Strong Uniqueness of Dirichlet Operators Related to Stochastic Quantization Under Exponential Interactions in One-Dimensional Infinite Volume

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Abstract In this survey paper, we discuss strong uniqueness of Dirichlet operators related to stochastic quantization under exponential (and polynomial) interactions in one-dimensional infinite volume based on joint works with Sergio Albeverio and Michael Röckner (Albeverio et al., J Funct Anal 262:602–638, 2012, [4], Kawabi and Röckner, J Funct Anal 242:486–518, 2007, [11]). We also raise an open problem.

Keywords Strong uniqueness · L^p-uniqueness · Essential self-adjointness Dirichlet operator · Stochastic quantization · Gibbs measure · Path space · SPDE

2010 AMS Classification Numbers 35R15 · 35R60 · 46N50 · 47D07

1 Introduction

Dirichlet form theory on infinite dimensional spaces plays a crucial role in many fields of mathematical physics including Euclidean quantum field theory and statistical mechanics. It is also indispensable in stochastic analysis on path and loop spaces over Riemannian manifolds. From an analytic point of view, it is very important to study L^p-uniqueness of the Dirichlet operator associated with a given Dirichlet form, that is, the question whether or not the Dirichlet operator restricted to some minimal domain uniquely determines a strongly continuous semigroup on the corresponding L^p-space. As is well known, in the case of $p = 2$, this uniqueness is equivalent to essential self-adjointness of the Dirichlet operator. This kind of uniqueness problem on infinite dimensional state spaces has been studied intensively by many authors. We refer to Eberle [6] and references therein for a detailed review. However, it is still understood very insufficiently in the sense that there are several important types of infinite dimensional Dirichlet operators for which it is not known whether uniqueness

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A. Eberle et al. (eds.), *Stochastic Partial Differential Equations and Related Fields*, Springer Proceedings in Mathematics & Statistics 229,
https://doi.org/10.1007/978-3-319-74929-7_31

holds or not. As in Michael Röckner's speech at the conference dinner of the SPDE conference 2016 in Bielefeld, the most prominent example in which essential self-adjointness is not known is the stochastic quantization of $P(\phi)_2$-quantum fields in *infinite volume* in the context of Euclidean quantum field theory. We refer to Albeverio–Ma–Röckner [5] for a concise overview on stochastic quantization. We should also mention that recently there has arisen a renewed interest in singular SPDEs, in connection with Hairer's groundbreaking work on regularity structures [8] and related work by Gubinelli, Imkeller and Perkowski [7]. By using these new theories, Mourrat and Weber [13] constructed a unique strong solution to the stochastic quantization equation associated with the $P(\phi)_2$-quantum fields in infinite volume, and Röckner, R. Zhu and X. Zhu [15] obtained restricted Markov uniqueness for the corresponding Dirichlet operator. Note that essential self-adjointness implies restricted Markov uniqueness. However, the converse does not hold in general.

On the other hand, even in the case of $P(\phi)_1$-quantum fields in infinite volume, essential self-adjointness of the Dirichlet operator has been open for many years and solved by the author and Röckner [11]. Moreover in that paper, it was shown that the corresponding dynamics coincides with the $P(\phi)_1$-*time evolution*, which had been constructed by Iwata [10] as a unique strong solution to the stochastic quantization equation (7) defined on the whole line $\mathbf{R}$.

In this survey paper, we discuss L^p-uniqueness of the Dirichlet operator on an infinite volume path space $C(\mathbf{R}, \mathbf{R}^d)$ with Gibbs measures obtained in [4, 11]. Important examples of the Gibbs measures are $P(\phi)_1$, $\exp(\phi)_1$, and trigonometric quantum fields in infinite volume. In particular, $\exp(\phi)_1$-quantum fields were introduced (for the case where $\mathbf{R}$ occurring in (1) below is replaced by a 2-dimensional Euclidean space-time $\mathbf{R}^2$ and where $d = 1$) by Albeverio and Høegh-Krohn in the early 1970s (cf. [1, 2]). More precisely, we are concerned with Gibbs measures on an infinite volume path space $C(\mathbf{R}, \mathbf{R}^d)$ given by the following formal expression:

$$\mu(dw) = Z^{-1} \exp\Big\{ -\frac{1}{2}\int_{\mathbf{R}} \big((-\Delta_x + m^2)w(x), w(x)\big)_{\mathbf{R}^d} dx \\ - \int_{\mathbf{R}} \Big(\int_{\mathbf{R}^d} e^{(w(x),\xi)_{\mathbf{R}^d}} \nu(d\xi) \Big) dx \Big\} \prod_{x \in \mathbf{R}} dw(x). \tag{1}$$

Here Z is a normalizing constant, $m > 0$ denotes mass, $\Delta_x := d^2/dx^2$, ν is a bounded positive measure on $\mathbf{R}^d$ with compact support, and $\prod_{x\in\mathbf{R}} dw(x)$ stands for a (heuristic) volume measure on the space of maps from $\mathbf{R}$ into $\mathbf{R}^d$. This has the interpretation of a quantized d-dimensional vector field with an interaction of exponential type in the 1-dimensional space-time $\mathbf{R}$, a model which is known as stochastic quantization of the $\exp(\phi)_1$-quantum field model (with weight measure ν).

