On the Links Between Argumentation-Based Reasoning and Nonmonotonic Reasoning

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Abstract. In this paper we investigate the links between instantiated argumentation systems and the axioms for non-monotonic reasoning described in [15] with the aim of characterising the nature of argument based reasoning. In doing so, we consider two possible interpretations of the consequence relation, and describe which axioms are met by ASPIC$^+$ under each of these interpretations. We then consider the links between these axioms and the rationality postulates. Our results indicate that argument based reasoning as characterised by ASPIC$^+$ is—according to the axioms of [15]—non-cumulative and non-monotonic, and therefore weaker than the weakest non-monotonic reasoning systems considered in [15]. This weakness underpins ASPIC$^+$'s success in modelling other reasoning systems. We conclude by considering the relationship between ASPIC$^+$ and other weak logical systems.

1 Introduction

The rationality postulates proposed by Caminada and Amgoud [4] have been influential in the development of instantiated argumentation systems. These postulates identify desirable properties for the conclusions drawn from an argument based reasoning process, and focus on the effects of non-defeasible rules within an argumentation system. However, these postulates provide no desiderata with regards to the conclusions drawn from the defeasible rules found within an argumentation system. This latter type of rule is critical to argumentation, and identifying postulates for such rules is therefore important. At the same time, a large body of work exists which deals with non-monotonic reasoning (NMR). Such NMR systems (exemplified by approaches such as circumscription [18], default logic [23] and auto-epistemic logic [21]) introduce various approaches to handling defeasible reasoning, and axioms have been proposed to categorise such systems [15].

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In this paper we seek to combine the rich existing body of work on NMR with structured argumentation systems. We aim to identify what axioms structured argument systems, exemplified by ASPIC$^+$ [19] meet[1]. In doing so, we also wish to investigate the links between NMR axioms and the rationality postulates. This latter strand of work will, in the future, potentially allow us to identify additional rationality postulates which have not been considered to date.

2 The ASPIC$^+$ Argumentation Framework

ASPIC$^+$ [19] is a widely used formalism for structured argumentation, which satisfies the rationality postulates of [4][2]. Arguments within ASPIC$^+$ are constructed by chaining two types of inference rules, beginning with elements of a knowledge base. The first type of inference rule is referred to as a *strict* rule, and represents rules whose conclusion can be unconditionally drawn from a set of premises. This is in contrast to *defeasible* inference rules, which allow for a conclusion to be drawn from a set of premises as long as no exceptions or contrary conclusions exist.

Definition 1. *An* argumentation system *is a triple* $AS = \langle \mathcal{L}, \mathcal{R}, n \rangle$ *where:*

- $\mathcal{L}$ *is a logical language.*
- $\bar{\cdot}$ *is a function from* $\mathcal{L}$ *to* $2^{\mathcal{L}}$*, such that:*
  - ϕ *is a contrary of* ψ *if* $\phi \in \overline{\psi}$, $\psi \notin \overline{\phi}$
  - ϕ *is a contradictory of* ψ *(denoted by '*$\phi = -\psi$*'), if* $\phi \in \overline{\psi}$, $\psi \in \overline{\phi}$
  - *each* $\phi \in \mathcal{L}$ *has at least one contradictory.*
- $\mathcal{R} = \mathcal{R}_s \cup \mathcal{R}_d$ *is a set of strict (*$\mathcal{R}_s$*) and defeasible (*$\mathcal{R}_d$*) inference rules of the form* $\phi_1, \ldots, \phi_n \rightarrow \phi$ *and* $\phi_1, \ldots, \phi_n \Rightarrow \phi$ *respectively (where* ϕ_i, ϕ *are meta-variables ranging over wff in* $\mathcal{L}$*), and* $\mathcal{R}_s \cap \mathcal{R}_d = \emptyset$*.*
- $n : \mathcal{R}_d \mapsto \mathcal{L}$ *is a naming convention for defeasible rules.*

We write $\phi_1, \ldots, \phi_n \rightsquigarrow \phi$ if $\mathcal{R}$ contains a strict rule $\phi_1, \ldots, \phi_n \rightarrow \phi$ or a defeasible rule $\phi_1, \ldots, \phi_n \Rightarrow \phi$.

Definition 2. *A* knowledge base *in an argumentation system* $\langle \mathcal{L}, \mathcal{R}, n \rangle$ *is a set* $\mathcal{K} \subseteq \mathcal{L}$ *consisting of two disjoint subsets* $\mathcal{K}_n$ *(the axioms) and* $\mathcal{K}_p$ *(the ordinary premises).*

An argumentation theory consists of an argumentation system and knowledge base.

Definition 3. *An* argumentation theory AT *is a pair* $\langle AS, \mathcal{K} \rangle$*, where* AS *is an argumentation system* AS *and* $\mathcal{K}$ *is a knowledge base.*

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[1] ASPIC$^+$ was selected for this study due to its popularity, and its ability to model a variety of other structured systems [20].

[2] While additional rationality postulates have been proposed [24], we do not consider them in this paper.

An argumentation theory is *strict* iff $\mathcal{R}_d = \emptyset$ and $\mathcal{K}_p = \emptyset$, and is *defeasible* otherwise.

To ensure that reasoning meets norms for rational reasoning according to the *rationality postulates* of [4], an ASPIC$^+$ argumentation system's strict rules must be closed under transposition. That is, given a strict rule with premises $\varphi = \{\phi_1, \ldots, \phi_n\}$ and conclusion ϕ (written $\varphi \rightarrow \phi$), a set of n additional rules of the following form must be present in the system: $\{\overline{\phi}\} \cup \varphi \backslash \{\phi_i\} \rightarrow \overline{\phi_i}$ for all $1 \leq i \leq n$.

Arguments are defined recursively in terms of sub-arguments and through the use of several functions: $\texttt{Prem}(A)$ returns all the premises of argument A; $\texttt{Conc}(A)$ returns A's conclusion, and $\texttt{TopRule}(A)$ returns the last rule used within the argument. $\texttt{Sub}(A)$ returns all of A's sub-arguments. Given this, arguments are defined as follows.

Definition 4. *An* argument A *on the basis of an argumentation theory* $AT = \langle\langle\mathcal{L}, \mathcal{R}, n\rangle, \mathcal{K}\rangle$ *is:*

1. ϕ *if* $\phi \in \mathcal{K}$ *with:* $\texttt{Prem}(A) = \{\phi\}$; $\texttt{Conc}(A) = \{\phi\}$; $\texttt{Sub}(A) = \{A\}$; $\texttt{TopRule}(A)$ $=$ *undefined.*
2. $A_1, \ldots, A_n \rightarrow/\Rightarrow \phi$ *if* A_i *are arguments such that there respectively exists a strict/defeasible rule* $\texttt{Conc}(A_1), \ldots, \texttt{Conc}(A_n) \rightarrow/\Rightarrow \phi$ *in* $\mathcal{R}_s/\mathcal{R}_d$. $\texttt{Prem}(A) = \texttt{Prem}(A_1) \cup \ldots \cup \texttt{Prem}(A_n)$; $\texttt{Conc}(A) = \phi$; $\texttt{Sub}(A) = \texttt{Sub}(A_1) \cup \ldots \cup \texttt{Sub}(A_n) \cup \{A\}$; $\texttt{TopRule}(A) = \texttt{Conc}(A_1), \ldots, \texttt{Conc}(A_n) \rightarrow/\Rightarrow \phi$.

