit**, and such cutting process is called an $S_{1,j}$-**zone-slit**;

• If a GML_6^n body is cut along its *SB*-surface ($\xrightarrow{SB}$), then the corresponding vanishing zone is called a *SB*-**slit**, and such cutting process is called an *SB*-**zone-slit**;

• If a GML_6^n body is cut along its $VS_{0,j}$-surface ($\xrightarrow{VS_{0,j}}$), where $j = 2, 3$, then the corresponding vanishing zone is called a $VS_{0,j}$-**slit**, and such cutting processes called an $VS_{0,j}$-**zone-slit**;

• If a GML_6^n body is cut along its $V_{0,2}$-surface ($\xrightarrow{V_{0,2}}$), then the corresponding vanishing zone is called a $V_{0,2}$-**slit**, and such cutting process is called an $V_{0,2}$-**zone-slit**;

• If a GML_6^n body is cut along its $VB_{0,3}$-surface ($\xrightarrow{V_{0,3}}$), then the corresponding vanishing zone is called a $VB_{0,3}$-**slit**, and such cutting process is called an $VB_{0,3}$-**zone-slit**;

Remark 2 **a**. For the complete review of all cases of $S_{1,j}$-**zone-slit** when $j = 2, 3$ or 4 some different variants should be considered separately :

I. $b_j^1 + b_j^2 < A$; II.$b_j^1 + b_j^2 = A$; III. $b_j^1 + b_j^2 > A$;

b. We consider separately the case IV., when the line $C_3^1 C_3^2$ contains a center of symmetry of the hexagon and correspondingly we have a *SB*-zone-slit.

Using the technique described in [12, 13] or in [11] we obtain the following theorems:

Theorem 1 *If the number of twisting is $n = 6\omega$, where $\omega \in Z$ (ω is a number of full rotations of radial cross section of a body around of basic line) and the GML_6^n body* *is cut along some of its slit-surfaces, then an object "bulk link-2"* $\{(2\omega)_1^2\}$ *(according to the classic tabulation of the links of small complexity [2, 6, 15]) of the two bulk link-1 appears, but both components of this bulk link-2 have 7 different geometric structures, more precisely:*

A. after an $S_{1,j}$-zone-slit for each $j = 2$, 3 or 4 of the GML_6^n body, an object bulk link-2 $\{(2\omega)_1^2\}$ appears of $GML_{(j+1)}^{(j+1)\omega}\{\omega\}$ and $GML_{(9-j)}^{(9-j)\omega}\{0\}$ (see Definition 6) bodies, whose radial cross sections are correspondingly $(j+1)$ and $(9-j)$ angular plane figures, i.e. for each natural $\omega = 0; 1; 2; \ldots;$

$$GML_6^{6\omega} \xrightarrow{S_{1,j}} link - 2\{(2\omega)_1^2\} \; of \; the \; GML_{(j+1)}^{(j+1)\omega}\{\mu\} \; and \; GML_{(9-j)}^{(9-j)\omega}\{\nu\} \tag{3}$$

• When $j = 2$ or 3 then always $\mu = \omega$ and $\nu = 0$ in formula (6).

• When $j = 4$ there are two fundamentally different subcases, that are associated with finding a center of symmetry figure after the cut; more precisely: If center of symmetry of the radial cross section (initial hexagon) after cutting remains in:

• a. one of the five-angular part of domain, then this figure has a characteristic 0 and other equal to ω, i.e. in formula (6) - $\mu = 0$ and $\nu = \omega$ or $\mu = \omega$ and $\mu = 0$

• b. the cutting line (*SB*-slit), then in formula (6) - both elements have characteristic $\mu = \nu = \omega$.

B. after an $VS_{0,j}$-zone-slit for each $j = 2; 3;$ of the GML_6^n body, an object bulk link-2 $\{(2\omega)_1^2\}$ appears of $GML_{(j+1)}^{(j+1)\omega}\{\omega\}$ and $GML_{(8-j)}^{(8-j)}\omega[0]$ bodies, whose radial cross section are correspondingly $(j+1)$ and $(8-j)$ angular plane figures, i.e. for each natural $\omega = 0; 1; 2; \ldots$ and $j = 2, 3;$

$$GML_6^{6\omega} \xrightarrow{VS_{0,j}} link - 2\{(2\omega)_1^2\} \; of \; the \; GML_{(j+1)}^{(j+1)\omega}\{\omega\} \; and \; GML_{(8-j)}^{(8-j)\omega}\{0\} \tag{4}$$

C. after an $V_{0,2}$ -zone-slit of the GML_6^n body, an object bulk link-2 $\{(2\omega)_1^2\}$ appears of $GML_3^{3\omega}\{\omega\}$ and $GML_5^{5\omega}\{0\}$ bodies, whose radial cross sections are correspondingly 3 and 4 angular plane figures, i.e. for each natural $\omega = 0; 1; 2; \cdots;$

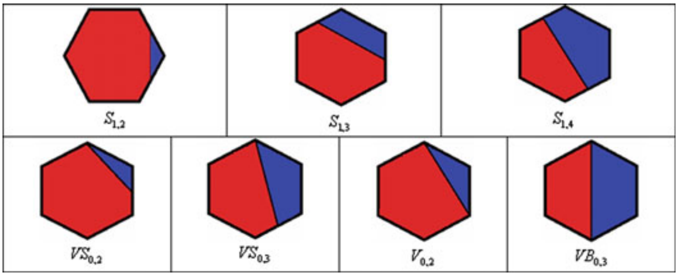
$$GML_6^{6\omega} \xrightarrow{S_{0,2}} link - 2\{(2\omega)_1^2\} \; of \; the \; GML_3^{3\omega}[\omega] \; and \; GML_5^{5\omega}[0] \tag{5}$$

D. after an $VB_{0,3}$-**zone-slit** of the GML_6^n body, an object bulk link-2 $\{(2\omega)_1^2\}$ appears of two identical $GML_4^{4\omega}\{\omega\}$ bodies, whose radial cross section are 4 angular plane figures, i.e. for each natural $\omega = 0; 1; 2; \ldots;$

$$GML_6^{6\omega} \xrightarrow{V_{0,3}} link - 2\{(2\omega)_1^2\} \; of \; the \; GML_4^{4\omega}\{\omega\} \tag{6}$$

Sketch of the proof. We draw one straight line connecting the center of symmetry with the corresponding vertices, so we obtain a star like hexagram, and this case of GML_6^n, generalized Möbius-Listing's surfaces, has been studied in (see. f.e. [12] or [11]). Slit surface for bulky link is converted in a corresponding slit-line, but in this

Table 1 All, different shapes of the cross-section of $GML_6^n\{0\}$, which appear after the cut of the body, and the number n is divisible by 6



situation we know all possible variants that appear after cutting. But after we return to the considered case and we count separately how many rotations some vertices of new bodies that appear after cutting make! All possible cases and corresponding shapes of the radial cross sections after the cutting process are given in Table 1.

Theorem 2 *If $n \equiv 6\omega + q$ is a number of twisting, where ω is an arbitrary integer number and $q = 1$ or 5, and the GML_6^n body is cut along some of its slit-surfaces, then 12 different cases of geometric forms of the radial cross sections appear and each case generates four possible twists of the GML bodies which appear after cutting. More precisely following results hold:*

Case A. Taking into account Remark 2 we have three different subcases:

• **Case A.I** If $b_2^1 + b_2^2 < A$ (see Remark 2.I), after an $S_{1,2}$-zone-slit of the GML_6^n body an object bulk link-2 appears, of $GML_{12}^{2(6\omega+q)}\{0\}$ and $GML_3^{3(6\omega+1)+(q-1))}\{\omega + q/6\}$ bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also, their radial cross sections are twelve and three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,2}} link - 2 \text{ of the } GML_{12}^{(12\omega+2q)}\{0\} \text{ and } GML_3^{(18\omega+3q+15)}\left\{\omega + \frac{q}{6}\right\} \quad (7)$$

• **Case A.II** if $b_2^1 + b_2^2 = A$ (see Remark 2.II), after an $S_{1,2}$-zone-slit of the GML_6^n body an object bulk link-2 appears, of $GML_6^{(6\omega+q)}\{0\}$ and $GML_3^{3(6\omega+1)+(q-1))}\{\omega + q/6\}$ bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also, their radial cross sections are six and three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,2}} link - 2 \text{ of the } GML_6^{(6\omega+q)}\{0\} \text{ and } GML_3^{(18\omega+3q+15)}\left\{\omega + \frac{q}{6}\right\} \quad (8)$$

• **Case A.III** if $b_2^1 + b_2^2 > A$ (see Remark 2.III), after an $S_{1,2}$-zone-slit of the GML_5^n body an object bulk link-3 appears, of $GML_6^{(6\omega+q)}\{0\}$, $GML_3^{3(6(\omega+1)+(q-1))}$

$\{\omega + q/6\}$ and $GML_5^{5(6\omega+1)+(q-1))}\{\omega + q/6\}$ bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also their radial cross sections are six, three and five angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,2}} link-3 \ of \ the \ GML_6^{(6\omega+q)}\{0\},$$
$$GML_3^{(18\omega+3q+15)}\left\{\omega + \frac{q}{6}\right\} \ and \ GML_5^{(30\omega+5q+25)}\left\{\omega + \frac{q}{6}\right\} \tag{9}$$

Case B. Taking into account Remark 2 we have three different subcases:

- **Case B.I** if $b_3^1 + b_3^2 < A$ (see Remark 2.I), after a $S_{1,3}$-zone-slit of the GML_6^n body an object bulk link-3 appears, of $GML_6^{(6(\omega+q)}\{0\}$ and two bodies $GML_4^{4(6\omega+1)+(q-1))}\{\omega + q/6\}$. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also their radial cross sections are six and four angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,3}} link-3 \ of \ the \ GML_6^{(6\omega+q)}\{0\}$$
$$and \ two \ GML_4^{(24\omega+4q+20)}\left\{\omega + \tfrac{q}{6}\right\} \tag{10}$$

- **Case B.II** if $b_3^1 + b_3^2 = A$ (see Remark 2.II), after a S_{13}-zone-slit of the GML_6^n body an object bulk link-3 appears, of $GML_6^{(6\omega+q)}\{0\}$, $GML_3^{3(6(\omega+1)+(q-1)})\{\omega + q/6\}$ and $GML_4^{4(6(\omega+1)+(q-1)})\{\omega + q/6\}$ bodies. The first one has the same structure $\{0\}$ as the initial body before cutting. Also their radial cross sections are six, three and four angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,3}} link-3 \ of \ GML_6^{(6\omega+q)}\{0\}, \quad GML_3^{(18\omega+3q+15)}\left\{\omega + \tfrac{q}{6}\right\}$$
$$and \ GML_4^{(24\omega+4q+20)}\left\{\omega + \tfrac{q}{6}\right\} \tag{11}$$

- **Case B.III** if $b_3^1 + b_3^2 > A$ (see Remark 2.III), after a $S_{1,3}$-zone-slit of the GML_6^n body an object bulk link-4 appears, of $GML_6^{(6\omega+q)}\{0\}$, two bodies $GML_3^{3(6(\omega+1)+(q-1))}\{\omega + q/6\}$ and $GML_6^{6(6(\omega+1)+(q-1))}\{\omega + q/6\}$ bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also their radial cross sections correspondingly are two times three and six angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$.

$$GML_6^{6\omega+q} \xrightarrow{S_{1,3}} link-4 \ of \ the \ GML_6^{(6\omega+q)}\{0\},$$
$$two \ GML_3^{(18\omega+3q+15)}\left\{\omega + \tfrac{q}{6}\right\} and \ GML_6^{(36\omega+6q+30)}\left\{\omega + \tfrac{q}{6}\right\} \tag{12}$$

Case C. Taking into account Remark 2 we have two different subcases:

- **Case C.I** after a $S_{1,4}$-zone-slit of the GML_6^n body an object bulk link-4 appears, of $GML_6^{(6\omega+q)}\{0\}$, $GML_3^{3(6(\omega+1)+(q-1))}\{\omega + 0.2q/6\}$, $GML_4^{4(6(\omega+1)+(q-1))}\{\omega + 0.2q/6\}$

and $GML_5^{5(6(\omega+1)+(q-1))}\{\omega+0.2q/6\}$ bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also their radial cross sections correspondingly are six, three, four and five angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,3}} link-4 \; of \; the \; GML_6^{(6\omega+q)}\{0\}, \; GML_3^{(18\omega+3q+15)}\left\{\omega+\tfrac{q}{6}\right\} \\ GML_4^{(24\omega+4q+20)}\left\{\omega+\tfrac{q}{6}\right\} \; and \; GML_5^{(30\omega+5q+25)}\left\{\omega+\tfrac{q}{6}\right\} \tag{13}$$

• **Case C.II** after a ***SB***-zone-slit of the GML_6^n body (see Remark 2), an object bulk link-1 (bulk-knot) appears, of $GML_4^{4(6\omega+1)+(q-1))}\{\omega+q/6\}$ body (**Möbius strip phenomenon**). Its radial cross section is a four angular plane figure, i.e. for each natural $\omega = 0, 1, 2, \ldots$.

$$GML_6^{6\omega+q} \xrightarrow{SB} link-1 \; of \; the \; GML_4^{(24\omega+4q+20)}\left\{\omega+\tfrac{q}{6}\right\} \tag{14}$$

• **Case D.I** after a $\boldsymbol{VS_{0,2}}$**-zone-slit** of the GML_6^n body, an object bulk link-3 appears, of $GML_6^{(6\omega+q)}\{0\}$, $GML_3^{3(6(\omega+1)+(q-1))}\{\omega+q/6\}$ and $GML_4^{4(6\omega+1)+(q-1))}$ $\{\omega+q/6\}$ bodies. Their radial cross sections are six, three and four angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{VS_{0,2}} link-3 \; of \; GML_6^{(6\omega+q)}\{0\}, \\ GML_3^{(18\omega+3q+15)}\left\{\omega+\tfrac{q}{6}\right\} \; and \; GML_4^{(24\omega+4q+20)}\left\{\omega+\tfrac{q}{6}\right\} \tag{15}$$

• **Case D.II** after a $\boldsymbol{VS_{0,3}}$**-zone-slit** of the GML_6^n body, an object bulk link-4 appears, of $GML_6^{(6\omega+q)}\{0\}$, two bodies $GML_3^{3(6(\omega+1)+(q-1))}\{\omega+q/6\}$ and $GML_5^{5(6\omega+1)+(q-1))}\{\omega+q/6\}$ bodies. Their radial cross sections are six, five and two times three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

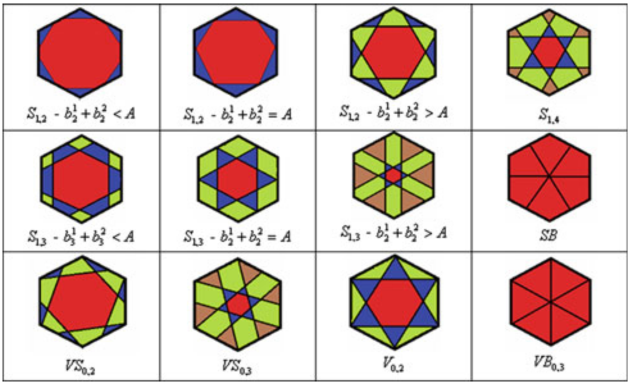
$$GML_6^{6\omega+q} \xrightarrow{VS_{0,3}} link-4 \; of \; the \; GML_6^{(6(\omega+q)}\{0\}, \\ two \; GML_3^{(18\omega+3q+15)}\left[\omega+\tfrac{q}{6}\right], GML_5^{(30\omega+5q+25)}\left[\omega+\tfrac{q}{6}\right] \tag{16}$$

• **Case E**. after a $\boldsymbol{V_{0,2}}$**-zone-slit** of the GML_6^n body, an object bulk link-3 appears, of $GML_6^{(6\omega+q)}\{0\}$ and two $GML_3^{3(6(\omega+1)+(q-1))}\{\omega+q/6\}$ bodies. Their radial cross sections are correspondingly six and three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{VS_{0,2}} link-3 \; of \; the \; GML_6^{(6\omega+q)}\{0\}, \\ and \; two \; GML_3^{(18\omega+3q+15)}\left\{\omega+\tfrac{q}{6}\right\} \tag{17}$$

• **Case F**. after a $\boldsymbol{VB_{0,3}}$**-zone-slit** of the GML_6^n body, an object bulk link-1 (or bulk knot)appears, of $GML_3^{3(6(\omega+1)+(q-1))}\{\omega+q/6\}$ body (**Möbius strip phenomenon**).

Table 2 All, different shapes of the cross-section of $GML_6^n\{0\}$, which appear after the cut of the body, and the number $n = 6\omega + q$ where $q = 1$ or 5



Radial cross section of this body is three angular plane figure, i.e. for each natural $\omega = 0, 1, 2, \dots$

$$GML_6^{6\omega+q} \xrightarrow{VB_{0,3}} link - 1 \ of \ the \ GML_3^{(18\omega+3q+15)} \left\{\omega + \frac{q}{6}\right\} \tag{18}$$

Sketch of the proof. The proof is absolutely similar to previous theorem and is based on the corresponding results for the surfaces (see. f.e. [10] or [11]). All possible cases and corresponding shapes of the radial cross sections after the cutting process are given in Table 2.

Theorem 3 *If $n \equiv 6\omega + 2q$ is a number of twisting, where ω is an arbitrary integer number and $q = 1$ or 2, and the GML_6^n body is cut along some of its slit-surfaces, then 10 different cases of geometric forms of the radial cross sections appear and each of these cases generates four possible twists of the GML bodies which appear after cutting. More precisely following results hold:*

- **Case A**. after a $S_{1,2}$-zone-slit of the GML_6^n body an object bulk link-2 appears, of $GML_9^{3(3\omega+q)}\{0\}$ and $GML_3^{3(3(\omega+1)+(q-1))}\{\omega + q/6\}$bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also their radial cross sections are nine and three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \dots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,2}} link - 2 \ of \ the \ GML_9^{(9\omega+3q)}\{0\} \ and \ GML_3^{(9\omega+3q+6)} \left\{\omega + \frac{q}{3}\right\} \tag{19}$$

Case B. Taking into account Remark 2 we have three different subcases:

- Case **B.I** if $b_3^1 + b_3^2 < A$ (see Remark 2.I), after a $S_{1,3}$-zone-slit of the GML_6^n body an object bulk link-2 appears, of $GML_9^{2(3\omega+q)}\{0\}$ and two bodies

$GML_3^{4(3\omega+1)+(q-1))}\{\omega + q/3\}$. The first one has the same structure $(0)_1$ as the initial body before cutting. Also their radial cross sections are six and four angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,3}} link - 2 \; of \; the \; GML_6^{(6\omega+2q)}\{0\} \; and \; GML_4^{(12\omega+4q+8)}\left\{\omega + \tfrac{q}{3}\right\} \tag{20}$$

• **Case B.II** if $b_3^1 + b_3^2 = A$ (see Remark 2.II), after a S_{13}-zone-slit of the GML_6^n body an object bulk link-2 appears, of $GML_3^{(3\omega+q)}\{0\}$ and $GML_4^{4(3(\omega+1)+(q-1))}\{\omega + q/3\}$ bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also their radial cross sections are six and three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,3}} link - 2 \; of \; the \; GML_3^{(3\omega+q)}\{0\} \; and \; GML_4^{(12\omega+4q+8)}\left\{\omega + \tfrac{q}{3}\right\} \tag{21}$$

• **Case B.III** if $b_3^1 + b_3^2 > A$ (see Remark 2.III), after a $S_{1,3}$-zone-slit of the GML_6^n body an object bulk link-3 appears, of $GML_3^{(3\omega+q)}\{0\}$, $GML_3^{3(3(\omega+1)+(q-1))}\{\omega + q/3\}$ and $GML_6^{6(3(\omega+1)+(q-1))}\{\omega + q/3\}$ bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also their radial cross sections correspondingly are five, two times three and six angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$\begin{gathered} GML_6^{6\omega+q} \xrightarrow{S_{1,3}} link - 3 \; of \; the \; GML_3^{(3\omega+q)}\{0\}, \\ GML_3^{(9\omega+3q+6)}\left\{\omega + \tfrac{q}{3}\right\} \; and \; GML_6^{(18\omega+6q+12)}\left\{\omega + \tfrac{q}{3}\right\} \end{gathered} \tag{22}$$

Case C. Taking into account Remark 2 we have two different subcases:

• **Case C.I** after a $S_{1,4}$-zone-slit of the GML_6^n body an object bulk link-3 appears, of $GML_3^{(3\omega+q)}\{0\}$, $GML_3^{3(3(\omega+1)+(q-1))}\{\omega + q/3\}$ and $GML_5^{5(3(\omega+1)+(q-1))}$ $\{\omega + q/3\}$, bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting. Also their radial cross sections correspondingly are six, three, four and five angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$\begin{gathered} GML_6^{6\omega+q} \xrightarrow{S_{1,3}} link - 3 \; of \; the \; GML_3^{(3\omega+q)}\{0\}, \\ GML_3^{(9\omega+3q+6)}\left\{\omega + \tfrac{q}{3}\right\} \; and \; GML_6^{(15\omega+6q+10)}\left\{\omega + \tfrac{q}{3}\right\} \end{gathered} \tag{23}$$

• **Case C.II** after a ***SB***-zone-slit of the GML_6^n body (see Remark 2) an object bulk link-2 appears, of two $GML_4^{4(3(\omega+1)+(q-1))}\{\omega + q/3\}$ bodies. Their radial cross sections are four angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{SB} link - 2 \; of \; the \; two \; GML_4^{(12\omega+4q+8)}\left\{\omega + \tfrac{q}{3}\right\} \tag{24}$$

• **Case D.I** after a $\boldsymbol{VS}_{0,2}$**-zone-slit** of the GML_6^n body, an object bulk link-2 appears, of $GML_6^{2(3\omega+q)}\{0\}$, and $GML_3^{3(3(\omega+1)+(q-1))}\{\omega+q/3\}$ bodies. Their radial cross sections are six and three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \stackrel{VS_{0,2}}{\longrightarrow} link-2 \; of \; the \; GML_6^{(6\omega+2q)}\{0\} \; and \; GML_3^{(9\omega+3q+6)}\left\{\omega+\tfrac{q}{3}\right\} \tag{25}$$

• **Case D.II** after a $\boldsymbol{VS}_{0,3}$**-zone-slit** of the GML_6^n body, an object bulk link-4 appears, of $GML_3^{(3\omega+q)}\{0\}$, two bodies $GML_3^{3(3(\omega+1)+(q-1))}\{\omega+q/3\}$ and $GML_5^{5(3(\omega+1)+(q-1))}\{\omega+q/3\}$ bodies. Their radial cross sections are six, five and two times three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$\begin{gathered} GML_6^{6\omega+q} \stackrel{VS_{0,3}}{\longrightarrow} link-3 \; of \; the \; GML_3^{(3\omega+q)}\{0\}, \\ GML_3^{(9\omega+3q+6)}\left\{\omega+\tfrac{q}{3}\right\} \; and GML_5^{(15\omega+5q+10)}\left\{\omega+\tfrac{q}{3}\right\} \end{gathered} \tag{26}$$

• **Case E**. after a $\boldsymbol{V}_{0,2}$**-zone-slit** of the GML_6^n body, an object bulk link-2 appears, of $GML_3^{(3\omega+q)}\{0\}$ and $GML_3^{3(3(\omega+1)+(q-1))}\{\omega+q/3\}$ bodies. Their radial cross sections are correspondingly three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \stackrel{V_{0,2}}{\longrightarrow} link-2 \; of \; the \; GML_3^{(3\omega+q)}\{0\} \; and \; GML_3^{(9\omega+3q+6)}\left\{\omega+\tfrac{q}{3}\right\} \tag{27}$$

• **Case F**. after a $\boldsymbol{VB}_{0,3}$**-zone-slit** of the GML_6^n body, an object bulk link-2 appears of $GML_3^{3(3(\omega+1)+(q-1))}\{\omega+q/3\}$ bodies. Their radial cross section are three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

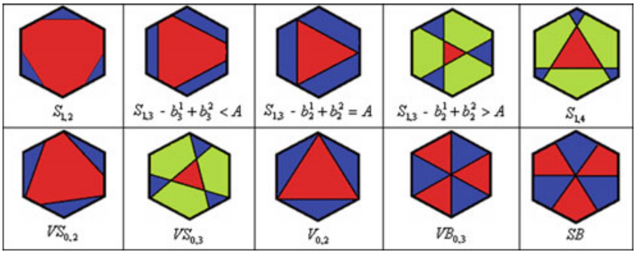
$$GML_6^{6\omega+q} \stackrel{VB_{0,3}}{\longrightarrow} link-2 \; of \; the \; GML_3^{(9\omega+3q+6)}\left\{\omega+\tfrac{q}{3}\right\} \tag{28}$$

Sketch of the proof. The proof is absolutely similar to the previous theorem and it is based on the corresponding results for the surfaces GML_6^n (see. f.e. [10] or [11]). All possible cases and corresponding shapes of the radial cross sections after the cutting process are given in Table 3.

Theorem 4 *If* $n \equiv 6\omega + 3$ *is a number of twisting, where* ω *is an arbitrary integer number and the* GML_6^n *body is cut along some of its slit-surfaces, then 7 different cases of geometric forms of the radial cross sections appear and each case generates four possible twists of the GML bodies which appear after cutting. More precisely the following results hold:*

• **Case A**. after a $S_{1,2}$-zone-slit of the GML_6^n body an object bulk link-2 appears, of $GML_8^{4(2\omega+1)}\{0\}$ and $GML_3^{3(2(\omega+1))}\{\omega+1/2\}$ bodies. The first one has the same structure $(0)_1$ as the initial body before cutting and the second $\{(2\omega+1)_1\}$. Also, their radial cross sections are eight and three angular plane figures, i.e. for each

Table 3 All, different shapes of the cross-section of $GML_6^n\{0\}$, which appear after the cut of the body, and the number $n = 6\omega + 2q$ where $q = 1$ or 2



natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,2}} link-2 \ of \ the \ GML_8^{(8\omega+4)}\{0\} \ and \ GML_3^{(6\omega+6)}\left\{\omega+\tfrac{1}{2}\right\} \tag{29}$$

Case B. after a $S_{1,3}$-zone-slit of the GML_6^n body an object bulk link-2 appears, of $GML_6^{3(2\omega+1)}\{0\}$ and $GML_4^{4(2(\omega+1))}\{\omega+1/2\}$ bodies. The first one has the same structure $\{(0)_1\}$ as the initial body before cutting and second $\{(2\omega+1)_1\}$. Also, their radial cross sections are six and four angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{S_{1,3}} link-2 \ of \ the \ GML_6^{(6\omega+3)}\{0\} \ and \ GML_4^{(8\omega+8)}\left\{\omega+\tfrac{1}{2}\right\} \tag{30}$$

Case C. Taking into account Remark 2 we have two different subcases:

• **Case C.I** after a $S_{1,4}$ -zone-slit of the GML_6^n body an object bulk link-2 appears, of $GML_4^{2(2\omega+1)}\{0\}$ and $GML_5^{5(2(\omega+1))}\{\omega+1/2\}$ bodies. The first one has the same structure $(0)_1$ as the initial body before cutting. Also, their radial cross sections correspondingly are four and five angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+3} \xrightarrow{S_{1,4}} link-2 \ of \ the \ GML_4^{(4\omega+2)}\{0\} \ and \ GML_5^{(10\omega+10)}\left\{\omega+\tfrac{1}{2}\right\} \tag{31}$$

• **Case C.II** after a ***SB***-zone-slit of the GML_6^n body an object bulk link-1 (or bulk knot) appears of $GML_5^{5(2(\omega+1))}\{\omega+1/2\}$ body (**Möbius strip phenomenon**). Its radial cross section is a five angular plane figure, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \xrightarrow{SB} link-1 \ of \ the \ GML_5^{(10\omega+10)}\{\omega+\tfrac{1}{2}\} \tag{32}$$

• **Case D.I** after a $\boldsymbol{VS_{0,2}}$**-zone-slit** of the GML_6^n body, an object bulk link-2 appears, of $GML_6^{3(2\omega+1)}\{0\}$ and $GML_3^{3(2(\omega+1))}\{\omega+1/2\}$ bodies. Their radial cross sections are six and three angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \overset{VS_{0,2}}{\longrightarrow} link - 2 \; of \; the \; GML_6^{(6\omega+3q)}\{0\} \; and \; GML_3^{(6\omega+6)}\left\{\omega+\tfrac{1}{2}\right\} \tag{33}$$

• **Case D.II** after a $VS_{0,3}$-**zone-slit** of the GML_6^n body, an object bulk link-2 appears, of $GML_4^{2(2\omega+1)}\{0\}$ and $GML_4^{4(2(\omega+1))}\{\omega+1/2\}$ bodies. Their radial cross sections are four angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+q} \overset{VS_{0,2}}{\longrightarrow} link - 2 \; of \; the \; GML_4^{(4\omega+2)}\{0\} \; and \; GML_4^{(8\omega+8)}\left\{\omega+\tfrac{1}{2}\right\} \tag{34}$$

• **Case E**. after a V_0, 2-**zone-slit** of the GML_6^n body, an object bulk link-2 appears, $GML_4^{2(2\omega+1)}\{0\}$ of and $GML_3^{3(2(\omega+1))}\{\omega+1/2\}$ bodies. Their radial cross sections are correspondingly three and four angular plane figures, i.e. for each natural $\omega = 0, 1, 2, \ldots$

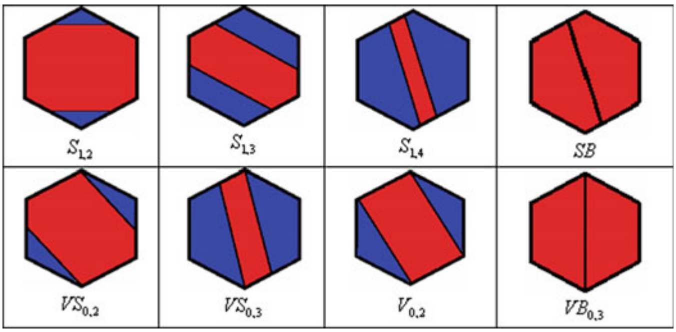
$$GML_6^{6\omega+3} \overset{V_{0,2}}{\longrightarrow} link - 2 \; of \; the \; GML_3^{4(4\omega+2)}\{0\} \; and \; GML_3^{(6\omega+6)}\left\{\omega+\tfrac{1}{2}\right\} \tag{35}$$

• **Case F**. After a $VB_{0,3}$-**zone-slit** of the GML_6^n body an object bulk link-1 (or bulk knot)appears, of the $GML_4^{4(2(\omega/+1))}\{\omega+1/2\}$ body. Its radial cross section is a four angular plane figure, i.e. for each natural $\omega = 0, 1, 2, \ldots$

$$GML_6^{6\omega+3} \overset{VB_{0,3}}{\longrightarrow} link - 1 \; of \; the \; GML_4^{(8\omega+8)}\left\{\omega+\tfrac{1}{2}\right\} \tag{36}$$

Sketch of the proof. The proof is absolutely similar to the previous theorem and it is based on the corresponding results for the surfaces GML_5^n (see. f.e. [10, 12] or [11]). All possible cases and corresponding shapes of the radial cross sections after the cutting process are given in Table 4.

Table 4 All, different shapes of the cross-section of $GML_6^n\{0\}$, which appear after the cut of the body, and the number $n = 6\omega + 3$



II. Similarity and difference. Relations between the sets of Generalized Möbius-Listing's Bodies and Surfaces with different numbers of symmetry

We return to the six questions formulated at the beginning of this article, and we try to answer each individually and to make some generalizations.

Answers to the first three questions are given correspondingly in Theorems 1.7 of [11] and 1.4 of the current article. This shows that, while solutions for GML_m^n surfaces are given for all cases, for any numbers m (of symmetry of the radial cross section) and n (of rotation of the ends of corresponding prism before identifying), instead similar problems for GML_m^n bodies are investigated, at this stage, only for some concrete values of the number m ($m = 2, 3, 4, 5, 6$ - see [11, 13].

Remark 2.1. Similarity and difference between corresponding ribbon and bulky links ($m = 3, 4, 5$ **or** 6).

• A. Similarity. If $n = m\omega$, then groups of bulk links which appear after cutting of the GML_m^n body are similar to the ribbon links which appear after an s-zone slit of the GML_m^n surface (see [12]; on these similarities the proofs of all Theorems in this article were based;

• B. Differences. If $n = m\omega$ then, after cutting of a GML_m^n body never bulk link appears which is analogous to the ribbon links which appear after a b-zone slit of the surface (see Theorem 1B. opposite of Theorems 3 and 6; see also [9, 11–13]);

⋆ These are **new phenomena and exist only** for GML_m^n surfaces with star like plane figures in the radial cross sections - after one cutting object with more then two independent components (i.e. link-j and $2 < j \leq m$) appears.

• C. Similarity. If $n = m\omega + q$, when $\gcd(m, q) = 1$, then sets of the bulk link –1 which appears after cutting of a GML_m^n bodies are similar to the ribbon link –1 which appears after an s-zone slit of the corresponding surfaces, i.e. these objects individually have characteristics similar the basic line $\{\omega + q/m\}$; The proofs of Theorem 2 were based on these similarities. (see also [12] or [13]);

• D. Differences. For arbitrary n after an s-zone slit of a GML_m^n surface never appears a geometric object with more then two independent components, i.e. ribbon link-3 or link-4 with three or four components (similar cases: link-3 in A.III., B.I., B.II., D.I, and E. and link-4 in B.III., C.I., D.II. of Theorem 2 and B.III., C.I., D.II G. of Theorem 3).

• E. Differences. For arbitrary n after an s-zone slit of the GML_m^n surface never ribbon link-j ($j > 2$) with three or more components. These are **new phenomena and exist only for** GML_m^n **bodies**, when after one cutting an object with more than two independent components appears. Unfortunately we have not yet found the general law for computing the full number of the different cases, for arbitrary numbers of m and n; more precisely we have some results but we do not know the general process of proving;

• F. Similarity. For arbitrary numbers m and n after an s-zone slit always one component of the arising link is similar to the initial GML_m^n body or surface.

• G. Similarity. For arbitrary numbers m and n after a b-zone slit of GML_m^n bodies or surfaces the initial body structure disappears, i.e. characteristics of the arising objects is never equal to zero ($\{\omega + q/m\}$, $q = 0, 1, 2, \ldots, m - 1$)!

• H. Differences. For arbitrary numbers m and n after a b-zone slit of GML_m^n bodies the number of independent components of the arising object is never more than two! But, for arbitrary numbers m and n after a b-zone slit of GML_m^n surfaces, the number of independent components of the arising object may be more than two. More precisely, this number equals the numbers m or $\varsigma \equiv \gcd(m, q)$ (see correspondingly in Theorem 1 or 2, case B)!

• I. Similarity. if number m is even then **Möbius phenomena** is preserved for GML_m^n bodies. In particularly see cases C.II and F. in Theorem 2 or 4.

Remark 2.2. Generalisations.

• A. If m is an even number, then for different n (more precisely, if $\gcd(m, n) = 1$) - after one full cutting of GML_m^n bodies, maximum $\frac{m}{2} + 1$ independent geometric objects appear (this number depends also on the geometric place of the cutting line in the cross section of body), i.e. link-$(m/2 + 1)$ appear and only one element has a structure similar to the figure before cutting (see cases B.III., C.I., and D.II about link –4, for $m = 6$, in Theorem 2, also cases A.III., B.I., and C. about link-3, for $m = 4$ in Theorem 4 [11]);

• B. If m is an odd number, then for different n (more precisely, if $\gcd(m, n) = 1$) - after one full cutting of GML_m^n bodies, maximum $\left[\frac{m}{2}\right] + 2$ independent geometric objects appear (this number depends also on the geometric place of the cutting line in the cross section of body), i.e. link-$([m/2] + 2)$ appear and only one element has a structure similar to the figure before cutting (see cases C., E3. and E4 about link-3, for $m = 3$ in Theorems 4, 5 [13] and cases B.III., B.IV., and E about link –4 for $m = 5$ in Theorem 7 [11];

• C. If m is even number, then always some values of n exist, such that after one full cutting of GML_m^n bodies only 1 independent geometric object appears (for this, the cutting line should include the center of symmetry of the radial cross section of the body), i.e. knot; see cases C.II and F in Theorem 2 or 4 for $m = 6$, also cases B.II and D. in Theorems 4 and 5 for $m = 4$ [11];

⋆ If m is an even number, then always some values of n are, and for which the phenomenon of the Möbius strip is realized!

• B. If m is an odd number, then there exist some values of n, such that after one cutting of GML_m^n at least 2 independent geometric object appear (for this the cutting line should include the center of symmetry of the radial cross section of body), i.e. (link-2 appears), whose index is defined by $\gcd(m, n) = 1$;

⋆ If m is an odd number, then for any value of the parameter n, the phenomenon of the Möbius band is never realized!

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Existence and Multiplicity of Periodic Solutions to Fractional p-Laplacian Equations

Lin Li and Stepan Tersian

Abstract This paper deals with the existence and multiplicity of periodic solutions for the fractional p-Laplacian equations. The minimization argument and extended Clark's theorem are applied to prove our results.

Keywords Periodic solutions · Fractional p-Laplacian equations · Clark's theorem

1 Introduction

In this paper we consider the fractional p-Laplacian problem

$$\begin{cases} {}_tD_T^{\alpha}(\varphi_p\left({}_0D_t^{\alpha}u(t)\right)) + a(t)\varphi_q(u(t)) = b(t)\varphi_r(u(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{P_T}$$

where $0 < \alpha < 1$, $p > 1$, $q > r > 1$, $\varphi_p(t) = |t|^{p-2}t$ and $a = a(t)$, $b = b(t)$ positive continuous T-periodic functions on $[0, T]$.

Recently, a great attention has been focused on the study of boundary value problems (BVP) for fractional differential equations. They appear in mathematical models in different branches in Science as physics, chemistry, biology, geology, as well as, control theory, signal theory, nanoscience and so on. The reader can find many applications in the books [1, 9, 12, 16] and references therein. The existence and multiplicity of solutions for BVP for nonlinear fractional differential equations is

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extensively studied using various tools of nonlinear analysis as fixed point theorems, degree theory and the method of upper and lower solutions [2]. Starting with the pioneering work of Jiao and Zhou [8], the variational methods are applied to fractional differential equations. The approach was extended by various authors as [4, 5, 7].

The purpose of our paper is to treat the quasilinear case $p \neq 2$ variationally and to prove existence and multiplicity results for problem $(\mathrm{P_T})$.

Our main result for the problem $(\mathrm{P_T})$ is as follows.

Theorem 1 *Let $p > 1$, $q > r > 1$, $\alpha > 1/p$ and $a = a(t)$, $b = b(t)$ be positive continuous T-periodic functions on $[0, T]$. Then the problem $(\mathrm{P_T})$ has at least one solution.*

If, in addition, we assume $p > r$ then $(\mathrm{P_T})$ has infinitely many pairs of solutions $(u_m, -u_m)$, $u_m \neq 0$, with $\max_{t\in[0,T]} |u_m(t)| \to 0$ as $m \to \infty$.

Remark 1 Theorem 1 can be extended to equations with more general nonlinear terms as

$$ {}_tD_T^{\alpha}(\varphi_p\left({}_0D_t^{\alpha}u(t)\right)) - f(t, u(t)) + h(t, u(t)) = 0. $$

Let

$$ F(t, u) = \int_0^u f(t, \sigma)\,\mathrm{d}\sigma, \qquad H(t, u) = \int_0^u h(t, \sigma)\,\mathrm{d}\sigma. $$

Suppose that functions $f(t, \sigma)$ and $h(t, \sigma)$ are continuous in (t, σ) and there exist positive constants $a_1, a_2, b_1, b_2, q > r > 1$ such that for all $u \in \mathbb{R}$,

$$ a_1|u|^q \leq F(t, u) \leq a_2|u|^q, \qquad b_1|u|^r \leq H(t, u) \leq b_2|u|^r. $$

With the same assumptions on p, q, r, the existence parts of Theorem 1 is valid. If, moreover, $f(t, \sigma)$ and $h(t, \sigma)$ are odd functions of σ, the multiplicity results are valid, too.

2 Preliminaries

In this section we introduce some basic definitions of fractional calculus which are used further in this paper. For more details we refer the reader to [9, 12, 14].

Let u be a function defined on $[a, b]$. The left (right) Riemann–Liouville fractional integral of order $\alpha > 0$ for function u is defined by

$$ {}_aI_t^{\alpha}u(t) = \tfrac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\mathrm{d}s,\ \ t \in [a, b], $$
$$ {}_tI_b^{\alpha}u(t) = \tfrac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)\mathrm{d}s,\ \ t \in [a, b]. $$

Here, $\Gamma(\cdot)$ is the Gamma function, provided in both cases that the right-hand side is pointwise defined on $[a, b]$.

The left and right Riemann–Liouville fractional derivatives of order $\alpha > 0$ for function u denoted by ${}_aD_t^\alpha u(t)$ and ${}_tD_b^\alpha u(t)$, respectively, are defined by

$$
\begin{aligned}
{}_aD_t^\alpha u(t) &= \tfrac{d^n}{dt^n}\, {}_aI_t^{n-\alpha} u(t),\\
{}_tD_b^\alpha u(t) &= (-1)^n \tfrac{d^n}{dt^n}\, {}_tI_b^{n-\alpha} u(t),
\end{aligned}
$$

where $t \in [a, b]$, $n - 1 \leq \alpha < n$ and $n \in \mathbb{N}$. Note that, $AC([a, b], \mathbb{R}^N)$ is the space of functions which are absolutely continuous on $[a, b]$ and $AC^k([a, b], \mathbb{R}^N)(k = 0, 1, \ldots)$ is the space of functions f such that $f \in C^{k-1}([a, b], \mathbb{R}^N)$ and $f^{k-1} \in AC([a, b], \mathbb{R}^N)$. So, the left and right Caputo fractional derivatives are defined via the above Riemann–Liouville fractional derivatives [9]. In particular, they are defined for the function belonging to the space of absolutely continuous functions. Namely, if $\alpha \in (n - 1, n)$ and $u \in AC^n[a, b]$, then the left and right Caputo fractional derivative of order α for function u denoted by ${}_a^cD_t^\alpha u(t)$ and ${}_t^cD_b^\alpha u(t)$ respectively, are defined by

$$
\begin{aligned}
{}_a^cD_t^\alpha u(t) &= {}_aI_t^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^n(s)\mathrm{d}s,\\
{}_t^cD_b^\alpha u(t) &= (-1)^n\, {}_tI_b^{n-\alpha} u^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} u^{(n)}(s)\mathrm{d}s.
\end{aligned}
$$

The Riemann–Liouville fractional derivative and the Caputo fractional derivative are connected with each other by the following relations (see [9, 12]):

Theorem 2 *Let $n \in \mathbb{N}$ and $n - 1 < \alpha < n$. If u is a function defined on $[a, b]$ for which the Caputo fractional derivatives ${}_a^cD_t^\alpha u(t)$ and ${}_t^cD_b^\alpha u(t)$ of order α exists together with the Riemann–Liouville fractional derivatives ${}_aD_t^\alpha u(t)$ and ${}_tD_b^\alpha u(t)$, then*

$$
\begin{aligned}
{}_a^cD_t^\alpha u(t) &= {}_aD_t^\alpha u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \quad t \in [a, b],\\
{}_t^cD_b^\alpha u(t) &= {}_tD_b^\alpha u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, \quad t \in [a, b].
\end{aligned}
$$

In particular, when $0 < \alpha < 1$, we have

$$
{}_a^cD_t^\alpha u(t) = {}_aD_t^\alpha u(t) - \frac{u(a)}{\Gamma(1-\alpha)} (t-a)^{-\alpha}, \quad t \in [a, b] \tag{1}
$$

and

$$
{}_t^cD_b^\alpha u(t) = {}_tD_b^\alpha u(t) - \frac{u(b)}{\Gamma(1-\alpha)} (b-t)^{-\alpha}, \quad t \in [a, b]. \tag{2}
$$

Now we consider some properties of the Riemann–Liouville fractional integral and derivative operators [9].

Theorem 3 *(1) We have*

$$
\begin{aligned}
{}_aI_t^\alpha({}_aI_t^\beta u(t)) &= {}_aI_t^{\alpha+\beta}u(t),\\
{}_tI_b^\alpha({}_tI_b^\beta u(t)) &= {}_tI_b^{\alpha+\beta}u(t), \quad \forall\alpha,\beta>0,
\end{aligned}
$$

(2) **Left inverse.** *Let $u\in L^1[a,b]$ and $\alpha>0$,*

$$
\begin{aligned}
{}_aD_t^\alpha({}_aI_t^\alpha u(t)) &= u(t),\ \text{a.e. } t\in[a,b],\\
{}_tD_b^\alpha({}_tI_b^\alpha u(t)) &= u(t),\ \text{a.e. } t\in[a,b].
\end{aligned}
$$

(3) For $n-1\le\alpha<n$, if the left and right Riemann–Liouville fractional derivatives ${}_aD_t^\alpha u(t)$ and ${}_tD_b^\alpha u(t)$, of the function u are integrable on $[a,b]$, then

$$
\begin{aligned}
{}_aI_t^\alpha({}_aD_t^\alpha u(t)) &= u(t)-\sum_{k=}^{n}[{}_aI_t^{k-\alpha}u(t)]_{t=a}\frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)},\\
{}_tI_b^\alpha({}_tD_b^\alpha u(t)) &= u(t)-\sum_{k=1}^{n}[{}_tI_n^{k-\alpha}u(t)]_{t=b}\frac{(-1)^{n-k}(b-t)^{\alpha-k}}{\Gamma(\alpha-k+1)},
\end{aligned}
$$

for $t\in[a,b]$.

(4) **Integration by parts**

$$
\int_a^b[{}_aI_t^\alpha u(t)]v(t)\mathrm{d}t=\int_a^b u(t){}_tI_b^\alpha v(t)\mathrm{d}t,\ \alpha>0, \tag{3}
$$

provided that $u\in L^p[a,b]$, $v\in L^q[a,b]$ and

$$
p\ge1,\ q\ge1\ \text{ and }\ \frac{1}{p}+\frac{1}{q}<1+\alpha\ \text{ or }\ p\ne1,\ q\ne1\ \text{ and }\ \frac{1}{p}+\frac{1}{q}=1+\alpha.
$$

$$
\int_a^b[{}_aD_t^\alpha u(t)]v(t)\mathrm{d}t=\int_a^b u(t){}_tD_b^\alpha v(t)\mathrm{d}t,\quad 0<\alpha\le1, \tag{4}
$$

provided the conditions

$$
\begin{aligned}
u(a)=u(b)=0,\ u'\in L^\infty[a,b],\ v\in L^1[a,b]\ \text{ or}\\
v(a)=v(b)=0,\ v'\in L^\infty[a,b],\ u\in L^1[a,b],
\end{aligned}
$$

are fulfilled.

(5) *Let* $0 < \frac{1}{p} < \alpha \leq 1$ *and* $u(x) \in L^p[0, T]$*, then* ${}_0I_t^{\alpha}u$ *is Hölder continuous on* $[0, T]$ *with exponent* $\alpha - \frac{1}{p}$ *and* $\lim_{t \to 0^+} {}_0I_t^{\alpha}u(t) = 0$*. Consequently,* ${}_0I_t^{\alpha}u$ *can be continuously extended by* 0 *in* $x = 0$*.*

2.1 Fractional Derivative Space

In order to establish a variational structure for problem ($\mathrm{P_T}$), it is necessary to construct appropriate function spaces. For this setting we take some results from [7, 8, 17].

Let us recall that for any fixed $t \in [0, T]$ and $1 \leq p < \infty$,

$$\|u\|_{L^p} = \left(\int_0^T |u(s)|^p \mathrm{d}s\right)^{1/p} \quad \text{and} \quad \|u\|_{\infty} = \max_{t \in [0,T]} |u(t)|.$$

Definition 1 Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative Sobolev space $E_0^{\alpha,p}$ is defined by

$$\begin{aligned} E_0^{\alpha,p} &= \{u \in L^p[0, T] : \ {}_0D_t^{\alpha}u \in L^p[0, T] \text{ and } u(0) = u(T) = 0\} \\ &= \overline{C_0^{\infty}[0, T]}^{\|.\|_{\alpha,p}}. \end{aligned}$$

where $\|.\|_{\alpha,p}$ is defined by

$$\|u\|_{\alpha,p}^p = \int_0^T |u(t)|^p \mathrm{d}t + \int_0^T |{}_0D_t^{\alpha}u(t)|^p \mathrm{d}t. \tag{5}$$

Remark 2 For any $u \in E_0^{\alpha,p}$, noting the fact that $u(0) = 0$, we have ${}_0^cD_t^{\alpha}u(t) = {}_0D_t^{\alpha}u(t)$, $t \in [0, T]$ according to (1).

Proposition 1 ([7]) *Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$*. The fractional derivative Sobolev space* $E_0^{\alpha,p}$ *is a reflexive and separable Banach space.*

We recall some properties of the space $E_0^{\alpha,p}$.

Proposition 2 ([8]) *Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$*. For all* $u \in E_0^{\alpha,p}$*, we have*

$$\|u\|_{L^p} \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|{}_0D_t^{\alpha}u\|_{L^p}. \tag{6}$$

If $\alpha > 1/p$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$*, then*

$$\|u\|_{\infty} \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)p'+1)^{1/p'}} \|{}_0D_t^{\alpha}u\|_{L^p}. \tag{7}$$

Remark 3 Let $1/p < \alpha \le 1$, if $u \in E_0^{\alpha,p}$, then $u \in L^s[0,T]$ for $s \in [p,+\infty]$. In fact

$$\int_0^T |u(t)|^s \mathrm{d}t = \int_0^T |u(t)|^{s-p}|u(t)|^p \mathrm{d}t \le \|u\|_\infty^{s-p}\|u\|_{L^p}^p.$$

In particular the embedding $E_0^{\alpha,p} \hookrightarrow L^s[0,T]$ is continuous for all $s \in [p,+\infty]$.

Remark 4 According to (6), we can consider in $E_0^{\alpha,p}$ the following norm

$$\|u\|_{\alpha,p} = \|{}_0D_t^\alpha u\|_{L^p}, \tag{8}$$

is equivalent to (5).

Proposition 3 ([8]) *Let* $0 < \alpha \le 1$ *and* $1 < p < \infty$. *Assume that* $\alpha > \frac{1}{p}$ *and* $u_k \rightharpoonup u$ *in* $E_0^{\alpha,p}$. *Then* $u_k \to u$ *in* $C[0,T]$, *i.e.*

$$\|u_k - u\|_\infty \to 0,\ k \to \infty.$$

We say that $u \in E_0^{\alpha,p}$ is a *weak solution* of ($\mathrm{P_T}$) if the integral identity

$$\int_0^T \left[\varphi_p\left({}_0D_t^\alpha u(t)\right){}_0D_t^\alpha v(t) + a(t)\varphi_q\left(u(t)\right)v(t) - b(t)\varphi_r\left(u(t)\right)v(t)\right]\mathrm{d}t = 0$$

holds for any function $v \in E_0^{\alpha,p}$.

Let $\Phi_s(\tau) = \frac{|\tau|^s}{s}$ be the antiderivative of $\varphi_s(\tau)$. We introduce the functional $I : E_0^{\alpha,p} \to \mathbb{R}$ associated with ($\mathrm{P_T}$) as follows:

$$I(u) := \int_0^T \left[\Phi_p\left({}_0D_t^\alpha u(t)\right) + a(t)\Phi_q\left(u(t)\right) - b(t)\Phi_r\left(u(t)\right)\right]\mathrm{d}t.$$

It's Gâteaux derivative at $u \in E_0^{\alpha,p}$ in the direction $v \in E_0^{\alpha,p}$ is given by

$$\left\langle I'(u), v\right\rangle = \int_0^T \left[\varphi_p\left({}_0D_t^\alpha u(t)\right){}_0D_t^\alpha v(t) + a(t)\varphi_q\left(u(t)\right)v(t) - b(t)\varphi_r\left(u(t)\right)v(t)\right]\mathrm{d}t.$$

Hence, critical points of I are in one-to-one correspondence with weak solutions of ($\mathrm{P_T}$).

Our approach is variational. The existence part of our result relies on the standard minimization argument (see, e.g., [3, 6, 11]) applied to I. We state it explicitly below for reader's convenience.

Theorem 4 (Minimization argument) *Let* $E : X \to \mathbb{R}$ *be weakly sequentially lower semicontinuous functional on a reflexive Banach space* X *and let* E *have a bounded* *minimizing sequence. Then E has a minimum on X, i.e., there exists $u_0 \in X$ such that $E(u_0) = \inf_{u \in X} E(u)$. If E is differentiable then u_0 is a critical point of E.*

Now we introduce more notations and some necessary definitions. Let X be a real Banach space, $E \in C^1(X, \mathbb{R})$ means that E is a continuously Fréchet differentiable functional defined on X.

Definition 2 $E \in C^1(X, \mathbb{R})$ is said to satisfy the Palais–Smale condition if any sequence $\{u_n\} \subset X$, for which $\{E(u_n)\}$ is bounded and $E'(u_n) \to 0$ as $n \to +\infty$, possesses a convergent subsequence in X.

Our multiplicity result in Theorem 1 relies on the generalization of Clark's theorem. See [13, pp. 53–54] for the original version of Clark's theorem which has been applied by many authors (see, e.g., [15]). In our paper we use the extension of Clark's theorem proved recently by Liu and Wang [10]. For reader's convenience, we present this extended version.