Furthermore, we also discuss a characterization of the stochastic dynamics corresponding to the above Dirichlet operator. Thanks to a general theory of Albeverio–Röckner [3], the stochastic dynamics constructed through the Dirichlet form approach solves the parabolic SPDE (7) below weakly. However, we obtain something much better, namely existence and uniqueness of a strong solution. We achieve this by first proving pathwise uniqueness for SPDE (7) and then applying the recent work on the Yamada–Watanabe theorem for mild solutions of SPDEs (cf. Ondreját [14]). As a consequence, we have the existence of a unique strong solution to SPDE (7) by using simple and straightforward arguments which do not rely on any finite volume approximations discussed as in [10]. However, as we will mention in Sect. 2, this uniqueness does not imply the L^p-uniqueness of the Dirichlet operator, and vice versa.

The rest of this paper is organized as follows: In Sect. 2, we present the framework and state our strong uniqueness results for both Dirichlet operators and corresponding stochastic dynamics. In Sect. 3, we raise an open problem.

2 Framework and Results

First of all, we introduce some notation and objects we will be working with. We define a weight function $\rho_r \in C^\infty(\mathbf{R}, \mathbf{R})$, $r \in \mathbf{R}$ by $\rho_r(x) := e^{r\chi(x)}$, $x \in \mathbf{R}$, where $\chi \in C^\infty(\mathbf{R}, \mathbf{R})$ is a positive symmetric convex function satisfying $\chi(x) = |x|$ for $|x| \geq 1$. We fix a positive constant $\bar{r}$ sufficiently small. We set $E := L^2(\mathbf{R}, \mathbf{R}^d; \rho_{-2\bar{r}}(x)dx)$. This space is a Hilbert space with its inner product defined by

$$(w, \tilde{w})_E := \int_{\mathbf{R}} \big(w(x), \tilde{w}(x)\big)_{\mathbf{R}^d} \rho_{-2\bar{r}}(x)dx, \quad w, \tilde{w} \in E.$$

Moreover, we set $H := L^2(\mathbf{R}, \mathbf{R}^d)$ and denote by $\|\cdot\|_E$ and $\|\cdot\|_H$ the corresponding norms in E and H, respectively. We regard the dual space E^* of E as $L^2(\mathbf{R}, \mathbf{R}^d; \rho_{2\bar{r}}(x)dx)$. We endow $C(\mathbf{R}, \mathbf{R}^d)$ with the compact uniform topology and introduce a family of Banach spaces

$$\mathscr{C}_r := \Big\{w \in C(\mathbf{R}, \mathbf{R}^d) | \lim_{|x|\to\infty} |w(x)|\rho_{-r}(x) < \infty\Big\}, \quad r > 0$$

with norms defined by $\|w\|_{r,\infty} := \sup_{x\in\mathbf{R}} |w(x)|\rho_{-r}(x)$, $w \in \mathscr{C}_r$. We also introduce a tempered subspace of $C(\mathbf{R}, \mathbf{R}^d)$ by $\mathscr{C} := \cap_{r>0}\mathscr{C}_r$. We note that $\mathscr{C}$ is a Fréchet space with respect to the system of norms $\{\|\cdot\|_{r,\infty}\}_{r>0}$ and the inclusion $\mathscr{C} \subset E \cap C(\mathbf{R}, \mathbf{R}^d)$ is dense with respect to the topology of E. Let $\mathscr{B}$ be the topological σ-field on $C(\mathbf{R}, \mathbf{R}^d)$. For $T_1 < T_2 \in \mathbf{R}$, we define by $\mathscr{B}_{[T_1,T_2]}$ and $\mathscr{B}_{[T_1,T_2],c}$ the sub-σ-fields of $\mathscr{B}$ generated by $\{w(x); T_1 \leq x \leq T_2\}$ and $\{w(x); x \leq T_1, x \geq T_2\}$, respectively. For $T_1, T_2 \in \mathbf{R}$ and $z_1, z_2 \in \mathbf{R}^d$, let $\mathscr{W}^{z_1,z_2}_{[T_1,T_2]}$ be the path space measure of the Brownian bridge such that $w(T_1) = z_1$, $w(T_2) = z_2$. We sometimes regard this measure as a probability measure on the measurable space $(C(\mathbf{R}, \mathbf{R}^d), \mathscr{B})$ by putting $w(x) = z_1$ for $x \leq T_1$ and $w(x) = z_2$ for $x \geq T_2$.

