Optimal Exploitation of Renewable Resources: Lessons in Sustainability from an Optimal Growth Model of Natural Resource Consumption

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Abstract We study an optimal growth model for a single resource based economy. The resource is governed by the standard model of logistic growth, and is related to the output of the economy through a Cobb-Douglas type production function with exogenously driven knowledge stock. The model is formulated as an infinite-horizon optimal control problem with unbounded set of control constraints and non-concave Hamiltonian. We transform the original problem to an equivalent one with simplified dynamics and prove the existence of an optimal admissible control. Then we characterize the optimal paths for all possible parameter values and initial states by applying the appropriate version of the Pontryagin maximum principle. Our main finding is that only two qualitatively different types of behavior of sustainable optimal paths are possible depending on whether the resource growth rate is higher than the social discount rate or not. An analysis of these behaviors yields general criterions for sustainable and strongly sustainable optimal growth (w.r.t. the corresponding notions of sustainability defined herein).

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1 Introduction

Following the first analysis conducted by Ramsey (1928), the mathematical problem of inter-temporal resource allocation has attracted a significant amount of attention over the past decades, and has driven the evolution of first exogenous, and then endogenous growth theory (see Acemoglu 2009; Barro and Sala-i-Martin 1995). Employed growth models are typically identified by the production of economic output, the dynamics of the inputs of production, and the comparative mechanism of alternate consumption paths. Our framework considers a renewable resource, whose reproduction is logistic in nature, as the only input to production. The relationship of the resource with the output of the economy is explained through a Cobb-Douglas type production function with an exogenously driven knowledge stock. Alternate consumption paths are compared via a discounted utilitarian approach. The question that we concern ourselves with for our chosen framework, is the following: what are the conditions of sustainability for optimal development?

In the context of sustainability, the discounted utilitarian approach may propose undesirable solutions in certain scenarios. For instance, discounted utilitarianism has been reported to force consumption asymptotically to zero even when sustainable paths with non-decreasing consumption are feasible (Asheim and Mitra 2010). The Brundtland Commission defines sustainable development as development that meets the needs of the present, without compromising the ability of future generations to meet their own needs (Brundtland Commission 1987). In this spirit, we employ the notion of sustainable development, as a consumption path ensuring a non-decreasing welfare for all future generations. This notion of sustainability is natural, and has also been used by various authors in their work. For instance, Valente (2005) evaluates this notion of sustainability for an exponentially growing natural resource, and derives a condition necessary for sustainable consumption. We extend this model by allowing the resource to grow at a declining rate (the logistic growth model). We build on the work presented previously in Manzoor et al. (2014) which proves the existence of an optimal path only in the case when the resource growth rate is higher than the social discount rate and admissible controls are uniformly bounded.

Our model is formulated as an infinite-horizon optimal control problem with logarithmic instantaneous utility. The problem involves unbounded controls and the non-concave Hamiltonian. These preclude direct application of the standard existence results and Arrow's sufficient conditions for optimality. We transform the original problem to an equivalent one with simplified dynamics and prove the general existence result. Then we apply a recently developed version of the maximum principle (Aseev and Veliov 2012, 2014, 2015) to our problem and describe the optimal paths for all possible parameter values and initial states in the problem. Our analysis of the Hamiltonian phase space reveals that there are only two qualitatively different types of behavior of the sustainable optimal paths in the model. In the first case the instantaneous utility is a non-decreasing function in the long run along the optimal path (we call such paths *sustainable*).

The second case corresponds to the situation when the optimal path is sustainable and in addition the resource stock is asymptotically nonvanishing (we call such paths *strongly sustainable*). We show that a strongly sustainable equilibrium is attainable only when the resource growth rate is higher than the social discount rate. When this condition is violated, we see that the optimal resource exploitation rate asymptotically follows the Hotelling rule of optimal depletion of an exhaustible resource (Hotelling 1974). In this case optimal consumption is sustainable only if the depletion of the resource is compensated by appropriate growth of the knowledge stock and/or decrease of the output elasticity of the resource.

The paper is organized as follows. Section 2 sets up the problem. Section 3 establishes the equivalence of the problem with a simpler version, and applies the maximum principle after proving the existence of an optimal control. Section 4 presents an analysis of the associated Hamiltonian system and formulates the optimal feedback law. We conclude in Sect. 5 where we develop conditions for sustainability and strong sustainability of the optimal paths in our model.

The paper draws on a companion working paper (Aseev and Manzoor 2016), which contains proofs for several auxiliary results related to our model.

2 Problem Formulation

Consider a society consuming a single renewable resource. The resource, whose quantity is given by $S(t) > 0$ at each instant of time $t \geq 0$, is governed by the standard model of logistic growth. In the absence of consumption, it regenerates at rate $r > 0$ and saturates at carrying capacity $K > 0$. The society consumes the resource by exerting effort (exploitation rate) $u(t) > 0$ resulting in a total consumption velocity of $u(t)S(t) > 0$ at time $t \geq 0$ respectively. The dynamics of the resource stock are then given by the following equation:

$$\dot{S}(t) = r\,S(t)\left(1 - \frac{S(t)}{K}\right) - u(t)S(t), \qquad u(t) \in (0, \infty).$$

The initial stock of the resource is $S(0) = S_0 > 0$.

We assume a single resource economy whose output $Y(t) > 0$ at instant $t \geq 0$ is related to the resource by the Cobb-Douglas type production function

$$Y(t) = A(t)\big(u(t)S(t)\big)^{\alpha}, \qquad \alpha \in (0, 1]. \tag{1}$$

Here $A(t) > 0$ represents an exogenously driven knowledge stock at time $t \geq 0$. We assume $\dot{A}(t) \leq \mu A(t)$, where $\mu \geq 0$ is a constant, and $A(0) = A_0 > 0$.

The whole output $Y(t)$ produced at each instant $t \geq 0$ is consumed and the corresponding instantaneous utility is measured by the logarithmic function $t \mapsto \ln Y(t) = \ln A(t) + \alpha\,[\ln S(t) + \ln u(t)]$, $t \geq 0$.

This leads to the following optimal control problem $(P1)$:

$$J(S(\cdot), u(\cdot)) = \int_0^{\infty} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt \to \max, \tag{2}$$

$$\dot{S}(t) = rS(t)\left(1 - \frac{S(t)}{K}\right) - u(t)S(t), \qquad S(0) = S_0, \tag{3}$$

$$u(t) \in (0, \infty), \tag{4}$$

where $\rho > 0$ is the subjective discount rate.

By an *admissible control* in problem $(P1)$ we mean a Lebesgue measurable locally bounded function $u\colon [0, \infty) \mapsto \mathbb{R}^1$ which satisfies the control constraint (4) for all $t \geq 0$. By definition, the corresponding to $u(\cdot)$ *admissible trajectory* is a (locally) absolutely continuous function $S(\cdot) : [0, \infty) \mapsto \mathbb{R}^1$ which is a Caratheodory solution (see Filippov 1988) to the Cauchy problem (3) on the whole infinite time interval $[0, \infty)$. Due to the local boundedness of the admissible control $u(\cdot)$ such admissible trajectory $S(\cdot)$ always exists and is unique (see Filippov 1988, Section 7). A pair $(S(\cdot), u(\cdot))$ where $S(\cdot)$ is an admissible control and $S(\cdot)$ is the corresponding admissible trajectory is called an *admissible pair* in problem $(P1)$.

Due to (3) for any admissible trajectory $S(\cdot)$ the following estimate holds:

$$S(t) \leq S_{\max} = \max\{S_0, K\}, \qquad t \geq 0. \tag{5}$$

The integral in (2) is understood in improper sense, i.e.

$$J(S(\cdot), u(\cdot)) = \lim_{T\to\infty} \int_0^{T} \mathrm{e}^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt \tag{6}$$

if the limit exists.

Using estimate (5) and control system (3) it can be easily shown that there is a decreasing function $\omega : [0, \infty) \mapsto (0, \infty)$ such that $\omega(t) \to +0$ as $t \to \infty$ and for any admissible pair $(S(\cdot), u(\cdot))$ the following inequality holds:

$$\int_T^{T'} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] \mathrm{d}t < \omega(T), \qquad 0 \leq T < T'. \tag{7}$$

This fact immediately implies that for any admissible pair $(S(\cdot), u(\cdot))$ the limit in (6) always exists and is either finite or equals $-\infty$ (see Aseev and Manzoor 2016 for details).

Due to (7) for any admissible pair $(S(\cdot), u(\cdot))$ the value $\sup_{(S(\cdot),u(\cdot))} J(S(\cdot), u(\cdot))$ is finite. This allows us to understand the optimality of an admissible pair $(S_*(\cdot), u_*(\cdot))$ in the strong sense (Carlson et al. 1991). By definition, an admissible pair $(S_*(\cdot), u_*(\cdot))$ is *strongly optimal* (or, for brevity, simply *optimal*) in the problem $(P1)$ if the functional (2) takes the maximal possible value on this pair, i.e.

$$J(S_*(\cdot), u_*(\cdot)) = \sup_{(S(\cdot),u(\cdot))} J(S(\cdot), u(\cdot)) < \infty.$$

Notice, that the set of control constraints in problem $(P1)$ (see (4)) is nonclosed and unbounded. Due to this circumstance the standard existence theorems (see e.g. Balder 1983; Cesari 1983) are not applicable to problem $(P1)$ directly. Moreover, the situation is complicated here by the fact that the Hamiltonian of problem $(P1)$ is non-concave in the state variable S. These preclude the usage of Arrow's sufficient conditions for optimality (see Carlson et al. 1991).