Theorem 5 ([10, Theorem 1.1]) *Let X be a Banach space, $E \in C^1(X, \mathbb{R})$. Assume that E satisfies the Palais–Smale condition, it is even and bounded from below, and $E(0) = 0$. If for any $k \in \mathbb{N}$, there exist a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} E < 0$, where $S_\rho = \{u \in X, \|u\|_X = \rho\}$, then at least one of the following conclusions holds.*

- *There exists a sequence of critical points $\{u_n\}$ satisfying $E(u_n) < 0$ for all n and $\|u_n\|_X \to 0$ as $n \to \infty$.*
- *There exists $r > 0$ such that for any $0 < \alpha < r$ there exists a critical point u such that $\|u\|_X = \alpha$ and $E(u) = 0$.*

In our approach, we use this assertion combined with the following remark.

Remark 5 It is already noted in [10], that Theorem 5 implies the existence of infinitely many pairs of critical points $(u_m, -u_m)$, $u_m \neq 0$, such that $E(u_m) \leq 0$, $E(u_m) \to 0$, and $\|u_m\|_X \to 0$ as $m \to \infty$.

3 Proofs of Main Results

We write the functional I as $I(u) = I_1(u) + I_2(u)$, where

$$I_1(u) = \int_0^T \Phi_p\left({}_0D_t^\alpha u(t)\right) \mathrm{d}t \text{ and } I_2(u) = \int_0^T \left[a(t)\Phi_q\left(u(t)\right) - b(t)\Phi_r\left(u(t)\right)\right] \mathrm{d}t.$$

Clearly, the functional I_1 is continuous, convex and hence weakly sequentially lower semicontinuous on $E_0^{\alpha,p}$. Due to the compact embedding $E_0^{\alpha,p} \hookrightarrow\hookrightarrow C[0, T]$ by Proposition 3, I_2 is weakly sequentially continuous on $E_0^{\alpha,p}$. Hence, I is weakly sequentially lower semicontinuous on $E_0^{\alpha,p}$.

Since a and b are positive continuous functions on $[0, T]$, there exist constants $a_i, b_i, i = 1, 2$, such that

$$0 < a_1 \le a(t) \le a_2, \qquad 0 < b_1 \le b(t) \le b_2. \tag{9}$$

We start with the proof of the *existence* of a solution of ($\mathrm{P_T}$). The plan is to apply Theorem 4 with $X = E_0^{\alpha,p}$ and $E = I$. We show that I is bounded from below on $E_0^{\alpha,p}$ and has a bounded minimizing sequence.

Consider the function $f(\tau) = \frac{1}{q}a_1\tau^q - \frac{1}{r}b_2\tau^r,\ \tau \ge 0$. Then

$$f(\tau) \ge \frac{r-q}{qr}\left(\frac{b_2^q}{a_1^r}\right)^{\frac{1}{q-r}} =: c_1.$$

Recall that by Remark 4 $\|u\|_{\alpha,p} = \|{}_0D_t^\alpha u\|_{L^p}$. Then we can estimate I from below on $E_0^{\alpha,p}$ as follows:

$$\begin{aligned} I(u) &\ge \int_0^T \Phi_p\left({}_0D_t^\alpha u(t)\right)\,\mathrm{d}t + \int_0^T \left(\frac{1}{q}a_1|u(t)|^q - \frac{1}{r}b_2|u(t)|^r\right)\mathrm{d}t \\ &\ge \frac{1}{p}\|u\|_{\alpha,p}^p + Tc_1. \end{aligned}$$

Hence, $\inf_{u\in E_0^{\alpha,p}} I(u) > -\infty$.

Let $\{u_n\} \subset E_0^{\alpha,p}$ be a minimizing sequence, $I(u_n) \to \inf_{u\in E_0^{\alpha,p}} I(u)$. Then there exists $c_2 \in \mathbb{R}$ such that

$$c_2 \ge I(u_n) \ge \frac{1}{p}\|u_n\|_{\alpha,p}^p + Tc_1.$$

Hence, $\{u_n\}$ is a bounded sequence in $E_0^{\alpha,p}$. Since I is weakly sequentially lower semicontinuous on $E_0^{\alpha,p}$, Theorem 4 implies that I has a critical point in $E_0^{\alpha,p}$. It follows from our discussions in Sect. 2 that this critical point is a solution of ($\mathrm{P_T}$). This concludes the proof of existence part of Theorem 1.

In order to prove the *multiplicity* result in Theorem 1, we need the following lemma.

Lemma 1 *The functional I satisfies the Palais–Smale condition on $E_0^{\alpha,p}$.*

Proof (Proof of Lemma 1) Let $\{u_n\}$ be a Palais–Smale sequence, i.e., $\{I(u_n)\}$ is bounded in $\mathbb{R}$ and $I'(u_n) \to 0$ in $(E_0^{\alpha,p})^*$, where $(E_0^{\alpha,p})^*$ is the dual space of $E_0^{\alpha,p}$. From the boundedness of $\{I(u_n)\}$, exactly as above, we deduce that $\{u_n\}$ is bounded in $E_0^{\alpha,p}$. Passing to a subsequence, if necessary, we may assume that there exists $u \in E_0^{\alpha,p}$ such that $u_n \rightharpoonup u$ weakly in $E_0^{\alpha,p}$ and $u_n \to u$ strongly in $C[0, T]$. By $I'(u_n) \to 0$ in $(E_0^{\alpha,p})^*$, we have

$$
\begin{aligned}
0 \leftarrow & \langle I'(u_n) - I'(u), u_n - u\rangle \\
= & \int_0^T \left[\varphi_p\left({}_0D_t^\alpha u_n(t)\right) - \varphi_p\left({}_0D_t^\alpha u(t)\right)\right]\left({}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)\right)\mathrm{d}t \\
& + \int_0^T a(t)\left[\varphi_q\left(u_n(t)\right) - \varphi_q\left(u(t)\right)\right](u_n(t) - u(t))\,\mathrm{d}t \\
& - \int_0^T b(t)\left[\varphi_r\left(u_n(t)\right) - \varphi_r\left(u(t)\right)\right](u_n(t) - u(t))\,\mathrm{d}t.
\end{aligned}
\tag{10}
$$

The last two terms in (10) tend to 0 due to the uniform convergence $u_n \to u$ in $C[0, T]$ by Proposition 3. Then, by (10) and Hölder's inequality, we obtain

$$
\begin{aligned}
0 = & \lim_{n\to\infty}\int_0^T \left[\varphi_p\left({}_0D_t^\alpha u_n(t)\right) - \varphi_p\left({}_0D_t^\alpha u(t)\right)\right]\left({}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u(t)\right)\mathrm{d}t \\
\geq & \lim_{n\to\infty}\Bigg\{\int_0^T |{}_0D_t^\alpha u_n(t)|^p\,\mathrm{d}t - \int_0^T |{}_0D_t^\alpha u(t)|^p\,\mathrm{d}t \\
& - \left(\int_0^T |{}_0D_t^\alpha u_n|^p\,\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_0^T |{}_0D_t^\alpha u|^p\,\mathrm{d}t\right)^{\frac{1}{p}} \\
& - \left(\int_0^T |{}_0D_t^\alpha u_n(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^T |{}_0D_t^\alpha u(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p'}}\Bigg\} \\
= & \lim_{n\to\infty}\left[\left(\int_0^T |{}_0D_t^\alpha u_n|^p\,\mathrm{d}t\right)^{\frac{1}{p}} - \left(\int_0^T |{}_0D_t^\alpha u|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\right] \\
& \left[\left(\int_0^T |{}_0D_t^\alpha u_n(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p'}} - \left(\int_0^T |{}_0D_t^\alpha u(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p'}}\right] \\
= & \lim_{n\to\infty}\left(\|u_n\|_{\alpha,p} - \|u\|_{\alpha,p}\right)\left(\|u_n\|_{\alpha,p}^{p-1} - \|u\|_{\alpha,p}^{p-1}\right) \geq 0,
\end{aligned}
$$

where $p' = \frac{p}{p-1}$ is the exponent conjugate to $p > 1$. This implies $\|u_n\|_{\alpha,p} \to \|u\|_{\alpha,p}$. Hence, the weak convergence $u_n \rightharpoonup u$ in $E_0^{\alpha,p}$ and the uniform convexity of $E_0^{\alpha,p}$ yield $u_n \to u$ in $E_0^{\alpha,p}$.

Now we verify the "geometric" assumptions of Theorem 5. Recall that the functional I is bounded from below on $E_0^{\alpha,p}$, even and $I(0) = 0$. Let $k \in \mathbb{N}$ be arbitrary and X^k be k-dimensional subspace of $E_0^{\alpha,p}$ spanned by the basis elements $\{\phi_1, \ldots, \phi_m\} \subset E_0^{\alpha,p}$. The separability of $E_0^{\alpha,p}$ allows for such construction. We use the fact that all norms $\|\cdot\|_{\alpha,p}$, $\|\cdot\|_{L^q}$ and $\|\cdot\|_{L^r}$ are equivalent on X^k, i.e., there exist positive constants $c_4, \ldots, c_7$ such that for all $u \in X^k$,

$$
c_4\|u\|_{L^q} \leq \|u\|_{\alpha,p} \leq c_5\|u\|_{L^q} \quad \text{and} \quad c_6\|u\|_{L^r} \leq \|u\|_{\alpha,p} \leq c_7\|u\|_{L^r}. \tag{11}
$$

Set

$$\mathscr{S}_\rho^k := \left\{ u = \alpha_1\phi_1 + \cdots + \alpha_k\phi_k : \sum_{j=1}^{k} |\alpha_j|^p = \rho^p \right\} \subset X^k.$$

$\mathscr{S}_\rho^k$ is clearly homeomorphic to the unit sphere $\mathscr{S}^{k-1} \subset \mathbb{R}^k$. Then for $u = \sum_{j=1}^k \alpha_j\phi_j$, the expression $\|u\|_{X^k} = \left(\sum_{j=1}^k |\alpha_j|^p\right)^{1/p}$ defines also a norm on X^k equivalent to $\|\cdot\|_{\alpha,p}$, i.e., there exist positive constants c_8 and c_9 such that for all $u \in X^k$,

$$c_8 \|u\|_{X^k} \le \|u\|_{\alpha,p} \le c_9 \|u\|_{X^k}. \tag{12}$$

We show that there is (sufficiently small) $\rho > 0$ such that

$$\sup_{u \in \mathscr{S}_\rho^k} I(u) < 0. \tag{13}$$

Indeed, due to (9) and (11), for any $u \in \mathscr{S}_\rho^k$, we have

$$\begin{aligned} I(u) &= \int_0^T \left(\frac{1}{p} |{}_0D_t^\alpha u(t)|^p + \frac{1}{q} a(t)|u(t)|^q - \frac{1}{r} b(t)|u(t)|^r \right) \mathrm{d}t \\ &\le \frac{1}{p} \|u\|_{\alpha,p}^p + \frac{a_2}{q} \|u\|_{L^q}^q - \frac{b_1}{r} \|u\|_{L^r}^r \\ &\le \frac{1}{p} \|u\|_{\alpha,p}^p + \frac{a_2}{q c_4^q} \|u\|_{\alpha,p}^q - \frac{b_1}{r c_7^r} \|u\|_{\alpha,p}^r \\ &= \|u\|_{\alpha,p}^r \left[\frac{1}{p} \|u\|_{\alpha,p}^{p-r} + \frac{a_2}{q c_4^q} \|u\|_{\alpha,p}^{q-r} - \frac{b_1}{r c_7^r} \right]. \end{aligned} \tag{14}$$

Recall our assumptions $1 < r < p$ and $r < q$. Then (13) follows from (12) and (14). This implies there exists $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} I(u) < 0$, where $S_\rho = \{u \in E_0^{\alpha,p}, \|u\| = \rho\}$.

We have verified all assumptions of Theorem 5. Taking into account Remark 5, the multiplicity result in Theorem 1 follows.

4 Extension to the Impulsive Problem

Denote $0 = t_0 < t_1 < \cdots < t_l < t_{l+1} = T$ and set $\mathscr{J} = \bigcup_{j=0}^l \mathscr{J}_j$, where $\mathscr{J}_j = (t_j, t_{j+1})$, $j = 0, \ldots, l$. Consider the impulsive problem

$$\begin{cases} {}_tD_T^\alpha(\varphi_p\left({}_0D_t^\alpha u(t)\right)) - a(t)\varphi_q(u(t)) + b(t)\varphi_r(u(t)) = 0 \quad \text{for } t \in \mathscr{J}, \\ u(0) - u(T) = 0, \\ \Delta\left({}_tD_T^{\alpha-1}\varphi_p({}_0D_t^\alpha u(t_j))\right) = g_j\left(u(t_j)\right) \quad \text{for } j = 1, \ldots, l, \end{cases} \tag{Q$_\mathrm{T}$}$$

where $\Delta\left({}_tD_T^{\alpha-1}\varphi_p({}_0D_t^\alpha u(t_j))\right) := {}_tD_T^{\alpha-1}\varphi_p({}_0D_t^\alpha u(t_j^+)) - {}_tD_T^{\alpha-1}\varphi_p({}_0D_t^\alpha u(t_j^-))$, ${}_tD_T^{\alpha-1}\varphi_p({}_0D_t^\alpha u(t_j^\pm)) = \lim_{t\to t_j^\pm} {}_tD_T^{\alpha-1}\varphi_p({}_0D_t^\alpha u(t))$, and $g_j : \mathbb{R} \to \mathbb{R}$ are given continuous functions.

We can prove an existence result for the impulsive problem ($\mathrm{Q_T}$) as follows.

Theorem 6 *Let $p > 1$, $q > r > 1$, $a = a(t)$, $b = b(t)$ be positive continuous T-periodic functions on $[0, T]$ and $g_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, \ldots, l)$ be continuous functions satisfying for all $\tau \in \mathbb{R}$ and $j = 1, \ldots, l$,*

$$\int_0^\tau g_j(\sigma)\, \mathrm{d}\sigma \ge c \tag{15}$$

with a given constant $c \in \mathbb{R}$. Then ($\mathrm{Q_T}$) has at least one solution.

If, in addition, $p > r$ and for all $\tau \in \mathbb{R}$ and $j = 1, \ldots, l$,

$$\int_0^\tau g_j(\sigma)\, \mathrm{d}\sigma \le 0, \tag{16}$$

and g_j are odd functions, then ($\mathrm{Q_T}$) has infinitely many pairs of solutions $(u_m, -u_m)$, $u_m \ne 0$, with $\max_{t\in[0,T]} |u_m(t)| \to 0$ as $m \to \infty$.

Let $G_j(\tau) = \int_0^\tau g_j(\sigma)\, \mathrm{d}\sigma$, $j = 1, \ldots, l$. Then the functional $J : X_p \to \mathbb{R}$ associated with ($\mathrm{Q_T}$), defined by

$$J(u) := \int_0^T \left[\Phi_p\left({}_0D_t^\alpha u(t)\right) + a(t)\Phi_q(u(t)) - b(t)\Phi_r(u(t))\right] \mathrm{d}t + \sum_{j=1}^l G_j\left(u(t_j)\right),$$

is Gâteaux differentiable at any $u \in E_0^{\alpha,p}$ and its critical points are in one-to-one correspondence with weak solutions of ($\mathrm{Q_T}$). We say that $u \in E_0^{\alpha,p}$ is a *weak solution* of impulsive problem ($\mathrm{Q_T}$) if the identity

$$\int_0^T \left[\varphi_p\left({}_0D_t^\alpha u(t)\right) {}_0D_t^\alpha v(t) + a(t)\varphi_q(u(t))\, v(t) - b(t)\varphi_r(u(t))\, v(t)\right] \mathrm{d}t + \sum_{j=1}^l g_j\left(u(t_j)\right) v(t_j) = 0$$

holds for any $v \in E_0^{\alpha,p}$.

Remark 6 Theorem 6 can be extended to equations with more general nonlinear terms as in Remark 1.

The proof of Theorem 6 is similar with Theorem 1. It follows from (15) that

$$J(u) \geq I(u) + cl \geq \frac{1}{p}\|u\|_{\alpha,p}^{p} + Tc_1 + cl,$$

i.e., J is bounded from bellow on $E_0^{\alpha,p}$. Due to (15), the boundedness of minimizing sequence is proved analogously as in the case of functional I. As mentioned above, J is weakly sequentially lower semicontinuous, and so the existence of a solution of (Q_T) follows again from Theorem 4. Furthermore, since every g_j ($j = 1, \ldots, l$) is odd, J is even and the assumptions (15) and (16) guarantee that the assertion of Lemma 1 holds also for the functional J. The assumption (16) also guarantees that analogue of (14) holds also for J. Thus the multiplicity result for (Q_T) follows again from Theorem 5. Let $l = 1$ and $k > 1$. An example of a function g_j which satisfies assumptions (15) and (16) is $g_1(t) = \frac{-2t}{(1+t^2)^k}$ for which $\frac{1}{1-k} < \int_0^{\tau} g_1(\sigma)\, d\sigma \leq 0$.

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Oscillation of Third Order Mixed Type Neutral Difference Equations

S. Selvarangam, M. Madhan, E. Thandapani and S. Pinelas

Abstract In this paper, we obtain some sufficient conditions for the oscillation of all solutions of the third order nonlinear neutral difference equation with mixed arguments of the form

$$\Delta^3(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2})^\alpha + q_n x^\beta_{n-\sigma_1} + p_n x^\gamma_{n+\sigma_2} = 0, n \geq n_0,$$

where α, β and γ are the ratios of odd positive integers. Some examples are provided to substantiate our results.

Keywords Third order · Nonlinear · Neutral · Difference equations · Oscillation · Mixed arguments

2010 AMS Classification 39A10 · 39A21 · 39A70

1 Introduction

In this paper, we study the oscillatory behavior of all solutions of the following type of third order nonlinear neutral difference equations with mixed arguments

$$\Delta^3(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2})^\alpha + q_n x^\beta_{n-\sigma_1} + p_n x^\gamma_{n+\sigma_2} = 0, \; n \geq n_0 \geq 0 \tag{1.1}$$

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under the following assumptions:

(B_1) $\{b_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers and there exist finite constants b and c such that $0 \le b_n < b$ and $0 \le c_n < c$;
(B_2) α, β and γ are ratios of odd positive integers;
(B_3) τ_1, τ_2, σ_1 and σ_2 are positive integers;
(B_4) $\{q_n\}$ and $\{p_n\}$ are sequences of nonnegative real numbers, and not both identically zero for infinitely many values of n.

Let $\theta = \max(\tau_1, \sigma_1)$. By a solution of equation (1.1), we mean a sequence $\{x_n\}$ of real numbers defined for all $n \ge n_0 - \theta$, and satisfying the Eq. (1.1) for all $n \ge n_0$. A nontrivial solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its nontrivial solutions are oscillatory. Further we assume that the Eq. (1.1) possesses such nontrivial solutions.

Recently there has been much interest on the study of oscillatory and asymptotic behavior of solutions of difference equations [1, 2]. The study of third order difference equations has received less attention when compared to first and second order difference equations. In [3–9] the authors studied third order difference equations with and without mixed arguments and obtained some results related to oscillation of all solutions. But these results cannot be applied to a nonlinear equation of the type (1.1), since that results are applicable only for linear mixed type difference equations. In [10], the authors considered the third order nonlinear difference equation with mixed arguments of the form

$$\Delta(a_n \Delta^2(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2})) + q_n x^{\beta}_{n-\sigma_1} + p_n x^{\beta}_{n+\sigma_2} = 0,\ n \ge n_0,$$

and obtained some sufficient conditions for the oscillation of all its solutions. In [11], the authors considered the third order nonlinear difference equation with mixed neutral terms of the form

$$\Delta^3(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2})^{\alpha} = q_n x^{\beta}_{n-\sigma_1} + p_n x^{\gamma}_{n+\sigma_2},\ n \ge n_0,$$

and derived several sufficient conditions for the oscillation of all its solutions. Motivated by this observation, in this paper we obtain some sufficient conditions for all the solutions of equation (1.1) to be oscillatory. In Sect. 2, we present some preliminary lemmas which will be used in the proofs of main theorems. In Sect. 3, we obtain some sufficient conditions for the oscillation of all solutions of equation (1.1). Examples are provided to illustrate the main results in Sect. 4.

2 Preliminary Lemmas

For simplicity, we use the following notations, without further mention:

$$z_n = \left(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2}\right)^{\alpha},$$
$$Q_n = \min\left\{q_{n-\tau_1}, q_n, q_{n-\tau_2}\right\},$$
$$P_n = \min\left\{p_{n-\tau_1}, p_n, p_{n-\tau_2}\right\},$$

and

$$R_n = P_n + Q_n.$$

For convenience, when we write a functional inequality without specifying its domain of validity, we assume that it holds for large values of n. We prove all the results only for positive solutions of equation (1.1), since the proof for the negative case is similar.

Lemma 2.1 *If $\{x_n\}$ is a positive solution of equation (1.1), then there are only the following two cases for the corresponding sequence $\{z_n\}$ for sufficiently large $n \geq n_0$:*

(I) $z_n > 0,\ \Delta z_n > 0,\ \Delta^2 z_n > 0$ *and* $\Delta^3 z_n \leq 0;$ (2.1)

(II) $z_n > 0,\ \Delta z_n < 0,\ \Delta^2 z_n > 0$ *and* $\Delta^3 z_n \leq 0.$ (2.2)

Proof Let $\{x_n\}$ be a positive solution of equation (1.1). Then there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\sigma_1} > 0$ and $x_{n-\tau_1} > 0$ for all $n \geq n_1$. From the definition of z_n, we have $z_n > 0$ for all $n \geq n_1$. Now Eq. (1.1), implies that

$$\Delta^3 z_n = -q_n x_{n-\sigma_1} - p_n x_{n+\sigma_2} \leq 0 \text{ for all } n \geq n_1.$$

Then $\Delta^2 z_n$ is nonincreasing for all $n \geq n_1$, and therefore $\Delta^2 z_n$ is eventually positive or eventually negative. We shall prove that $\Delta^2 z_n > 0$ for all $n \geq n_1$. If not, then $\Delta^2 z_n \leq 0$ for all $n \geq n_1$, and therefore there exists a negative constant M_1 and an integer $n_2 \geq n_1$ such that

$$\Delta^2 z_n \leq M_1 \text{ for all } n \geq n_2.$$

Summing the last inequality from n_2 to $n-1$, we obtain

$$\Delta z_n \leq \Delta z_{n_2} + M_1(n - n_2).$$

Letting $n \to \infty$ in the last inequality, we see that $\Delta z_n \to -\infty$, so there exists an integer $n_3 \geq n_2$, and a negative constant M_2 such that

$$\Delta z_n \leq M_2 \text{ for all } n \geq n_3.$$

Again summing the last inequality from n_3 to $n-1$, we get

$$z_n \leq z_{n_3} + M_2(n - n_3) \text{ for all } n \geq n_3.$$

Now letting $n \to \infty$ in the last inequality, we see that $z_n \to -\infty$, which is a contradiction to the positivity of z_n. Hence $\Delta^2 z_n > 0$ for all $n \geq n_1$. This completes the proof. ■

Lemma 2.2 *Let* $A \geq 0$ *and* $B \geq 0$. *Then*

$$A^\delta + B^\delta \geq (A+B)^\delta \quad \text{if } \; 0 < \delta \leq 1 \tag{2.3}$$

and

$$A^\delta + B^\delta \geq \frac{1}{2^{\delta-1}}(A+B)^\delta \quad \text{if } \; \delta \geq 1. \tag{2.4}$$

Proof The proof may be found in [12]. ■

Lemma 2.3 *Let* $\{x_n\}$ *be an eventually positive solution of equation* (1.1) *and suppose case (I) of Lemma 2.1 holds. Then for some constant* $k \in (0, 1)$ *and for some integer* $N \geq n_0$, *we have*

$$\frac{z_n}{\Delta z_n} \geq \frac{n-N}{2} \geq kn \quad \text{for all } \; n \geq N. \tag{2.5}$$

Proof The proof is similar to that of Lemma 2.2 of [13], and hence the details are omitted. ■

Lemma 2.4 *Let* $\{x_n\}$ *be an eventually positive solution of equation* (1.1) *and suppose case (I) of Lemma 2.1 holds. Then there exists an integer* $N \geq n_0$ *such that*

$$z_n \geq \theta n(n-1)\Delta^2 z_n, \qquad n \geq N$$

where $0 < \theta < 1$.

Proof The proof of the lemma is similar to that of Lemma 2.2 of [13], and hence the details are omitted. ■

Lemma 2.5 *Let* $\{x_n\}$ *be a positive solution of equation* (1.1), *and the corresponding sequence* $\{z_n\}$ *satisfies case (II) of Lemma 2.1. If*

$$\sum_{n=n_0}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}(q_t + p_t) = \infty, \tag{2.6}$$

then, $\lim_{n\to\infty} x_n = 0$.

Proof The proof of the lemma is similar to that of Lemma 2.4 of [10], and hence the details are omitted. ■

3 Oscillation Results

In this section, we present some new oscillation criteria for Eq. (1.1). We begin with the following theorem.

Theorem 3.1 *Assume that* $0 < \beta = \gamma \le 1$, $\sigma \ge \max\{\sigma_1, \sigma_2, \tau_2\}$ *and* (2.6) *hold. If the first order difference inequality*

$$\Delta y_n + R_n \left(\frac{k\sigma(n-2\sigma)}{1+b^\beta+c^\beta} \right)^{\frac{\beta}{\alpha}} y_{n-2\sigma}^{\frac{\beta}{\alpha}} \le 0, \;\; n \ge n_0, \tag{3.1}$$

has no positive solution, then every solution of equation (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof Let $\{x_n\}$ be a positive solution of equation (1.1). Then there exists an integer $n_1 \ge n_0$ such that $x_n > 0$, $x_{n-\sigma_1} > 0$ and $x_{n-\tau_1} > 0$ for all $n \ge n_1$. Define a sequence $\{y_n\}$ by

$$y_n = z_n + b^\beta z_{n-\tau_1} + c^\beta z_{n+\tau_2} \tag{3.2}$$

for all $n \ge n_1$. Then

$$\Delta^3 y_n = \Delta^3 z_n + b^\beta \Delta^3 z_{n-\tau_1} + c^\beta \Delta^3 z_{n+\tau_2}$$

for all $n \ge n_1$. Using Eq. (1.1), the last equation becomes

$$\Delta^3 y_n + q_n x_{n-\sigma_1}^\beta + p_n x_{n+\sigma_2}^\beta + b^\beta \left[q_{n-\tau_1} x_{n-\tau_1-\sigma_1}^\beta + p_{n-\tau_1} x_{n-\tau_1+\sigma_2}^\beta \right] + c^\beta \left[q_{n+\tau_2} x_{n+\tau_2-\sigma_1}^\beta + p_{n+\tau_2} x_{n+\tau_2+\sigma_2}^\beta \right] = 0$$

for all $n \ge n_1$. That is,

$$\Delta^3 y_n + Q_n \left[x_{n-\sigma_1}^\beta + b^\beta x_{n-\tau_1-\sigma_1}^\beta + c^\beta x_{n+\tau_2-\sigma_1}^\beta \right] + P_n \left[x_{n+\sigma_2}^\beta + b^\beta x_{n+\sigma_2-\tau_1}^\beta + c^\beta x_{n+\sigma_2+\tau_2}^\beta \right] \le 0 \tag{3.3}$$

for all $n \ge n_1$. Now using the inequality (2.3) twice, the last inequality becomes

$$\Delta^3 y_n + Q_n z_{n-\sigma_1}^{\frac{\beta}{\alpha}} + P_n z_{n+\sigma_2}^{\frac{\beta}{\alpha}} \le 0 \tag{3.4}$$

for all $n \ge n_1$. Since $\{x_n\}$ is a positive solution of equation (1.1), we have two cases for the corresponding sequence $\{z_n\}$ as stated in Lemma 2.1.

Case (I): In this case there exists an integer $n_2 \ge n_1$ such that $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$ and $\Delta^3 z_n \le 0$ for all $n \ge n_2$. Then, by definition of y_n, we have $y_n > 0$,

$\Delta y_n > 0$, $\Delta^2 y_n > 0$ and $\Delta^3 y_n \le 0$ for all $n \ge n_2$. From the inequality (3.4), we can write

$$\Delta^3 y_n + R_n z_{n-\sigma}^{\frac{\beta}{\alpha}} \le 0 \tag{3.5}$$

for all $n \ge n_2$. Since $\Delta^2 z_n > 0$, Δz_n is increasing and therefore

$$\begin{aligned} \Delta y_n &= \Delta z_n + b^\beta \Delta z_{n-\tau_1} + c^\beta \Delta z_{n+\tau_2} \\ &\le (1 + b^\beta + c^\beta)\Delta z_{n+\sigma}, \text{ for all } n \ge n_2. \end{aligned} \tag{3.6}$$

Using monotonicity of Δz_n, we get

$$z_{n+\sigma} = z_n + \sum_{s=n}^{n+\sigma-1} \Delta z_s \ge \sigma \Delta z_n \tag{3.7}$$

for all $n \ge n_2$. Now combining (3.5)–(3.7), we obtain

$$\Delta^3 y_n + \frac{R_n(\sigma)^{\frac{\beta}{\alpha}}}{(1 + b^\beta + c^\beta)^{\frac{\beta}{\alpha}}} (\Delta y_{n-2\sigma})^{\frac{\beta}{\alpha}} \le 0$$

for all $n \ge n_2$. Setting $w_n = \Delta y_n$, we see that $w_n = \Delta y_n > 0$, $\Delta w_n = \Delta^2 y_n > 0$, and

$$\Delta^2 w_n + \frac{R_n(\sigma)^{\frac{\beta}{\alpha}}}{(1 + b^\beta + c^\beta)^{\frac{\beta}{\alpha}}} w_{n-2\sigma}^{\frac{\beta}{\alpha}} \le 0$$

for all $n \ge n_2$. Now using Lemma 2.3, in the last inequality, we see that

$$\Delta^2 w_n + \frac{R_n(\sigma)^{\frac{\beta}{\alpha}}}{(1 + b^\beta + c^\beta)^{\frac{\beta}{\alpha}}} (k(n - 2\sigma))^{\frac{\beta}{\alpha}} (\Delta w_{n-2\sigma})^{\frac{\beta}{\alpha}} \le 0 \tag{3.8}$$

for all $n \ge n_2$. Now let $u_n = \Delta w_n > 0$, then we have from (3.8) that $\{u_n\}$ is a positive solution of the inequality (3.1), which is a contradiction.
Case (II): Suppose $\{z_n\}$ satisfies case (II) of Lemma 2.1. Then by Lemma 2.5, we get $\lim_{n\to\infty} x_n = 0$. This completes the proof. ■

Theorem 3.2 *Assume that* $\beta = \gamma \ge 1$, $\sigma \ge \max\{\sigma_1, \sigma_2, \tau_2\}$, *and* (2.6) *hold. If the first order difference inequality*

$$\Delta y_n + \frac{R_n}{4^{\beta-1}} \left(\frac{k\sigma(n - 2\sigma)}{(1 + b^\beta + c^\beta)} \right)^{\frac{\beta}{\alpha}} y_{n-2\sigma}^{\frac{\beta}{\alpha}} \le 0, \ n \ge n_0, \tag{3.9}$$

has no positive decreasing solution, then every solution of equation (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof Define a sequence $\{y_n\}$ by

$$y_n = z_n + b^\beta z_{n-\tau_1} + c^\beta z_{n+\tau_2}$$

for all $n \geq n_1 \geq n_0$. Then proceeding as in the proof of Theorem 3.1, and using the inequality (2.4) instead of the inequality (2.3) the result follows. ∎

Corollary 3.3 *Assume that* $\alpha = \beta = \gamma \leq 1$, $\sigma \geq \max\{\sigma_1, \sigma_2, \tau_2\}$, *and* (2.6) *hold. If*

$$\lim_{n\to\infty} \inf \sum_{s=n-2\sigma}^{n-1} R_s(s-2\sigma) > \left(\frac{2\sigma}{2\sigma+1}\right)^{2\sigma+1} \left(\frac{1+b^\beta+c^\beta}{k\sigma}\right), \tag{3.10}$$

then every solution of equation (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof By Theorem 7.5.1 of [14], and (3.10), the result follows from Theorem 3.1. ∎

Corollary 3.4 *Assume that* $1 \geq \gamma = \beta < \alpha$, $\sigma \geq \max\{\sigma_1, \sigma_2, \tau_2\}$, *and* (2.6) *hold. If*

$$\sum_{n=N}^{\infty} R_n(n-2\sigma)^{\frac{\beta}{\alpha}} = \infty, \ n \geq N \geq n_0, \tag{3.11}$$

then every solution of equation (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof By Theorem 1 of [7], and (3.11), the result follows from Theorem 3.1. ∎

Corollary 3.5 *Assume that* $1 \geq \gamma = \beta > \alpha$, $\sigma \geq \max\{\sigma_1, \sigma_2, \tau_2\}$, *and* (2.6) *hold. If there exists a* λ, *such that* $\lambda > \frac{1}{2\sigma} \log \frac{\beta}{\alpha}$ *and*

$$\lim_{n\to\infty} \inf[R_n \exp(-e^{n\lambda})] > 0, \tag{3.12}$$

then every solution of equation is either oscillatory or tends to zero as $n \to \infty$.

Proof By Theorem 2 of [7], and (3.12), the result follows from Theorem 3.1. ∎

Corollary 3.6 *Assume that* $\alpha = \beta = \gamma \geq 1$, $\sigma \geq \max\{\sigma_1, \sigma_2, \tau_2\}$, *and* (2.6) *hold. If*

$$\lim_{n\to\infty} \inf \sum_{s=n-2\sigma}^{n-1} R_s(s-2\sigma) > \left(\frac{2\sigma}{2\sigma+1}\right)^{2\sigma+1} 4^{\beta-1} \left(\frac{1+b^\beta+c^\beta}{k\sigma}\right), \tag{3.13}$$

then every solution of equation (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof By Theorem 7.5.1 of [14], and (3.13), the result follows from Theorem 3.2. ∎

Corollary 3.7 *Assume that* $1 \leq \gamma = \beta < \alpha$, $\sigma \geq \max\{\sigma_1, \sigma_2, \tau_2\}$, *and* (2.6) *hold. If*

$$\sum_{n=N}^{\infty} R_n (n-2\sigma)^{\frac{\beta}{\alpha}} = \infty, \ n \geq N \geq n_0, \tag{3.14}$$

then every solution of equation (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof By Theorem 1 of [7], and (3.14), the result follows from Theorem 3.2. ■

Corollary 3.8 *Assume that* $1 \leq \gamma = \beta > \alpha$, $\sigma \geq \max\{\sigma_1, \sigma_2, \tau_2\}$, *and* (2.6) *hold. If there exists a* λ, *such that* $\lambda > \frac{1}{2\sigma} \log \frac{\beta}{\alpha}$ *and*

$$\lim_{n\to\infty} \inf [R_n \exp(-e^{n\lambda})] > 0, \tag{3.15}$$

then every solution of equation is either oscillatory or tends to zero as $n \to \infty$.

Proof By Theorem 2 of [7], and (3.15), the result follows from Theorem 3.2. ■

Theorem 3.9 *Assume that* $0 < \beta \leq 1 < \gamma$, $\beta < \alpha < \gamma$, $b \leq 1$, $c \leq 1$, $\sigma \geq \max\{\sigma_1, \sigma_2, \tau_2\}$, *and* (2.6) *hold. If*

$$\lim_{n\to\infty} \inf \sum_{s=n-2\sigma}^{n-1} P_s^{\eta_1} Q_s^{\eta_2} (s-2\sigma)(s-2\sigma-1) > \left(\frac{2\sigma}{2\sigma+1}\right)^{2\sigma+1} \left(\frac{1+b^\beta+c^\beta}{\theta \eta_1^{-\eta_1} \eta_2^{-\eta_2}}\right) \tag{3.16}$$

where $\eta_1 = \frac{\alpha-\beta}{\gamma-\beta}$, $\eta_2 = \frac{\gamma-\alpha}{\gamma-\beta}$ *are satisfied and* $0 < \theta < 1$, *then every solution of equation* (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof Let $\{x_n\}$ be a positive solution of equation (1.1). Then there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\tau_1} > 0$ and $x_{n-\sigma_1} > 0$ for all $n \geq n_1$. Define a sequence $\{y_n\}$ by

$$y_n = z_n + b^\beta z_{n-\tau_1} + c^\beta z_{n+\tau_2} \tag{3.17}$$

for all $n \geq n_2 \geq n_1$. Then $y_n > 0$, and

$$\Delta^3 y_n = \Delta^3 z_n + b^\beta \Delta^3 z_{n-\tau_1} + c^\beta \Delta^3 z_{n+\tau_2}$$

for all $n \geq n_2$. That is,

$$\Delta^3 y_n + q_n x_{n-\sigma_1}^\beta + p_n x_{n+\sigma_2}^\gamma + b^\beta \left[q_{n-\tau_1} x_{n-\tau_1-\sigma_1}^\beta + p_{n-\tau_1} x_{n-\tau_1+\sigma_2}^\gamma \right] + c^\beta \left[q_{n+\tau_2} x_{n+\tau_2-\sigma_1}^\beta + p_{n+\tau_2} x_{n+\tau_2+\sigma_2}^\gamma \right] = 0$$

or

$$\Delta^3 y_n + Q_n \left[x_{n-\sigma_1}^\beta + b^\beta x_{n-\tau_1-\sigma_1}^\beta + c^\beta x_{n+\tau_2-\sigma_1}^\beta \right] + P_n \left[x_{n+\sigma_2}^\gamma + b^\beta x_{n-\tau_1+\sigma_2}^\gamma + c^\beta x_{n+\tau_2+\sigma_2}^\gamma \right] \leq 0.$$

Applying (2.3) twice in the second part of left hand side of last inequality, we have

$$\Delta^3 y_n + Q_n z_{n-\sigma_1}^{\frac{\beta}{\alpha}} + P_n \left[x_{n+\sigma_2}^{\gamma} + b^{\beta} x_{n-\tau_1+\sigma_2}^{\gamma} + c^{\beta} x_{n+\tau_2+\sigma_2}^{\gamma}\right] \le 0, \; n \ge n_2.$$

Since $b \le 1$, $c \le 1$, $\beta \le 1$ and $\gamma \ge 1$, the last inequality becomes

$$\Delta^3 y_n + Q_n z_{n-\sigma_1}^{\frac{\beta}{\alpha}} + P_n \left[x_{n+\sigma_2}^{\gamma} + b^{\gamma} x_{n-\tau_1+\sigma_2}^{\gamma} + \frac{c^{\gamma}}{2^{\gamma-1}} x_{n+\tau_2+\sigma_2}^{\gamma}\right] \le 0, \; n \ge n_2.$$

Now applying (2.4) twice in the third part of left hand side of last inequality, we obtain

$$\Delta^3 y_n + Q_n z_{n-\sigma_1}^{\frac{\beta}{\alpha}} + \frac{P_n z_{n+\sigma_2}^{\frac{\gamma}{\alpha}}}{4^{\gamma-1}} \le 0, \; n \ge n_2. \tag{3.18}$$

Since $\{x_n\}$ is a positive solution of equation (1.1), there are two cases for $\{z_n\}$ as stated in Lemma 2.1.

Case (I): In this case there exists an integer $n_3 \ge n_2$ such that $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$ and $\Delta^3 z_n \le 0$ for all $n \ge n_3$. Now from the inequality (3.18), we have

$$\Delta^3 y_n + Q_n z_{n-\sigma}^{\frac{\beta}{\alpha}} + \frac{P_n}{4^{\gamma-1}} z_{n-\sigma}^{\frac{\gamma}{\alpha}} \le 0. \tag{3.19}$$

for all $n \ge n_3$. Define $u_1 = \eta_1^{-1} \frac{P_n}{4^{\gamma-1}} z_{n-\sigma}^{\frac{\gamma}{\alpha}}$ and $u_2 = \eta_2^{-1} Q_n z_{n-\sigma}^{\frac{\beta}{\alpha}}$. Using arithmetic-geometric mean inequality

$$\frac{u_1\eta_1 + u_2\eta_2}{\eta_1 + \eta_2} \le \left(u_1^{\eta_1} u_2^{\eta_2}\right)^{\frac{1}{\eta_1+\eta_2}}$$

and the fact $\eta_1 + \eta_2 = 1$, in (3.19), we obtain

$$\Delta^3 y_n + \eta_1^{-\eta_1} \eta_2^{-\eta_2} \left(\frac{P_n}{4^{\gamma-1}}\right)^{\eta_1} Q_n^{\eta_2} z_{n-\sigma} \le 0, \; \text{for all} \; n \ge n_3. \tag{3.20}$$

Since z_n is monotonically increasing, we get

$$y_n \le (1 + b^{\beta} + c^{\beta}) z_{n+\sigma}, \; \text{for all} \; n \ge n_3. \tag{3.21}$$

Using the inequality (3.21) in (3.20), we have

$$\Delta^3 y_n + \frac{\eta_1^{-\eta_1} \eta_2^{-\eta_2}}{1 + b^{\beta} + c^{\beta}} \left(\frac{P_n}{4^{\gamma-1}}\right)^{\eta_1} Q_n^{\eta_2} y_{n-2\sigma} \le 0, \; n \ge n_3. \tag{3.22}$$

Now using Lemma 2.4, in the last inequality, we get

$$\Delta^3 y_n + \frac{\theta\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{1+b^\beta+c^\beta}\left(\frac{P_n}{4^{\gamma-1}}\right)^{\eta_1} Q_n^{\eta_2}(n-2\sigma)(n-2\sigma-1)\Delta^2 y_{n-2\sigma} \le 0 \quad (3.23)$$

where $0<\theta<1$, for all $n \ge n_3$. Setting $w_n = \Delta^2 y_n > 0$, the last inequality becomes

$$\Delta w_n + \frac{\theta\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{1+b^\beta+c^\beta}\left(\frac{P_n}{4^{\gamma-1}}\right)^{\eta_1} Q_n^{\eta_2}(n-2\sigma)(n-2\sigma-1) w_{n-2\sigma} \le 0 \quad (3.24)$$

for all $n \ge n_3$. Thus $\{w_n\}$ is a positive solution of the inequality (3.24). But by the condition (3.16) and Theorem 7.5.1 of [14], we see that the inequality (3.24) has no positive solution, which is a contradiction.

Case (II): Suppose $\{z_n\}$ satisfies case (II) of Lemma 2.1. Then by Lemma 2.5, we get $\lim_{n\to\infty} x_n = 0$. This completes the proof. ■

Theorem 3.10 *Assume that* $0<\gamma\le 1<\beta$, $\gamma<\alpha<\beta$, $b\le 1$, $c\le 1$, $\sigma\ge\max\{\sigma_1,\sigma_2,\tau_2\}$, *and* (2.6) *hold. If*

$$\liminf_{n\to\infty}\sum_{s=n-2\sigma}^{n-1} P_s^{\eta_1} Q_s^{\eta_2}(s-2\sigma)(s-2\sigma-1) > \left(\frac{2\sigma}{2\sigma+1}\right)^{2\sigma+1}\left(\frac{1+b^\beta+c^\beta}{\theta\eta_1^{-\eta_1}\eta_2^{-\eta_2}}\right) \quad (3.25)$$

where $\eta_1 = \frac{\beta-\alpha}{\beta-\gamma}$, $\eta_2 = \frac{\alpha-\gamma}{\beta-\gamma}$ *are satisfied and* $0<\theta<1$, *then every solution of equation* (1.1) *is either oscillatory or tends to zero as* $n\to\infty$.

Proof The proof is similar to that of Theorem 3.9, so the details are omitted. ■

Theorem 3.11 *Let* $0<\gamma\le 1<\beta, \gamma<\alpha<\beta, b\ge 1, c\ge 1, \sigma\ge\max\{\sigma_1,\sigma_2,\tau_2\}$ *and* (2.6) *hold. If*

$$\liminf_{n\to\infty}\sum_{s=n-2\sigma}^{n-1} Q_s^{\eta_1} P_s^{\eta_2}(s-2\sigma)(s-2\sigma-1) > \left(\frac{2\sigma}{2\sigma+1}\right)^{2\sigma+1}\left(\frac{1+b^\beta+\frac{c^\beta}{2^{\gamma-1}}}{\theta\eta_1^{-\eta_1}\eta_2^{-\eta_2}}\right) \quad (3.26)$$

where $\eta_1 = \frac{\beta-\alpha}{\beta-\gamma}$, $\eta_2 = \frac{\alpha-\gamma}{\beta-\gamma}$ *are satisfied and* $0<\theta<1$, *then every solution of equation* (1.1) *is either oscillatory or tends to zero as* $n\to\infty$.

Proof Let $\{x_n\}$ be a positive solution of equation (1.1). Then there exists an integer $n_1\ge n_0$ such that $x_n>0$, $x_{n-\sigma_1}>0$ and $x_{n-\tau_1}>0$ for all $n\ge n_1$. Now define a sequence $\{y_n\}$ by

$$y_n = z_n + b^\beta z_{n-\tau_1} + \frac{c^\beta}{2^{\gamma-1}} z_{n+\tau_2} \quad (3.27)$$

for all $n\ge n_1$. Then $y_n>0$, and

$$\Delta^3 y_n = \Delta^3 z_n + b^\beta\Delta^3 z_{n-\tau_1} + \frac{c^\beta}{2^{\gamma-1}}\Delta^3 z_{n+\tau_2}$$

for all $n\ge n_1$. That is,

$$\Delta^3 y_n + q_n x^{\beta}_{n-\sigma_1} + p_n x^{\gamma}_{n+\sigma_2} + b^{\beta}[q_{n-\tau_1} x^{\beta}_{n-\tau_1-\sigma_1} + p_{n-\tau_1} x^{\gamma}_{n-\tau_1+\sigma_2}] + \frac{c^{\beta}}{2^{\gamma-1}}[q_{n+\tau_2} x^{\beta}_{n+\tau_2-\sigma_1} + p_{n+\tau_2} x^{\gamma}_{n+\tau_2+\sigma_2}] \leq 0$$

or

$$\Delta^3 y_n + Q_n\left[x^{\beta}_{n-\sigma_1} + b^{\beta} x^{\beta}_{n-\tau_1-\sigma_1} + \frac{c^{\beta}}{2^{\gamma-1}} x^{\beta}_{n+\tau_2-\sigma_1}\right] + P_n\left[x^{\gamma}_{n+\sigma_2} + b^{\beta} x^{\gamma}_{n-\tau_1+\sigma_2} + \frac{c^{\beta}}{2^{\gamma-1}} x^{\gamma}_{n+\tau_2+\sigma_2}\right] \leq 0 \text{ for all } n \geq n_1.$$

Since $0 < \gamma \leq 1 \leq \beta$, $b \geq 1$, and $c \geq 1$, the last inequality becomes

$$\Delta^3 y_n + Q_n\left[x^{\beta}_{n-\sigma_1} + b^{\beta} x^{\beta}_{n-\tau_1-\sigma_1} + \frac{c^{\beta}}{2^{\beta-1}} x^{\beta}_{n+\tau_2-\sigma_1}\right] + P_n\left[x^{\gamma}_{n+\sigma_2} + b^{\gamma} x^{\gamma}_{n-\tau_1+\sigma_2} + c^{\gamma} x^{\gamma}_{n+\tau_2+\sigma_2}\right] \leq 0 \text{ for all } n \geq n_1.$$

Now using (2.3) and (2.4) twice each in the second and third part of left hand side of last inequality, respectively, we have

$$\Delta^3 y_n + \frac{Q_n}{4^{\beta-1}} z^{\frac{\beta}{\alpha}}_{n-\sigma_1} + P_n z^{\frac{\gamma}{\alpha}}_{n+\sigma_2} \leq 0,\ n \geq n_1. \tag{3.28}$$

Now we have two cases for $\{z_n\}$ as stated in Lemma 2.1, since $\{x_n\}$ is a positive solution of equation (1.1).

Case (I): In this case, there exists an integer $n_2 \geq n_1$ such that $z_n > 0$, $\Delta z_n > 0$, $\Delta^2 z_n > 0$ and $\Delta^3 z_n \leq 0$ for all $n \geq n_2$. From the inequality (3.28), we can write

$$\Delta^3 y_n + \frac{Q_n}{4^{\beta-1}} z^{\frac{\beta}{\alpha}}_{n-\sigma} + P_n z^{\frac{\gamma}{\alpha}}_{n-\sigma} \leq 0. \tag{3.29}$$

for all $n \geq n_2$. Define $u_1 = \eta_1^{-1} \frac{Q_n}{4^{\beta-1}} z^{\frac{\beta}{\alpha}}_{n-\sigma}$ and $u_2 = \eta_2^{-1} P_n z^{\frac{\gamma}{\alpha}}_{n-\sigma}$. Using arithmetic-geometric mean inequality in (3.29), we have

$$\Delta^3 y_n + \eta_1^{-\eta_1} \eta_2^{-\eta_2} \left(\frac{Q_n}{4^{\beta-1}}\right)^{\eta_1} P_n^{\eta_2} z_{n-\sigma} \leq 0, \quad \text{for all } n \geq n_2. \tag{3.30}$$

Since $\Delta^2 z_n > 0$, we have

$$\Delta y_n = \Delta z_n + b^{\beta} \Delta z_{n-\tau_1} + \frac{c^{\beta}}{2^{\gamma-1}} \Delta z_{n+\tau_2} \leq \left(1 + b^{\beta} + \frac{c^{\beta}}{2^{\gamma-1}}\right) \Delta z_{n+\sigma}, \quad \text{for all } n \geq n_2. \tag{3.31}$$

Using the inequality (3.31) in (3.30), we obtain

$$\Delta^3 y_n + \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{1+b^\beta+\frac{c^\beta}{2^{\gamma-1}}}\left(\frac{Q_n}{4^{\beta-1}}\right)^{\eta_1} P_n^{\eta_2} y_{n-2\sigma} \le 0, \quad \text{for all } n \ge n_2. \tag{3.32}$$

Using Lemma 2.4 in (3.32), we obtain

$$\Delta^3 y_n + \frac{\theta\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{1+b^\beta+\frac{c^\beta}{2^{\gamma-1}}}\left(\frac{Q_n}{4^{\beta-1}}\right)^{\eta_1} P_n^{\eta_2}(n-2\sigma)(n-2\sigma-1)\Delta^2 y_{n-2\sigma} \le 0. \tag{3.33}$$

By taking $\Delta^2 y_n = w_n > 0$, then we see that $\{w_n\}$ is positive solution of

$$\Delta w_n + \frac{\theta\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{1+b^\beta+\frac{c^\beta}{2^{\gamma-1}}}\left(\frac{Q_n}{4^{\beta-1}}\right)^{\eta_1} P_n^{\eta_2}(n-2\sigma)(n-2\sigma-1)w_{n-2\sigma} \le 0. \tag{3.34}$$

But by the condition (3.26) and Theorem 7.5.1 of [14], the inequality (3.34) has no positive solution, which is a contradiction.
Case (II): Suppose $\{z_n\}$ satisfies case (II) of Lemma 2.1. Then by Lemma 2.5, we get $\lim_{n\to\infty} x_n = 0$. This completes the proof. ■

Theorem 3.12 *Assume that* $0 < \beta \le 1 < \gamma$, $\beta < \alpha < \gamma$, $b \ge 1$, $c \ge 1$, $\sigma \ge \max\{\sigma_1, \sigma_2, \tau_2\}$, *and* (2.6) *hold. If*

$$\liminf_{n\to\infty} \sum_{s=n-2\sigma}^{n-1} Q_s^{\eta_1} P_s^{\eta_2}(s-2\sigma)(s-2\sigma-1) > (4^{\gamma-1})^{\eta_1}\left(\frac{2\sigma}{2\sigma+1}\right)^{\tau_2+\sigma+1}\left(\frac{1+\frac{b^\beta}{2^{\beta-1}}+c^\beta}{\theta\eta_1^{-\eta_1}\eta_2^{-\eta_2}}\right) \tag{3.35}$$

where $\eta_1 = \frac{\alpha-\beta}{\gamma-\beta}$, $\eta_2 = \frac{\gamma-\alpha}{\gamma-\beta}$ *and* $0 < \theta < 0$, *then every solution of equation* (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof The proof is similar to that of Theorem 3.5, and hence the details are omitted. ■

4 Examples

In this section, we present some examples to illustrate the main results.

Example 4.1 Consider the following third order neutral difference equation

$$\Delta^3\left(x_n+\frac{1}{2}x_{n-2}+\frac{1}{3}x_{n+1}\right)^{\frac{1}{3}}+10\left(\frac{7}{6}\right)^{\frac{1}{3}}x_{n-2}^{\frac{1}{3}}+2\left(\frac{7}{6}\right)^{\frac{1}{3}}x_{n+3}^{\frac{1}{3}}=0,\ n\geq 1. \tag{4.1}$$

Here $b_n=\frac{1}{2},\ c_n=\frac{1}{3},\ \tau_1=2,\ \tau_2=1,\ \sigma_1=2,\ \sigma_2=3,\ \alpha=\beta=\gamma=\frac{1}{3},\ q_n=10(\frac{7}{6})^{\frac{1}{3}},\ p_n=2(\frac{7}{6})^{\frac{1}{3}}$ and $R_n=12\left(\frac{7}{6}\right)^{\frac{1}{3}}$. Now we see that

$$\sum_{n=1}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}(p_t+q_t)=\sum_{n=1}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}12\left(\frac{7}{6}\right)^{\frac{1}{3}}=\infty$$

and by taking $\sigma=3$, we have

$$\liminf_{n\to\infty}\sum_{s=n-6}^{n-1}R_s(s-6)=\liminf_{n\to\infty}\sum_{s=n-6}^{n-1}12\left(\frac{7}{6}\right)^{\frac{1}{3}}(s-6)=\infty.$$

Therefore all the conditions of Corollary 3.3 are satisfied, and note that $\{x_n\}=\{(-1)^{3n}\}$ is an oscillatory solution of equation (4.1).