We write $\mathcal{A}(AT)$ to denote the set of arguments on the basis of the theory AT, and given a set of arguments $\mathbf{A}$, we write $\texttt{Concs}(\mathbf{A})$ to denote the conclusions of those arguments, that is:

$$\texttt{Concs}(\mathbf{A}) = \{\texttt{Conc}(A) | A \in \mathbf{A}\}$$

Like other argumentation systems, ASPIC$^+$ utilises conflict between arguments—represented through attacks—to determine what conclusions are justified.

An argument can be attacked in three ways: on its ordinary premises, on its conclusion, or on its inference rules. These three kinds of attack are called undermining, rebutting and undercutting attacks, respectively.

Definition 5. *An argument* A attacks *an argument* B *iff* A *undermines, rebuts or undercuts* B*, where:*

- A *undermines* B *(on* B'*) iff* $\texttt{Conc}(A) = \overline{\phi}$ *for some* $B' = \phi \in \texttt{Prem}(B)$ *and* $\phi \in \mathcal{K}_p$.
- A *rebuts* B *(on* B'*) iff* $\texttt{Conc}(A) = \overline{\phi}$ *for some* $B' \in \texttt{Sub}(B)$ *of the form* $B''_1, \ldots, B''_2 \Rightarrow \phi$.
- A *undercuts* B *(on* B'*) iff* $\texttt{Conc}(A) = \overline{n(r)}$ *for some* $B' \in \texttt{Sub}(B)$ *such that* $\texttt{TopRule}(B)$ *is a defeasible rule* r *of the form* $\phi_1, \ldots, \phi_n \Rightarrow \phi$.

Note that, in ASPIC$^+$ rebutting is *restricted*: an argument with a strict $\texttt{TopRule}$ can rebut an argument with a defeasible $\texttt{TopRule}$, but not vice versa. ([5,16]

introduce the ASPIC- and ASPIC^+_D systems which use unrestricted rebut). Finally, a set of arguments is said to be *consistent* iff there is no attack between any arguments in the set.

Attacks can be distinguished by whether they are preference-dependent (rebutting and undermining) or preference-independent (undercutting). The former succeed only when the attacker is preferred. The latter succeed whether or not the attacker is preferred. Within ASPIC$^+$ preferences over defeasible rules and ordinary premises are combined to obtain a preference ordering over arguments [19]. Here, we are not concerned about the means of combination, but, following [19], we only consider *reasonable* orderings. For our purposes, a reasonable ordering is one such that adding a strict rule or axiom to an argument will neither increase nor decrease its preference level.

Definition 6. *A preference ordering $\preceq$ is a binary relation over arguments, i.e., $\preceq \subseteq \mathcal{A} \times \mathcal{A}$, where $\mathcal{A}$ is the set of all arguments constructed from the knowledge base in an argumentation system.*

Combining these elements results in the following.

Definition 7. *A* structured argumentation framework *is a triple $\langle \mathcal{A}, att, \preceq \rangle$, where $\mathcal{A}$ is the set of all arguments constructed from the argumentation system, att is the attack relation, and $\preceq$ is a preference ordering on $\mathcal{A}$.*

Preferences over arguments interact with attacks such that *preference-dependent* attacks succeed when the attacking argument is preferred. In contrast *preference-independent* attacks always succeed. Attacks that succeed are called *defeats*. Using Definition 4 and the notion of defeat, we can instantiate an abstract argumentation framework from a structured argumentation framework.

Definition 8. *An* (abstract) argumentation framework *AF corresponding to a structured argumentation framework $SAF = \langle \mathcal{A}, att, \preceq \rangle$ is a pair $\langle \mathcal{A}, Defeats \rangle$ such that Defeats is the defeat relation on $\mathcal{A}$ determined by SAF.*

This abstract argumentation framework can be evaluated using standard argumentation semantics [8], defining the notion of an extension:

Definition 9. *Let $AF = \langle \mathcal{A}, Defeats \rangle$ be an argumentation framework, let $A \in \mathcal{A}$ and $E \subseteq \mathcal{A}$. E is said to be* conflict-free *iff there does not exist a $B, C \in E$ such that B defeats C. E is said to defend A iff for every $B \in \mathcal{A}$ such that B defeats A, there exists a $C \in E$ such that C defeats B. The characteristic function $F : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ is defined as $F(E) = \{A \in \mathcal{A} | E \text{ defends } A\}$. E is called (1) an* admissible *set iff E is conflict-free and $E \subseteq F(E)$; (2) a* complete *extension iff E is conflict-free and $E = F(E)$; (3) a* grounded *extension iff E is the minimal complete extension; (4) a* preferred *extension iff E is a maximal complete extension, where minimality and maximality are w.r.t. set inclusion; and (5) a* stable *extension iff E is a preferred extension which attacks all arguments in $\mathcal{A} - E$.*

We note in passing that other extensions have been defined and refer the reader to [1] for further details.

For a given semantics, if an argument is in an extension, it is said to be justified, given the information in the argumentation framework, and given the semantics that have been adopted. Dealing with structured arguments, we are not only interested in what arguments hold, but which propositions are the conclusions of arguments that hold, given some semantics. Thus we say that a proposition is a *justified conclusion* if it is the conclusion of an argument that is in an extension under some semantics. In fact, as [6] points out, the situation is more complex than that, since under some semantics there may be multiple extensions. Thus [6] defines the notions of sceptically, credulously and universally justified conclusions under a given semantics as follows.

Definition 10. *For $T \in \{admissible, complete, preferred, grounded, stable\}$, if $AF = \langle \mathcal{A}, Defeats \rangle$ is an argumentation framework. we say that:*

- *ϕ is a T credulously justified conclusion of AF iff there exists an argument A and a T extension E such that $A \in E$ and $\texttt{Conc}(A) = \phi$.*
- *ϕ is a T sceptically justified conclusion of AF iff for every T extension E, there exists an argument $A \in E$ such that $\texttt{Conc}(A) = \phi$.*
- *ϕ is a T universally justified conclusion of AF iff there exists an argument A for every T extension E, such that $A \in E$ and $\texttt{Conc}(A) = \phi$.*

3 Axiomatic Reasoning and ASPIC$^+$

Kraus *et al.* [15], building on earlier work by Gabbay [11], identified a set of axioms which characterise non-monotonic inference in logical systems, and studied the relationships between sets of these axioms. Their goal was to characterise different kinds of reasoning; to pin down what it means for a logical system to be monotonic or non-monotonic; and—in particular—to be able to distinguish between the two. Table 1 presents the axioms of [15], which we will use to characterise reasoning in ASPIC$^+$. The symbol $\mid\!\sim$ encodes a consequence relation, while $\models$ identifies the statements obtainable from the underlying theory. We have altered some of the symbols used in [15] to avoid confusion with the notation of ASPIC$^+$. Equivalence is denoted $\equiv$ (rather than $\leftrightarrow$), and $\hookrightarrow$ (rather than $\rightarrow$) denotes the existence of a strict or defeasible rule.

Consequence relations that satisfy Ref, LLE, RW, Cut and CM are said to be *cumulative*, and [15] describes them as being the weakest interesting logical system. Cumulative consequence relations which also satisfy CP are *monotonic*, while consequence relations that are cumulative and satisfy M are called *cumulative monotonic*. Such relations are stronger than cumulative but not monotonic in the usual sense.

To determine which axioms ASPIC$^+$ does or does not comply with, we must decide how different aspects of the axioms should be interpreted. We interpret the consequence relation $\mid\!\sim$ in two ways that are natural in the context of ASPIC$^+$—describing these in detail later—and which fit with the high level meaning of "if α is in the knowledge base, then β follows", or "β is a consequence of α".

Table 1. The axioms from [15] that we will consider.