Our analysis below is based on application of the recently developed normal form version of the Pontryagin maximum principle (Pontryagin et al. 1964) for infinite-horizon optimal control problems with adjoint variable specified explicitly via the Cauchy type formula (see Aseev and Veliov 2012, 2014, 2015). However, such approach assumes that the optimal control exists. So, the proof of the existence of an optimal admissible pair $(S_*(\cdot), u_*(\cdot))$ in problem $(P1)$ and establishing of the corresponding version of the maximum principle will be our primary goal in the next section.

3 Existence of an Optimal Control and the Maximum Principle

Let us transform problem $(P1)$ into a more appropriate equivalent form.

Due to (3) along any admissible pair $(S(\cdot), u(\cdot))$ we have

$$\frac{d}{dt}\left[e^{-\rho t}\ln S(t)\right] \overset{\text{a.e.}}{=} -\rho e^{-\rho t}\ln S(t) + re^{-\rho t} - e^{-\rho t}\left(\frac{r}{K}S(t) + u(t)\right), \quad t > 0.$$

Integrating this equality on arbitrary time interval $[0, T]$, $T > 0$, we obtain

$$\int_0^T e^{-\rho t}\ln S(t)\,dt = \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \frac{r}{\rho^2}\left(1 - e^{-\rho T}\right) - \int_0^T e^{-\rho t}\left(\frac{r}{\rho K}S(t) + \frac{u(t)}{\rho}\right)dt.$$

Hence, for any admissible pair $(S(\cdot), u(\cdot))$ and arbitrary $T > 0$ we have

$$\int_0^T e^{-\rho t}\left[\ln S(t) + \ln u(t)\right]dt = \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \frac{r}{\rho^2}\left(1 - e^{-\rho T}\right) - \frac{r}{\rho K}\int_0^T e^{-\rho t}S(t)\,dt + \int_0^T e^{-\rho t}\left(\ln u(t) - \frac{u(t)}{\rho}\right)dt. \qquad (8)$$

Here due to estimate (7) limits of the both sides in (8) as $T \to \infty$ exist and equal either a finite number or $-\infty$ simultaneously, and due to (5) either (i) $\lim_{T\to\infty} e^{-\rho T} \ln S(T) = 0$ or (ii) $\liminf_{T\to\infty} e^{-\rho T} \ln S(T) < 0$.

In the case (i) passing to the limit in (8) as $T \to \infty$ we get

$$\int_0^{\infty} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt = \frac{\ln S_0}{\rho} + \frac{r}{\rho^2} - \frac{r}{\rho K} \int_0^{\infty} e^{-\rho t} S(t)\, dt + \int_0^{\infty} e^{-\rho t} \left(\ln u(t) - \frac{u(t)}{\rho}\right) dt, \qquad (9)$$

where both sides in (9) are equal to a finite number or $-\infty$ simultaneously.

In the case (ii) condition $\liminf_{T\to\infty} e^{-\rho T} \ln S(T) < 0$ implies

$$\int_0^{\infty} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt = \lim_{T\to\infty} \int_0^{T} e^{-\rho t} \left[\ln S(t) + \ln u(t)\right] dt = -\infty$$

(see Aseev and Manzoor 2016 for details). Hence, in the case (ii) (9) also holds as $-\infty = -\infty$.

Neglecting now the constant terms in the right-hand side of (9) we obtain the following optimal control problem $(\tilde{P}1)$ which is equivalent to $(P1)$:

$$\tilde{J}(S(\cdot), u(\cdot)) = \int_0^{\infty} e^{-\rho t} \left[\ln u(t) - \frac{u(t)}{\rho} - \frac{r}{\rho K} S(t)\right] dt \to \max,$$

$$\dot{S}(t) = r S(t) \left(1 - \frac{S(t)}{K}\right) - u(t) S(t), \qquad S(0) = S_0, \qquad (10)$$

$$u(t) \in (0, \infty). \qquad (11)$$

Further, the function $u \mapsto ln\, u - u/\rho$ is increasing on $(0, \rho]$ and it reaches the global maximum at point $u_* = \rho$. Hence, any optimal control $u_*(\cdot)$ in $(\tilde{P}1)$ (if such exists) must satisfy to inequality $u_*(t) \geq \rho$ for almost all $t \geq 0$. Hence, without loss of generality the control constraint (11) in $(\tilde{P}1)$ (and hence the control constraint (4) in $(P1)$) can be replaced by the control constraint $u(t) \in [\rho, \infty)$. Thus we arrive to the following (equivalent) problem $(P2)$:

$$J(S(\cdot), u(\cdot)) = \int_0^{\infty} e^{-\rho t} \left[\ln u(t) + \ln S(t)\right] dt \to \max,$$

$$\dot{S}(t) = r S(t) \left(1 - \frac{S(t)}{K}\right) - u(t) S(t), \qquad S(0) = S_0,$$

$$u(t) \in [\rho, \infty). \qquad (12)$$

Here the class of admissible controls in problem $(P2)$ consists of all locally bounded functions $u(\cdot)$ satisfying the control constraint (12) for all $t \geq 0$.

To simplify dynamics in $(P2)$ let us introduce the new state variable $x(\cdot)$: $x(t) = 1/S(t)$, $t \geq 0$. As it can be verified directly, in terms of the state variable $x(\cdot)$ problem $(P2)$ can be rewritten as the following (equivalent) problem $(P3)$:

$$J(x(\cdot), u(\cdot)) = \int_0^{\infty} e^{-\rho t} \left[\ln u(t) - \ln x(t)\right] dt \to \max, \tag{13}$$

$$\dot{x}(t) = [u(t) - r]\, x(t) + a, \qquad x(0) = x_0 = \frac{1}{S_0}, \tag{14}$$

$$u(t) \in [\rho, \infty). \tag{15}$$

Here $a = r/K$. The class of admissible controls $u(\cdot)$ in $(P3)$ consists of all measurable locally bounded functions $u\colon [0, \infty) \mapsto [\rho, \infty)$.

Notice, that due to linearity of (14) for arbitrary admissible control $u(\cdot)$ the corresponding trajectory $x(\cdot)$ can be expressed via the Cauchy formula (see Hartman 1964):

$$x(t) = x_0 e^{\int_0^t u(\xi)\, d\xi - rt} + a e^{\int_0^t u(\xi)\, d\xi - rt} \int_0^t e^{-\int_0^s u(\xi)\, d\xi + rs}\, ds, \qquad t \geq 0. \tag{16}$$

Since the problems $(P1)$, $(P2)$ and $(P3)$ are equivalent we will focus our analysis below on problem $(P3)$ with simplified dynamics (see (14)) and the closed set of control constraints (see (15)).

The constructed problem $(P3)$ is a particular case of the following autonomous infinite-horizon optimal control problem $(P4)$ with exponential discounting:

$$\begin{gathered} J(x(\cdot), u(\cdot)) = \int_0^{\infty} e^{-\rho t} g(x(t), u(t))\, dt \to \max, \\ \dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0, \\ u(t) \in U. \end{gathered} \tag{17}$$

Here U is a nonempty closed subset of $\mathbb{R}^m$, $x_0 \in G$ is an initial state, G is an open convex subset of $\mathbb{R}^n$, $\rho > 0$ is the discount rate, and $f : G \times U \mapsto \mathbb{R}^n$ and $g : G \times U \mapsto \mathbb{R}^m$ are given functions. The class of admissible controls in $(P4)$ consists of all measurable locally bounded functions $u\colon [0, \infty) \mapsto U$. The optimality of admissible pair $(x_*(\cdot), u_*(\cdot))$ is understood in the strong sense (Carlson et al. 1991).

Problems of type $(P4)$ were intensively studied in last decades (see Aseev 2015a,b, 2016; Aseev et al. 2012; Aseev and Kryazhimskiy 2004; Aseev and Kryazhimskii 2007; Aseev and Veliov 2012, 2014, 2015). Here we will employ the existence result and the variant of the Pontryagin maximum principle for problem

($P4$) developed in Aseev (2015b, 2016) and Aseev and Veliov (2012, 2014, 2015) respectively.

We will need to verify validity of the following conditions (see Aseev 2015b, 2016; Aseev et al. 2012; Aseev and Veliov 2012, 2014, 2015).

(A1) *The functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ together with their partial derivatives $f_x(\cdot,\cdot)$ and $g_x(\cdot,\cdot)$ are continuous and locally bounded on $G \times U$.*

(A2) *There exists a number $\beta > 0$ and a nonnegative integrable function $\lambda : [0,\infty) \mapsto \mathbb{R}^1$ such that for every $\zeta \in G$ with $\|\zeta - x_0\| < \beta$ Eq. (17) with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) has a solution $x(\zeta;\cdot)$ on $[0,\infty)$ in G, and*

$$\max_{\theta\in[x(\zeta;t),x_*(t)]} \left| e^{-\rho t} \langle g_x(\theta, u_*(t)), x(\zeta;t) - x_*(t)\rangle \right| \overset{a.e.}{\leq} \|\zeta - x_0\|\lambda(t).$$

Here $[x(\zeta;t), x_*(t)]$ denotes the line segment with vertices $x(\zeta;t)$ and *$x_*(t)$.*

(A3) *There is a positive function $\omega(\cdot)$ decreasing on $[0,\infty)$ such that $\omega(t) \to +0$ as $t \to \infty$ and for any admissible pair $(x(\cdot), u(\cdot))$ the following estimate holds:*

$$\int_T^{T'} e^{-\rho t} g(x(t), u(t))\, dt \leq \omega(T), \qquad 0 \leq T \leq T'.$$

Obviously, condition ($A1$) is satisfied because $f(x,u) = [u - r]x + a$, $g(x,u) = \ln u - \ln x$, $f_x(x,u) = u - r$ and $g_x(x,u) = -1/x$, $x > 0$, $u \in [\rho,\infty)$, in ($P3$).