Example 4.2 Consider the following third order neutral difference equation

$$\Delta^3\left(x_n+2x_{n-2}+2x_{n+1}\right)^{\frac{1}{3}}+x_{n-1}^{\frac{1}{5}}+9x_{n+2}^{\frac{1}{5}}=0,\ n\geq 3. \tag{4.2}$$

Here $b_n=2,\ c_n=2,\ q_n=1,\ p_n=9,\ \tau_1=2,\ \tau_2=1,\ \sigma_1=1,\ \sigma_2=2,\ \alpha=\frac{1}{3},$ $\beta=\gamma=\frac{1}{5}$ and $R_n=10$. Now we see that

$$\sum_{n=3}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}(p_t+q_t)=\sum_{n=3}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}10=\infty$$

and by taking $\sigma=2$, we have

$$\sum_{n=5}^{\infty}R_n(n-4)^{\frac{1}{3}}=\sum_{n=5}^{\infty}10(n-4)^{\frac{1}{3}}=\infty.$$

Therefore all the conditions of Corollary 3.4 are satisfied, and note that $\{x_n\}=\{(-1)^{15n}\}$ is an oscillatory solution of equation (4.2).

Example 4.3 Consider the following third order neutral difference equation

$$\Delta^3\left(x_n+\frac{1}{4}x_{n-1}+\frac{1}{2}x_{n+1}\right)^3+\left(\frac{7}{8}\right)^3\frac{255}{2^{2n+7}}x_{n-1}+44\left(\frac{7}{8}\right)^3 4^n x_{n+1}^5=0,\ n\geq 6. \tag{4.3}$$

Here $b_n=\frac{1}{4}$, $c_n=\frac{1}{2}$, $q_n=\left(\frac{7}{8}\right)^3\frac{255}{2^{2n+7}}$, $p_n=44\left(\frac{7}{8}\right)^3 4^n$, $\alpha=3$, $\beta=1$, $\gamma=5$, $\tau_1=\tau_2=\sigma_1=\sigma_2=1,\ \eta_1=\frac{1}{2},\ \eta_2=\frac{1}{2},\ P_n=11(\frac{7}{8})^3 4^n,\ Q_n=(\frac{7}{8})^3\frac{255}{2^{2n+7}},$ and $\sigma=1$.

Now we see that

$$\sum_{n=6}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}(p_t+q_t)=\sum_{n=6}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}44\left(\frac{7}{8}\right)^3 4^n+\left(\frac{7}{8}\right)^3\frac{255}{2^{2n+7}}=\infty$$

and

$$\lim_{n\to\infty}\inf\sum_{s=n-2}^{n-1}P_s^{\eta_1}Q_s^{\eta_2}(s-2\sigma)(s-2\sigma-1)$$
$$=\lim_{n\to\infty}\inf\sum_{s=n-2}^{n-1}\left(\frac{2805}{128}\right)^{\frac{1}{2}}\left(\frac{7}{8}\right)^3(s-2)(s-3)=\infty.$$

Therefore all the conditions of Theorem 3.9 are satisfied, and note that $\{x_n\}=\{\frac{1}{2^n}\}$ is a nonoscillatory solution tending to zero as $n\to\infty$.

Example 4.4 Consider the following third order neutral difference equation

$$\Delta^3\left(x_n+2x_{n-1}+12x_{n+2}\right)^3+\left(\frac{3}{2}\right)^3 4^n x_{n-1}^5+\frac{470}{4^n}x_{n+1}=0,\ \ n\geq 6. \tag{4.4}$$

Here $b_n=2$, $c_n=12$, $\tau_1=1$, $\tau_2=2$, $\sigma_1=\sigma_2=1$, $\alpha=3$, $\beta=5$, $\gamma=1$, $p_n=\frac{470}{4^n}$, $q_n=\left(\frac{3}{2}\right)^3 4^n$, $\eta_1=\frac{1}{2}$, $\eta_2=\frac{1}{2}$, $P_n=\frac{470}{4^n}$, $Q_n=\frac{27}{2}4^n$, and $\sigma=2$. Now we see that

$$\sum_{n=6}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}(p_t+q_t)=\sum_{n=6}^{\infty}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\left(\left(\frac{470}{4^t}+\left(\frac{3}{2}\right)^3 4^t\right)\right)=\infty$$

and

$$\lim_{n\to\infty}\inf\sum_{s=n-6}^{n-1}P_s^{\eta_2}Q_s^{\eta_1}(s-2\sigma)(s-2\sigma-1)=\lim_{n\to\infty}\inf\sum_{s=n-6}^{n-1}(6345)^{\frac{1}{2}}(s-4)(s-5)=\infty.$$

Therefore all the conditions of Theorem 3.11 are satisfied, and note that $\{x_n\}=\{\frac{1}{2^n}\}$ is a nonoscillatory solution tending to zero as $n\to\infty$.

Remark 4.5 The results obtained in [3, 4, 6, 10, 11, 16] cannot be applied to Eqs. (4.1)–(4.4), since $\alpha\neq 1$ and $\beta\neq\gamma$. Further the results established in [5, 8, 9, 13, 15, 17–21] cannot be applied to Eqs. (4.1)–(4.4), since the neutral term contain both delay and advanced arguments. Therefore our results extend and complement to the known results mentioned above. Further the results of this paper can be improved in the sense that "all solutions of equation (1.1) are oscillatory", and we investigate this in our future study.

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The Fuzzy Henstock–Kurzweil Delta Integral on Time Scales

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Abstract We investigate properties of the fuzzy Henstock–Kurzweil delta integral (shortly, FHK Δ-integral) on time scales, and obtain two necessary and sufficient conditions for FHK Δ-integrability. The concept of uniformly FHK Δ-integrability is introduced. Under this concept, we obtain a uniformly integrability convergence theorem. Finally, we prove monotone and dominated convergence theorems for the FHK Δ-integral.

Keywords Fuzzy Henstock–Kurzwei integral · Convergence theorems · Time scales

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1 Introduction

The Lebesgue integral, with its convergence properties, is superior to the Riemann integral. However, a disadvantage with respect to Lebesgue's integral, is that it is hard to understand without substantial mathematical maturity. Also, the Lebesgue integral does not inherit the naturalness of the Riemann integral. Henstock [23] and Kurzweil [26] gave, independently, a slight, yet powerful, modification of the Riemann integral to get the now called Henstock–Kurzweil (HK) integral, which possesses all the convergence properties of the Lebesgue integral. For the fundamental results of HK integral, we refer to the papers [7, 23, 38, 39, 43, 45] and monographs [20, 27, 34]. As an important branch of the HK integration theory, the fuzzy Henstock–Kurzweil (FHK) integral has been extensively studied in [8, 13, 18, 19, 22, 30, 35, 36, 41, 42].

In 1988, Hilger introduced the theory of time scales in his Ph.D. thesis [24]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The aim is to unify and generalize discrete and continuous dynamical systems, see, e.g., [2–6, 9, 15, 21, 31]. In [32], Peterson and Thompson introduced a more general concept of integral, i.e., the HK Δ-integral, which gives a common generalization of the Riemann Δ and Lebesgue Δ-integral. The theory of HK integration for real-valued and vector-valued functions on time scales has been developed rather intensively, see, e.g., the papers [1, 11, 16, 28, 29, 33, 37, 40, 44] and references cited therein.

In 2015, Fard and Bidgoli introduced the FHK delta integral and presented some of its basic properties [14]. Nonetheless, to our best knowledge, there is no systematic theory for the FHK delta integral on time scales. In this work, in order to complete the FHK delta integration theory, we give two necessary and sufficient conditions of FHK delta integrability (see Theorems 3 and 7). Moreover, we obtain some convergence theorems for the FHK delta integral, in particular Theorem 9 of dominated convergence and Theorem 10 of monotone convergence.

After Sect. 2 of preliminaries, in Sect. 3 the definition of FHK delta integral is introduced, and our necessary and sufficient conditions of FHK delta integrability are proved. We also obtain some convergence theorems. Finally, in Sect. 4, we give conclusions and point out some directions that deserve further study.

2 Preliminaries

A fuzzy subset of the real axis $u : \mathbb{R} \to [0, 1]$ is called a fuzzy number provided that

(1) u is normal: there exists $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;
(2) u is fuzzy convex: $u(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{u(x_1), u(x_2)\}$ for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in (0, 1)$;
(3) u is upper semi-continuous;
(4) $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact.

Denote by $\mathbb{R}_{\mathscr{F}}$ the space of fuzzy numbers. We define the α-level set $[u]^\alpha$ by

$$[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}, \quad \alpha \in (0, 1].$$

From conditions (1)–(4), $[u]^\alpha$ is denoted by $[u]^\alpha = \left[\underline{u^\alpha}, \overline{u^\alpha}\right]$. For $u_1, u_2 \in \mathbb{R}_{\mathscr{F}}$ and $\lambda \in \mathbb{R}$, we define

$$[u_1 + u_2]^\alpha = [u_1]^\alpha + [u_2]^\alpha \text{ and } [\lambda \odot u_1]^\alpha = \lambda[u_1]^\alpha$$

for all $\alpha \in [0, 1]$. The Hausdorff distance between u_1 and u_2 is defined by

$$\mathbf{D}(u_1, u_2) = \sup_{\alpha\in[0,1]} \max \left\{|\underline{u_1^\alpha} - \underline{u_2^\alpha}|, |\overline{u_1^\alpha} - \overline{u_2^\alpha}|\right\}.$$

Then, the metric space $(\mathbb{R}_{\mathscr{F}}, \mathbf{D})$ is complete. Let $a, b \in \mathbb{T}$. We define the half-open interval $[a, b)_{\mathbb{T}}$ by

$$[a, b)_{\mathbb{T}} = \{x \in \mathbb{T} : a \leq x < b\}.$$

The open and closed intervals are defined similarly. For $x \in \mathbb{T}$, we denote by σ the forward jump operator, i.e., $\sigma(x) := \inf\{y > x : y \in \mathbb{T}\}$, and by ρ the backward jump operator, i.e., $\rho(x) := \sup\{y < x : y \in \mathbb{T}\}$. Here, we put $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, where $\sup \mathbb{T}$ and $\inf \mathbb{T}$ are finite. In this situation, $\mathbb{T}^\kappa := \mathbb{T}\backslash\{\sup \mathbb{T}\}$ and $\mathbb{T}_\kappa := \mathbb{T}\backslash\{\inf \mathbb{T}\}$, otherwise, $\mathbb{T}^\kappa := \mathbb{T}$ and $\mathbb{T}_\kappa := \mathbb{T}$. If $\sigma(x) > x$, then we say that x is right-scattered, while if $\rho(x) < x$, then we say that x is left-scattered. If $\sigma(x) = x$ and $x < \sup \mathbb{T}$, then x is called right-dense, and if $\rho(x) = x$ and $x > \inf \mathbb{T}$, then x is left-dense. The graininess functions μ and η are defined by $\mu(x) := \sigma(x) - x$ and $\eta(x) := x - \rho(x)$, respectively.

In what follows, all considered intervals are intervals in $\mathbb{T}$. A division D of $[a, b]_{\mathbb{T}}$ is a finite set of interval-point pairs $\{([x_{i-1}, x_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ such that

$$\bigcup_{i=1}^n [x_{i-1}, x_i]_{\mathbb{T}} = [a, b]_{\mathbb{T}}$$

and $\xi_i \in [a, b]_{\mathbb{T}}$ for each i. We write $\Delta x_i = x_i - x_{i-1}$. We say that

$$\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$$

is a Δ-gauge on $[a, b]_{\mathbb{T}}$ if $\delta_L(\xi) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_R(\xi) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$ and $\delta_R(\xi) \geq \mu(\xi)$ for any $\xi \in [a, b)_{\mathbb{T}}$. The symbol $\Gamma(\Delta, [a, b]_{\mathbb{T}})$ stands for the set of Δ-gauge on $[a, b]_{\mathbb{T}}$. Let $\delta^1(\xi)$ and $\delta^2(\xi)$ be Δ-gauges such that

$$0 < \delta_L^1(\xi) < \delta_L^2(\xi)$$

for any $\xi \in (a,b]_{\mathbb{T}}$ and $0 < \delta_R^1(\xi) < \delta_R^2(\xi)$ for any $\xi \in [a,b)_{\mathbb{T}}$. Then we call $\delta^1(\xi)$ finer than $\delta^2(\xi)$ and write $\delta^1(\xi) < \delta^2(\xi)$. We say that $D = \{([x_{i-1}, x_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ is a δ-fine HK division of $[a,b]_{\mathbb{T}}$ if $\xi_i \in [x_{i-1}, x_i]_{\mathbb{T}} \subset (\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i))_{\mathbb{T}}$ for each i. Let $\mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$ be the set of all δ-fine HK divisions of $[a,b]_{\mathbb{T}}$. Given an arbitrary $D \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$, $D = \{([x_{i-1}, x_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$, we write

$$S(f, D, \delta) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

for integral sums over D, whenever $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathscr{F}}$.

Lemma 1 (See [25]) *Suppose that* $u \in \mathbb{R}_{\mathscr{F}}$. *Then,*

(1) the interval $[u]^\alpha$ *is closed for* $\alpha \in [0,1]$*;*
(2) $[u]^{\alpha_1} \supset [u]^{\alpha_2}$ *for* $0 \le \alpha_1 \le \alpha_2 \le 1$*;*
(3) for any sequence $\{\alpha_n\}$ *satisfying* $\alpha_n \le \alpha_{n+1}$ *and* $\alpha_n \to \alpha \in (0,1]$, *we have* $\bigcap_{n=1}^{\infty}[u]^{\alpha_n} = [u]^\alpha$.

Conversely, if a collection of subsets $\{\mathrm{u}^\alpha : \alpha \in [0,1]\}$ *verify (1)–(3), then there exists a unique* $\mathrm{u} \in \mathbb{R}_{\mathscr{F}}$ *such that* $[u]^\alpha = \mathrm{u}^\alpha$ *for* $\alpha \in (0,1]$ *and* $[u]^0 = \overline{\bigcup_{\alpha \in (0,1]} \mathrm{u}^\alpha} \subset \mathrm{u}^0$.

Lemma 2 (See [17]) *Suppose that* $u \in \mathbb{R}_{\mathscr{F}}$. *Then,*

(1) $\underline{u^\alpha}$ *is bounded and nondecreasing;*
(2) $\overline{u^\alpha}$ *is bounded and nonincreasing;*
(3) $\underline{u^1} \le \overline{u^1}$*;*
(4) for $c \in (0,1]$, $\lim_{\alpha \to c^-} \underline{u^\alpha} = \underline{u^c}$ *and* $\lim_{\alpha \to c^-} \overline{u^\alpha} = \overline{u^c}$*;*
(5) $\lim_{\alpha \to 0^+} \underline{u^\alpha} = \underline{u^0}$ *and* $\lim_{\alpha \to 0^+} \overline{u^\alpha} = \overline{u^0}$.

Conversely, if $\underline{u^\alpha}$ *and* $\overline{u^\alpha}$ *satisfy items (1)–(5), then there exists* $\mathrm{u} \in \mathbb{R}_{\mathscr{F}}$ *such that*

$$[\mathrm{u}]^\alpha = \left[\underline{\mathrm{u}^\alpha}, \overline{\mathrm{u}^\alpha}\right] = \left[\underline{u^\alpha}, \overline{u^\alpha}\right].$$

3 The Fuzzy Henstock–Kurzweil Delta Integral

We introduce the concept of fuzzy Henstock–Kurzweil (FHK) delta integrability.

Definition 1 A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathscr{F}}$ is called FHK Δ-integrable on $[a,b]_{\mathbb{T}}$ with the FHK Δ-integral $\tilde{A} = (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x$, if for each $\varepsilon > 0$ there exists a $\delta \in \Gamma(\Delta, [a,b]_{\mathbb{T}})$ such that $\mathbf{D}\left(S(f, D, \delta), \tilde{A}\right) < \varepsilon$ for each $D \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$. The family of all FHK Δ-integrable functions on $[a,b]_{\mathbb{T}}$ is denoted by $\mathscr{FHK}_{[a,b]_{\mathbb{T}}}$.

Remark 1 It is clear that Definition 1 is more general than the HK Δ-integral introduced by Peterson and Thompson in [32] and more general than the FH integral introduced by Wu and Gong in [41, 42].

The proofs of Theorems 1 and 2 are straightforward and are left to the reader.

Theorem 1 *The FHK Δ-integral of $f(x)$ is unique.*

Theorem 2 *If $f(x), g(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$ and $\alpha, \beta \in \mathbb{R}$, then*

$$\alpha f(x) + \beta g(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$$

with

$$(FHK)\int_{[a,b]_{\mathbb{T}}} (\alpha f(x)+\beta g(x))\Delta x = \alpha(FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x + \beta(FHK)\int_{[a,b]_{\mathbb{T}}} g(x)\Delta x.$$

Follows a Cauchy–Bolzano condition for the FHK Δ-integral.

Theorem 3 (The Cauchy–Bolzano condition) *Function $f(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$ if and only if for each $\varepsilon > 0$ there exists a $\delta \in \Gamma(\Delta, [a,b]_{\mathbb{T}})$ such that*

$$\mathbf{D}\left(S(f, D_1, \delta), S(f, D_2, \delta)\right) < \varepsilon$$

for any $D_1, D_2 \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$.

Proof (Necessity) Let $\varepsilon > 0$. By hypothesis, there exists $\delta \in \Gamma(\Delta, [a,b]_{\mathbb{T}})$ such that

$$\mathbf{D}\left(S(f, D, \delta), (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x\right) < \frac{\varepsilon}{2}$$

for any $D \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$. Let $D_1, D_2 \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$. Then,

$$\begin{aligned}&\mathbf{D}\left(S(f, D_1, \delta), S(f, D_2, \delta)\right)\\ &\leq \mathbf{D}\left(S(f, D_1, \delta), (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x\right) + \mathbf{D}\left(S(f, D_2, \delta), (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x\right)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

(Sufficiency) For each n, choose a $\delta_n \in \Gamma(\Delta_n, [a,b]_{\mathbb{T}})$ such that

$$\mathbf{D}\left(S(f, D_1, \delta_n), S(f, D_2, \delta_n)\right) < \frac{1}{n}$$

for any $D_1, D_2 \in \mathfrak{D}(\delta_n, [a,b]_{\mathbb{T}})$. Replacing δ_n by $\bigcap_{j=1}^{n} \delta_j = \delta_n$, we may assume that $\delta_{n+1} \subset \delta_n$. For each n, fix a $D_n \in \mathfrak{D}(\delta_n, [a,b]_{\mathbb{T}})$. For $j > n$, we have $\delta_j \subset \delta_n$, so $D_j \in \mathfrak{D}(\delta_n, [a,b]_{\mathbb{T}})$. Thus, $\mathbf{D}\left(S(f, D_n, \delta_n), S(f, D_j, \delta_n)\right) < \frac{1}{n}$ and it follows that

$\{S(f, D_n, \delta_n)\}$ is a Cauchy sequence. We denote the limit of $\{S(f, D_n, \delta_n)\}$ by $\tilde{A}$ and let $\varepsilon > 0$. Choose $N > \frac{2}{\varepsilon}$ and let $D \in \mathfrak{D}(\delta_N, [a, b]_{\mathbb{T}})$. Then,

$$\begin{aligned}\mathbf{D}\left(S(f, D, \delta_N), \tilde{A}\right) &\leq \mathbf{D}\left(S(f, D, \delta_N), S(f, D_N, \delta_N)\right) + \mathbf{D}\left(S(f, D_N, \delta_N), \tilde{A}\right) \\ &< \frac{1}{N} + \frac{1}{N} \\ &< \varepsilon.\end{aligned}$$

Hence, $f(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$.

Theorem 4 *Let $c \in (a, b)_{\mathbb{T}}$. If $f(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,c]_{\mathbb{T}}} \bigcap \mathscr{F}\mathscr{H}\mathscr{K}_{[c,b]_{\mathbb{T}}}$, then*

$$f(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$$

with

$$(FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x = (FHK)\int_{[a,c]_{\mathbb{T}}} f(x)\Delta x + (FHK)\int_{[c,b]_{\mathbb{T}}} f(x)\Delta x.$$

Proof Let $\varepsilon > 0$. By assumption, there exist Δ-gauges

$$\delta^i(\xi) = (\delta^i_L(\xi), \delta^i_R(\xi)), \quad i = 1, 2,$$

such that

$$\mathbf{D}\left(S(f, D_1, \delta^1),\ (FHK)\int_{[a,c]_{\mathbb{T}}} f(x)\Delta x\right) < \varepsilon,$$
$$\mathbf{D}\left(S(f, D_2, \delta^2),\ (FHK)\int_{[c,b]_{\mathbb{T}}} f(x)\Delta x\right) < \varepsilon,$$

respectively for any $D_1 \in \mathfrak{D}(\delta^1, [a, c]_{\mathbb{T}})$, $D_1 = \{([x^1_{k-1}, x^1_\kappa]_{\mathbb{T}}, \xi^1_\kappa)\}^n_{k=1}$, and for any $D_2 \in \mathfrak{D}(\delta^2, [c, b]_{\mathbb{T}})$, $D_2 = \{([x^2_{k-1}, x^2_\kappa]_{\mathbb{T}}, \xi^2_\kappa)\}^m_{k=1}$. We define $\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$ on $[a, b]_{\mathbb{T}}$ by setting

$$\delta_L(\xi) = \begin{cases} \delta^1_L(\xi), & \text{if } \xi \in [a, c)_{\mathbb{T}}, \\ \delta^1_L(\xi), & \text{if } \xi = c = \rho(c), \\ \min\left\{\delta^1_L(\xi), \frac{\eta(c)}{2}\right\}, & \text{if } \xi = c > \rho(c), \\ \min\left\{\delta^2_L(\xi), \frac{\xi - c}{2}\right\}, & \text{if } \xi \in (c, b]_{\mathbb{T}}, \end{cases}$$

and

$$\delta_R(\xi) = \begin{cases} \min\left\{\delta_R^1(\xi), \max\left\{\mu(\xi), \frac{c-\xi}{2}\right\}\right\}, & \text{if } \xi \in [a,c)_{\mathbb{T}}, \\ \min\{\delta_R^2(\xi)\}, & \text{if } \xi \in [c,b]_{\mathbb{T}}. \end{cases}$$

Now, let $D \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$, $D = \{([x_{k-1}, x_k]_{\mathbb{T}}, \xi_k)\}_{k=1}^{p}$. It follows that

(i) either $c = \xi_q$ and $t_q > c$;
(ii) or $\xi_q = \rho(c) < c$ and $t_q = c$.

The case (ii) is straightforward. For (i), one has

$$\begin{aligned}
&\mathbf{D}\left(S(f, D, \delta),\ (FHK)\int_{[a,c]_{\mathbb{T}}} f(x)\Delta x + (FHK)\int_{[c,b]_{\mathbb{T}}} f(x)\Delta x\right) \\
&= \mathbf{D}\left(\sum_{k=1}^{p} f(\xi_k)\Delta x_k,\ (FHK)\int_{[a,c]_{\mathbb{T}}} f(x)\Delta x + (FHK)\int_{[c,b]_{\mathbb{T}}} f(x)\Delta x\right) \\
&\leq \mathbf{D}\left(\sum_{k=1}^{q-1} f(\xi_k)\Delta x_k + f(c)(c - t_{q-1}),\ (FHK)\int_{[a,c]_{\mathbb{T}}} f(x)\Delta x\right) \\
&\quad + \mathbf{D}\left(\sum_{k=q+1}^{p} f(\xi_k)\Delta x_k + f(c)(t_q - c),\ (FHK)\int_{[c,b]_{\mathbb{T}}} f(x)\Delta x\right) \\
&< \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

The intended result follows.

Corollary 1 *If $f \in \mathscr{FHK}_{[a,b]_{\mathbb{T}}}$, then $f \in \mathscr{FHK}_{[r,s]_{\mathbb{T}}}$ for any $[r,s]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$.*

Definition 2 (*See* [32]) Let $\mathbb{T}' \subset \mathbb{T}$. We say $\mathbb{T}'$ has delta measure zero if it has Lebesgue measure zero and contains no right-scattered points. A property $\mathscr{P}$ is said to hold Δ a.e. on $\mathbb{T}$ if there exists $\mathbb{T}'$ of measure zero such that $\mathscr{P}$ holds for every $t \in \mathbb{T} \setminus \mathbb{T}'$.

Theorem 5 *Let $f(x) = g(x)$ Δ a.e. on $[a,b]_{\mathbb{T}}$. If $f(x) \in \mathscr{FHK}_{[a,b]_{\mathbb{T}}}$, then so $g(x)$. Moreover,*

$$(FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x = (FHK)\int_{[a,b]_{\mathbb{T}}} g(x)\Delta x.$$

Proof Let $\varepsilon > 0$. Then there exists a $\delta \in \Gamma(\Delta, [a,b]_{\mathbb{T}})$ such that

$$\mathbf{D}\left(S(f, D, \delta), (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x\right) < \varepsilon$$

for any $D \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$, $D = \{([x_{i-1}, x_i]_{\mathbb{T}}, \xi_i)\}$. Set $E = \sum_{j=1}^{\infty} E_j$, where

$$E_j = \left\{x : j-1 < \mathbf{D}(f(x), g(x)) \le j,\ \ t \in [a,b]_{\mathbb{T}}\right\}_{j=1}^{\infty}.$$

For each j, there exists F_j consisting of a collection of open intervals with total length less than $\varepsilon \cdot 2^{-j} \cdot j^{-1}$, such that $E_j \subset F_j$. Define

$$\delta(\xi) = \begin{cases} (\delta_L^0(\xi), \delta_R^0(\xi)), & \text{if } \xi \in [a,b]_{\mathbb{T}} \backslash E, \\ (\delta_L^1(\xi), \delta_R^1(\xi)), & \text{if } \xi \in E_j \text{ satisfies } (\xi - \delta_L^1(\xi), \xi + \delta_R^1(\xi))_{\mathbb{T}} \subset F_j. \end{cases}$$

Then, for any $D \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$, $D = \{([x_{i-1}, x_i]_{\mathbb{T}}, \xi_i)\}$, one has

$$\begin{aligned} &\mathbf{D}\left(S(g, D, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x\right) \\ &= \mathbf{D}\left(\sum_{\xi_i \in [a,b]_{\mathbb{T}}} g(\xi_i)\Delta x_i,\ (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x\right) \\ &= \mathbf{D}\left(\sum_{\xi_i \in E} g(\xi_i)\Delta x_i + \sum_{\xi_i \in [a,b]_{\mathbb{T}} \backslash E} g(\xi_i)\Delta x_i,\ (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x\right) \\ &= \mathbf{D}\left(\sum_{\xi_i \in E} g(\xi_i)\Delta x_i + \sum_{\xi_i \in [a,b]_{\mathbb{T}} \backslash E} f(\xi_i)\Delta x_i + \sum_{\xi_i \in E} f(\xi_i)\Delta x_i,\right. \\ &\qquad \left.(FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x + \sum_{\xi_i \in E} f(\xi_i)\Delta x_i\right). \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathbf{D}\left(S(g, D, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x\right) \\ &\le \mathbf{D}\left(\sum_{\xi_i \in [a,b]_{\mathbb{T}}} f(\xi_i)\Delta x_i,\ (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x\right) \\ &\quad + \mathbf{D}\left(\sum_{\xi_i \in E} g(\xi_i)\Delta x_i,\ \sum_{\xi_i \in E} f(\xi_i)\Delta x_i\right) \\ &\le \varepsilon + \sum_{j=1}^{\infty} \sum_{\xi_i \in E_j} \mathbf{D}\left(f(\xi_i), g(\xi_i)\right)\Delta x_i \\ &\le 2\varepsilon. \end{aligned}$$

The proof is complete.

Theorem 6 (See [32]) *Let* $[a, b]_{\mathbb{T}}$ *be given. Assume*

(1) $\lim_{n\to\infty} f_n(x) = f(x)$ *holds* Δ *a.e.;*
(2) $G(x) \le f_n(x) \le H(x)$ *holds* Δ *a.e.;*
(3) $f_n(x),\ G(x),\ H(x) \in \mathscr{HK}_{[a,b]_{\mathbb{T}}}$.

Then $f(x) \in \mathscr{HK}_{[a,b]_{\mathbb{T}}}$. *Moreover,*

$$\lim_{n\to\infty} (HK) \int_{[a,b]_{\mathbb{T}}} f_n(x) \Delta x = (HK) \int_{[a,b]_{\mathbb{T}}} f(x) \Delta x.$$

Theorem 7 *Function* $f(x) \in \mathscr{FHK}_{[a,b]_{\mathbb{T}}}$ *if and only if* $\underline{f(x)^\alpha}, \overline{f(x)^\alpha} \in \mathscr{HK}_{[a,b]_{\mathbb{T}}}$ *for all* $\alpha \in [0, 1]$ *uniformly, i.e., the* Δ*-gauge in Definition 1 is independent of* α.

Proof (Necessity) Let $\tilde{A} = (FHK) \int_{[a,b]_{\mathbb{T}}} f(x) \Delta x$. Given $\varepsilon > 0$, there exists a

$$\delta \in \Gamma(\Delta, [a, b]_{\mathbb{T}})$$

such that $\mathbf{D}\left(S(f, D, \delta), \tilde{A}\right) < \varepsilon$ for any $D \in \mathfrak{D}(\delta, [a, b]_{\mathbb{T}})$. Then,

$$\begin{aligned} &\sup_{\alpha\in[0,1]} \max \left\{ \left| \underline{[S(f, D, \delta)]^\alpha} - \underline{\tilde{A}^\alpha} \right|, \left| \overline{[S(f, D, \delta)]^\alpha} - \overline{\tilde{A}^\alpha} \right| \right\} \\ &\quad = \sup_{\alpha\in[0,1]} \max \left\{ \left| S(\underline{f^\alpha}, D, \delta) - \underline{\tilde{A}^\alpha} \right|, \left| S(\overline{f^\alpha}, D, \delta) - \overline{\tilde{A}^\alpha} \right| \right\} \\ &\quad < \varepsilon \end{aligned}$$

and

$$\left| S(\underline{f^\alpha}, D, \delta) - \underline{\tilde{A}^\alpha} \right| < \varepsilon, \quad \left| S(\overline{f^\alpha}, D, \delta) - \overline{\tilde{A}^\alpha} \right| < \varepsilon$$

for any $\alpha \in [0, 1]$ and for any $D \in \mathfrak{D}(\delta, [a, b]_{\mathbb{T}})$. Thus, $\underline{f(x)^\alpha}, \overline{f(x)^\alpha} \in \mathscr{HK}_{[a,b]_{\mathbb{T}}}$ uniformly for any $\alpha \in [0, 1]$.

(Sufficiency) Let $\varepsilon > 0$. By assumption, there exists a $\delta \in \Gamma(\Delta, [a, b]_{\mathbb{T}})$ such that

$$\left| S(\underline{f^\alpha}, D, \delta) - \underline{\tilde{A}^\alpha} \right| < \varepsilon, \quad \left| S(\overline{f^\alpha}, D, \delta) - \overline{\tilde{A}^\alpha} \right| < \varepsilon$$

for any $D \in \mathfrak{D}(\delta, [a, b]_{\mathbb{T}})$ and for any $\alpha \in [0, 1]$, where

$$\underline{\tilde{A}^\alpha} = (FHK) \int_{[a,b]_{\mathbb{T}}} \underline{f^\alpha} \Delta x, \quad \overline{\tilde{A}^\alpha} = (FHK) \int_{[a,b]_{\mathbb{T}}} \overline{f^\alpha} \Delta x.$$

To prove that $\left\{\left[\underline{\tilde{A}^{\alpha}}, \overline{\tilde{A}^{\alpha}}\right], \alpha \in [0,1]\right\}$ represents a fuzzy number, it is enough to check that $\left[\underline{\tilde{A}^{\alpha}}, \overline{\tilde{A}^{\alpha}}\right]$ satisfies items (1)–(3) of Lemma 1:

(1) for $\alpha \in [0,1]$, if $\underline{f^{\alpha}} \le \overline{f^{\alpha}}$, then $\underline{\tilde{A}^{\alpha}} \le \overline{\tilde{A}^{\alpha}}$, i.e., the interval $\left[\underline{\tilde{A}^{\alpha}}, \overline{\tilde{A}^{\alpha}}\right]$ is closed.

(2) $\underline{f^{\alpha}}$ and $\overline{f^{\alpha}}$ are, respectively, nondecreasing and nonincreasing functions on $[0,1]$. For any $0 \le \alpha_1 \le \alpha_2 \le 1$ one has

$$\begin{aligned}(FHK)\int_{[a,b]_{\mathbb{T}}} \underline{f^{\alpha_1}}\Delta x &\le (FHK)\int_{[a,b]_{\mathbb{T}}} \underline{f^{\alpha_2}}\Delta x \\ &\le (FHK)\int_{[a,b]_{\mathbb{T}}} \overline{f^{\alpha_2}}\Delta x \\ &\le (FHK)\int_{[a,b]_{\mathbb{T}}} \overline{f^{\alpha_1}}\Delta x.\end{aligned}$$

This implies $\left[\underline{\tilde{A}^{\alpha_1}}, \overline{\tilde{A}^{\alpha_1}}\right] \supset \left[\underline{\tilde{A}^{\alpha_2}}, \overline{\tilde{A}^{\alpha_2}}\right]$.

(3) For any $\{\alpha_n\}$ satisfying $\alpha_n \le \alpha_{n+1}$ and $\alpha_n \to \alpha \in (0,1]$, we have

$$\bigcap_{n=1}^{\infty} [f]^{\alpha_n} = [f]^{\alpha},$$

that is,

$$\bigcap_{n=1}^{\infty}\left[\underline{f^{\alpha_n}}, \overline{f^{\alpha_n}}\right] = \left[\underline{f^{\alpha}}, \overline{f^{\alpha}}\right],$$

$\lim_{n\to\infty} \underline{f^{\alpha_n}} = \underline{f^{\alpha}}$ and $\lim_{n\to\infty} \overline{f^{\alpha_n}} = \overline{f^{\alpha}}$. Moreover,

$$\underline{f^{0}} \le \underline{f^{\alpha_n}} \le \underline{f^{1}}, \quad \overline{f^{1}} \le \overline{f^{\alpha_n}} \le \overline{f^{0}}.$$

Thanks to Theorem 6, we have $\underline{f^{\alpha}}, \overline{f^{\alpha}} \in \mathscr{HK}_{[a,b]_{\mathbb{T}}}$ and

$$\lim_{n\to\infty}(HK)\int_{[a,b]_{\mathbb{T}}} \underline{f^{\alpha_n}}\Delta x = (HK)\int_{[a,b]_{\mathbb{T}}} \underline{f^{\alpha}}\Delta x,$$

$$\lim_{n\to\infty}(HK)\int_{[a,b]_{\mathbb{T}}} \overline{f^{\alpha_n}}\Delta x = (HK)\int_{[a,b]_{\mathbb{T}}} \overline{f^{\alpha}}\Delta x.$$

Consequently,

$$\bigcap_{n=1}^{\infty}\left[\underline{\tilde{A}^{\alpha_n}}, \overline{\tilde{A}^{\alpha_n}}\right] = \left[\underline{\tilde{A}^{\alpha}}, \overline{\tilde{A}^{\alpha}}\right].$$

Define $\tilde{A}$ by $\left\{\left[\underline{\tilde{A}^\alpha}, \overline{\tilde{A}^\alpha}\right], \alpha \in [0, 1]\right\}$. Thus,

$$\mathbf{D}\left(S(f, D, \delta), \tilde{A}\right) < \varepsilon$$

for each $D \in \mathfrak{D}(\delta, [a, b]_{\mathbb{T}})$.

Definition 3 A sequence $\{f_n(x)\}$ of HK Δ-integrable functions is called uniformly FHK Δ-integrable on $[a, b]_{\mathbb{T}}$ if for each $\varepsilon > 0$ there exists a $\delta \in \Gamma(\Delta, [a, b]_{\mathbb{T}})$ such that

$$\mathbf{D}\left(S(f_n, D, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x\right) < \varepsilon$$

for any $D \in \mathfrak{D}(\delta, [a, b]_{\mathbb{T}})$ and for any n.

Theorem 8 *Let $f_n(x) \in \mathscr{FHK}_{[a,b]_{\mathbb{T}}}$, $n = 1, 2, \ldots$, satisfy:*

(1) $\lim_{n\to\infty} f_n(x) = f(x)$ *on* $[a, b]_{\mathbb{T}}$*;*
(2) $f_n(x)$ *are uniformly FHK Δ-integrable on* $[a, b]_{\mathbb{T}}$.

Then $f(x) \in \mathscr{FHK}_{[a,b]_{\mathbb{T}}}$ *and*

$$\lim_{n\to\infty}(FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x = (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x.$$

Proof Let $\varepsilon > 0$. By assumption, there exists a $\delta \in \Gamma(\Delta, [a, b]_{\mathbb{T}})$ such that

$$\mathbf{D}\left(S(f_n, D, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x\right) < \varepsilon$$

for any $D \in \mathfrak{D}(\delta, [a, b]_{\mathbb{T}})$ and for every n. Fix a $D_0 \in \mathfrak{D}(\delta, [a, b]_{\mathbb{T}})$. From (1) of Theorem 8, there exists N such that

$$\mathbf{D}\left(S(f_n, D_0, \delta),\ S(f_m, D_0, \delta)\right) < \varepsilon$$

for arbitrary $n, m > N$. Then,

$$\begin{aligned}
&\mathbf{D}\left((FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x,\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_m(x)\Delta x\right)\\
&\quad\le \mathbf{D}\left(S(f_n, D_0, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x\right) + \mathbf{D}\left(S(f_n, D_0, \delta),\ S(f_m, D_0, \delta)\right)\\
&\qquad + \mathbf{D}\left(S(f_m, D_0, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_m(x)\Delta x\right)\\
&\quad< 3\varepsilon
\end{aligned}$$

for any $n, m > N$ and, hence, $\left\{(FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x\right\}$ is a Cauchy sequence. Let

$$\lim_{n\to\infty}(FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x = \tilde{A}.$$

We now prove that

$$\tilde{A} = (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x.$$

Let $\varepsilon > 0$. By hypothesis, there exists a $\delta \in \Gamma(\Delta, [a,b]_{\mathbb{T}})$ such that

$$\mathbf{D}\left(S(f_n, D, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x\right) < \varepsilon$$

for any $D \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$ and for all n. Choose N that satisfies

$$\mathbf{D}\left((FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x,\ \tilde{A}\right) < \varepsilon$$

for all $n > N$. For the above D and N, there exists $N_0 > N$ satisfying

$$\mathbf{D}\left(S(f_{N_0}, D, \delta),\ S(f, D, \delta)\right) < \varepsilon.$$

Therefore,

$$\begin{aligned}
&\mathbf{D}\left(S(f, D, \delta),\ \tilde{A}\right)\\
&\quad\le \mathbf{D}\left(S(f, D, \delta),\ S(f_{N_0}, D, \delta)\right) + \mathbf{D}\left(S(f_{N_0}, D, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_{N_0}(x)\Delta x\right)\\
&\qquad + \mathbf{D}\left((FHK)\int_{[a,b]_{\mathbb{T}}} f_{N_0}(x)\Delta x,\ \tilde{A}\right)\\
&\quad< 3\varepsilon
\end{aligned}$$

and the result follows.

Definition 4 (*See* [10]) A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is called absolutely continuous on $[a,b]_{\mathbb{T}}$, if for each $\varepsilon > 0$ there exists $\gamma > 0$ such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \varepsilon$$

whenever $\bigcup_{i=1}^{n}[x_{i-1}, x_i]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$ and $\sum_{i=1}^{n} \Delta x_i < \gamma$.

Theorem 9 (Dominated convergence theorem) *Let the time scale interval* $[a,b]_{\mathbb{T}}$ *be given. If* $f_n(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$, $n = 1, 2, \ldots$, *satisfy*

(1) $\lim_{n\to\infty} f_n(x) = f(x)$ Δ *a.e.;*
(2) $G(x) \le f_n(t) \le H(x)$ Δ *a.e. and* $G(x),\ H(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$;

then sequence $\{f_n(x)\}$ *is uniformly FHK* Δ*-integrable. Thus,* $f(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$ *and*

$$\lim_{n\to\infty}(FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x = (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x.$$

Proof By hypothesis, one has

$$\begin{aligned}\mathbf{D}\left(f_p(x), f_q(x)\right) &= \sup_{\alpha\in[0,1]} \max\left\{|\underline{f_p(x)^\alpha} - \underline{f_q(x)^\alpha}|, |\overline{f_p(x)^\alpha} - \overline{f_q(x)^\alpha}|\right\}\\ &\le \sup_{\alpha\in[0,1]} \max\left\{|\underline{H(x)^\alpha} - \underline{G(x)^\alpha}|, |\overline{H(x)^\alpha} - \overline{G(x)^\alpha}|\right\}\\ &= \mathbf{D}\left(H(x), G(x)\right).\end{aligned}$$

Then, $\mathbf{D}\left(H(x), G(x)\right)$ is Lebesgue Δ-integrable. Let

$$\mathbf{D}(x) = \int_{[a,x]_{\mathbb{T}}} D\left(H(s), G(s)\right)\Delta s.$$

From [10], $\mathbf{D}(x)$ is absolutely continuous on $[a,b]_{\mathbb{T}}$. Let $\varepsilon > 0$. Then there exists $\gamma > 0$ such that

$$\sum_{i=1}^{n} |\mathbf{D}(x_i) - \mathbf{D}(x_{i-1})| < \frac{\varepsilon}{b-a}$$

whenever $\bigcup_{i=1}^n [x_{i-1}, x_i]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$ and $\sum_{i=1}^n \Delta x_i < \gamma$. The limit $\lim_{n\to\infty} f_n(x) = f(x)$ holds Δ a.e. on $[a,b]_{\mathbb{T}}$ and $\{\mathbf{D}(f_n(x), f(x))\}$ is a sequence of Δ-measurable functions. Thanks to Egorov's theorem, there exists an open set Ω with $m(\Omega) < \delta$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly for $x \in [a,b]_{\mathbb{T}}\backslash\Omega$. Thus, there exists N such that $\mathbf{D}(f_p(x), f_q(x)) < \frac{\varepsilon}{b-a}$ for any $p, q > N$ and for any $x \in [a,b]_{\mathbb{T}}\backslash\Omega$. Suppose that $\delta_1 \in \Gamma(\Delta, [a,b]_{\mathbb{T}})$ such that

$$\left|S(\mathbf{D}\left(H(x), G(x)\right), D, \delta_1) - \int_{[a,b]_{\mathbb{T}}} \mathbf{D}\left(H(x), G(x)\right)\Delta x\right| < \varepsilon$$

and

$$\mathbf{D}\left(S(f_n, D, \delta_1),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x\right) < \varepsilon$$

for $1 \le n \le N$ and for any $D \in \mathfrak{D}(\delta_1, [a,b]_{\mathbb{T}})$. Define $\delta \in \Gamma(\Delta, [a,b]_{\mathbb{T}})$ by

$$\delta(\xi) = \begin{cases} \delta_1(\xi) & \text{if } \xi \in [a,b]_{\mathbb{T}} \backslash \Omega, \\ \min\{\delta_1(\xi), \rho(\xi, \Omega)\}, & \text{if } \xi \in \Omega, \end{cases}$$

where $\rho(\xi, \Omega) = \inf\{|\xi - \xi'| : \xi' \in \Omega\}$. Fix $n > N$. One has

$$\mathbf{D}\left(S(f_n, D, \delta), S(f_N, D, \delta)\right) = \mathbf{D}\left(\sum_{\xi_i \in [a,b]_{\mathbb{T}}} f_n(\xi_i)\Delta x_i, \sum_{\xi_i \in [a,b]_{\mathbb{T}}} f_N(\xi_i)\Delta x_i\right)$$

$$\le \mathbf{D}\left(\sum_{\xi_i \in [a,b]_{\mathbb{T}} \backslash \Omega} f_n(\xi_i)\Delta x_i, \sum_{\xi_i \in [a,b]_{\mathbb{T}} \backslash \Omega} f_N(\xi_i)\Delta x_i\right)$$
$$+ \mathbf{D}\left(\sum_{\xi_i \in \Omega} f_n(\xi_i)\Delta x_i, \sum_{\xi_i \in \Omega} f_N(\xi_i)\Delta x_i\right)$$
$$\le \varepsilon + \sum_{\xi_i \in \Omega} \mathbf{D}(f_n(\xi_i), f_N(\xi_i))\Delta x_i$$
$$\le \varepsilon + \left|\sum_{\xi_i \in \Omega} \mathbf{D}(H(\xi_i), G(\xi_i))\Delta x_i - \int_\Omega \mathbf{D}(H(x), G(x))\Delta x\right| + \left|\int_\Omega \mathbf{D}(H(x), G(x))\Delta x\right|$$
$$\le 3\varepsilon$$

for any $D \in \mathfrak{D}(\delta, [a,b]_{\mathbb{T}})$. Hence,

$$\mathbf{D}\left(S(f_n, D, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x\right)$$
$$\le \mathbf{D}\left(S(f_n, D, \delta),\ S(f_N, D, \delta)\right) + \mathbf{D}\left(S(f_N, D, \delta),\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_N(x)\Delta x\right)$$
$$+ \mathbf{D}\left((FHK)\int_{[a,b]_{\mathbb{T}}} f_N(x)\Delta x,\ (FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x\right)$$
$$\le 5\varepsilon.$$

Our dominated convergence theorem is proved.

As a consequence of Theorem 9, we get the following monotone convergence theorem.

Theorem 10 (Monotone convergence theorem) *Let the time scale interval* $[a,b]_{\mathbb{T}}$ *be given. If* $f_n(x) \in \mathscr{FHK}_{[a,b]_{\mathbb{T}}}$, $n = 1, 2, \ldots$, *satisfy*

(1) $\lim_{n\to\infty} f_n(x) = f(x)$ Δ *a.e.;*
(2) $\{f_n(x)\}$ *is a monotone sequence and* $f_n(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$*;*

then $\{f_n(x)\}$ *is uniformly FHK* Δ*-integrable. Consequently,* $f(x) \in \mathscr{F}\mathscr{H}\mathscr{K}_{[a,b]_{\mathbb{T}}}$. *Moreover,*

$$\lim_{n\to\infty}(FHK)\int_{[a,b]_{\mathbb{T}}} f_n(x)\Delta x = (FHK)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta x.$$

4 Conclusion

We investigated the fuzzy Henstock–Kurzweil (FHK) delta integral on time scales. Our results give a common generalization of the classical FHK and HK integrals. For future researches, we will investigate the characterization of FHK delta integrable functions. Another interesting line of research consists to study the concept of fuzzy Henstock–Stieltjes integral on time scales.

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Oscillation of Sublinear Second Order Neutral Differential Equations via Riccati Transformation

Arun Kumar Tripathy and Abhay Kumar Sethi

Abstract In this work, we establish the necessary conditions for oscillation of the second order neutral delay differential equations of the form:

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\gamma})' + q(t)x^{\gamma}(\sigma(t)) + v(t)x^{\gamma}(\eta(t)) = 0$$

under the assumptions

$$\int_0^{\infty} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} dt = \infty$$

and

$$\int_0^{\infty} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} dt < \infty$$

for various ranges of $p(t)$, where $0 < \gamma \leq 1$ is a quotient of odd positive integers.

Keywords Oscillation · Nonoscillation · Neutral · Delay · Nonlinear

Mathematics Subject Classification(2010) 34 K · 34C10

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1 Introduction

Consider a class of nonlinear neutral delay differential equations of the form:

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\gamma})'+q(t)x^{\gamma}(\sigma(t))+v(t)x^{\gamma}(\eta(t))=0, \qquad (1)$$

where $0<\gamma\leq 1, r,\ q,\ v,\ \tau,\ \sigma, \eta\in C(\mathbf{R}_{+},\mathbf{R}_{+}),\ \ p\in C(\mathbf{R}_{+},\mathbf{R}), \tau(t)\leq t,\ \ \sigma(t)\leq t,\ \ \eta(t)\leq t$ with $\lim_{t\to\infty}\tau(t)=\infty=\lim\limits_{t\to\infty}\sigma(t)=\infty=\lim_{t\to\infty}\eta(t)$. The objective of our work is to establish the oscillation character of all solutions of (1) under the assumptions

(A_0) $$\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}dt=\infty$$

and

(A_{00}) $$\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}dt<\infty$$

for various range of $p(t)$ with $|p(t)|<\infty$ and γ is a quotient of odd positive integers. In [14, 15], Tripathy and Sethi have discussed the necessary and sufficient condition for oscillation of all solutions of

$$(r(t)((x(t)+p(t)x(\tau(t)))'))'+q(t)G(x(\sigma(t)))+v(t)H(x(\eta(t)))=0, \qquad (2)$$

where $G, H\in C(\mathbf{R}_{+},\mathbf{R})$. If $G(x)=x^{\gamma}=H(x)$, then (2) is a special case of (1). Hence it is interesting to study (1). Unlike the methods of [15], in this work we study (1) by means of Riccati transformation technique. Dzurina [7] has presented some sufficient conditions for oscillation of the second order differential equations with mixed arguments of the form:

$$\left(\frac{1}{r(t)}x'(t)\right)'+q(t)x(\sigma(t))+v(t)x(\eta(t))=0.$$

In an another work, Baculikova and Dzurina [4] have studied the oscillatory behaviour of solutions of

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\gamma})'+q(t)x^{\beta}(\sigma(t))=0,$$

where γ,β are the ratio of odd positive integers and $0\leq p(t)\leq p_0<\infty$, and they have established the sufficient conditions for oscillation by means comparison results. We find the same technique when Baculikova et al. [5] have studied the oscillation properties of the second order neutral differential equations of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))'))'+q(t)x(\sigma(t))+v(t)x(\eta(t))=0.$$

But, our work illustrates sufficient conditions for oscillation of all solutions of (1) without comparison results In recent years, there is constant interest in obtaining new sufficient conditions for the oscillation or nonoscillation of the solutions of second order neutral functional differential equations(see for e.g. [8, 9, 11, 12, 17]) which is a part of so called *Dynamical Systems*. As nonneutral differential equations are arising in various problems of physics, biology and economics, and these equations are special cases of neutral differential equations, a special attention has been given to the study of second order neutral differential equations.

In spite of the above fact, we find numerous applications of neutral equations in electric networks, where they are frequently used for the study of distributed networks containing lossless transmission lines which arise in high speed computers (see for e.g. [10]). With the preceding lines of motivation, an effort is made here to study (1) by applying Riccati transformation technique. In this direction, we refer the reader to some of the works ([1–3, 6, 13, 16]) and the references cited therein.

Definition 1 By a solution of (1), we mean a continuously differentiable function $x(t)$ which is defined for $t \geq T^* = \min\{\tau(t_0),\ \sigma(t_0),\ \eta(t_0)\}$ such that $x(t)$ satisfies (1) for all $t \geq t_0$. In the sequel, it will always be assumed that the solutions of (1) exist on some half line $[t_1, \infty),\ \ t_1 \geq t_0$. A solution of (1) is said to be oscillatory, if it has arbitrarily large zeros; otherwise, it is called non-oscillatory. Equation (1) is called oscillatory, if all its solutions are oscillatory.

