Chapter 4
Simplicial Toric Varieties Which Are Set-Theoretic Complete Intersections

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Abstract We say that a polynomial ideal I is set-theoretically generated by a family of elements $f_1, \ldots, f_k$ in I if the radical of I coincides with the radical of the ideal generated by $f_1, \ldots, f_k$. Over an algebraically closed field, the smallest number among all such possible k is the minimal number of equations needed to define the zero set of I. To find this number is a classical problem in both Commutative Algebra and Algebraic Geometry. This problem is even not solved for the defining ideals of toric varieties, whose zeros are given parametrically by monomials. In this lecture notes we study set-theoretically generation of the defining ideals of simplicial toric varieties, which are defined by the property that the exponents of the parametrizing monomials span a simplicial complex. We review and improve most of results on simplicial toric varieties which are set-theoretic complete intersections, previously obtained by the author in collaboration with M. Barile and A. Thoma.

4.1 Introduction

In the beginning of Algebraic Geometry, varieties were described by equations. However, such description is ambiguous. In order to be more precise, the notion of ideal (defining a variety) was introduced. But if we define a variety as the zero set of a polynomial ideal, there is still ambiguity because different ideals can have the same zero set. The famous Hilbert Nullstellensatz helps us to understand this phenomenon better.

More precisely, let $S := K[X_1, \ldots, X_n]$ be a polynomial ring over a field K. Let $\mathbb{A}^n_K$ be the affine n-dimensional space over K. Given a set $f_1, \ldots, f_k$ of polynomials, the zero set

$$Z(f_1, \ldots, f_k) = \{P \in \mathbb{A}^n_K \mid f_i(P) = 0\ \forall i = 1, \ldots, k\}$$

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is called an algebraic set. It is also the zero set $Z(I)$ of the ideal $I = (f_1, \dots, f_k)$.

For any subset $Y \subset \mathbb{A}^n_K$, we define the ideal of Y by

$$I(Y) = \{f \in S \mid f(P) = 0\ \forall P \in Y\}.$$

For an algebraic set V, the ideal $I(V)$ is called the defining ideal of V. It is clear that if $I(V) = (f_1, \dots, f_s)$, then $V = \cap_{i=1}^{s} Z(f_i)$, i.e. V is the intersection of the hypersurfaces $Z(f_i)$. However, there are many ways to define V as an intersection of hypersurfaces. An important problem in Algebraic Geometry is to determine the minimum number of equations needed to define an algebraic set V set-theoretically, that is the minimal number s such that $V = \cap_{i=1}^{s} Z(f_i)$ for a family of s polynomials $f_1, \dots, f_s \in K[X_1, \dots, X_n]$. An important tool in the study of this problem is:

Theorem 4.1.1 (Hilbert's Nullstellensatz) *Let K be an algebraically closed field. Then for any family of polynomials $f_1, \dots, f_s$, we have*

$$I(Z(f_1, \dots, f_s)) = \text{rad}\,(f_1, \dots, f_s).$$

This result leads to the following definition.

Definition 4.1.2 The arithmetical rank of an algebraic set $V \subset \mathbb{A}^n_K$ is the number

$$ara(V) = \min\{k \mid \exists f_1, \dots, f_k \in S : I(V) = \text{rad}\,(f_1, \dots, f_k)\},$$

and the arithmetical rank of an ideal I is

$$ara(I) = \min\{k \mid \exists f_1, \dots, f_k \in S : \text{rad}\, I = \text{rad}\,(f_1, \dots, f_k)\}.$$

Let $\mathbb{P}^n_K$ be the projective n-dimensional space over K. Similarly, one can define an algebraic set in $\mathbb{P}^{n-1}_K$ as the zero set of a family of homogeneous polynomials in S. For any subset $Y \subset \mathbb{P}^{n-1}_K$, one define $I(Y)$ to be the ideal generated by the homogeneous polynomials $f \in K[X_0, \dots, X_n]$ vanishing on Y. Then we also have the homogeneous Hilbert Nullstellensatz, and we can define the arithmetical rank of an algebraic set in $\mathbb{P}^n_K$ or of a homogeneous ideal in $K[X_0, \dots, X_n]$.

Thus, if K is an algebraically closed field, we have $ara(Z(I)) = ara(I)$ for any ideal I (homogeneous or not). However, it is more convenient to work over any field K and on set-theoretic generation of ideals. From now on, when we consider affine or projective algebraic sets, we only take care of their defining ideals. For an arbitrary ideal I, we always have the following inequalities:

$$ht(I) \le ara(I) \le \mu(I).$$

Here, $ht(I)$ denotes the height and $\mu(I)$ the minimal number of generators of I. When $h(I) = ara(I)$, the ideal I as well as the algebraic set $V = Z(I)$ is called a *set-theoretic complete intersection (s.t.c.i)*. When $ht(I) = \mu(I)$, it is called a *complete intersection*. It is called an *almost set-theoretic complete intersection* if $ara(I) \leq ht(I) + 1$.

In this lecture notes we focus on toric ideals and toric varieties whose precise definition will be given in Sect. 4.2. Toric ideals and toric varieties play an important role in both Commutative Algebra and Algebraic Geometry because they serve as models for general algebraic varieties. Toric ideals are generated by binomials. Moreover, each binomial is a difference of two monomials with coefficients equal to 1. A rather systematic study of binomial ideals (i.e. generated by binomials) was done by Eisenbud and Sturmfels in [7]. There are numerous publications on minimal generation of a binomial ideal or of its radical, see, for example, [12] Chapter V and [1, 2, 4, 9, 10, 13–15, 19, 22].

The *binomial arithmetical rank* $bar(I)$ of a binomial ideal I is the smallest integer s for which there exist binomials $f_1, \ldots, f_s$ in S such that $rad(I) = rad(f_1, \ldots, f_s)$. This intermediate invariant is, on one side, easier to compute. On the other side, it gives an upper bound for the arithmetical rank of a binomial ideal I as we always have:

$$ht(I) \leq ara(I) \leq bar(I) \leq \mu(I).$$

Using binomial arithmetic rank, one has obtained many results on set-theoretic complete intersections. In this lecture notes we review, and sometimes improve, some of these results.

The main results are (see Sects. 4.2, 4.3 for the used notations):

1. *In characteristic $p > 0$, every simplicial toric affine or projective variety with almost full parametrization is a set-theoretic complete intersection.* This extends previous results by Hartshorne [10], Moh [13], and Barile et al. [2].
2. *In any characteristic, every simplicial toric affine or projective variety with full parametrization is an almost set-theoretic complete intersection.* We give a more transparent proof of this result, which is due to Barile et al. [2].
3. *Let $\overline{V}(p, q, r)$ be the projective toric curve in $\mathbb{P}^3_K$ with parametrization*

$$w = u^r, x = u^{r-p}v^p, y = u^{r-q}v^q, z = v^r.$$

   Then $\overline{V}(p, q, r)$ in $\mathbb{P}^3$ is a set-theoretic complete intersection for $r \gg 0$.
4. *Let $p, q_0, q_1, \ldots, q_{n-1}$ be positive integers. Let $\overline{V}(p, q_0, q_1, \ldots, q_{n-2}) \subset \mathbb{P}^n_K$ be the projective toric curve with parametrization*

$$\begin{aligned} w &= u^{q_{n-2}}, \\ x &= u^{q_{n-2}-p}v^p, \\ y &= u^{q_{n-2}-q_0}v^{q_0}, \\ z_1 &= u^{q_{n-2}-q_1}v^{q_1}, \\ &\cdots \\ z_{n-2} &= v^{q_{n-2}}. \end{aligned}$$

Let $\overline{V}_1(p, q_0, q_1, \ldots, q_{n-2}, q_{n-1}) \subset \mathbb{P}_K^{n+1}$ *be the projective curve defined by*

$$\begin{aligned} w &= u^{q_{n-1}}, \\ x &= u^{q_{n-1}-p} v^{p}, \\ y &= u^{q_{n-1}-q_0} v^{q_0}, \\ z_1 &= u^{q_{n-1}-q_1} v^{q_1}, \\ &\ldots \\ z_{n-2} &= u^{q_{n-1}-q_{n-2}} v^{q_{n-2}}, \\ z_{n-1} &= v^{q_{n-1}}. \end{aligned}$$

Let $\gcd(p, q_{n-2}) = l$, $p' = p/l, q' = q_{n-2}/l$. *Assume that* $q_{n-1} \geq p'q'(q'-1) + q'l$. *If* $\overline{V}(p, q_0, q_1, \ldots, q_{n-2})$ *is a set-theoretic complete intersection, then so is* $\overline{V}_1(p, q_0, q_1, \ldots, q_{n-2}, q_{n-1})$.

Moreover, the proofs presented here are constructive. It should be mentioned that there is no general way to study set-theoretically generation of ideals. This is not surprising because one can not give an answer to this most famous problem on this subject, which deals a very simple case of projective curve in $\mathbb{P}_K^3$:

Question 4.1.3 Assume that K is a field of characteristic 0. Let $\overline{V}(1, 3, 4)$ be the projective toric curve with parametrization

$$w = u^4, x = u^3v^1, y = u^1v^3, z = v^4.$$

Is $\overline{V}(1, 3, 4)$ a set-theoretic complete intersection?

4.2 Definition of Toric Varieties by Parametrization, Semigroups or Lattices

There are several ways to introduce a toric variety, which is associated with a set of n vectors $\mathbf{a_i} = (a_{i,1}, \ldots, a_{i,m}) \in \mathbb{Z}^m, i = 1, \ldots, n$.

1. **Parametrization.** A *toric variety* $V \subset K^n$ is a variety having a following parametrization of the form

$$\begin{aligned} x_1 &= \underline{u}^{\mathbf{a_1}}, \\ &\vdots \\ x_n &= \underline{u}^{\mathbf{a_n}}, \end{aligned}$$

where $\underline{u}^{\mathbf{a_i}} = u_1^{a_{i,1}} \cdots u_m^{a_{i,m}}$, $i = 1, \ldots, n$, are monomials in a polynomial ring $K[u_1, \ldots, u_m]$. Sometimes we simply say that V is parametrized by $\underline{u}^{\mathbf{a_1}}, \ldots, \underline{u}^{\mathbf{a_n}}$.

2. **Semigroups.** The coordinate ring of the above toric variety is isomorphic to the subring $K[\underline{u}^{\alpha}, \alpha \in \Sigma_A] \subset K[u_1, \ldots, u_m]$. This subring can be considered as the semigroup ring $K[\Sigma_A]$ of the semigroup

$$\Sigma_A = \mathbb{N}\mathbf{a_1} + \ldots + \mathbb{N}\mathbf{a_n} \subset \mathbb{Z}^m.$$

Note that $K[\Sigma_A]$ is a domain and that $\dim K[\Sigma_A] = \operatorname{rank} A$, where A is the $m \times n$ matrix whose i-th column vector is $\mathbf{a_i}$.

There is a canonical surjective map $\Psi : S = K[X_1, \ldots, X_n] \to K[\Sigma_A]$. Let $I_A = \ker \Psi$. Then I_A is the defining ideal of the toric variety in S. One calls I_A a *toric ideal.*

We give now a short proof of the fact that I_A is generated by binomials. Observe that

- For any non zero monomial $M \in S$ its image $\Psi(M)$ is non zero.
- For any monomials $M_1, M_2 \in S$, if $\Psi(M_1) = \Psi(M_2)$ then $M_1 - M_2 \in I_A$.
- For any non zero monomials $M_1, M_2 \in S$, if $\Psi(M_1) \neq \Psi(M_2)$ then any linear combination $\alpha\Psi(M_1) + \beta\Psi(M_2)$ with $(\alpha, \beta) \in (K^2)^*$ is non zero.

Let $F = \sum_{i=1}^{t} \alpha_i M_i \in I_A$, where $\alpha_i \in K^*$ and M_i is a nonzero monomial, $i = 1, \ldots, t$. By the observation above we may assume that $\Psi(M_i) = \Psi(M_j)$ for any $i, j = 1, \ldots, t$. It is clear that this implies $\sum_{i=1}^{t} \alpha_i = 0$ and consequently $\alpha_1 = -\sum_{i=2}^{t} \alpha_i$. Hence $F = \sum_{i=2}^{t} \alpha_i (M_i - M_1)$. That shows that the toric ideal I_A is generated by binomials of the type $M - N$, where M, N are monomials with coefficients 1 without common divisor.

3. **Lattice of relations.** Note that any vector $\alpha \in \mathbb{Z}^n$ can be uniquely written as $\alpha = \alpha_+ - \alpha_-$, with $\alpha_+, \alpha_- \in \mathbb{N}^n$ such as $(\alpha_+)_i(\alpha_-)_i = 0$ for all $i = 1, \ldots, n$. Let

$$L_A := \{\alpha \in \mathbb{Z}^n \mid X^{\alpha_+} - X^{\alpha_-} \in I_A\}.$$

Then $L_A \subset \mathbb{Z}^n$ is a subgroup of finite rank. We call it the *lattice of relations* associated to I_A. It is easy to see that $L_A \subset \mathbb{Z}^n$ is the set of integer solutions of the linear system $AX = 0$.

In general, given a subgroup of finite rank (lattice) $L \subset \mathbb{Z}^n$, we can define the ideal $I_L \subset S$ generated by the binomials $X^{\alpha_+} - X^{\alpha_-}$, $\alpha \in L$. It is called the *lattice ideal* associated to L. We call L saturated if $dv \in L$ for some $d \in \mathbb{Z} \setminus \{0\}$, $v \in \mathbb{Z}^n$, implies $v \in L$.

Remark The lattice of relations of a toric ideal I_A is saturated and has the property

$$I_{L_A} = I_A.$$

For any vector $\alpha \in \mathbb{Z}^n$, we set $F_\alpha := X^{\alpha+} - X^{\alpha-}$. Note that F_α is a reduced binomial, that is it can't be factored by a monomial.

Lemma 4.2.1 *Let I_A be a toric ideal and $\mathbf{v_1}, \ldots, \mathbf{v_r}$ a basis of L_A. Let $F_{\mathbf{v_i}} \in I_A$ be the binomial associated to $\mathbf{v_i}$. Then*

$$Z(F_{\mathbf{v_1}}, \ldots, F_{\mathbf{v_r}}) \cap (K^*)^n = V(I_A) \cap (K^*)^n.$$

Proof We have only to prove the inclusion $Z(F_{\mathbf{v_1}}, \ldots, F_{\mathbf{v_r}}) \cap (K^*)^n \subset V(I_A)$.

