

Lower Bounds for Weak Approximation Errors for Spatial Spectral Galerkin Approximations of Stochastic Wave Equations

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Abstract Although for a number of semilinear stochastic wave equations existence and uniqueness results for corresponding solution processes are known from the literature, these solution processes are typically not explicitly known and numerical approximation methods are needed in order for mathematical modelling with stochastic wave equations to become relevant for real world applications. Therefore, the numerical analysis of convergence rates for such numerical approximation processes is required. A recent article by the authors proves upper bounds for weak errors for spatial spectral Galerkin approximations of a class of semilinear stochastic wave equations. The findings there are complemented by the main result of this work, that provides lower bounds for weak errors which show that in the general framework considered the established upper bounds can essentially not be improved.

Keywords Stochastic wave equations · Weak convergence · Lower bounds · Essentially sharp convergence rates · Spectral Galerkin approximations

Mathematics Subject Classification classes 60H15 · 65C30

1 Introduction

In this work we consider numerical approximation processes of solution processes of stochastic wave equations and examine corresponding weak convergence properties. As opposed to strong convergence, weak convergence even in the case of stochastic evolution equations with regular nonlinearities is still only poorly understood (see, e.g., [3, 6–8, 12] for several weak convergence results for stochastic wave equations and, e.g., the references in Sect. 1 in [4] for further results on weak convergence in the

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literature). Therefore, equations available to current numerical analysis are limited to model problems, such as the ones considered in the present article, that cannot take into account the full complexity of models for evolutionary processes under influence of randomness appearing in real world applications (see, e.g., the references in Sect. 1 in [4]). The recent article [4] by the authors provides upper bounds for weak errors for spatial spectral Galerkin approximations of a class of semilinear stochastic wave equations, including equations driven by multiplicative noise and, in particular, the hyperbolic Anderson model. The main result of this article, Theorem 1.1 below, in turn shows that the weak convergence rates for stochastic wave equations established in Theorem 1.1 in [4] can in the general setting there *essentially not be improved*. Theorem 1.1 is obtained by proving lower bounds for weak errors in the case of concrete examples of stochastic wave equations with additive noise and without drift nonlinearity (cf. Corollary 2.10 and (1.4) below). We argue similarly to the reasoning in Sect. 7 in Conus et al. [1] and Sect. 9 in Jentzen and Kurniawan [5]. First results on lower bounds for strong errors for two examples of stochastic heat equations were achieved in Davie and Gaines [2]. Furthermore, lower bounds for strong errors for examples and whole classes of stochastic heat equations have been established in Müller-Gronbach et al. [10] (see also the references therein) and in Müller-Gronbach and Ritter [9], respectively. Results on lower bounds for weak errors in the case of a few specific examples of stochastic heat equations can be found in Conus et al. [1] and in Jentzen and Kurniawan [5].