Let us show that ($A2$) also holds for any admissible pair $(x_*(\cdot), u_*(\cdot))$ in ($P3$). Set $\beta = x_0/2$ and define the nonnegative integrable function $\lambda : [0,\infty) \mapsto \mathbb{R}^1$ as follows: $\lambda(t) = 2e^{-\rho t}/x_0$, $t \geq 0$. Then, as it can be seen directly, for any real ζ: $|\zeta - x_0| < \beta$, the Cauchy problem (14) with $u(\cdot) = u_*(\cdot)$ and the initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) has a solution $x(\zeta;\cdot)$ on $[0,\infty)$ and

$$\max_{\theta\in[x(\zeta;t),x_*(t)]} \left| e^{-\rho t} g_x(\theta, u_*(t))\,(x(\zeta;t) - x_*(t)) \right| \overset{\text{a.e.}}{\leq} |\zeta - x_0|\lambda(t).$$

Hence, for any admissible pair $(x_*(\cdot), x_*(\cdot))$ condition ($A2$) is also satisfied.

Validity of ($A3$) for any admissible pair $(x_*(\cdot), u_*(\cdot))$ follows from (7) directly.

For an admissible pair $(x(\cdot), u(\cdot))$ consider the following linear system:

$$\dot{z}(t) = -[f_x(x(t), u(t))]^* z(t) = [-u(t) + r]z(t). \tag{18}$$

The normalized fundamental solution $Z(\cdot)$ to Eq. (18) is defined as follows:

$$Z(t) = e^{-\int_0^t u(\xi)\, d\xi + rt}, \qquad t \geq 0. \tag{19}$$

Due to (16) and (19) for any admissible pair $(x(\cdot), u(\cdot))$ we have

$$\left|e^{-\rho t} Z^{-1}(t) g_x(x(t), u(t))\right|$$
$$= \left|\frac{e^{-\rho t} e^{\int_0^t u(\xi)\,d\xi - rt}}{x_0 e^{\int_0^t u(\xi)\,d\xi - rt} + a e^{\int_0^t u(\xi)\,d\xi - rt} \int_0^t e^{-\int_0^s u(\xi)\,d\xi + rs}\,ds}\right| \le \frac{e^{-\rho t}}{x_0}, \quad t \ge 0.$$

Hence, for any $T > 0$ the function $\psi_T : [0, T] \mapsto \mathbb{R}^1$ defined as

$$\psi_T(t) = Z(t) \int_t^T e^{-\rho s} Z^{-1}(s) g_x(x(s), u(s))\,ds$$
$$= -e^{-\int_0^t u(\xi)\,d\xi + rt} \int_t^T \frac{e^{\int_0^s u(\xi)\,d\xi - rs} e^{-\rho s}}{x(s)}\,ds, \qquad t \in [0, T], \tag{20}$$

is absolutely continuous, and the function $\psi : [0, \infty) \mapsto \mathbb{R}^1$ defined as

$$\psi(t) = Z(t) \int_t^\infty e^{-\rho s} Z^{-1}(s) g_x(x(s), u(s))\,ds$$
$$= -e^{-\int_0^t u(\xi)\,d\xi + rt} \int_t^\infty \frac{e^{\int_0^s u(\xi)\,d\xi - rs} e^{-\rho s}}{x(s)}\,ds, \qquad t \ge 0, \tag{21}$$

is locally absolutely continuous.

Define the normal form Hamilton-Pontryagin function $\mathcal{H} : [0, \infty) \times (0, \infty) \times [\rho, \infty) \times \mathbb{R}^1 \mapsto \mathbb{R}^1$ and the normal-form Hamiltonian $H : [0, \infty) \times (0, \infty) \times \mathbb{R}^1 \mapsto \mathbb{R}^1$ for problem $(P3)$ in the standard way:

$$\mathcal{H}(t, x, u, \psi) = \psi f(x, u) + e^{-\rho t} g(x, u) = \psi[(u - r)x + a] + e^{-\rho t}[\ln u - \ln x],$$

$$H(t, x, \psi) = \sup_{u \ge \rho} \mathcal{H}(t, x, u, \psi),$$

$$t \in [0, \infty), \quad x \in (0, \infty), \quad u \in [\rho, \infty), \quad \psi \in \mathbb{R}^1.$$

Theorem 1 *There is an optimal admissible control $u_*(\cdot)$ in problem $(P3)$. Moreover, for any optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ we have*

$$u_*(t) \overset{a.e.}{\le} \left(1 + \frac{1}{K x_*(t)}\right)(r + \rho), \qquad t \ge 0. \tag{22}$$

Proof Let us show that there is a continuous function $M : [0, \infty) \mapsto \mathbb{R}^1$, $M(t) \ge 0$, $t \ge 0$, and a function $\delta : [0, \infty) \mapsto \mathbb{R}^1$, $\delta(t) > 0$, $t \ge 0$, $\lim_{t\to\infty} (\delta(t)/t) = 0$, such that for any admissible pair $(x(\cdot), u(\cdot))$, satisfying on a set $\mathfrak{M} \subset [0, \infty)$, meas $\mathfrak{M} > 0$, to inequality $u(t) > M(t)$, for all $t \in \mathfrak{M}$ we have

$$\inf_{T:\, T-\delta(T)\geq t} \left\{ \sup_{u\in[\rho, M(t)]} \mathcal{H}(t, x(t), u, \psi_T(t)) - \mathcal{H}(t, x(t), u(t), \psi_T(t)) \right\} > 0, \tag{23}$$

where the function $\psi_T(\cdot)$ is defined on $[0, T]$, $T > 0$, by equality (20).

Let $(x(\cdot), u(\cdot))$ be an arbitrary admissible pair in $(P3)$. Then due to (16) and (19), for any $T > 0$ and arbitrary $t \in [0, T]$ we get (see (20))

$$-x(t)\psi_T(t) = \left[x_0 + a\int_0^t e^{-\int_0^s u(\xi)\,d\xi + rs}\,ds \right] \int_t^T \frac{e^{-\rho s}}{x_0 + a\int_0^s e^{-\int_0^\tau u(\xi)\,d\xi + r\tau}\,d\tau}\,ds$$

$$\geq x_0 \int_t^T \frac{e^{-\rho s}}{x_0 + a\int_0^s e^{r\tau}\,d\tau}\,ds \geq \frac{r x_0 e^{-(r+\rho)t}}{(r x_0 + a)(r+\rho)} \left[1 - e^{-(r+\rho)(T-t)} \right]. \tag{24}$$

For a $\delta > 0$ define the function $M_\delta\colon [0, \infty) \mapsto \mathbb{R}^1$ by equality

$$M_\delta(t) = \frac{(r x_0 + a)(r+\rho)}{r x_0 \left[1 - e^{-(r+\rho)\delta}\right]} e^{rt} + \frac{1}{\delta}, \qquad t \geq 0. \tag{25}$$

Then for any $T\colon T - \delta > t$ and arbitrary $(x(\cdot), u(\cdot))$ the function $u \mapsto \mathcal{H}(t, x(t), u, \psi_T(t))$ reaches its maximal value on $[\rho, \infty)$ at the point (see (24))

$$u_T(t) = -\frac{e^{-\rho t}}{x(t)\psi_T(t)} \leq \frac{(r x_0 + a)(r+\rho)}{r x_0 \left[1 - e^{-(r+\rho)(T-t)}\right]} e^{rt} \leq M_\delta(t) - \frac{1}{\delta}. \tag{26}$$

Now, set $\delta(t) \equiv \delta$ and $M(t) \equiv M_\delta(t)$, $t \geq 0$. Let $(x(\cdot), u(\cdot))$ be an admissible pair such that inequality $u(t) > M_\delta(t)$ holds on a set $\mathfrak{M} \subset [0, \infty)$, meas $\mathfrak{M} > 0$. For arbitrary $t \in \mathfrak{M}$ define the function $\Phi\colon [t + \delta, \infty) \mapsto \mathbb{R}^1$ as follows

$$\Phi(T) = \sup_{u\in[\rho, M(t)]} \mathcal{H}(t, x(t), u, \psi_T(t)) - \mathcal{H}(t, x(t), u(t), \psi_T(t))$$

$$= \psi_T(t) u_T(t) x(t) + e^{-\rho t} \ln u_T(t) - \left[\psi_T(t) u(t) x(t) + e^{-\rho t} \ln u(t)\right], \qquad T \geq t + \delta.$$

Due to (26) we have

$$\Phi(T) = -e^{-\rho t} + e^{-\rho t}\left[-\rho t - \ln(-\psi_T(t)) - \ln x(t)\right]$$
$$- \left[\psi_T(t) u(t) x(t) + e^{-\rho t} \ln u(t)\right], \qquad T \geq t + \delta.$$

Hence, due to (20) and (26) for a.e. $T \geq t + \delta$ we get

$$\frac{d}{dT}\Phi(T) = -\frac{e^{-\rho t}}{\psi_T(t)}\frac{d}{dT}\left[\psi_T(t)\right] - u(t)x(t)\frac{d}{dT}\left[\psi_T(t)\right]$$
$$= x(t)\frac{d}{dT}\left[\psi_T(t)\right]\left[\frac{e^{-\rho t}}{-\psi_T(t)x(t)} - u(t)\right] = x(t)\frac{d}{dT}\left[\psi_T(t)\right](u_T(t) - u(t)) > 0.$$

Hence,

$$\inf_{T>0:\, t\leq T-\delta}\left\{\sup_{u\in[\rho,M(t)]} \mathcal{H}(t, x(t), u, \psi_T(t)) - \mathcal{H}(t, x(t), u(t), \psi_T(t))\right\}$$
$$= \inf_{T>0:\, t\leq T-\delta} \Phi(T) = \Phi(t+\delta) > 0.$$

Thus, for any $t \in \mathfrak{M}$ inequality (23) is proved.