Let $P \in Z(F_{\mathbf{v_1}}, \ldots, F_{\mathbf{v_r}}) \cap (K^*)^n$. Then $F_{\mathbf{v_1}}(P) = 0, \ldots, F_{\mathbf{v_r}}(P) = 0$. Let $F \in I_A$ be any reduced binomial then there exist $\mathbf{v} \in L_A$ such that $F = X^{\mathbf{v}+} - X^{\mathbf{v}-}$. Since $\mathbf{v_1}, \ldots, \mathbf{v_r}$ is a basis of L_A, we can write $\mathbf{v} = \alpha_1 \mathbf{v_1} + \cdots + \alpha_r \mathbf{v_r}$ for some integers α_i. Let $P = (x_1, \ldots, x_n) \in (K^*)^n$. We have $\underline{x}^{\mathbf{v_i}+} - \underline{x}^{\mathbf{v_i}-} = 0$ for $i = 1, \ldots, r$. Since $P = (x_1, \ldots, x_n) \in (K^*)^n$, this is equivalent to $\underline{x}^{\mathbf{v_i}} = 1$ for $i = 1, \ldots, r$, which implies $\underline{x}^{\alpha_i \mathbf{v_i}} = 1$ for $i = 1, \ldots, r$. Hence $1 = \underline{x}^{\sum_{i=1}^{r} \alpha_i \mathbf{v_i}} = \underline{x}^{\mathbf{v}}$, and so $F(P) = 0$.

The following result [7, Corollary 2.6] gives an exact relationship between binomial ideals and toric ideals.

Theorem 4.2.2 *Let K be an algebraically closed field. A binomial ideal is toric if and only if it is prime.*

For simplicity, we say that a binomial ideal is a set-theoretic complete intersection of binomials if $bar(I) = ht(I)$. We have the following theorem from [16].

Theorem 4.2.3 *Let K be a field of characteristic zero. A toric ideal is a set-theoretic complete intersection of binomials if and only if it is a complete intersection.*

By virtue of this theorem, we always assume that our toric ideal is not a complete intersection in the rest of the lecture notes.

4.3 Simplicial Toric Varieties Which Are Set-Theoretic Complete Intersections

Most of the results on set-theoretic complete intersections in this lecture notes concern the following class of toric varieties.

Let $\mathbf{e_1}, \ldots, \mathbf{e_n}$ denote the elements of the canonical basis of $\mathbb{Z}^n$. Let $\mathbf{a_i} = (a_{i,1}, \ldots, a_{i,n})$, $i = 1, \ldots, r$, be non zero vectors in $\mathbb{N}^n$.

Definition 4.3.1 Let A be a matrix with column vectors $d_1\mathbf{e_1}, \ldots, d_n\mathbf{e_n}, \mathbf{a_1}, \ldots, \mathbf{a_r}$, where $d_1 \ldots, d_n \in \mathbb{N}^*$, that is

$$A = \begin{pmatrix} d_1 & 0 & \ldots & 0 & a_{1,1} & \ldots & a_{r,1} \\ 0 & d_2 & \ldots & 0 & a_{1,2} & \ldots & a_{r,2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & d_n & a_{1,n} & \ldots & a_{r,n} \end{pmatrix}.$$

Then I_A is called the *simplicial toric ideal* associated to A and its affine variety $V_A = V(I_A)$ in K^{n+r} is called an *affine simplicial toric variety*.

In this case, the dimension of the affine semigroup ring $K[\Sigma_A]$ is n. Note that V_A has codimension $r \geq 2$ in K^{n+r} and has the following parametrization:

$$\begin{aligned} x_1 &= u_1^{d_1}, \\ &\vdots \\ x_n &= u_n^{d_n}, \\ y_1 &= u_1^{a_{1,1}} \cdots u_n^{a_{1,n}}, \\ &\vdots \\ y_r &= u_1^{a_{r,1}} \cdots u_n^{a_{r,n}}, \end{aligned}$$

One can define a *projective simplicial toric variety* similarly as above. For that we need to assume that $d_1 = \cdots = d_r = \deg u^{\mathbf{a_i}}$ for all $i = 1, \ldots, r$.

For any vector $\mathbf{v} \in \mathbb{Z}^m$, we set $\text{supp}\,(\mathbf{v}) = \{j \in \{1, \ldots, m\} \mid v_j \neq 0\}$ and call it the support of $\mathbf{v}$.

Definition 4.3.2 We say that the parametrization of V_A is *full* if $\text{supp}\,\mathbf{a_i} = \text{supp}\,\mathbf{a_j}$ for $i, j = 1, \ldots, r$. The parametrization of V_A is *almost full* if $\text{supp}\,\mathbf{a_1} \subset \text{supp}\,\mathbf{a_2} \subset \cdots \subset \text{supp}\,\mathbf{a_r}$.

Note that when working with full or almost full parametrization we may always assume that $\text{supp}\,\mathbf{a_r} = \{1, \ldots, m\}$.

In this section we extend the results on simplicial varieties with full parametrization of [2] to those with almost full parametrization. Namely, we will prove the following results.

1. *In characteristic $p > 0$, any simplicial toric affine or projective variety with almost full parametrization is a set-theoretic complete intersection* (see Theorem 4.3.8).
2. *In any characteristic, any simplicial toric affine or projective variety with full parametrization is an almost set-theoretic complete intersection* (see Theorem 4.3.11).

4.3.1 Lattice of Relations of Simplicial Toric Varieties

As we said above for toric ideals I_A the lattice L_A is the set of integer solutions of the linear system $AX = 0$. That is the problem of finding binomials in I_A is equivalent to finding solutions of $AX = 0$ or more generally of $AX = \mathbf{b}$. For any matrix with integer coefficients A, we set $\mid A \mid$ to be the greatest common divisor of all its maximal minors. We say that the matrix A, has *full rank* if at least one of its maximal minors is non null. Suppose that A has full rank. If there exists one column vector for which some integer multiple belongs to the lattice generated by the other column vectors, we can delete this column vector preserving our search of solutions for the equation $AX = \mathbf{b}$. That means that we can assume that all the maximal minors are non zero.

We have the following basic lemma in Number Theory (see [11], or for a modern presentation, [23, p. 51]):

Lemma 4.3.3 *Assume that $|A| \neq 0$. The linear Diophantine system $AX = \mathbf{b}$ has an integer solution if and only if $|A| \neq 0$ and $|A| = |A\mathbf{b}|$, where $A\mathbf{b}$ is the augmented matrix.*

Another important ingredient is given by the chapter IV of [5] about basis of Lattices. We learn in this chapter that we can find triangular basis of a lattice that we will describe thanks to Lemma 4.3.3 in the case of simplicial toric varieties.

With the notations of Definition 4.3.1, for all $i = 0 \ldots, r$, let A_i be the matrix:

$$A_i = \begin{pmatrix} d_1 & 0 & \ldots & 0 & a_{1,1} & \ldots & a_{i,1} \\ 0 & d_2 & \ldots & 0 & a_{1,2} & \ldots & a_{i,2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & d_n & a_{1,n} & \ldots & a_{i,n} \end{pmatrix}.$$

We denote by $\mathbf{d_i}$ the ith column vector of A for all $i = 1, \ldots, n$, and by $\mathbf{a_i}$ the $(n + i)$th column vector of A for all $i = 1, \ldots, r$. We set $D[j_1, \ldots, j_n]$ the determinant of the $n \times n$ submatrix consisting of the columns of A with the indices $j_1, \ldots, j_n$, where $\{j_1, \ldots, j_n\}$ is an n-subset of $\{1, 2, \ldots, n+r\}$. For all $i = 0, \ldots, r$ let $|A_i| := \gcd\{D[j_1, \ldots, j_n] : 1 \leq j_1 < j_2 < \cdots < j_n \leq n + i\}$; for the sake of simplicity we set $g_i = |A_i|$. Moreover, let $\zeta_i = g_{i-1}/g_i$, for all $i = 1, \ldots, r$.

Let us remark that any integer solution α of the linear system $A_i Z = 0$ gives rise to a binomial, more precisely, let write $\alpha = \beta + \gamma$, with $\operatorname{supp} \beta \subset \{1, 2, \ldots, n\}$, $\operatorname{supp} \gamma \subset \{n + 1, n + 2, \ldots, n + i\}$, then the binomial $F_{\alpha+\beta} = \underline{x}^{\beta_+} y^{\gamma_+} - x^{\beta_-} y^{\gamma_-}$ in the variables $x_1, \ldots, x_n, y_1, \ldots, y_i$ belongs to I_A.

In our situation we have the following corollary of Lemma 4.3.3 which can be seen as a generalization of [15, Remark 2.1.2]:

Theorem 4.3.4 *Keep the above notations. Then*

1. *For any $i = 1, \ldots, r$, the linear Diophantine system $A_{i-1} Z = \theta \mathbf{a_i}$ has an integer solution if and only if $\theta \in \zeta_i \mathbb{Z}$.*
2. *The lattice $L_A \subset \mathbb{Z}^{n+r}$ of rank r has a triangular basis:*

$$\{(\mathbf{w_1}, s_{(1,1)}, 0, \ldots, 0), \quad (\mathbf{w_2}, s_{(2,1)}, s_{(2,2)}, 0, \ldots, 0), \quad \ldots,$$
$$(\mathbf{w_r}, s_{(r,1)}, s_{(r,2)}, \ldots, s_{(r,r)})\},$$

where $\mathbf{w_1}, \ldots, \mathbf{w_r} \in \mathbb{Z}^n$ and $s_{(i,i)} = \zeta_i$.

3. *Let* $\mathbf{s_1} = (s_{(1,1)}, 0, \ldots, 0), \mathbf{s_1} = (s_{(2,1)}, s_{(2,2)}, 0, \ldots, 0), \ldots, \mathbf{s_r} = (s_{(r,1)}, s_{(r,2)}, \ldots, s_{(r,r)})$. *For* $i \in \{1, \ldots, r\}$ *we have the reduced binomials*

$$F_{\mathbf{w_i}+\mathbf{s_i}} := M_i - N_i y_i^{\zeta_i} \in I_A,$$

where M_i, N_i *are monomials in* $K[x_1, \ldots, x_n, y_1, \ldots, y_{i-1}]$.
4. $Z(F_{\mathbf{w_1}+\mathbf{s_1}}, \ldots, F_{\mathbf{w_r}+\mathbf{s_r}}) \cap (K^*)^{n+r} \subset V_A$.

Proof

1. We have $g_0 = d_0 d_1 \ldots d_n$ and for all $1 \le i \le r$, the numbers g_{i-1} are non null. On the other hand it holds:

$$g_i = \gcd\{g_{i-1}, D[j_1, \ldots, j_{n-1}, n+i] : 1 \le j_1 < j_2 < \cdots < j_n \le n+i-1\}, \tag{4.1}$$

which yields

$$1 = \gcd\{\frac{g_{i-1}}{g_i}, \frac{D[j_1, \ldots, j_{n-1}, n+i]}{g_i} : 1 \le j_1 < j_2 < \cdots < j_n \le n+i-1\}, \tag{4.2}$$

$$\begin{aligned}|A_{i-1}, \theta\mathbf{a_i}| &= \gcd\{g_{i-1}, \theta D[j_1, \ldots, j_{n-1}, n+i] : 1 \le j_k \le n+i-1\}\\ &= \gcd\{(\frac{g_{i-1}}{g_i})g_i, \theta D[j_1, \ldots, j_{n-1}, n+i] : 1 \le j_k \le n+i-1\}\\ &= g_i \gcd\{(\frac{g_{i-1}}{g_i}), \theta\frac{D[j_1, \ldots, j_{n-1}, n+i]}{g_i} : 1 \le j_k \le n+i-1\}\end{aligned}$$

Hence $|A_{i-1}, \theta\mathbf{a_i}| g_{i-1} = |A_{i-1}|$ if and only if

$$g_i \gcd\{(\frac{g_{i-1}}{g_i}), \theta\frac{D[j_1, \ldots, j_{n-1}, n+i]}{g_i} : 1 \le j_k \le n+i-1\} = g_{i-1},$$

or equivalently

$$\gcd\{\zeta_i, \theta\frac{D[j_1, \ldots, j_{n-1}, n+i]}{g_i} : 1 \le j_k \le n+i-1\} = \zeta_i.$$

Using (2) it implies that $|A_{i-1}, \theta\mathbf{a_i}| g_{i-1} = |A_{i-1}|$ if and only if $\theta \in \zeta_i\mathbb{Z}$.
2. By the first part, for every $i \in \{1, \ldots, r\}$ the Diophantine system $A_{i-1}\mathbf{x} = \zeta_i\mathbf{a_i}$ always has a solution. This means that the vector $\zeta_i\mathbf{a_i}$ can be expressed as a linear combination of the vectors $\mathbf{d_1}, \ldots, \mathbf{d_n}, \mathbf{a_1}, \ldots, \mathbf{a_{i-1}}$ with integer coefficients, i.e., one has

$$\zeta_i\mathbf{a_i} = w_{(i,1)}\mathbf{d_1} + \cdots + w_{(i,n)}\mathbf{d_n} + s_{(i,1)}\mathbf{a_1} + \cdots + s_{(i-1,i-1)}\mathbf{a_{i-1}}, \tag{4.3}$$

for some integers $w_{(i,j)}, \dots, s_{(i,j)}$. Setting for every $i \in \{1, \dots, r\}$ $\mathbf{w_i} = (w_{(i,1)}, \dots, w_{(i,n)})$, we have that

$$\{(\mathbf{w_1}, s_{(1,1)}, 0, \dots, 0), \ \ (\mathbf{w_2}, s_{(2,1)}, s_{(2,2)}, 0, \dots, 0), \ \ \dots,$$
$$(\mathbf{w_r}, s_{(r,1)}, s_{(r,2)}, \dots, s_{(r,r)})\},$$

is a triangular basis of L_A.

3. The expression (3) gives us monomials M_i, N_i in $K[x_1, \dots, x_n, y_1, \dots, y_{i-1}]$ such that $F_{\mathbf{w_i}+\mathbf{s_i}} := M_i - N_i y_i^{\zeta_i}$.
4. Follows from the above items and Lemma 4.2.1.

Triangular basis will give us some particular binomials which will play an important role in our proofs.

Remark For the sake of simplicity we shall set $\mathbf{s} = (s_1, \dots, s_{r-1})$, $\underline{y} = (y_1, \dots, y_{r-1})$. In particular, if $(\mathbf{w}, \mathbf{s}, t) \in L_A$, then $t \in \zeta_r\mathbb{Z}$ and, conversely, for all multiples t of ζ_r there is $\mathbf{s} \in \mathbb{Z}^{r-1}$, $\mathbf{w} \in \mathbb{Z}^n$ such that $(\mathbf{w}, \mathbf{s}, t) \in L_A$.