Theorem 1.1 *For all real numbers $\eta, T \in (0, \infty)$, every \mathbb{R} -Hilbert space $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$, every orthonormal basis $(e_n)_{n \in \mathbb{N} = \{1, 2, 3, \dots\}}: \mathbb{N} \rightarrow H$ of H , every probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $(\mathbb{F}_t)_{t \in [0, T]}$, and every id_H -cylindrical $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ -Wiener process $(W_t)_{t \in [0, T]}$ there exist a strictly increasing sequence $(\lambda_n)_{n \in \mathbb{N}}: \mathbb{N} \rightarrow (0, \infty)$, a linear operator $A: D(A) \subseteq H \rightarrow H$ with $D(A) = \{v \in H: \sum_{n=1}^{\infty} |\lambda_n \langle e_n, v \rangle_H|^2 < \infty\}$ and $\forall v \in D(A): Av = \sum_{n=1}^{\infty} -\lambda_n \langle e_n, v \rangle_H e_n$, a family of interpolation spaces $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, associated to $-A$ (cf., e.g., [11, Section 3.7]), a family of \mathbb{R} -Hilbert spaces $(\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r})$, $r \in \mathbb{R}$, with $\forall r \in \mathbb{R}: (\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r}) = (H_{r/2} \times H_{r/2-1/2}, \langle \cdot, \cdot \rangle_{H_{r/2} \times H_{r/2-1/2}}, \|\cdot\|_{H_{r/2} \times H_{r/2-1/2}})$, families of functions $P_N: \bigcup_{r \in \mathbb{R}} H_r \rightarrow \bigcup_{r \in \mathbb{R}} H_r$, $N \in \mathbb{N} \cup \{\infty\}$, and $\mathbf{P}_N: \bigcup_{r \in \mathbb{R}} \mathbf{H}_r \rightarrow \bigcup_{r \in \mathbb{R}} \mathbf{H}_r$, $N \in \mathbb{N} \cup \{\infty\}$, with $\forall N \in \mathbb{N} \cup \{\infty\}, r \in \mathbb{R}, u \in H_r, (v, w) \in \mathbf{H}_r: (P_N(u) = \sum_{n=1}^N (\lambda_n)^{-r} \langle e_n, u \rangle_{H_r} (\lambda_n)^{-r} e_n$ and $\mathbf{P}_N(v, w) = (P_N(v), P_N(w))$), a linear operator $\mathbf{A}: D(\mathbf{A}) \subseteq \mathbf{H}_0 \rightarrow \mathbf{H}_0$ with $D(\mathbf{A}) = \mathbf{H}_1$ and $\forall (v, w) \in \mathbf{H}_1: \mathbf{A}(v, w) = (w, Av)$, real numbers $\gamma, c \in (0, \infty)$, a vector $\xi \in \mathbf{H}_\gamma$, and functions $\varphi \in C_b^2(\mathbf{H}_0, \mathbb{R})$, $\mathbf{F} \in C_b^2(\mathbf{H}_0, \mathbf{H}_0)$, $\mathbf{B} \in C_b^2(\mathbf{H}_0, \text{HS}(H, \mathbf{H}_0))$, and $(C_\varepsilon)_{\varepsilon \in (0, \infty)}: (0, \infty) \rightarrow [0, \infty)$ with $\forall \beta \in (\gamma/2, \gamma]: (-A)^{-\beta/2} \in \text{HS}(H)$, $\mathbf{F}(\mathbf{H}_0) \subseteq \mathbf{H}_\gamma$, $(\mathbf{H}_0 \ni v \mapsto \mathbf{F}(v) \in \mathbf{H}_\gamma) \in C_b^2(\mathbf{H}_0, \mathbf{H}_\gamma)$, $\forall v \in \mathbf{H}_0, u \in H: \mathbf{B}(v)u \in \mathbf{H}_\gamma$, $\forall v \in \mathbf{H}_0: (H \ni u \mapsto \mathbf{B}(v)u \in \mathbf{H}_\gamma) \in L(H, \mathbf{H}_\gamma)$, and $(\mathbf{H}_0 \ni v \mapsto (H \ni u \mapsto \mathbf{B}(v)u \in \mathbf{H}_\gamma) \in L(H, \mathbf{H}_\gamma)) \in C_b^2(\mathbf{H}_0, L(H, \mathbf{H}_\gamma))$ such that*

- (i) *it holds that there exist up to modifications unique $(\mathbb{F}_t)_{t \in [0, T]}$ -predictable stochastic processes $\mathbf{X}^N: [0, T] \times \Omega \rightarrow \mathbf{P}_N(\mathbf{H}_0)$, $N \in \mathbb{N} \cup \{\infty\}$, which satisfy for all $N \in \mathbb{N} \cup \{\infty\}, t \in [0, T]$ that $\sup_{s \in [0, T]} \mathbb{E}[\|\mathbf{X}_s^N\|_{\mathbf{H}_0}^2] < \infty$ and \mathbb{P} -a.s. that*

$$\mathbf{X}_t^N = e^{t\mathbf{A}}\mathbf{P}_N\xi + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{P}_N\mathbf{F}(\mathbf{X}_s^N) ds + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{P}_N\mathbf{B}(\mathbf{X}_s^N) dW_s \quad (1.1)$$

