Generalized Solutions to Nonlinear Fokker–Planck Equations with Linear Drift

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Dedicated to Michael Röckner at his 60th birthday.

Abstract Existence and long-time behaviour of solutions to nonlinear Fokker–Planck equations (NFPEs) with linear drift are studied.

Keywords Fokker–Planck equation · Entropy · Accretive · Mild solution · Lyapunov function

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1 The Problem

Here, we shall consider the equation

$$\begin{aligned} &u_t(t,x) + \mathrm{div}_x(D(x)u(t,x)) - \Delta_x \beta(u(t,x)) = 0 \text{ in } (0,\infty)\times\mathbb{R}^d,\\ &u(0,x) = u_0(x), \quad x\in\mathbb{R}^d,\ d\ge 1, \end{aligned} \tag{1.1}$$

where

(i) $D \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$, div $D \in L^\infty(\mathbb{R}^d)$.

β is continuous, monotonically nondecreasing, $\beta(0) = 0$,

$|\beta(r)| \le C_1|r|^m + C_2$, $\forall r \in \mathbb{R}$, where $1 \le m < \infty$.

Equation (1.1) describes the evolution of a probability density $u = P$ associated to the Markovian stochastic processes with drift coefficients $(D_i)_{i=1}^d = D$ and diffusion

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$\sigma_{ij} = \delta_{ij}$. Moreover, it is related to the anomalous diffusion which describes particle transport in irregular media. In the special case $D \equiv 0$, (1.1) reduces to the nonlinear porous media equation in $\mathbb{R}^d$.

In $1D$, Eq. (1.1) can be derived from the entropy functional

$$S[u] = \int_{\mathbb{R}} \Phi[u(x)]dx, \tag{1.2}$$

where

$$\Phi \in C^\infty(0, \infty), \ \lim_{r \to 0} \Phi'(r) = \infty \text{ and } \Phi''(r) < 0 \text{ for } r > 0. \tag{1.3}$$

The corresponding Fokker–Planck equation is

$$P_t + \left(H(x)P - \frac{1}{\alpha} (\Phi(P) - P\Phi'(P))_x \right)_x = 0. \tag{1.4}$$

Here the drift function H is the gradient of a potential V (i.e., $H = -\frac{dV}{dx}$) and the constant α represents the strength of fluctuations [5]. A similar approach applies to higher dimensions.

In the special case of the Boltzmann–Gibbs entropy

$$S[u] = -\int u(x) \log u(x) dx,$$

Equation (1.4) reduces to

$$P_t + P_x - \frac{1}{\alpha} P_{xx} = 0, \tag{1.5}$$

while, for the entropy functional

$$S[u] = \frac{1}{p-1} \int (|u|^p - u)dx, \ p > 1, \tag{1.6}$$

Equation (1.4) with $H \equiv 1$ reads as the Plastino and Plastino model [7]

$$P_t + P_x - \frac{1}{\alpha} ((P)^p)_{xx} = 0.$$

Assumption (i) agrees with the key entropy condition (1.3). Indeed, if $\Phi \in C^1(0, \infty) \cap C[0, \infty)$ is a solution to the equation

$$\Phi(r) - r\Phi'(r) = \beta(r), \ \forall r > 0; \ \Phi'(0) = \infty, \tag{1.7}$$

such that

$$\Phi''(r) < 0, \ \Phi'(r) \geq 0, \ \forall r \in \mathbb{R},$$

where β satisfies (i), the NFPE reduces to (1.1) such that (1.3) holds. We note that, in particular, assumption (i) is satisfied for $\beta(u) = \frac{1}{\alpha}\ln(1+u)$, that is, for the Fokker–Planck equation of classical bosons (see [4, 5])

$$P_t + (DP)_x + \frac{1}{\alpha}(\ln(1+P))_{xx} = 0.$$

In [1], Eq. (1.1), was studied the existence of an entropy solution for the Fokker–Planck equation

$$\begin{aligned} &u_t + \operatorname{div}(D(x,u)u) - \Delta\beta(u) = 0, \text{ in } (0,T)\times\mathbb{R}^d,\\ &u(0,x) = u_0(x), \end{aligned} \tag{1.8}$$

where $D(x,u) \equiv b(u)$, with b continuous. In this work, we shall confine to the case of linear drift $D(x,u) \equiv D(x)$.