For all $\mathbf{s} \in \mathbb{Z}^{r-1}$, we can write $\mathbf{s} = \mathbf{s}_+ - \mathbf{s}_-$. Fix an element $(\mathbf{w}, \mathbf{s}, s_r) \in L_A$. Let $\mathbf{w} = \mathbf{w}_+ - \mathbf{w}_-$. Then the binomial corresponding to $(\mathbf{w}, \mathbf{s}, s_r) \in L_A$ is

$$\underline{y}^{\mathbf{s}_+} \underline{x}^{\mathbf{w}_+} - y_r^{-s_r} \underline{y}^{\mathbf{s}_-} \underline{x}^{\mathbf{w}_-},$$

provided $s_r \le 0$; otherwise it is

$$\underline{y}^{\mathbf{s}_+} y_r^{s_r} \underline{x}^{\mathbf{w}_+} - \underline{y}^{\mathbf{s}_-} \underline{x}^{\mathbf{w}_-}.$$

Remark Let

$$J = I_A \cap K[x_1, \dots, x_n, y_1, \dots, y_{r-1}].$$

Then J is the defining ideal of the simplicial toric variety of codimension $r - 1$ having the following parametrization:

$$\begin{aligned} x_1 &= u_1^{d_1}, \\ &\vdots \\ x_n &= u_n^{d_n}, \\ y_1 &= u_1^{a_{1,1}} \cdots u_n^{a_{1,n}}, \\ &\vdots \\ y_{r-1} &= u_1^{a_{r-1,1}} \cdots u_n^{a_{r-1,n}}. \end{aligned}$$

Note that if the parametrization of the variety defined by I_A is full (resp. almost full), then the parametrization of the variety defined by J satisfies the same property.

4.3.2 Simplicial Toric Varieties in Characteristic $p > 0$

We introduce one more piece of notation. Let M_1, M_2 be monomials, and let $h = M_1 - M_2$. For all positive integers q we set

$$h^{(q)} = M_1^q - M_2^q.$$

Lemma 4.3.5 *Let* $J = I_A \cap K[x_1, \ldots, x_n, y_1, \ldots, y_{r-1}]$, *and* $\delta > 0$ *an integer for which there is a binomial*

$$f_r = y_r^{\zeta_r \delta} - \underline{y}^{\mathbf{s}_\delta} x_1^{l_1} \cdots x_n^{l_n} \in I_A.$$

Then for any binomial h in I_A *we have*

$$h^{(\delta)} \in (J, f_r).$$

Proof Let $h \in I_A$ be a binomial. Since I_A is a prime ideal, we may assume that

$$h = y_r^{\zeta_r \rho} g_1 - g_2$$

for some monomials $g_1, g_2 \in K[x_1, \ldots, x_n, y_1, \ldots, y_{r-1}]$. Then

$$\begin{aligned} h^{(\delta)} &= y_r^{\zeta_r \rho \delta} g_1^\delta - g_2^\delta \\ &= (f_r^{(\rho)} + (\underline{y}^{\mathbf{s}_\delta} x_1^{l_1} \cdots x_n^{l_n})^\rho) g_1^\delta - g_2^\delta \\ &\in (J, f_r). \end{aligned}$$

Lemma 4.3.6 *Suppose that* $\operatorname{supp} \mathbf{a_r} = \{1, \ldots, m\}$. *For all sufficiently large integers* $\delta > 0$ *there is a binomial*

$$f_r = y_r^{\zeta_r \delta} - \underline{y}^{\mathbf{s}} x_1^{l_1} \cdots x_n^{l_n} \in I_A.$$

Proof Let $\delta > 0$. There is $\mathbf{s}'$ such that $(\mathbf{s}', -\zeta_r) \in Ker\Phi$. Hence there are integers $r_1', \ldots, r_n'$ such that for all i

$$\sum_{j=1}^{r-1} s'{}_j a_{j,i} - \zeta_r a_{r,i} = r_i' d_i$$

for all i. Multiplying this relation by $\delta > 0$ we get

$$\sum_{j=1}^{r-1} \delta s'{}_j a_{j,i} - \zeta_r \delta a_{r,i} = \delta r_i' d_i$$

for all i. Let $d = \text{lcm}\{d_1, \ldots, d_n\}$. Replacing $\delta s'_j$ by its residue s_j modulo d, we get a relation

$$\sum_{j}^{r-1} s_j a_{j,i} - \zeta_r \delta a_{r,i} = r_i d_i.$$

Thus, if δ is sufficiently large, we will have $r_i < 0$ for all i. Then $f_r = y_r^{\zeta_r \delta} - \underline{y}^{\mathbf{s}} x_1^{-r_1} \cdots x_n^{-r_n} \in I_A$ as required.

As an immediate consequence we have:

Corollary 4.3.7 *Suppose that* $\text{supp}\, \mathbf{a_r} = \{1, \ldots, m\}$. *Let* p *be a prime number. For any sufficiently large integer* m *there is a binomial*

$$f_r = y_r^{\zeta_r p^m} - \underline{y}^{\mathbf{s}} x_1^{l_1} \cdots x_n^{l_n} \in I_A.$$

The next theorem improves [2, Theorem 1], where the case of full parametrization was considered.

Theorem 4.3.8 *Suppose that char* $K = p > 0$. *Then every simplicial toric variety having an almost full parametrization is a set-theoretic complete intersection.*

Proof We proceed by induction on $r \geq 1$. Since the polynomial ring $K[x_1, \ldots, x_n, y_1]$ is an UFD the claim is true for $r = 1$.

Suppose that $r \geq 2$ and the claim is true in codimension $r - 1$. Let $h \in I_A$ be a binomial, then by Corollary 4.3.7 and Lemma 4.3.6, for m sufficiently large we get

$$h^{p^m} = h^{(p^m)} \in (f_r, J).$$

By the induction hypothesis the ideal J is set-theoretically generated by $r - 1$ binomials $f_1, \ldots, f_{r-1}$. Hence some power of h lies in $(f_1, \ldots, f_r)$.

Remark Note that the proof of the preceding result yields an effective and recursive construction of the defining equations of a simplicial toric variety having almost full parametrization over any field K of characteristic $p > 0$.

Exercise 4.3.9 Assume that K is a field of characteristic p. Let $\overline{V}(1, 3, 4)$ be the projective toric curve in $\mathbb{P}^3$ with parametrization

$$w = u^4, x = u^3 v^1, y = u^1 v^3, z = v^4.$$

1. Write the matrix A corresponding to $\overline{V}(1, 3, 4)$.
2. Use Theorem 4.3.4 to find a triangular basis of the Lattice L_A.
3. Give f_1, f_2 such that $\text{rad}\,(f_1, f_2) = I(\overline{V}(1, 3, 4))$ showing that in characteristic p, $\overline{V}(1, 3, 4)$ is a set-theoretic complete intersection.

4.3.3 Almost Set-Theoretic Complete Intersections

We have studied the case where the field K is of characteristic $p > 0$, so now we assume that the field K is algebraically closed of characteristic 0, since we will use the Hilbert's Nullstellensatz.

In this section we show that simplicial toric varieties having a full parametrization are almost set-theoretic complete intersections.

If the parametrization of V_A is full we will improve the triangular basis of I_A founded in Theorem 4.3.4.

Lemma 4.3.10 *Let V_A be a simplicial toric variety. If the parametrization of V_A is full, then for every $i = 2, \ldots, r$ there exists a binomial*

$$F_i = y_{i-1}^{\mu_i} - x_1^{\nu_{i,1}} \cdots x_n^{\nu_{i,n}} y_1^{\mu_{i,1}} \cdots y_{i-2}^{\mu_{i,i-2}} y_i^{\zeta_i} \in I_A,$$

and there also exists a binomial

$$F_1 = y_1^{\zeta_1} - x_1^{\nu_{1,1}} \cdots x_n^{\nu_{1,n}} \in I_A,$$

for some strictly positive integers μ_i, $\mu_{i,j}$ and $\nu_{i,j}$.

Proof In this proof, for all $i = 1, \ldots, n$, $\mathbf{d_i}$ will denote the ith column vector of A and for all $i = 1, \ldots, r$, $\mathbf{a_i}$ will denote the $(n+i)$th column vector of A.

Set $\mu = \gcd(d_1, \ldots, d_n)$ and $q_i = \gcd(\mu, a_{i,1}, \ldots, a_{i,n})$ for all $i = 1, \ldots, r$. For all $i = 1, \ldots, r$ and all $j = 1, \ldots, n$ let $\rho_{i,j} = a_{i,j}\mu/d_j q_j$. Then, for all $i = 1, \ldots, r$, one has that

$$G_i = y_i^{\mu/q_i} - x_1^{\rho_{i,1}} \cdots x_n^{\rho_{i,n}} \in I_A.$$

It is easy to see that $\zeta_1 = \mu/q_1$, then for $i = 1$ the preceding formula yields the required binomial F_1.

As we have seen in Theorem 4.3.4, for all $i = 1, \ldots, r$ the vector $\zeta_i \mathbf{a_i}$ can be expressed as a linear combination of the vectors $\mathbf{d_1}, \ldots, \mathbf{d_n}, \mathbf{a_1}, \ldots, \mathbf{a_{i-1}}$ with integer coefficients, i.e., one has

$$\zeta_i \mathbf{a_i} = w_{(i,1)}\mathbf{d_1} + \cdots + w_{(i,n)}\mathbf{d_n} + s_{(i,1)}\mathbf{a_1} + \cdots + s_{(i-1,i-1)}\mathbf{a_{i-1}}, \tag{4.2}$$

for some integers $w_{(i,j)}, \ldots, s_{(i,j)}$ and this expression gives us monomials M_i, N_i in $K[x_1, \ldots, x_n, y_1, \ldots, y_{i-1}]$ such that $M_i - N_i y_i^{\zeta_i} \in I_A$.

Now suppose that the parametrization of V_A is full. From the binomial G_j we see that for each $\mathbf{a_j}$ there exist positive integers $\rho_j = \mu/q_i, \rho_{j,1}, \ldots, \rho_{j,n}$ such that $\rho_j \mathbf{a_j} = \rho_{j,1}\mathbf{d_1} + \cdots + \rho_{j,n}\mathbf{d_n}$. Furthermore, for all $1 \leq j \leq i-2$ there exists a positive integer ν_j such that, after adding all the zero vectors $\nu_j(\rho_{j,1}\mathbf{d_1} + \cdots + \rho_{j,n}\mathbf{d_n} - \rho_j \mathbf{a_j})$ to the right-hand side of (2), the new coefficient $-\mu_{i,j}$ of $\mathbf{a_j}$ is negative for all $j = 1, \ldots, i-2$. There also exists a large positive integer ν_{i-1}

such that after adding the zero vector $\nu_{i-1}(\rho_{i-1}\mathbf{a_{i-1}} - (\rho_{i-1,1}\mathbf{d_1} + \cdots + \rho_{i-1,n}\mathbf{d_n}))$ on the right-hand side of the new equation, for all $j = 1, \ldots, n$ the new coefficient $-\nu_{i,j}$ of $\mathbf{d_j}$ is negative and the new coefficient μ_i of $\mathbf{a_{i-1}}$ is positive. It follows that for all $i = 2 \ldots, r$ we have a binomial

$$F_i = y_{i-1}^{\mu_i} - x_1^{\nu_{i,1}} \cdots x_n^{\nu_{i,n}} y_1^{\mu_{i,1}} \ldots y_{i-2}^{\mu_{i,i-2}} y_i^{\zeta_i} \in I_A.$$

Theorem 4.3.11 *Assume that K is algebraically closed field of characteristic 0. Let V_A be a simplicial toric variety having a full parametrization. Then $r \leq bar(I_A)) \leq r + 1$. In fact $bar(I_A) = r + 1$ unless I_A is a complete intersection.*

Proof Consider the r binomials $F_1, F_2, \ldots, F_r$ which were defined in Lemma 3 and let F_{r+1} be any binomial monic in y_r, for example G_r. We claim that $I_A = \mathrm{rad}\,(F_1, \ldots, F_{r+1})$.

By virtue of Hilbert Nullstellensatz the claim is proved once it has been shown that every point $\mathbf{x} = (x_1, \ldots, x_n, y_1, \ldots, y_r)$ which is a common zero of $F_1, \ldots, F_{r+1}$ in K^{n+r} is also a point of V_A. First of all note that if $x_k = 0$ for some index k, then $y_j = 0$ for all indices j. It is then easy to find $u_1, \ldots, u_n \in K$ which allow us to write $\mathbf{x}$ as a point of V_A. Now suppose that $x_k \neq 0$ for all indices k, $F_1(\mathbf{x}) = 0, \ldots, F_{r+1}(\mathbf{x}) = 0$, we have inductively that $y_1 \neq 0, \ldots, y_r \neq 0$. So we can assume that all the coordinates of $\mathbf{x}$ are non zero. Note that the vectors in L_A corresponding to $F_1, F_2, \ldots, F_r$ form a triangular basis of L_A, hence by applying Theorem 4.3.4 we have that $\mathbf{x}$ is a point of V_A.

Exercise 4.3.12 Assume that K is an algebraically closed field of characteristic 0. Let $V(1, 3, 4)$ be the projective toric curve in $\mathbb{P}^3$ with parametrization

$$w = u^4, x = u^3v^1, y = u^1v^3, z = v^4.$$

Use Exercise 4.3.9 and the above section to give F_1, F_2, F_3 binomials such that $\mathrm{rad}\,(F_1, F_2, F_3) = I(V(1, 3, 4))$.

4.4 Equations in Codimension 2

This section is an English shorten version of the results in [15].

In this section we suppose that $r = 2$, i.e., V_A is a simplicial toric variety of codimension 2 in K^{n+2}. The parametrization of V_A now is:

$$x_1 = u_1^{d_1},$$

$$\vdots$$

$$x_n = u_n^{d_n},$$

$$y_1 = u_1^{a_{1,1}} \cdots u_n^{a_{1,n}},$$
$$y_2 = u_1^{a_{2,1}} \cdots u_n^{a_{2,n}},$$

where the vectors $\mathbf{a_1}, \mathbf{a_2}$ may have zero components.

4.4.1 The Lattice Associated in Codimension Two

In this section, we introduce the reduced lattice associated to V_A, which determines the associated lattice L_A, in this particular case.

Consider the morphism of groups:

$$\Phi : \mathbb{Z}^2 \longrightarrow \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z} \quad (s, p) \mapsto (sb_1 - pc_1, \ldots, sb_n - pc_n)$$

Definition 4.4.1 The reduced lattice associated to V_A is

$$Ker(\Phi) := \{(s, p) \in \mathbb{Z}^2 \mid sb_i - pc_i \equiv 0 \, mod \, d_i, \forall i = 1, \ldots, n\}.$$

Remark $Ker(\Phi)$ is not the lattice of V_A in the sense given in Sect. 4.3, but it determines the lattice of V_A. For any $i = 1, \ldots, n$ there exists integers numbers l_i such that $sb_i - pc_i = l_i d_i$. To the vector $(s, p) \in Ker(\Phi)$ corresponds the vectors $(-l_1, \ldots, -l_n, s, -p)$ in the Lattice L_A. As a consequence, we associate to the vector $(s, p) \in Ker(\Phi)$ with $s \geq 0$ a binomial $F_{(-l_1,\ldots,-l_n,s,-p)} \in I_A$ and we call it the binomial associated to (s, p). Reciprocally, any vector $(\mathbf{w}, s, -p) \in L_A$, with $s \geq 0$, determines a unique $(s, p) \in Ker(\Phi)$.