(ii) and it holds for all $\varepsilon \in (0, \infty)$, $N \in \mathbb{N}$ that

$$c \cdot (\lambda_N)^{-\eta} \leq |\mathbb{E}[\varphi(\mathbf{X}_T^\infty)] - \mathbb{E}[\varphi(\mathbf{X}_T^N)]| \leq C_\varepsilon \cdot (\lambda_N)^{\varepsilon-\eta}. \quad (1.2)$$

Here and below we denote for every non-trivial \mathbb{R} -Hilbert space $(V, \langle \cdot, \cdot \rangle_V, \|\cdot\|_V)$ and every \mathbb{R} -Hilbert space $(W, \langle \cdot, \cdot \rangle_W, \|\cdot\|_W)$ by $C_b^2(V, W)$ the set of all globally bounded twice continuously Fréchet differentiable functions from V to W with globally bounded derivatives. In the following we provide a few further comments regarding the statement and the proof of Theorem 1.1. The initial value ξ and the functions \mathbf{F} and \mathbf{B} in the setting of Theorem 1.1 can be chosen in such a way that there exist appropriate $\xi_0 \in H$, $\xi_1 \in H_{-1/2}$ and appropriate functions $F: \mathbf{H}_0 \rightarrow H_{-1/2}$, $B: \mathbf{H}_0 \rightarrow \text{HS}(H, H_{-1/2})$ such that $\xi = (\xi_0, \xi_1)$, $\mathbf{F} = (0, F)$, and $\mathbf{B} = (0, B)$. In this case, for every $N \in \mathbb{N} \cup \{\infty\}$ the first component process $X^N: [0, T] \times \Omega \rightarrow P_N(H)$ of \mathbf{X}^N is, roughly speaking, a mild solution of the stochastic wave-type evolution equation

$$\ddot{X}_t = AX_t + P_N F(X_t, \dot{X}_t) + P_N B(X_t, \dot{X}_t) \dot{W}_t \quad (1.3)$$

with $X_0 = P_N \xi_0$, $\dot{X}_0 = P_N \xi_1$ for $t \in [0, T]$. Theorem 1.1 is a direct consequence of Theorem 1.1 in [4] (with $\gamma = 2\eta$, $\beta = \min\{\eta + \varepsilon, 2\eta\}$, $\rho = 0$ in the notation of Theorem 1.1 in [4]) and Corollary 2.10 below (with $p = 1/\eta$, $\delta = 1/2 - \eta$ in the notation of Corollary 2.10 below). In the case $\eta = 1/2$, the lower bound in (1.2) is obtained, for example, for the stochastic wave equations

$$\ddot{X}_t(x) = \frac{\partial^2}{\partial x^2} X_t(x) + P_N \dot{W}_t(x) \quad (1.4)$$

with $X_0(x) = \dot{X}_0(x) = 0$ and $X_t(0) = X_t(1) = 0$ for $x \in (0, 1)$, $t \in [0, T]$, $N \in \mathbb{N} \cup \{\infty\}$, corresponding to the choices $H = L^2((0, 1); \mathbb{R})$, $\forall n \in \mathbb{N}: e_n = \sqrt{2} \sin(n\pi(\cdot)) \in H, \forall n \in \mathbb{N}: \lambda_n = \pi^2 n^2, \xi = 0, \mathbf{F} = 0, \mathbf{B} = (\mathbf{H}_0 \ni (v, w) \mapsto (H \ni u \mapsto (0, u) \in \mathbf{H}_0) \in \text{HS}(H, \mathbf{H}_0))$ in the setting of Theorem 1.1 (cf. Corollary 2.11 below). Inequality (1.2) reveals that the weak convergence rates in Theorem 1.1 in [4] are essentially sharp. More details and further lower bounds for weak approximation errors for stochastic wave equations can be found in Corollary 2.8 and Corollary 2.10 below.