2 The Existence and Uniqueness of a Generalized Solution

To (1.1) we associate the operator $A : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ defined as the closure $\overline{A}_1$ in $L^1(\mathbb{R}^d)\times L^1(\mathbb{R}^d)$ of the operator

$$\begin{aligned} A_1 u &= -\Delta\beta(u) + \operatorname{div}(D(x)u),\ \forall u \in D(A_1),\\ D(A_1) &= \{u \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d),\ \beta(u)\in H^1(\mathbb{R}^d),\ A_1u \in L^1(\mathbb{R}^d)\}. \end{aligned} \tag{2.1}$$

We have also

Lemma 2.1 *The operator A_1 is accretive in $L^1(\mathbb{R}^d)$ and*

$$L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d) \subset R(I+\lambda A_1),\ \forall\lambda > 0. \tag{2.2}$$

$$(I+\lambda A_1)^{-1} f \geq 0 \textit{ in } \mathbb{R}^d \textit{ if } f \geq 0 \textit{ in } \mathbb{R}^d \tag{2.3}$$

$$\int_{\mathbb{R}^d} (I+\lambda A_1)^{-1} f\,dx = \int_{\mathbb{R}^d} f\,dx \textit{ in } \mathbb{R}^d. \tag{2.4}$$

Proof The accretivity of A_1 follows by multiplying the equation

$$u - \bar{u} + \lambda(A_1 u - A_1\bar{u}) = f - \bar{f},\ \ u,\bar{u}\in D(A_1),$$

in the duality pair ${}_{H^{-1}(\mathbb{R}^d)}\langle\cdot,\cdot\rangle_{H^1(\mathbb{R}^d)}$ with $\mathcal{X}_\varepsilon(u-\bar{u})$ and integrate over $\mathbb{R}^d$, where $\mathcal{X}_\varepsilon$ is a smooth approximation of the sign function for $\varepsilon \to 0$.

More precisely, χ_ε is defined by

$$\chi_\varepsilon(r) = \begin{cases} -1 & \text{for } r < -\varepsilon, \\ \frac{1}{\varepsilon} r & \text{for } |r| < \varepsilon, \\ 1 & \text{for } r > \varepsilon. \end{cases}$$

To prove (2.2), we fix $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ and consider the equation $u + \lambda A_1 u = f$, that is,

$$u - \lambda\Delta\beta(u) + \lambda\,\mathrm{div}(Du) = f \text{ in } \mathcal{D}'(\mathbb{R}^d),\ \ \lambda > 0. \tag{2.5}$$

(Here, $\mathcal{D}'(\mathbb{R}^d)$ is the space of distributions on $\mathbb{R}^d$.)

We set $\beta_\varepsilon = \frac{1}{\varepsilon}\beta(I + \varepsilon\beta)^{-1}$) and approximate (2.5) by

$$u - \lambda\Delta(\beta_\varepsilon(u) + \varepsilon u) + \lambda\,\mathrm{div}(Du) = f \text{ in } \mathbb{R}^d, \tag{2.6}$$

Equivalently,

$$(\varepsilon + \beta_\varepsilon)^{-1}(v) - \lambda\Delta v + \lambda\,\mathrm{div}(D(\varepsilon + \beta_\varepsilon)^{-1}(v)) = f \text{ in } \mathbb{R}^d. \tag{2.7}$$

The operator $v \to \beta_\varepsilon^{-1}(v) - \lambda\Delta v$ is coercive and maximal monotone in $H^1(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$ and so, for each $w \in L^2(\mathbb{R}^d)$,

$$(\varepsilon + \beta_\varepsilon)^{-1}(v) - \lambda\Delta v = -\lambda\,\mathrm{div}(D(\varepsilon + \beta_\varepsilon)^{-1}(w)) + f \quad \text{in } \mathbb{R}^d$$

has a unique solution $v = F(w) \in H^1(\mathbb{R}^d)$.

By the contraction principle, for $\lambda > \frac{1}{2L}\|D\|_\infty$, Eq. (2.7) has a unique solution $v_\varepsilon \in H^1(\mathbb{R}^d)$. This extends to all $\lambda > 0$.

We have by (2.6)

$$\begin{aligned} &|u_\varepsilon|_p \le |f|_p,\ \forall\varepsilon > 0,\ \ p \in [1, \infty), \\ &|\nabla\beta_\varepsilon(u_\varepsilon)|_2^2 + \varepsilon|\nabla u_\varepsilon|_2^2 \le C,\ \forall\varepsilon > 0. \end{aligned}$$

(Here, $|\cdot|_p$, $1 \le p \le \infty$, is the norm of $L^p(\mathbb{R}^d)$.)