Proposition 4.4.2 *We will define a fan decomposition of* $Ker(\Phi)$ *in* $\mathbb{R}^2_+$*, i.e. we will determine a family of vectors* $\epsilon_{-1}, \epsilon_0, \ldots, \epsilon_{m+1} \in Ker(\Phi) \cap \mathbf{Z}^2_+$ *such that* $\epsilon_i, \epsilon_{i+1}$ *is a base of* $Ker(\Phi)$*, with* $det(\epsilon_i, \epsilon_{i+1}) > 0$.

Proof We use the notion of base adapted to a lattice used in [5] p. 67. This allows us to determine a base $\epsilon_{-1}, \epsilon_0$ of $Ker(\Phi)$. Precisely $\epsilon_{-1} = (s_{-1}, 0)$, $\epsilon_0 = (s_0, p_0)$ where s_{-1} is the smallest positive integer $s \neq 0$ such that $(s, 0) \in Ker(\Phi)$ and p_0 is the smallest positive integer $p \neq 0$ such that there is a vector $(s, p) \in Ker(\Phi)$, s_0 is unique defined such that $s_0 < s_{-1}$.

Consider Euclide's algorithm, with negative rest, for the computation of the greatest common divisor, gcd (s_{-1}, s_0):

$$s_{-1} = q_1 s_0 - s_1$$
$$s_0 = q_2 s_1 - s_2$$
$$\ldots$$
$$s_{m-1} = q_{m+1} s_m$$

$$s_{m+1} = 0$$

$$\forall i \; q_i \geq 2 \; , \; s_i \geq 0.$$

Let us define the sequence: $p_i \; (-1 \leq i \leq m+1)$, by $p_{-1} = 0$ and:

$$p_{i+1} = p_i q_{i+1} - p_{i-1} \; , (0 \leq i \leq m).$$

We set $\epsilon_i = (s_i, p_i)$. By induction it is easy to check that $s_i p_{i+1} - s_{i+1} p_i = p_0 s_{-1}$ for all $-1 \leq i \leq m+1$, completing the proof.

In particular we have defined two sequences $\{s_i\}$, $\{p_i\}$.

Example 4.4.3 Let consider the projective monomial curve with parametrization:

$$X = s^{10}, Y = s^7 t^3 Z = s^3 t^7, W = t^{10}.$$

The lattice $Ker(\Phi)$ is given by the vectors (s, p) such that (r, r', s, p) is an integer solution of the system:

$$7s - 3p = 10r$$

$$s - 7p = 10r'$$

Note that the Lattice L_A is given by the vectors $(-r, -r', s, -p)$ such that (s, p, r, r') is an integer solution of the above system.

We have the following table

i	s_i	p_i	r_i	r'_i	q_i
−1	10	0	7	3	0
0	9	1	6	2	0
1	8	2	5	1	2
2	7	3	4	0	2
3	6	4	3	−1	2
4	5	5	2	−2	2
5	4	6	1	−3	2
6	3	7	0	−4	2
7	2	8	−1	−5	2
8	1	9	−2	−6	2
9	0	10	−3	−7	2

Corollary 4.4.4 *For* $i = -1, \ldots, m+1$ *we set* $\epsilon_i = (s_i, p_i)$. *With the above notations, the vectors*

$$\epsilon_{m+1}, \ldots, \epsilon_0, \epsilon_{-1}, \epsilon_{-1} - \epsilon_0, \ldots, \epsilon_{-1} - (q_1 - 1)\epsilon_0 = \epsilon_0 - \epsilon_1, \ldots, \epsilon_0 - (q_2 - 1)\epsilon_1 = \epsilon_1 - \epsilon_2, \ldots,$$

$$\epsilon_{m-2} - (q_{m-1} - 1)\epsilon_{m-1} = \epsilon_{m-1} - \epsilon_m, \ldots, \epsilon_{m-1} - (q_m - 1)\epsilon_m = \epsilon_m - \epsilon_{m+1}, -\epsilon_{m+1}$$

are a fan decomposition of $\mathbb{R}_+ \times \mathbb{R}$. *The determinant of two consecutive vectors is* $-p_0 s_{-1}$.

Proof The conclusion is a consequence of the above Proposition, since :

$$\det(\epsilon_{i-1} - j\epsilon_i, \epsilon_{i-1} - (j+1)\epsilon_i) = -\det(\epsilon_{i-1}, \epsilon_i).$$

The fan decomposition of $\mathbb{R}_+ \times \mathbb{R}$ is represented in Fig. 4.1:

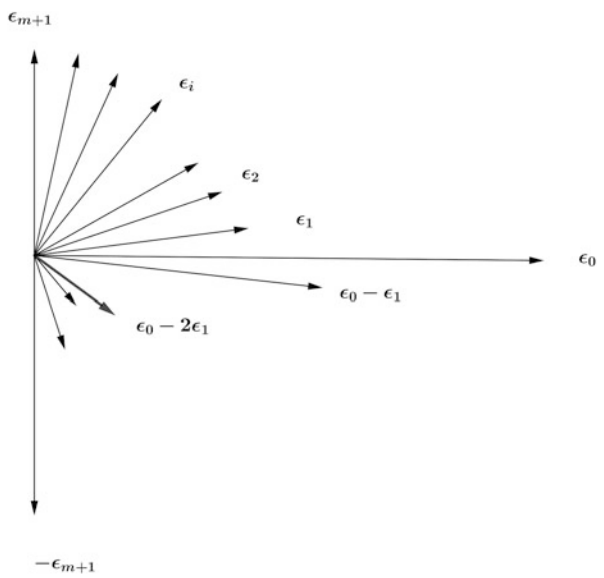
Corollary 4.4.5 *The set of binomials associated to the vectors*

$$\epsilon_{m+1}, \ldots, \epsilon_0, \epsilon_{-1}, \epsilon_{-1} - \epsilon_0, \ldots, \epsilon_{-1} - (q_1 - 1)\epsilon_0 = \epsilon_0 - \epsilon_1, \ldots, \epsilon_0 - (q_2 - 1)\epsilon_1 = \epsilon_1 - \epsilon_2, \ldots,$$

$$\epsilon_{m-2} - (q_{m-1} - 1)\epsilon_{m-1} = \epsilon_{m-1} - \epsilon_m, \ldots, \epsilon_{m-1} - (q_m - 1)\epsilon_m = \epsilon_m - \epsilon_{m+1}, -\epsilon_{m+1}$$

is a Universal Grobner Basis of I_A.

Fig. 4.1 Fan decomposition



4.4.2 Effective Computation of the Fan Associated to the Universal Grobner Basis of I_A

We can assume that d_i, b_i, c_i are coprime.

Lemma 4.4.6 *For any i let $\delta_i = \gcd(d_i, b_i)$, and*

$$\Phi_i : \mathbb{Z}^2 \longrightarrow \mathbb{Z}/d_i\mathbb{Z} \quad (s, p) \mapsto (sb_i - pc_i)$$

Then $Ker(\Phi_i)$ is a $\mathbb{Z}$–free submodule of $\mathbb{Z}^2$ generated by the vectors $(d_i/\delta_i, 0)$, $(\tilde{s}_{i,0}, \delta_i)$ where $\tilde{s}_{i,0}$ is the unique integer such that $\tilde{s}_{i,0} b_i - (\delta_i)c_i \equiv 0 \ mod\ d_i$ and $0 \le \tilde{s}_{i,0} < d_i/\delta_i$.

The proof is elementary. We have the following consequence:

Lemma 4.4.7 *Let*

$$\begin{aligned}
\rho_i &= \gcd(d_1/\delta_1, d_i/\delta_i)\ ,\ \ \chi_i = \gcd(\delta_1, \delta_i), \\
\kappa_i &= \gcd((\delta_1 s_{i,0})/(\chi_i) - (\delta_i s_{1,0})/(\chi_i), \rho_i), \\
s_{-1} &= \mathrm{lcm}(d_1/\delta_1, \ldots, d_n/\delta_n), \quad p_{-1} = 0, \quad \text{and} \\
p_0 &= \mathrm{lcm}_{2 \le i \le n}((\rho_i/\kappa_i)\mathrm{lcm}(\delta_1, \delta_i)).
\end{aligned}$$

Then $Ker\Phi$ is a subgroup of $\mathbb{Z}^2$ generated by the vectors: (s_{-1}, p_{-1}) (s_0, p_0) where s_0 is the unique integer such that:

$$0 \le s_0 < \mathrm{lcm}(d_1/\delta_1, \ldots, d_n/\delta_n) \quad \text{and}$$

$$\forall i \in \{1, \ldots, n\} \ \ s_0 \equiv s_{i,0} p_0/\delta_i \ mod\ d_i/\delta_i.$$

For the proof we refer to [15].

Definition 4.4.8 We define the sequences of integers $\{s_i\}$, $\{p_i\}$ as in Proposition 4.4.2. That is $\{s_i\}$ is defined by Euclid algorithm and $\{p_i\}$ by $p_{-1} = 0$ and:

$$p_{i+1} = p_i q_{i+1} - p_{i-1}\ , (0 \le i \le m).$$

For all $j \in \{1, \ldots, n\}$ we define the sequences $\{r_{j,i}\}$ by

$$r_{j,i} = (s_i b_j - p_i c_j)/d_j \quad -1 \le i \le m+1,\ 1 \le j \le n.$$

Lemma 4.4.9

1) The sequences $\{s_i\}$*,* $\{p_i\}$*,* $\{r_{j,i}\}$*,* $1 \leq j \leq n$ *satisfy the following recurrent relations:*
$v_{i+2} = q_{i+2}v_{i+1} - v_i$ *for* $-1 \leq i \leq m-1$*.*
2) $r_{j,-1} = s_{-1}b_i/d_i.\forall 1 \leq j \leq n$
3) For any index i *such that* $-1 \leq i \leq m$*, we have:*

$$i)\ s_i p_{i+1} - s_{i+1}p_i = s_{-1}p_0$$

$$ii)\ s_{i+1}r_{j,i} - s_i r_{j,i+1} = s_{-1}p_0 c_i/d_i$$

$$iii)\ p_{i+1}r_{j,i} - p_i r_{j,i+1} = s_{-1}p_0 b_i/d_i$$

Lemma 4.4.10 *For all* j *the sequences* $\{s_i\}$*,* $\{r_{j,i}\}$ *are strictly decreasing, and the sequence* $\{p_i\}$ *is strictly increasing.*

Definition 4.4.11

1) Let D_j be the line with equation $sb_j - pc_j = 0$. By changing if necessary the order of the variables x_j we can assume that the slopes of the lines D_j are in increasing order.
2) Let ν (resp. μ) the unique integer such that $r_{1,\nu} \geq 0 > r_{1,\nu+1}$, (resp. $r_{n,\mu} > 0 \geq r_{n,\mu+1}$).
3) Suppose that $\mu \neq \nu$. For $1 \leq i \leq \mu - \nu$ let k_i be the smallest integer $j \leq n-1$ such that $r_{j,\nu+i} < 0$. We set $k_{\mu-\nu+1} = n$.

Lemma 4.4.12 *We have:*

i) $-1 \leq \nu \leq \mu \leq m$,
ii) let $1 \leq i \leq \mu - \nu$*. If* $l \leq k_i$ *then* $r_{l,\nu+i} < 0$ *and if* $l > k_i$ *then* $r_{l,\nu+i} \geq 0$*,*
iii) if $r_{j,\mu+1} = 0$ *then* $r_{n,\mu+1} = 0$*, and*
iv) $\mu = \nu$ *if and only if* $r_{j,u} \leq 0$ *for all* $j \in \{1, \ldots, n\}$ *and* $u \geq \nu + 1$*.*

Theorem 4.4.13 ([15]) *Let* V_A *be a simplicial toric variety of codimension 2.* V_A *is arithmetically Cohen-Macaulay if and only if* $\mu = \nu$*. If* V_A *is not arithmetically Cohen-Macaulay the ideal* I_A *is minimally generated by the binomials associated to the vectors*

$$\begin{array}{c} \epsilon_\nu\ ,\ \epsilon_{\nu+1}\ ,\ \epsilon_\nu - \epsilon_{\nu+1}, \\ \epsilon_\nu - 2\epsilon_{\nu+1}\ ,\ \ldots\ ,\ \epsilon_\nu - q_{\nu+2}\epsilon_{\nu+1}\ ,\ \epsilon_{\nu+2}, \\ \ldots \\ \epsilon_{\mu-1} - 2\epsilon_\mu\ ,\ \ldots\ ,\ \epsilon_{\mu-1} - q_{\mu+1}\epsilon_\mu\ ,\ \epsilon_{\mu+1}. \end{array}$$

The proof consist to check that the mentioned binomials are a Grobner basis of I_A.

Example 4.4.14 We consider again the Toric variety of Example 4.4.3, with parametrization

$$X = s^{10}, Y = s^7 t^3 Z = s^3 t^7, W = t^{10}.$$

Its defining ideal is generated by the polynomials:

$$\begin{aligned} F_1 &= Z^7 - Y^3 W^4, \\ F_2 &= YZ - XW, \\ F_3 &= Y^4 W^3 - XZ^6, \\ F_4 &= Y^5 W^2 - X^2 Z^5, \\ F_5 &= Y^6 W - X^3 Z^4, \\ F_6 &= Y^7 - X^4 Z^3. \end{aligned}$$

Theorem 4.4.15 *[15] Let V_A be a simplicial toric variety of codimension 2. Assume that V_A is arithmetically Cohen-Macaulay that is $\mu = \nu$. The ideal I_A is generated by three binomials $F_{\epsilon_\nu}, F_{\epsilon_{\nu+1}}, F_{\epsilon_\nu - \epsilon_{\nu+1}}$ associated to the vectors*

$$\epsilon_\nu, \epsilon_{\nu+1}, \epsilon_\nu - \epsilon_{\nu+1}.$$

That is

$$\begin{aligned} F_{\epsilon_\nu} &= z^{s_\nu} \; - \; y^{p_\nu} x_1^{r_{1,\nu}} \dots x_n^{r_{n,\nu}}, \\ F_{\epsilon_{\nu+1}} &= y^{p_{\nu+1}} \; - \; z^{s_{\nu+1}} x_1^{-r_{1,\nu+1}} \dots x_n^{-r_{n,\nu+1}}, \\ F_{\epsilon_\nu - \epsilon_{\nu+1}} &= z^{s_\nu - s_{\nu+1}} y^{p_{\nu+1} - p_\nu} \; - \; x_1^{r_{1,\nu} - r_{1,\nu+1}} \dots x_n^{r_{n,\nu} - r_{n,\nu+1}}. \end{aligned}$$

In fact $F_{\epsilon_\nu}, F_{\epsilon_{\nu+1}}, F_{\epsilon_\nu - \epsilon_{\nu+1}}$ are the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_1^{r_{1,\nu}} \dots x_n^{r_{n,\nu}} & y^{p_\nu} & z^{s_\nu - s_{\nu+1}} \\ y^{p_{\nu+1} - p_\nu} & z^{s_{\nu+1}} & x_1^{-r_{1,\nu+1}} \dots x_n^{-r_{n,\nu+1}} \end{pmatrix}.$$

Moreover I_A is a complete intersection if and only if either $p_\nu = 0$ or $s_{\nu+1} = 0$.