2 Lower Bounds for Weak Errors

2.1 Setting

Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be a separable \mathbb{R} -Hilbert space, for every set A let $\mathcal{P}(A)$ be the power set of A , let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal fil-

tration $(\mathbb{F}_t)_{t \in [0, T]}$, let $(W_t)_{t \in [0, T]}$ be an id_H -cylindrical $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ -Wiener process, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H , let $\lambda: \mathbb{H} \rightarrow \mathbb{R}$ be a function with $\sup_{h \in \mathbb{H}} \lambda_h < \infty$, let $A: D(A) \subseteq H \rightarrow H$ be the linear operator which satisfies $D(A) = \{v \in H: \sum_{h \in \mathbb{H}} |\lambda_h \langle h, v \rangle_H|^2 < \infty\}$ and $\forall v \in D(A): Av = \sum_{h \in \mathbb{H}} \lambda_h \langle h, v \rangle_H h$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $(\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r})$, $r \in \mathbb{R}$, be the family of \mathbb{R} -Hilbert spaces which satisfies for all $r \in \mathbb{R}$ that $(\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r}) = (H_{r/2} \times H_{r/2-1/2}, \langle \cdot, \cdot \rangle_{H_{r/2} \times H_{r/2-1/2}}, \|\cdot\|_{H_{r/2} \times H_{r/2-1/2}})$, let $P_I: \bigcup_{r \in \mathbb{R}} H_r \rightarrow \bigcup_{r \in \mathbb{R}} H_r$, $I \in \mathcal{P}(\mathbb{H})$, and $\mathbf{P}_I: \bigcup_{r \in \mathbb{R}} \mathbf{H}_r \rightarrow \bigcup_{r \in \mathbb{R}} \mathbf{H}_r$, $I \in \mathcal{P}(\mathbb{H})$, be the functions which satisfy for all $I \in \mathcal{P}(\mathbb{H})$, $r \in \mathbb{R}$, $u \in H_r$, $(v, w) \in \mathbf{H}_r$ that $P_I(u) = \sum_{h \in I} \langle |\lambda_h|^{-r} h, u \rangle_{H_r} |\lambda_h|^{-r} h$ and $\mathbf{P}_I(v, w) = (P_I(v), P_I(w))$, let $\mathbf{A}: D(\mathbf{A}) \subseteq \mathbf{H}_0 \rightarrow \mathbf{H}_0$ be the linear operator which satisfies $D(\mathbf{A}) = \mathbf{H}_1$ and $\forall (v, w) \in \mathbf{H}_1: \mathbf{A}(v, w) = (w, Av)$, let $\mu: \mathbb{H} \rightarrow \mathbb{R}$ be a function which satisfies $\sum_{h \in \mathbb{H}} \frac{|\mu_h|^2}{|\lambda_h|} < \infty$, let $\mathbf{B} \in \text{HS}(H, \mathbf{H}_0)$ be the linear operator which satisfies for all $v \in H$ that $\mathbf{B}v = (0, \sum_{h \in \mathbb{H}} \mu_h \langle h, v \rangle_H h)$, and let $\mathbf{X}^I = (X^{I,1}, X^{I,2}): \Omega \rightarrow \mathbf{P}_I(\mathbf{H}_0)$, $I \in \mathcal{P}(\mathbb{H})$, be random variables which satisfy for all $I \in \mathcal{P}(\mathbb{H})$ that it holds \mathbb{P} -a.s. that $\mathbf{X}^I = \int_0^T e^{(T-s)\mathbf{A}} \mathbf{P}_I \mathbf{B} dW_s$.

2.2 Lower Bounds for the Squared Norm

Lemma 2.1 *Assume the setting in Sect. 2.1. Then for all $I \in \mathcal{P}(\mathbb{H})$ it holds \mathbb{P} -a.s. that*

$$\begin{aligned} \mathbf{X}^I &= \mathbf{P}_I \mathbf{X}^{\mathbb{H}} = \begin{pmatrix} X^{I,1} \\ X^{I,2} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{h \in I} \left(\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \sin(|\lambda_h|^{1/2}(T-s)) d\langle h, W_s \rangle_H \right) h \\ \sum_{h \in I} \left(\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \cos(|\lambda_h|^{1/2}(T-s)) d\langle h, W_s \rangle_H \right) |\lambda_h|^{1/2} h \end{pmatrix}. \end{aligned} \tag{2.1}$$