Since the instantaneous utility in (13) is concave in u, the system (14) is affine in u, the set U is closed (see (15)), conditions $(A1)$ and $(A3)$ are satisfied, and since $(A2)$ also holds for any admissible pair $(x_*(\cdot), u_*(\cdot))$ in $(P3)$, all conditions of the existence result in Aseev (2016) are fulfilled (see also Aseev 2015b, Theorem 1). Hence, there is an optimal admissible control $u_*(\cdot)$ in $(P3)$ and, moreover, $u_*(t) \overset{a.e.}{\leq} M_\delta(t)$, $t \geq 0$. Passing to a limit in this inequality as $\delta \to \infty$ we get (see (25))

$$u_*(t) \overset{\text{a.e.}}{\leq} \left(1 + \frac{1}{Kx_0}\right)(r+\rho)e^{rt}, \qquad t \geq 0. \tag{27}$$

Further, it is easy to see that for any $\tau > 0$ the pair $(\tilde{x}_*(\cdot), \tilde{u}_*(\cdot))$ defined as $\tilde{x}_*(t) = x_*(t+\tau)$, $\tilde{u}_*(t) = u_*(t+\tau)$, $t \geq 0$, is an optimal admissible pair in the problem $(P3)$ taken with initial condition $x(0) = x_*(\tau)$. Hence, using the same arguments as above we get the following inequality for $(\tilde{x}_*(\cdot), \tilde{u}_*(\cdot))$ (see (27)):

$$\tilde{u}_*(t) \overset{\text{a.e.}}{\leq} \left(1 + \frac{1}{K\tilde{x}_*(0)}\right)(r+\rho)e^{rt}, \qquad t \geq 0.$$

Hence, for arbitrary fixed $\tau > 0$ we have

$$u_*(t) = \tilde{u}_*(t-\tau) \overset{\text{a.e.}}{\leq} \left(1 + \frac{1}{Kx_*(\tau)}\right)(r+\rho)e^{r(t-\tau)}, \qquad t \geq \tau.$$

Due to arbitrariness of $\tau > 0$ this implies (22). □

Theorem 2 *Let $(x_*(\cdot), u_*(\cdot))$ be an optimal admissible pair in problem $(P3)$. Then the function $\psi : [0, \infty) \mapsto \mathbb{R}^1$ defined for pair $(x_*(\cdot), u_*(\cdot))$ by formula (21)* *is (locally) absolutely continuous and satisfies the conditions of the normal form maximum principle, i.e.* $\psi(\cdot)$ *is a solution of the adjoint system*

$$\dot{\psi}(t) = -\mathcal{H}_x\left(x_*(t), u_*(t), \psi(t)\right), \tag{28}$$

and the maximum condition holds:

$$\mathcal{H}(x_*(t), u_*(t), \psi(t)) \overset{a.e.}{=} H(x_*(t), \psi(t)). \tag{29}$$

Proof As it already have been shown above condition $(A1)$ is satisfied and $(A2)$ holds for any admissible pair $(x_*(\cdot), u_*(\cdot))$ in $(P3)$. Hence, due to the variant of the maximum principle developed in Aseev and Veliov (2012, 2014, 2015) the function $\psi : [0, \infty) \mapsto \mathbb{R}^1$ defined for pair $(x_*(\cdot), u_*(\cdot))$ by formula (21) satisfies the conditions (28) and (29). □

Notice, that the Cauchy type formula (21) implies (see (16) and (19))

$$\psi(t) = -e^{-\int_0^t u_*(\xi)\,d\xi + rt} \int_t^\infty \frac{e^{-\rho\tau} e^{\int_0^\tau u_*(\xi)\,d\xi - r\tau}}{e^{\int_0^\tau u_*(\xi)\,d\xi - r\tau}\left[x_0 + a\int_0^\tau e^{-\int_0^\theta u_*(\xi)\,d\xi + r\theta}\,d\theta\right]}\,d\tau$$

$$> -\frac{e^{-\int_0^t u_*(\xi)\,d\xi + rt}}{x_0 + a\int_0^t e^{-\int_0^\theta u_*(\xi)\,d\xi + r\theta}\,d\theta} \int_t^\infty e^{-\rho\tau}\,d\tau = -\frac{e^{-\rho t}}{\rho x_*(t)}, \qquad t \geq 0. \tag{30}$$

Thus, due to (21) the following condition holds:

$$0 < -\psi(t)x_*(t) < \frac{e^{-\rho t}}{\rho}, \qquad t \geq 0. \tag{31}$$

Note also, that due to (Aseev 2015a, Corollary to Theorem 3) formula (21) implies the following stationarity condition for the Hamiltonian (see Aseev and Kryazhimskii 2007; Michel 1982):

$$H(t, x_*(t), \psi(t)) = \rho \int_t^\infty e^{-\rho s} g(x_*(s), u_*(s))\,ds, \qquad t \geq 0. \tag{32}$$

It can be shown directly that if an admissible pair (not necessary optimal) $(x(\cdot), u(\cdot))$ together with an adjoint variable $\psi(\cdot)$ satisfies the core conditions (28) and (29) of the maximum principle and $\lim_{t\to\infty} H(t, x(t), \psi(t)) = 0$ then condition (32) holds for the triple $(x(\cdot), u(\cdot), \psi(\cdot))$ as well (see Aseev and Kryazhimskii 2007, Section 3).

Further, due to the maximum condition (29) for a.e. $t \geq 0$ we have

$$u_*(t) = \arg\max_{u\in[\rho,\infty)} \left[\psi(t)x_*(t)u + e^{-\rho t}\ln u\right].$$

This implies (see (31))

$$u_*(t) \overset{a.e.}{=} -\frac{e^{-\rho t}}{\psi(t)x_*(t)} > \rho, \qquad t \in [0, \infty). \tag{33}$$

Substituting this formula for $u_*(\cdot)$ in (14) and in (28) due to Theorem 2 we get that any optimal trajectory $x_*(\cdot)$ together with the corresponding adjoint variable $\psi(\cdot)$ must satisfy to the Hamiltonian system of the maximum principle:

$$\begin{aligned} \dot{x}(t) &= -rx(t) - \frac{e^{-\rho t}}{\psi(t)} + a, \\ \dot{\psi}(t) &= r\psi(t) + \frac{2e^{-\rho t}}{x(t)}. \end{aligned} \tag{34}$$

Moreover, estimate (31) and condition (32) must hold as well.

In the terms of the current value adjoint variable $\lambda(\cdot)$, $\lambda(t) = e^{\rho t}\psi(t)$, $t \geq 0$, one can rewrite system (34) as follows:

$$\begin{aligned} \dot{x}(t) &= -rx(t) - \frac{1}{\lambda(t)} + a, \\ \dot{\lambda}(t) &= (\rho + r)\lambda(t) + \frac{2}{x(t)}. \end{aligned} \tag{35}$$

In terms of variable $\lambda(\cdot)$ estimate (31) takes the following form:

$$0 < -\lambda(t)x_*(t) < \frac{1}{\rho}, \qquad t \geq 0. \tag{36}$$

Accordingly, the optimal control $u_*(\cdot)$ can be expressed as follows (see (33)):

$$u_*(t) \overset{a.e.}{=} -\frac{1}{\lambda(t)x_*(t)}, \qquad t \geq 0. \tag{37}$$

Define the normal form current value Hamiltonian $M : (0, \infty) \times \mathbb{R}^1 \mapsto \mathbb{R}^1$ for problem $(P3)$ in the standard way (see Aseev and Kryazhimskii 2007, Section 3):

$$M(x, \lambda) = e^{\rho t} H(t, x, \psi), \quad x \in (0, \infty), \quad \lambda \in \mathbb{R}^1. \tag{38}$$

Then in the current value terms the stationarity condition (32) takes the form

$$M(x_*(t), \lambda(t)) = \rho e^{\rho t} \int_t^\infty e^{-\rho s} g(x_*(s), u_*(s))\, ds, \qquad t \geq 0. \tag{39}$$

In the next section we will analyze the system (35) coupled with the estimate (36) and the stationarity condition (39). We will show that there are only two qualitatively different types of behavior of the optimal paths that are possible. If $r > \rho$ then the optimal path asymptotically approaches an optimal nonvanishing steady state while the corresponding optimal control tends to $(r + \rho)/2$ as $t \to \infty$. If $r \leq \rho$ then the optimal path $x_*(\cdot)$ goes to infinity, while the corresponding optimal control $u_*(\cdot)$ tends to ρ as $t \to \infty$, i.e. asymptotically it follows the Hotelling rule of optimal depletion of an exhaustible resource (Hotelling 1974).