We now introduce a (U-)Gibbs measure μ on $C(\mathbf{R}, \mathbf{R}^d)$ based on Iwata [9]. Let $U \in C(\mathbf{R}^d, \mathbf{R})$ be a (self-interaction) potential function which can be written as

$$U(z) = U_0(z) + U_1(z), \qquad z \in \mathbf{R}^d,$$

where $U_0 \in C(\mathbf{R}^d, \mathbf{R})$ is convex and $U_1 \in C^1(\mathbf{R}^d, \mathbf{R})$. We impose the following three conditions on U:
(U1): There exists a constant $K_1 \in \mathbf{R}$ such that

$$\big(\widetilde{\nabla} U(z_1) - \widetilde{\nabla} U(z_2), z_1 - z_2\big)_{\mathbf{R}^d} \geq K_1 |z_1 - z_2|^2, \qquad z_1, z_2 \in \mathbf{R}^d,$$

where $\widetilde{\nabla} U(z) := \partial_0 U_0(z) + \nabla U_1(z)$, $z \in \mathbf{R}^d$ and $\partial_0 U_0$ is the minimal section of the subdifferential ∂U_0. (We note that $\widetilde{\nabla} U$ coincides with the usual gradient ∇U provided $U \in C^1(\mathbf{R}^d, \mathbf{R})$.)
(U2): There exist $K_2 > 0$, $R > 0$ and $\alpha > 0$ such that $U_1(z) \geq K_2 |z|^\alpha$, $|z| > R$.

(U3): There exist $K_3, K_4 > 0$ and $0 < \beta < 1 + \frac{\alpha}{2}$ such that

$$|\widetilde{\nabla} U(z)| \leq |\partial_0 U_0(z)| + |\nabla U_1(z)| \leq K_3 \exp(K_4 |z|^\beta), \qquad z \in \mathbf{R}^d.$$

Let $H_U := -\frac{1}{2}\Delta_z + U$ be the Schrödinger operator on $L^2(\mathbf{R}^d, \mathbf{R})$, where $\Delta_z := \sum_{i=1}^d \partial^2 / \partial z_i^2$ is the d-dimensional Laplacian. Then condition **(U2)** assures that H_U has purely discrete spectrum and a complete set of eigenfunctions. We denote by $\lambda_0 (> \min U)$ the minimal eigenvalue and by Ω the corresponding normalized eigenfunction in $L^2(\mathbf{R}^d, \mathbf{R})$. This eigenfunction is called ground state and it can be chosen to be strictly positive. Moreover, it has exponential decay at infinity. To be precise, there exist some positive constants D_1, D_2 such that

$$0 < \Omega(z) \leq D_1 \exp\big(- D_2 |z|\, U_{\frac{1}{2}|z|}(z)^{1/2} \big), \quad z \in \mathbf{R}^d, \tag{2}$$

where $U_{\frac{1}{2}|z|}(z) := \inf\{U(y) \mid |y - z| \leq \frac{1}{2}|z|\}$.

For $T_1 < T_2$, and for all $T_1 \leq x_1 < x_2 < \cdots < x_n \leq T_2$, $A_1, A_2, \ldots, A_n \in \mathscr{B}(\mathbf{R}^d)$, we define a cylinder set $A \in \mathscr{B}_{[T_1, T_2]}$ by $A := \{w \in C(\mathbf{R}, \mathbf{R}^d) \mid w(x_1) \in A_1, w(x_2) \in A_2, \ldots, w(x_n) \in A_n\}$. Next, we set

$$\begin{aligned}
\mu(A) &:= \Big(\Omega, e^{-(x_1 - T_1)(H_U - \lambda_0)}\big(\mathbf{1}_{A_1} e^{-(x_2 - x_1)(H_U - \lambda_0)}\big(\mathbf{1}_{A_2} \cdots \\
&\qquad\qquad e^{-(x_n - x_{n-1})(H_U - \lambda_0)}\big(\mathbf{1}_{A_n} e^{-(T_2 - x_n)(H_U - \lambda_0)}\Omega\big)\big)\big)\Big)_{L^2(\mathbf{R}^d, \mathbf{R})} \\
&= e^{\lambda_0 (T_2 - T_1)} \int_{\mathbf{R}^d} \Omega(z_1) \Big\{ \int_{\mathbf{R}^d} \Omega(z_2) p(T_2 - T_1, z_1, z_2) \\
&\quad \times \Big(\int_{C(\mathbf{R}, \mathbf{R}^d)} \mathbf{1}_A(w) \exp\Big(- \int_{T_1}^{T_2} U(w(x)) dx \Big) \mathscr{W}^{z_1, z_2}_{[T_1, T_2]}(dw) \Big) dz_2 \Big\} dz_1,
\end{aligned} \tag{3}$$