2 Oscillation Results with (A_0)

This section deals with the sufficient conditions for oscillation of all solutions of (1) under the assumption (A_0). Throughout our discussion, we use the following notation

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{1}$$

Lemma 1 ([6]) *Assume that* (A_0) *holds and* $r(t) \in C'[(t_0, \infty), \mathbf{R})$ *such that* $r'(t) > 0$. *Let* $x(t)$ *be an eventually positive solution of (1) such that* $(r(t)(x'(t))^\gamma)' \leq 0$ *for* $t \geq t_0$. *Then* $x'(t) > 0$ *and* $x''(t) < 0$ *for* $t \geq t_1 > t_0$, *where* $0 < \gamma \leq 1$ *is a quotient of odd positive integers.*

Lemma 2 ([9]) *Assume that the assumptions of Lemma 1 hold. Then there exists a* $t^* \in [t_0, \infty)$ *sufficiently large so that*
$(i)\ x(t) > tx'(t)\ for\ t \in [t^*, \infty)$
$(ii) \frac{x(t)}{t}\ is\ strictly\ decreasing\ on\ [t^*, \infty)$,
where $0 < \gamma \leq 1$ *is a quotient of odd positive integers.*

Lemma 3 ([4]) *Assume that* $A \geq 0$, $B \geq 0$ *and* $\lambda \geq 0$. *Then*

$$(A + B)^\lambda \leq 2^{\lambda-1}(A^\lambda + B^\lambda). \tag{2}$$

If $0 < \gamma \leq 1$, *then*

$$(A + B)^\lambda \leq (A^\lambda + B^\lambda). \tag{3}$$

Theorem 1 *Let* $0 \leq p(t) \leq a < \infty,\ \tau(\sigma(t)) = \sigma(\tau(t))$ *and* $\tau(\eta(t)) = \eta(\tau(t))$ *be hold for* $t \in [t_0, \infty)$. *Assume that* (A_0) *holds, and* $r'(t) > 0,\ \tau'(t) \geq 1$ *for large* t. *Furthermore, assume that there exists a positive differentiable function* $\delta(t)$ *such that*
(A_1) $\int_{t_0}^{\infty}\left[\delta(s)Q(s)\left(\frac{\sigma(s)}{s}\right)^{\gamma} + \delta(s)V(s)\left(\frac{\eta(s)}{s}\right)^{\gamma} - \frac{(1+a^{\gamma})r(s)((\delta'(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]ds = \infty,$
where $Q(t) = min\{q(t), q(\tau(t))\}, V(t) = min\{v(t), v(\tau(t))\}$ *and* $(\delta'(t))_+ = \max\{\delta'(t), 0\}$. *Then every solution of (1) oscillates.*

Proof Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0$. Hence, there exists $t_1 > t_0$ such that $x(t) > 0,\ x(\tau(t)) > 0,\ x(\sigma(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. Using (1), (1) becomes

$$(r(t)(z'(t))^{\gamma})' = -q(t)x^{\gamma}(\sigma(t)) - v(t)x^{\gamma}(\eta(t)) \leq 0,\ \not\equiv 0\ for\ t \geq t_1. \tag{4}$$

So, $r(t)(z'(t))^{\gamma}$ is nonincreasing on $[t_1, \infty)$, that is, either $z'(t) > 0$ or $z'(t) < 0$ for $t \geq t_2 > t_1$. By Lemma 1, it follows that $z'(t) > 0$ for $t \geq t_2$. From (1), it is easy to see that

$$\begin{gathered}(r(t)(z'(t))^{\gamma})' + q(t)x^{\gamma}(\sigma(t)) + v(t)x^{\gamma}(\eta(t)) + \\ a^{\gamma}(r(\tau(t))(z'(\tau(t))^{\gamma})' + a^{\gamma}q(\tau(t))x^{\gamma}(\sigma(\tau(t)) + a^{\gamma}v(\tau(t))x^{\gamma}(\eta(\tau(t)) = 0\end{gathered}$$

for $t \geq t_2$. By using Lemma 3, the above relation yields that

$$(r(t)(z'(t))^{\gamma})' + a^{\gamma}(r(\tau(t))(z'(\tau(t))^{\gamma})' + Q(t)z^{\gamma}(\sigma(t)) + V(t)z^{\gamma}(\eta(t)) \leq 0 \tag{5}$$

and therefore,

$$\frac{(r(t)(z'(t))^{\gamma})'}{z^{\gamma}(t)} + \frac{a^{\gamma}(r(\tau(t))(z'(\tau(t))^{\gamma})'}{z^{\gamma}(t)} + \frac{Q(t)z^{\gamma}(\sigma(t))}{z^{\gamma}(t)} + \frac{V(t)z^{\gamma}(\eta(t))}{z^{\gamma}(t)} \leq 0. \tag{6}$$

Let $\delta(t)$ be the positive differentiable function and consider the general Riccati substitution

$$w(t) = \delta(t)r(t)\left(\frac{z'(t)}{z(t)}\right)^{\gamma} \tag{7}$$

and

$$u(t) = \delta(t)r(\tau(t))\left(\frac{z'(\tau(t))}{z(\tau(t))}\right)^{\gamma}. \tag{8}$$

Due to Lemma 1, $w(t) > 0$ and $u(t) > 0$ on $[t_2, \infty)$. Now,

$$w'(t) = \delta'(t)r(t)\left(\frac{z'(t)}{z(t)}\right)^{\gamma} + \delta(t)\left(r(t)\left(\frac{z'(t)}{z(t)}\right)^{\gamma}\right)' \tag{9}$$

and

$$u'(t) = \delta'(t)r(\tau(t))\left(\frac{z'(\tau(t))}{z(\tau(t))}\right)^{\gamma} + \delta(t)\left(r(\tau(t))\left(\frac{z'(\tau(t))}{z(\tau(t))}\right)^{\gamma}\right)'. \quad (10)$$

Using (7) and (8) in (9) and (10), we get

$$w'(t) + a^{\gamma}u'(t) = \frac{\delta'(t)}{\delta(t)}\left[w(t) + a^{\gamma}u(t)\right]$$
$$+\delta(t)\left[\left(r(t)\left(\frac{z'(t)}{z(t)}\right)^{\gamma}\right)' + a^{\gamma}\left(r(\tau(t))\left(\frac{z'(\tau(t))}{z(\tau(t))}\right)^{\gamma}\right)'\right]. \quad (11)$$

Upon using the fact

$$\left(r(t)\left(\frac{z'(t)}{z(t)}\right)^{\gamma}\right)' = \frac{(r(t)(z'(t))^{\gamma})'}{z^{\gamma}(t)} - \frac{r(t)(z'(t))^{\gamma}(z^{\gamma}(t))'}{(z^{\gamma}(t))^2}$$

and

$$\left(r(\tau(t))\left(\frac{z'(\tau(t))}{z(\tau(t))}\right)^{\gamma}\right)' = \frac{(r(\tau(t))(z'(\tau(t))^{\gamma})'}{z^{\gamma}(\tau(t))} - \frac{r(\tau(t))(z'(\tau(t))^{\gamma}(z^{\gamma}(\tau(t))'}{(z^{\gamma}(\tau(t)))^2}$$
$$\leq \frac{(r(\tau(t))(z'(\tau(t))^{\gamma})'}{z^{\gamma}(t)} - \frac{r(\tau(t))(z'(\tau(t))^{\gamma}(z^{\gamma}(\tau(t))'}{(z^{\gamma}(\tau(t)))^2},$$

where $\tau(t) \leq t$ and $z(t)$ is nondecreasing on $[t_2, \infty)$ in (11) and then applying (6), we obtain

$$w'(t) + a^{\gamma}u'(t) \leq \frac{\delta'(t)}{\delta(t)}[w(t) + a^{\gamma}u(t)] - Q(t)\delta(t)\frac{z^{\gamma}(\sigma(t))}{(z^{\gamma}(t)} - V(t)\delta(t)\frac{z^{\gamma}(\eta(t))}{(z^{\gamma}(t)}$$
$$-\delta(t)\left[\frac{r(t)(z'(t))^{\gamma}(z^{\gamma}(t))'}{(z^{\gamma}(t))^2} + a^{\gamma}\frac{r(\tau(t))(z'(\tau(t))^{\gamma}(z^{\gamma}(\tau(t))'}{(z^{\gamma}(\tau(t))^2}\right]. \quad (12)$$

By Lemma 2, let there exist $t_3 > t_2$ such that $\frac{z(t)}{t}$ is decreasing on $[t_3, \infty)$ and hence

$$\frac{z(\sigma(t))}{z(t)} \geq \frac{\sigma(t)}{t}, \ t \in [t_3, \infty). \quad (13)$$

Using (13) in (12), we obtain

$$\begin{aligned} w'(t) + a^{\gamma}(t) &\leq \frac{\delta'(t)}{\delta(t)}[w(t) + a^{\gamma}u(t)] - Q(t)\delta(t)\left(\frac{\sigma(t)}{t}\right)^{\gamma} - V(t)\delta(t)\left(\frac{\eta(t)}{t}\right)^{\gamma} \\ &\quad -\delta(t)\left[\frac{r(t)(z'(t))^{\gamma}\gamma z'(t)(z(t))^{\gamma}}{z^{\gamma}(t)z^{\gamma}(t)z(t)} + a^{\gamma}\frac{r(\tau(t))(z'(\tau(t))^{\gamma}\gamma z'(\tau(t))(z(\tau(t))^{\gamma}}{z^{\gamma}(\tau(t))z^{\gamma}(\tau(t))z(\tau(t))}\right] \\ &= \frac{\delta'(t)}{\delta(t)}[w(t) + a^{\gamma}u(t)] - Q(t)\delta(t)\left(\frac{\sigma(t)}{t}\right)^{\gamma} - V(t)\delta(t)\left(\frac{\eta(t)}{t}\right)^{\gamma} \\ &\quad -\gamma\delta(t)\left[\frac{r(t)(z'(t))^{\gamma}z'(t)}{z^{\gamma}(t)z(t)} + a^{\gamma}\frac{r(\tau(t))(z'(\tau(t))^{\gamma}z'(\tau(t))\tau'}{z^{\gamma}(\tau(t))z(\tau(t))}\right] \end{aligned}$$

$$= \frac{\delta'(t)}{\delta(t)}[w(t) + a^\gamma u(t)] - Q(t)\delta(t)\left(\frac{\sigma(t)}{t}\right)^\gamma - V(t)\delta(t)\left(\frac{\eta(t)}{t}\right)^\gamma$$
$$- \gamma\delta(t)\left[\frac{r^{(1+\frac{1}{\gamma})}}{r^{\frac{1}{\gamma}}}\left(\frac{z'(t)}{z(t)}\right)^{\gamma+1} + a^\gamma \frac{r^{(1+\frac{1}{\gamma})}(\tau(t))}{r^{\frac{1}{\gamma}}(\tau(t))}\left(\frac{z'(\tau(t))}{z(\tau(t))}\right)^{\gamma+1}\right]$$
$$\le \frac{(\delta'(t)_+)}{\delta(t)}[w(t) + a^\gamma u(t)] - Q(t)\delta(t)\left(\frac{\sigma(t)}{t}\right)^\gamma - V(t)\delta(t)\left(\frac{\eta(t)}{t}\right)^\gamma$$
$$- \gamma\delta(t)\left[\left(\frac{w(t)}{\delta(t)}\right)^\alpha \frac{1}{r^{\frac{1}{\gamma}}} + \frac{a^\gamma}{r^{\frac{1}{\gamma}}(\tau(t))}\left(\frac{u(t)}{\delta(t)}\right)^\alpha\right], \tag{14}$$

where $\alpha = \frac{(\gamma+1)}{\gamma}$. Let's define

$$A^\alpha = \frac{\gamma\delta}{r^{\frac{1}{\gamma}}}\left(\frac{w}{\delta}\right)^\alpha > 0,\ B^{\alpha-1} = \frac{r^{\frac{1}{(\gamma+1)}}}{\alpha(\gamma\delta)^{\frac{1}{\alpha}}}((\delta')_+) \ge 0.$$

Then using the inequality

$$\alpha A B^{\alpha-1} - A^\alpha \le (\alpha - 1)B^\alpha$$

we obtain that

$$\left(\frac{w(t)}{\delta(t)}\right)((\delta'(t))_+) - \frac{\gamma\delta(t)}{r^{\frac{1}{\gamma}}(t)}\left(\frac{w(t)}{\delta(t)}\right)^\alpha$$
$$= \alpha A B^{\alpha-1} - A^\alpha \le (\alpha-1)B^\alpha \le \frac{r(t)((\delta'(t))_+)^{\gamma+1}}{\delta^\gamma(t)(\gamma+1)^{\gamma+1}}, \tag{15}$$

and similarly

$$\left(\frac{u(t)}{\delta(t)}\right)((\delta'(t))_+) - \frac{\gamma\delta(t)}{r^{\frac{1}{\gamma}}(\tau(t))}\left(\frac{u(t)}{\delta(t)}\right)^\alpha \le \frac{r(\tau(t))((\delta'(t))_+)^{\gamma+1}}{\delta^\gamma(t)(\gamma+1)^{\gamma+1}}. \tag{16}$$

Using (15) and (16) in (14), we find

$$w'(t) + a^\gamma u'(t) \le \frac{r(t)((\delta'(t))_+)^{\gamma+1}}{\delta^\gamma(t)(\gamma+1)^{\gamma+1}}$$
$$+ a^\gamma \frac{r(\tau(t))((\delta'(t))_+)^{\gamma+1}}{\delta^\gamma(t)(\gamma+1)^{\gamma+1}} - \delta(t)Q(t)\left(\frac{\sigma(t)}{t}\right)^\gamma - \delta(t)V(t)\left(\frac{\eta(t)}{t}\right)^\gamma. \tag{17}$$

Integrating (17) from $t_4(> t_3)$ to t, we get

$$-w(t_4) - a^\gamma u(t_4) \le w(t) + a^\gamma u(t) - w(t_4) - a^\gamma u(t_4)$$
$$\le \int_{t_4}^t \left[\frac{(1+a^\gamma)r(s)((\delta'(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)} - \delta(s)\left\{Q(s)\left(\frac{\sigma(s)}{s}\right)^\gamma + V(s)\left(\frac{\eta(s)}{s}\right)^\gamma\right\}\right] ds$$

which is a contradiction to (A_2). This completes the proof of the theorem.

Theorem 2 *Let* $0 \le p(t) \le a < 1,\ t \in [t_0, \infty)$. *Assume that* (A_0) *holds and* $r'(t) > 0$. *Furthermore, assume that there exists a positive differentiable function* $\delta(t)$ *such that*

(A_2) $\int_{t_0}^{\infty} \left[\delta(s)q(s)\left(\frac{\sigma(s)}{s}\right)^{\gamma} + \delta(s)v(s)\left(\frac{\eta(s)}{s}\right)^{\gamma} - \frac{r(s)((\delta'(s))_+)^{\gamma+1}}{(1-a)(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} ds\right] = \infty.$

Then every solution of (1) oscillates.

Proof Proceeding as in the proof of Theorem 1, we get (4) and by Lemma 1 $z(t)$ is nondecreasing on $[t_2, \infty)$. Hence there exists $t_3 > t_2$ such that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &= x(t) + p(t)x(\tau(t)) - p(t)x(\tau(t)) \\ &\quad - p(t)p(\tau(t))p(\tau(\tau(t))) \\ &= x(t) - p(t)p(\tau(t))p(\tau(\tau(t))) \\ &\le x(t), \end{aligned}$$

that is, $x(t) \ge (1-a)z(t)$ on $[t_3, \infty)$. Consequently, (1) reduces to

$$(r(t)(z'(t))^{\gamma})' + (1-a)q(t)z^{\gamma}(\sigma(t)) + (1-a)v(t)z^{\gamma}(\eta(t)) \le 0$$

$t \in [t_3, \infty)$.

The rest of the proof follows from the proof of Theorem 1 without Riccati substitution (8) and hence the details are omitted. The proof of the theorem is complete.

Theorem 3 *Let* $-1 < a \le p(t) \le 0$ *for* $t \in [t_0, \infty)$. *Assume that* (A_0) *holds and* $r'(t) > 0$. *Furthermore, assume that there exists a positive differentiable function* $\delta(t)$ *such that*

(A_3) $\int_{t_0}^{\infty} \left[\delta(s)q(s)\left(\frac{\sigma(s)}{s}\right)^{\gamma} + \delta(s)v(s)\left(\frac{\eta(s)}{s}\right)^{\gamma} - \frac{r(s)((\delta'(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} ds\right] = \infty$

and

(A_4) $\int_{t_0}^{\infty} \left[\frac{1}{r(\theta)} \int_{t_0}^{\theta} [q(s) + v(s)]ds\right]^{\frac{1}{\gamma}} d\theta = \infty$

hold. Then every solution of (1) either oscillates or converges to zero as $t \to \infty$.

Proof Proceeding as in the proof of Theorem 1 we get (4) for $t \in [t_2, \infty)$. Thus $z(t)$ and $z'(t)$ are monotonic functions on $[t_2, \infty)$. In what follows, we consider the following four possible cases:

$$\begin{aligned} &(i)\ \ z(t) > 0,\ z'(t) > 0,\ (ii)\ \ z(t) < 0,\ z'(t) > 0, \\ &(iii)\ \ z(t) > 0,\ z'(t) < 0,\ (iv)\ \ z(t) < 0,\ z'(t) < 0. \end{aligned}$$

Case(i) In this case, $z(t) \le x(t)$ and $\lim_{t\to\infty} r(t)z'(t)$ exists. Therefore, (1) reduces to

$$\frac{(r(t)(z'(t))^{\gamma})'}{z^{\gamma}(t)} + \frac{q(t)z^{\gamma}(\sigma(t))}{z^{\gamma}(t)} + \frac{v(t)z^{\gamma}(\eta(t))}{z^{\gamma}(t)} \le 0 \qquad (18)$$

for $t \geq t_3 > t_2$. Upon using the positive differentiable function $\delta(t)$, we consider the general Riccati substitution (7) and hence

$$\begin{aligned} w'(t) &= \frac{\delta'(t)}{\delta(t)}[w(t)] + \delta(t)\left(r(t)\left(\frac{z'(t)}{z(t)}\right)^{\gamma}\right)' \\ &\leq \frac{\delta'(t)}{\delta(t)}[w(t)] + \delta(t)\left[\frac{(r(t)(z'(t))^{\gamma})'}{z^{\gamma}(t)} - \frac{(r(t)(z'(t))^{\gamma})(z^{\gamma}(t))'}{z^{\gamma}(t)}\right]. \end{aligned} \tag{19}$$

Due to (18), (19) becomes

$$w'(t) \leq \frac{\delta'(t)}{\delta(t)}[w(t)] - \delta(t)\left[\frac{q(t)z^{\gamma}(\sigma(t))}{z^{\gamma}(t)} + \frac{v(t)z^{\gamma}(\eta(t))}{z^{\gamma}(t)} + \frac{r(t)(z'(t))^{\gamma}(z^{\gamma}(t))'}{(z^{\gamma}(t))^2}\right].$$

Using the same type of argument as in the proof of Theorem 1, the last inequality yields

$$w'(t) \leq \frac{\delta'(t)}{\delta(t)}[w(t)] - \gamma\delta(t)\left[q(t)\left(\frac{\sigma(t)}{t}\right)^{\gamma} + v(t)\left(\frac{\eta(t)}{t}\right)^{\gamma} + \left(\frac{w(t)}{\delta(t)}\right)^{\alpha}\frac{1}{r^{\frac{1}{\gamma}}}\right],$$

where $\alpha = \frac{(\gamma+1)}{\gamma}$. The rest of this case is similar to Theorem 1.

Case(ii) Let $\lim_{t\to\infty} z(t) = \beta, \beta \in (-\infty, 0]$. We claim that $x(t)$ is bounded. If not, there exists a sequence $\{\alpha_n\}$ such that $\alpha_n \to \infty$ as $n \to \infty$ and $x(\alpha_n) = max\{x(t) : t_3 \leq t \leq \alpha_n\}$. Therefore,

$$\begin{aligned} z(\alpha_n) &= x(\alpha_n) + p(\alpha_n)x(\tau(\alpha_n)) \\ &\geq x(\alpha_n) + ax(\tau(\alpha_n)) \\ &\geq x(\alpha_n) + ax(\alpha_n) \\ &= (1+a)x(\alpha_n) \ (because 1 + a > 0) \\ &\to +\infty \ \ as\ n \to \infty \end{aligned}$$

gives a contradiction. Hence,

$$\begin{aligned} 0 \geq \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty} z(t) \\ &\geq \limsup_{t\to\infty}(x(t) + ax(\tau(t))) \\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}(ax(\tau(t))) \\ &= \limsup_{t\to\infty} x(t) + a\limsup_{t\to\infty} x(\tau(t)) \\ &= (1+a)\limsup_{t\to\infty} x(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$.

Case(iii) Proceeding as in $Case(ii)$, we may show that $x(t)$ is bounded. Let $\lim_{t\to\infty} z(t) = \beta$, $\beta \in [0, \infty)$. We assert that $\beta = 0$. If not, there exist $t_3 > t_2$ and $l > 0$ such that $z(\sigma(t)) \geq z(t) > l$ and $z(\eta(t)) \geq z(t) > l$ for $t \geq t_3$. From (1) it follows that $z(t) \leq x(t)$ and hence (4) yields

$$(r(t)(z'(t))^\gamma)' \leq -l^\gamma \left[q(t) + v(t)\right], t \geq t_3.$$

Integrating the above inequality from t_3 to t, we get

$$z'(t) < -l \left[\frac{1}{r(t)} \int_{t_3}^{t} [q(s) + v(s)]ds\right]^{\frac{1}{\gamma}},$$

that is,

$$z(t) < z(t_3) - l \int_{t_3}^{t} \left[\frac{1}{r(\theta)} \int_{t_3}^{\theta} [q(s) + v(s)]ds\right]^{\frac{1}{\gamma}} d\theta < 0,$$

for large t due to (A_4). Hence $l = 0$. Using the same type of reasoning as in $Case(ii)$ we can show that $\lim_{t\to\infty} x(t) = 0$.

Case(iv) We have $r(t)(z'(t))^\gamma$ is nonincreasing and $z(t) < 0$ for $t \geq t_2$. If $x(t)$ is unbounded, then by $Case(ii)$ it follows that $z(t) > 0$ for large t which is absurd. Hence, $x(t)$ is bounded. Consequently, $z(t)$ is bounded and $\lim_{t\to\infty} z(t)$ exists. Since $z(t) < 0$ and nonincreasing, then we can find $\beta > 0$ and a $t_3 > t_2$ such that $z(t) < \beta$ for $t \geq t_3$. Proceeding as in $Case(iii)$, we obtain the fact that $\lim_{t\to\infty} z(t) = -\infty$ due to (A_4). This contradiction argues against the $Case(iv)$. This completes the proof of the theorem.

Remark 1 In Theorem 3, it is learnt that $x(t)$ is bounded when $z(t) < 0$. Also, $x(t)$ is bounded when $z(t) > 0$ in $Case(iii)$. Hence for unbounded $x(t)$, $Cases(ii)$, (iii) and (iv) are not existing ultimately. Therefore, we have proved the following result:

Theorem 4 *Let $-1 < a \leq p(t) \leq 0$ for $t \in [t_0, \infty)$. Assume that $r'(t) > 0$, (A_0) and (A_3) hold. Then every unbounded solution of (1) oscillates.*

Theorem 5 *Let $-\infty < a \leq p(t) \leq d < -1$, $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ be hold for all $t \in [t_0, \infty)$. Assume that all conditions of Theorem 3 hold. If*
(A_5) $\int_{t_0}^{\infty} [q(\tau(s)) + v(\tau(s))]\, ds = \infty$,
then every bounded solution of (1) either oscillates or converges to zero as $t \to \infty$.

Proof Let $x(t)$ be a bounded nonoscillatory solution of (1). Then proceeding as in the proof of Theorem 3, we have four possible cases for $t \in [t_2, \infty)$. Among these cases, $Cases(i)$, (iii) and (iv) are similar. For $Case(ii)$, we claim that $\beta = 0$. If not, then there exist $l < 0$ and $t_3 > t_2$ such that $z(\sigma(t)) \leq z(t) < l$, $z(\eta(t)) \leq z(t) < l$ for $t \geq t_3$. From (1), it follows that $z(t) > ax(\tau(t))$ and hence $x(\tau(\sigma(t)) > \frac{1}{a} z(\sigma(t))$, that is, $x(\sigma(\tau(t)) > \left(\frac{l}{a}\right)$ for $t \geq t_3$. Also, $x(\eta(\tau(t)) > \left(\frac{l}{a}\right)$ for $t \geq t_3$. Since (1) can be written as

$$(r(\tau(t))(z'(\tau(t))^{\gamma})' + q(\tau(t))x^{\gamma}(\sigma(\tau(t)) + v(\tau(t))x^{\gamma}(\eta(\tau(t)) = 0,$$

then for $t \geq t_3$, it follows that

$$(r(\tau(t))(z'(\tau(t))^{\gamma})' + \left(\frac{l}{a}\right)^{\gamma} q(\tau(t)) + \left(\frac{l}{a}\right)^{\gamma} v(\tau(t)) \leq 0.$$

Consequently,

$$\begin{aligned}\left(\frac{l}{a}\right)^{\gamma}\left[\int_{t_3}^{t} q(\tau(s)) + \int_{t_3}^{t} v(\tau(s))\right] ds \quad &\leq -\left[(r(\tau(t))(z'(\tau(t))^{\gamma})'\right]_{t_3}^{t} \\ &< -r(\tau(t))(z'(\tau(t)))^{\gamma} < \infty \; as \; t \to \infty\end{aligned}$$

contradicts (A_5). So, our claim holds. Therefore,

$$\begin{aligned}0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty} z(t) \\ &\leq \liminf_{t\to\infty}(x(t) + dx(\tau(t))) \\ &\leq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}(dx(\tau(t))) \\ &= \limsup_{t\to\infty} x(t) + d \limsup_{t\to\infty} x(\tau(t)) \\ &= (1+d)\limsup_{t\to\infty} x(t) \; (because (1+d) < 0)\end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$. Hence the proof of the theorem is complete.

3 Oscillation Criteria with (A_{00})

This section deals with the sufficient conditions for oscillation of all solutions of (1) under the assumption (A_{00}).

Lemma 4 ([16]) *Assume that (A_{00}) holds. Let $u(t)$ be an eventually positive continuous function on $[t_0, \infty)$, $t_0 \geq 0$ such that $r(t)u'(t)$ is continuous and differentiable function with $(r(t)u'(t))^{\gamma})' \leq 0, \not\equiv 0$ for large $t \in [t_0, \infty)$, where $r(t)$ is positive and continuous function defined on $[t_0, \infty)$. Then the following statements hold:*
(i) If $u'(t) > 0$, then there exists a constant $C > 0$ such that $u(t) > CR(t)$ for large t.

(ii) If $u'(t) < 0$, then $u(t) \geq -(r(t)(u'(t))^{\gamma})^{\frac{1}{\gamma}} R(t)$, where $R(t) = \int_t^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} ds$.

Theorem 6 *Let $0 \leq p(t) \leq a < \infty$, $\tau(\eta(t)) = \eta(\tau(t))$ and $\tau(\sigma(t)) = \sigma(\tau(t))$ for $t \in [t_0, \infty)$. Assume that (A_{00}) holds, and $r'(t) > 0$, $\tau'(t) \geq \tau_0 > 0$ for any large t. Furthermore, assume that*
(A_6) $\int_{t_0}^{\infty} [R^{\gamma}(\sigma(t))Q(t) + R^{\gamma}(\eta(t))V(t)]\,dt = \infty$
and
(A_7) $\int_T^{\infty} \left[Q(s)R^{\gamma}(s) + V(s)R^{\gamma}(s) + \gamma(1+\tau_0 a^{\gamma})A^{\frac{\gamma+1}{\gamma}} \frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)} - \gamma(1+a^{\gamma})\frac{(R(s))^{-1}}{r^{\frac{1}{\gamma}}(s)}\right] ds = \infty$
hold for any constants $A < 0$ and $T > 0$, where $Q(t)$ and $V(t)$ are defined in Theorem 1. Then every solution of (1) oscillates.

Proof Proceeding as in the proof of Theorem 1, we obtained (4) and (5) for $t \geq t_2$. In what follows, we consider two possible cases $z'(t) > 0$ or $z'(t) < 0$ for $t \geq t_3 > t_2$. If $z'(t) > 0$ for $t \geq t_3$, then $z(t) \geq CR(t)$ due to Lemma 4(i). Therefore, (5) implies that

$$C^{\gamma}R^{\gamma}(\sigma(t))Q(t) + C^{\gamma}R^{\gamma}(\eta(t))V(t) \leq -(r(t)(z'(t))^{\gamma})' - (a^{\gamma}r(\tau(t))(z'(\tau(t)))^{\gamma})' \tag{1}$$

for $t \geq t_3$. Integrating (1) from t_3 to t, we get

$$\begin{aligned}
&\int_{t_3}^{t} C^{\gamma}R^{\gamma}(\sigma(s))Q(s) + \int_{t_3}^{t} C^{\gamma}R^{\gamma}(\eta(t))V(s)ds \\
&\leq -\left[(r(s)(z'(s))^{\gamma}) + (a^{\gamma}r(\tau(s))(z'(\tau(s)))^{\gamma}ds\right]_{t_3}^{t} \\
&\leq r(t_3)z'(t_3)^{\gamma} + a^{\gamma}r(\tau(t_3))(z'(\tau(t_3)))^{\gamma} < \infty,
\end{aligned}$$

a contradiction to (A_6). Ultimately, $z'(t) < 0$ for $t \geq t_2$. We consider the Riccati substitutions

$$w(t) = r(t)(z'(t)/z(t))^{\gamma} \tag{2}$$

and

$$u(t) = r(\tau(t))(z'(\tau(t))/z(\tau(t)))^{\gamma} \tag{3}$$

such that $w(t) < 0$ and $u(t) < 0$ for $t \geq t_3 > t_2$. From Lemma 4(ii), it is easy to verify that

$$-1 \leq w(t)R^{\gamma}(t) \leq 0 \tag{4}$$

for $t \geq t_3$. On the other hand, $w(t) \leq u(t)$ implies that

$$-1 \leq u(t)R^{\gamma}(t) \leq 0 \tag{5}$$

for $t \geq t_3$, where we have used the fact that both $z(t)$ and $r(t)(z'(t))^{\gamma}$ are nonincreasing functions on $[t_3, \infty)$. Since

$$w'(t) = \frac{(r(t)(z'(t))^\gamma)'}{z^\gamma(t)} - \frac{r(t)(z'(t))^\gamma(z^\gamma(t))'}{(z^\gamma(t))^2}$$
$$= \frac{(r(t)(z'(t))^\gamma)'}{z^\gamma(t)} - \frac{\gamma r(t)(z'(t))^{\gamma+1}}{z(t)z^\gamma(t)}$$
$$= \frac{(r(t)(z'(t))^\gamma)'}{z^\gamma(t)} - \frac{\gamma w^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)}$$

and

$$u'(t) = \frac{(r(\tau(t))(z'(\tau(t))^\gamma)'}{z^\gamma(\tau(t))} - \frac{r(\tau(t))(z'(\tau(t))^\gamma(z^\gamma(\tau(t))'}{(z^\gamma(\tau(t)))^2}$$
$$\leq \frac{(r(\tau(t))(z'(\tau(t)))^\gamma)'}{z^\gamma(\tau(t))} - \frac{\tau_0\gamma r(t)(z'(\tau(t)))^{\gamma+1}}{z(\tau(t))z^\gamma(\tau(t))}$$
$$\leq \frac{(r(\tau(t))(z'(\tau(t)))^\gamma)'}{z^\gamma(\tau(t))} - \frac{\tau_0\gamma u^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)}.$$

Consequently,

$$w'(t) + a^\gamma u'(t) \leq \frac{(r(t)(z'(t))^\gamma)'}{z^\gamma(t)} + a^\gamma \frac{(r(\tau(t))(z'(\tau(t)))^\gamma)'}{z^\gamma(\tau(t))} - \frac{\gamma w^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} - \frac{\tau_0\gamma a^\gamma u^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \tag{6}$$

$$\leq -Q(t)\frac{z^\gamma(\sigma(t))}{z^\gamma(t)} - V(t)\frac{z^\gamma(\eta(t))}{z^\gamma(t)} - \frac{\gamma w^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} - \frac{\tau_0\gamma a^\gamma u^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \tag{7}$$

for $t \geq t_3$. Since $\sigma(t) \leq t$, then $(z(t))/(z(\sigma(t)))^\gamma \geq 1$ and hence the inequality (6) yields

$$w'(t) + a^\gamma u'(t) \leq -Q(t) - V(t) - \frac{\gamma w^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} - \frac{\tau_0\gamma a^\gamma u^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} \tag{8}$$

for $t \geq t_3$. Therefore, we find (8) as

$$w'(t)R^\gamma(t) + a^\gamma u'(t)R^\gamma(t) \leq -Q(t)R^\gamma(t) - V(t)R^\gamma(t) - R^\gamma(t)\left[\frac{\gamma w^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)} + \frac{\tau_0\gamma a^\gamma u^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)}\right].$$

Indeed, $w^{\frac{1+\gamma}{\gamma}}(t) > 0$ for any $\gamma > 0$ and $w(t) < 0$, and so also $u^{\frac{1+\gamma}{\gamma}}(t) > 0$. Therefore, $w'(t) < 0$ and $u'(t) < 0$ on $[t_3, \infty)$. Hence there exists a constant $A < 0$ and $t_4 > t_3$ such that $u(t) \leq A$ and $A \geq u(t) = w(\tau(t)) \geq w(t)$ on $[t_4, \infty)$. Integrating the preceding inequality from t_4 to t, we obtain

$$w(t)R^\gamma(t) - w(t_4)R^\gamma(t_4) - \int_{t_4}^t (R^\gamma(s))'w(s)ds$$
$$+ a^\gamma u(t)R^\gamma(t) - a^\gamma u(t_4)R^\gamma(t_4) - \int_{t_4}^t a^\gamma(R^\gamma(s))'u(s)ds$$

$$\leq -\int_{t_4}^{t} Q(s)R^{\gamma}(s)ds - \int_{t_4}^{t} V(s)R^{\gamma}(s)ds - \gamma(1+\tau_0 a^{\gamma})A^{\frac{\gamma+1}{\gamma}} \int_{t_4}^{t} \left[\frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)}\right] ds,$$

that is,

$$w(t)R^{\gamma}(t) - w(t_4)R^{\gamma}(t_4) + \gamma \int_{t_4}^{t} (R(s))^{\gamma-1} \frac{w(s)}{r^{\frac{1}{\gamma}}(s)} ds$$
$$+ a^{\gamma}u(t)R^{\gamma}(t) - a^{\gamma}u(t_4)R^{\gamma}(t_4) + \gamma \int_{t_4}^{t} a^{\gamma}(R(s))^{\gamma-1} \frac{u(s)}{r^{\frac{1}{\gamma}}(s)} ds$$
$$\leq -\int_{t_4}^{t} Q(s)R^{\gamma}(s)ds - \int_{t_4}^{t} V(s)R^{\gamma}(s)ds - \gamma(1+\tau_0 a^{\gamma})A^{\frac{\gamma+1}{\gamma}} \int_{t_4}^{t} \left[\frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)}\right] ds.$$

As a result,

$$\gamma \int_{t_4}^{t} (R(s))^{\gamma-1} \frac{w(s)}{r^{\frac{1}{\gamma}}(s)} ds + \gamma(1+\tau_0 a^{\gamma})A^{\frac{\gamma+1}{\gamma}} \int_{t_4}^{t} \left[\frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)}\right] ds + \gamma \int_{t_4}^{t} \frac{a^{\gamma}(R(s))^{\gamma-1}u(s)}{r^{\frac{1}{\gamma}}(s)} ds$$
$$\leq -w(t)R^{\gamma}(t) + w(t_4)R^{\gamma}(t_4)$$
$$-\int_{t_4}^{t} Q(s)R^{\gamma}(s)ds - \int_{t_4}^{t} V(s)R^{\gamma}(s)ds + a^{\gamma}u(t_4)R^{\gamma}(t_4) - a^{\gamma}u(t)R^{\gamma}(t). \quad (9)$$

Upon using (4) and (5), (9) reduces to

$$-\gamma \int_{t_4}^{t} \frac{(R(s))^{-1}}{r^{\frac{1}{\gamma}}(s)} ds + \gamma(1+\tau_0 a^{\gamma})A^{\frac{\gamma+1}{\gamma}} \int_{t_4}^{t} \left[\frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)}\right] ds - \gamma \int_{t_4}^{t} \frac{a^{\gamma}(R(s))^{-1}}{r^{\frac{1}{\gamma}}(s)} ds$$
$$\leq -w(t)R^{\gamma}(t) + w(t_4)R^{\gamma}(t_4) - \int_{t_4}^{t} Q(s)R^{\gamma}(s)ds - \int_{t_4}^{t} V(s)R^{\gamma}(s)ds$$
$$+ a^{\gamma}u(t_4)R^{\gamma}(t_4) - a^{\gamma}u(t)R^{\gamma}(t),$$

that is,

$$-\gamma(1+a^{\gamma}) \int_{t_4}^{t} \frac{(R(s))^{-1}}{r^{\frac{1}{\gamma}}(s)} ds + \gamma(1+\tau_0 a^{\gamma})A^{\frac{\gamma+1}{\gamma}} \int_{t_4}^{t} \frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)} ds + \int_{t_4}^{t} Q(s)R^{\gamma}(s)ds$$
$$+ \int_{t_4}^{t} V(s)R^{\gamma}(s)ds \leq -w(t)R^{\gamma}(t) + w(t_4)R^{\gamma}(t_4) + a^{\gamma}u(t_4)R^{\gamma}(t_4) - a^{\gamma}u(t)R^{\gamma}(t).$$

Therefore,

$$\int_{t_4}^{\infty} \left[Q(s)R^{\gamma}(s) + V(s)R^{\gamma}(s) + \gamma(1+\tau_0 a^{\gamma})A^{\frac{\gamma+1}{\gamma}} \frac{R^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)} - \gamma(1+a^{\gamma}) \frac{(R(s))^{-1}}{r^{\frac{1}{\gamma}}(s)}\right] ds < \infty,$$

a contradiction to (A_7). This completes the proof of the theorem.

Theorem 7 *Let* $-1 < a \leq p(t) \leq 0,\ t \in [t_0, \infty)$. *Assume that* (A_{00}) *holds and* $r'(t) > 0$ *for any large* t. *If* (A_5),

(A_8) $\int_T^\infty [R^\gamma(\sigma(t))q(t) + R^\gamma(\eta(t))v(t)]\,dt = \infty$
and
(A_9) $\int_T^\infty \left[q(s)R^\gamma(s) + v(s)R^\gamma(s) + \gamma\left\{A^{\frac{\gamma+1}{\gamma}} \frac{R^\gamma(s)}{r^{\frac{1}{\gamma}}(s)} - \frac{(R(s))^{-1}}{r^{\frac{1}{\gamma}}(s)}\right\}\right] ds = \infty.$
hold for any constants $A < 0$ and $T > 0$, then every solution of (1) either oscillates or converges to zero as $t \to \infty$.

Proof Proceeding as in the proof of Theorem 6, we have (4). Thus $z(t)$ and $(r(t)z'(t))^\gamma$ are monotonic function on $[t_2, \infty)$. Here, we consider the four possible cases of Theorem 3. It is easy to verify the cases following to Theorems 6 and 3. Hence the details are omitted. This completes the proof of the theorem.

Theorem 8 *Let $-\infty < a \le p(t) \le d < -1$, $\tau(\sigma(t)) = \sigma(\tau(t))$ and $\tau(\eta(t)) = \eta(\tau(t))$ be hold for all $t \in [t_0, \infty)$. Assume that all conditions of Theorem 7 hold. If (A_4) hold, then every bounded solution of (1) either oscillates or converges to zero as $t \to \infty$.*

Proof The proof of the theorem follows from the proof of Theorems 5 and 7, Hence the details are omitted. The proof of the theorem is complete.

Theorem 9 *Let $-1 < a \le p(t) \le 0$ for $t \in [t_0, \infty)$. Assume that $r'(t) > 0$, (A_{00}) and (A_8) hold. Then every unbounded solution of (1) oscillates.*

Proof The proof of the theorem follows from Remark 1 and the proof of Theorems 7 and 4. Hence the details are omitted.

4 Discussion and Examples

In this work, our objective was to establish the sufficient conditions for oscillation of all solutions of (1). But, our method fails to provide the conclusion in the range $-\infty < a \le p(t) \le d < -1$. However, we could manage in Theorems 5 and 8 with bounded solution. In the literature, we don't find the discussion concerning the oscillation of neutral equations when $-\infty < p(t) \le -1$. So, it is interesting to study the oscillation property of neutral equations in this range, and at the same time it would be interesting to see *an all solution oscillatory problem.*

In our next problem, we study the oscillatory behaviour of solutions of (1) under the key assumptions (A_0) and (A_{00}) in which $\gamma \ge 1$ is a quotient of odd positive integers. We conclude this section with the following examples to illustrate our mail results:

Example (1):
Consider

$$((t^\gamma((x(t) + (1 + t^{-1})x(t-2)))')^\gamma)' + (t+2)^\gamma x^\gamma(t-2) + (t+2)^\gamma x^\gamma(t-2) = 0 \quad (1)$$

on $[2, \infty)$, where $a = 2$. If we choose $\delta(t) = 1$, then all conditions of Theorem 1 are hold true. Hence, (1) is oscillatory.

Example (2):
Consider

$$((e^{\frac{t}{3}}((x(t)+(1+t^{-1})x(t-1)))')^{\frac{1}{3}})'+e^{t}x^{\frac{1}{3}}(t-1)+\left(e^{t}+\frac{1+2^{\frac{1}{3}}}{3}e^{\frac{t+1}{3}}\right)x^{\frac{1}{3}}(t-1)=0 \tag{2}$$

on $[2, \infty)$, where $a = 2$. Clearly, $Q(t) = e^{t-1}$, $V(t) = (e^{t-1} + \frac{1+2^{\frac{1}{3}}}{3}e^{\frac{t}{3}})$ and all conditions of Theorem 6 are hold true. Hence, (2) is oscillatory.

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Steady and Unsteady Navier–Stokes Flow with Lagrangian Differences

Werner Varnhorn

Abstract The motion of a viscous incompressible fluid flow in bounded domains with a smooth boundary can be described by the nonlinear Navier–Stokes system (N). This description corresponds to the so-called Eulerian approach. We develop a new approximation method for (N) in both the steady and the nonsteady case by a suitable coupling of the Eulerian and the Lagrangian representation of the flow, where the latter is defined by the trajectories of the particles of the fluid. The method leads to a sequence of uniquely determined approximate solutions with a high degree of regularity, which contains a convergent subsequence with limit function v such that v is a weak solution on (N).

Keywords Navier–Stokes equations · Weak solutions · Lagrangian differences

1 Introduction

For the description of fluid flow there are in principle two approaches, the Eulerian approach and the Lagrangian approach. The first one describes the flow by its velocity $v = (v_1(t, x), v_2(t, x), v_3(t, x)) = v(t, x)$ at time t in every point $x = (x_1, x_2, x_3)$ of the domain G containing the fluid. The second one uses the trajectory $x = (x_1(t), x_2(t), x_3(t)) = x(t) = X(t, 0, x_0)$ of a single particle of fluid, which at initial time $t = 0$ is located at some point $x_0 \in G$. The second approach is of great importance for the numerical analysis and computation of fluid flow also involving different media with interfaces [2, 3, 5, 9], while the first one has also often been used in connection with theoretical investigations concerning regularity and uniqueness [4, 6–8, 10].

It is the aim of the present note to develop a new approximation method for the nonlinear Navier–Stokes equations by coupling both the Lagrangian and the Eulerian

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approach. This method avoids fixed point considerations and leads to a sequence of approximate systems with solution being unique and having a high degree of regularity, important at least for numerical purposes. Moreover, we can show that our method allows the construction of (global in time in the unsteady case) weak solutions of the Navier–Stokes equations (compare [2, 4] for a local theory): The sequence of approximate solutions has at least one accumulation point satisfying the Navier–Stokes equations in a weak sense [6].

2 The Steady Navier–Stokes System

We consider the stationary motion of a viscous incompressible fluid in a bounded domain $G \subset \mathbb{R}^3$ with a sufficiently smooth boundary S. Because for steady flow the streamlines and the trajectories of the fluid particles coincide, both approaches mentioned above are correlated by the autonomous system of characteristic ordinary differential equations

$$x'(t) = v(x(t)), \quad x(0) = x_0 \in G, \tag{1}$$

which is an initial value problem for

$$t \longrightarrow x(t) = X(t, 0, x_0) = X(t, x_0),$$

if the velocity field $x \longrightarrow v(x)$ is known in G.

To determine the velocity, in the present case we have to solve the steady nonlinear equations

$$\begin{aligned} -\nu \Delta v + v \cdot \nabla v + \nabla p &= F \quad \text{in} \quad G, \\ \operatorname{div} v = 0 \quad \text{in} \quad G, \quad v &= 0 \quad \text{on} \quad S \end{aligned} \tag{2}$$

of Navier–Stokes. Here $x \longrightarrow p(x)$ is an unknown kinematic pressure function. The constant $\nu > 0$ (kinematic viscosity) and the external force density F are given data. The incompressibility of the fluid is expressed by $\operatorname{div} v = 0$, and on the boundary S we require the no-slip condition $v = 0$.

3 The Lagrangian Approach - Autonomous Case

Throughout the paper, we use the same notation for scalar- and vector-valued functions. Let us start recalling some facts, which concern existence and uniqueness for the solution of the initial value problem (1): If the function v belongs to the space $C_0^{\text{lip}}(\overline{G})$ of vector fields being Lipschitz continuous in the closure $\overline{G} = G \cup S$ and vanishing on the boundary S, then for all $x_0 \in G$ the solution

$$t \longrightarrow x(t) = X(t, x_0)$$

of (1) is uniquely determined and exists for all $t \in \mathbb{R}$ (because $v = 0$ on the boundary S, the trajectories remain in G for all times). Due to the uniqueness, the set of mappings

$$\Re = \{X(t, \cdot) : G \to G \mid t \in \mathbb{R}\}$$

defines a commutative group of C^1- diffeomorphisms on G. In particular, for $t \in \mathbb{R}$ the inverse mapping $X(t, \cdot)^{-1}$ of $X(t, \cdot)$ is given by $X(-t, \cdot)$, i.e.

$$\begin{aligned} X(t, \cdot) \circ X(-t, \cdot) &= X(t, X(-t, \cdot)) \\ &= X(t - t, \cdot) = X(0, \cdot) = \text{id}, \end{aligned}$$

or, equivalently,

$$X(t, X(-t, x)) = x$$

for all $x \in G$ and all $t \in \mathbb{R}$. Moreover, for the functional determinant we obtain $\det \nabla X(t, x) = 1$ if

$$v \in C^{\text{lip}}_{0,\sigma}(\overline{G}) = \{u \in C^{\text{lip}}_0(\overline{G}) | \operatorname{div} u = 0\},$$

in addition. This important measure preserving property implies

$$\langle f, g\rangle = \langle f \circ X(t, \cdot), g \circ X(t, \cdot)\rangle$$

for all functions $f, g \in L^2(G)$, where $\langle\cdot, \cdot\rangle$ denotes the usual scalar product in $L^2(G)$.

4 The Eulerian Approach

Next let us consider the Navier–Stokes boundary value problem (2). It is well known that, given $F \in L^2(G)$, there is at least one function v satisfying (2) in some weak sense [6]. To define such a weak solution we need the space $V(G)$, being the closure of $C^\infty_{0,\sigma}(G)$ (smooth divergence free vector functions with compact support in G) with respect to the Dirichlet-norm $\|\nabla u\| = \sqrt{\langle \nabla u, \nabla u\rangle}$, where we define

$$\langle \nabla u, \nabla v\rangle = \sum_{i,j=1}^{3} \langle D_j u_i, D_j v_i\rangle.$$

Let us recall the following

Definition 1 Let $F \in L^2(G)$ be given. A function $v \in V(G)$ satisfying for all $\Phi \in C^{\infty}_{0,\sigma}(G)$ the identity

$$\nu\langle \nabla v,\ \nabla\Phi\rangle \ - \ \langle v\cdot\nabla\Phi, v\rangle \ = \ \langle F,\ \Phi\rangle \tag{3}$$

is called a weak solution of the Navier–Stokes equations (2), and (3) is called the weak form of (2).

For a suitable approximation of the nonlinear term let us keep in mind its physical deduction. It is a convective term arising from the total or substantial derivative of the velocity vector v. Thus it seems to be reasonable to use a total difference quotient for its approximation.

To do so, let $v \in C^{lip}_{0,\sigma}(\overline{G})$ be given. Then for any $\varepsilon \in \mathbb{R}$ the mapping $X(\varepsilon,\cdot):$ $G \to G$ and its inverse $X(-\varepsilon,\cdot)$ are well defined. Consider for some $u \in C^1(G)$ ($C^m(G)$ denotes the space of continuous functions having continuous partial derivatives up to and including order $m \in \mathbb{N}$ in G) and $x \in G$ the two one-sided Lagrangian difference quotients

$$L^{\varepsilon}_{+}\, u(x) = \frac{1}{\varepsilon}\,[u(X(\varepsilon,\cdot)) \ - \ u(x)]\,,$$
$$L^{\varepsilon}_{-}\, u(x) = \frac{1}{\varepsilon}\,[u(x) \ - \ u(X(-\varepsilon,\cdot))]\,,$$

and the central Lagrangian difference quotient

$$L^{\varepsilon}\, u(x) = \frac{1}{2}\left(L^{\varepsilon}_{+}\, u(x) \ + \ L^{\varepsilon}_{-}\, u(x)\right). \tag{4}$$

Since for sufficiently regular functions

$$L^{\varepsilon}_{-}\, u(x) \longrightarrow v(x)\cdot\nabla u(x)$$

and

$$L^{\varepsilon}_{+}\, u(x) \longrightarrow v(x)\cdot\nabla u(x)$$

as $\varepsilon \to 0$, the above quotients can all be used for the approximation of the convective term $v\cdot\nabla v$. There is, however, an important advantage of the central quotient (4) with respect to the conservation of the energy:

Let $v \in C^{lip}_{0,\sigma}(G)$ and $u,\ w \in L^2(G)$. Let $X(\varepsilon,\cdot)$ and $X(-\varepsilon,\cdot)$ denote the mappings constructed from the solution of (1). Then, using the measure preserving property from above, we obtain only for the central quotient the following orthogonality relations:

$$\langle L^{\varepsilon}\, u,\ w\rangle = -\langle u,\ L^{\varepsilon}\, w\rangle, \qquad \langle L^{\varepsilon}\, u,\ u\rangle = 0. \tag{5}$$

5 The Approximate Steady System

To establish an approximation procedure in the steady case we assume that some approximate velocity field v^n has been found. To construct v^{n+1} we proceed as follows:

(1) Construct $X^n = X(\frac{1}{n}, \cdot)$ and its inverse $X^{-n} = X(-\frac{1}{n}, \cdot)$ from the initial value problem

$$x'(t) = v^n(x(t)), \qquad x(0) = x_0 \in G. \tag{6}$$

(2) Construct v^{n+1} and p^{n+1} from the linear boundary value problem

$$\begin{aligned} -\nu \Delta v^{n+1} + \frac{n}{2}[v^{n+1} \circ X^n - v^{n+1} \circ X^{-n}] + \nabla p^{n+1} &= F \quad \text{in} \quad G, \\ \text{div } v^{n+1} &= 0 \quad \text{in} \quad G, \\ v^{n+1} &= 0 \quad \text{on} \quad S. \end{aligned} \tag{7}$$

Concerning the existence and uniqueness for the solution of (6) and (7) we need the usual Sobolev Hilbert spaces $H^m(G)$, $m \in \mathbb{N}$, which denote the closure of $C^m(G)$ with respect to the norm $\| \cdot \|_{H^m}$ (see [1]). A first main result is now stated in the following

Theorem 1 **(a)** *Assume* $v^n \in H^3(G) \cap V(G)$ *and* $F \in H^1(G)$. *Then for all* $x_0 \in G$ *the initial value problem (6) is uniquely solvable, and the mappings*

$$X^n : G \to G, \qquad X^{-n} : G \to G$$

are measure preserving C^1- *diffeomorphisms in* G.
Moreover, there is a uniquely determined solution

$$v^{n+1} \in H^3(G) \cap V(G), \; \nabla p^{n+1} \in H^1(G)$$

of the equations (7).
The velocity field v^{n+1} *satisfies the energy equation* $\nu \|\nabla v^{n+1}\|^2 = \langle F, v^{n+1} \rangle$.
(b) *Assume* $v^0 \in H^3(G) \cap V(G)$ *and* $F \in H^1(G)$. *Let* (v^n) *denote the sequence of solutions constructed in view of Part (a). Then* (v^n) *is bounded in* $V(G)$ *i.e.* $\|\nabla v^n\|^2 \leq C_{G,F,\nu}$ *for all* $n \in \mathbb{N}$, *where the constant* $C_{G,F,\nu}$ *does not depend on* n. *Moreover,* (v^n) *has an accumulation point* $v \in V(G)$ *satisfying (3), i.e.* v *is a weak solution of the Navier–Stokes equations (2).*

6 The Unsteady Navier–Stokes System

Let us consider now the motion of a nonstationary viscous incompressible fluid flow in a bounded cylindrical domain $G_T := (0, T) \times G$ with $T > 0$ and $G \subset \mathbb{R}^3$ with a sufficiently smooth boundary S. Without loss of generality, in this section we assume conservative external forces and consider the following Navier–Stokes initial boundary value problem:

Construct in G_T a velocity field $v = v(t, x)$ and some pressure function $p = p(t, x)$ as a solution of the system

$$\begin{aligned} v_t - \nu \Delta v + \nabla p + v \cdot \nabla v &= 0 \\ & \quad \text{in } (0, T) \times G, \\ \nabla \cdot v &= 0 \\ v &= 0 \ \text{ on } (0, T) \times S, \\ v &= v_0 \text{ in } \ \{0\} \times G. \end{aligned} \tag{N}$$

Here v_0 is a suitable prescribed initial velocity distribution and $\nu > 0$ the kinematic viscosity.

The existence of a classical solution global in time of this problem without any smallness restriction on the data has not been proved up to now (Millennium Problem). Hence also a globally stable approximation scheme does not exist for this system. In order to construct classically solvable equations, as in the steady-state case, an approximation of the nonlinear convective term $v(t, x) \cdot \nabla v(t, x)$, which is responsible for the non-global existence (up to now) of the solution, by means of Lagrangian difference quotients seems to be reasonable.

In the following we show that the nonstationary Navier–Stokes system (N) can also be approximated by means of Lagrangian differences. The resulting approximate system (N_ε) is uniquely solvable for all $\varepsilon > 0$, its solution exists globally in time, has a high degree of regularity and satisfies the corresponding energy equation.

7 The Lagrangian Approach - Non-autonomous Case

Let J be a compact time interval, and let $\tilde{v} \in C(J, H^3(G) \cap V(G))$ be a given velocity field being strongly H^3-continuous. Consider the non-autonomous initial value problem

$$\begin{aligned} \dot{x}(t) &= \tilde{v}(t, x(t)) \\ x(s) &= x_0 \end{aligned}, \quad (s, x_0) \in J \times G \tag{A}$$

concerning the trajectory $x(t) = X(t, s, x_0)$ of a fluid particle, which at time $t = s$ is located in $x_0 \in G$. Due to well-known results on ordinary differential equations, as in the autonomous case, the uniquely determined general solution $X(t, s, x_0)$ of (A) exists for all times, and the mapping

$$X(t, s, \cdot) : G \to G, \qquad t, s \in J$$

is a measure preserving C^1-diffeomorphism with inverse function

$$X(t, s, \cdot)^{-1} = X(s, t, \cdot).$$

Now, as in the steady case we approximate the time dependent nonlinear convective term $v(t, x) \cdot \nabla v(t, x)$ by a central Lagrangian difference quotient as follows:

$$v(t, x) \cdot \nabla v(t_0, x) \quad \sim \quad \frac{1}{2\varepsilon} \Big(v(t_0, X(t + \varepsilon, t, x)) - v(t_0 , X(t, t + \varepsilon, x)) \Big). \tag{8}$$

Here $\sim$ means that for a sufficiently regular function v the right hand side converges to the expression on the left hand side as $\varepsilon \to 0$.

The main advantage of the central quotient in (8), which we denote by

$$\frac{1}{2\varepsilon} \, (v \circ X - v \circ X^{-1})$$

for abbreviation, is again the validity of an analogon to the orthogonality relation of Hopf [6]:

Using $\langle \cdot, \cdot \rangle$ as $L^2(G)$-scalar product Hopf obtains the global (in time) existence of weak solutions to the Navier–Stokes system (N) due to the important orthogonality relation

$$(v \cdot \nabla v, v) = 0, \qquad v \in V(G).$$

Using the measure preserving property of the mapping X, we analogously obtain

$$\frac{1}{2\varepsilon} \, (v \circ X - v \circ X^{-1}, v) \; = \; \frac{1}{2\varepsilon} \Big((v \circ X, v) - (v \circ X^{-1}, v) \Big) \quad =$$
$$\frac{1}{2\varepsilon} \Big((v \circ X, v) - (v \circ X^{-1} \circ X, v \circ X) \Big) \; = \; \frac{1}{2\varepsilon} \Big((v \circ X, v) - (v, v \circ X) \Big) \; = \; 0,$$

which implies the validity of the energy equation for all sufficiently regular solutions of the approximate system, if central Lagrangian differences instead of one-sided quotients are used.