Abbr.	Axiom	Name
Ref	$\alpha \mid\sim \alpha$	Reflexivity
LLE	$\dfrac{\models \alpha \equiv \beta \qquad \alpha \mid\sim \gamma}{\beta \mid\sim \gamma}$	Left Logical Equivalence
RW	$\dfrac{\models \alpha \hookrightarrow \beta \qquad \gamma \mid\sim \alpha}{\gamma \mid\sim \beta}$	Right Weakening
Cut	$\dfrac{\alpha \wedge \beta \mid\sim \gamma \qquad \alpha \mid\sim \beta}{\alpha \mid\sim \gamma}$	Cut
CM	$\dfrac{\alpha \mid\sim \beta \qquad \alpha \mid\sim \gamma}{\alpha \wedge \beta \mid\sim \gamma}$	Cautious Monotonicity
M	$\dfrac{\models \alpha \hookrightarrow \beta \qquad \beta \mid\sim \gamma}{\alpha \mid\sim \gamma}$	Monotonicity
T	$\dfrac{\alpha \mid\sim \beta \qquad \beta \mid\sim \gamma}{\alpha \mid\sim \gamma}$	Transitivity
CP	$\dfrac{\alpha \mid\sim \beta}{\overline{\beta} \mid\sim \overline{\alpha}}$	Contraposition

Assuming such an interpretation of $\alpha \mid\sim \beta$ we can consider the meaning of the axioms. Some axioms are clear. For example, axiom T says that if β is a consequence of α, and γ is a consequence of β, then γ is a consequence of α. Other axioms are more ambiguous. Does $\alpha \wedge \beta \mid\sim \gamma$ in Cut mean that γ is a consequence of the conjunction $\alpha \wedge \beta$, or a consequence of α and β together? In other words is $\wedge$ a feature of the language underlying the reasoning system, or a feature of the meta-language in which the properties are written? Similarly, given the distinction between strict and defeasible rules, is $\alpha \hookrightarrow \beta$ a strict rule in ASPIC$^+$, a defeasible rule, or some statement in the property meta-language?

We interpret the symbols found in the axioms as follows:

- $\models \alpha$ means that α is an element of the relevant knowledge base.
- $\alpha \wedge \beta$ means both α and β, in particular in Cut and CM, $\wedge$ means that both α and β are in the knowledge base.
- $\alpha \equiv \beta$ is taken—as usual—to abbreviate the formula $(\alpha \hookrightarrow \beta) \wedge (\beta \hookrightarrow \alpha)$. We assume $\alpha \hookrightarrow \beta$ and $\beta \hookrightarrow \alpha$ have the same interpretation, i.e., both or neither are strict.
- $\alpha \hookrightarrow \beta$ has two interpretations. We have the *strict* interpretation in which $\alpha \hookrightarrow \beta$ denotes a strict rule $\alpha \rightarrow \beta$ in ASPIC$^+$, and the *defeasible* interpretation in which $\alpha \hookrightarrow \beta$ denotes either a strict or defeasible rule. We denote the latter interpretation by writing $\alpha \rightsquigarrow \beta$.

4 Axioms and Consequences in ASPIC$^+$

In this section we examine which of the axioms ASPIC$^+$ satisfies. Before doing so however, we must further pin down some aspects of ASPIC$^+$ rules.

4.1 Preliminaries

To evaluate ASPIC$^+$, we have to be a bit more precise about exactly what we are evaluating. We start by saying that we assume an arbitrary ASPIC$^+$ argumentation theory $AT = \langle\langle\mathcal{L}, \mathcal{R}, n\rangle, \mathcal{K}\rangle$, in the sense that we say nothing about the contents of the knowledge base, or what domain-specific rules it contains. However, we distinguish between two classes of theory, with respect to the *base logic* that the theory contains.

The idea we capture by this is that in addition to domain specific rules—rules, for example, about birds and penguins flying—an ASPIC$^+$ theory might also contain rules for reasoning in some logic. For example, we might equip an ASPIC$^+$ theory with the axioms and inference rules of classical logic. Such a theory would be able to construct arguments using all the rules of classical logic, as well as all the domain-specific rules in the theory. The two base logics that we consider are classical logic, and what we call the "empty" base logic, where the ASPIC$^+$ theory only contains domain-specific rules. (We make some observations about other base logics—intuitionistic logic and defeasible logic [2], but show no formal results for them.)

For each of the base logics, we consider the two different interpretations of the non-monotonic consequence relation $\mid\!\sim$ described above, identifying which axioms each interpretation satisfies. For our theory AT, we write AT_x to denote an extension of this augmentation theory also containing proposition x: $AT_x = \langle\langle\mathcal{L}, \mathcal{R}, n\rangle, \mathcal{K} \cup \{x\}\rangle$. An argument present in the latter, but not former, theory is denoted A^x.

4.2 Argument Construction

We begin by considering the consequence relation as representing argument construction. In other words, we interpret $\alpha \mid\!\sim \beta$ as meaning that if α is in the axioms or ordinary premises of a theory, we can construct an argument for β. More precisely:

Definition 11. *We write $\alpha \mid\!\sim_{B,a} \beta$, if for every* ASPIC$^+$ *argumentation theory $AT = \langle\langle\mathcal{L}, \mathcal{R}, n\rangle, \mathcal{K}\rangle$ with base logic B such that $\beta \notin \texttt{Concs}(\mathcal{A}(AT))$, it is the case that $\beta \in \texttt{Concs}(\mathcal{A}(AT_\alpha))$, where $B = \{\emptyset, c\}$, representing the empty and classical base logics respectively.*

Proposition 1. *Ref, LLE, RW, Cut and CM hold for $\mid\!\sim_{\emptyset,a}$ in strict and defeasible theories.*

Proof. *Consider an arbitrary theory* $AT = \langle\langle\mathcal{L},\mathcal{R},n\rangle,\mathcal{K}\rangle$. ***[Ref]*** *Given a theory* AT_α, *we have an argument* $A^\alpha = [\alpha]$, *so Ref holds for* $\mid\!\sim_{\emptyset,a}$. ***[LLE]*** *Since* $\alpha \mid\!\sim_{\emptyset,a} \gamma$, AT_α *contains a chain of arguments* $A_1^\alpha, A_2^\alpha, \ldots, A_n^\alpha$ *with* $A_1^\alpha = [\alpha]$ *and* $\texttt{Conc}(A_n^\alpha) = \gamma$. *Given* $\models \alpha \equiv \beta$, *we have that both* $\alpha \rightsquigarrow \beta$ *and* $\beta \rightsquigarrow \alpha$ *are in the theory* AT, *so are in the theory* AT_β. *Within* AT_β, *we obtain a chain of arguments* $B_0^\beta = [\beta], B_1^\beta = [B_0^\beta \rightsquigarrow \alpha], A_2^\beta, \ldots, A_n^\beta$. *That is* $\beta \mid\!\sim_{\emptyset,a} \gamma$. *Therefore, both strict and defeasible versions of LLE hold for* $\mid\!\sim_{\emptyset,a}$. ***[RW]*** *Since* $\gamma \mid\!\sim_{\emptyset,a} \alpha$ *in theory* AT_γ, *there is a chain of arguments* $A_1^\gamma, A_2^\gamma, \ldots, A_n^\gamma$ *with* $A_1^\gamma = [\gamma]$ *and* $\texttt{Conc}(A_n^\gamma) = \alpha$. *Given* $\models \alpha \hookrightarrow \beta$, *theory* AT *must contain* $\alpha \rightsquigarrow \beta$, *as must* AT_γ. *In* AT_γ, *we have a chain of arguments* $A_1^\gamma, \ldots, A_n^\gamma, A_{n+1}^\gamma = [A_n^\gamma \Rightarrow \beta]$. *Thus,* $\gamma \mid\!\sim_{\emptyset,a} \beta$, *and both strict and defeasible versions of RW hold for* $\mid\!\sim_{\emptyset,a}$. ***[Cut]*** *Since* $\alpha \wedge \beta \mid\!\sim_{\emptyset,a} \gamma$, *there is a chain of arguments* $A_1^{\alpha,\beta}, A_2^{\alpha,\beta}, \ldots, A_n^{\alpha,\beta}$ *with* $A_1^{\alpha,\beta} = [\alpha]$, $A_2^{\alpha,\beta} = [\beta]$ *in theory* $AT_{\alpha,\beta}$, *and* $\texttt{Conc}(A_n^{\alpha,\beta}) = \gamma$. *In theory* AT_α, *since* $\alpha \mid\!\sim_{\emptyset,a} \beta$, *there is a chain of arguments* $B_1^\alpha, B_2^\alpha, \ldots, B_m^\alpha$ *with* $B_1^\alpha = [\alpha]$ *and* $\texttt{Conc}(B_m^\alpha) = \beta$. *There is also a chain of arguments* $B_1^\alpha, B_2^\alpha, \ldots, B_m^\alpha, A_3^\alpha, \ldots, A_n^\alpha$. *That is* $\alpha \mid\!\sim_{\emptyset,a} \gamma$. *Therefore, cut holds for* $\mid\!\sim_{\emptyset,a}$. ***[CM]*** *Since* $\alpha \mid\!\sim_{\emptyset,a} \gamma$ AT_α *has a chain of arguments* $A_1^\alpha, \ldots, A_n^\alpha$ *with* $A_1^\alpha = [\alpha]$ *and* $\texttt{Conc}(A_n^\alpha) = \gamma$. $AT_{\alpha,\beta}$ *has a similar chain of arguments* $A_1^{\alpha,\beta}, \ldots, A_n^{\alpha,\beta}$, *so* $\alpha \wedge \beta \mid\!\sim_{\emptyset,a} \gamma$. *CM thus holds for* $\mid\!\sim_{\emptyset,a}$.