where $p(t, z_1, z_2)$, $t > 0$, $z_1, z_2 \in \mathbf{R}^d$ is the transition probability density of standard Brownian motion $(B_t)_{t \geq 0}$ on $\mathbf{R}^d$, and we used the Feynman–Kac formula for the second line. Then by recalling that $e^{-t H_U}\Omega = e^{-t\lambda_0}\Omega$, $\|\Omega\|_{L^2(\mathbf{R}^d, \mathbf{R})} = 1$ and by the Markov property of the d-dimensional Brownian motion, (3) defines a consistent family of probability measures, and hence μ can be extended to a probability measure on $C(\mathbf{R}, \mathbf{R}^d)$. We mention that the Gibbs measure μ coincides with the probability law of the $P(\phi)_1$-process associated with the potential U, that is, the stationary solution of the following SDE on $\mathbf{R}^d$:

$$dz_t = \nabla \log \Omega(z_t)dt + dB_t.$$

Carrying out the standard moment estimates of Brownian motion, we see that the Gibbs measure μ is supported on the tempered path space $\mathscr{C}$. Thus we may regard $\mu \in \mathscr{P}(E)$ by identifying it with its image measure under the inclusion map of $\mathscr{C}$ into E. Furthermore, μ satisfies the following DLR-equation:

$$\mathbf{E}^{\mu}\big[\mathbf{1}_A|\mathscr{B}_{[T_1,T_2],c}\big](\xi) = Z^{-1}_{[T_1,T_2]}(\xi) \int_A \exp\left(-\int_{T_1}^{T_2} U(w(x))dx\right) \mathscr{W}^{\xi(T_1),\xi(T_2)}_{[T_1,T_2]}(dw),$$
$$\mu\text{-a.e. } \xi \in C(\mathbf{R}, \mathbf{R}^d), \ \text{for all } A \in \mathscr{B}_{[T_1,T_2]}, T_1 < T_2, \qquad (4)$$

where $Z_{[T_1,T_2]}(\xi) := \mathbf{E}^{\mathscr{W}^{\xi(T_1),\xi(T_2)}_{[T_1,T_2]}}[\exp(-\int_{T_1}^{T_2} U(w(x))dx)]$ is a normalizing constant. Although generally there exist other probability measures on $C(\mathbf{R}, \mathbf{R}^d)$ satisfying the DLR-equation (4), we only consider the Gibbs measure μ which has been constructed in (3).

Remark 1 If condition **(U2)** holds with $\alpha > 2$, we obtain the following *Fernique type estimate*:

$$\begin{aligned}
\mathbf{E}^{\mu}\big[\exp(p\|w\|_E^2)\big] &= \sum_{n=0}^{\infty} \frac{p^n}{n!} \mathbf{E}^{\mu}\big[\|w\|_E^{2n}\big] \\
&\le \sum_{n=0}^{\infty} \left(\frac{p}{r}\right)^n \frac{1}{n!} \int_{\mathbf{R}^d} |z|^{2n}_{\mathbf{R}^d} \Omega(z)^2 dz \\
&\le D_1^2 \sum_{n=0}^{\infty} \left(\frac{p}{r}\right)^n \frac{1}{n!} \int_{\mathbf{R}^d} |z|^{2n}_{\mathbf{R}^d} e^{-c|z|^{1+\alpha/2}} dz \\
&= \frac{C_1 S_{d-1}}{(1+\alpha/2)} \sum_{n=0}^{\infty} \left(\frac{p}{r}\right)^n \frac{1}{n!} c^{\frac{1+\alpha/2}{2n+d}} \Gamma\left(\frac{2n+d}{1+\alpha/2}\right) \\
&\le C_2 \exp\left\{C_3 \left(\frac{p}{r}\right)^{\frac{\alpha+2}{\alpha-2}}\right\}, \quad p > 0,
\end{aligned}$$

where c and C_i $(i = 1, 2, 3)$ are positive constants and S_{d-1} denotes the area of the $(d-1)$-dimensional unit sphere.