8 Time Delay and Compatibility Conditions

To avoid fixed point considerations for the solution of the regularized approximate system – the velocity vector v as well as the mappings X are unknown – by means of a time delay we replace $v \cdot \nabla v$ by $\frac{1}{2\varepsilon}$ $(v \circ X - v \circ X^{-1})$ with trajectories X constructed at earlier time points, where the velocity v is known already.

To do so, on the given time interval $[0, T]$ we define a time grid by

$$t_k = k \cdot \varepsilon,\ k = 0, \ldots, N \in \mathbb{N},$$

where $\varepsilon := \frac{T}{N} > 0$. Setting

$$X_k := X(t_k, t_{k-1}, x),$$

for $t \in (t_k, t_{k+1}]$ we could use e.g. the approximation

$$v(t, x) \cdot \nabla v(t, x) \sim \frac{1}{2\varepsilon} (v(t, X_k) - v(t, X_k^{-1})). \tag{9}$$

To initiate this procedure we extend the initial value v_0 continuously to a start function

$$v_s \in C([-\varepsilon, 0], H^3(G) \cap V(G)).$$

Then, indeed, on the subintervals $(t_k, t_{k+1}]$ we can successively construct the mappings X_k from the given velocity field v and vice versa. Nevertheless, we do not obtain a global on $[0, T]$ existing solution of a problem regularized by (9). This is due to a certain unpleasant compatibility condition, which always occurs in parabolic problems at the corner of the space time cylinder:
For the unique construction of the mapping X_k, if integer order Sobolev spaces are used, we need a velocity field

$$v \in C([t_{k-1}, t_k], H^3(G) \cap V(G)),$$

i.e.

$$v_t \in C([t_{k-1}, t_k], V(G)).$$

Using

$$P : L^2(G) \to H(G) := \overline{C^\infty_{0,\sigma}(G)}^{\|\cdot\|}$$

as orthogonal projection we obtain, in particular, the condition

$$v_t(t_k) = \nu P \Delta v(t_k) - \frac{1}{2\varepsilon} P\Big((v(t_k, X_k) - v(t_k, X_k^{-1})\Big) \in V(G). \tag{10}$$

Due to $v(t_k) \in H^3(G) \cap V(G)$ we find that the right hand side of (10) is contained in $H^1(G) \cap H(G)$, only. Hence an approximation of the type (9) is not possible since the condition $v_t(t_k) \in V(G)$ implies the condition

$$\nu P \Delta v(t_k) - \frac{1}{2\varepsilon} P\Big((v(t_k, X_k) - v(t_k, X_k^{-1})\Big) = 0 \text{ on } S, \tag{11}$$

which can not be satisfied in general.

9 The Approximate System (N_ε)

Instead of a system regularized by (9) we consider for $\varepsilon > 0$ an approximate Navier–Stokes system in G_T defined by

$$\begin{aligned} v_t - \nu\Delta v + \nabla p + Z_\varepsilon v &= 0 \\ & \quad \text{in } (0,T)\times G, \\ \nabla\cdot v &= 0 \\ v &= 0 \quad \text{on } (0,T)\times S \\ v_t &= a_0 \text{ in } \{0\} \times G, \end{aligned} \tag{N_ε}$$

where $a_0 \in V(G)$ is a prescribed initial acceleration, and where for $t \in [t_k, t_{k+1}]$

$$\begin{aligned} Z_\varepsilon v(t,x) := \frac{1}{2\varepsilon}\Big(& (t-t_k)(v(t,X_k) - v(t,X_k^{-1}))+ \\ & + (t_{k+1}-t)(v(t,X_{k-1}) - v(t,X_{k-1}^{-1}))\Big) \end{aligned}$$

is continuously defined on $[0,T]$.

In this case all compatibility conditions are satisfied: The condition for $t = 0$ can be fulfilled following a hint of V. A. Solonnikov by prescribing $v_t(0) = a_0 \in V(G)$ instead of $v(0) = v_0$:
For a given start function

$$v_s \in C([-2\varepsilon, -\varepsilon], H^3(G) \cap V(G)) \tag{12}$$

we solve the problem (A) and obtain the mapping X_{-1}. Then we consider the steady problem

$$\nu P\Delta v_0 - \frac{1}{2\varepsilon} P(v_0 \circ X_{-1} - v_0 \circ X_{-1}^{-1}) = a_0 \quad \text{in } G, \tag{13}$$

and obtain by well-known existence and regularity results a uniquely determined solution

$$v_0 \in H^3(G) \cap V(G),$$

which, since functions in $V(G)$ vanish on the boundary S, satisfies the required compatibility condition (11). By linear interpolation between $v_s(-\varepsilon)$ and v_0 we then obtain an extended start function

$$v_s \in C([-2\varepsilon, 0], H^3(G) \cap V(G)). \tag{14}$$

Since the compatibility condition in all the following grid points t_k are automatically satisfied due to the continuity of the function

$$t \to Z_\varepsilon v(t),$$

we finally obtain, by successively constructing the mappings form the velocity field v and vice versa, the following main result:

Theorem 2 **(a)** *Let* $[0, T]$ *and* $\varepsilon := \frac{T}{N} > 0$ *be given, let* $a_0 \in V(G)$ *and* $v_s \in C([-2\varepsilon, -\varepsilon], H^3(G) \cap V(G))$. *Let the initial construction for* $v_0 \in H^3(G) \cap V(G)$ *be carried out as described above. Then there exist uniquely determined functions*

$$v \in C([0, T], H^3(G) \cap V(G)), \qquad \nabla p \in C([0, T], H^1(G))$$

as the solution of the system (N_ε). *For* v *holds on* $[0, T]$ *the energy equation*

$$\|v(t)\|^2 + 2\nu \int_0^t \|v(s)\|^2 \, ds \; = \; \|v_0\|^2,$$

and H^3*–Norm estimates can be constructed uniformly on* $[0, T]$ *depending on the data,* T *and* ε.

(b) *Let* $\varepsilon := \frac{T}{N} > 0$ *and let* (v^N) *denote the sequence of solutions constructed in view of Part (a). Then from the energy equation it follows that* (v^N) *is bounded in* $L^\infty(0, T; L^2(G))$ *and* $L^2(0, T; H^1(G))$. *Moreover,* (v^N) *has an accumulation point* $v \in L^\infty(0, T; L^2(G)) \cap L^2(0, T; H^1(G))$ *satisfying the Navier–Stokes system (N) in the weak sense of Leray–Hopf.*

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On Some Discrete Boundary Value Problems in Canonical Domains

Alexander V. Vasilyev and Vladimir B. Vasilyev

Abstract We study some discrete boundary value problems for discrete elliptic pseudo-differential equations in a half-space. These statements are related with a special periodic factorization of an elliptic symbol and a number of boundary conditions depends on an index of periodic factorization. This approach was earlier used by authors for studying special types of discrete convolution equations. Here we consider more general equations and functional spaces.

Keywords Disctrete operator · Periodic factorization · Discrete boundary value problem

1 Introduction

We will consider a certain class of discrete operators and equations in some so-called canonical domains from Euclidean space $\mathbf{R}^m$. These operators are defined by a given function on the m-dimensional cube $\mathbf{T}^m = [-\pi, \pi]^m$, such a function is called a symbol of the discrete operator. Simple examples of such operators have the form

$$u_d(\tilde{x}) \longmapsto \sum_{\tilde{x} \in D_d} A_d(\tilde{x} - \tilde{y}) u_d(\tilde{y}), \quad \tilde{x} \in D_d,$$

where $D_d = \mathbf{Z}^m \cap D$, D is a domain $D \subset \mathbf{R}^m$, A_d, u_d are functions of a discrete variable $\tilde{x} \in \mathbf{Z}^m$, and the given function $A_d(\tilde{x})$ is called a kernel of the operator. Such operators and related ones are called discrete convolutions and were studied from different points of view in a lot of papers (see, for example, [1–9]).

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This paper is devoted to more general operators and equations related to the special canonical domain $D = \mathbf{R}^m_+ = \{x \in \mathbf{R}^m : x = (x_1, \dots, x_m), x_m > 0\}$ although there are some first results for other canonical domains, for example $D = C^a_+ = \{x \in \mathbf{R}^m : x = (x', x_m), x_m > a|x'|, a > 0\}$ [10–12].

2 Discrete Pseudo-differential Operators

2.1 Discrete Fourier Transform and Symbols

Let $u_d(\tilde{x})$ be a function of a discrete variable $\tilde{x} \in h\mathbf{Z}^m$, $h > 0$. The discrete Fourier transform F_d of the function u_d is called the following series

$$(F_d u_d)(\xi) \equiv \tilde{u}(\xi) \equiv \sum_{\tilde{x} \in h\mathbf{Z}^m} e^{-i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^m, \quad \xi \in \hbar\mathbf{T}^m, \quad \hbar \equiv h^{-1},$$

if the series converges.

Evidently the function $\tilde{u}(\xi)$ is defined on $\mathbf{R}^m$, ant it is a periodic function with basic cube of periods $\hbar\mathbf{T}^m$; such functions we call periodic functions.

The Fourier transform is an isomorphism between $L_2(h\mathbf{Z}^m)$ and $L_2(\hbar\mathbf{T}^m)$, moreover

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi} \tilde{u}(\xi) d\xi, \quad \tilde{x} \in h\mathbf{Z}^m.$$

Example 1 Let

$$(\Delta_k^{(1)} u_d)(\tilde{x}) = \frac{u_d(\tilde{x}_1, \dots, \tilde{x}_k + h, \dots, \tilde{x}_m) - u_d(\tilde{x}_1, \dots, \tilde{x}_k, \dots, \tilde{x}_m)}{h}$$

be the divided difference of first order on $\tilde{x}_k$, then for its Fourier transform we have

$$(\widetilde{\Delta_k^{(1)} u_d})(\xi) = \frac{e^{-i\xi_k} - 1}{h}, \quad \xi \in \mathbf{T}^m.$$

If we consider the divided difference of second order on $\tilde{x}_k$

$$(\Delta_k^{(2)} u_d)(\tilde{x}) = \hbar^2 (u_d(\tilde{x}_1, \dots, \tilde{x}_k + 2, \dots, \tilde{x}_m) - 2u_d(\tilde{x}_1, \dots, \tilde{x}_k + 1, \dots, \tilde{x}_m) +$$

$$u_d(\tilde{x}_1, \dots, \tilde{x}_k, \dots, \tilde{x}_m)),$$

then

$$(\widetilde{\Delta_k^{(2)} u_d})(\xi) = \hbar^2 (e^{-i\xi_k} - 1)^2 \tilde{u}_d(\xi), \quad \xi \in \hbar\mathbf{T}^m.$$

Thus, if we introduce so called discrete Laplacian for a function of a discrete variable

$$(\Delta_d u_d)(\tilde{x}) \equiv \left(\sum_{k=1}^{m} \Delta_k^{(2)} u_d\right)(\tilde{x}), \quad \tilde{x} \in h\mathbf{Z}^m$$

we obtain

$$\widetilde{(\Delta_d u_d)}(\xi) = \hbar^2 \sum_{k=1}^{m} (e^{-i\xi_k} - 1)^2 \tilde{u}_d(\xi), \quad \xi \in \hbar\mathbf{T}^m.$$

2.2 Functional Spaces

Definition 1 Discrete Sobolev–Slobodetskii space $H^s(h\mathbf{Z}^m), s \in \mathbf{R}$, consists of functions for which the following norm

$$||u_d||_s = \left(\int_{\hbar\mathbf{T}^m} (1 + |\hat{\xi}^2|)^s |\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}$$

is finite,

$$\hat{\xi}^2 \equiv \hbar^2 \sum_{k=1}^{m} (e^{-ih\xi_k} - 1)^2.$$

Definition 2 The discrete space $H^s(hD_d)$ consists of functions from $H^s(h\mathbf{Z}^m)$ for which their supports belong to $\overline{hD_d}$. A norm in the space $H^s(hD_d)$ is induced by the norm of $H^s(h\mathbf{Z}^m)$. The space $H^s_+(hD_d)$ consists of functions of a discrete variable defined in hD_d which admit continuation on the whole $H^s(h\mathbf{Z}^m)$. The norm in such a space is given by the formula

$$||u_d||_s^+ = \inf ||\ell u_d||_s,$$

where infimum is taken over all continuations ℓ.

We will denote by $\widetilde{H}^s(hD_d)$, $\widetilde{H}^s(h\mathbf{Z}^m \setminus hD_d)$ images of the spaces $H^s(hD_d)$, $H^s(h\mathbf{Z}^m \setminus hD_d)$ under discrete Fourier transform F_d.

Similar functional spaces were introduced and studied in the paper [13], there are a lot of their useful properties.

2.3 Periodic Symbols and Discrete Operators

Definition 3 The function $\widetilde{A}_d(\xi) \in C(\hbar\mathbf{T}^m)$ is called a symbol of discrete pseudo-differential operator A_d, which is defined by the formula

$$(A_d u_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \sum_{\tilde{y}\in h\mathbf{Z}^m} \int_{\hbar\mathbf{T}^m} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \widetilde{A}_d(\xi)\tilde{u}_d(\xi)d\xi, \quad \tilde{x} \in h\mathbf{Z}^m.$$

The symbol $\widetilde{A}_d(\xi)$ is called an elliptic symbol if $\widetilde{A}_d(\xi) \neq 0, \forall \xi \in \hbar\mathbf{T}^m$.

We denoted by E_α the class of periodic symbols satisfying the condition

$$c_1(1+|\hat{\xi}^2|)^{\frac{\alpha}{2}} \le |A_d(\xi)| \le c_2(1+|\hat{\xi}^2|)^{\frac{\alpha}{2}} \tag{1}$$

with constants c_1, c_2 non-depending on h.

Remark 1 We use this definition taking into account in future limit transfer from discrete structure to continue one, and

$$|\hat{\xi}^2| \sim |\xi|^2, h \to 0.$$

Theorem 1 *A discrete pseudo-differential operator with symbol $A_d(\xi) \in E_\alpha$ is a linear bounded operator $A_d : H^s(h\mathbf{Z}^m) \to H^{s-\alpha}(h\mathbf{Z}^m)$ with a norm non-depending on h.*

Each such operator corresponds to the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{2}$$

and we will seek the solution $u_d \in H^s(hD_d)$ for the given right-hand side $v_d \in H^s_+(hD_d)$ and given operator A_d with symbol $A_d(\xi) \in E_\alpha$.

3 Discrete Equations in a Half-Space

In this section we study an auxiliary technique for studying solvability of the Eq. (2) for the special case $D = \mathbf{R}^m_+$.

3.1 Periodic Hilbert Transform

We will remind here the classical Hilbert transform and its connections with boundary properties of analytic functions [14–16] and will describe same properties of its periodic analogue.

The classical Hilbert transform is defined by the following one-dimensional singular integral

$$(Hu)(x) = v.p. \int_{-\infty}^{+\infty} \frac{u(y)dy}{x-y}, \quad x \in \mathbf{R}.$$

This transform plays key role under studying solvability of model elliptic pseudo-differential equations in a multidimensional half-space $\mathbf{R}^m_+ = \{x \in \mathbf{R}^m : x = (x_1, \ldots, x_m), x_m > 0\}$. Its periodic analogue is the following

$$(H^{per}u)(x) = \frac{1}{2\pi i} v.p. \int_{-\pi}^{\pi} \cot \frac{x-y}{2} u(y)dy, \quad x \in [-\pi, \pi].$$

It was shown [6] that this periodic singular integral appears under studying discrete equations in the discrete half-space $\mathbf{Z}^m_+ = \mathbf{Z}^m \cap \mathbf{R}^m_+$, also such integrals appear under summation of Fourier series [17].

3.2 Periodic Riemann Boundary Value Problem

Let us denote by P_+, P_- projection operators on $hD_d, h\mathbf{Z}^m \setminus hD_d$ respectively. To apply the discrete Fourier transform F_d to the Eq. (2) we need to know what are the operators $F_d P_+, F_d P_-$. It was done in papers [4, 6], and here we will briefly describe these constructions.

One can define a discrete analogue of the Schwartz space $S(h\mathbf{Z}^m)$ (see for example [13]) and introduce for such functions the following operators which are generated by periodic analogue of the Hilbert transform, $\xi = (\xi', \xi_m)$,

$$(H^{per}_{\xi'} \tilde{u}_d)(\xi) = \frac{1}{2\pi i} v.p. \int_{-\hbar\pi}^{\hbar\pi} \cot \frac{\hbar(\xi_m - \eta_m)}{2} \tilde{u}_d(\xi', \eta_m) d\eta_m, \quad \xi' \in \hbar\mathbf{T}^{m-1},$$

$$P^{per}_{\xi'} = 1/2(I + H^{per}_{\xi'}), \quad Q^{per}_{\xi'} = 1/2(I - H^{per}_{\xi'}).$$

Lemma 1 *We have the following relations*

$$F_d P_+ = P^{per}_{\xi'} F, \quad F_d P_- = Q^{per}_{\xi'} F.$$

The Lemma 1 implies that a solvability of the Eq. (2) is closely related to a solvability of one-dimensional singular integral equation with the periodic Hilbert transform and a parameter $\xi' \in \hbar\mathbf{T}^{m-1}$. The last equation can be solved with a help of so called periodic Riemann problem [6] which is formulated as followings. Let us denote by $\Pi_\pm$ the upper and lower half-strips in a complex plane $\mathbf{C}$,

$$\Pi_\pm = \{z \in \mathbf{C} : z = t + is, t \in [-\pi, \pi], \ \pm s > 0\}.$$

The problem is the following. Finding two functions $\Phi^\pm(t), t \in [-\pi, \pi]$ (from appropriate functional spaces), which admit an analytical continuation into $\Pi_\pm$ and satisfy the linear relation

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \tag{3}$$

where $G(t), g(t)$ are given functions on $[-\pi, \pi]$, $G(-\pi) = G(\pi)$, $g(-\pi) = g(\pi)$. If $G(t) \equiv 1$ then the problem (3) is called a jump problem.

Lemma 2 *For $|s| < 1/2$, the operators $P_{\xi'}^{per}$, $Q_{\xi'}^{per}$ are bounded projectors $P_{\xi'}^{per} : \widetilde{H}^s(h\mathbf{Z}^m) \to \widetilde{H}^s(hD_d)$, $Q_{\xi'}^{per} : \widetilde{H}^s(h\mathbf{Z}^m) \to \widetilde{H}^s(h\mathbf{Z}^m \setminus hD_d)$, and a jump problem has unique solution $\Phi^+ \in \widetilde{H}^s(hD_d)$, $\Phi^- \in \widetilde{H}^s(h\mathbf{Z}^m \setminus hD_d)$ for arbitrary $g \in \widetilde{H}^s(h\mathbf{Z}^m)$,*

$$\Phi^+ = P_{\xi'}^{per} g, \qquad \Phi^- = -Q_{\xi'}^{per} g.$$

3.3 Periodic Factorization

To study the general Riemann boundary value problem (3) we will use the following concept.

Definition 4 Periodic factorization of an elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form

$$A_d(\xi) = A_{d,+}(\xi)A_{d,-}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ admit an analytical continuation into half-strips $\hbar\Pi_\pm$ on the last variable ξ_m for all fixed $\xi' \in \hbar\mathbf{T}^{m-1}$ and satisfy the estimates

$$|A_{d,+}^{\pm 1}(\xi)| \le c_1(1 + |\hat{\zeta}^2|)^{\pm\frac{æ}{2}}, \quad |A_{d,-}^{\pm 1}(\xi)| \le c_2(1 + |\hat{\zeta}^2|)^{\pm\frac{\alpha - æ}{2}},$$

with constants c_1, c_2 non-depending on h,

$$\hat{\zeta}^2 \equiv \hbar^2 \left(\sum_{k=1}^{m-1} (e^{-ih\xi_k} - 1)^2 + (e^{-ih(\xi_m + i\tau)} - 1)^2 \right), \quad \xi_m + i\tau \in \hbar\Pi_\pm.$$

The number $æ \in \mathbf{R}$ is called an index of periodic factorization.

For some simple cases one can use the topological formula

$$æ = \frac{1}{2\pi} \int_{-\hbar\pi}^{\hbar\pi} d \arg A_d(\cdot, \xi_m),$$

where $A_d(\cdot, \xi_m)$ means that $\xi' \in \hbar\mathbf{T}^{m-1}$ is fixed, and the integral is the integral in Stieltjes sense. It means that we need to calculate divided by 2π variation of the argument of the symbol $A_d(\xi)$ when ξ_m varies from $-\hbar\pi$ to $\hbar\pi$ under fixed ξ'.

Example 2 Let $A_d(\xi) = k^2 + \hat{\xi}^2$, $k \in \mathbf{R}$, such that the condition (1) is satisfied, in other words A_d is the discrete Laplacian plus $k^2 I$. The variation of an argument mentioned above can be calculated immediately, and it equals to 1.

4 Solvability

As we will see the index of factorization very influences on the solvability picture of the Eq. (3).

4.1 Existence and Uniqueness Theorem

Theorem 2 *If the elliptic symbol $\tilde{A}_d(\xi) \in E_\alpha$ admits periodic factorization with index æ so that $|æ - s| < 1/2$ then the Eq. (2) has unique solution in the space $H^s(hD_d)$ for arbitrary right-hand side $v_d \in H^{s-\alpha}(hD_d)$.*

Proof Let ℓv_d be an arbitrary continuation of v_d on the whole $h\mathbf{Z}^m$ so that $\ell v_d \in H^{s-\alpha}(h\mathbf{Z}^m)$. Let

$$w_d(\tilde{x}) = (\ell v_d)(\tilde{x}) - (A_d u_d)(\tilde{x})$$

and rewrite

$$(A_d u_d)(\tilde{x}) + w_d(\tilde{x}) = (\ell v_d)(\tilde{x}).$$

Further applying the discrete Fourier transform F_d and using the periodic factorization we write

$$\tilde{A}_{d,+}(\xi)\tilde{u}_d(\xi) + \tilde{A}^{-1}_{d,-}(\xi)\tilde{w}_d(\xi) = \tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi).$$

According to Theorem 1 we have $\tilde{A}_{d,+}(\xi)\tilde{u}_d(\xi) \in \widetilde{H}^{s-æ}(h\mathbf{Z}^m)$, $\tilde{A}^{-1}_{d,-}(\xi)\tilde{w}_d(\xi) \in \widetilde{H}^{s-\alpha+\alpha-æ}(h\mathbf{Z}^m)$ and analogously $\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi) \in \widetilde{H}^{s-æ}(h\mathbf{Z}^m)$. Moreover, really $\tilde{A}_{d,+}(\xi)\tilde{u}_d(\xi) \in \widetilde{H}^{s-æ}(hD_d)$ in view of a holomorphy property, and accurate considerations with supports of $A_{d,-}(\xi)$ and $\tilde{w}_d(\xi)$ show that in fact $\tilde{A}^{-1}_{d,-}(\xi)\tilde{w}_d(\xi) \in \widetilde{H}^{s-æ}(h\mathbf{Z}^m \setminus hD_d)$.

Thus we obtain a variant of a jump problem for the space $\widetilde{H}^{s-æ}(h\mathbf{Z}^m)$ which can be solved by the Lemma 2. According to this lemma we have

$$\tilde{A}_{d,+}(\xi)\tilde{u}_d(\xi) = P^{per}_{\xi'}(\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi))$$

or finally

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi)P^{per}_{\xi'}(\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)).$$

It finishes the proof. □

Remark 2 It is easy to see that the solution does not depend on choice of continuation ℓv_d.

4.2 A General Solution of the Discrete Equation

Here we consider more complicated case when the condition $|æ - s| < 1/2$ does not hold. There are two possibilities in this situation, and we consider one case which leads to typical boundary value problems.

Theorem 3 *Let* $æ - s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$. *Then a general solution of the Eq. (2) in Fourier images has the following form*

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi)X_n(\xi)P^{per}_{\xi'}(X^{-1}_n(\xi)\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)) + \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}^k_m,$$

where $X_n(\xi)$ *is an arbitrary polynomial of order* n *of variables* $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1), k = 1, \dots, m$, *satisfying the condition (1),* $c_k(\xi'), j = 0, 1, \dots, n-1$, *are arbitrary functions from* $H_{s_k}(h\mathbf{T}^{m-1}), s_k = s - æ + k - 1/2$.

The Theorem 3 implies that if we want to have a unique solution in the case $æ - s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$, we need some additional conditions to determine uniquely unknown functions $c_k(\xi'), k = 0, 1, \dots, n-1$. This case we will discuss in the next section.

Corollary 1 *Let* $æ - s = n + \delta, \in \mathbf{N}, |\delta| < 1/2, \ v_d \equiv 0$. *A general solution of the equation (2) has the following form*

$$\tilde{u}_d(\tilde{x}', \tilde{x}_m) = \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}^k_m.$$

5 Boundary Value Problems

This section is a direct continuation of the previous one and gives a statement of simple boundary value problem for the Eq. (2). We start from a formula for general solution for the Eq. (2) including unknown functions $c_k(\xi')$, $k = 0, 1\ldots, n-1$. For simplicity we consider a homogeneous equation (2) although all results will be valid for inhomogeneous case without additional special requirements.

Let us introduce the following boundary conditions

$$(B_j u_d)(\tilde{x}', 0) = b_j(\tilde{x}'), \quad j = 0, 1\ldots, n-1, \tag{4}$$

where $B_{d,j}$ be a discrete pseudo-differential operators of order $\alpha_j \in \mathbf{R}$ with symbols $\widetilde{B}_j(\xi) \in C(\hbar\mathbf{T}^m)$

$$(B_{d,j}u_d)(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbf{T}^m} \sum_{\tilde{y}\in h\mathbf{Z}^m} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \widetilde{B}_j(\xi)\tilde{u}_d(\xi)d\xi.$$

One can rewrite boundary conditions (4) in Fourier images

$$\int\limits_{-h^{-1}\pi}^{h^{-1}\pi} \widetilde{B}_j(\xi', \xi_m)\tilde{u}_d(\xi', \xi_m)d\xi_m = \tilde{b}_j(\xi'), \quad j = 0, 1\ldots, n-1, \tag{5}$$

so that according to properties of pseudo-differential operators (Theorem 1) and trace properties [13] we need to require $b_j(\tilde{x}') \in H^{s-\alpha_j-1/2}(h\mathbf{Z}^{m-1})$.

Let us denote

$$s_{jk}(\xi') = \int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{A}^{-1}_{d,+}(\xi)\widetilde{B}_j(\xi', \xi_m)\hat{\zeta}^k_m d\xi_m.$$

Now we can formulate the following result.

Theorem 4 *If* $æ - s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$, *then the boundary value problem (2) and (4) has a unique solution in the space* $H^s(hD_d)$ *for arbitrary* $b_j \in H^{s-\alpha_j-1/2}(h\mathbf{Z}^{m-1})$, $j = 0, \ldots, n-1$, *iff*

$$\det(s_{kj}(\xi'))^{æ}_{k,j=0} \neq 0, \quad \forall\, \xi' \in \mathbf{T}^{m-1}. \tag{6}$$

A priori estimate holds

$$||u_d||_s \leq c \sum_{j=0}^{n-1} [b_j]_{s-\alpha_j-1/2},$$

where c does not depend on h, and $[\cdot]_s$ *denotes* H^s*-norm in the space* $H^s(h\mathbf{Z}^{m-1})$.

Proof Substituting the general solution of the Eq. (2) into boundary conditions (5) we have

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{A}_{d,+}^{-1}(\xi)\widetilde{B}_j(\xi',\xi_m)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}_m^k d\xi_m = \tilde{b}_j(\xi'), \quad j = 0, 1. \dots, n-1,$$

and further

$$\sum_{k=0}^{n-1} c_k(\xi') \int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{A}_{d,+}^{-1}(\xi)\widetilde{B}_j(\xi',\xi_m)\hat{\zeta}_m^k d\xi_m = \tilde{b}_j(\xi'), \quad j = 0, 1. \dots, n-1,$$

Thus, we obtain the following system of linear algebraic equations

$$\sum_{k=0}^{n-1} s_{jk}(\xi')c_k(\xi') = \tilde{b}_j(\xi'), \quad j = 0, 1. \dots, n-1,$$

with respect to unknown functions $c_k(\xi')$, $k = 0, 1. \dots, n-1$. The condition (6) is necessary and sufficient for a unique solvability of inhomogeneous system.

A priori estimates can be easily obtained using properties of pseudo-differential operators and appropriate properties of discrete H^s-spaces. □

The condition (6) is a variant of Shapiro–Lopatinskii condition [18].

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Asymptotic Behaviour in a Certain Nonlinearly Perturbed Heat Equation: Non Periodic Perturbation Case

Carlos Ramos, Ana Santos and Sandra Vinagre

Abstract We consider a system described by the linear heat equation with adiabatic boundary conditions. We impose a nonlinear perturbation determined by a family of interval maps characterized by a certain set of parameters. The time instants of the perturbation are determined by an additional dynamical system, seen here as part of the external interacting system. We analyse the complex behaviour of the system, through the scope of symbolic dynamics, and the dependence of the behaviour on the time pattern of the perturbation, comparing it with previous results in the periodic case.

Keywords Heat equation · Chaotic dynamics · Iteration theory · Topological entropy

1 Introduction and Preliminaries

In the present work, we consider the linear heat equation describing the temperature on a wire with adiabatic endpoints. We assume that in most of the time the system evolves continuously, described by the usual linear heat equation, and in this regime we have the explicit time dependent solution. In certain time instants there is a perturbation which changes the system nonlinearly. After this perturbation the system evolves again continuously and the process repeats. The nonlinear perturbation is described by a nonlinear iterated interval map f, which is introduced below.

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Considering the temperature function on the ideal wire, the physical interpretation is presented as a motivation for the study of certain infinite dimensional systems which have, nonetheless, much properties strongly related with iteration of interval maps (one dimensional systems). These infinite dimensional dynamical systems, despite its simple definition, may present very complex behaviour showing some features of real turbulence regarding the balance between coherence versus disorder. In particular, depending on the parameters, the systems present chaotic behaviour, that is, non-periodic evolution and sensitivity to initial conditions, with exponential divergence of two arbitrary close initial temperature functions.

This work was motivated by the following: first, by the papers of Romanenko et al. [7], Sharkovsky [10] and Shakovsky et al. [13], where classes of nonlinear boundary value problems are reduced to difference equations with continuous argument, and whose solutions are, basically, characterized by the iteration of a nonlinear function. It is considered a certain space $\mathscr{A}$ and an operator T_f, induced by an interval map f, so that for $\varphi \in \mathscr{A}$ we have $T_f(\varphi) = f \circ \varphi$. The particular feature of the system $(\mathscr{A}, T_f)$, for certain choices of the map f, is that the number of different critical values will always grow with the iteration of T_f, for almost all initial conditions. Second, by the works of Sharkovsky [11, 12] concerning the notion of ideal turbulence. For a certain classes of real maps f the system $(\mathscr{A}, T_f)$ exhibits ideal turbulence. The concept of ideal turbulence is of particular interest since it characterizes the complexity of certain infinite dimensional systems. Finally, by the approach and the methods from discrete dynamical systems developed by Sousa Ramos and his collaborators, see [8, 9, 14, 15].

On the contrary, for the systems presented in this work, which were first introduced in [2, 6], the number of critical points does not grow exponentially. Here we analyse the behaviour of the systems and the dependence of the behaviour on the time pattern of the perturbation, comparing it with previous results in the periodic case.

Before giving in detail our systems, we present some preliminaries.

Consider a m-modal map f in the class $C^1(I, I)$, for a certain interval I, and the class of differentiable functions

$$\mathscr{A} = \left\{\varphi \in C^1([0,1]) : \varphi'(0) = \varphi'(1) = 0, |cp(\varphi)| < \infty\right\},$$

where $|cp(\varphi)|$ denotes the number of critical points of φ. That is, a function belongs to the class $\mathscr{A}$ if it is differentiable, its derivatives at the endpoints are 0 and its number of critical points are finite. Consider also the operator T_f defined by

$$\begin{aligned} T_f : \mathscr{A} &\to \quad \mathscr{A} \\ \varphi &\mapsto f \circ \varphi. \end{aligned}$$

Note that this operator is well defined since $(f \circ \varphi)'(0) = (f \circ \varphi)'(1) = 0$. Moreover, if $\phi \in \mathscr{A}$ and $Im(\phi) \subset I$, then $Im(T_f^k \phi) \subset I$, for every $k \in \mathbb{N}$. Therefore, we obtain a discrete infinite dimension dynamical system $(\mathscr{A}, T_f)$ in the sense that we have a set $\mathscr{A}$ (with additional structure, a topology or a metric, for now not specified) and a self map T_f, which characterizes the discrete time evolution.

The considered interval maps, modeling the perturbation, belongs to the well studied quadratic family defined by $f_\mu(x) = 1 - \mu x^2$, with $\mu \in (0, 2]$. There is a maximal invariant interval, $[-1, 1]$, where the relevant dynamics occurs, that is, the iterates $f_\mu^k(x_0) := f_\mu(\ldots f_\mu(x_0))$ (k times) of initial points x_0 in $[-1, 1]$ will belong to $[-1, 1]$, for every k. For initial points x_0 outside $[-1, 1]$, the iterates $f_\mu^k(x_0)$, for $k \in \mathbb{N}$, diverge to infinity. Therefore, we consider the one parameter family of operators T_{f_μ}, induced by f_μ on $\mathscr{A}$.

2 Symbolic Dynamics for Unimodal Maps

The main idea behind Milnor and Thurston's kneading theory [5] is to provide a classification of modal maps in the interval using the symbolic itineraries of its critical points. Next, we present a brief description of the symbolic dynamics for the particular case of unimodal maps.

Consider a unimodal map f in the interval $I = [-1, 1]$, with 0 being the unique critical point where f gets the maximal value. Assume that $f \in C^1(I, I)$. We assign the symbols L (left) and R (right) to each point x of the subintervals of monotonicity $[-1, 0)$ and $(0, 1]$, respectively, and the symbol C to the critical point 0. This assignment is called the *address* of x and it is denoted by $ad(x)$. The address of the point x, $ad(x)$, is thus given by

$$ad(x) = \begin{cases} L & if \quad x < 0, \\ C & if \quad x = 0, \\ R & if \quad x > 0. \end{cases}$$

As usual, we get a correspondence between orbits of points and symbolic sequences of the alphabet $\mathscr{A} = \{L, C, R\}$, the itinerary of x under the map f,

$$it_f(x) := ad(x)\ ad(f(x))\ ad\left(f^2(x)\right) \cdots \in \{L, C, R\}^{\mathbb{N}}.$$

The orbit, under f, of the critical point is of special importance, in particular, its itinerary. The *kneading sequence* of an unimodal map f is the itinerary of the image of the critical point, that is,

$$\mathscr{K} := it_f(f(0)) = K_1 K_2 \cdots \in \{L, C, R\}^{\mathbb{N}}.$$

The significance of this symbolic topological invariant was made clear when Guckenheimer [3] presented a classification theorem of modal maps in the interval based on its kneading data, showing how close it is from its topological classification.

An *admissible sequence* is a sequence in $\{L, C, R\}^{\mathbb{N}}$ which occurs as an itinerary for some point $x \in [-1, 1]$ and an *admissible word* is some word occurring in an admissible sequence. The *sequence space* is the set of all infinite admissible sequences in $\{L, C, R\}^{\mathbb{N}}$ and is denoted by Σ.

In the sequence space Σ, we define the usual shift map $\sigma : \Sigma \to \Sigma$ by

$$\sigma(P_1 P_2 P_3 \dots) = P_2 P_3 \dots,$$

where $P_1 P_2 P_3 \dots \in \Sigma$, and we have

$$\sigma\left(it_f(x)\right) = it_f(f(x)).$$

Therefore, we obtain the symbolic system (Σ, σ) associated with the unimodal map.

Given a finite symbolic sequence K of $\mathscr{A}^{\mathbb{N}}$, we define its parity as $\rho(K) := (-1)^{n_R(K)} = \varepsilon(K_1)\varepsilon(K_2)\dots\varepsilon(K_k)$, where $n_R(K)$ denotes the number of symbols R in K and $\varepsilon(L) = 1$ and $\varepsilon(R) = -1$. We define the parity of the empty sequence as $+1$. As we shall see, this parity function will play an important role on the definition of a dynamical meaningful symbolic order relation.

From the order relation $L \prec C \prec R$, inherited from the order of the interval, we introduce an order relation between sequences as follows: given any distinct sequences P and Q of $\mathscr{A}^{\mathbb{N}}$, admitting that they have a common initial subsequence, i.e., there is a $k \geq 0$ such that $P_1 \dots P_k = Q_1 \dots Q_k$ and $P_{k+1} \neq Q_{k+1}$, we will say that $P \prec Q$ if and only if $P_{k+1} \prec Q_{k+1}$ and $\rho(P_1 \dots P_k) = +1$, or $Q_{k+1} \prec P_{k+1}$ and $\rho(P_1 \dots P_k) = -1$.

In what following, we present the transition matrix for $m-$modal maps, that allows us to determine the topological entropy of the map f. Let f be $m-$modal map in the interval $I = [a, b]$ with kneading invariant $\mathscr{K} = (\mathscr{K}_1, \dots, \mathscr{K}_m)$, such that the orbits of the critical points are all periodic with periods $p_1, \dots, p_m$, respectively, that is,

$$(\mathscr{K}_i)_{p_i} = C_i, \quad \text{for} \quad i = 1, \dots, m \quad \text{and} \quad p_i > 0,$$

where C_i is the symbol that corresponds to the critical point c_i, for $i = 1, \dots, m$.

Let $\{X_i\}_{i=1}^{p_1+\dots+p_m}$ be the set of itineraries given by the union of the sets

$$\left\{\sigma^i(\mathscr{K}_1)\right\}_{i=1}^{p_1}, \dots, \left\{\sigma^i(\mathscr{K}_m)\right\}_{i=1}^{p_m},$$

where σ is shift-operator, and let $\{x_i\}_{i=1}^{p_1+\dots+p_m}$ be the set of the points of the interval such that

$$it_f(x_i) = X_i.$$

Denoting by ρ a permutation in the set $\{1, 2, \dots, p_1 + \dots + p_m\}$ such that

$$x_{\rho(1)} < x_{\rho(2)} < \dots < x_{\rho(p_1+\dots+p_m)}$$

and doing $z_i = x_{\rho(i)}$ and $J_i = \left[z_i, z_{i+1}\right]$, for $i = 1, 2, \ldots, p_1 + \ldots + p_m$, we obtain a partition of the interval I determined by the orbits of the m critical points of the map. In this conditions, we can define the following matrix.

The transition matrix associated to the kneading invariant $\mathscr{K} = (\mathscr{K}_1, \ldots, \mathscr{K}_m)$, denoted by $A_{\mathscr{K}}$, is the square matrix, with dimension $p_1 + \cdots + p_m - 1$, whose elements a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if} \quad J_j \subset f(J_i), \\ 0 & \text{otherwise.} \end{cases}$$

We can calculate the topological entropy of a piecewise monotonic map in the interval through the corresponding transition matrix. This result, which proof is in [4], can be stated as follows.

Proposition 1 *Let f be a $m-$modal map with kneading invariant $\mathscr{K} = \mathscr{K}(f)$. Let $A_{\mathscr{K}}$ be the transition matrix associated to $\mathscr{K}$. Then, the topological entropy of f is given by*

$$h_t(f) = \log\left(\lambda_{\max}(A_{\mathscr{K}})\right),$$

where $\lambda_{\max}(A_{\mathscr{K}})$ is the spectral radius of $A_{\mathscr{K}}$.

3 Nonlinear Perturbed Heat Equation

We consider the unit interval representing an ideal wire. The temperature function at each point $x \in [0, 1]$ and each time instant $t \in \mathbb{R}_0^+$ is denoted by $\psi(x, t)$. We consider also that the wire is such that the time evolution of the temperature function is described by the linear heat equation

$$\frac{\partial \psi}{\partial t} = \lambda \frac{\partial^2 \psi}{\partial x^2}, \tag{1}$$

where λ is a constant, the *diffusion coefficient*. If there is no heat exchange in the endpoints $x = 0$ and $x = 1$, we have adiabatic boundary conditions

$$\frac{\partial \psi}{\partial x}(0, t) = \frac{\partial \psi}{\partial x}(1, t) = 0. \tag{2}$$

The initial condition $\psi(x, 0) = \phi_0(x)$ is chosen from the class $\mathscr{A}$, introduced in the preliminaries. The solution can be written as follows

$$\psi(x, t) = \sum_{n=0}^{\infty} c_n e^{-n^2\pi^2\lambda t} \cos(n\pi x), \tag{3}$$

where the coefficients c_n are determined by the initial condition written as a cosine Fourier series

$$\phi_0(x) = \sum_{n=0}^{\infty} c_n \cos(n\pi x). \tag{4}$$

Example 1 Let us consider $\lambda = 0.005$ and

$$\begin{aligned}\psi(x,0) = \phi_0(x) &= 0.2 + 0.1\cos(\pi x) - 0.2\cos(2\pi x) + 0.1\cos(3\pi x) + \\ &+ 0.1\cos(4\pi x) - 0.1\cos(5\pi x) + 0.2\cos(6\pi x) + 0.1\cos(7\pi x) + \\ &+ 0.2\cos(8\pi x) + 0.2\cos(9\pi x).\end{aligned}$$

In Fig. 1, we show the evolution of the system in a linear continuous regime for the initial condition $\psi(x,0)$.

Suppose the system is perturbed in time instants $t_1, t_2, \ldots$ through a certain non-linear process. Being the temperature distribution along the wire initially given by the function $\psi_0(x,t)$, for $t_0 < t < t_1$, after the perturbation the temperature function is $\psi_1(x,t)$, for $t > t_1$. We have continuous time evolution for $t \in]t_j, t_{j+1}[$ and discrete time evolution for $t = t_j$. We assume that the perturbation is characterized by a non-linear map f so that $\psi_{j+1}(x, t_{j+1}) = f(\psi_j(x, t_{j+1}))$, with $\psi_1(x, t_1) = f(\psi_0(x, t_1))$.

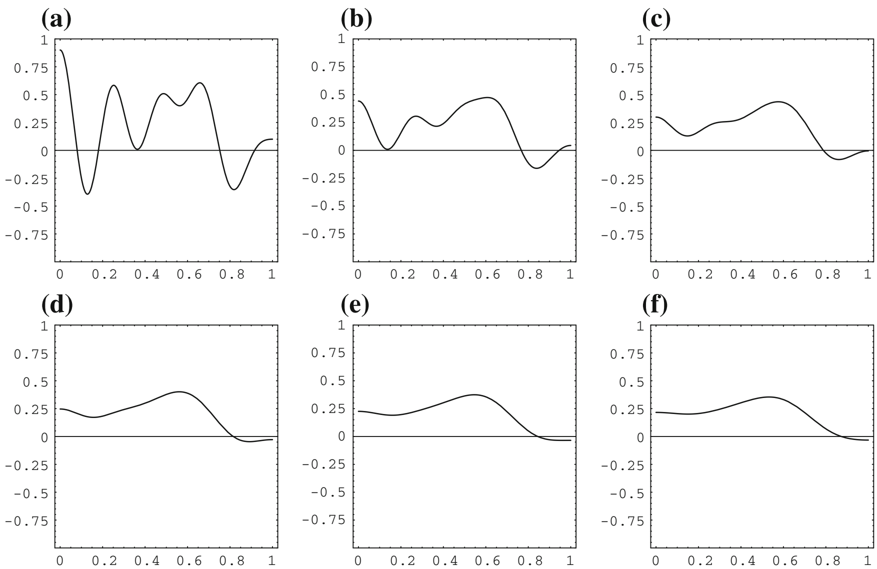

Fig. 1 Graphs of **a** $\psi(x,0)$, **b** $\psi(x,0.4)$, **c** $\psi(x,0.8)$, **d** $\psi(x,1.2)$, **e** $\psi(x,1.6)$ and **f** $\psi(x,2)$, with $\lambda = 0.005$ and $\psi(x,0) = 0.2 + 0.1\cos(\pi x) - 0.2\cos(2\pi x) + 0.1\cos(3\pi x) + 0.1\cos(4\pi x) - 0.1\cos(5\pi x) + 0.2\cos(6\pi x) + 0.1\cos(7\pi x) + 0.2\cos(8\pi x) + 0.2\cos(9\pi x)$

If the time instants are $t_k = k \in \mathbb{N}$, the time evolution of the system is described by the sequence of functions

$$\{\psi_0, \psi_1, \psi_2, \ldots, \psi_k, \ldots\}, \tag{5}$$

each function ψ_k satisfying the heat equation for $x \in [0, 1]$, $t \in [k, k+1[$, $k \in \mathbb{N}_0$, with initial conditions determined by

$$\psi_{k+1}(x, k+1) = f\left(\psi_k(x, k+1)\right), \quad \text{for } k \in \mathbb{N}_0,$$

and $\psi_0(x, 0) = \phi_0(x)$ a given function from $\mathscr{A}$.

The following example illustrates what occurs to the system when we introduce a perturbation at certain time instants.

Example 2 Consider $f_\mu(x) = 1 - \mu x^2$, with $\mu = 2$, $\lambda = 0.00005$ and

$$\begin{aligned}\psi_0(x, 0) = 0.2 &+ 0.1\cos(\pi x) - 0.2\cos(2\pi x) + 0.1\cos(3\pi x) + \\ &+ 0.1\cos(4\pi x) - 0.1\cos(5\pi x) + 0.2\cos(6\pi x) + 0.1\cos(7\pi x) + \\ &+ 0.2\cos(8\pi x) + 0.2\cos(9\pi x).\end{aligned}$$

In the Fig. 2, we show the evolution of the system described by the heat equation, which is perturbed in time instants $t = 1, 2, 3, 4, 56, 57, 58, 59, 60$.

The discrete dynamical system used in this work is the following. We consider the state space $\mathscr{A}$, the operator T_{f_μ} and an operator $U_{\lambda,\varepsilon} : \mathscr{A} \to \mathscr{A}$, which gives the time evolution under the unperturbed regime, with diffusion coefficient λ. The operator $U_{\lambda,\varepsilon}$ is defined implicitly by

$$U_{\lambda,\varepsilon}\psi(x, t) := \psi(x, t + \varepsilon).$$

Let us consider the operator $V_{\mu,\lambda,\varepsilon} : \mathscr{A} \to \mathscr{A}$ defined by

$$V_{\mu,\lambda,\varepsilon} := T_{f_\mu} \circ U_{\lambda,\varepsilon}.$$

If the system is perturbed in natural time instants, with fixed increments, it is sufficient to consider a natural value for ε. In fact, if the time increment is $a \notin \mathbb{N}$, we can rescale through the parameter λ. Therefore, we set $\varepsilon = 1$ and we define $V_{\mu,\lambda} \equiv V_{\mu,\lambda,1}$. Our discrete dynamical system is, then, defined by the pair $\left(\mathscr{A}, V_{\mu,\lambda}\right)$. When we iterate a function $\phi_0(x)$ in $\mathscr{A}$, under $V_{\mu,\lambda}$, the obtained iterates $\phi_k(x) = V^k_{\mu,\lambda}(\phi_0(x))$ will correspond to the solution given by the sequence of functions (5) in the time instants $\phi_k(x) = \psi_k(x, k)$. If, for some reason, we need to obtain the temperature function at a non integer time instant t' we simply use the solution presented in (3) with initial condition given by $\psi(x, 0) = V^k_{\mu,\lambda}(\phi_0(x))$, where $k = \left[t'\right]$ is the integer part of t'. Then, we evaluate the function for the time instant $t' - k$, that is, $\psi\left(x, t' - k\right)$.

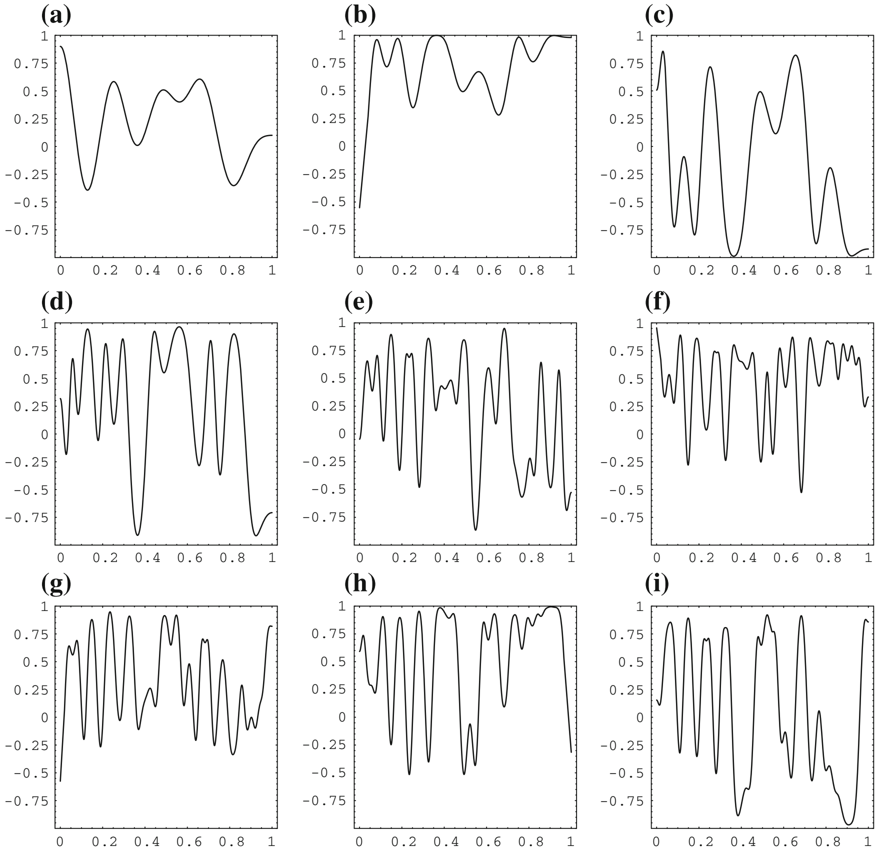

Fig. 2 Graphs of **a** $\psi_0(x,1)$, **b** $\psi_1(x,2)$, **c** $\psi_2(x,3)$, **d** $\psi_3(x,4)$, **e** $\psi_{55}(x,56)$, **f** $\psi_{56}(x,57)$, **g** $\psi_{57}(x,58)$, (h) $\psi_{58}(x,59)$ and (i) $\psi_{59}(x,60)$, with $\lambda = 0.00005$, $f_\mu(x) = 1 - \mu x^2$, $\mu = 2$ and $\psi_0(x,0) = 0.2 + 0.1\cos(\pi x) - 0.2\cos(2\pi x) + 0.1\cos(3\pi x) + 0.1\cos(4\pi x) - 0.1\cos(5\pi x) + 0.2\cos(6\pi x) + 0.1\cos(7\pi x) + 0.2\cos(8\pi x) + 0.2\cos(9\pi x)$

4 The Evolution of Critical Points of the Iterates $\phi_k = V^k_{\mu,\lambda}(\phi_0)$ for Topological Entropy of f_μ Equal to Zero

The topological entropy, $h_t(g)$, of an interval map, g, is an important measure for the characterization of the complex behaviour of the map under iteration, see for details [5]. Among other aspects, the topological entropy measures the growth rate of the periodic orbits for g. For the infinite dimensional system $(\mathscr{A}, T_g)$, the topological entropy of g measures the growth rate of the number of critical points for the functions in $\mathscr{A}$, see [1]. For positive topological entropy, the iterates will have increasingly number of critical points, growing exponentially under iteration. When the topological entropy is equal to zero the growth rate is polynomial or there is no growth at all, for almost every initial functions.

For the cases in which the quadratic map f_μ has entropy equal to zero, every point in the interval, under iteration of f_μ, will be attracted to the critical orbit which will have period 2^k, for a certain natural k. Moreover, the repulsive coexisting periodic orbits are of period 2^j, with $j < k$. This is verified for values of the parameter μ below the Feigenbaum point μ_{2^∞} for which the map f_μ has a nonperiodic critical orbit, has zero topological entropy and has coexisting repulsive periodic orbits of period 2^j, for every natural j, see [5] for more details.

Example 3 Consider $f_\mu(x) = 1 - \mu x^2$, with $\mu = 1.3107\ldots$. The *kneading sequence is given by* $\mathscr{K} = RLRC$, that is, the critical point of f_μ is a periodic point with period 4, and the transition matrix is given by

$$A_{\mathscr{K}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The spectral radius of $A_{\mathscr{K}}$ is equal to 1 and the topological entropy is zero.

4.1 Perturbation in Fixed Time Increments

In the works [2, 6] was established that, in certain conditions – namely positive topological entropy of f_μ – the iterates, under $V_{\mu,\lambda}$, have an exponential growth of number of critical points up to a certain level. After attaining a certain number of critical points, which depends on the parameters, this number oscillates and becomes limited. On the other hand, for the cases in which the quadratic map f_μ has topological entropy equal to zero, that is, for the cases in which μ gives $h_t\left(f_\mu\right) = 0$, we have periodic orbits, since every point under iteration of f_μ converges to the critical periodic orbit. In our context, a periodic orbit will be a sequence of functions $\{\psi_0, \psi_1, \ldots, \psi_k, \psi_0, \ldots\}$ so that $V^k_{\mu,\lambda}(\psi_0) = \psi_0$, for a certain k.

In the following example we show an attracting periodic orbit of period 4.