Since Ref, LLE, RW, Cut and CM hold, $\mid\!\sim_{\emptyset,a}$ is cumulative for both strict and defeasible theories.

Proposition 2. *M and T hold for* $\mid\!\sim_{\emptyset,a}$ *in strict and defeasible theories.*

Proof. *Consider an arbitrary theory* $AT = \langle\langle\mathcal{L},\mathcal{R},n\rangle,\mathcal{K}\rangle$. ***[M]*** *Since* $\beta \mid\!\sim_{\emptyset,a} \gamma$, *in the theory* AT_β, *there is a chain of arguments* $A_1^\beta, A_2^\beta, \ldots, A_n^\beta$ *with* $A_1^\beta = [\beta]$ *and* $\texttt{Conc}(A_n^\beta) = \gamma$. *Given* $\models \alpha \hookrightarrow \beta$, *we have* $\alpha \rightsquigarrow \beta$ *in the theory* AT, *and also in the theory* AT_α. *In the latter, there is a chain of arguments* $B_0^\alpha = [\alpha], B_1^\alpha = [B_0^\alpha \rightsquigarrow \beta], A_2^\alpha, \ldots, A_n^\alpha$. *That is* $\alpha \mid\!\sim_{\emptyset,a} \gamma$. *Therefore, both strict and defeasible versions of M hold for* $\mid\!\sim_{\emptyset,a}$. ***[T]*** *Since* $\beta \mid\!\sim_{\emptyset,a} \gamma$, *in* AT_β, *there is a chain of arguments* $B_1^\beta, B_2^\beta, \ldots, B_m^\beta$ *with* $B_1^\beta = [\beta]$ *and* $\texttt{Conc}(B_m^\beta) = \gamma$. *Similarly, since* $\alpha \mid\!\sim_{\emptyset,a} \beta$, *in* AT_α, *there is a chain of arguments* $A_1^\alpha, A_2^\alpha, \ldots, A_n^\alpha$ *with* $A_1^\alpha = [\alpha]$ *and* $\texttt{Conc}(A_n^\alpha) = \beta$. *Combining this with* $B_1^\alpha, B_2^\alpha, \ldots, B_m^\alpha$, *we obtain the combined chain of arguments* $A_1^\alpha, A_2^\alpha, \ldots, A_n^\alpha, B_2^\alpha, \ldots, B_m^\alpha$. *That is* $\alpha \mid\!\sim_{\emptyset,a} \gamma$. *Therefore, T holds for* $\mid\!\sim_{\emptyset,a}$.

Thus $\mid\!\sim_{\emptyset,a}$ is cumulative monotonic for strict or defeasible theories. It is not, however, monotonic.

Proposition 3. *CP does not hold for* $\mid\!\sim_{\emptyset,a}$ *in strict or defeasible theories.*

Proof. *Consider an* ASPIC$^+$ *theory which contains:* $\mathcal{K} = \{c\}$, $\mathcal{R}_s = \{\alpha, c \rightarrow d; \alpha, \overline{d} \rightarrow \overline{c}; c, \overline{d} \rightarrow \overline{\alpha}; \alpha \rightarrow e; \overline{e} \rightarrow \overline{\alpha}; d, e \rightarrow \beta; d, \overline{\beta} \rightarrow \overline{e}; \overline{\beta}, e \rightarrow \overline{d}\}$ *We have* $\alpha \mid\!\sim_{\emptyset,a} \beta$ *but not* $\overline{\beta} \mid\!\sim_{\emptyset,a} \overline{\alpha}$. *Therefore, CP does not hold for* $\mid\!\sim_{\emptyset,a}$.

Having characterised $\mid\sim_{\emptyset,a}$, we consider $\mid\sim_{c,a}$. Clearly this will satisfy all the properties that are satisfied by $\mid\sim_{\emptyset,a}$, since it includes all the inference rules of $\mid\sim_{\emptyset,a}$. In addition, we have the following.

Proposition 4. *CP holds for* $\mid\sim_{c,a}$ *in strict theories.*

Proof. Any strict ASPIC$^+$ *theory with a classical base logic will generate the same set of consequences as classical logic. Furthermore, we know that CP is satisfied under classical logic. Therefore, the consequence relation* $\mid\sim_{c,a}$ *satisfies CP for any strict theory.*

Thus $\mid\sim_{c,a}$ is monotonic for strict theories. However:

Proposition 5. *CP does not hold for* $\mid\sim_{c,a}$ *in defeasible theories.*

Proof. Consider the counter-example from Proposition 3 where all rules are defeasible. Since the defeasible portion of the theory does not contain a rule of the form $\overline{\beta} \rightarrow \overline{d} \vee \overline{e}$*, CP will not be satisfied.*

4.3 Justified Conclusions

Next we interpret $\alpha \mid\sim \beta$ as meaning that if α is in a theory, we can construct an argument for β such that β is in the set of justified conclusions (regardless of preferences). We will consider only the grounded and preferred semantics, but, as we will see, we have to bring in the ideas from Definition 10 since different kinds of justified conclusion lead to $\alpha \mid\sim \beta$ satisfying different properties. We start with:

Definition 12. *Let* $AF = \langle \mathcal{A}, Defeats \rangle$ *be an abstract argumentation framework, we define*

$$\mathtt{Just}_g(\mathcal{A}(AT)) = \{\phi|\phi \text{ is a grounded justified conclusion}\}$$
$$\mathtt{Just}_p^c(\mathcal{A}(AT)) = \{\phi|\phi \text{ is a preferred credulously justified conclusion}\}$$
$$\mathtt{Just}_p^s(\mathcal{A}(AT)) = \{\phi|\phi \text{ is a preferred sceptically justified conclusion}\}$$
$$\mathtt{Just}_p^u(\mathcal{A}(AT)) = \{\phi|\phi \text{ is a preferred universally justified conclusion}\}$$

Note that we don't have to distinguish between different classes of grounded justified conclusion because, since there is exactly one grounded extension, the three different classes of grounded justified conclusion coincide. Then:

Definition 13. *We write* $\alpha \mid\sim_{B,j}^{g} \beta$*, if for every* ASPIC$^+$ *argumentation theory* $AT = \langle\langle \mathcal{L}, \mathcal{R}, n\rangle, \mathcal{K}\rangle$ *with the B base logic such that* $\beta \notin \mathtt{Just}_g(\mathcal{A}(AT))$*, it is the case that* $\beta \in \mathtt{Just}_g(\mathcal{A}(AT_\alpha))$*, where* $B = \{\emptyset, c\}$.