4 Analysis of the Hamiltonian System

Due to Theorem 2 it is sufficient to analyze the behavior of system (35) only in the open set $\Gamma = \{ (x, \lambda) \colon x > 0, \lambda < 0\}$ in the phase plane $\mathbb{R}^2$.

Let us introduce functions $y_1 \colon (1/K, \infty) \mapsto (-\infty, 0)$ and $y_2 \colon (0, \infty) \mapsto (-\infty, 0)$ as follows (recall that $a = r/K$):

$$y_1(x) = \frac{1}{a - rx}, \quad x \in \left(\frac{1}{K}, \infty\right), \qquad y_2(x) = -\frac{2}{(\rho + r)x}, \quad x \in (0, \infty).$$

Due to (35) the curves $\gamma_1 = \{(x, \lambda) \colon \lambda = y_1(x), x \in (1/K, \infty)\}$ and $\gamma_2 = \{(x, \lambda) \colon \lambda = y_2(x), x \in (0, \infty)\}$ are the nullclines at which the derivatives of variables $x(\cdot)$ and $\lambda(\cdot)$ vanish respectively.

Two qualitatively different cases are possible: (i) $r > \rho$ and (ii) $r \leq \rho$.

Consider case (i). In this case the nullclines γ_1 and γ_2 have a unique intersection point $(\hat{x}, \hat{\lambda})$ which is a unique equilibrium of system (35) in Γ:

$$\hat{x} = \frac{2r}{(r - \rho)K}, \qquad \hat{\lambda} = \frac{(\rho - r)K}{(\rho + r)r}. \tag{40}$$

The corresponding equilibrium control $\hat{u}(\cdot)$ is

$$\hat{u}(t) \equiv \hat{u} = \frac{\rho + r}{2}, \qquad t \geq 0. \tag{41}$$

The eigenvalues of the system linearized around the equilibrium are given by

$$\sigma_{1,2} = \frac{\rho}{2} \pm \frac{1}{2}\sqrt{2r^2 - \rho^2},$$

which are real and distinct with opposite signs when $r > \rho$. Hence, by the Grobman-Hartman theorem in a neighborhood Ω of the equilibrium state $(\hat{x}, \hat{\lambda})$ the system (35) is of saddle type (see Hartman 1964, Chapter 9).

The nullclines γ_1 and γ_2 divide the set Γ in four open regions:

$$\Gamma_{-,-} = \left\{(x, \lambda) \in \Gamma:\ \lambda < y_1(x),\ \frac{1}{K} < x \le \hat{x}\right\} \bigcup \left\{(x, \lambda) \in \Gamma:\ \lambda < y_2(x), \hat{x} < x < \infty\right\},$$

$$\Gamma_{+,-} = \left\{(x, \lambda) \in \Gamma:\ \lambda < y_2(x), 0 < x \le \frac{1}{K}\right\} \bigcup \left\{(x, \lambda) \in \Gamma:\ y_1(x) < \lambda < y_2(x),\ \frac{1}{K} < x < \hat{x}\right\},$$

$$\Gamma_{+,+} = \left\{(x, \lambda) \in \Gamma:\ y_2(x) < \lambda < 0, 0 < x \le \hat{x}\right\} \bigcup \left\{(x, \lambda) \in \Gamma:\ y_1(x) < \lambda < 0, \hat{x} < x < \infty\right\},$$

$$\Gamma_{-,+} = \left\{(x, \lambda) \in \Gamma:\ y_2(x) < \lambda < y_1(x), x > \hat{x}\right\}.$$

Any solution $(x(\cdot), \lambda(\cdot))$ of (35) in Γ has definite signs of derivatives of its (x, λ)-coordinates in the sets $\Gamma_{-.-}$, $\Gamma_{-.+}$, $\Gamma_{+,+}$, and $\Gamma_{-,+}$. These signs are indicated by the corresponding subscripts.

The behavior of the flows is shown in Fig. 1 through the phase portrait.

For any initial state $(\xi, \beta) \in \Gamma$ there is a unique solution $(x_{\xi,\beta}(\cdot), \lambda_{\xi,\beta}(\cdot))$ of the system (35) satisfying initial conditions $x(0) = \xi$, $\lambda(0) = \beta$, and due to the standard extension result this solution is defined on some maximal time interval $[0, T_{\xi,\beta})$ in Γ where $0 < T_{\xi,\beta} \le \infty$ (see Hartman 1964, Chapter 2).

Let us consider behaviors of solutions $(x_{\xi,\beta}(\cdot), \lambda_{\xi,\beta}(\cdot))$ of system (35) in Γ for all possible initial states $(\xi, \beta) \in \Gamma$ as $t \to T_{\xi,\beta}$.

The standard analysis of system (35) shows that only three types of behavior of solutions $(x_{\xi,\beta}(\cdot), \lambda_{\xi,\beta}(\cdot))$ of (35) in Γ as $t \to T_{\xi,\beta}$ are possible:

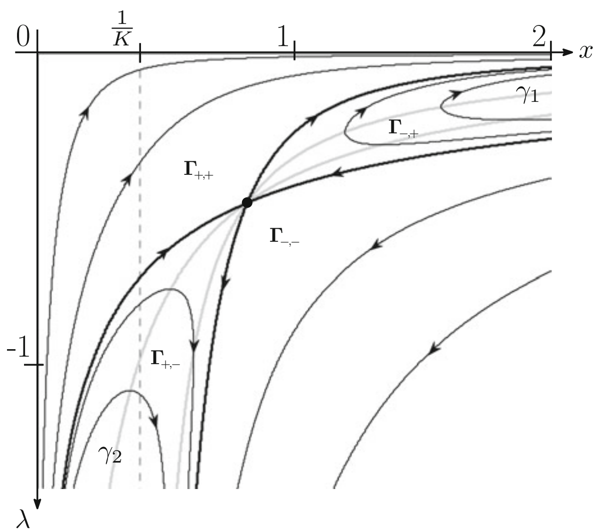

Fig. 1 Phase portrait of (35) around $(\hat{x}, \hat{\lambda})$. Here $r = 5$, $\rho = 0.1$, and $K = 2.5$

1. $(x_{\xi,\beta}(t), \lambda_{\xi,\beta}(t)) \in \Gamma_{-,-}$ or $(x_{\xi,\beta}(t), \lambda_{\xi,\beta}(t)) \in \Gamma_{+,-}$ for all sufficiently large times t. In this case $T_{\xi,\beta} = \infty$ and $\lim_{t\to\infty} \lambda_{\xi,\beta}(t) = -\infty$ while $\lim_{t\to\infty} x_{\xi,\beta}(t) = 1/K$. Due to Theorem 2 such asymptotic behavior does not correspond to an optimal path because it contradicts the necessary condition (36).
2. $(x_{\xi,\beta}(t), \lambda_{\xi,\beta}(t)) \in \Gamma_{+,+}$ for all sufficiently large times t. In this case $\lim_{t\to T_{\xi,\beta}} x_{\xi,\beta}(t) = \infty$ and $\lim_{t\to T_{\xi,\beta}} \lambda_{\xi,\beta}(t) = 0$. If $(x_{\xi,\beta}(\cdot), \lambda_{\xi,\beta}(\cdot))$ corresponds to an optimal pair $(x_*(\cdot), u_*(\cdot))$ in $(P3)$ then due to Theorem 2 $x_*(\cdot) \equiv x_{\xi,\beta}(\cdot)$, $T_{\xi,\beta} = \infty$, $\lim_{t\to\infty} x_*(t) = \infty$, and $\lim_{t\to\infty} \lambda_{\xi,\beta}(t) = 0$. Set $\lambda_*(\cdot) \equiv \lambda_{\xi,\beta}(\cdot)$ in this case and define the function $\phi_*(\cdot)$ by equality $\phi_*(t) = \lambda_*(t)x_*(t)$, $t \in [0, \infty)$.