Now we are in a position to introduce the pre-Dirichlet form $(\mathscr{E}, \mathscr{F}\mathscr{C}_b^{\infty})$. Let $\mathscr{F}\mathscr{C}_b^{\infty}$ be the space of all smooth cylinder functions on E having the form

$$F(w) = f(\langle w, \varphi_1\rangle, \dots, \langle w, \varphi_n\rangle), \quad w \in E,$$

with $n \in \mathbf{N}$, $f = f(\alpha_1, \ldots, \alpha_n) \in C_b^\infty(\mathbf{R}^n, \mathbf{R})$ and $\varphi_1, \ldots, \varphi_n \in C_0^\infty(\mathbf{R}, \mathbf{R}^d)$. Here we set $\langle w, \varphi \rangle := \int_{\mathbf{R}} (w(x), \varphi(x))_{\mathbf{R}^d} dx$ if the integral converges absolutely. Note that $\mathscr{F}\mathscr{C}_b^\infty$ is dense in $L^p(\mu)$ for all $p \geq 1$. For $F \in \mathscr{F}\mathscr{C}_b^\infty$, we define the H-Fréchet derivative $D_H F : E \to H$ by

$$D_H F(w) := \sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}(\langle w, \varphi_1 \rangle, \ldots, \langle w, \varphi_n \rangle)\varphi_j.$$

Then we consider the pre-Dirichlet form $(\mathscr{E}, \mathscr{F}\mathscr{C}_b^\infty)$ which is given by

$$\mathscr{E}(F, G) = \frac{1}{2} \int_E \big(D_H F(w), D_H G(w) \big)_H \mu(dw), \qquad F, G \in \mathscr{F}\mathscr{C}_b^\infty.$$

Proposition 1 ([4, Proposition 2.7]):

$$\mathscr{E}(F, G) = -\int_E \mathscr{L}_0 F(w) G(w) \mu(dw), \qquad F, G \in \mathscr{F}\mathscr{C}_b^\infty,$$

where $\mathscr{L}_0 F \in L^p(\mu),\ p \geq 1,\ F \in \mathscr{F}\mathscr{C}_b^\infty$ *is given by*

$$\begin{aligned}
\mathscr{L}_0 F(w) &= \frac{1}{2}\mathrm{Tr}(D_H^2 F(w)) + \frac{1}{2}\big\langle w, \Delta_x D_H F(w(\cdot)) \big\rangle - \frac{1}{2}\big\langle (\widetilde{\nabla} U)(w(\cdot)), D_H F(w) \big\rangle \\
&= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j}\big(\langle w, \varphi_1 \rangle, \ldots, \langle w, \varphi_n \rangle \big) \langle \varphi_i, \varphi_j \rangle \\
&\quad + \frac{1}{2} \sum_{i=1}^n \frac{\partial f}{\partial \alpha_i}\big(\langle w, \varphi_1 \rangle, \ldots, \langle w, \varphi_n \rangle \big) \cdot \big\{ \langle w, \Delta_x \varphi_i \rangle - \langle (\widetilde{\nabla} U)(w(\cdot)), \varphi_i \rangle \big\}.
\end{aligned}$$

This proposition means that the operator $\mathscr{L}_0$ is the pre-Dirichlet operator which is associated with the pre-Dirichlet form $(\mathscr{E}, \mathscr{F}\mathscr{C}_b^\infty)$. In particular, $(\mathscr{E}, \mathscr{F}\mathscr{C}_b^\infty)$ is closable in $L^2(\mu)$. Let us denote by $\mathscr{D}(\mathscr{E})$ the completion of $\mathscr{F}\mathscr{C}_b^\infty$ with respect to the $\mathscr{E}_1^{1/2}$-norm. By the standard theory of Dirichlet forms, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a Dirichlet form and the operator $\mathscr{L}_0$ has a self-adjoint extension $(\mathscr{L}_\mu, \mathrm{Dom}(\mathscr{L}_\mu))$, called the Friedrichs extension, corresponding to the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. The semigroup $\{e^{t\mathscr{L}_\mu}\}_{t \geq 0}$ generated by $(\mathscr{L}_\mu, \mathrm{Dom}(\mathscr{L}_\mu))$ is Markovian, i.e., $0 \leq e^{t\mathscr{L}_\mu} F \leq 1$, μ-a.e. whenever $0 \leq F \leq 1$, μ-a.e. Moreover, since $\{e^{t\mathscr{L}_\mu}\}_{t \geq 0}$ is symmetric on $L^2(\mu)$, the Markovian property implies that $\|e^{t\mathscr{L}_\mu} F\|_{L^1(\mu)} \leq \|F\|_{L^1(\mu)}$ holds for $F \in L^2(\mu)$, and $\{e^{t\mathscr{L}_\mu}\}_{t \geq 0}$ can be extended as a family of C_0-semigroup of contractions in $L^p(\mu)$ for all $p \geq 1$.