Example 4 Consider $f_\mu(x) = 1 - \mu x^2$, with $\mu = 1.3107\ldots$, $\lambda = 0.00005$ and

$$\begin{aligned}\psi_0(x, 0) = 0.2 + 0.1\cos(\pi x) - 0.2\cos(2\pi x) + 0.1\cos(3\pi x) + \\ + 0.1\cos(4\pi x) - 0.1\cos(5\pi x) + 0.2\cos(6\pi x) + \\ + 0.1\cos(7\pi x) + 0.2\cos(8\pi x) + 0.2\cos(9\pi x).\end{aligned}$$

In the Fig. 3, we present the graphs of $V^k_{\mu,\lambda}(\psi_0)$, with $k = 95, \ldots, 100$, when the critical point of f_μ is a periodic point with period 4.

For the considered parameter μ, the topological entropy of f_μ is equal to zero and, for k large enough, the system approaches a periodical orbit.

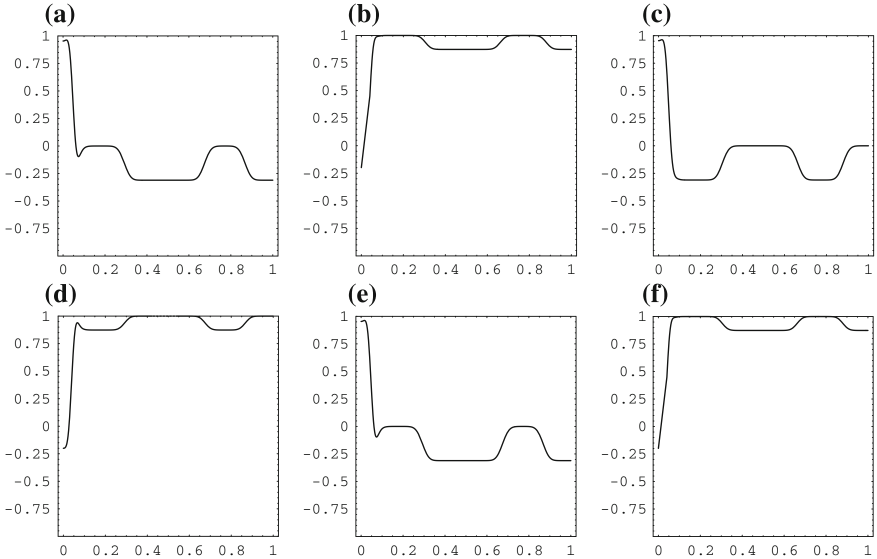

Fig. 3 Graphs of **a** $V^{95}_{\mu,\lambda}(\psi_0)$, **b** $V^{96}_{\mu,\lambda}(\psi_0)$, **c** $V^{97}_{\mu,\lambda}(\psi_0)$, **d** $V^{98}_{\mu,\lambda}(\psi_0)$, **e** $V^{99}_{\mu,\lambda}(\psi_0)$ and **f** $V^{100}_{\mu,\lambda}(\psi_0)$, with $\lambda = 0.00005$, $f_\mu(x) = 1 - \mu x^2$, $\mu = 1.3107\ldots$ and $\psi_0(x, 0) = 0.2 + 0.1\cos(\pi x) - 0.2\cos(2\pi x) + 0.1\cos(3\pi x) + 0.1\cos(4\pi x) - 0.1\cos(5\pi x) + 0.2\cos(6\pi x) + 0.1\cos(7\pi x) + 0.2\cos(8\pi x) + 0.2\cos(9\pi x)$

We can, after a systematic analysis, state the following result.

Numerical Result 1 Let f_μ be a quadratic map, such that $\mu < \mu_{2^\infty}$, $\psi_0 \in \mathscr{A}$ and $\psi_k = V^k_{\mu,\lambda}(\psi_0)$, with $k \in \mathbb{N}_0$. Then,

$$\lim_n \left| V^{n+2^m}_{\mu,\lambda}(\psi_0) - V^n_{\mu,\lambda}(\psi_0) \right| = 0,$$

for a certain $m \in \mathbb{N}$.

4.2 *Perturbation in Non Periodic Time Instants*

Suppose the system is perturbed in non periodic time instants $t_1, t_2, \ldots$. We choose the time instants to be $t_{k+1} = k + 1 + g_b(s_k)$, where $t_0, s_0 = 0$, $s_{k+1} = g_b(s_k)$, g is a surjective map such that zero is not a fixed point and $k \in \mathbb{N}_0$. The map g determines the time increment between the perturbation instants, it can gives fixed time increment, recovering the previous section case, can be periodic or non periodic. As previously, the time evolution of the system is described by the sequence of functions

$$\{\psi_0, \psi_1, \psi_2, \ldots, \psi_k, \ldots\},$$

each function ψ_k satisfying the heat equation for $x \in [0, 1]$, $t \in [k, k+1[$ and $k \in \mathbb{N}_0$, with initial conditions determined by

$$\psi_{k+1}(x, t_{k+1}) = f\left(\psi_k(x, t_{k+1})\right), \text{ for } k \in \mathbb{N}_0,$$

and $\psi_0(x, 0)$ a given function from $\mathscr{A}$.

In this case the operator $V_{f,\lambda,g} : \mathscr{A} \to \mathscr{A}$ is defined by

$$V_{f,\lambda,g} := T_{f_\mu} \circ U_{\lambda,g},$$

where the operator $U_{\lambda,g}$ is defined by

$$U_{\lambda,g}\psi\left(x, t\right) := \psi\left(x, g\left(t\right)\right),$$

depending on the time instant map g.

In the following example we show an attracting periodic orbit of period 12.

Example 5 Consider $f_\mu\left(x\right) = 1 - \mu x^2$, with $\mu = 1.3107\ldots$, $g_b\left(x\right) = -b^2x^2 + 2(-x + x^2)s + 2x - b^2$, with $b = 1.7548\ldots$, $\lambda = 0.00005$ and

$$\psi_0\left(x, 0\right) = 0.03\cos(2\pi x) + 0.2\cos(3\pi x) - 0.2\cos(5\pi x) + 0.3\cos(6\pi x) + \\ + 0.3\cos(12\pi x).$$

In the Fig. 4, we present the graphs of $V^k_{\mu,\lambda}(\psi_0)$, with $k = 95, \ldots, 119$, when the critical point of f_μ is a periodic point with period 4. For this parameter μ, the topological entropy of f_μ is equal to zero and, for k large enough, the system approaches a periodical function of period 12. Note that, for this parameter b, the function that defines the time instants, g_b, is periodic with period 3.

We can, after a systematic analysis, state the following result.

Numerical Result 2 Let f_μ be a quadratic map, such that $\mu = \mu_{2^m} < \mu_{2^\infty}$, g_b a periodic map with period p, $\psi_0 \in A$ and $\psi_k = V^k_{\mu,\lambda}(\psi_0)$, with $k \in \mathbb{N}_0$. Then,

$$\lim_n \left| V^{n+r}_{\mu,\lambda}(\psi_0) - V^n_{\mu,\lambda}(\psi_0) \right| = 0,$$

where r is the least common multiple of 2^m and p.

5 Conclusions

Our results, for the case in which f_μ has topological entropy equal to zero, which means attracting periodic orbits of period 2^m, shows that independently of the initial condition ϕ_0 the time evolution $\psi\left(t\right)$ approaches a periodic orbit, that is, a sequence

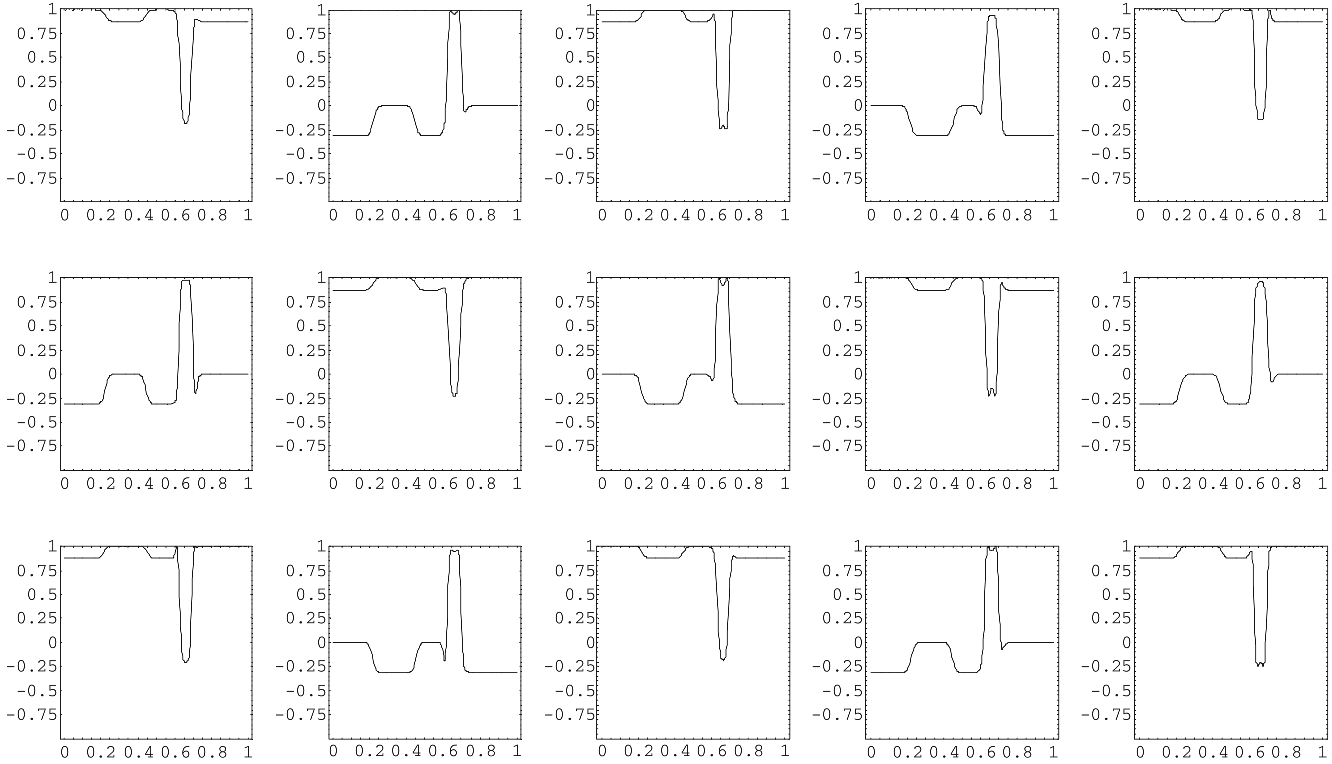

Fig. 4 Graphs of $V^k_{f,\lambda,g}(\psi_0)$, with $k = 105, \ldots, 119$, $\lambda = 0.00005$, $f_\mu(x) = 1 - \mu x^2$, $\mu = 1.3107\ldots$ and $\psi_0(x, 0) = 0.03\cos(2\pi x) + 0.2\cos(3\pi x) - 0.2\cos(5\pi x) + 0.3\cos(6\pi x) + 0.3\cos(12\pi x)$

of functions $\left\{\widetilde{\psi}_0, \widetilde{\psi}_1, \ldots, \widetilde{\psi}_k\right\}$. In the case where the perturbation occurs in fixed time increment, the period k is equal to the period of f_μ. In the case where the perturbation oscilates through a map increment g, with period p, then the period k is the least common multiple of 2^m and p.

What is remarkable is that the precise form of the functions $\widetilde{\psi}_0$ depends on the ϕ_0. Nevertheless, the periods are always the same.

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Mathematical Model for Optimising Bi-Enzyme Biosensors

Qi Wang and Yupeng Liu

Abstract It is often the case that the equilibrium values are the only piece of information required for the solution of a practical problem (although, sometimes, time to achieve equilibrium or size of the device is the real issue) and in such situations it is important to identify the conditions under which a complex partial differential equations model can be replaced with a simpler one. In this paper, we study a flow injection analysis of a bi-enzyme electrode, with the aim of finding the ratio of the two enzymes involved which yields the highest current amplitude.

Keywords Biosensors · Cascade reactions · Equilibrium · Dynamical systems

1 Introduction

The problem we study here is motivated by a series of experiments conducted at the National Centre for Sensor Research (NCSR) and the Biomedical Diagnostics Institute (BDI) at Dublin City University over the past few years by a group of researchers interested in building a biosensing platform based on a bi-enzyme electrode. For more details, we refer the reader to [1–3]. This study investigates a model biosensor system which consists of two enzymes immobilised onto an electrode modified with a conducting polymer. The first enzyme, glucose oxidase (GOX), acts as the source of the substrate for the second enzyme, horseradish peroxidase (HRP), producing hydrogen peroxide from the oxidation of glucose to gluconolactone. Horseradish peroxidase is in direct electronic communication with the electrode via the conducting polymer and facilitates the electrocatalytic reduction of hydrogen peroxide,

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which can be measured amperometrically at moderate reducing potentials. Cascade schemes, where an enzyme is catalytically linked to another enzyme, can produce signal amplification and therefore increase the biosensor efficiency. HRP and GOX have very different kinetic characteristics (which have been studied extensively) and so obtaining the optimum performance of this biosensing system will depend on the correct ratio of the two enzymes.

2 Modelling Strategies

In attempting to construct a mathematical model for this problem, we make the following simplifying assumptions:

1. The immobilisation mechanisms of the two enzymes are equally efficient and hence the distribution of immobilised HRP and GOX molecules on the surface of the electrode is proportional to that of the solution used.
2. Immobilisation of HRP and GOX produces a geometrically close-packed spherical monolayer which is spatially homogeneous.
3. The electron transfer process is 100% efficient, since this parameter only affects the magnitude of the signals, and not their relative responses.

A cascade reaction takes place at the electrode. Glucose oxidase catalyses the oxidation reaction of glucose to gluconic acid, with production of H_2O_2. HRP is oxidised by hydrogen peroxide and then subsequently reduced by electrons provided by the electrode, as shown in the following abbreviated reaction. These reactions may be summarised as follows:

$$\begin{gathered}\beta\text{-D-glucose} + O_2 + H_2O \xrightarrow{\text{GOX}} \text{gluconic acid} + H_2O_2,\\ H_2O_2 + \text{HRP} \longrightarrow \text{Compound I} + H_2O,\\ \text{Compound I} + 2e^- + H^+ \longrightarrow \text{HRP} + H_2O.\end{gathered}$$

We can see that these reactions taking place at the electrode is a biochemical cascade reaction, since the product of the first reaction feeds into the second reaction as the substrate and is then consumed. We are going to use the standard Michaelis–Menten equations to model these reactions,

$$E_1 + S_1 \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} C_1 \xrightarrow{k_2} E_1 + S_2, \qquad E_2 + S_2 \underset{k_{-3}}{\overset{k_3}{\rightleftarrows}} C_2 \xrightarrow{k_4} E_2 + P, \tag{1}$$

where E_1 denotes the first enzyme GOX, E_2 denotes the second enzyme HRP, S_1 denotes the first substrate glucose, S_2 denotes the second substrate hydrogen peroxide, C_1 and C_2 are the two complexes and P is the final product. k_1, k_{-1}, k_2, k_3, k_{-3} and k_4 are constant parameters which represent the rate of the reactions.

A detailed stability analysis on a mathematical model is presented here which shows that the system displays different behaviour for different values of the enzyme ratio ζ. We also use results from geometric singular perturbation theory and monotone dynamical systems in order to achieve a good understanding of this model.

3 Formation of the Model

In this mathematical model, we neglect diffusion of both substrates in the cascade reactions (1), and construct a one-point model. An ordinary differential equation model for a cascade reaction was defined as

$$\begin{cases} \dfrac{ds_1}{dt} = \varepsilon_1 \left(\dfrac{k_{-1}}{k_1 s_0} c_1 - \left(\dfrac{\zeta}{1+\zeta} - c_1 \right) s_1 \right) & \text{(2a)} \\ \dfrac{ds_2}{dt} = \varepsilon_1 \left(\dfrac{k_2}{k_1 s_0} c_1 - \dfrac{k_3}{k_1} \left(\dfrac{1}{1+\zeta} - c_2 \right) s_2 + \dfrac{k_{-3}}{k_1 s_0} c_2 \right) & \text{(2b)} \\ \dfrac{dc_1}{dt} = \left(\dfrac{\zeta}{1+\zeta} - c_1 \right) s_1 - \dfrac{K_m^1}{s_0} c_1 & \text{(2c)} \\ \dfrac{dc_2}{dt} = \dfrac{k_3}{k_1} \left(\dfrac{1}{1+\zeta} - c_2 \right) s_2 - \dfrac{K}{s_0} c_2, & \text{(2d)} \end{cases}$$

If we assume that glucose is present in constant supply at the reaction point (i.e., $s_1(t) = s_0$), the non-dimensional system (2) can be further simplified to the following system

$$\begin{cases} \dfrac{dc_1}{dt} = \dfrac{\zeta}{1+\zeta} - \left(1 + \dfrac{K_m^1}{s_0} \right) c_1 & \text{(3a)} \\ \dfrac{dc_2}{dt} = \dfrac{k_3}{k_1} \left(\dfrac{1}{1+\zeta} - c_2 \right) s_2 - \dfrac{K}{s_0} c_2 & \text{(3b)} \\ \dfrac{ds_2}{dt} = \varepsilon_1 \left(\dfrac{k_2}{k_1 s_0} c_1 - \dfrac{k_3}{k_1} \left(\dfrac{1}{1+\zeta} - c_2 \right) s_2 + \dfrac{k_{-3}}{k_1 s_0} c_2 \right), & \text{(3c)} \end{cases}$$

with initial conditions $c_1(0) = 0$, $c_2(0) = 0$ and $s_2(0) = 0$, where

$$\varepsilon_1 = \frac{e}{s_0}, \quad K_m^1 = \frac{k_{-1} + k_2}{k_1}, \quad K = \frac{k_{-3} + k_4}{k_1}.$$

Instead of studying the behaviour of a single reaction (defined by the initial conditions specified above), we have decided to look at the global behaviour of the dynamical system (3). We also anticipate that all solutions of this system will display the exact same asymptotic behaviour as $t \to \infty$.

Given the interpretation of c_1, c_2 and s_2 as concentrations, it is important to establish the positivity of solutions for this system. More precisely, we consider the positive octant

$$\Gamma = \left\{(c_1, c_2, s_2) \in R^3 \mid c_1 \geq 0, c_2 \geq 0, s_2 \geq 0\right\},$$

and prove it is a positively invariant region (which means that trajectories entering this region cannot leave it in forward time). Hence, a solution with a positive initial condition will stay positive for all $t \geq 0$. This is easily done if we show that the flow points inwards on all three boundaries of the region Γ. In particular, we have to check that

$$\frac{dc_1}{dt} \geq 0, \text{ on } c_1 = 0, c_2 \geq 0, s_2 \geq 0,$$
$$\frac{dc_2}{dt} \geq 0, \text{ on } c_2 = 0, c_1 \geq 0, s_2 \geq 0,$$
$$\frac{ds_2}{dt} \geq 0, \text{ on } s_2 = 0, c_1 \geq 0, c_2 \geq 0,$$

and these conditions can be easily verified in system (3). Next, we are going to find the equilibrium points of system (3). At equilibrium, from Eqs. (3a) and (3b), we obtain

$$c_1^* = \frac{\zeta}{(1+\zeta)\left(1+\frac{K_m^1}{s_0}\right)}, \tag{4}$$

$$c_2^* = \frac{s_2^*}{(1+\zeta)\left(s_2^*+\frac{K_m^2}{s_0}\right)}, \tag{5}$$

with

$$K_m^2 = \frac{k_{-3}+k_4}{k_3}.$$

Then from Eqs. (3c), (4) and (5), we find the equilibrium value for $s_2(t)$ is,

$$s_2^* = \frac{\zeta k_2 K_m^2}{s_0\left(k_4\left(1+\frac{K_m^1}{s_0}\right)-\zeta k_2\right)}, \tag{6}$$

which is positive if and only if

$$\zeta < \frac{k_4}{k_2}\left(1+\frac{K_m^1}{s_0}\right).$$

We let

$$\zeta^* = \frac{k_4}{k_2}\left(1 + \frac{K_m^1}{s_0}\right), \tag{7}$$

as we will use this parameter frequently in the remainder of this chapter. Therefore, we conclude that when $\zeta < \zeta^*$, the equilibrium values are

$$\begin{cases} c_1^* = \dfrac{k_4\zeta}{k_2\zeta^*(1+\zeta)} & \text{(8a)} \\ c_2^* = \dfrac{\zeta}{\zeta^*(1+\zeta)} & \text{(8b)} \\ s_2^* = \dfrac{\zeta K_m^2}{s_0(\zeta^* - \zeta)}, & \text{(8c)} \end{cases}$$

and when $\zeta > \zeta^*$, we do not have an equilibrium value for $s_2(t)$ in the positive octant Γ. In order to better visualise the behaviour of the solution of system (3) starting at (0, 0, 0), we now show that it is confined to an invariant set. We define the set Ω_1, such that

$$\Omega_1 = \left\{(c_1, c_2, s_2) \in R^3 \mid 0 \le c_1 \le c_1^*, 0 \le c_2 \le \frac{1}{1+\zeta}, 0 \le s_2 \le \infty\right\}.$$

Next we are going to show the set Ω_1 is an invariant set, by showing that all the trajectories point inwards when crossing the boundary of the set Ω_1, i.e., we need to show that:

On	***Need to show***
$c_1 = 0$	$\frac{dc_1}{dt} > 0$
$c_1 = c_1^*$	$\frac{dc_1}{dt} = 0$
$c_2 = 0$	$\frac{dc_2}{dt} > 0$
$c_2 = 1/(1+\zeta)$	$\frac{dc_2}{dt} < 0$
$s_2 = 0$	$\frac{ds_2}{dt} > 0$

We can easily see that, on $c_1 = 0$, we have

$$\frac{dc_1}{dt} = \frac{\zeta}{1+\zeta} > 0.$$

On $c_1 = c_1^*$, we have

$$\frac{dc_1}{dt} = \frac{\zeta}{1+\zeta} - \left(1 + \frac{K_m^1}{s_0}\right)\frac{\zeta}{(1+\zeta)\left(1 + \frac{K_m^1}{s_0}\right)} = 0.$$

On $c_2 = 0$, we have

$$\frac{dc_2}{dt} = \frac{k_3}{k_1}\left(\frac{1}{1+\zeta}\right)s_2 > 0.$$

On $c_2 = 1/(1+\zeta)$, we have

$$\frac{dc_2}{dt} = -\frac{K}{s_0(1+\zeta)} < 0.$$

On $s_2 = 0$, we have

$$\frac{ds_2}{dt} = \varepsilon_1\left(\frac{k_2}{k_1 s_0}c_1^* + \frac{k_{-3}}{k_1 s_0}c_2\right) > 0,$$

since $c_1^* > 0$ and $c_2 \geq 0$. Thus, the set Ω_1 is an invariant set, and thus the solution starting at (0, 0, 0) stays in this invariant set.

In the next three subsections, we are going to investigate in detail the long term behaviour of this solution and prove that

$$\lim_{t\to\infty} c_1(t) = c_1^*, \quad \text{for all} \quad \zeta,$$

$$\lim_{t\to\infty} c_2(t) = \begin{cases} c_2^*, & \text{if } \zeta \leq \zeta^* \\ \dfrac{1}{1+\zeta}, & \text{if } \zeta \geq \zeta^*, \end{cases}$$

$$\lim_{t\to\infty} s_2(t) = \begin{cases} s_2^*, & \text{if } \zeta < \zeta^* \\ \infty, & \text{if } \zeta \geq \zeta^*. \end{cases}$$

These results are easy to interpret in the context of the cascade reactions (1). If $\zeta < \zeta^*$, there is a relatively small amount of $e_1(0)$ compared to $e_2(0)$ which means that the production of s_2 in the first reaction is somehow balanced by its consumption in the second reaction and an equilibrium state can be reached. On the other hand, if $\zeta \geq \zeta^*$, the relatively large amount of $e_1(0)$ can facilitate the production of s_2 which is then not consumed fast enough in the second reaction so its concentration can grow indefinitely. (Recall that we have assumed an unlimited supply of s_1!)

We now present briefly a local stability analysis for this equilibrium point. The Jacobian matrix for system (3) can be constructed as follows:

$$\begin{pmatrix} \frac{\partial f_1}{\partial c_1^*} & \frac{\partial f_1}{\partial c_2^*} & \frac{\partial f_1}{\partial s_2^*} \\ \frac{\partial f_2}{\partial c_1^*} & \frac{\partial f_2}{\partial c_2^*} & \frac{\partial f_2}{\partial s_2^*} \\ \frac{\partial f_3}{\partial c_1^*} & \frac{\partial f_3}{\partial c_2^*} & \frac{\partial f_3}{\partial s_2^*} \end{pmatrix} = \begin{pmatrix} -1 - \frac{K_m^1}{s_0} & 0 & 0 \\ 0 & -\frac{k_3}{k_1}s_2^* + \frac{K}{s_0} & \frac{k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2^*\right) \\ \frac{\varepsilon_1 k_2}{k_1 s_0} & \varepsilon_1\left(\frac{k_3}{k_1}s_2^* + \frac{k_{-3}}{k_1 s_0}\right) & -\frac{\varepsilon_1 k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2^*\right) \end{pmatrix}$$

Then denoting the eigenvalues of the Jacobian matrix by λ_1, λ_2 and λ_3, we obtain

$$\lambda_1 = -1 - \frac{K_m^1}{s_0} < 0,$$

$$\lambda_2 + \lambda_3 = -\frac{k_3 s_0 s_2^* + k_{-3} + k_4 + \varepsilon_1 k_3 s_0 \left(\frac{1}{1+\zeta} - c_2^*\right)}{k_1 s_0} < 0,$$

$$\text{and} \quad \lambda_2 \lambda_3 = \frac{\varepsilon_1 k_3 k_4 \left(\frac{1}{1+\zeta} - c_2^*\right)}{k_1^2 s_0} > 0;$$

this shows that we have three negative eigenvalues, which tells us the equilibrium point is locally stable. Also, a global stability analysis will be presented later in this section.

4 Slow-Fast Dynamics

With the notation introduced in (3), that system can be written as

$$\begin{cases} \dfrac{dc_1}{dt} = f_1(c_1) & \text{(9a)} \\ \dfrac{dc_2}{dt} = f_2(c_2, s_2) & \text{(9b)} \\ \dfrac{ds_2}{dt} = \varepsilon_1 f_3(c_1, c_2, s_2), & \text{(9c)} \end{cases}$$

where $\varepsilon_1 = e/s_0$ is a small parameter, i.e., $\varepsilon_1 \ll 1$. Posed this way, c_1 and c_2 are fast variables, s_2 is the slow variable and t is the fast time. System (9) is called a **slow-fast system** (also known as **a singularly perturbed system** or **system with multiple scales**).

If we let $\tau = \varepsilon_1 t$, the system can be written in the form

$$\begin{cases} \varepsilon_1 \dfrac{dc_1}{d\tau} = f_1(c_1) & \text{(10a)} \\ \varepsilon_1 \dfrac{dc_2}{d\tau} = f_2(c_2, s_2) & \text{(10b)} \\ \dfrac{ds_2}{d\tau} = f_3(c_1, c_2, s_2). & \text{(10c)} \end{cases}$$

Then in system (10), c_1 and c_2 are still the fast variables, and s_2 is still the slow variable, but τ is the slow time. Regardless of how time is scaled, as long as $f_1(c_1) \neq 0$ and $f_2(c_2, s_2) \neq 0$, we have

$$\left|\frac{dc_1}{dt}\right| \gg \left|\frac{ds_2}{dt}\right| \quad \text{and} \quad \left|\frac{dc_2}{dt}\right| \gg \left|\frac{ds_2}{dt}\right|.$$

Thus, it is the relative rates which makes c_1 and c_2 fast and s_2 slow.

The **fast subsystem (or unperturbed system)** corresponding to system (9) is defined as

$$\begin{cases} \dfrac{dc_1}{dt} = f_1(c_1) & \text{(11a)} \\ \dfrac{dc_2}{dt} = f_2(c_2, s_2) & \text{(11b)} \\ \dfrac{ds_2}{dt} = 0, & \text{(11c)} \end{cases}$$

and the equilibrium set of system (11) is given by

$$f_1(c_1) = 0, \text{ which implies } c_1^* = \frac{\zeta}{(1+\zeta)\left(1+\frac{K_m^1}{s_0}\right)}, \tag{12}$$

$$f_2(c_2, s_2) = 0, \text{ which implies } c_2^* = \frac{s_2^*}{(1+\zeta)\left(s_2^*+\frac{K_m^2}{s_0}\right)}; \tag{13}$$

since ds_2/dt is identically zero (hence s_2 is constant), this set defines a one-dimensional curve of fixed points M_0, which can be thought of as a **trivially** invariant manifold. Also, it can be shown that in the fast system each of these fixed points is stable. Moreover, since the eigenvalues were shown to be strictly negative, each of these equilibrium points is hyperbolic. The manifold M_0 is then said to be **normally hyperbolic** and it is these manifolds that occupy an important place in geometric singular perturbation theory (refer to [4]).

On the other hand, the **slow subsystem (or layer system)** is obtained by letting $\varepsilon_1 = 0$ in system (10), which gives

$$\begin{cases} 0 = f_1(c_1) & \text{(14a)} \\ 0 = f_2(c_2, s_2) & \text{(14b)} \\ \dfrac{ds_2}{d\tau} = f_3(c_1, c_2, s_2). & \text{(14c)} \end{cases}$$

The first two equations in system (14) give

$$c_1^* = \frac{\zeta}{(1+\zeta)\left(1+\frac{K_m^1}{s_0}\right)}, \tag{15}$$

$$c_2^* = \frac{s_2^*}{(1+\zeta)\left(s_2^*+\frac{K_m^2}{s_0}\right)}. \tag{16}$$

This defines a one-dimensional curve in the (c_1, c_2, s_2) space, called the **slow manifold**. Unlike the case of the fast system, there is now (slow) flow along this manifold, which is derived from Eq. (14c), and is given by

$$\frac{ds_2}{d\tau} = F(s_2) = \frac{k_2\zeta}{(1+\zeta)\left(1+\frac{K_m^1}{s_0}\right)} - \frac{(k_{-3}+k_4)\,s_2 - k_{-3}s_2}{(1+\zeta)\left(s_2+\frac{K_m^2}{s_0}\right)}.$$

Note that, by letting $F(s_2) = 0$, we obtain

$$s_2 = \frac{k_2\zeta K_m^2}{s_0\left(k_4\left(1+\frac{K_m^1}{s_0}\right) - k_2\zeta\right)},$$

so there is an equilibrium point on this slow manifold provided the same condition as before, namely $\zeta < \zeta^*$, is satisfied. Unlike regular perturbed systems, neither the slow nor fast subsystem is sufficient for understanding the behaviour of system (9). The dynamics of the original system are then often explained by combining the information obtained from the fast and slow systems.

5 Slow Invariant Manifold

The main question at this point is whether the normally hyperbolic manifold M_0 given by Eqs. (12) and (13) obtained in the fast (unperturbed) subsystem persists for the original system (9) with the perturbation added. The conditions for the persistence of this manifold are given by a theorem due to Fenichel (refer to [5]), and other results in geometric singular perturbation theory (refer to [4] for a review of this theory). A rigorous analysis of the fast-slow dynamics of system (9) is beyond the aim of this work so will not be given. Instead, we use an approximation method (refer to [6]) for the qualitative asymptotic analysis of singular differential equations by reducing the order of the differential system under consideration. The method relies on the theory of invariant manifolds, which essentially replaces the original system by another system on an invariant manifold with dimension equal to that of the slow subsystem.

Definition 1 *A system of differential equations is called* ***autonomous*** *if it maps into itself under arbitrary translations along the time axis. In other words a system is autonomous if its right-hand side is independent of time* (refer to [7]).

Theorem 1 (refer to [6]) *A smooth surface* $y = h(x, \varepsilon)$, $(x \in R^m, y \in R^n)$ *in* $R^m \times R^n$ *is a* ***slow invariant manifold*** *of the autonomous system*

$$\dot{x} = f(x, y, \varepsilon), \quad \varepsilon\dot{y} = g(x, y, \varepsilon), \tag{17}$$

if any trajectory $x = x(t, \varepsilon)$, $y = y(t, \varepsilon)$ *of the system* (17) *that has at least one point* $x = x_0$, $y = y_0$ *in common with the surface* $y = h(x, \varepsilon)$, *i.e.,* $y_0 = h(x_0, \varepsilon)$, *lies entirely in this surface, i.e.,* $y(t, \varepsilon) = h\,(x(t, \varepsilon), \varepsilon)$.

The motion along an invariant manifold of the system (17) *is governed by the equation*

$$\dot{x} = f\,(x, h(x, \varepsilon), \varepsilon)\,.$$

If $x(t, \varepsilon)$ *is a solution of this equation, then the pair* $(x(t, \varepsilon), y(t, \varepsilon))$, *where* $y(t, \varepsilon) = h\,(x(t, \varepsilon), \varepsilon)$, *is a solution of the original system* (17), *since it defines a trajectory on the invariant manifold.*

Substituting the function $h(x, \varepsilon)$ instead of y into system (17) gives the following first order invariance equation for $h(x, \varepsilon)$,

$$\varepsilon\frac{\partial h}{\partial x} f\,(x, h(x, \varepsilon), \varepsilon) = g(x, h, \varepsilon)$$

(refer to [6]).

Now we are going to restate the slow invariant manifold defined in Theorem 1 in the notation of our system (10). *A smooth surface* $c_1 = h_1(s_2, \varepsilon_1)$ *and* $c_2 = h_2(s_2, \varepsilon_1)$ *is a slow invariant manifold of system* (10) *if any trajectory* $s_2 = s_2(t, \varepsilon_1)$, $c_1 = c_1(t, \varepsilon_1)$ *and* $c_2 = c_2(t, \varepsilon_1)$ *of the system that has at least one point* $s_2 = s_{20}$, $c_1 = c_{10}$ *and* $c_2 = c_{20}$ *in common with the surface* $c_1 = h_1(s_2, \varepsilon_1)$ *and* $c_2 = h_2(s_2, \varepsilon_1)$ *(i.e.,* $c_{10} = h_1(s_{20}, \varepsilon_1)$ *and* $c_{20} = h_2(s_{20}, \varepsilon_1)$*), and lies entirely in this surface (i.e.,* $c_1(t, \varepsilon_1) = h_1\,(s_2(t, \varepsilon_1), \varepsilon_1)$ *and* $c_2(t, \varepsilon_1) = h_2\,(s_2(t, \varepsilon_1), \varepsilon_1)$*).*

The motion along an invariant manifold of system (10) *is governed by the equation*

$$\dot{s_2} = f_3\,(s_2, h_1(s_2, \varepsilon_1), h_2(s_2, \varepsilon_1))\,.$$

Hence by substituting $h_1(s_2, \varepsilon_1)$ *and* $h_2(s_2, \varepsilon_1)$ *instead of* c_1, c_2 *into system* (10) *yields the following invariance equations*,

$$\varepsilon_1\frac{\partial h_1}{\partial s_2} f_3\,(s_2, h_1(s_2, \varepsilon_1), h_2(s_2, \varepsilon_1)) = f_1(c_1), \tag{18}$$

$$\varepsilon_1\frac{\partial h_2}{\partial s_2} f_3\,(s_2, h_1(s_2, \varepsilon_1), h_2(s_2, \varepsilon_1)) = f_2(c_2, s_2). \tag{19}$$

To calculate an approximation to the one-dimensional slow invariant manifold, we let

$$c_1 = h_1(s_2, \varepsilon_1) = \phi_0(s_2) + \varepsilon_1 \phi_1(s_2) + O(\varepsilon_1^2), \tag{20}$$

$$c_2 = h_2(s_2, \varepsilon_1) = \psi_0(s_2) + \varepsilon_1 \psi_1(s_2) + O(\varepsilon_1^2). \tag{21}$$

Then if we substitute Eqs. (20) and (21) into (18) and (19), the invariant equations become

$$\begin{aligned}\varepsilon_1 \frac{\partial \phi_0}{\partial s_2} \left(\frac{k_2}{k_1 s_0} (\phi_0(s_2) + \varepsilon_1 \phi_1(s_2)) \right. & - \frac{k_3}{k_1} \left(\frac{1}{1+\zeta} - \psi_0(s_2) - \varepsilon_1 \psi_1(s_2) \right) s_2 \\ & \left. + \frac{k_{-3}}{k_1 s_0} \left(\psi_0(s_2) + \varepsilon_1 \psi_1(s_2) \right) \right) \\ & = \frac{\zeta}{1+\zeta} - \left(1 + \frac{K_m^1}{s_0} \right) \left(\phi_0(s_2) + \varepsilon_1 \phi_1(s_2) \right),\end{aligned} \tag{22}$$

$$\begin{aligned}\varepsilon_1 \frac{\partial \psi_0}{\partial s_2} \left(\frac{k_2}{k_1 s_0} (\phi_0(s_2) + \varepsilon_1 \phi_1(s_2)) \right. & - \frac{k_3}{k_1} \left(\frac{1}{1+\zeta} - \psi_0(s_2) - \varepsilon_1 \psi_1(s_2) \right) s_2 \\ & \left. + \frac{k_{-3}}{k_1 s_0} \left(\psi_0(s_2) + \varepsilon_1 \psi_1(s_2) \right) \right) \\ & = \frac{k_3}{k_1} \left(\frac{1}{1+\zeta} - \psi_0(s_2) - \varepsilon_1 \psi_1(s_2) \right) s_2 \\ & - \frac{K}{s_0} \left(\psi_0(s_2) + \varepsilon_1 \psi_1(s_2) \right).\end{aligned} \tag{23}$$

Now, from Eqs. (22) and (23), at $O(1)$, we obtain

$$\phi_0(s_2) = \frac{\zeta}{(1+\zeta)\left(1 + \frac{K_m^1}{s_0}\right)},$$

$$\psi_0(s_2) = \frac{s_2}{(1+\zeta)\left(s_2 + \frac{K_m^2}{s_0}\right)},$$

and at $O(\varepsilon_1)$, we obtain

$$\phi_1(s_2) = \frac{k_3 K_m^2}{k_2(1+\zeta)\left(1 + \frac{K_m^1}{s_0}\right)\left(s_2 + \frac{K_m^2}{s_0}\right)},$$

$$\psi_1(s_2) = \frac{k_1 K_m^2}{k_2(1+\zeta)\left(s_2 + \frac{K_m^2}{s_0}\right)^2}.$$

Therefore, the approximation of the slow invariant manifold is given as

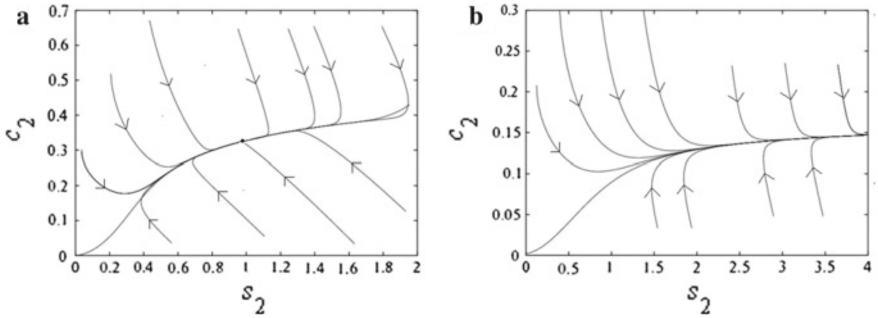

Fig. 1 Phase portrait of system (3) showing c_2 against s_2 in the cases of: **a** $\zeta < \zeta^*$, **b** $\zeta \geq \zeta^*$

$$c_1 = h_1(s_2, \varepsilon_1) = \frac{1}{(1+\zeta)\left(\zeta + \frac{K_m^1}{s_0}\right)} \left(1 + \varepsilon_1 \frac{k_3 K_m^2}{k_2\left(s_2 + \frac{K_m^2}{s_0}\right)}\right) + O(\varepsilon_1^2), \quad (24)$$

$$c_2 = h_2(s_2, \varepsilon_1) = \frac{1}{(1+\zeta)\left(s_2 + \frac{K_m^2}{s_0}\right)} \left(s_2 + \varepsilon_1 \frac{k_1 K_m^2}{k_2\left(s_2 + \frac{K_m^2}{s_0}\right)}\right) + O(\varepsilon_1^2). \quad (25)$$

Note that, when $\varepsilon_1 = 0$, $c_1 = h_1(s_2, 0)$ and $c_2 = h_2(s_2, 0)$, Eqs. (24) and (25) reduce to the equations of the slow manifold (15) and (16). Figure 1 displays a two dimensional phase diagram of the perturbed system (3) showing the dynamics in the variables c_2 and s_2. These graphs were obtained using the dynamical systems software XPP created by Prof. Bard Ermentrout at the University of Pittsburgh (available online at [8]). The existence of the slow manifold is clearly visible in these diagrams with an equilibrium point present in Fig. 1a (if $\zeta < \zeta^*$), and no equilibrium point in Fig. 1b (if $\zeta \geq \zeta^*$).

6 Dynamical Systems Analysis

In this section, we are going to give an alternative analysis of the system (3), which does not use the assumption that ε_1 is a small parameter. From Eq. (8c) we notice that the equilibrium value for s_2^* is positive if $\zeta < \zeta^*$ and negative (therefore irrelevant to our study) if $\zeta \geq \zeta^*$. Here, we are going to analyse these two cases in more detail.

Case 1: When $\zeta < \zeta^*$, we know that in this case there is a unique equilibrium point, (c_1^*, c_2^*, s_2^*), which is positive and stable. However, the stability established previously by linear analysis is only local. That is to say, to determine whether an equilibrium of a system is stable or not, we have only considered infinitesimal perturbations around the nominal solution. This analysis is adequate for linear systems, since linear systems have identical local and global properties, but it is not adequate for non-linear systems. Therefore, in this model, the local stability does not establish that the solution starting at (0, 0, 0) converges to the equilibrium point (c_1^*, c_2^*, s_2^*).

To prove the equilibrium point (c_1^*, c_2^*, s_2^*) is globally stable, we are going to use LaSalle's Invariance Principle (refer to [9]) which is stated below.

Theorem 2 (LaSalle's Invariance Principle) *Consider the autonomous system*

$$\dot{x} = f(x), \quad x(0) = x_0, \tag{26}$$

defined on the domain $D \subset R^n$. Let $\Omega \subset D$ be a compact (i.e., closed and bounded) set that is positively invariant with respect to the dynamics given by (26). Let $V(\cdot)$ be a continuously differentiable function on D such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$ and let M be the largest invariant set contained in E. Then every solution starting in Ω converges to M as $t \to \infty$.

A set M is called an **invariant set** with respect to the dynamics (26) if

$$x(0) \in M, \text{ implies } x(t) \in M, \ \forall\, t \in R.$$

A set M is called **positively invariant** if

$$x(0) \in M, \text{ implies } x(t) \in M, \ \forall\, t \geq 0.$$

By definition, trajectories can neither enter nor leave an invariant set; trajectories may enter a positively invariant set; however, they just cannot leave it in forward time.

In our model, the set Ω is defined as a cube which has the origin and the equilibrium point, (c_1^*, c_2^*, s_2^*) as opposite corner points. More precisely we have

$$\Omega = \left\{(c_1, c_2, s_2) \in R^3 \mid 0 \leq c_1 \leq c_1^*, 0 \leq c_2 \leq c_2^*, 0 \leq s_2 \leq s_2^*\right\}.$$

In order to show Ω is a positively invariant set with respect to the dynamics of system (3), we need to show that all the flow trajectories starting in the set Ω stay in the set Ω forever.

The six faces in this cube are characterised by $c_1 = c_1^*$, $c_2 = c_2^*$, $s_2 = s_2^*$, $c_1 = 0$, $c_2 = 0$ and $s_2 = 0$. In order to show that all the trajectories point inwards through each side of the cube and do not leave it in forward time, we need to show:

On	*Need to show*
$c_1 = c_1^*$	$\frac{dc_1}{dt} \leq 0$
$c_2 = c_2^*$	$\frac{dc_2}{dt} \leq 0$
$s_2 = s_2^*$	$\frac{ds_2}{dt} \leq 0$
$c_1 = 0$	$\frac{dc_1}{dt} \geq 0$
$c_2 = 0$	$\frac{dc_2}{dt} \geq 0$
$s_2 = 0$	$\frac{ds_2}{dt} \geq 0$

From system (3), on the face defined by $c_1 = c_1^*$, we have

$$\frac{dc_1}{dt} = \frac{\zeta}{1+\zeta} - \left(1 + \frac{K_m^1}{s_0}\right) c_1^* = 0,$$

since $dc_1/dt = 0$ at equilibrium. This means that all trajectories starting in this plane remain in this plane. On the face defined by $c_2 = c_2^*$, we have

$$\frac{dc_2}{dt} = \frac{k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2^*\right) s_2 - \frac{K}{s_0} c_2^*. \tag{27}$$

We also know that at equilibrium

$$\frac{k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2^*\right) s_2^* - \frac{K}{s_0} c_2^* = 0; \tag{28}$$

subtracting Eq. (28) from (27) yields

$$\frac{dc_2}{dt} = \frac{k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2^*\right)\left(s_2 - s_2^*\right) \le 0,$$

since $0 \le s_2 \le s_2^*$ in Ω and $0 < c_2^* < 1/(1+\zeta)$. On the face defined by $s_2 = s_2^*$, we have

$$\frac{ds_2}{dt} = \varepsilon_1 \left(\frac{k_2}{k_1 s_0} c_1 - \frac{k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2\right) s_2^* + \frac{k_{-3}}{k_1 s_0} c_2\right). \tag{29}$$

Similarly, we also know that at equilibrium

$$\frac{ds_2}{dt} = \varepsilon_1 \left(\frac{k_2}{k_1 s_0} c_1^* - \frac{k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2^*\right) s_2^* + \frac{k_{-3}}{k_1 s_0} c_2^*\right) = 0, \tag{30}$$

and subtracting Eq. (30) from (29), we get

$$\frac{ds_2}{dt} = \varepsilon_1 \left(\frac{k_2}{k_1 s_0}\left(c_1 - c_1^*\right) + \frac{k_{-3}}{k_1 s_0}\left(c_2 - c_2^*\right) + \frac{k_3}{k_1}\left(c_2 - c_2^*\right) s_2^*\right) \le 0,$$

since $0 \le c_1 \le c_1^*$, $0 \le c_2 \le c_2^*$ and $s_2^* \ge 0$ in Ω. On the face defined by $c_1 = 0$, we have

$$\frac{dc_1}{dt} = \frac{\zeta}{1+\zeta} > 0.$$

On the face defined by $c_2 = 0$, we have

$$\frac{dc_2}{dt} = \frac{k_3}{k_1} \frac{1}{1+\zeta} s_2 \ge 0,$$

since $0 \le s_2 \le s_2^*$ in Ω. On the face defined by $s_2 = 0$, we have

$$\frac{ds_2}{dt} = \varepsilon_1 \left(\frac{k_2}{k_1 s_0} c_1 + \frac{k_{-3}}{k_1 s_0} c_2 \right) \ge 0,$$

since $0 \le c_1 \le c_1^*$ and $0 \le c_2 \le c_2^*$ in Ω.

Thus, we have shown that Ω is a positively invariant set with respect to the dynamics of system (3). Next, we need to define the **Lyapunov function** for this model.

The Lyapunov function is simply a continuous scalar function of the state variables, with continuous partial derivatives. The original motive for the development of Lyapunov's direct method was based on the physical concept of the energy content of a system, which, in the usual dissipative case, is naturally a decreasing function of time, and this is often a fruitful source of Lyapunov functions in practice. But on the other hand, there is no reason why we should be restricted to using a function of this type, and indeed it may not be appropriate in many cases. There is, unfortunately, no completely general systematic procedure for obtaining Lyapunov functions; refer to [10] for more details on how to construct the Lyapunov functions.

In our model, the Lyapunov function is a function of the three state variables $c_1(t)$, $c_2(t)$ and $s_2(t)$ defined as

$$V(\cdot) = V(c_1(t), c_2(t), s_2(t)).$$

We let

$$V(c_1(t), c_2(t), s_2(t)) = \alpha(c_1^* - c_1) + \beta(c_2^* - c_2) + \gamma(s_2^* - s_2), \tag{31}$$

where α, β and γ are arbitrary constants which will be estimated in later calculations, subject to the condition that $\dot{V}(c_1(t), c_2(t), s_2(t)) \le 0$. We have

$$\dot{V}(c_1(t), c_2(t), s_2(t)) = \frac{\partial V}{\partial c_1} \cdot \frac{\partial c_1}{\partial t} + \frac{\partial V}{\partial c_2} \cdot \frac{\partial c_2}{\partial t} + \frac{\partial V}{\partial s_2} \cdot \frac{\partial s_2}{\partial t}, \tag{32}$$

and since

$$\frac{\partial V}{\partial c_1} = -\alpha, \tag{33}$$

$$\frac{\partial V}{\partial c_2} = -\beta, \tag{34}$$

$$\frac{\partial V}{\partial s_2} = -\gamma, \tag{35}$$

we get

$$\dot{V}(c_1(t), c_2(t), s_2(t)) = -\alpha\left(\frac{\zeta}{1+\zeta} - \left(1+\frac{K_m^1}{s_0}\right)c_1\right) - \beta\left(\frac{k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2\right)s_2 - \frac{K}{s_0}c_2\right)$$
$$-\gamma\left(\varepsilon_1\left(\frac{k_2}{k_1 s_0}c_1 - \frac{k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2\right)s_2 + \frac{k_{-3}}{k_1 s_0}c_2\right)\right),$$

giving

$$\dot{V}(c_1(t), c_2(t), s_2(t)) = -\frac{\alpha\zeta}{1+\zeta} + \left(\alpha + \frac{\alpha K_m^1}{s_0} - \frac{\gamma\varepsilon_1 k_2}{k_1 s_0}\right)c_1$$
$$+\left(\frac{\beta K}{s_0} + \frac{\gamma\varepsilon_1 k_{-3}}{k_1 s_0}\right)c_2 + \left(\frac{\gamma\varepsilon_1 k_3}{k_1} - \frac{\beta k_3}{k_1}\right)s_2\left(\frac{1}{1+\zeta} - c_2\right).$$

If we let

$$\frac{\gamma\varepsilon_1 k_3}{k_1} - \frac{\beta k_3}{k_1} = 0,$$

$\dot{V}(c_1(t), c_2(t), s_2(t))$ simplifies to

$$\dot{V}(c_1(t), c_2(t), s_2(t)) = -\frac{\alpha\zeta}{1+\zeta} + \frac{\alpha k_1 s_0 + \alpha(k_{-1}+k_2) - \beta k_2}{k_1 s_0}c_1 + \frac{\beta k_4}{k_1 s_0}c_2,$$

and then by letting $\alpha k_1 s_0 + \alpha(k_{-1}+k_2) - \beta k_2 = 0$, we obtain

$$\alpha = \frac{\beta k_2}{k_1 s_0 + k_{-1} + k_2},$$

which yields

$$\dot{V}(c_1(t), c_2(t), s_2(t)) = -\frac{\beta k_2 \zeta}{(k_1 s_0 + k_{-1} + k_2)(1+\zeta)} + \frac{\beta k_4}{k_1 s_0}c_2$$

$$= -\frac{\beta k_4}{k_1 s_0}\left(\frac{k_2\zeta}{k_4\left(1+\frac{K_m^1}{s_0}\right)(1+\zeta)} - c_2\right)$$
$$= -\frac{\beta k_4}{k_1 s_0}\left(c_2^* - c_2\right).$$

Now if we take $\beta = 1$, we get

$$\dot{V}(c_1(t), c_2(t), s_2(t)) = -\frac{k_4}{k_1 s_0}(c_2^* - c_2) \le 0,$$

since $0 \leq c_2 \leq c_2^*$ in Ω. We can see that $\dot{V}(c_1(t), c_2(t), s_2(t)) = 0$ if and only if $c_2 = c_2^*$.

Therefore, the function V defined in Eq. (31) satisfies the conditions of LaSalle's Invariance Principle if we take

$$\alpha = \frac{k_2}{k_1 s_0 + k_{-1} + k_2}, \quad \beta = 1, \quad \gamma = \frac{1}{\varepsilon_1}.$$

Now let E denote all the points in the set Ω, where $\dot{V}(c_1(t), c_2(t), s_2(t)) = 0$ (one side of the cube Ω is defined by $c_2 = c_2^*$), the largest invariant set contained in this plane is the equilibrium point (c_1^*, c_2^*, s_2^*) itself, which corresponds to the set M mentioned in the LaSalle's Invariance Principle. Thus, every solution starting in the cube Ω converges to the equilibrium point (c_1^*, c_2^*, s_2^*) as $t \to \infty$. In particular, this proves that the solution with initial conditions $c_1(0) = 0$, $c_2(0) = 0$, $s_2(0) = 0$ converges to the equilibrium point.

Case 2: When $\zeta \geq \zeta^*$, there is no equilibrium point in the positive octant where $c_1 \geq 0$, $c_2 \geq 0$ and $s_2 \geq 0$.

In this case we need to show that

$$\begin{cases} \lim_{t\to\infty} c_1(t) = c_1^* = \dfrac{\zeta}{(1+\zeta)\left(1 + \frac{K_m^1}{s_0}\right)} & \text{(36a)} \\ \lim_{t\to\infty} s_2(t) = \infty & \text{(36b)} \\ \lim_{t\to\infty} c_2(t) = \dfrac{1}{1+\zeta}, & \text{(36c)} \end{cases}$$

where the first limit is easily verified since the first equation only depends on $c_1(t)$. In what follows we will use results from the theory of monotone dynamical systems (refer to [11]) to show Eqs. (36b) and (36c) are valid.

We first define some order relations on R^n as follows. For $u, v \in R^n$, we write

$$u \leq v \;\Leftrightarrow\; u_i \leq v_i,$$
$$u < v \;\Leftrightarrow\; u_i \leq v_i,\; u \neq v,$$
$$u \ll v \;\Leftrightarrow\; u_i < v_i,$$

where $i = 1, \ldots n$ (refer to [12]).