Exercise 4.4.16 Let K be any field. Let $V(1, 3, 4)$ be the projective toric curve in $\mathbb{P}^3$ with parametrization

$$w = u^4, x = u^3 v^1, y = u^1 v^3, z = v^4.$$

1. Draw the fan decomposition of $V(1, 3, 4)$.
2. Use Theorem 4.4.13 to find a minimal generating set F_1, F_2, F_3, F_4 of the ideal I_A.
3. Use the fact that we have an explicit formulation of $Ker(\Phi)$, and so of L_A, together with the fan decomposition to prove directly Theorem 4.4.13 for this example. (Hint. Binomials are represented by plane vectors.)

The material developed in this section help to understand not only generators but also syzygies for codimension two simplicial toric ideals. See for example [6].

4.5 Almost-Complete Intersections and Set-Theoretic Complete Intersections

From now on, we assume that the field K is algebraically closed of characteristic 0, since we will use the Hilbert's Nullstellensatz.

4.5.1 Almost-Complete Intersections: The General Case

Lemma 4.5.1 *Assume that we have r binomials in* $K[x_1, \ldots, x_n, y_1, \ldots, y_r]$*:*

$$\begin{aligned} F_1 &= y_1^{\rho_1} - y_2^{\beta_{1,2}} \cdots y_r^{\beta_{1,r}} h_1(\underline{x}), \\ F_2 &= y_2^{\rho_2} - y_1^{\beta_{2,1}} y_3^{\beta_{2,3}} \cdots y_r^{\beta_{2,r}} h_2(\underline{x}), \\ F_3 &= y_3^{\rho_3} - y_1^{\beta_{3,1}} y_4^{\beta_{3,4}} \cdots y_r^{\beta_{3,r}} h_3(\underline{x}), \\ &\cdots \\ F_{r-1} &= y_{r-1}^{\rho_{r-1}} - y_1^{\beta_{r-1,1}} y_r^{\beta_{r-1,r}} h_{r-1}(\underline{x}), \\ F_r &= y_r^{\rho_r} - y_1^{\beta_{r,1}} h_r(\underline{x}), \end{aligned}$$

where $h_1(\underline{x}), \ldots, h_r(\underline{x})$ *are monomials in* $x_1, \ldots, x_n$, $\rho_1 > \sum_{k=2}^{r} \beta_{k,1}$, *and for* $j = 2, \ldots, r$, $\rho_j \geq \sum_{k=1}^{j-1} \beta_{k,j}$. *Let* $\sigma = \rho_2 \cdots \rho_r$. *Then we have*

$$F_1^{\sigma} = y_1^{\sum_{j=2}^{r} \alpha_{j,\sigma} \beta_{j,1}} \widetilde{F_1^{\sigma}}, \quad \bmod (F_2, \ldots, F_r),$$

with

$$\widetilde{F_1^{\sigma}} = \sum_{k=0}^{\sigma} (-1)^k \binom{\sigma}{k} y_1^{\gamma_{k,1}} y_2^{\delta_{2,k}} y_3^{\delta_{3,k}} \cdots y_r^{\delta_{r,k}} h_1^{\alpha_{1,k}} h_2^{\alpha_{2,k}} \cdots h_2^{\alpha_{r,k}},$$

where all exponents are non negative integer numbers such that $0 \leq \delta_{j,k} < \rho_j$, $\alpha_{j,0} = \delta_{j,0} = 0$, $\delta_{j,\sigma} = 0$ *for* $j = 2, \ldots, r$, $k = 0, \ldots, \sigma$, *and* $\gamma_{0,1} > \gamma_{1,1} > \cdots > \gamma_{\sigma,1} = 0$.

Proof We have

$$(y_1^{\rho_1} - y_2^{\beta_{1,2}} \cdots y_r^{\beta_{1,r}} h_1(\underline{x}))^\sigma = \sum_{k=0}^{\sigma} (-1)^k \binom{\sigma}{k} y_1^{(\sigma-k)\rho_1} y_2^{k\beta_{1,2}} \cdots y_r^{k\beta_{1,r}} h_1^k(\underline{x}).$$

Let $\alpha_{1,k} = k$, we define $\alpha_{2,k}$, $\delta_{2,k}$ by the relation

$$\alpha_{1,k}\beta_{1,2} = \alpha_{2,k}\rho_2 + \delta_{2,k},\ \alpha_{2,k} \geq 0,\ 0 \leq \delta_{2,k} < \rho_2.$$

Note that $\alpha_{1,0} = 0$, hence $\alpha_{2,0} = \delta_{2,0} = 0$, and $\alpha_{1,\sigma} = \sigma$, hence $\alpha_{2,\sigma} = (\sigma/\rho_2)\beta_{1,2}$, $\delta_{2,\sigma} = 0$. By using F_2 we get:

$$F_1^\sigma = \sum_{k=0}^{\sigma} (-1)^k \binom{\sigma}{k} y_1^{(\sigma-k)\rho_1 + \alpha_{2,k}\beta_{2,1}} y_2^{\delta_{2,k}} \cdots y_3^{k\beta_{1,3} + \alpha_{2,k}\beta_{2,3}} + \cdots$$
$$+ y_r^{k\beta_{1,r} + \alpha_{2,k}\beta_{2,r}} h_1^{\alpha_{1,k}}(\underline{x}) \mod F_2.$$

We define $\alpha_{3,k}$, $\delta_{3,k}$ by the relation:

$$\alpha_{1,k}\beta_{1,3} + \alpha_{2,k}\beta_{2,3} = \alpha_{3,k}\rho_3 + \delta_{3,k},\ \alpha_{3,k} \geq 0,\ 0 \leq \delta_{3,k} < \rho_3.$$

Note that $\alpha_{3,0} = \delta_{3,0} = 0$, and $\delta_{3,\sigma} = 0$. By using F_3 we get:

$$F_1^\sigma = \sum_{k=0}^{\sigma} (-1)^k \binom{\sigma}{k} y_1^{(\sigma-k)\rho_1 + \alpha_{2,k}\beta_{2,1} + \alpha_{3,k}\beta_{3,1}} y_2^{\delta_{2,k}} y_3^{\delta_{3,k}} \cdots y_r^{k\beta_{1,r} + \alpha_{2,k}\beta_{2,r} + \alpha_{3,k}\beta_{3,r}}$$
$$\times h_1^{\alpha_{1,k}} h_2^{\alpha_{2,k}}$$

modulo the ideal (F_2, F_3). We can inductively define the numbers $\alpha_{j,k}$, $\delta_{j,k}$ by the relation:

$$\alpha_{1,k}\beta_{1,j} + \alpha_{2,k}\beta_{2,j} + \cdots + \alpha_{j-1,k}\beta_{j-1,j} = \alpha_{j,k}\rho_j + \delta_{j,k},\ \alpha_{j,k} \geq 0,\ 0 \leq \delta_{j,k} < \rho_j.$$

Note that $\alpha_{j,0} = \delta_{j,0} = 0$, and $\delta_{j,\sigma} = 0$. Hence

$$F_1^\sigma = \sum_{k=0}^{\sigma} (-1)^k \binom{\sigma}{k} y_1^{(\sigma-k)\rho_1 + \sum_{j=2}^{r} \alpha_{j,k}\beta_{j,1}} y_2^{\delta_{2,k}} y_3^{\delta_{3,k}} \cdots y_r^{\delta_{r,k}} h_1^{\alpha_{1,k}} h_2^{\alpha_{2,k}} \cdots$$
$$\times h_2^{\alpha_{r,k}} \mod (F_2, \ldots, F_r).$$

It is easy to prove by induction that

$$\forall k = 0, \ldots, \sigma - 1,\ 0 \leq \alpha_{1,k+1} - \alpha_{1,k} \leq 1.$$

Hence

$$\begin{aligned}
&(\sigma - k)\rho_1 + \sum_{j=2}^{r} \alpha_{j,k}\beta_{j,1} - (\sigma - k - 1)\rho_1 + \sum_{j=2}^{r} \alpha_{j,k+1}\beta_{j,1} \\
&\quad = \rho_1 + \sum_{j=2}^{r} (\alpha_{j,k} - \alpha_{j,k+1})\beta_{j,1} \\
&\quad > \sum_{j=2}^{r} \beta_{j,1} + \sum_{j=2}^{r} (\alpha_{j,k} - \alpha_{j,k+1})\beta_{j,1} = \sum_{j=2}^{r} (1 + \alpha_{j,k} - \alpha_{j,k+1})\beta_{j,1} \geq 0
\end{aligned}$$

We can factor by $y_1^{\sum_{j=2}^{r} \alpha_{j,\sigma}\beta_{j,1}}$ and finally get

$$\begin{aligned}
F_1^{\sigma} &= y_1^{\sum_{j=2}^{r} \alpha_{j,\sigma}\beta_{j,1}} \Big(\sum_{k=0}^{\sigma} (-1)^k \binom{\sigma}{k} y_1^{\gamma_{k,1}} y_2^{\delta_{2,k}} y_3^{\delta_{3,k}} \cdots y_r^{\delta_{r,k}} h_1^{\alpha_{1,k}} h_2^{\alpha_{2,k}} \cdots h_2^{\alpha_{r,k}}\Big) \\
&\quad \mod (F_2, \ldots, F_r).
\end{aligned}$$

with $\gamma_{k,1} > \gamma_{k+1,1}$.

Theorem 4.5.2 *Let V_A be a simplicial toric variety. Let*

$$\begin{aligned}
F_1 &= y_1^{\rho_1} - y_2^{\beta_{1,2}} \cdots y_r^{\beta_{1,r}} h_1(\underline{x}), \\
F_2 &= y_2^{\rho_2} - y_1^{\beta_{2,1}} y_3^{\beta_{2,3}} \cdots y_r^{\beta_{2,r}} h_2(\underline{x}), \\
F_3 &= y_3^{\rho_3} - y_1^{\beta_{3,1}} y_4^{\beta_{3,4}} \cdots y_r^{\beta_{3,r}} h_3(\underline{x}), \\
&\cdots \\
F_{r-1} &= y_{r-1}^{\rho_{r-1}} - y_1^{\beta_{r-1,1}} y_r^{\beta_{r-1,r}} h_{r-1}(\underline{x}), \\
F_r &= y_r^{\rho_r} - y_1^{\beta_{r,1}} h_r(\underline{x}), \\
F_{r+1} &= y_1^{\rho_1 - \sum_{k=2}^{r} \beta_{k,1}} y_2^{\rho_2 - \beta_{1,2}} y_3^{\rho_3 - \sum_{k=1}^{2} \beta_{k,3}} \cdots y_r^{\rho_r - \sum_{k=1}^{r-1} \beta_{k,r}} - h_1(\underline{x}) \cdots h_r(\underline{x}),
\end{aligned}$$

be $r + 1$ binomials in $I_A \subset K[x_1, \ldots, x_n, y_1, \ldots, y_r]$, where $h_1(\underline{x}), \ldots, h_r(\underline{x})$ are monomials in $x_1, \ldots, x_n$, $\rho_1 > \sum_{k=2}^{r} \beta_{k,1}$, and for $j = 2, \ldots, r$, $\rho_j \geq \sum_{k=1}^{j-1} \beta_{k,j}$. Note that if for $i = 1, \ldots, r$, F_i corresponds to the vector v_i in the lattice L_A, then F_{r+1} corresponds to the vector $v_1 + \cdots + v_r$. Suppose that $I_A = J + (F_1, \ldots, F_{r+1})$

and $J \subset \operatorname{rad}(F_1, \ldots, F_{r+1})$. *Then,* $I_A = \operatorname{rad}(F_2, \ldots, F_r, \widetilde{F_1^\sigma})$; *in particular* V_A *is a set-theoretic complete intersection.*

Proof Since

$$F_1^\sigma = y_1^{\sum_{j=2}^r \alpha_{j,\sigma}\beta_{j,1}} \widetilde{F_1^\sigma} \mod (F_2, \ldots, F_r),$$

and I_A is a prime ideal, we have that $\widetilde{F_1^\sigma} \in I_A$. So we only need to prove that if $P = (x_1, \ldots, x_n, y_1, \ldots, y_r)$ is a zero of $F_2, \ldots, F_r, \widetilde{F_1^\sigma}$, then P is also a zero of I_A. We note that $F_1^\sigma(P) = 0$. Since $J \subset \operatorname{rad}(F_1, \ldots, F_{r+1})$, we have $H(P) = 0$ for any $H \in J$. So we only have to check that $F_{r+1}(P) = 0$.

Note that for $i = 1, \ldots, r$, $\rho_i \neq 0$. Let examine the terms of $F_{r+1}(P)$. We have four cases:

1. Suppose that $h_i(P) = 0$ for some $i = 1, \ldots, r$. Since $F_i(P) = 0$, we have $y_i = 0$. If $\rho_i - \sum_{k=1}^{i-1} \beta_{k,i} > 0$, we have $F_{r+1}(P) = 0$. If $\rho_i - \sum_{k=1}^{i-1} \beta_{k,i} = 0$, let $1 \leq k_1 \leq i-1$ be the smallest integer such that $\beta_{k_1,i} \neq 0$. Since $F_{k_1}(P) = 0$ we have $y_{k_1} = 0$. If $\rho_{k_1} - \sum_{k=1}^{k_1-1} \beta_{k,k_1} > 0$, we have $F_{r+1}(P) = 0$. If $\rho_{k_1} - \sum_{k=1}^{k_1-1} \beta_{k,k_1} = 0$, there exists $1 \leq k_2 \leq k_1 - 1$ such that $\beta_{k_2,k_1} \neq 0$, a contradiction.
2. If $y_1 = 0$, then $\widetilde{F_1^\sigma}(P) = 0$ implies $h_i(P) = 0$ for some i, so we are done.
3. If $y_j = 0$ for some $j > 1$, let $i > 1$ be the biggest one such that $y_i = 0$. Then from $F_i(P) = 0$ we have either $h_i(P) = 0$, or $y_1 = 0$. We are done.
4. If for all $i = 1, \ldots, r$, $h_i(P) \neq 0$ and $y_i \neq 0$. For $i = 1, \ldots, r$, assume that F_i corresponds to the vector v_i in the lattice L_A, then F_{r+1} corresponds to the vector $v_1 + \ldots + v_r$. Since $F_i(P) = 0$ for $i = 1, \ldots, r$, the assertion follows trivially.