On a subsequence $\varepsilon \to 0$, we have

$$\begin{aligned} u_\varepsilon &\to u && \text{weakly in } L^p,\ \ p \in (1, \infty), \\ \beta_\varepsilon(u_\varepsilon) &\to \eta && \text{weakly in } H^1, \\ \Delta(\beta_\varepsilon(u_\varepsilon) + \varepsilon u_\varepsilon) &\to \Delta\eta && \text{weakly in } H^{-1}, \\ \mathrm{div}(Du_\varepsilon) &\to \mathrm{div}(Du) && \text{in } H^{-1}. \end{aligned}$$

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^d} \beta_\varepsilon(u_\varepsilon)u_\varepsilon\,dx \le -\lambda\int_{\mathbb{R}^d} |\nabla\eta|^2 dx - \int_{\mathbb{R}^d} fu\,dx = \int_{\mathbb{R}^d} \eta u\,dx,$$

$$u - \lambda \Delta \eta + \lambda \,\mathrm{div}(Du) = f \text{ in } \mathcal{D}'(\mathbb{R}^d).$$

Hence $\eta = \beta(u)$, a.e. in $\mathbb{R}^d$ and $u + \lambda A_1 u = f$, as claimed. ∎

Proposition 2.2 *Under assumption* (i), *the operator* $A = \overline{A}_1$ *is m-accretive in* $L^1(\mathbb{R}^d)$. *Moreover, one has for all* $\lambda > 0$

$$(I + \lambda A)^{-1} = (I + \lambda A_1)^{-1} \;\; \textit{on } L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \tag{2.8}$$

$$(I + \lambda A)^{-1} f \geq 0 \;\; \textit{if } f \geq 0 \;\; \textit{in } \mathbb{R}^d \tag{2.9}$$

$$\int_{\mathbb{R}^d} (I + \lambda A)^{-1} f \, dx = \int_{\mathbb{R}^d} f \, dx, \quad \forall \lambda > 0. \tag{2.10}$$

It follows also that $\overline{D(A)} = L^1(\mathbb{R}^d)$.

By Proposition 2.2, the finite difference scheme

$$\begin{aligned} u_h(t) + hAu_h(t) &= u_h(t-h), \;\; h > 0, \;\; t \geq 0, \\ u_h(t) &= u_0, \quad \text{for } t \leq 0, \end{aligned} \tag{2.11}$$

has a unique solution u_h and, by then, by the Crandall and Liggett exponential formula (see [3]),

$$u_h(t) \to u(t) \text{ strongly in } L^1(\mathbb{R}^d), \tag{2.12}$$

uniformly on compact intervals.

The function u is called *the mild solution to Eq.* (1.1).

Now, we can formulate the main existence result.

Theorem 2.3 *Under assumptions* (i), *for each* $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, *Eq.* (1.1) *has a unique mild solution* $u \in C([0, T]; L^1(\mathbb{R}^d))$, $\forall T > 0$. *Moreover,* $S(t)u_0 = u(t)$ *is a continuous semigroup of contractions in* $L^1(\mathbb{R}^d)$,

$$|S(t)u_0|_p \leq |u_0|_p, \;\; \forall u_0 \in L^p(\mathbb{R}^d), \;\; 1 \leq p \leq \infty \tag{2.13}$$

$$u(t, x) \geq 0, \;\; \textit{a.e. } x \in \mathbb{R}^d \;\; \textit{if } u_0(x) \geq 0, \;\; \textit{a.e. } x \in \mathbb{R}^d, \tag{2.14}$$

$$\int_{\mathbb{R}^d} u(t, x) dx = \int_{\mathbb{R}^d} u_0(x) dx, \;\; \forall t \geq 0. \tag{2.15}$$

Definition 2.4 The function $u \in C([0, T]; L^1(\mathbb{R}^d))$ is said to be a generalized solution to (1.1) if

$$\begin{aligned} &\frac{\partial u}{\partial t} + \mathrm{div}_x(D(x)u) - \Delta_x \beta(u) = 0 \;\; \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^d), \\ &u(0, x) = u_0(x) \qquad\qquad\qquad\qquad\quad \text{in } \mathbb{R}^d. \end{aligned} \tag{2.16}$$

By (2.11)–(2.12), we see that

Theorem 2.5 *Under assumption* (i)*, for each* $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$*, the mild solution* u *to* (2.1) *is a generalized solution. Moreover, there is at most one generalized solution* $u \in C([0,T]; L^1(\mathbb{R}^d) \cap L^\infty((0,T)\times\mathbb{R}^d)$.

The uniqueness part of Theorem 2.5 follows as in [1] and it will be omitted.