On the other hand, it is a fundamental question whether the Friedrichs extension is the only closed extension generating a C_0-semigroup on $L^p(\mu)$, $p \geq 1$, which for $p = 2$ is equivalent to the fundamental problem of essential self-adjointness of $\mathscr{L}_0$ in quantum physics. Even if $p = 2$, in general there are many lower bounded self-adjoint extensions $\widetilde{\mathscr{L}}$ of $\mathscr{L}_0$ in $L^2(\mu)$ which therefore generate different symmetric strongly continuous semigroups $\{e^{t\widetilde{\mathscr{L}}}\}_{t\geq 0}$. If, however, we have $L^p(\mu)$-uniqueness of $\mathscr{L}_0$ for some $p \geq 2$, there is hence only one semigroup which is strongly continuous and with generator extending $\mathscr{L}_0$. Consequently, in this case, only one such L^p-, hence only one such L^2-dynamics exists, associated with the Gibbs measure μ.

Before stating our main results of this paper, we recall the notion of "capacity" for the convenience. For an open set $O \subset E$, we define

$$\mathrm{Cap}(O) := \inf\{\mathscr{E}_1(F, F) |\ F \in \mathscr{D}(\mathscr{E}),\ F \geq 1 \text{ on } O,\ \mu\text{-a.e.}\}$$

and for an arbitrary subset $A \subset E$, we define $\mathrm{Cap}(A) := \inf\{\mathrm{Cap}(O) |\ A \subset O, O$ open$\}$.

The following two theorems are the main results of this survey paper.

Theorem 1 ([11, Theorem 2.4], [4, Theorem 2.8]): (1) *The pre-Dirichlet operator* $(\mathscr{L}_0, \mathscr{F}\mathscr{C}_b^\infty)$ *is* $L^p(\mu)$*-unique for all* $p \geq 1$*, i.e., there exists exactly one* C_0*-semigroup in* $L^p(\mu)$ *such that its generator extends* $(\mathscr{L}_0, \mathscr{F}\mathscr{C}_b^\infty)$.

(2) *There exists a diffusion process* $\mathbf{M} := (\Theta, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \{X_t\}_{t\geq 0}, \{\mathbf{P}_w\}_{w\in E})$ *such that the semigroup* $\{P_t\}_{t\geq 0}$ *generated by the unique (self-adjoint) extension of* $(\mathscr{L}_0, \mathscr{F}\mathscr{C}_b^\infty)$ *satisfies the following identity for any bounded measurable function* $F : E \to \mathbf{R}$*, and* $t > 0$*:*

$$P_t F(w) = \int_\Theta F(X_t(\omega))\mathbf{P}_w(d\omega), \quad \mu\text{-}a.s.\ w \in E. \tag{5}$$

Moreover, $\mathbf{M}$ *is the unique diffusion process solving the following "componentwise" SDE:*

$$\langle X_t, \varphi\rangle = \langle w, \varphi\rangle + \langle W_t, \varphi\rangle + \frac{1}{2}\int_0^t \big\{\langle X_s, \Delta_x\varphi\rangle - \langle(\widetilde{\nabla} U)(X_s(\cdot)), \varphi\rangle\big\}ds,$$
$$t > 0,\ \varphi \in C_0^\infty(\mathbf{R}, \mathbf{R}^d),\ \mathbf{P}_w\text{-}a.s., \tag{6}$$

for quasi-every $w \in E$ *and such that its corresponding semigroup given by (5) consists of locally uniformly bounded (in t) operators on* $L^p(\mu)$*,* $p \geq 1$*, where* $\{W_t\}_{t\geq 0}$ *is an* $\{\mathscr{F}_t\}_{t\geq 0}$*-adapted* H*-cylindrical Brownian motion starting at origin defined on* $(\Theta, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbf{P}_w)$.

Theorem 2 ([4, Theorem 2.9]): *For quasi-every* $w \in E$*, the parabolic SPDE*

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\big\{\Delta_x X_t(x) - (\widetilde{\nabla} U)(X_t(x))\big\} + \dot{W}_t(x), \quad x \in \mathbf{R},\ t > 0 \tag{7}$$

has a unique strong solution $X = \{X_t^w(\cdot)\}_{t\geq 0}$ *living in* $C([0,\infty), E) \cap C((0,\infty), \mathscr{C}_{\bar r})$. *Namely, there exists a set* $S \subset E$ *with* $\mathrm{Cap}(S) = 0$ *such that for any* H*-cylindrical Brownian motion* $\{W_t\}_{t\geq 0}$ *starting at origin defined on a filtered probability space* $(\Theta, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbf{P})$ *satisfying the usual conditions and for any initial datum*

$w \in E \setminus S$, there exists a unique $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process $X = \{X_t^w(\cdot)\}_{t\geq 0}$ living in $C([0,\infty), E) \cap C((0,\infty), \mathscr{C}_{\bar{r}})$ satisfying (6).