The following examples are applications of the above Theorem 4.5.2.

Example 4.5.3 Let V be the projective toric curve in $\mathbb{P}^4$ with parametrization

$$w = t^7, x = s^7, y = t^3s^4, z = t^4s^3, a = t^2s^5.$$

Then $I(V)$ is generated by

$$F_{v_1} = a^2 - xz, F_{v_2} = y^2 - az, F_{v_3} = z^3 - yaw, F_{v_1+v_2+v_3} = yz - xw.$$

Example 4.5.4 Let V be the toric surface in K^4 with parametrization

$$w = t^9, x = s^9, y = ts^5, z = t^2s^7, a = ts^8.$$

Then $I(V)$ is generated by

$$F_{v_1} = y^3 - az, F_{v_2} = a^2 - xz, F_{v_3} = z^5 - x^3aw, F_{v_1+v_2+v_3} = y^3z^3 - x^4w.$$

Example 4.5.5 Let V be the projective toric surface in $\mathbb{P}^5$ with parametrization

$$x = s^9, w = t^9, v = u^9, y = t^4 s^4 u, z = t^5 s^2 u^2, a = t^3 s^6.$$

Then $I(V)$ is generated by

$$F_{v_1} = y^2 - az, F_{v_2} = a^3 - wx^2, F_{v_3} = z^5 - vw^2 ya, F_{v_1+v_3} = yz^4 - vw^2 a^2.$$

Example 4.5.6 Let V be the projective toric curve in $\mathbb{P}^4$ with parametrization

$$\begin{aligned} x &= u^{11}, \\ w &= v^{11}, \\ y &= u^6 v^5, \\ z &= u^7 v^4, \\ a &= u^3 v^8. \end{aligned}$$

Then the ideal $I(V)$ is generated by:

$$\begin{aligned} F_{v_1} &= y^3 - wxz, \\ F_{v_2} &= -wa + z^2, \\ F_{v_3} &= -xy + a^2, \\ F_{v_1+v_2+v_3} &= -w^2 x^2 + y^2 az. \end{aligned}$$

$I(V)$ is a set-theoretic complete intersection.

We can compute $F_{v_1}^4$ modulo F_{v_2}, F_{v_3}, and we get:

$$F_{v_1}^4 = y(y^{11} - 4y^8 wxz + 6y^5 w^3 x^2 z - 4y^2 w^4 x^3 za + w^6 x^5) \text{ modulo } (F_{v_2}, F_{v_3}).$$

Let $F := y^{11} - 4y^8 wxz + 6y^5 w^3 x^2 z - 4y^2 w^4 x^3 za + w^6 x^5$. Our theorem says that $I(V) = \text{rad}\,(F_{v_2}, F_{v_3}, F)$.

Example 4.5.7 Let the projective surface $\overline{V}$ with parametrization

$$x = s^{15}, w = t^{15}, v = u^{15}, y = t^4 s^2 u^9, z = t^6 s^3 u^6, a = t^{10} s^5.$$

We have

$$I(V) = (y^2 a - z^3, y^3 - vz^2, w^2 x - a^3, -va + yz).$$

Note that if $y^2 a - z^3$ corresponds to a vector v_1, $y^3 - vz^2$ corresponds to a vector v_2, then $-va + yz$ corresponds to the vector $v_2 - v_1$. So V is a stci.

Example 4.5.8 Let V be the projective toric curve in $\mathbb{P}^4$ with parametrization

$$x = s^5, w = t^5, y = t^4 s, z = t^3 s^2, a = t^2 s^3.$$

The ideal $I(V)$ is generated by

$$xy - a^2, -wx + az, -ya + z^2, -wa + yz, y^2 - wz.$$

It is a Gorenstein projective curve in $\mathbb{P}^4$. We prove now that $I(V)$ is a set-theoretic complete intersection. We follow the ideas of Brezinsky [3]:

First note that $z(-wa + yz) = y(-ya + z^2) + a(y^2 - wz)$ implies $-wa + yz \in \text{rad}\,(xy - a^2, -wx + az, -ya + z^2, y^2 - wz)$. Next if $a^2 - xy$ corresponds to a vector v_1, $z^2 - ya$ corresponds to a vector v_2, $y^2 - wz$ corresponds to a vector v_3, then $az - wx$ corresponds to the vector $v_1 + v_2 + v_3$, so by our Theorem 4.5.2, $\text{rad}\,(xy - a^2, -wx + az, -ya + z^2, y^2 - wz) = \text{rad}\,(xy - a^2, -ya + z^2, y^2 - wz)$.

Now let $\alpha_1, \ldots, \alpha_5$ be any positive numbers, $\alpha := \alpha_1 + \cdots + \alpha_5 > 0$. Let us consider the variety W:

$$\begin{aligned} x &= s^{5\alpha}, \\ w &= t^{5\alpha}, \\ y &= t^{4\alpha} s^{\alpha}, \\ z &= t^{3\alpha} s^{2\alpha}, \\ a &= t^{2\alpha} s^{3\alpha}, \\ b &= t^{5\alpha_2+4\alpha_3+3\alpha_4+2\alpha_5} s^{5\alpha_1+\alpha_3+2\alpha_4+3\alpha_5}. \end{aligned}$$

Then W is a set-theoretic complete intersection. Note that the ideal $I(\overline{W})$ is generated by: $xy - a^2, -wx + az, -ya + z^2, -wa + yz, y^2 - wz, b^{\alpha} - x^{\alpha_1} w^{\alpha_2} y^{\alpha_3} z^{\alpha_4} a^{\alpha_5}$. $I(\overline{W}) = \text{rad}\,(xy - a^2, -ya + z^2, y^2 - wz, b^{\alpha} - x^{\alpha_1} w^{\alpha_2} y^{\alpha_3} z^{\alpha_4} a^{\alpha_5})$.

Example 4.5.9 Let V be the projective toric curve in $\mathbb{P}^3$, with parametrization

$$w = t^9, x = s^9, y = t^8 s, z = t^4 s^5.$$

V_A is arithmetically Cohen-Macaulay. Let $\overline{V}$ be the projective toric curve in $\mathbb{P}^4$, with parametrization

$$w = t^9, x = s^9, y = t^8 s, z = t^4 s^5, a = t^6 s^3.$$

Its ideal is generated by five elements: $-y^3 + w^2 a, -y^2 a + w^2 z, -w^2 x + yaz, a^2 - yz, -xy + z^2$ but is not Gorenstein, However we can still apply the method used by Brezinsky [3]. First note that $a(-y^2 a + w^2 z) = y^2(yz - a^2) + z(-y^3 + w^2 a)$ implies by studying both cases when $a = 0$ or when $a \neq 0$ that $-y^2 a + w^2 z \in$

rad $(xy - a^2, -wx + az, -ya + z^2, y^2 - wz)$. Secondly if $-y^3 + w^2 a$ corresponds to a vector v_1, $a^2 - yz$ corresponds to a vector v_2, $-xy + z^2$ corresponds to a vector v_3, then $-w^2 x + yaz$ corresponds to the vector $v_1 + v_2 + v_3$, so by our Theorem 4.5.2, rad $(-y^3 + w^2 a, -y^2 a + w^2 z, -w^2 x + yaz, a^2 - yz, -xy + z^2) =$ rad $((-y^3 + w^2 a)^4, a^2 - yz, -xy + z^2)$.

We have the following open question:

Question 4.5.10 : Let V be the toric variety with parametrization

$$w = t^d, x = s^d, y = \underline{s}^{\mathbf{a_1}}, z = \underline{s}^{\mathbf{a_2}}$$

and let V_1 be the toric variety with parametrization

$$w = t^d, x = s^d, y = \underline{s}^{\mathbf{a_1}}, z = \underline{s}^{\mathbf{a_2}}, a = \underline{s}^{\frac{\mathbf{a_1}+\mathbf{a_2}}{2}},$$

where we assume that $\frac{\mathbf{a_1}+\mathbf{a_2}}{2}$ has integer coordinates. We know by Theorem 4.5.12, that if V is arithmetically Cohen-Macaulay then it is a set-theoretic complete intersection. Can we say if $I(V_1)$ is a set-theoretic complete intersection?

We can answer to this question in Theorem 4.6.2 if one of the components of $\mathbf{a_1} + \mathbf{a_2}$ is odd.

Example 4.5.11 Let the projective curve with parametrization

$$w = t^5, x = s^5, y = t^3 s^2, z = t^1 s^4,$$

it is arithmetically Cohen-Macaulay. The projective curve with parametrization

$$w = t^5, x = s^5, y = t^3 s^2, z = t^1 s^4, a = t^2 s^3$$

is Gorenstein and generated by five elements.

4.5.2 Almost-Complete Intersections, The Codimension Two Case

In this subsection we apply Theorem 4.5.2 in the case of simplicial monomial varieties of codimension two which are arithmetically Cohen-Macaulay:

Theorem 4.5.12 *Let V_A be a simplicial toric variety of codimension 2, such that is arithmetically Cohen-Macaulay. Then V_A is a set-theoretic complete intersection.*

Proof By Theorem 4.4.15, the defining ideal of a simplicial monomial variety of codimension two arithmetically Cohen-Macaulay, is generated by three elements

$$
\begin{aligned}
F &= z^{s_\mu} - y^{p_\mu} x_1^{r_{1,\mu}} \cdots x_n^{r_{n,\mu}},\\
G &= y^{p_{\mu+1}} - z^{s_{\mu+1}} x_1^{-r_{1,\mu+1}} \cdots x_n^{-r_{n,\mu+1}},\\
H &= z^{s_\mu - s_{\mu+1}} y^{p_{\mu+1}-p_\mu} - x_1^{r_{1,\mu}-r_{1,\mu+1}} \cdots x_n^{r_{n,\mu}-r_{n,\mu+1}},
\end{aligned}
$$

for some positive integer exponents with $s_\mu > s_{\mu+1}$, $p_{\mu+1} > p_\mu$.

It is clear that we can apply the Theorem 4.5.2. Indeed let F_1 be the polynomial obtained from $(z^{s_\mu} - y^{p_\mu} \underline{x}^{r_\mu})^{p_{\mu+1}}$ by reduction modulo G. That is $F^{p_{\mu+1}} = AG + z^{p_\mu(s_{\mu+1})} F_1$. Then $I = \text{rad}\,(G, F_1)$.

Example 4.5.13 Let V_A be the projective toric surface in $\mathbb{P}^5$ with parametrization

$$
\begin{aligned}
v &= u^{10},\\
x &= s^{10},\\
w &= t^{10},\\
y &= t^5 s^5,\\
z &= t^4 s^2 u^4,\\
a &= t^2 s^6 u^2.
\end{aligned}
$$

The ideal I_A is generated by:

$$
\begin{aligned}
F_{v_1} &= z^3 - vwa,\\
F_{v_2} &= a^2 - xz,\\
F_{v_3} &= -y^2 + wx,\\
F_{v_1+v_2+v_3} &= vy^2 - az^2.
\end{aligned}
$$

Then I_A is a set-theoretic complete intersection. In fact we can compute $F_{v_1}^4$ modulo F_{v_2}, F_{v_3}, and we get:

$$
F_{v_1}^4 = z^2(v^4w^4x^2 - 4v^3w^3axz^2 + 6v^2w^2xz^5 - 4vwaz^7 + z^{11}) \mod (F_{v_2}, F_{v_3}).
$$

Let $F := v^4w^4x^2 - 4v^3w^3axz^2 + 6v^2w^2xz^5 - 4vwaz^7 + z^{11}$. By Theorem 4.5.2 we have $I(V) = \text{rad}\,(F_{v_2}, F_{v_3}, F)$.

Another proof: Let us consider the variety W:

$$
\begin{aligned}
v &= u^5,\\
x &= s^5,
\end{aligned}
$$

$$
\begin{aligned}
w &= t^5, \\
z &= t^2 s u^2, \\
a &= t s^3 u.
\end{aligned}
$$

By the trick developed in Sect. 4.6.1, $I_A = (I(W) + (y^2 - xw))$. $I(W)$ has codimension 2 and is arithmetically Cohen-Macaulay. Hence $I(W)$ is a set-theoretic complete intersection and so is I_A.

Remark For an arithmetically Cohen-Macaulay projective curve, the shape of the equations and the above theorem was known, by Stuckrad and Vogel [22] and by Robbiano and Valla [19]. For an arithmetically Cohen-Macaulay simplicial toric variety of codimension two, in [15] it was proved that its equations are given by the 2×2 minors of a 2×3 matrix, so the above theorem can be also proved by using [22], or the next theorem. Our proof is simpler, it gives us the ideal I up to radical in one step, while the next theorem needs several steps.

Theorem 4.5.14 ([19]) *Let R be a commutative ring with identity, let m, n be non negative integers, and let J be the ideal generated by the 2×2 minors of the matrix* $M = \begin{pmatrix} a & b^m & c \\ b^n & d & e \end{pmatrix}$*, with entries in R. Then we can construct two elements $f, g \in J$, such that*

$$\operatorname{rad}(J) = \operatorname{rad}(f, g).$$

4.6 Some Set-Theoretic Complete Intersection Toric Varieties

4.6.1 Tricks on Toric Varieties

The following theorem was originally stated and proved in [14], in the case of numerical semigroups, but it can be extended in general and the proofs are unchanged.

Theorem 4.6.1 *Let H be the semigroup of $\mathbb{N}^m$ generated by $\mathbf{a_1}, \ldots, \mathbf{a_n}$. Let $I_H \subset K[x_1, \ldots, x_n]$ be the toric ideal associated to H.*

1. *Let $l \in \mathbb{N}^*$, and $H^{(l)}$ be the semigroup generated by $l\mathbf{a_1}, \ldots, l\mathbf{a_{n-1}}, \mathbf{a_n}$. Then the ideal $I_{H^{(l)}}$ is generated by $\widetilde{f}(x_1, \ldots, x_n) := f(x_1, \ldots, x_{n-1}, x_n^l)$, where f runs over all the generators of I_H.*
2. *Let $l_1, \ldots, l_n \in \mathbb{N}, l = l_1 + \cdots + l_n > 0$, let $\overline{H}^{(l_1,\ldots,l_n)}$ be the semigroup generated by $l\mathbf{a_1}, \ldots, l\mathbf{a_{n-1}}, l\mathbf{a_n}, \mathbf{a_{n+1}} := l_1\mathbf{a_1} + \cdots + l_n\mathbf{a_n}$. If l is relatively prime to a component of $\mathbf{a_{n+1}}$ then $I_{\overline{H}^{(l_1,\ldots,l_n)}} = I_H + (x_{n+1}^l - x_1^{l_1} \cdots x_n^{l_n}) \subset K[x_1, \ldots, x_{n+1}]$.*

The following theorem follows from [14], Lemmas 1.3, 1.4, and 1.5. See also [21] Corollary 2.5 and [16] Theorem 2.6.