3 Long-Time Dynamical Behavior

Let $S(t)$ be the semigroup of contractions generated on $\overline{D(A)}$ under assumption (i).

Definition 3.1 A real valued function $\psi : L^1(\mathbb{R}^d) \to \mathbb{R}^+$ is a Lyapunov function for $S(t)$ if $\psi(S(t)u_0) \le \psi(u_0),\ \forall t \ge 0,\ \forall u_0 \in D(\psi) \cap \overline{D(A)}$, where $D(\psi) = \{u_0 \in L^1(\mathbb{R}^d);\ \psi(u_0) < \infty\}$.

As we shall see later on, the free energy of the system (the so-called H-functional) is the best candidate for the Lyapunov functions.

Again, by (2.11) and (2.12), we see that

Proposition 3.2 *Assume that* $\psi : L^1(\mathbb{R}^d) \to \mathbb{R}^+$ *is lower semicontinuous and, for all* $\lambda > 0$,

$$\psi((I+\lambda A)^{-1}u_0) \le \psi(u_0),\ \forall u_0 \in D(\psi) \cap \overline{D(A)}. \tag{3.1}$$

Then, ψ *is a Lyapunov function for the semigroup* $S(t)$.

Here, $D(\psi) = \{u;\ \psi(u) < \infty\}$.

We shall look at Lyapunov functions of the form

$$\psi(u) = \int_{\mathbb{R}^d} j(u(x))dx,\ \ \forall u \in L^1(\mathbb{R}^d), \tag{3.2}$$

where $j : \mathbb{R} \to \mathbb{R}^+$ is convex, lower semicontinuous and $j(0) = 0$. Then, as well known, $\psi : L^1(\mathbb{R}^d) \to \mathbb{R}$ is convex, lower semicontinuous and

$$D(\psi) = \{u \in L^1(\mathbb{R}^d);\ j(u) \in L^1(\mathbb{R}^d)\}.$$

We have (see [1]).

Theorem 3.3 *Under hypothesis* (i)*, the function* ψ *is a Lyapunov function for* $S(t)$.

Remark 3.4 In particular, it follows by Proposition 3.2 that the operator A is m-completely accretive in sense of [2].

4 The *H*-Theorem

The so-called H-theorem amounts to saying that, for $t \to \infty$, $S(t)u_0 \to v$, where v is a stationary solution to Eq. (1.1), that is, $Av = 0$ (see [5, 6]).

Since, as easily seen by (2.11), for each ℓ and $u_0 \in \overline{D(A)}$,

$$\int_{\mathbb{R}^d} |(S(t)u_0)(x+\ell) - (S(t)u_0)(x)|dx \le \int_{\mathbb{R}^d} |u_0(x+\ell) - u_0(x)|dx,$$

by the Kolmogorov compactness theorem, the trajectory $\{S(t)u_0;\ t \ge 0\}$ is compact in $L^1_{loc}(R^d)$ and in every $L^p_{loc}(R^d)$, $1 \le p < \infty$, if $u_0 \in L^1(R^d) \cap L^\infty(R^d)$.

Hence the ω-limit set

$$\omega(u_0) = \left\{ v = \lim_{t_n \to \infty} S(t_n)u_0 \text{ in } L^1_{loc}(\mathbb{R}^d) \right\}$$

is nonempty and, if $u_0 \in L^1(R^d) \cap L^\infty(R^d)$, we have also by (2.13) that

$$\omega(u_0) = \left\{ v = \underset{t_n \to \infty}{\text{weak limit}}\ \ S(t_n)u_0 \text{ in each } L^p(\mathbb{R}^d) \right\}.$$

By weak lower-semicontinuity of ψ, we have

$$\psi(v) \le \lim_{t \to \infty} \psi(S(t)u_0), \quad \forall v \in \omega(u_0).$$

If ψ is continuous on L^1 and $\{S(t)u_0,\ t \ge 0\}$ is compact, then we have

$$\psi(S(t)v) \equiv \psi(v),\ \forall t \ge 0,\ v \in \omega(u_0), \tag{4.1}$$

which is a weak form of the H-theorem.