Definition 14. *We write* $\alpha \mid\sim_{B,j}^{p,Sem} \beta$*, if for every* ASPIC$^+$ *argumentation theory* $AT = \langle\langle \mathcal{L}, \mathcal{R}, n\rangle, \mathcal{K}\rangle$ *with the B base logic such that* $\beta \notin \mathtt{Just}_p^{Sem}(\mathcal{A}(AT))$*, it is the case that* $\beta \in \mathtt{Just}_p^{Sem}(\mathcal{A}(AT_\alpha))$*, where* $B = \{\emptyset, c\}$ *and* $Sem \subseteq \{c, s, u\}$.

We also write $\mid\!\sim^{p,*}_{\emptyset,j}$ to denote the union of $\mid\!\sim^{p,c}_{\emptyset,j}$, $\mid\!\sim^{p,s}_{\emptyset,j}$ and $\mid\!\sim^{p,u}_{\emptyset,j}$. Thus, $\alpha \mid\!\sim^{p,*}_{\emptyset,j} \beta$ means that a conclusion holds under (at least one of) the consequence relations. We write $\mid\!\sim^{p,\cap\{S\}}_{\emptyset,j}$ to denote that a conclusion holds under all the consequence relations in S. Thus, for example if we have that $\alpha \mid\!\sim^{p,\cap\{c,s\}}_{\emptyset,j} \beta$, then it is the case that $\alpha \mid\!\sim^{p,c}_{\emptyset,j} \beta$ and $\alpha \mid\!\sim^{p,s}_{\emptyset,j} \beta$. Similarly, when we say an axiom holds for $\mid\!\sim^{p,*}_{\emptyset,j}$, it means that the axiom holds for at least one of $\mid\!\sim^{p,s}_{\emptyset,j}$, $\mid\!\sim^{p,u}_{\emptyset,j}$, and $\mid\!\sim^{p}_{\emptyset,j}$. The same interpretation applies for axioms holding with respect to $\mid\!\sim^{p,\cap\{s\}}_{\emptyset,j}$.

It is worth noting the following result.

Proposition 6. *If $\alpha \mid\!\sim^{g}_{B,j} \beta$ or $\alpha \mid\!\sim^{p,*}_{B,j} \beta$ then $\alpha \mid\!\sim_{B,a} \beta$.*

Proof. Follows immediately from the definitions—for β to be a justified conclusion, there must first be an argument with β as a conclusion.

Since there are, in general, less justified conclusions of a theory than there are arguments, $\mid\!\sim^{g}_{\emptyset,j}$ and $\mid\!\sim^{p,*}_{\emptyset,j}$ are more restrictive notions of consequence than $\mid\!\sim_{\emptyset,a}$. It is therefore no surprise to find that fewer of the axioms from [15] hold. We have the following.

Proposition 7. *Ref, and the defeasible versions of LLE and RW, do not hold for $\mid\!\sim^{g}_{\emptyset,j}$, $\mid\!\sim^{p,*}_{\emptyset,j}$ in defeasible theories.*

Proof. ***[Ref]*** *Consider an* ASPIC$^+$ *theory that contains: $\mathcal{K}_n = \{\overline{\alpha}\}$ and $\mathcal{R} = \emptyset$. Here, we have an argument $A = [\overline{\alpha}]$. If a is in the knowledge base $\mathcal{K}_p$, we have another argument $B = [a]$. However, B is defeated by A, but not vice versa. So B is not in any extension. Thus, Ref does not hold for either $\mid\!\sim^{g}_{\emptyset,j}$ or $\mid\!\sim^{p,*}_{\emptyset,j}$.* ***[LLE (defeasible version)]*** *Consider an* ASPIC$^+$ *theory that contains $\mathcal{K}_n = \{c\}$ and $\mathcal{R} = \{\alpha \Rightarrow \beta; \beta \Rightarrow \alpha; \alpha \Rightarrow \gamma; c \rightarrow \overline{n_1}\}$ where $n(\beta \Rightarrow \alpha) = n_1$. Here, $\alpha \mid\!\sim^{g}_{\emptyset,j} \gamma$ and $\alpha \mid\!\sim^{p,*}_{\emptyset,j} \gamma$, but, $\beta \not\mid\!\sim^{g}_{\emptyset,j} \gamma$ and $\beta \not\mid\!\sim^{p,*}_{\emptyset,j} \gamma$. Therefore, the defeasible version of LLE does not hold for either $\mid\!\sim^{g}_{\emptyset,j}$ or $\mid\!\sim^{p,*}_{\emptyset,j}$.* ***[RW (defeasible version)]*** *Consider an* ASPIC$^+$ *theory that contains $\overline{\beta}$ in its axioms. For such a theory, β will not appear in any justified conclusions. Therefore, the defeasible version of RW does not hold for either $\mid\!\sim^{g}_{\emptyset,j}$ or $\mid\!\sim^{p,*}_{\emptyset,j}$.*

Proposition 8. *The strict version of LLE and RW hold for $\mid\!\sim^{g}_{\emptyset,j}$ and $\mid\!\sim^{p,*}_{\emptyset,j}$ in strict and defeasible theories.*

Proof. Consider an arbitrary theory $AT = \langle\langle\mathcal{L}, \mathcal{R}, n\rangle, \mathcal{K}\rangle$. ***[RW (strict version)]*** *Consider the extension E_γ in AT_γ containing an argument A^γ with* `Conc`*$(A^\gamma) = \alpha$. Since $\models \alpha \rightsquigarrow \beta$, under the strict interpretation, we know that $\alpha \rightarrow \beta$ is in AT_γ. Therefore, we can construct an argument $B^\gamma = A^\gamma \rightarrow \beta$. Furthermore, the attackers of B are the attackers of A because* `TopRule`*(B) is a strict rule. Since A^γ is in the extension E_γ, B^γ is in the same extension E_γ. Therefore the strict version of RW holds for $\mid\!\sim^{g}_{\emptyset,j}$ and $\mid\!\sim^{p,*}_{\emptyset,j}$.* ***[LLE (strict version)]*** *Since $\models \alpha \equiv \beta$, under the strict interpretation, the rules $\beta \rightarrow \alpha$ and $\alpha \rightarrow \beta$ are in AT, AT_α, AT_β and $AT_{\alpha,\beta}$. Thus AT_α, AT_β, $AT_{\alpha,\beta}$ have the* *same extensions, just as for RW(strict version). If* $\alpha \mid\!\sim^{p,*}_{\emptyset,j} \gamma$*, then* $\beta \mid\!\sim^{p,*}_{\emptyset,j} \gamma$*. If* $\alpha \mid\!\sim^{g}_{\emptyset,j} \gamma$*, then* $\beta \mid\!\sim^{g}_{\emptyset,j} \gamma$*. Therefore, the strict version of LLE holds for* $\mid\!\sim^{g}_{\emptyset,j}$ *and* $\mid\!\sim^{p,*}_{\emptyset,j}$*.*