Remark 2 Obviously, the uniqueness result in Theorem 2 implies the (thus weaker) uniqueness stated for the diffusion process **M** in Theorem 1. However, it does not imply the $L^p(\mu)$-uniqueness of the Dirichlet operator. This is obvious, since a priori the latter might have extensions which generate non-Markovian semigroups which thus have no probabilistic interpretation as transition probabilities of a process. Therefore, neither of the two uniqueness results in Theorems 1 and 2, i.e., $L^p(\mu)$-uniqueness of the Dirichlet operator and strong uniqueness of the corresponding SPDE respectively, implies the other.

We give three examples which satisfy our conditions **(U1)**, **(U2)** and **(U3)**.

Example 1 ($P(\phi)_1$-quantum fields): We consider the case where the potential function U is written as the following potential function on $\mathbf{R}^d$:

$$U(z) = \sum_{j=0}^{2n} a_j |z|^j, \quad a_{2n} > 0,\ n \in \mathbf{N}.$$

Especially, in the case $U(z) = \frac{m^2}{2}|z|^2$, $m > 0$, the corresponding Gibbs measure μ coincides with the Gaussian measure on $\mathscr{C}$ with mean 0 and covariance operator $(-\Delta_x + m^2)^{-1}$. It is just the (space-time) free field of mass m in terms of Euclidean quantum field theory. A double-well potential $U(z) = a(|z|^4 - |z|^2)$, $a > 0$, is also particularly important from the point of view of physics.

We should mention that the Gibbs measure μ is supported by a smaller subset of $C(\mathbf{R}, \mathbf{R}^d)$ than $\mathscr{C}$. Actually, it holds

$$\mu\left(\left\{ w \,\Big|\, \limsup_{|x|\to\infty} \frac{|w(x)|_{\mathbf{R}^d}}{(\log|x|)^{1/(m+1)}} \leq C \right\}\right) = 1 \tag{8}$$

with a suitable constant $C > 0$. See e.g., Rosen–Simon [16] and Lörinczi–Hiroshima–Betz [12]. Following Remark 3 below, we can also show (8) easily.

Example 2 (*$exp(\phi)_1$-quantum fields*): We introduce a Gibbs measure μ with the formal expression (1). Let us consider an exponential type potential function $U: \mathbf{R}^d \to \mathbf{R}$ (with weight ν) given by

$$U(z) = \frac{m^2}{2}|z|^2 + V(z) := \frac{m^2}{2}|z|^2 + \int_{\mathbf{R}^d} e^{(\xi,z)_{\mathbf{R}^d}} \nu(d\xi), \quad z \in \mathbf{R}^d,$$

where ν is a bounded positive measure with $\mathrm{supp}(\nu) \subset \{\xi \in \mathbf{R}^d \,|\, |\xi| \leq L\}$ for some $L > 0$. We note that U is a smooth strictly convex function (i.e., $\nabla^2 U \geq m^2$). Hence we can take $K_1 = m^2$, $K_2 = \frac{m^2}{2}$ and $\alpha = 2$. Moreover,

$$|U(z)| \le \frac{m^2}{2}|z|^2 + \nu(\mathbf{R}^d)e^{L|z|} \le \left(\frac{m^2}{2L^2} + \nu(\mathbf{R}^d)\right)e^{2L|z|}, \quad z \in \mathbf{R}^d,$$

and

$$|\nabla U(z)| \le m^2|z| + \int_{\mathbf{R}^d} |\xi| e^{(\xi,z)_{\mathbf{R}^d}} \nu(d\xi) \le \left(\frac{m^2}{L} + L\nu(\mathbf{R}^d)\right)e^{L|z|}, \quad z \in \mathbf{R}^d.$$

Thus we can take $\beta = 1$, which satisfies $\beta < 1 + \frac{\alpha}{2}$ in condition **(U3)**.

Remark 3 Now we consider a simple example of $\exp(\phi)_1$-quantum fields in the case $d = 1$. This example has been discussed in the 2-dimensional space-time case (e.g., $\exp(\phi)_2$-quantum fields) in Albeverio–Høegh-Krohn [2]. Let δ_a be the Dirac measure at $a \in \mathbf{R}$ and we consider $\nu(d\xi) := \frac{1}{2}\big(\delta_{-a}(d\xi) + \delta_a(d\xi)\big),\ a > 0$. Then the corresponding potential function is $U(z) = \frac{m^2}{2}z^2 + \cosh(az)$, and (2) implies that the Schrödinger operator H_U has a ground state Ω satisfying

$$0 < \Omega(z) \le D_1 \exp\left(-\frac{D_2}{\sqrt{2}}|z|\, e^{\frac{a}{4}|z|}\right), \quad z \in \mathbf{R} \tag{9}$$

for some $D_1, D_2 > 0$. By the translation invariance of the Gibbs measure μ and (9), there exist positive constants M_1 and M_2 such that