By direct differentiation for a.e. $t \in [0, \infty)$ we get (see (35))

$$\dot{\phi}_*(t) \overset{\text{a.e.}}{=} (\rho + r)\lambda_*(t)x_*(t) + 2 - r\lambda(t)x_*(t) - 1 + a\lambda_*(t) = \rho\phi_*(t) + 1 + a\lambda_*(t).$$

Hence,

$$\phi_*(t) = e^{\rho t}\left[\phi_*(0) + \int_0^t e^{-\rho s}\,(1 + a\lambda_*(s))\,ds\right], \qquad t \in [0, \infty). \tag{42}$$

Since $\lim_{t\to\infty} \lambda_*(t) = 0$ the improper integral $\int_0^\infty e^{-\rho s}\,(1 + a\lambda_*(s))\,ds$ converges, and due to (36) we have $0 > \phi_*(t) = \lambda_*(t)x_*(t) > -1/\rho$ for all $t > 0$. Due to (42) this implies

$$\phi_*(0) = -\int_0^\infty e^{-\rho s}\,(1 + a\lambda_*(s))\,ds = -\frac{1}{\rho} - a\int_0^\infty e^{-\rho s}\lambda_*(s)\,ds.$$

Substituting this expression for $\phi_*(0)$ in (42) we get

$$\phi_*(t) = -\frac{1}{\rho} - ae^{\rho t}\int_t^\infty e^{-\rho s}\lambda_*(s)\,ds, \qquad t \in [0, \infty).$$

Due to the L'Hospital rule we have

$$\lim_{t\to\infty} e^{\rho t}\int_t^\infty e^{-\rho s}\lambda_*(s)\,ds = \lim_{t\to\infty} \frac{\int_t^\infty e^{-\rho s}\lambda_*(s)\,ds}{e^{-\rho t}} = \lim_{t\to\infty} \frac{\lambda_*(t)}{\rho} = 0.$$

Hence,

$$\lim_{t\to\infty} u_*(t) = \lim_{t\to\infty} \frac{-1}{\lambda_*(t)x_*(t)} = \lim_{t\to\infty} \frac{-1}{\phi_*(t)} = \rho.$$

But due to the system (35) and the inequality $r > \rho$ this implies $\lim_{t\to\infty} x_*(t) \le a < \infty$ that contradicts the equality $\lim_{t\to\infty} x_*(t) = \infty$. So, all these trajectories of (35) are the blow up ones. Thus, there are not any trajectories of (35) that correspond to optimal admissible pairs due to Theorem 2 in the case 2).

3. $\lim_{t\to\infty}(x(t), \lambda(t)) = (\hat{x}, \hat{\lambda})$ as $t \to \infty$. In this case, since the equilibrium $(\hat{x}, \hat{\lambda})$ is of saddle type, there are only two trajectories of (35) which tend to the equilibrium point $(\hat{x}, \hat{\lambda})$ asymptotically as $t \to \infty$ and lying on the stable manifold of $(\hat{x}, \hat{\lambda})$. One such trajectory $(x_1(\cdot), \lambda_1(\cdot))$ approaches the point $(\hat{x}, \hat{\lambda})$ from the left from the set $\Gamma_{+,+}$ (we call this trajectory *the left equilibrium trajectory*), while the second trajectory $(x_2(\cdot), \lambda_2(\cdot))$ approaches the point $(\hat{x}, \hat{\lambda})$ from the right from the set $\Gamma_{-,-}$ (we call this trajectory *the right equilibrium trajectory*). It is easy to see that both these trajectories are fit to estimate (36) and stationarity condition (39). Hence, $(x_1(\cdot), \lambda_1(\cdot))$, $(x_2(\cdot), \lambda_2(\cdot))$ and the stationary trajectory $(\hat{x}(\cdot), \hat{\lambda}(\cdot))$, $\hat{x}(\cdot) \equiv \hat{x}$, $\hat{\lambda}(\cdot) \equiv \hat{\lambda}$, $t \geq 0$, are unique trajectories of (35) which can correspond to the optimal pairs in problem $(P3)$ due to Theorem 2.

Due to Theorem 1 for any initial state $x_0 > 0$ an optimal control $u_*(\cdot)$ in problem $(P3)$ exists. Hence, for any initial state $\xi \in (0, \hat{x})$ there is a unique $\beta < 0$ such that the corresponding trajectory $(x_{\xi,\beta}(\cdot), \lambda_{\xi,\beta}(\cdot))$ coincides (up to a shift in time) with the left equilibrium trajectory $(x_1(\cdot), \lambda_1(\cdot))$ on time interval $[0, \infty)$. Analogously, for any initial state $\xi > \hat{x}$ there is a unique $\beta < 0$ such that the corresponding trajectory $(x_{\xi,\beta}(\cdot), , \lambda_{\xi,\beta}(\cdot))$ coincides (up to a shift in time) with the right equilibrium trajectory $(x_2(\cdot), \lambda_2(\cdot))$ on $[0, \infty)$. The corresponding optimal control is defined uniquely by (37). Thus, for any initial state $x_0 > 0$ the corresponding optimal pair $(x_*(\cdot), u_*(\cdot))$ in $(P3)$ is unique, and due to Theorem 2 the corresponding current value adjoint variable $\lambda_*(\cdot)$ is also unique.

Further, to the left of the point $(\hat{x}, \hat{\lambda})$ in the set $\Gamma_{+,+}$, the function $x_1(\cdot)$ increases. Therefore, while $(x_1(\cdot), \lambda_1(\cdot))$ lies in $\Gamma_{+,+}$, the time can be uniquely expressed in terms of the first coordinate of the trajectory $(x_1(\cdot), \lambda_1(\cdot))$ as a smooth function $t = t_1(x)$, $x \in (0, \hat{x})$. Changing the time variable $t = t_1(x)$ on interval $(0, \hat{x})$, we find that the function $\lambda_-(x) = \lambda_1(t_1(x))$, $x \in (0, \hat{x})$, is a solution to the following differential equation on $(0, \hat{x})$:

$$\frac{d\lambda(x)}{dx} = \frac{d\lambda(t_1(x))}{dt} \times \frac{dt_1(x)}{dx} = \frac{\lambda(x)\,((\rho + r)\lambda(x)x + 2)}{x\,(-r\lambda(x)x - 1 + a\lambda(x))} \tag{43}$$

with the boundary condition

$$\lim_{x\to\hat{x}-0} \lambda(x) = \hat{\lambda}. \tag{44}$$

Obviously, the curve $\lambda_- = \{(x, \lambda) : \lambda = \lambda_-(x), x \in (0, \hat{x})\}$ corresponds to the region of the stable manifold of $(\hat{x}, \hat{\lambda})$ where $x < \hat{x}$.

Analogously, to the right of the point $(\hat{x}, \hat{\lambda})$ in the set $\Gamma_{-,-}$, while $(x_1(\cdot), \lambda_1(\cdot))$ lies in $\Gamma_{-,-}$,the function $x_1(\cdot)$ decreases. Hence, the time can be uniquely expressed in terms of the first coordinate of the trajectory $(x_1(\cdot), \lambda_1(\cdot))$ as a smooth function $t = t_2(x)$, $x \in (\hat{x}, \infty)$. Changing the time variable $t = t_2(x)$ on interval $(\hat{x}, \infty)$, we find that the function $\lambda_+(x) = \lambda_2(t_2(x))$, $x > \hat{x}$, is a solution to the differential equation (43) on $(\hat{x}, \infty)$ with the boundary condition

$$\lim_{x \to \hat{x}+0} \lambda(x) = \hat{\lambda}. \tag{45}$$

As above, the curve $\lambda_+ = \{(x, \lambda) \colon \lambda = \lambda_+(x), x \in (\hat{x}, \infty)\}$ corresponds to the region of the stable manifold of $(\hat{x}, \hat{\lambda})$ where $x > \hat{x}$.

Using solutions $\lambda_-(\cdot)$ and $\lambda_+(\cdot)$ of differential equation (43) along with (37) we can get an expression for the optimal feedback law as follows

$$u_*(x) = \begin{cases} -\frac{1}{\lambda_-(x)x}, & \text{if } x < \hat{x}, \\ \frac{\rho+r}{2}, & \text{if } x = \hat{x}, \\ -\frac{1}{\lambda_+(x)x}, & \text{if } x > \hat{x}. \end{cases}$$

Now, consider the case (ii) when $r \le \rho$. In this case $y_2(x) > y_1(x)$ for all $x > 1/K$ and hence the nullclines γ_1 and γ_2 do not intersect in Γ. Accordingly, the system (35) does not have an equilibrium point in Γ.

The nullclines γ_1 and γ_2 divide the set Γ in three open regions:

$$\hat{\Gamma}_{-,-} = \left\{(x, \lambda) \in \Gamma \colon \lambda < y_1(x), x > \frac{1}{K}\right\},$$

$$\hat{\Gamma}_{+,-} = \left\{(x, \lambda) \in \Gamma \colon \lambda < y_2(x), 0 < x \le \frac{1}{K}\right\} \bigcup \left\{(x, \lambda) \in \Gamma \colon y_1(x) < \lambda < y_2(x), x > \frac{1}{K}\right\},$$

$$\hat{\Gamma}_{+,+} = \left\{(x, \lambda) \in \Gamma \colon y_2(x) < \lambda < 0, 0 < x \le \hat{x}\right\} \bigcup \left\{(x, \lambda) \in \Gamma \colon y_1(x) < \lambda < 0, \hat{x} < x < \infty\right\},$$

The behavior of the flows is shown in Fig. 2 through the phase portrait.

Any solution $(x(\cdot), \lambda(\cdot))$ of (35) in Γ has the definite signs of derivatives of its (x, λ) coordinates in each set $\hat{\Gamma}_{-.-}$, $\hat{\Gamma}_{+,+}$, and $\hat{\Gamma}_{-,+}$ as indicated by the subscripts.