Proposition 9. *Cut holds for* $\mid\!\sim^{g}_{\emptyset,j}$ *and* $\mid\!\sim^{p,s}_{\emptyset,j}$ *in strict and defeasible theories.*

Proof. Since $\alpha \mid\!\sim^{g}_{\emptyset,j} \beta$*, the grounded justified conclusions of* AT_α *contain* α *and* β*. By adding* β *into the knowledge base, the grounded justified conclusions will not change – if the newly added* β *is not justified, then it has not effect; if the newly added* β *is justified, it will remain in the justified conclusions. The same argument applies for* $\mid\!\sim^{p,s}_{\emptyset,j}$*.*

Proposition 10. *Cut does not hold for either* $\mid\!\sim^{p,c}_{\emptyset,j}$ *or* $\mid\!\sim^{p,u}_{\emptyset,j}$ *in defeasible theories.*

Proof. We will give a counter-example. Consider the ASPIC$^+$ *theory that include* $\mathcal{K} = \emptyset$ *and* $\mathcal{R} = \{a \Rightarrow c; c \Rightarrow b; b \Rightarrow \overline{c}; \overline{c} \Rightarrow r;\}$*. The credulous or universal justified conclusions of* AT_α *are* $\{a, b, c\}$*. The credulous or universal justified conclusions of* $AT_{\alpha,\beta}$ *are* $\{a, b, \overline{c}, r, c\}$*. That is* $a \wedge b \mid\!\sim^{p}_{\emptyset,j} r$*,* $a \mid\!\sim^{p}_{\emptyset,j} b$*, but* $a \not\mid\!\sim^{p}_{\emptyset,j} r$*. Therefore Cut does not hold for either* $\mid\!\sim^{p,c}_{\emptyset,j}$ *or* $\mid\!\sim^{p,u}_{\emptyset,j}$*.*

Proposition 11. *CM holds for* $\mid\!\sim^{g}_{\emptyset,j}$ *in strict and defeasible theories.*

Proof. Since $\alpha \mid\!\sim^{g}_{\emptyset,j} \gamma$*, the grounded justified conclusions of* AT_α *contain* α *and* γ*. By adding* β *into the knowledge base, the grounded justified conclusions will not change. The justification is same as in the proof of Proposition 9.*

Proposition 12. *CM does not hold for* $\mid\!\sim^{p,*}_{\emptyset,j}$ *in defeasible theories.*

Proof. We will give counter-examples. Consider an ASPIC$^+$ *theory that include* $\mathcal{K} = \emptyset$ *and* $\mathcal{R} = \{a \Rightarrow b; a \Rightarrow r; b \rightarrow \overline{n1}; r \rightarrow \overline{n2};\}$*, where* $n(a \Rightarrow b) = n1$ *and* $n(a \Rightarrow r) = n2$*. The credulous or universal justified conclusions of* AT_α *are* $\{a, r, \overline{n1}, b, \overline{n2}\}$*. And the credulous or universal justified conclusions of* $AT_{\alpha,\beta}$ *are* $\{a, b, \overline{n2}\}$*. That is* $a \mid\!\sim^{p}_{\emptyset,j} b$*,* $a \mid\!\sim^{p}_{\emptyset,j} r$*, but* $a \wedge b \not\mid\!\sim^{p}_{\emptyset,j} r$*. Therefore CM does not hold for either* $\mid\!\sim^{p,c}_{\emptyset,j}$ *or* $\mid\!\sim^{p,u}_{\emptyset,j}$*. Now, consider an* ASPIC$^+$ *theory that include* $\mathcal{K} = \emptyset$*,* $\mathcal{R} = \{a \Rightarrow r; r \Rightarrow b; b \Rightarrow \overline{r}\}$*. The sceptical justified conclusions of* AT_α *are* $\{a, b, r\}$*. And the sceptical justified conclusions of* $AT_{\alpha,\beta}$ *are* $\{a, b\}$*.* $a \mid\!\sim^{p}_{\emptyset,j} b$*,* $a \mid\!\sim^{p}_{\emptyset,j} r$*, but* $a \wedge b \not\mid\!\sim^{p}_{\emptyset,j} r$*. Therefore CM does not hold for* $\mid\!\sim^{p,s}_{\emptyset,j}$*.*

Proposition 13. *M, T and CP do not hold for* $\mid\!\sim^{g}_{\emptyset,j}$ *or* $\mid\!\sim^{p,*}_{\emptyset,j}$ *in defeasible theories.*

Proof. We will give counter-examples. ***[M]*** *Consider an* ASPIC$^+$ *theory that contains* $\mathcal{K}_n = \{\overline{\alpha}\}$ *and* $\mathcal{R} = \{\alpha \rightarrow \beta; \overline{\beta} \rightarrow \overline{\alpha}; \beta \Rightarrow \gamma\}$*. Thus,* $\beta \mid\!\sim^{g}_{\emptyset,j} \gamma$ *and* $\beta \mid\!\sim^{p,*}_{\emptyset,j} \gamma$*, however,* $\alpha \not\mid\!\sim^{g}_{\emptyset,j} \gamma$ *and* $\alpha \not\mid\!\sim^{p,*}_{\emptyset,j} \gamma$*. Therefore, M does not hold for* $\mid\!\sim^{g}_{\emptyset,j}$ *or* $\mid\!\sim^{p,*}_{\emptyset,j}$*.* ***[T]*** *Consider an* ASPIC$^+$ *theory which includes* $\mathcal{K} = \emptyset$ *and*

*$\mathcal{R} = \{\alpha \Rightarrow \beta; \beta \Rightarrow c; c \Rightarrow \gamma; \alpha \Rightarrow \overline{n_1}\}$ where $n(c \Rightarrow \gamma) = n_1$. Thus, $a \mid\!\sim^{g}_{\emptyset,j} b$, $b \mid\!\sim^{g}_{\emptyset,j} r$, $a \mid\!\sim^{p,*}_{\emptyset,j} b$ and $b \mid\!\sim^{p,*}_{\emptyset,j} r$, but $a \not\mid\!\sim^{g}_{\emptyset,j} r$ and $a \not\mid\!\sim^{p,*}_{\emptyset,j} r$. Therefore, T does not hold for $\mid\!\sim^{g}_{\emptyset,j}$ or $\mid\!\sim^{p,*}_{\emptyset,j}$. **[CP]** Since contraposition does not hold for $\mid\!\sim_{\emptyset,a}$, by Proposition 3 it cannot hold for $\mid\!\sim^{g}_{\emptyset,j}$ or $\mid\!\sim^{p,*}_{\emptyset,j}$.*

If we consider only strict theories, the following holds.

Proposition 14. *Ref, CM, M and T hold for $\mid\!\sim^{g}_{\emptyset,j}$ and $\mid\!\sim^{p,*}_{\emptyset,j}$ in strict theories.*

Proof. If the theory is strict, then for any argumentation theory, all conclusions are justified. Therefore, for any strict theory, if $\alpha \mid\!\sim_{\emptyset,a} \beta$, then $\alpha \mid\!\sim^{g}_{\emptyset,j} \beta$ and $\alpha \mid\!\sim^{p,*}_{\emptyset,j} \beta$. We know that $\mid\!\sim_{\emptyset,a}$ holds for Ref, CM, M and T, therefore, $\mid\!\sim^{g}_{\emptyset,j}$ and $\mid\!\sim^{p,*}_{\emptyset,j}$ holds for Ref, CM, M and T in strict theories.