Theorem 4.6.2

1. *If I_H is Cohen-Macaulay, Gorenstein, complete intersection or set-theoretic complete intersection then the same property holds for $I_{H^{(l)}}$.*
2. *If I_H is Cohen-Macaulay, Gorenstein, complete intersection or set-theoretic complete intersection and l is relatively prime to a component of $\mathbf{a_{n+1}}$ then the same property holds for $I_{\overline{H}^{(l_1,\ldots,l_n)}}$.*

We deduce a positive answer to Question 4.5.10 if one of the components of $\mathbf{a_1}+\mathbf{a_2}$ is odd. The following example shows that the hypothesis l is relatively prime to a component of $\mathbf{a_{n+1}}$ is necessary. We thank Mesut Sahin to pointed us this problem.

Example 4.6.3 Consider the projective surface with parametrization

$$x = s^9, w = t^9, v = u^9, z = t^5s^2u^2, a = t^3s^4u^2.$$

It is a complete intersection but the projective surface with parametrization

$$x = s^9, w = t^9, v = u^9, y = t^4s^3u^2, z = t^5s^2u^2, a = t^3s^4u^2$$

is not arithmetically Cohen-Macaulay. Its defining ideal is generated by six elements.

These tricks can be applied to the projective case using the following

Theorem 4.6.4 *Let H be the semigroup of $\mathbb{N}^m$ generated by $\mathbf{a_1},\ldots,\mathbf{a_n}$, which are not necessarily homogeneous with respect to the standard graduation. Suppose that $I_H = \operatorname{rad}(F_1,\ldots,F_r)$. Let $d = max\{\deg\mathbf{a_1},\ldots,\deg\mathbf{a_n}\}$, where $\deg\mathbf{a_i}$ is the sum of its components. Let H_1 be the semigroup in $\mathbb{Z}^{m+1}$ generated by $\mathbf{b_1},\ldots,\mathbf{b_{n+1}}$, where for $i = 1,\ldots,n$, $\mathbf{b_i} = \mathbf{a_i} + (d - \deg\mathbf{a_i})\mathbf{e_{m+1}}$ and $\mathbf{b_{n+1}} = d\mathbf{e_{m+1}}$. Let x_{n+1} be a new variable and let $F_1^h,\ldots,F_r^h$ be the homogenization of $F_1,\ldots,F_r$ with respect to x_{n+1}.*

Let $P = (x_1,\ldots,x_n,x_{n+1})$ be a zero of $F_1^h,\ldots,F_r^h$. If $x_{n+1} = 0$ implies that $F(P) = 0$ for all $F \in I_H^h$, then $I_{H_1} = \operatorname{rad}(F_1^h,\ldots,F_r^h)$.

Proof For projective closure and parametrization of toric varieties we refer to [4]. Let V_A be the zero set of I_H, then the projective closure $\overline{V}$ is the zero set of I_H^h and since both ideals I_H^h, I_{H_1}, are prime of the same height they coincide. This implies that $\operatorname{rad}(F_1^h,\ldots,F_r^h) \subset I_{H_1}$.

Let $P = (x_1,\ldots,x_n,x_{m+1})$ be a zero of $F_1^h,\ldots,F_r^h$. By hypothesis, if $x_{m+1} = 0$ then $P \in \overline{V}$. If $x_{m+1} \neq 0$ then $P \in \overline{V}$ since $\overline{V} \cap (x_{m+1} = 1) = \{(Q,1) \mid Q \in V\}$, by general arguments on the projective closure.

Example 4.6.5 Let V_A be the affine surface with parametrization

$$b = t^7, x = s^7, y = t^3 s^2, z = t^4 s^3, a = t^2 s^5,$$

$\overline{V}$ be the projective surface with parametrization

$$b = t^7, x = s^7, w = u^7, y = t^3 s^2 u^2, z = t^4 s^3, a = t^2 s^5.$$

Then

$$I(V) = (-a^2 + xz, z^4 - xab^2, -az^3 + x^2 b^2, y^7 - x^2 b^3),$$

and

$$I(\overline{V}) = (-a^2 + xz, z^4 - xab^2, -az^3 + x^2 b^2, y^7 - w^2 x^2 b^3).$$

Applying the proof of Theorem 4.5.2, we have that $\overline{V}$ is a set-theoretic complete intersection. Indeed, let $F_{\mathbf{v_1}} = z^4 - xab^2$, $F_{\mathbf{v_2}} = a^2 - xz$ then $F_{\mathbf{v_1}+\mathbf{v_2}} = az^3 - x^2 b^2$ and $F^2_{\mathbf{v_1}} = z(z^7 - 2z^3 ab^2 x + b^4 x^3) \mod F_{\mathbf{v_2}}$. Hence $I(\overline{V}) = \text{rad}\,(-a^2 + xz, z^7 - 2z^3 ab^2 x + b^4 x^3, y^7 - x^2 b^3)$.

4.6.2 Toric Curves in $\mathbb{P}^3$

In this section we consider curves, that is V_A is a simplicial toric variety of dimension 1 in K^3. The parametrization of V_A is:

$$x = v^p,$$
$$y = v^q,$$
$$z = v^r,$$

where $p < q < r$ are positive integers. We simply denote this curve by V or $V(p, q, r)$. Let $\overline{V}$ be the projective toric curve in $\mathbb{P}^3$, with parametrization

$$w = u^r,$$
$$x = u^{r-p} v^p,$$
$$y = u^{r-q} v^q,$$
$$z = v^r.$$

We simply denote this curve by $\overline{V}(p, q, r)$.

Theorem 4.6.6 ([20]) *Let a, b, p, q, r be natural integer numbers such that $r = ap + bq$. If $b \geq a(q - p - 1) + 1$, then $\overline{V}(p, q, r)$ is a set-theoretic complete intersection. Moreover $\overline{V}(p, q, r)$ is the zero set of the polynomials $F_1 := x^q - y^p w^{q-p}$, $F_2 = ((z - x^a y^b)^q)^h_{y^p = x^q}$, where $(H)_{y^p = x^q}$ means substitution when possible x^q by y^p, and H^h is the homogenization of H with respect to w.*

Proof This proof is more or less the proof given by Sahin [20].

Let us consider

$$(z - x^a y^b)^q = z^q + \sum_{k=1}^{q-1} (-1)^k \binom{q}{k} z^{q-k} x^{ka} y^{kb} + x^{qa} y^{qb}.$$

By setting $ka = s_k q + r_k$,with $0 \leq s_k, 0 \leq r_k < q$, we can write

$$((z - x^a y^b)^q)_{y^p = x^q} = z^q + \sum_{k=1}^{q-1} (-1)^k \binom{q}{k} z^{q-k} x^{r_k} y^{s_k p + kb} + y^{pa+qb}.$$

Note that for $k = 1, \dots, q - 1$, $q - k + r_k + s_k p + kb < q - k + ka + kb$, so it is enough to check the condition $q - k + ka + kb \leq pa + qb$ for $k = 0, \dots, q - 1$. This is equivalent to $q - k + ka - pa \leq qb + kb = q - k + (q - k)(b - 1)$, i.e., equivalent to $(k - p)a \leq (q - k)(b - 1)$ for $k = 0, \dots, q - 1$. This last condition is equivalent to $(k - p)a \leq (q - k)(b - 1)$ for $k = p + 1, \dots, q - 1$. We remark that if $b - 1 \geq a(q - p - 1)$, then $(k - p)a \leq a(q - p - 1) \leq b - 1 \leq (q - k)(b - 1)$. We can write:

$$((z - x^a y^b)^q)_{y^p = x^q} = \sum_{k=0}^{q-1} (-1)^k \binom{q}{k} z^{q-k} x^{r_k} y^{s_k + kb} w^{r - (q - k + r_k + s_k + kb)} + y^r.$$

By the preceding discussion $q - k + r_k + s_k p + kb < q - k + ka + kb \leq r$ if $b - 1 \geq a(q - p - 1)$. In conclusion the exponent of w in the monomial $z^{q-k} x^{r_k} y^{s_k + kb} w^{r - (q - k + r_k + s_k + kb)}$ is strictly positive.

Let $P = (w : x : y : z) \in Z(F_1, F_2)$. If $w = 0$, then $F_2(P) = 0$ implies $y = 0$, and $F_1(P) = 0$ implies $x = 0$. Hence $P = (0 : 0 : 0 : 1)$ belongs to $\overline{V}$. If $w \neq 0$, we can assume that $w = 1$. Hence $F_1(P) = 0$ implies that there exists $v \in K$ such that $x = v^p$, $y = v^q$ and $F_2(P) = 0$ implies $(z - v^r)^q = 0$, which finally implies $z = v^r$; that is $P = (1 : v^p : v^q : v^r)$ belongs to $\overline{V}$.

The next theorem uses a trick that improves Sahin's theorem in some cases:

Theorem 4.6.7 ([17])

1. *Let p, q, r be natural integer numbers and $\overline{V}(p, q, r)$ be the projective toric curve in $\mathbb{P}^3$, with parametrization $(u^r, u^{r-p} v^p, u^{r-q} v^q, v^r)$. Suppose that $r = ap + bq$, with $a, b \in \mathbb{N}$,*

a. if $p = 1$ *and* $0 \le a \le q-1, b \ge q-a$ *, or*
b. if $p > 1$ *and* $0 \le a \le q-1, b \ge (q-a-1)p$*,*

then $\overline{V}(p,q,r)$ *is a set-theoretic complete intersection. Moreover* $\overline{V}(p,q,r)$ *is the zero set of the polynomials* $F_1^h := x^q - y^p w^{q-p}$, F_2^h*, where* F_2^h *is obtained from* $((z - x^a y^b)^q)_{x^q - y^p}$*, by a trick, explained in the proof.*

2. *Let* l *be a natural number, let* $\overline{V}(lp, lq, r)$ *be the projective toric curve in* $\mathbb{P}^3$*. Suppose that* $r = ap + bq$*, with* $a, b \in \mathbb{N}$*,*

a. if $p = 1$ *and* $0 \le a \le q-1, b \ge q-a-1+l$*, or*
b. if $p > 1$ *and* $0 \le a \le q-1, b \ge q-a-p+l, b \ge (q-a-1)p$*,*

then $\overline{V}(lp, lq, r)$ *is a set-theoretic complete intersection. Moreover* $\overline{V}(lp, lq, r)$ *is the zero set of the polynomials* $F_1^h := x^q - y^p w^{q-p}, \overline{F}_2$ *where* $\overline{F}_2(w,x,y,z) = (F_2(x,y,z^l))^h$*, by the trick developed in Sect. 4.6.1.*

Proof We prove only the first claim, the second claim follows from the proof of the first and the trick developed in Sect. 4.6.1. The proof is more or less the one given in [17].

Let us consider

$$(z - x^a y^b)^q = z^q + \sum_{k=1}^{q-1} (-1)^{q-k} \binom{q}{k} z^k x^{(q-k)a} y^{(q-k)b} + x^{qa} y^{qb}.$$

By setting $(q-k)a = k(q-a) + q(a-k)$ and by using $y^p = x^q$, we get the polynomial

$$F_2 := z^q + \sum_{k=1}^{q-1} (-1)^{q-k} \binom{q}{k} z^k x^{k(q-a)} y^{r-k(b+p)} + y^r,$$

For $k = 1, \ldots, q-1$, the exponent of x in F_2 is $x^{k(q-a)}$ which is strictly positive. For $k = 1, \ldots, q-1$, the exponent of y in F_2 is $y^{r-k(b+p)}$ which is positive if and only if $b \ge (q-a-1)p$. Finally $\deg F_2 = r$ if and only if $b \ge q-a-p+1$.

It is easy to show that these conditions are equivalent to

1. if $p = 1$ and $0 \le a \le q-1, b \ge q-a$, or
2. if $p > 1$ and $0 \le a \le q-1, b \ge (q-a-1)p$.

We also remark that the affine curve $V(p,q,r)$ is a complete intersection by the trick developed in Sect. 4.6.1, so is clear that $V(p,q,r)$ is a set-theoretic complete intersection defined by (F_1, F_2).

Now we can prove that $\overline{V}(p, q, r)$ is a set-theoretic complete intersection defined by (F_1^h, F_2^h), where

$$F_1^h := x^q - y^p w^{q-p},$$

$$F_2^h := z^q w^{r-q} + \sum_{k=1}^{q-1} (-1)^{q-k} \binom{q}{k} z^k x^{k(q-a)} y^{r-k(b+p)} w^{k(b+p+a-q-1} + y^r.$$

Let $P = (w : x : y : z) \in Z(F_1, F_2)$. If $w = 0$ then $F_1(P) = 0$ implies $x = 0$, and $F_2(P) = 0$ implies $y = 0$. Hence $P = (0 : 0 : 0 : 1)$ and it is clear that it belongs to $\overline{V}$. If $w \neq 0$, we can assume that $w = 1$, the claim follows from the fact that $V(p, q, r)$ is a set-theoretic complete intersection defined by (F_1, F_2).

Remark We can compare the bounds on b given in Theorems 4.6.6 and 4.6.7. We assume that $0 \leq a \leq q - 1$

1. If $p = 1$ then the bound given by Theorem 4.6.7 is better, that is $b \geq q - a$.
2. If $p > 1$ and $p \leq a$ then the bound given by Theorem 4.6.7 is better, that is $b \geq (q - a - 1)p$,
3. If $p > 1$ and $p > a$ then the bound given by Theorem 4.6.6 is better, that is $b \geq a(q - p - 1) + 1$.

Proof We need a proof.

1. If $p = 1$ $q - a \leq a(q - 1 - 1) + 1 \Leftrightarrow (a - 1)(q - 1) \geq 0$.
2. If $p > 1$ $(q - a - 1)p \geq a(q - p - 1) + 1 \Leftrightarrow (p - a)(q - 1) \geq 1$.

Note also that the bound given in Theorem 4.6.7 is the best one given by the methods used, but the bound given by Theorem 4.6.6 is not the best obtained by the methods used. We sometimes can get better bounds by applying the proof of Theorem 4.6.6.

Theorem 4.6.8 *Suppose that* gcd $(p, q) = l$. *We set* $p' = p/l, q' = q/l$. *If* $r \geq p'q'(q' - 1) + q'l$, *then* $\overline{V}(p, q, r)$ *is a set-theoretic complete intersection.*

In particular given two positive numbers p, q *there is only a finite number of positive integers* r *for which we don't know if the projective toric curve* $\overline{V}(p, q, r)$ *in* $\mathbb{P}^3$ *is a set-theoretic complete intersection.*

Proof The Frobenius number for the semigroup generated by p', q' is $(p' - 1)(q' - 1)$, since $r \geq p'q'(q' - 1) + q'l \geq (p' - 1)(q' - 1)$, we have that r belongs to the semigroup generated by p', q', and we can find a, b integers such that $r = ap' + bq', 0 \leq a \leq q - 1, b \geq 1$. We will check the conditions for b in Theorem 4.6.7.