To give a specific example, we assume that

$$D = -\nabla V,\ V \in C^1(\mathbb{R}^d),\ V \ge 0, \tag{4.2}$$

$$\beta \in C^1(\mathbb{R}),\ \beta(0) = 0,\ \beta'(r) > 0,\ \forall r > 0, \tag{4.3}$$

$$\inf\{\|D(x)\|;\ x \in \mathbb{R}^d\} > 0. \tag{4.4}$$

Then, as easily seen by (4.2), (4.3), the energy functional

$$E(u) = \int_{\mathbb{R}^d} (V(x)u(x) - \Phi(u(x)))dx,$$

where Φ is given by (1.7), and so

$$\Phi''(r) = -\frac{\beta'(r)}{r} \ \forall r > 0.$$

We note that E is convex and is a Lyapunov functional for the semigroup $S(t)$. Indeed, taking into account that $\partial E(u) = V - \Phi'(u)$, we get that

$$\begin{aligned} \langle \partial E(u), A_1 u \rangle &= \int_{\mathbb{R}^d} (V - \Phi'(u))(-\Delta\beta(u) + \operatorname{div}(Du))dx \\ &= \int_{\mathbb{R}^d} \left(\frac{(\beta'(u))^2}{u} |\nabla u|^2 + u|\nabla V|^2 \right) dx \geq 0, \\ &\qquad\qquad \forall u \in D(A_1),\ u \geq 0, \end{aligned} \tag{4.5}$$

and, by density, this implies that

$$E(S(t)u_0) \leq \mathbb{E}(u_0),\ \forall t \geq 0. \tag{4.6}$$

Moreover, by (4.4), (4.6) and (2.11), we see that

$$\begin{aligned} &E(u_h(ih)) - E((u_h(i-1)h)) \\ &\quad \leq -h \int_{\mathbb{R}^d} \left(\frac{(\beta'(u_h(ih)))^2}{u_h(ih)} \right) |\nabla u_h(ih)|^2 + u_h(ih)|\nabla V|^2 dx \\ &\quad \leq -\rho \int_{\mathbb{R}^d} u_h(ih)dx,\ \forall i = 1, 2, \ldots, h > 0, \end{aligned}$$

where $\rho > 0$. This yields

$$\frac{1}{t-s}\,(E(S(t)u_0) - E(S(s)u_0)) \leq \rho|S(t)u_0|_1,\ \forall t > s > 0,$$

and, therefore,

$$E(S(t)u_0) \leq \exp(-\rho t)|u_0|_1,\ \forall t \geq 0.$$

Hence, if $u_\infty \in \omega(u_0)$, we have $E(u_\infty) = 0$. Assume further that

$$\inf_{x \in \mathbb{R}^d} V(x) > \sup_{r>0} \frac{\Phi(r)}{r}. \tag{4.7}$$

Then the latter implies that $u_\infty = 0$. We have, therefore,

Theorem 4.1 *Under assumptions* (4.2)–(4.6)*, we have*

$$\lim_{t\to\infty} S(t)u_0 = 0 \text{ in } L^1(\mathbb{R}^d) \text{ for each } u_0 \in L^1(\mathbb{R}^d).$$

5 Final Remarks

For $D = -\nabla V$, we associate with Eq. (1.1) the free energy functional

$$F(u) = \int_{\mathbb{R}^d} \Phi(u)dx + \int_{\mathbb{R}^d} Vu\,dx, \tag{5.1}$$

where Φ satisfies (1.7) and $V \in C^\infty(\mathbb{R}^d)$ is such that

$$V(x) \geq 0,\ |\nabla V(x)| \leq C(V(x)+1),\ \forall x \in \mathbb{R}^d.$$

On the set

$$K = \left\{u : \mathbb{R}^d \to [0,\infty) \text{ measurable}, \int_{\mathbb{R}^d} u(x)dx = 1, \int_{\mathbb{R}^d} |x|^2 u(x)dx < \infty\right\},$$

consider the iterative scheme

$$u_{k+1} = \arg\min_{u\in K} \left\{\frac{1}{2h} d^2(u, u_k) + F(u)\right\}, \tag{5.2}$$

where d is the Wasserstein distance (see, e.g., [6]).

Consider the sequence $u^h : [0, T] \to \mathbb{R}^d$ of the step functions

$$u^h(t) = u_k \text{ for } t \in [kh, (k+1)h],\ k = 0, 1. \tag{5.3}$$

Problem *Does the sequence* $\{u^h\}$ *strongly converge to* $S(t)u_0$ *for* $h \to 0$?

The answer is positive (see [6]) if $\Phi(u) = u \log u$ (the case of Gibbs–Boltzmann entropy), that is, for the Fokker–Planck equation

$$u_t + \operatorname{div}(Du) - \Delta u = 0 \text{ in } (0, T) \times \mathbb{R}^d$$

and one might suspect that it is true in this case for other functions Φ satisfying (1.7).

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