Proposition 15. *CP does not hold for $\mid\!\sim^{g}_{\emptyset,j}$ or $\mid\!\sim^{p,*}_{\emptyset,j}$ in strict theories.*

Proof. Since CP does not hold for $\mid\!\sim_{\emptyset,a}$ under strict theories, CP can not hold for $\mid\!\sim^{g}_{\emptyset,j}$ or $\mid\!\sim^{p,*}_{\emptyset,j}$.

This completes the characterisation of $\mid\!\sim^{g}_{\emptyset,j}$, $\mid\!\sim^{p,s}_{\emptyset,j}$, $\mid\!\sim^{p,c}_{\emptyset,j}$ and $\mid\!\sim^{p,u}_{\emptyset,j}$. As we argued above, adding classical logic as a base logic will create consequence relations that satisfy the same properties as each of these since they will includes all the same inference rules. In addition, we have the following:

Proposition 16. *CP holds for $\mid\!\sim^{g}_{c,j}$ and $\mid\!\sim^{p,*}_{c,j}$ in strict theories.*

Proof. As above, $\mid\!\sim_{c,a}$ satisfies CP in strict theories. Since the strict part of the theory is always consistent, any conclusions from the argument construction are justified. Therefore, the consequence relation $\mid\!\sim^{g}_{c,j}$ and $\mid\!\sim^{p,*}_{c,j}$ satisfies CP for strict theories.

4.4 Summary

The results for the two forms of consequence and the two base logics are summarized in Table 2. This shows, for example, that Ref is satisfied by $\mid\!\sim_{c,j}$ for strict theories whether the proposition in question is a premise or an axiom; that for defeasible theories, Ref is never satisfied by $\mid\!\sim_{c,j}$ for propositions that are premises, but is always satisfied for propositions that are axioms. Similarly, the table shows that CP does not hold for $\mid\!\sim_{\emptyset,a}$ for either strict or defeasible theories; that CP holds for $\mid\!\sim_{c,a}$ for strict theories, but not for defeasible theories.

Recall from Sect. 3 that a consequence relation which satisfies axioms Ref, LLE, RW, Cut and CM is said to be "cumulative", a cumulative consequence relation that also satisfies M is said to be "cumulative monotonic", and a consequence relation that satisfies CP is monotonic. Given this, it is clear that Table 2 is telling us that $\mid\!\sim_{\emptyset,a}$ is cumulative monotonic for both strict and defeasible theories, while $\mid\!\sim_{c,a}$ is monotonic for strict theories and cumulative monotonic for defeasible theories. Similarly, $\mid\!\sim^{g}_{\emptyset,j}$ is cumulative monotonic for strict theories, and cumulative for the strict portions of defeasible theories (if Ref is applied to axioms and LLE, RW and M are applied to strict rules only), but not even cumulative for the defeasible parts of defeasible theories (if Ref is applied to ordinary premises and LLE, RW or M are applied to defeasible rules, none of them hold). $\mid\!\sim^{p,s}_{\emptyset,j}$ is weaker than $\mid\!\sim^{g}_{\emptyset,j}$, since CM doesn't hold, and $\mid\!\sim^{p,\{c,u\}}_{\emptyset,j}$ is weaker still since Cut doesn't hold. Adding classical logic as a base logic means that CP holds, so $\mid\!\sim^{g}_{c,j}$ is monotonic for strict theories, and behaves exactly like $\mid\!\sim^{g}_{\emptyset,j}$ for defeasible theories. Again $\mid\!\sim^{p,s}_{c,j}$ is weaker than $\mid\!\sim^{g}_{c,j}$, since CM doesn't hold, and $\mid\!\sim^{p,\{c,u\}}_{c,j}$ is weaker still since Cut doesn't hold.

4.5 Discussion

What light do the results in Table 2 shine on ASPIC$^+$ and argumentation-based reasoning in general? We will answer that question by considering each of the consequence relations in turn.

Starting with $\mid\!\sim_{\emptyset,a}$, it is no surprise that the relation is cumulative monotonic and satisfies the axiom M which captures a form of monotonicity. It is clear from the detail of ASPIC$^+$, and indeed any argumentation system, that the number of arguments grows over time, and that once introduced, arguments do not disappear. However, the fact that $\mid\!\sim_{\emptyset,a}$ is not monotonic in the same strict sense as classical logic, and so is strictly weaker, as a result of not satisfying CP, is a bit more interesting. This is, of course, because arguments are not subject to the law of the excluded middle—it is perfectly possible for there to be arguments for α and $\overline{\alpha}$ from the same theory.

Turning to the various versions of consequence built around justified conclusions, they are perhaps more reasonable notions of consequence for ASPIC$^+$ than $\mid\!\sim_{\emptyset,a}$. If β is a justified conclusion of α, then there is an argument for β which holds despite any attacks (in the scenario we have considered, where all attacks may be defeats for some preference ordering—and therefore succeed—there can still be attacks on the argument for β, but the attacking arguments must themselves be defeated). This is quite a restrictive notion of consequence in a representation that allows for conflicting information, and as Table 2 makes clear, even $\mid\!\sim^{g}_{\emptyset,j}$, which is the strongest of the consequence relations based on justified conclusions, is a relatively weak notion of consequence and obeys less of the axioms than the non-monotonic logics analysed in [15], for example. For defeasible theories $\mid\!\sim^{g}_{\emptyset,j}$ is not cumulative, and only satisfies LLE and RW if the rules applied in those axioms are strict. As we pointed out above, at the time that [15] was published, cumulativity was considered the minimum requirement of a useful logic[3]. Whether or not one accepts this, it is clear that ASPIC$^+$ is weak. But is it too weak? To answer this, we should consider reason that $\mid\!\sim^{g}_{\emptyset,j}$ is not cumulative, which as Table 2 shows is due to LLE, RW and Ref.

[3] This position was doubtless a side-effect of the fact that at that time there were no logics that did not obey cumulativity. The subsequent discovery of logics of causality that are not cumulative suggests that this view should be revised.

Table 2. Summary of axioms satisfied by the argumentation-based consequence relations. Each row indicates which axiom is satisfied by a given consequence relation or not, and what conditions, if any, are required. See text for an explanation.

		Strict theories						Defeasible theories							
		$\mid\!\sim_{\emptyset,a}$	$\mid\!\sim_{c,a}$	$\mid\!\sim^{g}_{\emptyset,j}$	$\mid\!\sim^{p,*}_{\emptyset,j}$	$\mid\!\sim^{g}_{c,j}$	$\mid\!\sim^{p,*}_{c,j}$	$\mid\!\sim_{\emptyset,a}$	$\mid\!\sim_{c,a}$	$\mid\!\sim^{g}_{\emptyset,j}$	$\mid\!\sim^{p,s}_{\emptyset,j}$	$\mid\!\sim^{p,\{c,u\}}_{\emptyset,j}$	$\mid\!\sim^{g}_{c,j}$	$\mid\!\sim^{p,s}_{c,j}$	$\mid\!\sim^{p,\{c,u\}}_{c,j}$
Ref	Premise	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N
	Axiom									Y	Y	Y	Y	Y	Y
LLE	Defeasible rule	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N
	Strict rule									Y	Y	Y	Y	Y	Y
RW	Defeasible rule	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N
	Strict rule									Y	Y	Y	Y	Y	Y
Cut		Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	N
CM		Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	Y	N	N
M		Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N
T		Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N
CP		N	Y	N	N	Y	Y	N	N	N	N	N	N	N	N

LLE and RW only hold in the case of strict rules, either because the theory is strict, or because the case in question is of a strict rule in a defeasible theory. For both LLE and RW, the effect of the axiom is to extend an existing argument, either switching one premise for another (LLE), or adding a rule to the conclusion of an argument (RW). While having these axioms hold for defeasible rules would allow $\mid\!\sim^{g}_{\emptyset,j}$ to be cumulative for defeasible theories, this is not reasonable. Using LLE or RW to extend arguments with defeasible rules—by definition—means that the new arguments created by this extension can be defeated. Thus their conclusions may not be justified, and $\mid\!\sim^{g}_{\emptyset,j}$ must not be cumulative for defeasible rules. In other words $\mid\!\sim^{g}_{\emptyset,j}$ is not cumulative for defeasible rules exactly because it makes no sense for a system of defeasible rules to be cumulative.