The standard analysis of the behaviors of solutions $(x(\cdot), \lambda(\cdot))$ of system (35) in each of sets $\hat{\Gamma}_{-.-}$, $\hat{\Gamma}_{+.-}$ and $\Gamma_{+,+}$ shows that there are only two types of asymptotic behavior of solutions $(x(\cdot), \lambda(\cdot))$ of (35) that are possible:

1. $\lim_{t\to\infty} x(t) = 1/K$, $\lim_{t\to\infty} \lambda(t) = -\infty$. In this case $(x(t), \lambda(t)) \in \hat{\Gamma}_{-,-}$ for all sufficiently large times $t \ge 0$. Due to Theorem 2 such asymptotic behavior does not correspond to an optimal admissible pair because in this case $\lim_{t\to\infty} \lambda(t)x(t) = -\infty$ that contradicts condition (36). Thus this case can be eliminated from the consideration.
2. $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} \lambda(t) = 0$. In this case $(x(t), \lambda(t)) \in \hat{\Gamma}_{+,+}$ for all $t \ge 0$. Since the case (1) can be eliminated from the consideration, we conclude that the case (2) is the only one that can be realized for an optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ (which exists) in $(P3)$ due to the maximum principle (Theorem 2).

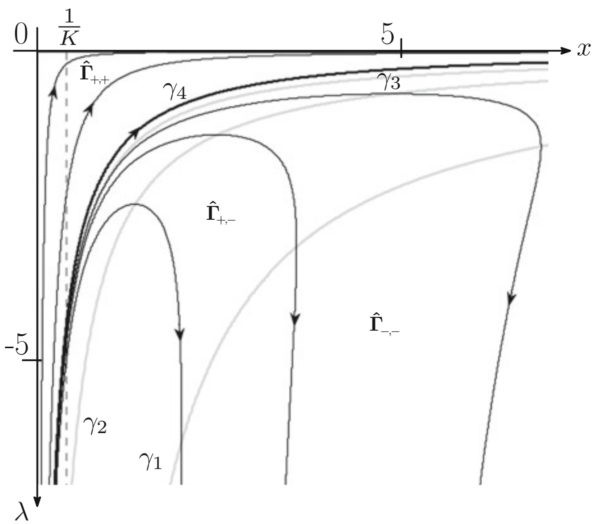

Fig. 2 Phase portrait of (35) in the case $r < \rho$. Here $r = 0.1$, $\rho = 0.5$, and $K = 2.5$

Let us consider behavior of trajectory $(x_*(\cdot), \lambda_*(\cdot))$ of system (35) that corresponds to the optimal pair $(x_*(\cdot), u_*(\cdot))$ in the set $\hat{\Gamma}_{+,+}$ in more details.

As in the subcase (b) of case (i) above, define the function $\phi_*(\cdot)$ as follows:

$$\phi_*(t) = \lambda_*(t)x_*(t), \qquad t \in [0, \infty).$$

Repeating the calculations presented in the subcase (b) of case (i) we get

$$\phi_*(t) = -\frac{1}{\rho} - ae^{\rho t} \int_t^\infty e^{-\rho s} \lambda_*(s)\, ds, \qquad t \in [0, \infty).$$

As in the subcase (b) of case (i) above, due to the L'Hospital rule this implies

$$\lim_{t\to\infty} e^{\rho t} \int_t^\infty e^{-\rho s} \lambda_*(s)\, ds = \lim_{t\to\infty} \frac{\int_t^\infty e^{-\rho s} \lambda_*(s)\, ds}{e^{-\rho t}} = \lim_{t\to\infty} \frac{\lambda_*(t)}{\rho} = 0.$$

Hence,

$$\lim_{t\to\infty} u_*(t) = \lim_{t\to\infty} \frac{-1}{\lambda_*(t)x_*(t)} = \lim_{t\to\infty} \frac{-1}{\phi_*(t)} = \rho.$$

Thus, asymptotically, any optimal admissible control $u_*(\cdot)$ satisfies the Hotelling rule (Hotelling 1974) of optimal depletion of an exhaustible resource in the case (ii).

Now let us show that the optimal control $u_*(\cdot)$ is defined uniquely by Theorem 2 in the case (ii).

Define the function $y_3\colon (0,\infty) \mapsto \mathbb{R}^1$ and the curve $\gamma_3 \subset \Gamma$ as follows:

$$y_3(x) = -\frac{1}{\rho x}, \quad x \in (0,\infty), \qquad \gamma_3 = \{(x,\lambda)\colon \lambda = y_3(x), x \in (0,\infty)\}.$$

It is easy to see that $y_3(x) \geq y_2(x)$ for all $x > 0$ and $y_3(x) > y_1(x)$ for all $x > 1/K$ in the case (ii). Hence, the curve γ_3 is located not below γ_2 and strictly above γ_1 in $\hat{\Gamma}_{+,+}$ (see Fig. 2). Notice that if $r = \rho$ then γ_3 coincide with γ_2 while if $r < \rho$ then γ_3 lies strictly above γ_2 in $\hat{\Gamma}_{+,+}$. It can be demonstrated directly that any trajectory $(x(\cdot), \lambda(\cdot))$ of system (35) can intersect curve γ_3 only one time and only in the upward direction.

Due to (36) a trajectory $(x_*(\cdot), \lambda_*(\cdot))$ of system (35) that corresponds to the optimal pair $(x_*(\cdot), u_*(\cdot))$ lies strictly above γ_3. Since the system (35) is autonomous by virtue of the theorem on uniqueness of a solution of first-order ordinary differential equation (see Hartman 1964, Chapter 3) trajectories of system (35) that lie above γ_3 do not intersect the curve $\gamma_4 = \{(x,\lambda)\colon x = x_*(t), \lambda = \lambda_*(t), t \geq 0\}$ which is the graph of the trajectory $(x_*(\cdot), \lambda_*(\cdot))$.

Further, trajectory $(x_*(\cdot), \lambda_*(\cdot))$ is defined on infinite time interval $[0,\infty)$. This implies that all trajectories $\big(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot)\big)$, $\beta \in (-1/(\rho x_0), \lambda_*(0))$, are also defined on the whole infinite time interval $[0,\infty)$, i.e. $T_{x_0,\beta} = \infty$ for all $\beta \in (-1/(\rho x_0), \lambda_*(0))$. Thus, we have proved that there is a nonempty set (a continuum) of trajectories $\big\{(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot))\big\}$, $\beta \in (-1/(\rho x_0), \lambda_*(0))$, $t \in [0,\infty)$, of system (35) lying strictly between the curves γ_3 and γ_4. All these trajectories are defined on the whole infinite time interval $[0,\infty)$ and, hence, all of them correspond to some admissible pairs $\big\{(x_{x_0,\beta}(\cdot), u_{x_0,\beta}(\cdot))\big\}$. Since these trajectories are located above γ_3 they satisfy also the estimate (36).

Consider the current value Hamiltonian $M(\cdot,\cdot)$ for (x,λ) lying above γ_3 in $\hat{\Gamma}_{+,+}$ (see (38)):

$$\begin{aligned} M(x,\lambda) &= \sup_{u \geq \rho}\{u\lambda x + \ln u\} + (a - rx)\lambda - \ln x \\ &= -1 - \ln(-\lambda x) + (a - rx)\lambda - \ln x, \qquad -\frac{1}{\rho x} < \lambda < 0. \end{aligned} \tag{46}$$

For any trajectory $(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot))$ of system (35) lying above γ_3 in $\hat{\Gamma}_{+,+}$ we have

$$x_{x_0,\beta}(t) \geq e^{(\rho-r)t}x_0, \quad t \geq 0.$$

On the other hand for any trajectory $(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot))$ of system (35) lying between γ_3 and γ_4 in $\hat{\Gamma}_{+,+}$ we have

$$\frac{1}{2(r+\rho)} < -\lambda_{x_0,\beta}(t)x_{x_0,\beta}(t) < \frac{1}{\rho} \quad \text{if} \quad x_{x_0,\beta}(t) > \frac{1}{K}.$$

These imply that for any trajectory $(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot))$ of system (35) lying between γ_3 and γ_4 in $\hat{\Gamma}_{+,+}$ and for corresponding adjoint variable $\psi_{x_0,\beta}(\cdot)$, $\psi_{x_0,\beta}(t) = e^{-\rho t}\lambda_{x_0,\beta}(t)$, $t \geq 0$, we have

$$\lim_{t\to\infty} H(t, x_{x_0,\beta}(t), \psi_{x_0,\beta}(t)) = \lim_{t\to\infty} \left\{e^{-\rho t} M(x_{x_0,\beta}(t), \lambda_{x_0,\beta}(t))\right\} = 0.$$

Hence, for any such trajectory $(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot))$ of system (35) we have (see (39))

$$M(x_{x_0,\beta}(t), \lambda_{x_0,\beta}(t)) = \rho e^{\rho t} \int_t^{\infty} e^{-\rho s} g(x_{x_0,\beta}(t), \lambda_{x_0,\beta}(t))\, ds, \qquad t \geq 0.$$

Let $u_{x_0,\beta}(\cdot)$ be the control corresponding to $x_{x_0,\beta}(\cdot)$, i.e. $u_{x_0,\beta}(t) = -1/(x_{x_0,\beta}(t)\lambda_{x_0,\beta}(t))$. Then taking in the last equality $t = 0$ we get

$$J(x_{x_0,\beta}(\cdot), u_{x_0,\beta}(\cdot)) = \int_0^{\infty} e^{-\rho s} g(x_{x_0,\beta}(t), \lambda_{x_0,\beta}(t))\, ds = \frac{1}{\rho} M(x_{x_0,\beta}(0), \lambda_{x_0,\beta}(0)).$$

For any $t \geq 0$ function $M(x_*(t), \cdot)$ (see (46)) increases on $\{\lambda\colon -1/(\rho x_*(t)) < \lambda < 0\}$. Hence, $M(x_*(t), \cdot)$ reaches its maximal value in λ on the set $\{\lambda\colon -1/(\rho x) < \lambda \leq \lambda_*(t)\}$ at the point $\lambda_*(t)$ that correspond to the optimal path $x_*(\cdot)$. Thus, all trajectories $(x_{x_0,\beta}(\cdot), \lambda_{x_0,\beta}(\cdot))$ of system (35) lying between γ_3 and γ_4 in $\hat{\Gamma}_{+,+}$ do not correspond to optimal admissible pairs in $(P3)$.