Next, we define monotone and cooperative systems, following [11].

Definition 2 *Consider the autonomous system of ordinary differential equations*

$$x' = f(x), \tag{37}$$

where f is continuously differentiable on an open subset $D \subset R^n$. Let $\phi_t(x)$ denotes the solution of system (37) that starts at the point x_0 at $t = 0$. *The function ϕ_t will be referred to as the flow corresponding to system (37).*

*Let $x_0, y_0 \in D$, and let $<_r$ denote any one of the relations $\leq, <, \ll$; then the dynamical system (37) is said to be **monotone** if $x_0 <_r y_0$ implies that $\phi_t(x_0) <_r \phi_t(y_0)$, for all $t > 0$.*

Definition 3 *We say that D is p-convex if $tx + (1-t)y \in D$ for all $t \in [0, 1]$ whenever $x, y \in D$ and $x \leq y$. If D is a convex set then it is also p-convex. Then the system (37) is said to be a **cooperative system** if*

$$\frac{\partial f_i}{\partial x_j}(x) \geq 0, \;\; i \neq j, \;\; x \in D$$

holds on the p-convex domain D.

In our system (3), we let

$$D = \Omega_1 = \left\{ (c_1, c_2, s_2) \in R^3 \mid 0 \leq c_1 \leq c_1^*, 0 \leq c_2 \leq \frac{1}{1+\zeta}, 0 \leq s_2 \leq \infty \right\}.$$

Recall that the infinite rectangular box Ω_1 was shown to be a positive invariant set for system (3) in Sect. 3, and it is clearly a p-convex set. We can also easily show that

$$\frac{\partial f_1}{\partial c_2} = 0, \;\; \frac{\partial f_1}{\partial s_2} = 0,$$
$$\frac{\partial f_2}{\partial c_1} = 0, \;\; \frac{\partial f_2}{\partial s_2} \geq 0,$$
$$\frac{\partial f_3}{\partial c_1} \geq 0, \;\; \frac{\partial f_3}{\partial c_2} \geq 0,$$

so that system (3) is a cooperative system, and a cooperative system generates a monotone dynamical system.

Proposition 1 *If $f(x)$ is cooperative and $<_r$ is as stated in Definition 1 above, then*

$$P_+ = \{x \in D \mid 0 <_r f(x)\}$$

is a positive invariant set, and any solution starting in this set is monotone so that any bounded solution here must converge to an equilibrium. (refer to [11] for the proof.)

It is easy to show that, in the case of our system, the point $(0, 0, 0)$ is in P_+, so the solution starting at the origin will be contained in P_+ for all $t > 0$. Thus, this solution is monotone but it cannot be bounded as the result above states that it would then converge to an equilibrium point inside $P_+ \subset D$. This contradicts the fact that there is no equilibrium point inside the domain D in this case. Hence, the solution starting at (0, 0, 0) is unbounded, so we must have

$$\lim_{t\to\infty} s_2(t) = \infty.$$

The component $c_2(t)$ is, however, both monotone and bounded and so must converge to a finite limit. We have $dc_2/dt = 0$, which implies

$$\frac{k_3}{k_1}\left(\frac{1}{1+\zeta} - c_2\right)s_2 - \frac{K}{s_0}c_2 = 0,$$

and as

$$\lim_{t\to\infty} s_2(t) = \infty,$$

we must have

$$\lim_{t\to\infty} c_2(t) = \frac{1}{1+\zeta}.$$

7 Results and Future Work

From the analysis presented in the previous subsections we conclude that the long term behaviour of system (3) is as follows,

$$\lim_{t\to\infty} c_1(t) = c_1^*, \quad \text{for all} \quad \zeta, \tag{38}$$

$$\lim_{t\to\infty} c_2(t) = \begin{cases} c_2^* = \dfrac{\zeta}{\zeta^*(1+\zeta)}, & \text{if } \zeta \le \zeta^* \quad (39) \\ \dfrac{1}{1+\zeta}, & \text{if } \zeta \ge \zeta^*, \quad (40) \end{cases}$$

$$\lim_{t\to\infty} s_2(t) = \begin{cases} s_2^*, & \text{if } \zeta < \zeta^* \quad (41) \\ \infty, & \text{if } \zeta \ge \zeta^*, \quad (42) \end{cases}$$

where c_1^*, c_2^* and s_2^* were defined in Eqs. (4), (5) and (6). We plot the steady state current, $k_4c_2(\infty)$, as a function of ζ for various values of s_0 (Fig. 2) and k_4/k_2 (Fig. 3). Note that the overlaying of curves in Fig. 2 for ζ values of 1 to 6 and in Fig. 3 for ζ values of 0 to 1, also note that from Eqs. (39) and (40), that the optimal GOX:HRP ratio is always given by

$$\zeta^* = \frac{k_4}{k_2}\left(1 + \frac{K_m^1}{s_0}\right),$$

as $c_2(\infty)$ achieves its maximum value of $1/(1+\zeta^*)$ when $\zeta = \zeta^*$. Hence, in this simple model which ignores the diffusion of the two substrates, it is possible to obtain an explicit formula which gives the optimal value of ζ in terms of the system parameters.

Note the agreement between the results as shown in Figs. 2 and 3.

In this paper, we have studied the behaviour of a bi-enzyme biosensor based on a cascade reaction, with particular emphasis placed on determining the value of the enzyme ratio which leads to optimal performance (characterised by maximum signal amplitude). Other mathematical models with vary complexity for analysing this optimisation problem will be constructed in the future, such as a "comprehensive model", where diffusion effects will be included for both substrates, glucose and hydrogen peroxide; and a "intermediate model", which will only consider the diffusion of the second substrate. A detailed comparison will be given of the three mathematical models (each neglecting different aspects of the biosensor functionality) and we will try to recommend the best modelling strategy under various physical conditions.

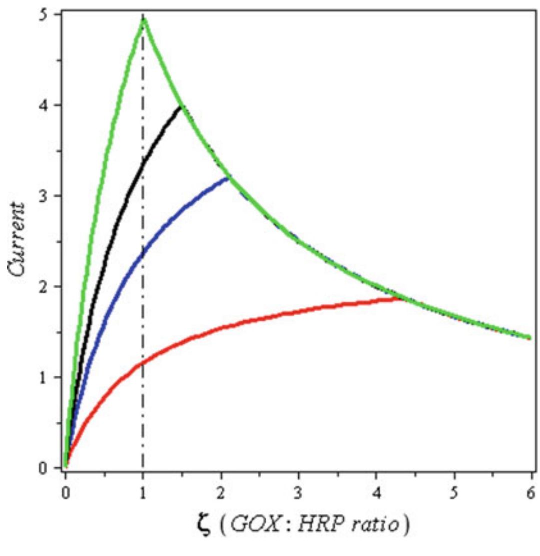

Fig. 2 Dependence of current on ζ for different initial concentrations of s_0. The curves correspond to $s_0 = 0.03$, 0.09, 0.2 and 5 mM from the bottom to top. Typical values for constants used in this simulation are: $k_1 = 10^2$, $k_{-1} = 10^{-1}$, $k_2 = 10$ and $k_4 = 10$

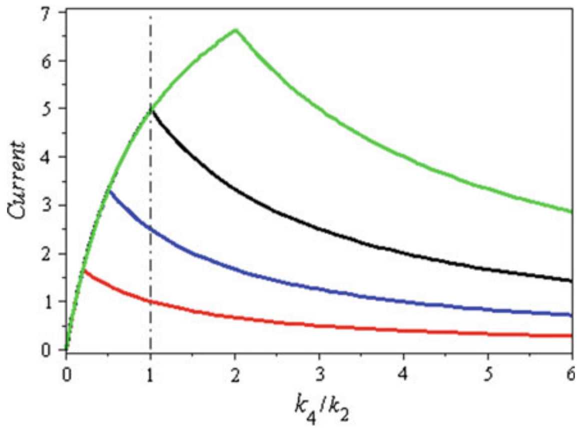

Fig. 3 Dependence of current on ζ for different values of k_4/k_2. The curves correspond to $k_4/k_2 =$ 0.2, 0.5, 1 and 2 from the bottom to top. Typical values for constants used in this simulation are the same as in Fig. 2

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Fibonacci Series with Several Parameters

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Abstract In this paper, we introduce higher order difference operator and its inverse by which we obtain x-Fibonacci sequence and its series with several theorems and results. Suitable examples verified by MATLAB are provided to illustrate our main results.

Keywords Higher order x-difference operator · Fibonacci sequence · Closed form solution · Fibonacci summation formula

Mathematics Subject Classification 2010 39A70 · 39A10 · 47B39 · 65J10 65Q10

1 Introduction

Fibonacci and Lucas numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy [5] and Vajda [9]. The k-Fibonacci sequence introduced by Falcon and Plaza [2] depends only on one integer parameter k and is defined as

$$F_{k,0} = 1, \quad F_{k,1} = 1 \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad \text{where} \quad n \geq 1,\ k \geq 1.$$

In particular, if $k = 2$, the Pell sequence is obtained as

$$P_0 = 0, \quad P_1 = 1 \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad \text{for} \quad n \geq 1.$$

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For $x=(x_1, x_2) \in R^2$, the Fibonacci sequence with two parameter is defined as $F_{(x,0)} = 1$, $F_{(x,1)} = x_1$ and $F_{(x,n)} = x_1 F_{(x,n-1)} + x_2 F_{(x,n-2)}$, where $n \geq 2$. When $x = (2, 1)$ and $F_0 = 0$, $F_1 = 1$ we get the Pell sequence. For third order Fibonacci sequence, let $x=(x_1, x_2, x_3) \in R^3$, the Fibonacci sequence with three parameters is defined as $F_{(x,0)} = 1$, $F_{(x,1)} = x_1 F_{(x,0)}$, $F_{(x,2)} = x_1 F_{(x,1)} + x_2 F_{(x,0)}$, $F_{(x,3)} = x_1 F_{(x,2)} + x_2 F_{(x,1)} + x_3 F_{(x,0)}$. These definitions motivate us to define x-Fibonacci sequence with several parameters using generalized x-difference operator. For more details on generalized difference operator one can refer [1, 3, 6–8].

2 x-Difference Operator and Fibonacci Sequence

In this section, we derive the sum of the terms in x-Fibonacci sequence using the inverse of x-difference operator on polynomial and geometric functions.

Definition 1 Let $x = (x_1, \cdots, x_m) \in R^m$, where R is the set of all real numbers. For each positive integer n, the higher order x-Fibonacci sequence is defined as $F_{(x,-n)} = 0$, $F_{(x,0)} = 1$, $F_{(x,1)} = x_1 F_{(x,0)}$, $F_{(x,2)} = x_1 F_{(x,1)} + x_2 F_{(x,0)}$ and in general

$$F_{(x,n)} = \sum_{i=1}^{m} x_i F_{(x,n-i)}, \quad F_{(x,0)} = 1 \quad and \quad F_{(x,-n)} = 0 \tag{1}$$

The following x-Fibonacci sequence

$F_{(x,n)} = \{1, 2, 3, 2, 0, 3, 25, 83, 167, 237, 252, \cdots\}$ is obtained by taking $m = 10$ and $x = (2, -1, -2, 3, 5, 4, 7, -4, 6, 7)$ in (1). Similarly, by taking other values for x, we can obtain corresponding x-Fibonacci sequence.

Definition 2 The x-difference operator $\underset{x}{\Delta}$ on $v(k)$ is defined as

$$\underset{x}{\Delta} v(k) = v(k) - x_1 v(k-1) - x_2 v(k-2) - \cdots - x_m v(k-m), \ k \in (-\infty, \infty). \tag{2}$$

$\underset{x}{\Delta} v(k) = u(k)$ is called x-difference equation and its inverse is defined as

$$if \quad \underset{x}{\Delta} v(k) = u(k), \quad then \quad v(k) = \overset{-1}{\underset{x}{\Delta}} u(k). \tag{3}$$

Lemma 1 *If* $a_{s,x} = 1 - \sum_{r=1}^{m} x_r a^{rs}$ *and* $a > 0$*, then*

$$\overset{-1}{\underset{x}{\Delta}} a^{-sk} = a_{s,x}^{-1} a^{-sk}, \qquad s, k \in (-\infty, \infty). \tag{4}$$

Proof Since $\underset{x}{\Delta}$ is linear, the proof follows from the Definition 2 and the relation $\underset{x}{\Delta} a^{-sk} = a^{-sk} a_{s,x}$.

Theorem 1 (Higher Order Fibonacci Summation Formula) *If $v(k)$ is a solution to the equation $\underset{x}{\Delta} v(k) = u(k)$, $k \in (-\infty, \infty)$, then $F_{(x,n)}$ in (1) satisfies the relation*

$$v(k) - \sum_{i=1}^{m}\sum_{j=i}^{m} x_j F_{(x,n+i-j)} v(k-(n+i)) = \sum_{i=0}^{n} F_{(x,i)} u(k-i), \tag{5}$$

where $F_{(x,0)} = 1$, $F_{(x,n)} = \sum_{j=1}^{m} x_j \; F_{(x,n-j)}$ *and* $F_{(x,s)} = 0$ *for* $s < 0$.

Proof For simplicity, we denote $x = (x_1, \cdots, x_m)$ and the relation $F_{(x,n)}$ by F_n in this theorem and proof.

From (2) and (3), we arrive

$$v(k) = u(k) + x_1 v(k-1) + x_2 v(k-2) + x_3 v(k-3) + \cdots + x_{m-1} v(k-(m-1)) + x_m v(k-m). \tag{6}$$

Replacing k by $k-1$ and then substituting the value of $v(k-1)$ in (6), we get

$$\begin{aligned} v(k) &= u(k) + x_1 u(k-1) + (x_1x_1 + x_2F_0)v(k-2) + (x_2x_1 + x_3F_0)v(k-3) \\ &+ (x_3x_1 + x_4F_0)v(k-4) + \cdots + (x_{m-1}x_1 + x_mF_0)v(k-m) + x_mx_1v(k-(m+1)), \end{aligned} \tag{7}$$

which gives, From (1),

$$\begin{aligned} v(k) &= \sum_{i=0}^{1} F_i u(k-i) + F_2 v(k-2) \\ &+ \sum_{i=2}^{m-1} (x_iF_1 + x_{i+1}F_0)v(k-(i+1)) + x_mF_1v(k-(m+1)) \end{aligned} \tag{8}$$

Replacing k by $k-2$ in (6) and then substituting $v(k-2)$ in (8), we obtain

$$\begin{aligned} v(k) &= \sum_{i=0}^{2} F_i u(k-i) + F_3 v(k-3) + \sum_{i=2}^{m-2} (x_iF_2 + x_{i+1}F_1 + x_{i+2}F_0)v(k-(i+2)) \\ &+ (x_{m-1}F_2 + x_mF_1)v(k-(m+1)) + x_mF_2v(k-(m+2)) \end{aligned}$$

Repeating this process again and again, we get (5).

Corollary 1 *Let $s \in (-\infty, \infty)$ and $a_{s,x} = 1 - \sum_{i=0}^{m} x_i a^{is} \neq 0$. Then we have*

$$1 - \sum_{i=1}^{m}\sum_{j=i}^{m} x_j F_{(x,n+i-j)} a^{s(n+i)} = a_{s,x} \sum_{i=0}^{n} a^{is} F_{(x,i)} \tag{9}$$

Proof The proof of (9) follows by taking $u(k) = a^{-sk}$ and applying (4) in (5).

The following example is an verification of (9).

Example 1 Taking $m = 4$, $n = 9$, $a = 3$, $x = (3, -5, 2, -1)$, $s = 2$, $3_{2,x} = 5482$, $F_x = \{1, 3, 4, -1, -18, -44, -48, 41, 293, 622, 531, -972, \cdots\}$ in (9), we get
$3_{2,x} \sum_{i=0}^{9} 3^{2i} F_{(x,i)} = 1 - F_{10}\, 3^{20} - (2F_8 - 5F_9 - F_7)\, 3^{22} - (2F_9 - F_8)\, 3^{24} + F_9\, 3^{26}$
$= 1391090991407362$.

```
syms a k s i n x1 x2 x3 x4 real
x1=3; x2=-5; x3=2; x4=-1; s=3; n=9; a=3;
fibf(1) = 1;    fibf(2) = x1*fibf(1);   fibf(3) = x1*fibf(2)+x2*fibf(1);
fibf(4) = x1*fibf(3)+x2*fibf(2)+x3*fibf(1);
for n = 5:20
fibf(n) = x1*fibf(n-1)+x2*fibf(n-2)+x3*fibf(n-3)+x4*fibf(n-4);    end
```

$LHS = (1 - 9 * x1 - 81 * x2 - 729 * x3 - 6561 * x4) * (fibf(1) + fibf(2) * a.\wedge 2 + fibf(3) * a.\wedge 4 + fibf(4) * a.\wedge 6 + fibf(5) * a.\wedge 8 + fibf(6) * a.\wedge 10. + fibf(7) * a.\wedge 12 + fibf(8) * a.\wedge 14 + fibf(9) * a.\wedge 16 + fibf(10) * a.\wedge 18)$
$RHS = (1 - (x1 * fibf(10) + x2 * fibf(9) + x3 * fibf(8) + x4 * fibf(7)) * a.\wedge 20 - (x2 * fibf(10) + x3 * fibf(9) + x4 * fibf(8)) * a.\wedge 22 - (x3 * fibf(10) + x4 * fibf(9)) * a.\wedge 24 - x4 * fibf(10) * a.\wedge 26)$

Corollary 2 *Let $x = (1, 1, \ldots, 1, 1)$ (m times) and $F_n = F_{(x,n)}$ be higher order Fibonacci numbers with respect to x given in (1). Then*

$$F_{n+1} + \sum_{j=0}^{m}(m - 1 - j)F_{n-i} - 1 = (m - 1)\sum_{i=0}^{n} F_i. \qquad (10)$$

Proof The proof follows by taking $a = 1$ and $x_j = 1$ in (4).

Theorem 2 *Assume that $1_{0,x} = 1 - \sum_{j=1}^{m} x_j \neq 0$ and $i_x = \sum_{j=1}^{m} x_j j^i$. Then a closed form solution of the x-difference equation $\underset{x}{\Delta}\, v(k) = k^p$ is given by*

$$1_{0,x} \overset{-1}{\underset{x}{\Delta}} k^p = k^p + \sum_{i=1}^{p} (-1)^i pC_i(i_x) \overset{-1}{\underset{x}{\Delta}} k^{p-i}, \quad p \geq 1, \qquad (11)$$

$$\overset{-1}{\underset{x}{\Delta}} k^0 = (1 - 0_x)^{-1}. \qquad (12)$$

Proof By taking $v(k) = k$ and k^2 in (2), we arrive

$$\underset{x}{\Delta} k = 1_{0,x}\, k + 1_x\, k^0, \quad \underset{x}{\Delta} k^2 = 1_{0,x}\, k^2 + 2(1_x)k - (2_x)k^0 \tag{13}$$

Taking $v(k) = k^3$ in (2) yields $\underset{x}{\Delta} k^3 = 1_{0,x}\, k^3 + 3(1_x)k^2 - 3(2_x)k + (3_x)k^0$.

Similarly, we arrive

$$\underset{x}{\Delta} k^p = 1_{0,x}\, k^p + \sum_{i=1}^{p} (-1)^{i+1} pC_i (i_x) k^{p-i}. \tag{14}$$

The proof of (11) follows by applying $\overset{-1}{\underset{x}{\Delta}}$ on both sides of above and using (12).

Corollary 3 *If* $1_{0,x} \neq 0$, *then* $\underset{x}{\Delta} v(k) = k^4$ *has a closed form solution of the difference equation*

$$\overset{-1}{\underset{x}{\Delta}} k^4 = \frac{k^4}{1_{0,x}} - \frac{4(1_x)}{1_{0,x}^2} k^3 + 6\left[\frac{2(1_x)^2}{1_{0,x}^3} + \frac{(2_x)}{1_{0,x}^2}\right] k^2 - 4\left[\frac{6(1_x)^3}{1_{0,x}^4} + \frac{6(1_x)(2_x)}{1_{0,x}^3} + \frac{(3_x)}{1_{0,x}^2}\right] k$$
$$+ \left[\frac{24(1_x)^4}{1_{0,x}^5} + \frac{36(1_x)^2(2_x)}{1_{0,x}^4} + \frac{8(1_x)(3_x)}{1_{0,x}^3} + \frac{6(2_x)^2}{1_{0,x}^3} + \frac{(4_x)}{1_{0,x}}\right] k^0. \tag{15}$$

Proof By putting $m = 4$ and $p = 4$ in (11), we derive

$$\overset{-1}{\underset{x}{\Delta}} k^4 = \frac{k^4}{1_{0,x}} - \frac{4(1_x)}{1_{0,x}} \overset{-1}{\underset{x}{\Delta}} k^3 + \frac{6(2_x)}{1_{0,x}} \overset{-1}{\underset{x}{\Delta}} k^2 - \frac{4(3_x)}{1_{0,x}} \overset{-1}{\underset{x}{\Delta}} k - \frac{4_x}{1_{0,x}} \overset{-1}{\underset{x}{\Delta}} k^0. \tag{16}$$

Applying (3) on the Eq. (13) and using (12), we find

$$\overset{-1}{\underset{x}{\Delta}} k = \frac{k}{1_{0,x}} - \frac{(1_x)}{1_{0,x}^2}, \quad \overset{-1}{\underset{x}{\Delta}} k^2 = \frac{k^2}{1_{0,x}} - \frac{2(1_x)k}{1_{0,x}^2} + \frac{2(1_x)^2}{1_{0,x}^3} + \frac{2_x}{1_{0,x}^2} \quad \text{and}$$
$$\overset{-1}{\underset{x}{\Delta}} k^3 = \frac{k^3}{1_{0,x}} - \frac{3(1_x)k^2}{1_{0,x}^2} + \frac{6(1_x)^2 k}{1_{0,x}^3} + \frac{3(2_x)k}{1_{0,x}^2} - \frac{6(1_x)^3}{1_{0,x}^4} - \frac{6(1_x)(2_x)}{1_{0,x}^3} - \frac{(3_x)}{1_{0,x}^2}.$$

Substituting the above values in (16) and using (12), we get (15).

Corollary 4 *If* $v(k) = \overset{-1}{\underset{x}{\Delta}} k^p$ *is the closed form solution given in (11), then*

$$v(k) - F_{(x,n+1)} v(k-(n+1)) - \sum_{i=2}^{m}\sum_{j=i}^{m} x_j F_{(x,n+i-j)} v(k-(n+i)) = \sum_{i=0}^{n} F_{(x,i)} (k-i)^p \tag{17}$$

Proof The proof follows by taking $u(k) = k^p$ in Theorem 1.

Example 2 By taking $m = 5$, $p = 4$, $k = 11$, $x = (-2, 3, 7, 5, 4)$, $n = 9$ and $F_x = \{1, -2, 7, -13, 38, -72, 194, -375, 966, -1907, 4769, \cdots\}$ in (17), we obtain

$$\sum_{i=0}^{9} F_{(x,i)}(11-i)^4 =$$
$$v(11) - 4769v(1) + 58v(0) + 10019v(-1) + 5671v(-2) + 7628v(-3),$$

$v(k) = -\frac{1}{16}k^4 - \frac{65}{64}k^3 - \frac{6603}{1024}k^2 - \frac{166483}{8192}k - \frac{3785627}{131072}$ is given in Corollary 3.

The Fibonacci summation formula can also be obtained from (17).

Theorem 3 *Let $u(k)$ and $v(k)$ be any two- real valued functions. Then*

$$\overset{-1}{\underset{x}{\Delta}}\big(u(k)v(k)\big) = u(k)\overset{-1}{\underset{x}{\Delta}}v(k) - \sum_{i=1}^{m} x_i \overset{-1}{\underset{x}{\Delta}}\big(\overset{-1}{\underset{x}{\Delta}}v(k-i)\underset{e_i}{\Delta}u(k)\big). \tag{18}$$

where $e_i = (0, 0, 0, \cdots, 1, \cdots, 0)$, ith component is 1 other component are zero.

Proof From (2) and taking $x = (x_1, \cdots, x_m)$ we arrive the relation

$$\underset{x}{\Delta}\big(u(k)w(k)\big) = u(k)w(k) - \sum_{i=1}^{m} x_i u(k-i)w(k-i). \tag{19}$$

Adding and subtracting $x_1u(k)w(k-1), x_2u(k)w(k-2), \cdots, x_mu(k)w(k-m)$ on the right side, we get

$$\underset{x}{\Delta}\big(u(k)w(k)\big) = u(k)\underset{x}{\Delta}w(k) + \sum_{i=1}^{m} x_i w(k-i)\underset{e_i}{\Delta}u(k) \tag{20}$$

Taking $w(k) = \overset{-1}{\underset{x}{\Delta}}v(k)$ in (20) and applying $\overset{-1}{\underset{x}{\Delta}}$ on both sides, give (18).

Corollary 5 *A closed form solution of the higher order difference equation $v(k) - x_1v(k-1) - x_2v(k-2) - x_3v(k-3) - \cdots - x_mv(k-m) = k^2a^{-sk}$ is given by*

$$\overset{-1}{\underset{x}{\Delta}}k^2a^{-sk} = a^{-sk}a_{s,x}^{-1}k^2 - 2a_{s,x}^{-2}\sum_{j=1}^{m} jx_ja^{-s(k-j)}k$$
$$+ 2a_{s,x}^{-3}\sum_{i=1}^{m} ix_i\sum_{j=1}^{m} jx_ja^{-s(k-i-j)} + a_{s,x}^{-2}\sum_{j=1}^{m} j^2x_ja^{-s(k-j)}. \tag{21}$$

Proof Taking $u(k) = k$ and $v(k) = a^{-sk}$ in (18), using (4) and (13), we find

$$\overset{-1}{\underset{x}{\Delta}} ka^{-sk} = ka^{-sk}a_{s,x}^{-1} - a_{s,x}^{-2}\sum_{i=1}^{m} ix_i \; a^{-s(k-i)}. \tag{22}$$

Again, taking $u(k) = k^2$ and $v(k) = a^{-sk}$ in (18), using (4) and (13), we arrive

$$\begin{aligned}\overset{-1}{\underset{x}{\Delta}} k^2a^{-sk} &= a_{s,x}^{-1}k^2a^{-sk}a_{s,x} \\ &\quad - 2a_{s,x}^{-1}a_{s,x}(x_1a^s + 2x_2a^{2s} + 3x_3a^{3s} + \cdots + mx_ma^{ms}) \overset{-1}{\underset{x}{\Delta}} ka^{-sk} \\ &\quad + a_{s,x}^{-1}a_{s,x}\left\{(1)^2x_1a^s + (2)^2x_2a^{2s} + (3)^2x_3a^{3s} + \cdots + (m)^2x_ma^{ms}\right\} \overset{-1}{\underset{x}{\Delta}} k^0a^{-sk}\end{aligned} \tag{23}$$

Substituting (22) in (23) and using (4), we get (21).

Corollary 6 *If* $v(k) = \overset{-1}{\underset{x}{\Delta}} k^2a^{-sk}$ *is the closed form solution given in (21), then*

$$a^{sk}\left\{v(k) - \sum_{i=1}^{m}\sum_{j=i}^{m} x_j F_{(x,n+i-j)}v(k-(n+i))\right\} = \sum_{i=0}^{n} F_{(x,i)}(k-i)^2a^{is}. \tag{24}$$

Proof The proof follows by taking $u(k) = k^2a^{-sk}$ in Theorem 1.

Example 3 By taking $m = 5$, $k = 7$, $a = 3$, $s = 3$, $n = 5$, $x = (4, -5, 2, 3, 7)$ and $F_x = \{1, 4, 11, 26, 60, 151, 417, 1188, 3331, 9091, \cdots\}$ in Corollary 6, we arrive

$$\begin{aligned}&\sum_{i=0}^{5} F_{(x,i)}(7-i)^2 3^{3i} = \\ &3^{-21}\big(v(7) - 417v(1) + 480v(0) - 664v(-1) - 873v(-2) - 1057v(-3)\big) \\ &\qquad = 8962110508 \text{ here } v(k) \text{ is taken from (21).}\end{aligned}$$

Corollary 7 *Let p be any real number. Then closed form solutions of the equations* $\underset{x}{\Delta} v(k) = a^{-sk}\cos pk$ *and* $\underset{x}{\Delta} v(k) = a^{-sk}\sin pk$ *are respectively*

$$\overset{-1}{\underset{x}{\Delta}} a^{-sk}\cos pk = a^{-sk}cp_{k,k+i}(cp_{0,i}^2 - sp_{0,-i}^2)^{-1} \tag{25}$$

and

$$\overset{-1}{\underset{x}{\Delta}} a^{-sk} \sin pk = a^{-sk} sp_{k,k+i} (cp_{0,i}^2 - sp_{0,-i}^2)^{-1}. \tag{26}$$

where $cp_{k,k+i} = \cos pk - \sum_{i=1}^{m} x_i \cos p(k+i)$, $sp_{k,k+i} = \sin pk - \sum_{i=1}^{m} x_i \sin p(k+i)$.

Proof Replacing $u(k)$ by $\sin pk$ and $\cos pk$ in (2) and using (3) we derive

$$\underset{x}{\Delta} a^{-sk} \cos pk = a^{-sk} cp_{k,k+i} \tag{27}$$

$$\underset{x}{\Delta} a^{-sk} \sin pk = a^{-sk} sp_{k,k+i} \tag{28}$$

Now, then multiplying on both sides (27) by $cp_{0,i}^2$, (28) by $-sp_{0,-i}^2$ and adding new two equation we arrive

$$\underset{x}{\Delta} a^{-sk} \big(cp_{0,i}^2 \cos pk - sp_{0,-i} \sin pk\big) = a^{-sk} (cp_{0,i}^2 - sp_{0,-i}^2) \cos pk. \tag{29}$$

Now, the proof of (25) follows by taking $\overset{-1}{\underset{x}{\Delta}}$ on both sides.

Similarly, (26) follows by multiplying (28) by $cp_{0,i}^2$ and (27) by $-sp_{0,-i}^2$ and solving new equation by subtraction.

Corollary 8 *If* $v(k) = \overset{-1}{\underset{x}{\Delta}} a^{-sk} \cos pk$ *is the closed form solution given in (25), then we have a Fibonacci summation formula*

$$a^{sk} \left\{ v(k) - \sum_{i=1}^{m} \sum_{j=i}^{m} x_j F_{(x,n+i-j)} v(k-(n+i)) \right\} = \sum_{i=0}^{n} F_{(x,i)} a^{is} \cos p(k-i) \tag{30}$$

and $v(k) = \overset{-1}{\underset{x}{\Delta}} a^{-sk} \sin pk$ *is the closed form solution given in (26), then we have*

$$a^{sk} \left\{ v(k) - \sum_{i=1}^{m} \sum_{j=i}^{m} x_j F_{(x,n+i-j)} v(k-(n+i)) \right\} = \sum_{i=0}^{n} F_{(x,i)} a^{is} \sin p(k-i) \tag{31}$$

Proof The proof follows by taking $u(k) = a^{-sk} \cos pk$ and $u(k) = a^{-sk} \sin pk$ in Theorem 1.

Corollary 9 *Let p be any real number. Then* $\underset{x}{\Delta} v(k) = \cos pk$ *is given by*

$$\overset{-1}{\underset{x}{\Delta}} \cos pk = cp_{k,k+i} (cp_{0,i}^2 - sp_{0,-i}^2)^{-1} \tag{32}$$

and $\underset{x}{\Delta} v(k) = \sin pk$ *is given by*

$$\underset{x}{\overset{-1}{\Delta}} a^{-sk} \sin pk = a^{-sk} sp_{k,k+i} (cp_{0,i}^2 - sp_{0,-i}^2)^{-1}. \tag{33}$$

Proof The proof follows by taking $a = 1$ in Corollary 7 for (25) and (26).

Corollary 10 *If* $v(k) = \underset{x}{\overset{-1}{\Delta}} \cos pk$ *is the closed form solution given in (32), then*

$$v(k) - \sum_{i=1}^{m}\sum_{j=i}^{m} x_j F_{(x,n+i-j)} v(k-(n+i)) = \sum_{i=0}^{n} F_{(x,i)} \cos p(k-i) \tag{34}$$

and $v(k) = \underset{x}{\overset{-1}{\Delta}} \sin pk$ *is the closed form solution given in (33), then*

$$v(k) - \sum_{i=1}^{m}\sum_{j=i}^{m} x_j F_{(x,n+i-j)} v(k-(n+i)) = \sum_{i=0}^{n} F_{(x,i)} \sin p(k-i) \tag{35}$$

Proof The proof follows by taking $u(k) = \cos pk$ and $\sin pk$ in Theorem 1.

Example 4 By taking $v(k)$ is as given in (25), $m = 5$, $k = 11$, $s = 3$, $p = -5$, $n = 6$, $a = 5$ and $x = (-3, 5, 9, 3, 7)$ in Corollary 8, we derive

$$5^{33}\big(v(11) + 9557v(4) - 7045v(3) - 22645v(2) - 2989v(1) - 18179v(0)\big)$$
$$= \sum_{i=0}^{6} F_{(x,i)} 5^{3i} \cos(-5(11-i)) = 9816345970062038.8990965070872297.$$

where
$F_x = \{1, -3, 14, -48, 190, -686, 2597, -9557, 35716, -132288, 492420, \cdots\}$

Similarly, by taking $m = 4$, $k = -9$, $s = 3$, $p = 5$, $n = 7$, $a = 3$, $x = (5, 2, -7, 6)$ and $F_x = \{1, 5, 27, 138, 715, 3692, 19086, 98637, \cdots\}$ in Corollary 8, we obtain

$$5^{-27}\big(v(-9) - 509803v(-17) - 85824v(-18) + 575943v(-19) - 591822v(-20)\big)$$
$$= \sum_{i=0}^{7} F_{(x,i)} 3^{3i} \sin(-5(11-i)) = 1028298372860206.625 \ (v(k) \text{ is given in } (26))$$

Conclusion: We obtained summation formula to higher order Fibonacci sequence by introducing higher order x-difference operator and have derived certain results on closed and summation form solution of higher order difference equation which will be used to our further research.

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Comparison Theorems for Second-Order Damped Nonlinear Differential Equations

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Abstract In this paper, we present comparison theorems for the oscillation of solutions of second-order damped nonlinear differential equations with p-Laplacian. Proof is given by means of phase plane analysis of systems. Moreover, combining the comparison theorem and (non)oscillation criteria for the generalized Euler differential equation, we give new (non)oscillation criteria for the damped equations.

Keywords Oscillation · Comparison theorem · Phase plane analysis · p-Laplacian

1 Introduction

Consider the nonlinear differential equation

$$(t^{\alpha-1}\Phi(x'))' + t^{\alpha-1-p} f(x) = 0, \quad t > 0, \tag{1}$$

where $\Phi(x)$ is a real-valued function defined by $\Phi(x) = |x|^{p-2}x$ with $p > 1, \alpha \neq p$, and $f(x)$ is a continuous function on $\mathbb{R}$ satisfying the signum condition

$$xf(x) > 0 \quad \text{if} \quad x \neq 0, \tag{2}$$

and a suitable smoothness condition to ensure the uniqueness of solutions of Eq. (1) to the initial value problem. Then each solution of Eq. (1) and its derivative exist in the future (for the proof, see [11, Theorem C] and [13, Proposition A]). Hence, we can discuss the asymptotic behavior near infinity of all solutions of Eq. (1).

A nontrivial solution of Eq. (1) is said to be oscillatory if there exists a sequence $\{t_n\}$ tending to infinity such that $x(t_n) = 0$. Otherwise, it is said to be nonoscillatory. Hence, a nonoscillatory solution of Eq. (1) satisfies that $x(t) > 0$ for t sufficiently large or $x(t) < 0$ for t sufficiently large.

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It is known that the differential equation

$$(\Phi(x'))' + \frac{1}{t^p} f(x) = 0 \tag{3}$$

and the damped differential equation

$$(\Phi(x'))' + \frac{2(p-1)}{t}\Phi(x') + \frac{1}{t^p} f(x) = 0 \tag{4}$$

play an essential role in oscillation theory for Eq. (1). In fact, changing variable $s = t^{|p-\alpha|/(p-1)}$, we can transform equation (1) into the nonlinear equations of the form (3) (resp., (4)) if $p > \alpha$ (resp., $p < \alpha$). For example, see the proof of Lemma 3.1 in [4]. Hence, in order to discuss the oscillation problem for Eq. (1), it suffices to consider Eqs. (3) and (4).

As for the Eq. (3), the research for the oscillatory behavior was started by Sugie and Hara [6]. They discussed oscillation problem for Eq. (3) with $p = 2$ and gave (non)oscillation criteria. After that their results were improved by many authors. For example, those results can be found in [1, 2, 4–15] and the references cited therein. In particular, the author [14] presented a comparison theorem for the oscillation of Eq. (3). In order to give the comparison theorem, we consider the equation

$$(\Phi(x'))' + \frac{1}{t^p} g(x) = 0, \tag{5}$$

where $g(x)$ is a continuous function on $\mathbb{R}$ satisfying the signum condition

$$xg(x) > 0 \quad \text{if} \quad x \neq 0. \tag{6}$$

Theorem 1 *Assume* (2) *and* (6). *Suppose that there exists $L > 0$ such that*

$$\int_{L \operatorname{sgn} x}^{x} f(\chi)d\chi \leq \int_{L \operatorname{sgn} x}^{x} g(\chi)d\chi \tag{7}$$

for $x > L$ (resp., $x < -L$). If Eq. (5) *has a positive solution (resp., negative solution), then all nontrivial solutions of Eq.* (3) *are nonoscillatory.*

From Lemmas 3.1 and 3.3 in [4], we see that all nontrivial solutions of Eq. (3) (resp., (4)) are unbounded (resp., tend to zero as $t \to \infty$). Therefore, the asymptotic behavior of Eqs. (3) and (4) are substantially different. Here a natural question now arises. What is a comparison theorem for the oscillation of Eq. (4)? The purpose of this paper is to answer the question. To this end, let us consider the damped nonlinear differential equation

$$(\Phi(x'))' + \frac{2(p-1)}{t}\Phi(x') + \frac{1}{t^p} g(x) = 0. \tag{8}$$

Then we have the following result.

Theorem 2 *Assume* (2) *and* (6). *Suppose that there exists* $\varepsilon > 0$ *such that*

$$\int_0^x f(\chi)d\chi \le \int_0^x g(\chi)d\chi \tag{9}$$

for $0 < x < \varepsilon$ *(resp.,* $-\varepsilon < x < 0$*). If Eq.* (8) *has a positive solution (resp., negative solution), then all nontrivial solutions of Eq.* (4) *are nonoscillatory.*

We suppose that $f(x)$ and $g(x)$ satisfy (9) for $|x|$ sufficiently small and that Eq. (8) has a nonoscillatory solution. Then the nonoscillatory solution is eventually positive or negative. Hence, by Theorem 2, we see that all nontrivial solutions of Eq. (4) are nonoscillatory. Therefore, we also have the following result.

Corollary 1 *Assume* (2) *and* (6). *Suppose that there exists* $\varepsilon > 0$ *such that* (9) *holds for* $0 < |x| < \varepsilon$. *If Eq.* (4) *has a nontrivial oscillatory solution, then all nontrivial solutions of Eq.* (8) *are oscillatory.*

Remark 1 In the case that $f(x) \equiv g(x)$, $f(x)$ and $g(x)$ satisfy (9) for $x \in \mathbb{R}$. It follows from Theorem 2 and Corollary 1 that oscillatory solutions and nonoscillatory solutions of Eq. (4) do not coexist if (2) holds. Hence, all nontrivial solutions of Eq. (4) are either oscillatory or nonoscillatory.

2 Proof of the Main Theorem

Let $s = \log t$ and $u(s) = x(t)$. Then Eq. (4) is transformed into the equation

$$(\Phi(\dot{u}))^{\cdot} + (p-1)\Phi(\dot{u}) + f(u) = 0, \qquad {}^{\cdot} = \frac{d}{ds},$$

which is equivalent to the system

$$\begin{cases} \dot{u} = \Phi^{-1}(v), \\ \dot{v} = -(p-1)v - f(u). \end{cases} \tag{10}$$

Here Φ^{-1} is the inverse function of Φ, i.e., $\Phi^{-1}(v) = |v|^{q-2}v$, where q is the number satisfying $(p-1)(q-1) = 1$.

To prove Theorem 2, we use phase plane analysis. Phase plane analysis is frequently carried out for the purpose of examining the asymptotic behavior of solutions of system (10). We call the projection of a positive semi-trajectory of system (10) onto the phase plane a *positive orbit*.

Using the following lemmas, we can prove Theorem 2.

Lemma 1 ([4, Lemma 3.1]) *Assume* (6) *and suppose that Eq.* (8) *has a positive solution. Then the solution is decreasing for t sufficiently large and it tends to zero as $t \to \infty$.*

Lemma 2 ([4, Lemmas 3.2 and 3.3]) *Assume* (2) *and suppose that Eq.* (4) *has a nontrivial oscillatory solution. Then the positive orbit of system* (10) *corresponding to the solution rotates around the origin in the clockwise direction as s increases, and it tends to the origin as $s \to \infty$.*

We are now ready to prove our main theorem.

Proof (*Proof of Theorem* 2) We give the proof only the case that there exists $\varepsilon > 0$ such that

$$\int_0^x f(\chi)d\chi \le \int_0^x g(\chi)d\chi \quad \text{for } \; 0 < x < \varepsilon, \tag{11}$$

and that Eq. (8) has a positive solution, because the other case is carried out in the same manner.

Let $x(t)$ be a positive solution of Eq. (8). Then, from Lemma 1, we have $0 < x(t) < \varepsilon$ and $x'(t) < 0$ for t sufficiently large. Furthermore, let $(u(s), v(s))$ be the solution of the system

$$\begin{cases} \dot{u} = \Phi^{-1}(v), \\ \dot{v} = -(p-1)v - g(u) \end{cases} \tag{12}$$

corresponding to $x(t)$. Then, we have $0 < u(s) = x(t) < \varepsilon$ and $v(s) = \Phi(\dot{u}(s)) = \Phi(tx'(t)) < 0$, and therefore, we see that

$$(u(s), v(s)) \in D := \{(u, v) : 0 < u < \varepsilon \; \text{ and } \; v < 0\}$$

for s sufficiently large. We next show that $(u(s), v(s))$ tends to the origin O as $s \to \infty$. Let

$$V(u, v) = \frac{1}{q}|v|^q + \int_0^u g(\chi)d\chi.$$

Then we see that

$$\frac{d}{ds}V(u(s), v(s)) = \Phi^{-1}(v(s))\dot{v}(s) + g(u(s))\dot{u}(s) = -(p-1)|v(s)|^q < 0$$

for s sufficiently large. Since $V(u(s), v(s))$ is nonnegative and decreasing, the function $V(u(s), v(s))$ has the limit as $s \to \infty$. Hence, $v(s)$ also has the limit $v_0 \le 0$ as $s \to \infty$ because $u(s) \to 0$ as $s \to \infty$. Suppose that $v_0 < 0$. Then we have $v(s) < v_0/2$ for s sufficiently large. Hence, we get

$$\dot{u}(s) = \Phi^{-1}(v(s)) < \Phi^{-1}(v_0/2) < 0$$

for s sufficiently large, and therefore, $u(s) \to -\infty$ as $s \to \infty$. This is a contradiction to the assumption that $u(s)$ is positive for s sufficiently large. Thus, we see that the positive orbit Γ corresponding to $(u(s), v(s))$ stay in the domain D for s sufficiently large, and it tends to the origin O as $s \to \infty$.

By way of contradiction, we suppose that Eq. (4) has a nontrivial oscillatory solution. Then, from Lemma 2, the positive orbit γ corresponding to the oscillatory solution rotates around the origin clockwise and it tends to the origin O. Hence, the positive orbit γ meets the positive orbit Γ at the point $P = (u_1, v_1) \in D$. After that the positive orbit γ crosses the negative v-axis at the point $Q = (0, v_2)$.

Taking into account the vector fields of (10) and (12) on D, the positive orbits γ and Γ can be regarded as the graphs of $v = \xi(u)$ and $v = \eta(u)$, which are solutions of the equations

$$\frac{d}{du}\xi(u) = \frac{-(p-1)\xi(u) - f(u)}{\Phi^{-1}(\xi(u))} \quad \text{and} \quad \frac{d}{du}\eta(u) = \frac{-(p-1)\eta(u) - g(u)}{\Phi^{-1}(\eta(u))},$$

respectively. Future, we see that

$$\xi(u) < \eta(u) < 0 \quad \text{for} \quad 0 < u < u_1, \tag{13}$$

$$\xi(u_1) = \eta(u_1), \tag{14}$$

because the arc OP of Γ lies above the arc QP of γ for $0 < u < u_1$, and γ and Γ cross at the point P. Using $\lim_{u\to+0} \xi(u) = v_2$, $\lim_{u\to+0} \eta(u) = 0$, (11), and (13), we have

$$\begin{aligned}
\frac{1}{q}\left(|\xi(u_1)|^q - |v_2|^q\right) &= \frac{1}{q}\left(|\xi(u_1)|^q - \lim_{u\to+0}|\xi(u)|^q\right) = \int_0^{u_1} \Phi^{-1}(\xi(u))\frac{d}{du}\xi(u)du \\
&= -\int_0^{u_1}(p-1)\xi(u)du - \int_0^{u_1} f(u)du \\
&\geq -\int_0^{u_1}(p-1)\eta(u)du - \int_0^{u_1} g(u)du \\
&= \int_0^{u_1} \Phi^{-1}(\eta(u))\frac{d}{du}\eta(u)du = \frac{1}{q}\left(|\eta(u_1)|^q - \lim_{u\to+0}|\eta(u)|^q\right) \\
&= \frac{1}{q}|\eta(u_1)|^q.
\end{aligned}$$

From (14) and $v_2 < 0$, we obtain $0 \leq -|v_2|^q/q < 0$ which is a contradiction. The proof is now complete.

3 Oscillation and Nonoscillation Criteria

In this section, using Theorem 2 and Corollary 1, we give (non)oscillation criteria for Eq. (4).

To begin with, we consider a simple case. Let $f(x) = \delta\Phi(x)$, where $\delta > 0$. Then Eq. (4) becomes the half-linear differential equation

$$(\Phi(x'))' + \frac{2(p-1)}{t}\Phi(x') + \frac{\delta}{t^p}\Phi(x) = 0, \tag{15}$$

which is called the generalized Euler differential equation. It is known that all nontrivial solutions of Eq. (15) are oscillatory if

$$\delta > \gamma_p := \left(\frac{p-1}{p}\right)^p;$$

otherwise they are nonoscillatory, see e.g., [3, 4]. These (non)oscillation criteria are very important and influence various oscillation criteria for nonlinear differential equations of the form (4). In fact, using our results and the oscillation criteria for Eq. (15), we can derive a pair of an oscillation theorem and a nonoscillation theorem for Eq. (4).

Corollary 2 *Assume* (2) *and suppose that there exist* $\varepsilon > 0$ *and* $\lambda > \gamma_p/p$ *such that*

$$\frac{1}{|x|^p}\int_0^x f(\chi)d\chi \geq \lambda \tag{16}$$

for $0 < |x| < \varepsilon$. *Then all nontrivial solutions of equation* (4) *are oscillatory.*

Proof Using (16), we see that

$$\int_0^x f(\chi)d\chi \geq \lambda|x|^p = \int_0^x p\lambda\Phi(\chi)d\chi$$

for $0 < |x| < \varepsilon$. Here we replace $f(x)$ and $g(x)$ in Corollary 1 with $p\lambda\Phi(x)$ and $f(x)$, respectively. Note that $p\lambda > \gamma_p$. Since all nontrivial solutions of Eq. (15) with $\delta > \gamma_p$ are oscillatory, all conditions for Corollary 1 hold, and therefore, we see that all nontrivial solutions of Eq. (4) are oscillatory.

Corollary 3 *Assume* (2) *and suppose that there exists* $\varepsilon > 0$ *such that*

$$\frac{1}{|x|^p}\int_0^x f(\chi)d\chi \leq \frac{\gamma_p}{p} \tag{17}$$

for $0 < x < \varepsilon$ *or* $-\varepsilon < x < 0$. *Then all nontrivial solutions of Eq.* (4) *are nonoscillatory.*

Proof We prove only the case that condition (17) holds for $0 < x < \varepsilon$, because the other case is carried out in the same manner. From (17), we have

$$\int_0^x f(\chi)d\chi \le \frac{\gamma_p}{p}|x|^p = \int_0^x \gamma_p \Phi(\chi)d\chi$$

for $0 < x < \varepsilon$. Furthermore, Eq. (15) with $\delta = \gamma_p$ has the positive solution $x(t) = t^{-(p-1)/p}$. Hence, from Theorem 2, all nontrivial solutions of Eq. (4) are nonoscillatory.

We next consider the case that $f(x)$ satisfies

$$\frac{f(x)}{\Phi(x)} = \gamma_p + \frac{\lambda}{(\log^2 |x|^{p/(p-1)})}$$

for $|x|$ sufficiently small. Let

$$\mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

Then, Došlý and the author [4] showed that all nontrivial solutions of Eq. (4) are oscillatory (resp., nonoscillatory) if $\lambda > \mu_p$ (resp., $\lambda \le \mu_p$). Thus, using Theorem 2 and Corollary 1, we have the following (non)oscillation criteria (we omit the proof).

Corollary 4 *Assume* (2) *and suppose that there exist* $\varepsilon > 0$ *and* $\lambda > \mu_p$ *such that*

$$\int_0^x f(\chi)d\chi \ge \int_0^x \left(\gamma_p + \frac{\lambda}{(\log^2 |\chi|^{p/(p-1)})}\right)\Phi(\chi)d\chi$$

for $0 < |x| < \varepsilon$. *Then all nontrivial solutions of Eq.* (4) *are oscillatory.*

Corollary 5 *Assume* (2) *and suppose that there exists* $\varepsilon > 0$ *such that*

$$\int_0^x f(\chi)d\chi \le \int_0^x \left(\gamma_p + \frac{\mu_p}{(\log^2 |\chi|^{p/(p-1)})}\right)\Phi(\chi)d\chi$$

for $0 < |x| < \varepsilon$. *Then all nontrivial solutions of Eq.* (4) *are nonoscillatory.*

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On Conditions for Weak Conservativeness of Regularized Explicit Finite-Difference Schemes for 1D Barotropic Gas Dynamics Equations

A. Zlotnik and T. Lomonosov

Abstract We consider explicit two-level three-point in space finite-difference schemes for solving 1D barotropic gas dynamics equations. The schemes are based on special quasi-gasdynamic and quasi-hydrodynamic regularizations of the system. We linearize the schemes on a constant solution and derive the von Neumann type necessary condition and a CFL type criterion (necessary and sufficient condition) for weak conservativeness in L^2 for the corresponding initial-value problem on the whole line. The criterion is essentially narrower than the necessary condition and wider than a sufficient one obtained recently in a particular case; moreover, it corresponds most well to numerical results for the original gas dynamics system.

Keywords Gas dynamics · 1D barotropic quasi-gas dynamics system of equations · Explicit finite-difference schemes · Stability criterion · Weak conservativeness

1 Introduction

The stability theory for finite-difference schemes for model problems in gas dynamics is well presented in the literature [1, 4–7, 11, 12]. In this paper we consider some finite-difference schemes for solving 1D barotropic gas dynamics equations. The schemes are explicit, two-level in time and use a symmetric three-point stencil in space. Their construction is based on special quasi-gasdynamic and quasi-hydrodynamic [3, 8, 13, 15, 20] regularizations of the original equations (without a regularization, the schemes are unstable). The schemes of this kind were successfully applied in numerous and various practical applications, in particular, see [2, 9, 10, 21], but their theory is not developed so well.

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We linearize the schemes on a constant solution and derive both the von Neumann type necessary condition and a CFL type criterion for weak conservativeness in L^2 for the corresponding initial-value problem on the whole line. The weak conservativeness in L^2 means the uniform in time bound for the norm of scaled solution by the norm of initial data instead of the energy conservation law for the linearized original system, i.e., the acoustics system of equations. Our numerical experience show that validity of the weak conservativeness property is important since it prevents numerical solutions from the well-known possible spurious oscillations. The property guarantees the uniform in time L^2 stability with respect to initial data.

Since in practice necessary conditions are often in use (a derivation of sufficient conditions is much more complicated in general), it is important to know to what extent this is lawful to do. The criterion turns out to be essentially narrower than the necessary condition and at the same time wider than a sufficient condition obtained recently in a particular case in [14]. Moreover, namely the criterion corresponds most well to results of numerical experiments for the original gas dynamics system. Therefore the criterion (but not the necessary condition or sufficient one) is most adequate and useful for practical purposes.

2 Systems of Equations, Finite-Difference Schemes and Their Linearization

The 1D barotropic gas dynamics (Euler) system of equations consists in the mass and momentum balance equations

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad \partial_t(\rho u) + \partial_x p(\rho) = 0, \tag{1}$$

where $\rho > 0$, u and p are the gas density and velocity (the sought functions) and pressure. We assume that $p'(\rho) > 0$ and consider the equations for $x \in \mathbb{R}$ and $t > 0$.

The 1D barotropic quasi-gas dynamics (QGD) system of equations consists in the regularized mass and momentum balance equations

$$\partial_t \rho + \partial_x j = 0, \quad \partial_t(\rho u) + \partial_x\big(ju + p(\rho) - \Pi\big) = 0, \tag{2}$$

$$j = \rho(u - w), \quad w = \frac{\tau}{\rho} u \partial_x(\rho u) + \hat{w}, \quad \hat{w} = \frac{\tau}{\rho}[\rho u \partial_x u + p'(\rho)], \tag{3}$$

$$\Pi = \Pi_{NS} + \rho u \hat{w} + \tau p'(\rho)\partial_x(\rho u), \quad \Pi_{NS} = \mu(\rho)\partial_x u. \tag{4}$$

Here j and Π are the regularized mass flux and stress, w and $\hat{w}$ are the regularizing velocities, $\tau = \tau(\rho) > 0$ is a regularization parameter and Π_{NS} is the Navier–Stokes viscous stress with $\mu(\rho) \geq 0$ being proportional to the viscosity coefficient. In the barotropic case, quasi-gasdynamic and quasi-hydrodynamic systems were introduced and investigated (in multidimensional case) in [16, 17, 20].