1. Suppose that $b < q' - a - p' - l$, then

$$r = ap' + bq' < r = ap' + q'(q' - a - p' - l) = q'(q' - p' - l) - a(q' - p')$$
$$\leq q'(q' - p' - l),$$

   and $q'(q' - p' - l) \leq p'q'(q' - 1) + q'l$ is equivalent to $q' \leq p'q'$, so we get a contradiction.

2. Suppose that $b < (q' - a - 1)p'$, then

$$r = ap' + bq' < r = ap' + q'((q' - a - 1)p') = q'((q' - 1)p') - a(q' - p')$$
$$\leq p'q'(q' - 1) \leq p'q'(q' - 1) + q'l,$$

we get again a contradiction.

We conclude that the conditions for b in Theorem 4.6.7 are satisfied, hence $\overline{V}(p, q, r)$ is a set-theoretic complete intersection.

Example 4.6.9 Let $\overline{V}(1, 2, r)$ be the projective toric curve in $\mathbb{P}^3$. Then $\overline{V}$ is a set-theoretic complete intersection for all integers $r \geq 3$, by applying the proof of the above theorem.

Example 4.6.10 Let $\overline{V}(1, 3, r)$ be the projective toric curve in $\mathbb{P}^3$. Then $\overline{V}$ is a set-theoretic complete intersection for all integers $r \geq 5$, by applying the proof of the above theorem.

Remark that in this case the only unsolved example is the famous projective quartic $\overline{V}(1, 3, 4)$.

Example 4.6.11 Let $\overline{V}(1, 4, r)$ be the projective toric curve in $\mathbb{P}^3$. Then $\overline{V}$ is a set-theoretic complete intersection for all integers $r \in \{7, 8, 10, \ldots\}$. By applying the proof of the above theorem, we get that $r \in \{7, 10, 11, 13, \ldots\}$. Now by a direct computation using [15], we get that $\overline{V}$ is a complete intersection for $r = 8, 12$.

The unsolved cases are $\overline{V}(1, 4, 5)$, $\overline{V}(1, 4, 6)$ and $\overline{V}(1, 4, 9)$.

Example 4.6.12 Let $\overline{V}(2, 3, r)$ be the projective toric curve in $\mathbb{P}^3$. Then $\overline{V}$ is a set-theoretic complete intersection for all integers $r \geq 4$. By applying the proof of the above theorem, we get that $r \in \{4, 7, 8, 10, 11, 12, \ldots\}$. Now by direct computation using [15], we get that $\overline{V}$ is a arithmetically Cohen-Macaulay for $r = 5$ and a complete intersection for $r = 6, 9$.

4.6.3 Toric Curves in $\mathbb{P}^n$

Let K be an algebraically closed field. In this subsection we consider curves in K^n, that is $V(p, q_0, q_1, \ldots, q_{n-2})$ is an affine simplicial toric variety of dimension 1. The parametrization of $V := V(p, q_0, q_1, \ldots, q_{n-2})$ is:

$$\begin{aligned} x &= v^p, \\ y &= v^{q_0}, \\ z_1 &= v^{q_1}, \\ &\ldots \\ z_{n-2} &= v^{q_{n-2}}. \end{aligned}$$

Theorem 4.6.13 *Let $p, q_0, q_1, \dots, q_{n-2}$ be positive integers. Let $\overline{V}(p, q_0, q_1, \dots, q_{n-2})$ be the projective toric curve in $\mathbb{P}^n$ with parametrization*

$$\begin{aligned} w &= u^{q_{n-2}}, \\ x &= u^{q_{n-2}-p} v^p, \\ y &= u^{q_{n-2}-q_0} v^{q_0}, \\ z_1 &= u^{q_{n-2}-q_1} v^{q_1}, \\ &\dots \\ z_{n-2} &= v^{q_{n-2}}. \end{aligned}$$

Suppose that $\overline{V}(p, q_0, q_1, \dots, q_{n-2})$ is a set-theoretic complete intersection, defined by $F_1, \dots, F_{n-1}$. Let $q_{n-1} \in \mathbb{N}$, and $\overline{V}_1$ the projective curve defined by

$$\begin{aligned} w &= u^{q_{n-1}}, \\ x &= u^{q_{n-1}-p} v^p, \\ y &= u^{q_{n-1}-q_0} v^{q_0}, \\ z_1 &= u^{q_{n-1}-q_1} v^{q_1}, \\ &\dots \\ z_{n-2} &= u^{q_{n-1}-q_{n-2}} v^{q_{n-2}}, \\ z_{n-1} &= v^{q_{n-1}}. \end{aligned}$$

If $q_{n-1} = ap + bq_{n-2}$, with $0 \le a \le q_{n-2} - 1$, $b \ge q_{n-2} - a$ when $p = 1$, or $0 \le a \le q_{n-2} - 1$, $b \ge (q_{n-2} - a - 1)p$ when $p > 1$, then $\overline{V}_1(p, q_0, q_1, \dots, q_{n-2}, q_{n-1})$ is a set-theoretic complete intersection.

In particular, let $\gcd(p, q_{n-2}) = l$. *We set $p' = p/l, q' = q_{n-2}/l$. If $q_{n-1} \ge p'q'(q'-1) + q'l$, then $\overline{V}_1(p, q_0, q_1, \dots, q_{n-2}, q_{n-1})$ is a set-theoretic complete intersection.*

Proof By the hypothesis $\overline{V}$ is a set-theoretic complete intersection, defined by $F_1, \dots, F_{n-1}$. We will prove that $\overline{V}_1$ is a set-theoretic complete intersection, defined by $F_1, \dots, F_{n-1}, F_n$, where F_n is the polynomial

$$z_{n-1}^{q_{n-2}} w^{q_{n-1}-q_{n-2}} + \sum_{k=1}^{q_{n-2}-1} (-1)^{q_{n-2}-k} \binom{q_{n-2}}{k} z_{n-1}^k x^{k(q_{n-2}-a)} z_{n-2}^{q_{n-1}-k(b+p)} \times w^{k(b+p+a-q_{n-2}-1)} + z_{n-2}^{q_{n-1}},$$

obtained from $((z_{n-1} - x^a z_{n-2}^b)^{q_{n-2}})_{z_{n-2}^p = x^{q_{n-2}}}$ by the trick used in the proof of the Theorem 4.6.7. Note that also by Theorem 4.6.7, all exponents are positive with our hypothesis.

First note that $F_n \in I(\overline{V}_1)$. Let $P = (w, x, y, z_1, \dots, z_{n-1}) \in \overline{V}_1$. If $w = 0$ then from the parametrization we get $x = y = z = \cdots = z_{n-2} = 0$, hence $F_n(P) = 0$. If $w \neq 0$, we can assume that $w = 1$, there exists $v \in K$ such that

$$x = v^p, y = v^{q_0}, z_1 = v^{q_1}, \dots, z_{n-1} = v^{q_{n-1}}.$$

If $v = 0$ then $x = y = z = \cdots = z_{n-1} = 0$, and $F_n(P) = 0$. If $v \neq 0$, we can perform the trick used in the proof of the Theorem 4.6.7, and we get that $F_n(P) = (z_{n-1} - x^a z_{n-2}^b)^{q_{n-2}} = 0$.

Secondly we prove that $F_1, \dots, F_{n-1} \in I(\overline{V})_1$. For $i = 1, \dots, n-1$, $F_i \in I(\overline{V})$. This implies $F_i^{deh} \in I(V)$, where F_i^{deh} is the dehomogenized polynomial, that is setting $w = 1$ in F_i, hence $F_i(1, v^p, v^q, v^{q_1}, \dots, v^{q_{n-2}}) = 0$, so $F_i^{deh} \in I(V_1)$ and finally $F_i \in I(\overline{V}_1)$. As a conclusion, the zero set of $F_1, \dots, F_{n-1}, F_n$, is included in $\overline{V}_1$.

Third, we have to prove that if $P = (w, x, y, z_1, \dots, z_{n-1})$ is a zero of $F_1, \dots, F_{n-1}, F_n$, then $P \in \overline{V}_1$. Let $P' = (w, x, y, z_1, \dots, z_{n-2})$, since $F_1(P') = \cdots = F_{n-1}(P') = 0$, there exist $u, v \in K$ such that

$$w = u^{q_{n-2}}, x = u^{q_{n-2}-p} v^p, y = u^{q_{n-2}-q} v^{q_0},$$
$$z_1 = u^{q_{n-2}-q_1} v^{q_1}, \dots, z_{n-2} = v^{q_{n-2}}.$$

Suppose that $w = 0$, then $x = y = z = \cdots = z_{n-3} = 0$. Hence $F_n(P) = 0$ implies $z_{n-2} = 0$, that is $P = (0, \dots, 0, 1)$, which is a point of $\overline{V}_1$. Suppose that $w \neq 0$, we can assume that $w = 1$, hence there exists $v \in K$ such that

$$x = v^p, y = v^{q_0}, z_1 = v^{q_1}, \dots, z_{n-2} = v^{q_{n-2}}.$$

In particular $x^{q_{n-2}} = (v^p)^{q_{n-2}} = z_{n-2}^p$. From $F_n(P) = 0$, we get $(z_{n-1} - x^a z_{n-2}^b)^{q_{n-2}} = 0$, that is $z_{n-1} = x^a z_{n-2}^b = v^{q_{n-1}}$.

Example 4.6.14 Consider the projective curve $\overline{V}(p, q_0, q_1, \dots, q_{n-2})$. Let $q_{n-1} = bq_{n-2}$ for a natural number $b \geq 2$. Then $\overline{V}_1(p, q_0, q_1, \dots, q_{n-2}, q_{n-1})$ is the zero set of $I(\overline{V}(p, q_0, q_1, \dots, q_{n-2}))$ and $F^h := z_{n-1} w^{b-1} - z_{n-2}^b$. In particular if $\overline{V}(p, q_0, q_1, \dots, q_{n-2})$ is a set-theoretic complete intersection, then $\overline{V}_1(p, q_0, q_1, \dots, q_{n-2}, q_{n-1})$ is a set-theoretic complete intersection.

Example 4.6.15 Let $\overline{V}(1, 2, 3, r)$ be the projective toric curve in $\mathbb{P}^4$ with parametrization

$$w = u^r, x = u^{r-1} v^1, y = u^{r-2} v^2, z_1 = u^{r-3} v^3, z_2 = v^r.$$

Then $\overline{V}$ is a set-theoretic complete intersection for all integers $r \geq 4$. By the above theorem we have that $\overline{V}(1, 2, 3, r)$ is a set-theoretic complete intersection for $r \geq 5$. The case $r = 4$ was done in [19]. Note that the case $r = 5$ follows also from [8]. This example was independently studied in [18].

Example 4.6.16 Let $\overline{V}(1, 3, 5, r)$ be the projective toric curve in $\mathbb{P}^4$ with parametrization

$$w = u^r, x = u^{r-1}v^1, y = u^{r-3}v^3, z_1 = u^{r-5}v^5, z_2 = v^r.$$

Then by using the Theorem 4.6.7 $\overline{V}$ is a set-theoretic complete intersection for all integers $r \in \{9, 13, 14, 17, 18, 19, 21, 22, \ldots\}$, and by Example 4.6.14, for all $r = 5b, b \geq 2$.

The trick used above can be improved. Let us consider the following example. Let $\overline{V}(1, 3, 5, 11)$ be the projective toric curve in $\mathbb{P}^4$, then $\overline{V}(1, 3, 5, 11)$ is a set-theoretic complete intersection on $I(\overline{V}(1, 3, 5))$ and F, where F is obtained from $(z_2 - y^2 z_1)^5 = 0$ working modulo $y^5 - z_1^3$.

In conclusion the only unknown cases are for $r = 6, 7, 8, 12$.

Example 4.6.17 Let $\overline{V}(2, 3, 5, r)$ be the projective toric curve in $\mathbb{P}^4$. We have seen in Example 4.6.14, that $\overline{V}(2, 3, 5, r)$ is a set-theoretic complete intersection for $r = 5b, b \geq 2$. By using the method in Theorem 4.6.6, we can see that $\overline{V}(2, 3, 5, r)$ is a set-theoretic complete intersection for $r = 12 + 5b, 14 + 5b$, and by using the methods in Theorem 4.6.7, that $\overline{V}(2, 3, 5, r)$ is a set-theoretic complete intersection for $r = 8 + 5b, 16 + 5b$. In conclusion $\overline{V}(2, 3, 5, r)$ is a set-theoretic complete intersection for all positive integers, except possibly for $r \in \{6, 7, 11\}$. Note that the case $\overline{V}(2, 3, 5, 9)$ was solved in [24].

Theorem 4.6.18 *Let $p, q_0, q_1, \ldots, q_{n-2}$ be positive integers. Let $\overline{V}$ be the projective toric curve in $\mathbb{P}^n$, with parametrization*

$$\begin{aligned} w &= u^{q_{n-2}}, \\ x &= u^{q_{n-2}-p}v^p, \\ y &= u^{q_{n-2}-q_0}v^{q_0}, \\ z_1 &= u^{q_{n-2}-q_1}v^{q_1}, \\ &\ldots \\ z_{n-2} &= v^{q_{n-2}}. \end{aligned}$$

For $i = 0, \ldots, q_{n-3}$ let $\gcd(p, q_i) = l_i$. We set $p' = p/l, q_i' = q_i/l_i$. Suppose that for $i = 1, \ldots, n-2$, $q_i \geq q_{i-1}'(q_{i-1}' - 1)(q_{i-1}' - p' - 1) + q_{i-1}' l_i$. Then $\overline{V}$ is a set-theoretic complete intersection.

Proof The proof is by induction, the case $n = 3$ is Theorem 4.6.8. The case $n - 1$ implies n follows from Theorem 4.6.13. In the case where $l_i = 1$ for all i we have that for $i = 1, \ldots, n-2$, there exist positive integers a_i, b_i such that $q_i = a_i p' + b_i q'_{i-1}, 0 \le a_i \le q'_{i-1} - 1$. $\overline{V}$ is the zero set of the polynomials

$$F_1 := x^{q_0} - y^p w^{q_0 - p}, F_2, \ldots, F_{n-1},$$

where F_{i-1} is obtained, by applying the trick used in the proof of Theorem 4.6.7, from

$$((z_i - x^{a_i} z_{i-1}^{b_i})^{q_{i-1}})^h_{z_{i-1}^p = x^{q_{i-1}}},$$

where $(H)_{y^p = x^{q_0}}$ means substitution when possible x^{q_0} by y^p, and H^h is the homogenization of H with respect to w.

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