From this we can also conclude that all trajectories $(x(\cdot), \lambda(\cdot))$ of system (35) lying above γ_4 also do not correspond to optimal admissible pars in $(P3)$. Indeed, if such trajectory $(x(\cdot), \lambda(\cdot))$ corresponds to an optimal pair $(x(\cdot), u(\cdot))$ in $(P3)$ then it must satisfy to condition (39). But in this case we have $\lambda(0) > \lambda_*(0)$ and

$$J(x(\cdot), u(\cdot)) = \frac{1}{\rho} M(x_0, \lambda(0)) = \frac{1}{\rho} M(x_0, \lambda_*(0)) = J(x_*(\cdot), \lambda_*(\cdot)),$$

that contradicts the fact that function $M(x_0, \cdot)$ increases on $\{\lambda\colon -1/(\rho x) < \lambda < 0\}$.

Thus, for any initial state x_0 there is a unique optimal pair $(x_*(\cdot), u_*(\cdot))$ in $(P3)$ in the case (ii). The corresponding current value adjoint variable $\lambda_*(\cdot)$ is also defined uniquely as the maximal negative solution to equation (see (35))

$$\dot{\lambda}(t) = (\rho + r)\lambda(t) + \frac{2}{x_*(t)} \tag{47}$$

on the whole infinite time interval $[0, \infty)$.

The function $x_*(\cdot)$ increases on $[0, \infty)$. Therefore, the time can be uniquely expressed as a smooth function $t = t_*(x)$, $x \in (0, \infty)$. Changing the time variable $t = t_*(x)$, we find that the function $\lambda_0(x) = \lambda_*(t_*(x))$ is solution to the differential equation (43) on the infinite interval $(0, \infty)$.

Using solution $\lambda_0(\cdot)$ of differential equation (43) along with (37) we can get an expression for the optimal feedback law as follows

$$u_*(x) = -\frac{1}{\lambda_0(x)x}, \qquad x > 0.$$

Thus, to find the optimal feedback, we must determine for an initial state $x_0 > 0$ the corresponding initial state $\lambda_0 < 0$ such that solution $(x_*(\cdot), \lambda_*(\cdot))$ of system (35) with initial conditions $x(0) = x_0$ and $\lambda(0) = \lambda_0$ exists on $[0, \infty)$ and $\lambda_*(\cdot)$ is the maximal negative function among all such solutions.

Let us summarize the results obtained in this section in the following theorem.

Theorem 3 *For any initial state $x_0 > 0$ there is a unique optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ in problem $(P3)$, and there is a unique adjoint variable $\psi(\cdot)$ that corresponds $(x_*(\cdot), u_*(\cdot))$ due to the maximum principle (Theorem 2).*

If $r > \rho$ then there is a unique equilibrium $(\hat{x}, \hat{\lambda})$ (see (40)) in the corresponding current value Hamiltonian system (35) and the optimal synthesis is defined as follows

$$u_*(x) = \begin{cases} -\frac{1}{\lambda_-(x)x}, & \text{if } x < \hat{x}, \\ \frac{r+\rho}{2}, & \text{if } x = \hat{x}, \\ -\frac{1}{\lambda_+(x)x}, & \text{if } x > \hat{x}, \end{cases}$$

where $\lambda_-(\cdot)$ and $\lambda_+(\cdot)$ are the unique solutions of (43) that satisfy the boundary conditions (44) and (45) respectively. In this case optimal path $x_*(\cdot)$ is either decreasing, or increasing on $[0, \infty)$, or $x_*(t) \equiv \hat{x}$, $t \geq 0$, depending on the initial state x_0. For any optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ we have $\lim_{t\to\infty} x_*(t) = \hat{x}$ and $\lim_{t\to\infty} u_*(t) = \hat{u}$ (see (41)).

If $r \leq \rho$ then for any initial state x_0 the corresponding optimal path $x_*(\cdot)$ in problem $(P3)$ is an increasing function, $\lim_{t\to\infty} x_*(t) = \infty$, and the corresponding optimal control $u_*(\cdot)$ satisfies asymptotically to the Hotelling rule of optimal depletion of an exhaustible resource, i.e. $\lim_{t\to\infty} u_*(t) = \rho$. The corresponding current value adjoint variable $\lambda_*(\cdot)$ is defined uniquely as the maximal negative solution to Eq. (47) on $[0, \infty)$. The optimal synthesis is defined as

$$u_*(x) = -\frac{1}{\lambda_0(x)x}, \qquad x > 0,$$

where $\lambda_0(x) = \lambda_*(t_*(x))$ is the corresponding solution of (43).

In the next section we discuss the issue of sustainability of optimal paths for different values of the parameters in the model.

5 Conclusion

Following Solow (1956) we assume that the knowledge stock $A(\cdot)$ grows exponentially, i.e. $A(t) = A_0 e^{\mu t}$, $t \geq 0$, where $\mu \geq 0$ and $A_0 > 0$ are constants.

Similar to Valente (2005) we say that an admissible pair $(S(\cdot), u(\cdot))$ is *sustainable* in our model if the corresponding instantaneous utility function $t \mapsto \ln Y(t)$, $t \geq 0$, non-decreases in the long run, i.e.

$$\lim_{T\to\infty} \inf_{t\geq T} \frac{d}{dt} \ln Y(t) = \lim_{T\to\infty} \inf_{t\geq T} \frac{\dot{Y}(t)}{Y(t)} \geq 0.$$

Substituting $Y(t) = A(t)\,(u(t)S(t))^{\alpha}$, $A(t) = A_0 e^{\mu t}$, $t \geq 0$, (see (1)) we get the following characterization of sustainability of an admissible pair $(S(\cdot), u(\cdot))$:

$$\frac{\mu}{\alpha} + \lim_{T\to\infty} \inf_{t\geq T} \left[\frac{\dot{u}(t)}{u(t)} + \frac{\dot{S}(t)}{S(t)} \right] \geq 0. \tag{48}$$

We call an admissible pair $(S(\cdot), u(\cdot))$ *strongly sustainable* if it is sustainable and, moreover, the resource stock $S(\cdot)$ is non-vanishing in the long run, i.e.

$$\lim_{T\to\infty} \inf_{t\geq T} S(t) = S_{\infty} > 0. \tag{49}$$

Consider case (i) when $r > \rho$. In this case due to Theorem 3 there is a unique optimal equilibrium pair in the problem (see (40) and (41)): $\hat{u}(t) \equiv \hat{u} = (r + \rho)/2$, $\hat{S}(t) \equiv \hat{S} = (r - \rho)K/(2r) > 0$, $t \geq 0$, and for any initial state S_0 the corresponding optimal path $S_*(\cdot)$ approaches asymptotically to the optimal equilibrium state $\hat{S}$ while the corresponding optimal exploitation rate $u_*(\cdot)$ approaches asymptotically to the optimal equilibrium value $\hat{u}$. Hence, both conditions (48) and (49) are satisfied. Thus the optimal admissible pair $(S_*(\cdot), u_*(\cdot))$ is strongly sustainable in this case.

Consider case (ii) when $r \leq \rho$. In this case due to Theorem 3 for any initial state S_0 the corresponding optimal control $u_*(\cdot)$ asymptotically satisfies the Hotelling rule of optimal depletion of an exhaustible resource (Hotelling 1974), i.e. $\lim_{t\to\infty} u_*(t) = \rho$, and $\lim_{t\to\infty} \dot{u}_*(t)/u_*(t) = 0$. The corresponding optimal path $S_*(\cdot)$ is asymptotically vanishing, and

$$\lim_{t\to\infty} \dot{S}_*(t)/S_*(t) = \lim_{t\to\infty} (r - u_*(t) - rS_*(t)/K) = r - \rho.$$

Hence, in the case (ii) the sustainability condition (48) takes the following form:

$$\frac{\mu}{\alpha} + r \geq \rho. \tag{50}$$

Notice, that in the case $\alpha = 1$ condition (50) coincides with Valente's necessary condition for sustainability in his capital-resource model with a renewable resource growing exponentially (see Valente 2005).

Since in the case (i) the inequality (50) holds automatically we conclude that (50) is a necessary and sufficient condition (a criterion) for sustainability of the optimal pair $(S_*(\cdot), u_*(\cdot))$ in our model while the stronger inequality

$$r > \rho$$

gives a criterion of its strong sustainability.

The criterion (50) gives the following guidelines for sustainable optimal growth: (1) Take measures to increase growth rate r; (2) Increase ratio of growth rate of knowledge stock μ to output elasticity α; and (3) Decrease social discount ρ i.e., plan long term. The sustainability criterion (50) gives a relationship between the state of technology (depicted by α), the environment (depicted by r), accumulation of knowledge (depicted by μ) and foresight of the social planner (depicted by ρ). According to the guideline 2 above, it is the *ratio* between μ and α that matters but not the individual quantities.

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