$$\begin{aligned} A_T &:= \mu\left(\left\{w \in C(\mathbf{R},\mathbf{R}) \mid |w(T)| > \frac{4}{a}\log\log T\right\}\right) \\ &= \int_{|z| > \frac{4}{a}\log\log T} \Omega(z)^2 dz \\ &\le M_1 \exp\big\{ - M_2(\log T)(\log\log T)\big\} = M_1 T^{-M_2 \log\log T} \end{aligned}$$

for T large enough, and it implies $\sum_{T=1}^{\infty} A_T < \infty$. Then the first Borel–Cantelli lemma yields

$$\mu\left(\left\{w \in C(\mathbf{R},\mathbf{R}) \mid \limsup_{|x|\to\infty} \frac{|w(x)|}{\log\log|x|} \le \frac{4}{a}\right\}\right) = 1.$$

This means that μ is supported by a much smaller subset of $C(\mathbf{R},\mathbf{R})$ than $\mathscr{C}$.

Example 3 (*Trigonometric quantum fields*): We consider a trigonometric type potential function $U : \mathbf{R}^d \to \mathbf{R}$ (with weight ν) given by

$$U(z) = \frac{m^2}{2}|z|^2 + V(z) := \frac{m^2}{2}|z|^2 + \int_{\mathbf{R}^d} \cos\big\{(\xi, z)_{\mathbf{R}^d} + \alpha\big\}\nu(d\xi), \quad z \in \mathbf{R}^d,$$

where $\alpha \in \mathbf{R}$, $m > 0$ and ν is a bounded signed measure with finite second absolute moment, i.e.,

$$|\nu|(\mathbf{R}^d) < \infty, \quad K(\nu) := \int_{\mathbf{R}^d} |\xi|^2 |\nu|(d\xi) < \infty.$$

This potential function is smooth, and it can be regarded as a bounded perturbation of a quadratic function. Moreover, $\nabla^2 U \geq m^2 - K(\nu)$ and

$$|\nabla U(z)| \leq m^2 |z| + K(\nu)^{1/2} |\nu|(\mathbf{R}^d)^{1/2}, \quad z \in \mathbf{R}^d.$$

This type of potential functions corresponds to quantum field models with "trigonometric interaction" and has been discussed especially in the 2-dimensional space-time case (cf. [1]).

3 An Open Problem

Finally, we raise an open problem which is concerned with this paper.

If the potential function U is a C^1-function with polynomial growth at infinity, Iwata [10] proves that SPDE (7) has a unique strong solution $X^w = \{X_t^w(\cdot)\}_{t\geq 0}$ living in $C([0,\infty), \mathscr{C})$ for *every* initial datum $w \in \mathscr{C}$. On the other hand, it should be remarked that $(\nabla U)(w(\cdot)) \notin \mathscr{C}$ for $w \in \mathscr{C}$ in the case of $\exp(\phi)_1$-quantum fields. Thus if U has exponential growth at infinity, we cannot apply Iwata's argument directly to solve SPDE (7) in $C([0,\infty), \mathscr{C})$ for *every* initial datum $w \in \mathscr{C}$. Can we overcome this difficulty?

It is natural to think that we can easily construct a unique strong solution to SPDE (7) living in $C([0,\infty), \mathscr{C}_e)$ for *every* initial datum $w \in \mathscr{C}_e$ by only replacing the state space $\mathscr{C}$ by a much smaller tempered subspace $\mathscr{C}_e$ and then by applying Iwata's argument. One might guess that a possible candidate for $\mathscr{C}_e$ is a subspace of $\mathscr{C}$ such that $(\nabla U)(w(\cdot)) \in \mathscr{C}_e$ holds for $w \in \mathscr{C}_e$, which is the space of all paths behaving like

$$|w(x)| \sim \log(\log(\log(\log(\cdots x)))) =: \rho_e(x)^{-1}$$

at infinity. However, we cannot follow all arguments in the papers [4, 10, 11] if we replace $\rho_{-2r}(x)$ by $\rho_e(x)$ because of $\int^{\infty} \rho_e(x)^{-2} dx = \infty$. Hence it seems that this approach is not valid for our problem.

On the other hand, in the case $d = 1$, by applying a comparison theorem for parabolic SPDEs (cf. Shiga [17]), we might construct a unique strong solution to SPDE (7) with exponentially growing drift which lives in $C([0,\infty), \mathscr{C})$ for *every* initial datum $w \in \mathscr{C}$. However, this approach does not work in the case $d \geq 2$.

Hence we still need to find a new approach to tackle this problem. It should be a preliminary step toward a construction of a unique strong solution to the stochastic quantization equation associated with the $\exp(\phi)_2$-quantum fields in *infinite volume*.

Acknowledgements The author was partially supported by JSPS Grant-in-Aid for Scientific Research (C) No. 26400134.

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