This weakness raises the question of whether reasoning in ASPIC^{+} can be strengthened. When we add classical logic as a base logic, we get a family of consequence relations that satisfy CP. Thus $\mid\!\sim^{g}_{c,j}$ is monotonic, but only if all elements are strict. For theories with defeasible elements, $\mid\!\sim^{g}_{c,j}$ cannot guarantee that CP will hold for arbitrary α and β, and, as above, LLE and RW will only hold for strict rules. Adding a base logic that is weaker than classical logic does not help in strengthening conclusions. If we add intuitionistic logic, for example, we don't get CP, because intuitionistic logic explicitly rejects this pattern of reasoning. A similar argument applies to Ref. Proposition 14 tells us that Ref holds for $\mid\!\sim^{g}_{\emptyset,j}$ and $\mid\!\sim^{p,*}_{\emptyset,j}$ for strict theories, meaning that α has to be an axiom[4]. If Ref were to hold for defeasible theories, α could be a premise. But premises can be defeated, again by definition, so it is not appropriate to directly conclude that any premise is a justified conclusion (it is necessary to go through the whole process of constructing arguments and establishing extensions to determine this).

From this we conclude that although $\mid\!\sim^{g}_{\emptyset,j}$ and $\mid\!\sim^{g}_{c,j}$ are not cumulative, and hence ASPIC^{+} is, in some sense, weaker than non-monotonic logics like circumscription [18] and default logic [23], it is not clear that it is too weak. That is strengthening $\mid\!\sim^{g}_{\emptyset,j}$ or $\mid\!\sim^{g}_{c,j}$ so that they would be cumulative for defeasible theories would allow for conclusions that make no sense from the point of view of argumentation-based reasoning. Whether there are other ways to strengthen ASPIC^{+} that do make sense is an open question, and one we intend to investigate in the future.

5 The Rationality Postulates

Finally, we consider the three postulates of [4] (which ASPIC^{+} complies with), namely (1) closure under strict rules; and (2) direct and (3) indirect consistency. We ask whether the axioms discussed in this paper are equivalent to any of these postulates. In what follows, we assume that strict rules are consistent.

[4] This is exactly how defeasible logic [2] satisfies Ref.

5.1 Closure Under Strict Rules

Proposition 17. *An argumentation framework meets closure under strict rules if and only if the consequence relation for strict rules complies with right weakening (RW) with regards to justified conclusions.*

Proof. *Given an argumentation framework AF, assume that α is in the justified conclusions. Therefore $\top \mid\sim_j \alpha$, and assume that there is a strict rule $\models \alpha \rightarrow \beta$. Using RW, we obtain $\top \mid\sim_j \beta$. Therefore RW implies closure under strict rules. Furthermore, having $\gamma \mid\sim_j \alpha$, as well as a strict rule $\alpha \rightarrow \beta$ results in $\gamma \mid\sim_j \beta$, i.e., the strict form of RW.*

5.2 Direct Consistency

Direct consistency with regards to $\mid\sim_j$ requires that no extension contains inconsistent arguments (and therefore inconsistent conclusions). This is equivalent to the following axiom, unobtainable from the axioms discussed previously.

$$\frac{\alpha \mid\sim_j \beta}{\alpha \not\mid\sim_j \overline{\beta}}$$

5.3 Indirect Consistency

Proposition 18. *Assume we have direct consistency, and that strict rules are consistent. Any system which satisfies monotonicity under strict rules will satisfy indirect consistency, and vice-versa.*

Proof. *From [4, Proposition 7], direct consistency and closure yield indirect consistency. We assume direct consistency, and monotonicity gives closure.*

In this section we have shown that the rationality postulates described in [4] can be described using axioms from classical logic and non-monotonic reasoning. In future work, we intend to determine whether these axioms can help identify additional rationality postulates. In addition, we will investigate whether these axioms can represent the additional rationality postulates described in [24].

6 Related Work

There are several papers describing work that is similar in some respects to what we report here. Billington [2] describes Defeasible Logic, a logic that, as its name implies, differs from classical logic in that it deals with defeasible reasoning. In addition to introducing the logic, [2] shows that defeasible logic satisfies the axioms of reflexivity, cut and cautious monotonicity suggested in [11], thus satisfying what [11] describes as the basic requirements for a non-monotonic system (such a system is equivalent to a cumulative system in [15]). [13] subsequently established significant links between reasoning in defeasible logic and argumentation-based reasoning. To do this, [13] provides an argumentation system that makes use of defeasible logic as its underlying logic, and shows that the system is compatible with Dung's semantics [8]. Given Defeasible Logic's close relation to Prolog [22], this line of work is closely related to Defeasible Logic Programming (DeLP) [12], a formalism combining results of Logic Programming and Defeasible Argumentation. As a rule-based argumentation system, DeLP also has strict/defeasible rules and a set of facts. DeLP differs from ASPIC$^+$ in the types of attack relation it permits (no undermining) and in the way that it computes conclusions (it does not implement Dung's semantics).

[17] first introduce an argument system, containing two kinds of inference rules, namely, monotonic inference rules and non-monotonic inference rules. They show that most well-known non-monotonic systems, such as default logic, autoepistemic logic, negation as failure and circumscription, can be formulated as instances of their argument system. [3] continues this line of work, presenting an abstract framework for default reasoning which includes Theorist, default logic, logic programming, autoepistemic logic, non-monotonic modal logics, and certain instances of circumscription as special cases. [13] subsequently established significant links between reasoning in defeasible logic and argumentation-based reasoning. To do this, [13] provides an argumentation system that makes use of defeasible logic as its underlying logic, and shows that the system is compatible with Dung's semantics [8]. Similar to the current work, [14] investigates various consequence relations of deductive argumentation and their satisfaction of various properties. However, [14] focuses entirely on argument construction and says nothing about justified conclusions.

Also related are [9, 10], which investigate cumulativity of ASPIC-like structured argumentation frameworks. Finally, [7] analyzes cautious monotonicity and cumulative transitivity with respect to Assumption-Based Argumentation.

7 Conclusions

In this paper we considered which of the axioms of [15] ASPIC$^+$ meets based on two different interpretations of the consequence relation. We demonstrated that, in terms of those axioms, the most natural forms of consequence in ASPIC$^+$ are rather weak. This is the case even when we assume ASPIC$^+$ theories contain all the inference rules of classical logic. However, as we discuss, strengthening the consequence relation (to, for example, be cumulative) neither makes sense in terms of argumentation-based reasoning, nor can easily be achieved by adding additional inference rules to ASPIC$^+$ theories. We also investigated the relationship between the axioms of [15] and the rationality postulates, and suggested an alternative, axiom based formulation of the latter.

As mentioned above, in the future we will investigate whether additional axioms can encode the rationality postulates described in [24]. We will also examine the properties of different interpretations of the logical symbols. For example, we assumed that ≡ encodes the presence of two rules, but says nothing about their preferences or defeaters. Finally, we may consider other interpretations of the consequence relation. This paper therefore opens up several significant avenues of future investigation.

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