The QGD system is simplified into the original system (1) for $\tau = \mu = 0$ and the Navier-Stokes system of equations for viscous compressible barotropic gas flow for $\tau = 0$ and $\mu > 0$.

System (1) can be linearized on a constant solution $\rho_* \equiv \text{const} > 0$ and $u_* = 0$. Substituting the solution in the form $\rho = \rho_* + \Delta\rho$ and $u = u_* + \Delta u$ in the equations and neglecting the terms having the second order of smallness with respect to $\Delta\rho$ and Δu and their derivatives leads us to the following system of equations:

$$\partial_t \Delta\rho + \rho_* \partial_x \Delta u = 0, \quad \rho_* \partial_t \Delta u + p'(\rho_*) \partial_x \Delta\rho u = 0. \tag{5}$$

For the dimensionless unknowns $\tilde{\rho} = \frac{\Delta\rho}{\rho_*}$ and $\tilde{u} = \frac{\Delta u}{\sqrt{p'(\rho_*)}}$ we gain the acoustics system of equations:

$$\partial_t \tilde{\rho} + c_* \partial_x \tilde{u} = 0, \quad \partial_t \tilde{u} + c_* \partial_x \tilde{\rho} = 0. \tag{6}$$

Hereafter $c_* = \sqrt{p'(\rho_*)}$ is the background velocity of sound. Given the initial data $\tilde{\rho}|_{t=0} = \tilde{\rho}_0$ and $\tilde{u}|_{t=0} = \tilde{u}_0$ (that one can consider complex-valued), for the solution to the last system the following energy conservation law holds

$$\|\tilde{\rho}(\cdot, t)\|^2_{L^2(\mathbb{R})} + \|\tilde{u}(\cdot, t)\|^2_{L^2(\mathbb{R})} = \|\tilde{\rho}_0\|^2_{L^2(\mathbb{R})} + \|\tilde{u}_0\|^2_{L^2(\mathbb{R})} \ \text{ for } \ t \geq 0. \tag{7}$$

Now we pass to discretization. Let ω_h be a uniform mesh on $\mathbb{R}$ with the nodes $x_k = kh$, $k \in \mathbb{Z}$, and step $h = X/N$. Let ω_h^* be an auxiliary mesh with the nodes $x_{k+1/2} = (k + 0.5)h$, $k \in \mathbb{Z}$. Define a uniform mesh in t with the nodes $t_m = m\Delta t$, $m \geq 0$, and step $\Delta t > 0$. We define the shift, averaging and difference quotient operators

$$v_{\pm,k} = v_{k\pm 1}, \quad (sv)_{k-1/2} = \frac{v_k + v_{k+1}}{2}, \quad (\delta v)_{k-1/2} = \frac{v_k - v_{k-1}}{h},$$
$$(\delta^* y)_k = \frac{y_{k+1/2} - y_{k-1/2}}{h}, \quad \delta_t v = \frac{v^+ - v}{\Delta t}, \quad v^{+,m} = v^{m+1}.$$

We first consider a standard explicit two-level in time and three-point symmetric in space discretization of the QGD equations (2)–(4):

$$\delta_t \rho + \delta^* j = 0, \quad \delta_t(\rho u) + \delta^*\big(jsu + p(s\rho) - \Pi\big) = 0 \ \text{ on } \ \omega_h, \tag{8}$$
$$j = (s\rho)su - (s\rho)w, \ (s\rho)w = (s\tau)\big[\delta(\rho u)\big]su + (s\rho)\hat{w}, \tag{9}$$
$$(s\rho)\hat{w} = (s\tau)\big[(s\rho)(su)\delta u + \delta p(\rho)\big], \tag{10}$$
$$\Pi = \mu\delta u + (su)(s\rho)\hat{w} + (s\tau)\big[p'(s\rho)\big]\delta(\rho u). \tag{11}$$

The main unknown functions $\rho > 0$, u and the parameter τ are defined on ω_h whereas $j, w, \hat{w}, \Pi$ and μ are defined on ω_h^*.

In [18] two non-standard spatial discretizations of the QGD equations (2)–(4) were constructed which are weakly conservative in energy (see their generalization to a multidimensional case in [19]). One of them has the "enthalpy" form

$$\delta_t\rho+\delta^* j=0,\ \ \delta_t(\rho u)+\delta^*(jsu-\Pi)+s^*\big[(s\rho)\delta \mathrm{h}(\rho)\big]=0, \tag{12}$$

$$j=s\rho\cdot su-s\rho\cdot w,\ \ s\rho\cdot w=\big[(\tau\partial_x)_h(\rho u)\big]su+(s\rho)\hat{w}, \tag{13}$$

$$\hat{w}=(s\tau)\big[(su)\delta u+\delta \mathrm{h}(\rho)\big],\ \ \Pi=\mu\delta u+(su)(s\rho)\hat{w}+p'(s\rho)(\tau\partial_x)_h(\rho u), \tag{14}$$

$$(\tau\partial_x)_h(\rho u)=\Big(s\frac{\tau}{\mathrm{h}'(\rho)}\Big)\big\{[\delta \mathrm{h}(\rho)]su+p'(s\rho)\delta u\big\}, \tag{15}$$

where $\mathrm{h}(\rho)=\int_{r_0}^{\rho}\frac{p'(r)}{r}\,dr$, with some $r_0>0$, is the gas *enthalpy* and thus $\mathrm{h}'(\rho)=\frac{p'(\rho)}{\rho}$. In the isentropic case $p(\rho)=p_1\rho^{\gamma}$ with $\gamma>1$, one can take $r_0=0$ and then $\mathrm{h}(\rho)=\frac{\gamma}{\gamma-1}\frac{p(\rho)}{\rho}$ and $\mathrm{h}'(\rho)=\gamma\frac{p(\rho)}{\rho^2}$. Notice the non-standard $\mathrm{h}(\rho)$-dependent discretizations of $\partial_x p(\rho)$ in (12) and (14) and $\tau\partial_x(\rho u)$ in (13) and (14), see (15).

We linearize scheme (8)–(11) on a constant solution $\rho_*\equiv \text{const}>0$ and $u_*=0$. To do that, we write its solution in the form $\rho=\rho_*+\Delta\rho$ and $u=u_*+\Delta u$, neglect terms having the second order of smallness with respect to $\Delta\rho$ and Δu and obtain

$$\delta_t\Delta\rho+\rho_*\delta^* s\Delta u-\tau(\rho_*)p'(\rho_*)\delta^*\delta\Delta\rho=0,$$

$$\rho_*\delta_t\Delta u+p'(\rho_*)\delta^* s\Delta\rho-\big[\mu(\rho_*)+\tau(\rho_*)\rho_* p'(\rho_*)\big]\delta^*\delta\Delta u=0.$$

For the dimensionless unknowns $\tilde{\rho}=\frac{\Delta\rho}{\rho_*}$ and $\tilde{u}=\frac{\Delta u}{c_*}$ we get equations

$$\delta_t\tilde{\rho}+c_*\delta^* s\tilde{u}-\tau(\rho_*)c_*^2\delta^*\delta\tilde{\rho}=0, \tag{16}$$

$$\delta_t\tilde{u}+c_*\delta^* s\tilde{\rho}-\Big[\frac{\mu(\rho_*)}{\rho_*}+\tau(\rho_*)c_*^2\Big]\delta^*\delta\tilde{u}=0 \tag{17}$$

(cf. systems (5) and (6)). The linearization of scheme (12)–(15) is the same.

Notice that since $u_*=0$, the linearization result remains the same if it would be $w=\hat{w}$ and the terms dependent on u were omitted in the definition of these variables, i.e., for example, $w=\hat{w}=\frac{\tilde{\tau}}{s\rho}\delta p(\rho)$ instead of formulas in (9) and (10).

We assume that the regularization parameter and viscosity coefficient are given by usual QGD-formulas

$$\tau(\rho)=\frac{\alpha h}{\sqrt{p'(\rho)}},\ \ \mu(\rho)=\alpha_s\tau(\rho)\rho p'(\rho),$$

where $\alpha>0$ and $\alpha_s\geq 0$ are parameters. Then omitting tildes above ρ and u, equations (16) and (17) can be rewritten in the following recurrent form

$$\rho^+ = \rho - \frac{\beta}{2}(u_+ - u_-) + \alpha\beta(\rho_- - 2\rho + \rho_-), \tag{18}$$

$$u^+ = u - \frac{\beta}{2}(\rho_+ - \rho_-) + \varkappa\alpha\beta(u_+ - 2u + u_-) \tag{19}$$

with three parameters α, $\beta := c_* \frac{\Delta t}{h}$ and $\varkappa := \alpha_s + 1 \geq 1$. The functions ρ^0 and u^0 are given, i.e., we consider the initial-value problem for the scheme. Below it is convenient to consider ρ and u as complex-valued mesh functions.

3 Weak Conservativeness Analysis

Let $\mathbf{y}^m = (\rho^m \ u^m)^T$, $m \geq 0$, be a column-vector function on ω_h and the linearized difference scheme (18) and (19) be rewritten in a matrix form

$$\mathbf{y}^+ = \begin{pmatrix} \alpha\beta & \frac{\beta}{2} \\ \frac{\beta}{2} & \varkappa\alpha\beta \end{pmatrix} \mathbf{y}_- + \begin{pmatrix} 1 - 2\alpha\beta & 0 \\ 0 & 1 - 2\varkappa\alpha\beta \end{pmatrix} \mathbf{y} + \begin{pmatrix} \alpha\beta & -\frac{\beta}{2} \\ -\frac{\beta}{2} & \varkappa\alpha\beta \end{pmatrix} \mathbf{y}_+. \tag{20}$$

Let H be a Hilbert space of complex valued square-summable on ω_h vector functions, i.e. having a finite norm

$$\|\mathbf{y}\|_H = \left(h \sum_{k=-\infty}^{\infty} |\mathbf{y}_k|^2\right)^{1/2}.$$

For $\mathbf{y}^0 = (\rho^0 \ u^0)^T \in H$ we have that $\mathbf{y}^m \in H$ for all $m \geq 1$. We define a *weak conservativeness* of scheme (20) as validity of the bound

$$\sup_{m \geq 0} \|\mathbf{y}^m\|_H \leq \|\mathbf{y}^0\|_H \quad \forall \mathbf{y}^0 \in H. \tag{21}$$

This definition is motivated by the energy conservation law (7) for the acoustics system of equation (6). It is essential to notice that for the linearized QGD-system (2)–(4) namely the corresponding inequality holds in place of equality (7) so that it is natural to study the bound for schemes based on such a system. Of course, estimate (21) guarantees the uniform in time stability in H with respect to initial data.

We first substitute a partial solution in the form $\mathbf{y}_k^m = e^{\mathbf{i}k\xi}\mathbf{v}^m(\xi)$, $k \in \mathbb{Z}$, $m \geq 0$, where $\mathbf{i}$ is the imaginary unit and $0 \leq \xi \leq 2\pi$ is a parameter, into (20) and obtain

$$\mathbf{v}^+(\xi) = G(\xi)\mathbf{v}(\xi), \quad G(\xi) = \begin{pmatrix} 1 - \omega_1 & -\mathbf{i}\omega_2 \\ -\mathbf{i}\omega_2 & 1 - \varkappa\omega_1 \end{pmatrix}, \tag{22}$$

where we denote $\omega_1 = 4\alpha\beta\theta, \theta = \sin^2 \frac{\xi}{2} \in [0, 1]$ and $\omega_2 = \beta \sin \xi$ for brevity. Below it is important that $\omega_2^2 = 4\beta^2\theta(1-\theta)$.

It is known (see similar formulas in [6]) that if $\mathbf{y}^0 = (\rho^0 \ u^0)^T \in H$, then there exists a function $\mathbf{v}^0 \in L^2(0, 2\pi)$ such that

$$\mathbf{v}^0(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \mathbf{y}_k^0 e^{-\mathbf{i}k\xi},$$

and we can write the solution to scheme (20) in an integral form

$$\mathbf{y}_k^m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathbf{v}^m(\xi) e^{\mathbf{i}k\xi}\, d\xi, \quad k \in \mathbb{Z},$$

where $\mathbf{v}^m \in L^2(0, 2\pi)$ due to (22). The following Parseval identity also holds

$$\|\mathbf{y}^m\|_H = \sqrt{h}\, \|\mathbf{v}^m\|_{L^2(0,2\pi)}, \quad m \geq 0. \tag{23}$$

The von Neumann type spectral condition

$$\max_{0 \leq \xi \leq 2\pi} \max_l \left| \lambda_l\big(G(\xi)\big) \right| \leq 1 \tag{24}$$

is known to be *a necessary condition* for property (21) to hold (see similar result in [6]). Hereafter $\lambda_l(A)$ are eigenvalues of a matrix A.

Let us determine the spectral form of property (21).

Lemma 1 *Validity of the spectral bound*

$$\max_{0 \leq \xi \leq 2\pi} \max_l \lambda_l\big((G^* G)(\xi)\big) \leq 1 \tag{25}$$

is necessary and sufficient for the weak conservativeness property (21) to hold.

Proof Due to the Parseval identity (23) and formula (22) we have

$$h^{-1} \|\hat{\mathbf{y}}\|_H^2 = \|\hat{\mathbf{v}}\|_{L^2(0,2\pi)}^2 = \|G\mathbf{v}\|_{L^2(0,2\pi)}^2 = (G^* G\mathbf{v}, \mathbf{v})_{L^2(0,2\pi)}.$$

Since $(G^* G)(\xi) \geq 0$ is a Hermitian matrix, it has a spectral decomposition $(G^* G)(\xi) = U^*(\xi) \Lambda(\xi) U(\xi)$, where $U(\xi)$ is a unitary matrix, and $\Lambda(\xi)$ is a diagonal matrix with numbers $\lambda_l\big((G^* G)(\xi)\big) \geq 0$ forming its diagonal. Hence for $z(\xi) := U(\xi)\mathbf{v}(\xi)$ we have

$$(G^* G\mathbf{v}, \mathbf{v})_{L^2(0,2\pi)} = (\Lambda \mathbf{z}, \mathbf{z})_{L^2(0,2\pi)} = \|\Lambda^{1/2} \mathbf{z}\|_{L^2(0,2\pi)}^2.$$

Thus $\|\mathbf{y}^m\|_H^2 = h \|\Lambda^{m/2} \mathbf{z}^0\|_{L^2(0,2\pi)}^2$ for $m \geq 0$, and bound (21) is equivalent to the following one

$$\sup_{m \geq 0} \|\Lambda^{m/2} \mathbf{z}^0\|_{L^2(0,2\pi)}^2 \leq \|\mathbf{z}^0\|_{L^2(0,2\pi)}^2 \quad \forall \mathbf{z}^0 \in L^2(0, 2\pi).$$

It holds if and only if the spectral bound (25) holds.

Remark 1 Under validity of the spectral bound (25), the norm $\|\mathbf{y}^m\|_H$ is actually non-increasing in $m \geq 0$ that serves as a stronger property than (21).

In the proof of this lemma, the specific form and dimension of the matrix G are clearly inessential, and actually it holds in general case.

In our case the matrix G^*G has the form

$$G^*G = \begin{pmatrix} (1-\omega_1)^2 + \omega_2^2 & -\mathbf{i}(1-\varkappa)\omega_1\omega_2 \\ \mathbf{i}(1-\varkappa)\omega_1\omega_2 & (1-\varkappa\omega_1)^2 + \omega_2^2 \end{pmatrix}.$$

Note that $G^*G = \left[(1-\omega_1)^2 + \omega_2^2\right]I$ in the simplest case $\varkappa = 1$, where I is a unit matrix.

Theorem 1 *The necessary spectral condition (24) holds if and only if*

$$\beta \leq \min\left\{(\varkappa+1)\alpha, \frac{1}{2\varkappa\alpha}\right\}. \tag{26}$$

Proof The characteristic polynomial for the matrix G has the following form

$$q_1(\lambda) = \lambda^2 - (\operatorname{tr} G)\lambda + \det G = \lambda^2 + [(\varkappa+1)\omega_1 - 2]\lambda + \\ + \left[\varkappa\omega_1^2 + \omega_2^2 + 1 - (\varkappa+1)\omega_1\right]. \tag{27}$$

We set $a_0 := q_1(1) = 1 - (\operatorname{tr} G) + \det G = \varkappa\omega_1^2 + \omega_2^2 \geq 0$ and notice that $a_0 = q_1(1) = 0$ if and only if $G = I$. We transform the unit circle $\{|\lambda| \leq 1\}$ with a punctured point $(1, 0)$ on $\mathbb{C}$ into the closed left half-plane $\{\operatorname{Re} z \leq 0\}$ and put

$$\hat{q}_1(z) := (z-1)^2 q_1\left(\frac{z+1}{z-1}\right) = a_0 z^2 + 2a_1 z + a_2,$$

where $a_1 = 1 - \det G$, $a_2 = 1 + \operatorname{tr} G + \det G$. It is well known that for $a_0 > 0$ the roots $\hat{q}_1(z)$ lie in $\{\operatorname{Re} z \leq 0\}$ under the conditions $a_1 \geq 0$ and $a_2 \geq 0$, i.e.

$$\varkappa\omega_1^2 + \omega_2^2 - (\varkappa+1)\omega_1 \leq 0, \quad \varkappa\omega_1^2 + \omega_2^2 - 2(\varkappa+1)\omega_1 + 4 \geq 0.$$

We rewrite these conditions as

$$\beta(4\varkappa\alpha^2\theta + 1 - \theta) - (\varkappa+1)\alpha \leq 0 \quad \text{for} \quad 0 \leq \theta \leq 1, \tag{28}$$

$$r(\theta) := \beta^2(4\varkappa\alpha^2 - 1)\theta^2 - \beta\big(2(\varkappa+1)\alpha - \beta\big)\theta + 1 \geq 0 \quad \text{for} \quad 0 \leq \theta \leq 1. \tag{29}$$

The left-hand side of (28) is linear in θ, thus it suffices to test it for $\theta = 0, 1$ that leads us to the condition

$$\beta \leq \min\left\{(\varkappa+1)\alpha, \frac{\varkappa+1}{4\varkappa\alpha}\right\}. \tag{30}$$

Next we analyze condition (29). Notice that $r(0) = 1$ and due to (30) we have $2(\varkappa + 1)\alpha - \beta > 0$. For $a = 4\varkappa\alpha^2 - 1 \neq 0$ the vertex of the parabola $r(\theta)$ is given by

$$\theta_v = \frac{2(\varkappa + 1)\alpha - \beta}{2\beta(4\varkappa\alpha^2 - 1)}.$$

For $4\varkappa\alpha^2 - 1 > 0$ the property $\theta_v > 1$ means that $\beta < \frac{(\varkappa+1)\alpha}{4\varkappa\alpha^2 - 0.5}$, and it holds due to (30). Hence condition (29) reduces to $r(1) \geq 0$, i.e.

$$4\varkappa\alpha^2\beta^2 - 2(\varkappa + 1)\alpha\beta + 1 = 4\varkappa\alpha^2\Big(\beta - \frac{1}{2\varkappa\alpha}\Big)\Big(\beta - \frac{1}{2\alpha}\Big) \geq 0. \tag{31}$$

For $4\varkappa\alpha^2 - 1 < 0$ we have $\theta_v < 0$, so that condition (29) reduces again to $r(1) \geq 0$. For $4\varkappa\alpha^2 - 1 = 0$ the condition also reduces to $r(1) \geq 0$ (since $r(0) = 1$).

Since $\varkappa \geq 1$, inequality (31) means that either of the conditions

$$\beta \leq \frac{1}{2\varkappa\alpha}, \quad \beta \geq \frac{1}{2\alpha} \tag{32}$$

holds. Combining them with (30), we obtain (26).

Now we turn to the spectral criterion (25).

Theorem 2 *The spectral criterion (25) holds if and only if*

$$\beta \leq \min\Big\{2\alpha, \frac{1}{2\varkappa\alpha}\Big\}. \tag{33}$$

Proof The characteristic polynomial of G^*G has the following form

$$q_2(\lambda) = \lambda^2 - \operatorname{tr}(G^*G)\lambda + (\det G)^2.$$

Since $\lambda_l(G^*G) \geq 0$, the property $|\lambda_l(G^*G)| \leq 1$ means validity of the conditions

$$\frac{1}{2}\operatorname{tr}(G^*G) \leq 1, \quad q_2(1) = 1 - \operatorname{tr}(G^*G) + (\det G)^2 \geq 0. \tag{34}$$

The first of them has the form

$$\frac{\varkappa^2 + 1}{2}\omega_1^2 + \omega_2^2 - (\varkappa + 1)\omega_1 \leq 0$$

and can be specified as

$$8\alpha^2\beta^2(\varkappa^2 + 1)\theta^2 + 4\beta^2\theta(1 - \theta) - 4\alpha\beta(\varkappa + 1)\theta \leq 0 \text{ for } 0 \leq \theta \leq 1. \tag{35}$$

After dividing by $4\beta\theta$ we get that it suffices to confine ourselves with the values $\theta = 0, 1$ that leads to the condition

$$\beta \le \min\left\{(\varkappa+1)\alpha, \frac{\varkappa+1}{2(\varkappa^2+1)\alpha}\right\}. \tag{36}$$

In order to transform the second condition (34) we notice that

$$\mathrm{tr}(G^*G) = 2(b+1) + (\varkappa-1)^2\omega_1^2, \quad (\det G)^2 = (b+1)^2,$$

where $b := \varkappa\omega_1^2 + \omega_2^2 - (\varkappa+1)\omega_1$, see (27). Hence the following factorization holds

$$q_2(1) = b^2 - (\varkappa-1)^2\omega_1^2 = \big(b - (\varkappa-1)\omega_1\big)\big(b + (\varkappa-1)\omega_1\big),$$

that is decisive for the simplicity of our analysis. Since $\varkappa \ge 1$, the condition $q_2(1) \ge 0$ is equivalent to validity of either of the conditions

$$\varkappa\omega_1^2 + \omega_2^2 - 2\omega_1 \le 0, \quad \varkappa\omega_1^2 + \omega_2^2 - 2\varkappa\omega_1 \ge 0,$$

i.e., more specifically, to validity of either of the conditions

$$\beta(4\varkappa\alpha^2\theta + 1 - \theta) - 2\alpha \le 0 \;\text{ for }\; 0 \le \theta \le 1,$$

$$\beta(4\varkappa\alpha^2\theta + 1 - \theta) - 2\varkappa\alpha \ge 0 \;\text{ for }\; 0 \le \theta \le 1.$$

As above they respectively mean that the inequalities

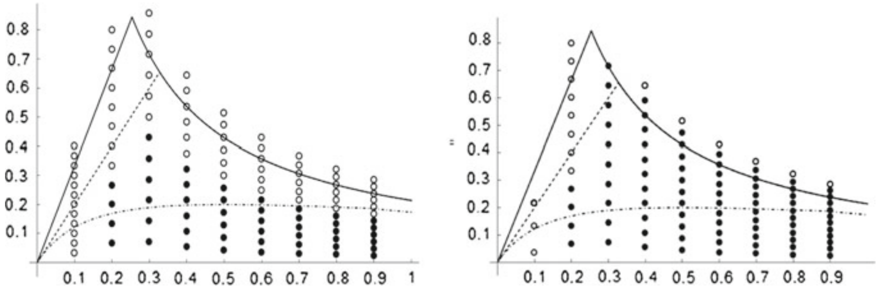
$$\beta \le \min\left\{2\alpha, \frac{1}{2\varkappa\alpha}\right\}, \quad \beta \ge \max\left\{2\varkappa\alpha, \frac{1}{2\alpha}\right\} \tag{37}$$

hold. Combining them with (36) we complete the proof.

It is essential that the function on the right-hand side of condition (33) reaches its maximal value at $\alpha = \alpha_* := \frac{1}{2\sqrt{\varkappa}} \le \frac{1}{2}$ and the maximal value equals $\frac{1}{\sqrt{\varkappa}} \le 1$. Hence the criterion coincides with the standard CFL stability condition $\beta \le 1$ if and only if $\alpha = \alpha_*$ and $\varkappa = 1$. The criterion gives an important information on the optimal choice of α since in practice for the original non-linear problem α is normally sought experimentally. Note also that criterion (33) and the necessary condition (26) coincide only in the case $\alpha \ge \alpha_*$.

We call attention to a paradoxical moment: criterion (33) becomes *stronger* as the coefficient of "effective viscosity" $\varkappa$ increases (it is harder to say that about the necessary condition (26)). Therefore the best choice in the present bounds is $\alpha_s = 0$, i.e. $\varkappa = 1$. But this conclusion is not universal in practice and it is known that in some situations $\alpha_s > 0$ has to be taken (see, for example, [21]).

In Fig. 1 we compare the necessary condition, the criterion of stability and the sufficient condition as well as the results of numerical experiments for the original



(a) The standard scheme (b) The "enthalpy" scheme

Fig. 1 The weak conservativeness analysis: the necessary condition (solid line), the criterion (dash line), the sufficient condition (dotdash line) together with conservative (painted balls) and non-conservative (unpainted balls) computations for the Riemann problem in dependence with α

system (1) for $p(\rho) = \rho^2$ (the scaled case of the shallow water equations) and $\varkappa = \frac{7}{3}$. The sufficient condition was obtained in [14] only for these $p(\rho)$ and $\varkappa$ by the energy method and has the form

$$\beta \le \min\left\{\frac{2\alpha}{1+6\alpha+4\alpha^2}, \frac{4\alpha}{1+6\alpha+16\alpha^2}\right\}.$$

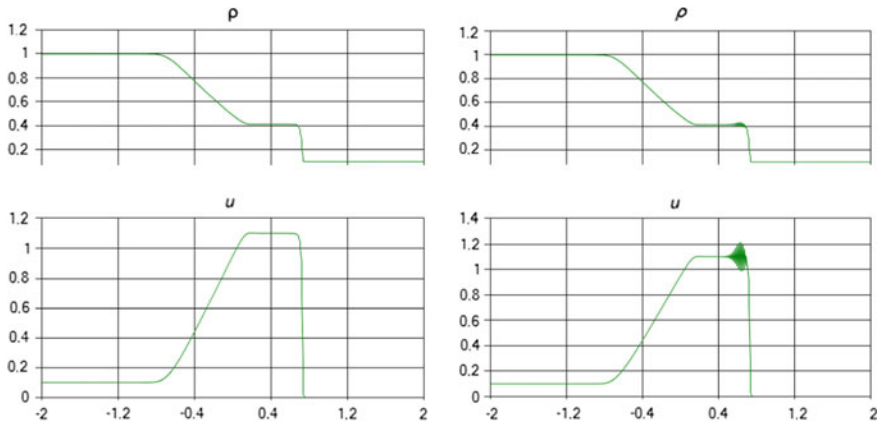
Notice that the first fraction in it is less than the second one for $0 < \alpha < \frac{3+\sqrt{17}}{8} \approx 0.890$. The corresponding graph is almost flat for $0.3 \le \alpha \le 0.9$ in contrast to the cases of necessary condition and criterion. The computations are accomplished for $0 \le t \le 0.5$ for the Riemann problem with the discontinuous initial data

$$\rho_0(x) = \begin{cases} 1, & x < 0 \\ 0.1, & x > 0 \end{cases}, \quad u_0(x) = \begin{cases} 0.1, & x < 0, \\ 0, & x > 0 \end{cases}$$

for both schemes (8)–(11) and (12)–(15) with $h = 1/125$.

We observe a good correspondence of the obtained criterion with the experimental results, and that the sufficient condition underestimates the criterion up to several times in the most interesting region $\alpha \approx \alpha_*$. Also the results for the "enthalpy" scheme (12)–(15) are clearly different from and better than for the standard one (8)–(11) though the above linearized analysis gives the same results for them.

We have identified non-conservative computations by noticeable well-known oscillations of the numerical solutions (some of computations have not even been completed yet due to overflow). In Fig. 2 we give an example of conservative and non-conservative solutions ρ and u for the "enthalpy" scheme (at time $t = 0.5$) for $\alpha = 0.4$ and two neighboring values of β from Fig. 1b.



(a) The conservative solution, $\beta \approx 0.589$

(b) The non-conservative solution, $\beta \approx 0.643$

Fig. 2 The examples of conservative and non-conservative solutions for the "enthalpy" scheme for $\alpha = 0.4$ (at time $t = 0.5$)

4 The Case of the Schemes Based on a Simplified Regularization

We also consider a simplified (quasi-hydrodynamic [8, 13, 15]) regularization (2)–(4), where the terms with $\partial_x(\rho u)$ are omitted, in particularly, it becomes $w = \hat{w}$. Correspondingly in schemes (8)–(11) and (12)–(15) we have to omit both terms with respectively $\delta(\rho u)$ and $(\tau\partial_x)_h(\rho u)$. In the linearized scheme the term $\tau(\rho_*)c_*^2$ disappears from Eq. (17), hence now $\varkappa = \alpha_s$. Notice that usually $0 < \alpha_s \le 1$ though in specific cases $\alpha_s > 1$ can be also taken.

Theorem 3 *For the simplified scheme based on the quasi-hydrodynamic regularization the following results are valid:*

(1) in the case $0 \le \alpha_s \le 1$ the necessary condition (24) and criterion (25) hold if and only if respectively

$$\beta \le \min\left\{(\alpha_s + 1)\alpha, \frac{1}{2\alpha}\right\}, \tag{38}$$

$$\beta \le \min\left\{2\alpha_s\alpha, \frac{1}{2\alpha}\right\}; \tag{39}$$

(2) in the case $\alpha_s \ge 1$ the results of Theorems 1 and 2 remain valid with $\varkappa = \alpha_s$.

Proof The above given analysis holds true except for some changes in the case $0 \le \varkappa = \alpha_s \le 1$. In this case, inequalities (32) are replaced by the following ones

$$\beta \le \frac{1}{2\alpha}, \quad \beta \ge \frac{1}{2\varkappa\alpha};$$

they being combined with (30) lead to (38). Also inequalities (37) are replaced by the following ones

$$\beta \le \min\left\{2\varkappa\alpha, \frac{1}{2\alpha}\right\}, \quad \beta \ge \max\left\{2\alpha, \frac{1}{2\varkappa\alpha}\right\},$$

they being combined with (36) lead to (39).

The maximal value of the function on the right-hand side of criterion (39) is reached at $\alpha = \alpha_* := \frac{1}{2\sqrt{\alpha_s}} \ge \frac{1}{2}$ and equals $\sqrt{\alpha_s} \le 1$. We notice that the necessary condition (38) is especially rough compared to criterion (39) for $\alpha_s \approx 0$, including the case $\alpha_s = 0$ when actually the stability is absent at all.

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Geometric Versus Automorphic Correspondence for Vertex Operator Algebra Modules

Alexander Zuevsky

Abstract We propose geometric and automorphic sides of a general correspondence for vertex operator algebras and review supporting examples.

Keywords Automorphic forms · Vertex algebras · Geometric correspondence

1 Introduction

The analogy between conformal field theory and the theory of automorphic representations was first observed in [45]. The purpose of these notes is to announce a way to relate the space of flat connections in certain vector bundles and automorphic representations for modular groups over Riemann surfaces. This non-trivial relation is due to the algebraic nature of underlying vertex operator algebras. The main idea is that the algebraic (computational) approach both on differential (algebraic-geometrical) [14, 22, 37, 43] and automorphic sides [24–31, 38–42] of correlation functions computation for vertex operator algebra modules plays an intermediate role in establishing the above mentioned relation. Note that neither CFT [2, 14] nor algebraic-geometry [9, 12, 22, 33–35, 37, 43] approaches are able to reconstruct rigorously, starting from the algebraic nature, the partition functions for vertex operator algebras on algebraic curves. Modular properties of the partition and correlation functions are assumed as a CFT fundamental feature. In contrast to that, the algebraic method (see, e.g., [32]) of correlation functions calculation does not postulate modular invariance as well as permits to compute the partition and n-point functions explicitly and prove their automorphic properties. In these notes we provide supporting examples for two sides of a general correspondence relating determinantal and differential representations for the normalized correlation functions.

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In the theory of linear difference equations (polynomial's of degree 0 or 1), solving the homogeneous equation $x_t = a_1 x_{t-1} + \cdots + a_n x_{t-n}$, involves first solving its characteristic equation $\lambda^n = a_1 \lambda^{n-1} + \cdots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n$, one solves characteristics equations for its characteristic roots λ_i, λ_i, $i = 1, \ldots, n$. In algebraic geometry, these equations determine also algebraic curves that we deal with in the theory of vertex algebras. On the other hand, difference equations represent a finite size element version of differential equations which can be also connected to algebraic curves and associated to algebraic structures considered on such curves. In our paper we deal with automorphic and differential side counterparts for vertex algebra structures constructed on algebraic curves which can be related to characteristic equations for linear difference equations.

2 Examples

2.1 Automorphic Side

Previously, (see, e.g., [9, 22, 33–35, 43]) formal expressions for VOA correlation functions were usually derived using their general, in particular, analytic properties, while expressions for the partition functions were simply postulated. At the same time, modular properties of correlation function were assumed from the very beginning.

2.1.1 Example: The Partition Functions for Heisenberg VOA

Let $Z_M^{(2)}$ be the genus two partition function for the rank one free Heisenberg VOA M [28]. The section of the corresponding bundle on the genus two Riemann surface can be tied together with two genus one section by means of the matrix related by to the period matrix $\Omega^{(2)}$. It has been computed in [28, 29]

$$\det\left(I - A_1^{(1)} A_2^{(1)}\right)^{-1/2} = \frac{Z_M^{(2)}\left(\Omega^{(2)}\right)}{Z_M^{(2)}(\tau_1, \tau_2)}, \tag{1}$$

where

$$Z_M^{(2)}(\tau_1, \tau_2) = Z_M^{(1)}(\tau_1)\, Z_M^{(1)}(\tau_2),$$

$Z_M^{(1)}(\tau) = \eta(\tau)^{-1}$ is the Heisenberg VOA torus partition function, τ is the torus modular parameter, and $\eta(\tau)$ is the Dedekind eta-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/24} \det \mathbf{Q},$$

$$\det \mathbf{Q} = \frac{Z_M^{(1)}(\tau)}{Z_L(\tau)},$$

where $Z_L = q^{1/24}$. Here $A_a^{(1)}$ for $a = 1, 2$ are infinite matrices with components indexed by $k, l \geq 1$ [27, 28],

$$A_a(k, l, \tau_a, \epsilon) = \epsilon^{(k+l)/2} \frac{(-1)^{k+1}(k+l-1)!}{\sqrt{kl}(k-1)!(l-1)!} E_{k+l}(\tau_a), \tag{2}$$

where ϵ is Riemann surfaces sewing scheme parameter [27, 44], and $E_n(\tau)$ are the Eisenstein series [36]. We note that $E_n(\tau)$ appear explicitly as a result of vertex operator algebra computation.

At genus $g_1 + g_2$, in the two curves sewing formalism, we obtain [38, 42] similar formula for the rank two Heisenberg VOA module

$$\det \left(I - A^{(g_1, g_2)}\right)^{-1} = \frac{Z_{M_2}^{(g_1+g_2)}\left(\Omega^{(g_1+g_2)}\right)}{Z_{M_2}^{(g_1, g_2)}\left(\Omega^{(g_1)}, \Omega^{(g_2)}\right)}, \tag{3}$$

$$Z_{M_2}^{(g_1, g_2)}\left(\Omega^{(g_1)}, \Omega^{(g_2)}\right) = Z_{M_2}^{(g_1)}\left(\Omega^{(g_1)}\right) Z_{M_2}^{(g_2)}\left(\Omega^{(g_2)}\right),$$

where $Z_{M_2}^{(g_1+g_2)}$ is a non-vanishing holomorphic function on the sewing domain and is automorphic with respect to $SL(2, Z) \times \cdots \times SL(2, Z) \subset Sp(2(g_1 + g_2), Z)$ with automorphy factor $\det(C\Omega^{(g_1+g_2)} + D)^{-1}$, and a multiplier system. Here $A^{(g_1, g_2)}$ is an infinite moment matrix containing genus $g_1 + g_2$ sewing data, and $\Omega^{(g_1+g_2)}$ is a genus $g_1 + g_2$ Riemann surface period matrix. Automorphic properties of the above mentioned genus two partition functions follow [38, 41] from the structure of the determinant and the lower genus partition functions in (3).

2.1.2 Example: The Free Fermion Partition Functions

The genus one free fermion twisted partition function is given by [21, 32]

$$Z_{\mathscr{V}}^{(1)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau) = q^{\alpha^2/2 - 1/24} \prod_{l \geq 1} \left(1 - \theta^{-1} q^{l - \frac{1}{2} + \alpha}\right) \left(1 - \theta q^{l - \frac{1}{2} - \alpha}\right) \tag{4}$$

$$= q^{\alpha^2/2 - 1/24} \det \mathbf{q}, \tag{5}$$

where f, g are twisting automorphism of the VOSA module corresponding to periodicities θ and α. It is clearly a modular invariant object. We can rewrite (5) as

$$\det \mathbf{q} = \frac{Z_{\mathscr{V}}^{(1)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau)}{Z_{L_\alpha}(\tau)}, \tag{6}$$

where $Z_{L_\alpha}(\tau) = q^{\alpha^2/2-1/24}$, and

$$\det \mathbf{q} = \prod_{l\geq 1} \left(1 - \theta^{-1} q^{l-\frac{1}{2}+\alpha}\right) \left(1 - \theta q^{l-\frac{1}{2}-\alpha}\right).$$

This partition function can be also obtained [38, 39] in a form analogous to (3) as a determinant of an infinite matrix containing moments of corresponding genus zero differentials when considering a self-sewing of the sphere.

At genus g curve case the following formula for the partition function can be also conjectured:

$$\det \left(I - Q^{(g_1+g_2)}\right)^{-1} = \frac{Z_{\mathscr{V}}^{(g_1+g_2)} \begin{bmatrix} f^{(g_1+g_2)} \\ g^{(g_1+g_2)} \end{bmatrix} \left(\Omega^{(g)}\right)}{Z_{\mathscr{V}}^{(g_1,g_2)} \begin{bmatrix} f^{(g_1,g_2)} \\ g^{(g_1,g_2)} \end{bmatrix} \left(\Omega^{(g_1)}, \Omega^{(g_2)}\right)}, \tag{7}$$

$$Z_{\mathscr{V}}^{(g_1,g_2)} \begin{bmatrix} f^{(g_1,g_2)} \\ g^{(g_1,g_2)} \end{bmatrix} \left(\Omega^{(g_1)}, \Omega^{(g_2)}\right) = Z_{\mathscr{V}}^{(g_1)} \begin{bmatrix} f^{(g_1)} \\ g^{(g_1)} \end{bmatrix} \left(\Omega^{(g_1)}\right) Z_{\mathscr{V}}^{(g_2)} \begin{bmatrix} f^{(g_2)} \\ g^{(g_2)} \end{bmatrix} \left(\Omega^{(g_2)}\right),$$

where $g = g_1 + g_2$, $Q^{(g)}$ is an infinite matrix, and $\Omega^{(g_1+g_2)}$, $\Omega^{(g_1)}$, $\Omega^{(g_2)}$ are period matrices for a sewn genus $g_1 + g_2$ curve and lower genus curves of genuses g_1, g_2.

2.1.3 Example: Genus One Generating *n*-Point Functions for Free Fermion VOSA

Let us denote by $Z_{V,h}^{(1)}(f;\tau)$ and $G_{2n,h}^{(1)}$ the shifted partition and n-point generating functions [32]. We proved the following formula for rank two free fermion twisted generating differential for $(\theta, \phi) \neq (1, 1)$:

$$\det \mathbf{P} = \frac{G_{2n,h}^{(1)}(f; \mathbf{x}; \mathbf{y}; \tau)}{Z_{V,h}^{(1)}(f;\tau)\, d\mathbf{x}\, d\mathbf{y}}, \tag{8}$$

where $\mathbf{P}$ is the $n \times n$ matrix:

$$\mathbf{P} = \left(P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau, x_i - y_j) \right), \quad (1 \leq i, j \leq n), \tag{9}$$

with θ, ϕ parameterizing vertex operator superalgebra automorphisms f, g. Here $P_1 \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau, z)$ is the deformed [8] Weierstrass series (in [32] it was called "twisted").

2.1.4 Generalizations of Classical Identities for Modular Forms

By comparison of the direct computation (via algebraic technique [25–31, 38–42]) of the partition and n-point functions with the bosonization technique [25], we obtain [38, 41] the genus two version of the Jacobi product and Frobenius–Fay trisecant identities. In particular, the genus two partition function for the free fermion vertex operator algebra can be written in these two ways resulting in a genus two counterpart of Jacobi product identity [38, 41]

$$\det(I - Q^{(2)}) \ \det\left(I - A_1^{(1)} A_2^{(1)}\right)^{1/2} = \frac{Z^{(2)}\left(\Omega^{(2)}\right)}{Z^{(1)}(\tau_1)\, Z^{(1)}(\tau_2)}, \tag{10}$$

$$Z^{(2)}\left(\Omega^{(2)}\right) = \Theta^{(2)} \begin{bmatrix} a \\ b \end{bmatrix} \left(\Omega^{(2)}\right),$$

$$Z^{(1)}(\tau_i) = \vartheta^{(1)} \begin{bmatrix} a_i \\ b_i \end{bmatrix} (\tau_i).$$

Here $\vartheta^{(1)} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} (\tau)$, $\Theta^{(2)} \begin{bmatrix} a \\ b \end{bmatrix} \left(\Omega^{(2)}\right)$, are genus one and genus two theta-functions correspondingly, $\Omega^{(2)}$ is a genus two period matrix, and $Q^{(2)}$, $A^{(i)}$, $i = 1, 2$ are infinite matrices that encode differential data of two tori. Similar formulas in the self-sewing of a Riemann surface are given in [41]. Higher genus identities are also available.

2.2 *Differential Side*

Using methods of algebraic geometry one is able [22, 37, 43] to determine formally general differential structure of correlation functions in certain bundles over Riemann surfaces.

2.2.1 Example: The Partition Functions for Free Fermionic VOSA

The partition function for the rank one Heisenberg vertex operator algebra is given by $Z_M^{(1)} = \eta(\tau)^{-1}$, [25]. The genus one twisted partition function (see (5) in Sect. 2.1.2) for the rank two free fermionic vertex operator superalgebra (see, e.g., [21, 32]) can be also be expressed as the normalized (i.e., divided by corresponding Heisenberg partition function)

$$\vartheta^{(1)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = e^{2\pi i\alpha\beta}\frac{Z^{(1)}_{\mathscr{V}}\begin{bmatrix}f\\ g\end{bmatrix}(\tau)}{Z^{(1)}_{M}(\tau)}, \tag{11}$$

for the torus theta function with characteristics α, β. This formula can be considered as an alternative definition of a θ-function via CFT ingredients. The genus two normalized twisted partition function can be computed in bosonization formalism to obtain [25, 38]

$$\vartheta^{(2)}\begin{bmatrix}\alpha^{(2)}\\ \beta^{(2)}\end{bmatrix}\left(\Omega^{(2)}\right) = e^{2\pi i\alpha^{(2)}\cdot\beta^{(2)}}\frac{Z^{(2)}_{\mathscr{V}}\begin{bmatrix}f^{(2)}\\ g^{(2)}\end{bmatrix}}{Z^{(2)}_{M}(\Omega^{(2)})}, \tag{12}$$

for the genus two Riemann theta function with characteristics $\alpha^{(2)} = (\alpha_1, \alpha_2)$, $\beta^{(2)} = (\beta_1, \beta_2)$ where $Z^{(2)}_M$ is the genus two Heisenberg partition function. Natural formulas relating the free fermion normalized twisted partition functions and corresponding theta-functions are also available at higher genus.

2.2.2 Example: Free Fermion Genus Two Twisted Generating *n*-Point Funcitons

Let $\mathscr{V}$ be a free fermionic vertex operator superalgebra V-module [32]. For $v_i \in V$, and points z_i, $i = 1, \ldots n$, on genus g curve, let us introduce [38] the following differentials:

$$\mathscr{F}^{(g)}_{\mathscr{V},n}\begin{bmatrix}f^{(g)}\\ g^{(g)}\end{bmatrix}(\mathbf{v}, \mathbf{z}) \equiv F^{(g)}_{\mathscr{V},n}\begin{bmatrix}f^{(g)}\\ g^{(g)}\end{bmatrix}(\mathbf{v}, \mathbf{z})\ d\mathbf{z}^{wt(\mathbf{v})}, \tag{13}$$

where $F^{(g)}_{\mathscr{V},n}$ denotes the functional part and $wt(v_i)$ are weights of the states v_i, [21]. In [38] we proved that for $g = 2$ the generating differential for all rank two free fermion VOSA n-point functions is given by the differential

$$\mathscr{G}^{(2)}_{\mathscr{V},n}\begin{bmatrix}f^{(2)}\\ g^{(2)}\end{bmatrix}(\mathbf{x}, \mathbf{y}) \equiv F^{(2)}_{\mathscr{V},n}\begin{bmatrix}f^{(2)}\\ g^{(2)}\end{bmatrix}((\psi^+, \mathbf{x}_1), (\psi^-, \mathbf{y}_1)\ d\mathbf{x}^{1/2}d\mathbf{y}^{-1/2}, \tag{14}$$

where $F^{(2)}_{\mathscr{V},n}\begin{bmatrix}f^{(2)}\\ g^{(2)}\end{bmatrix}((\psi^+, \mathbf{x}), (\psi^-, \mathbf{y}))$ is the genus two $2n$-point function in coordinates $\mathbf{x}, \mathbf{y}$, $i = 1, \ldots, n$, $d\mathbf{x}^{1/2}d\mathbf{y}^{-1/2} = \prod_{i,j=1}^{n} dx_i^{1/2}dy_j^{-1/2}$, $\psi^{\pm}$ are generating states, and $f^{(2)}$, $g^{(2)}$ are vectors containing pairs of V-twisting automorphisms. Then the normalized (i.e., divided by corresponding Heisenberg partition function) form of it is given by

$$\det S^{(2)} \cdot \vartheta^{(2)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\Omega^{(2)}\right) = e^{2\pi i \alpha \cdot \beta} \frac{\mathscr{G}^{(2)}_{\mathscr{V},n} \begin{bmatrix} f \\ g \end{bmatrix} (x_1, y_1, \ldots, x_n, y_n)}{Z^{(2)}_M(\Omega^{(2)})}, \tag{15}$$

where elements of the matrix $S^{(2)} = \left[S^{(2)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (x_i, y_j) \right]$, $i, j = 1, \ldots, n$ are genus two Szegő kernels,

$$S^{(g)} \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (x_i, y_j) = K^{(g)} \begin{bmatrix} \theta^{(g)} \\ \phi^{(g)} \end{bmatrix} (x_i, y_j)\, dx_i^{1/2}\, dy_j^{1/2}, \tag{16}$$

where $K^{(g)} \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z_1, z_2)$ is the functional part of the genus g prime form [10].

2.2.3 Example: Genus Two One-Point Virasoro Vector Function

In [38] (see also [41]) we give an example of a genus two generating function relation with a flat connection over a Riemann surface. In [38] we computed the one-point function for the Virasoro vector $\widetilde{\omega} = \frac{1}{2} a[-1]a$, where a is the Heisenberg element in the bosonized version of the rank two free fermion VOSA on the genus two Riemann surface obtained in the sewing procedure of two tori. The Virasoro vector can be written as $\widetilde{\omega} = \frac{1}{2}(\psi^+[-2]\psi^- + \psi^-[-2]\psi^+)$ for the rank two free fermion generators $\psi^\pm$ in the Zhu equivalent VOSA formulation. Let w, z be two distinct points on a genus two Riemann surface. To compute a one point function we use the normalized generating form

$$S^{(2)} \begin{bmatrix} \theta^{(2)} \\ \phi^{(2)} \end{bmatrix} (w, z) = \frac{\mathscr{G}^{(2)}_{\mathscr{V},1} \begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix} (w, z)}{Z^{(2)}_{\mathscr{V}} \begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix} (\Omega^{(2)})}.$$

Then we find [40] it follows that the Virasoro vector one-point differential form is

$$\mathscr{S}^{(2)} = \frac{\mathscr{F}^{(2)}_{\mathscr{V},1} \begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix} (\widetilde{\omega}, z)}{Z^{(2)}_{\mathscr{V}} \begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix} (\Omega^{(2)})}, \tag{17}$$

where $\mathscr{S}^{(2)} = dz^2 \lim_{w \to z} \left(\frac{1}{(w-z)^2} + \frac{1}{2} \left(\partial_w - \partial_z\right) K^{(2)}(w, z) \right)$. An alternative expression for this is given below (18), [38].

2.2.4 Example: Genus Two Ward Identity

It turns out that the genus two Riemann surface Virasoro one-point differential form can be represented by means of the Ward identity as a flat connection in certain bundle. Let $Z_M^{(2)}$ be the genus two partition function for the rank one Heisenberg VOA M (see (1) for explicit formula). Using results of [29] we proved in [38] that the Virasoro one-point normalized differential form for the rank two fermion VOSA satisfies the genus two Ward identity

$$\delta \cdot \vartheta^{(2)} \begin{bmatrix} \alpha^{(2)} \\ \beta^{(2)} \end{bmatrix} \left(\Omega^{(2)}\right) = e^{2\pi i \alpha^{(2)} \cdot \beta^{(2)}} \frac{\mathscr{F}^{(2)}_{\mathscr{V},1} \begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix} (\tilde{\omega}, z)}{Z_M^{(2)}(\Omega^{(2)})}, \tag{18}$$

where $\delta = \left[\mathscr{D}^{(2)} + \frac{1}{12} s^{(2)}(z)\right]$. Here the expression for the normalized genus two one-point differential form is represented as an action of a differential operator on an automorphic function. One can generalize (18) to genus g formula.

3 Appendix: Correlation Functions for Vertex Operator Algebras on Riemann Surfaces

The algebraic approach to determination of the analytic and automorphic structures of vertex algebra characters was developed along recent years in particular in [25–32, 38–42]. Vertex operator algebras (VOAs) [1, 3, 7, 15, 16, 19, 21] give rise to non-trivial relations among the representation theory, analytic number theory, and algebraic geometry. They also serve as an algebraic tool to generate automorphic forms as higher characters over VOA modules.

Complete descriptions of all n-point functions for the Heisenberg and lattice VOAs in [25–31] and for free fermionic vertex operator superalgebras (VOSAs) in [32, 38, 41] have been given. In these cases, modularity properties of the partitions and all n-point functions with respect to appropriate groups follow from algebraic features of vertex operator algebras. A modification of the Zhu reduction procedure [46] expressing $n+1$-point correlation functions via a finite sum of n-point functions with coefficients being twisted higher Weierstrass functions was introduced in [32]. In this algebraic approach we can find closed formulas and explicitly establish modular properties for bosonic and fermionic characters on the torus [32, 42]:

$$Z^{(1)}_{\mathscr{V}}(\mathbf{v}, \mathbf{z}; q) = \langle \mathscr{Y}_f(v_1, z_1) \ldots \mathscr{Y}_f(v_n, z_n) \rangle, \tag{19}$$

where $\mathscr{V}$ is a vertex operator superalgebra V-module, $q = e^{2\pi i \tau}$ for the torus modular parameter τ, $v_i \in V$, and formal VOSA parameters z_i being associated to local coordinates around n points on the torus, and

$$\langle \mathscr{Y}_f(v_1, z_1) \ldots \mathscr{Y}_f(v_n, z_n) \rangle = \mathrm{STr}_{\mathscr{V}} \left(\mathscr{Y}_f(v_1, z_1) \ldots \mathscr{Y}_f(v_n, z_n)\, q^{L_f(0) - C/24} \right),$$

where $L(0)$ is the zero mode of the Virasoro algebra, and C its central charge. The Zhu reduction procedure [46] shows that (19) is then equivalent to

$$\det M = \frac{Z_{\mathscr{V}}^{(1)}(\mathbf{v}, \mathbf{z}; q)}{Z_{\mathscr{V}}^{(1)}(q)},$$

where $Z_{\mathscr{V}}^{(1)}(q)$ is the torus partition function and for some matrix M.

A definition of genus two characters was given in [38, 41]. We can define [42] the genus g partition function by inductively sewing together lower genus 1-point functions $Z_{\mathscr{V}}^{(g_i)}(u, z_i)$, $i = 1, 2$, $z_i \in \Sigma^{(g_i)}$, on genus g_i Riemann surfaces $\Sigma^{(g_i)}$, $g = g_1 + g_2$, which are used to form a genus g Riemann surface $\Sigma^{(g)}$ in a sewing scheme (for a review of Riemann surface sewing schemes [44], (half-order) differentials on Riemann surfaces, properties of (twisted) elliptic functions, prime form and Szegő kernels see [39]). Here u belongs to the nth grading subspace of V, for a V-module $\mathscr{V}$:

$$Z_{\mathscr{V}}^{(g)}(\epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in V_{[n]}} Z_{\mathscr{V}}^{(g_1)}(u, z_1)\; Z_{\mathscr{V}}^{(g_2)}(\bar{u}, z_2),$$

$$Z_{\mathscr{V}}^{(g_1, g_2)}(\epsilon) = \langle Z_{\mathscr{V}}^{(g_1)}(u, z_1)\; Z_{\mathscr{V}}^{(g_2)}(\bar{u}, z_2)\rangle,$$

where z_1, z_2 are insertion points, ϵ is a sewing parameter, and $\bar{u}, u$ are related by a non-degenerate bilinear invariant form on V. Genus g n-point functions can be defined by similar formulas. For further discussion on computation of a vertex operator superalgebra partition and n-point functions on Riemann surfaces and determination of their modular properties see [31, 38–42].

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