Proof of Lemma 2.1. Lemma 2.5 in [4] proves that it holds \mathbb{P} -a.s. that

$$\begin{aligned} \mathbf{X}^{\mathbb{H}} &= \int_0^T e^{(T-s)\mathbf{A}} \mathbf{B} dW_s = \sum_{h \in \mathbb{H}} \int_0^T e^{(T-s)\mathbf{A}} \mathbf{B} h d\langle h, W_s \rangle_H \\ &= \sum_{h \in \mathbb{H}} \begin{pmatrix} \left(\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \sin(|\lambda_h|^{1/2}(T-s)) h d\langle h, W_s \rangle_H \right) \\ \left(\mu_h \int_0^T \cos(|\lambda_h|^{1/2}(T-s)) h d\langle h, W_s \rangle_H \right) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{h \in \mathbb{H}} \left(\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \sin(|\lambda_h|^{1/2}(T-s)) d\langle h, W_s \rangle_H \right) h \\ \sum_{h \in \mathbb{H}} \left(\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \cos(|\lambda_h|^{1/2}(T-s)) d\langle h, W_s \rangle_H \right) |\lambda_h|^{1/2} h \end{pmatrix}. \end{aligned} \tag{2.2}$$

Furthermore, Lemma 2.7 in [4] shows for all $I \in \mathcal{P}(\mathbb{H})$ that it holds \mathbb{P} -a.s. that

$$\mathbf{P}_I \mathbf{X}^{\mathbb{H}} = \int_0^T \mathbf{P}_I e^{(T-s)\mathbf{A}} \mathbf{B} dW_s = \int_0^T e^{(T-s)\mathbf{A}} \mathbf{P}_I \mathbf{B} dW_s = \mathbf{X}^I. \tag{2.3}$$

This and (2.2) complete the proof of Lemma 2.1. \square

Lemma 2.2 *Assume the setting in Sect. 2.1 and let $I \in \mathcal{P}(\mathbb{H})$. Then*

- (i) *it holds that $\langle h, X^{I,1} \rangle_{H_0}$, $h \in \mathbb{H}$, is a family of independent centred Gaussian random variables,*
- (ii) *it holds that $\langle |\lambda_h|^{1/2} h, X^{I,2} \rangle_{H_{-1/2}}$, $h \in \mathbb{H}$, is a family of independent centred Gaussian random variables, and*
- (iii) *it holds for all $h \in \mathbb{H}$ that*

$$\text{Var}(\langle h, X^{I,1} \rangle_{H_0}) = \mathbb{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \frac{1}{2} \left(T - \frac{\sin(2|\lambda_h|^{1/2}T)}{2|\lambda_h|^{1/2}} \right), \quad (2.4)$$

$$\text{Var}(\langle |\lambda_h|^{1/2} h, X^{I,2} \rangle_{H_{-1/2}}) = \mathbb{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \frac{1}{2} \left(T + \frac{\sin(2|\lambda_h|^{1/2}T)}{2|\lambda_h|^{1/2}} \right), \quad (2.5)$$

$$\text{Cov}(\langle h, X^{I,1} \rangle_{H_0}, \langle |\lambda_h|^{1/2} h, X^{I,2} \rangle_{H_{-1/2}}) = \mathbb{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \left(\frac{1 - \cos(2|\lambda_h|^{1/2}T)}{4|\lambda_h|^{1/2}} \right). \quad (2.6)$$

Proof of Lemma 2.2. Observe that Lemma 2.1 implies (i) and (ii). It thus remains to prove (iii). Lemma 2.1 assures for all $h \in \mathbb{H}$ that it holds \mathbb{P} -a.s. that

$$\langle h, X^{I,1} \rangle_{H_0} = \mathbb{1}_I(h) \frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \sin(|\lambda_h|^{1/2}(T-s)) \, d\langle h, W_s \rangle_H, \quad (2.7)$$

$$\langle |\lambda_h|^{1/2} h, X^{I,2} \rangle_{H_{-1/2}} = \mathbb{1}_I(h) \frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \cos(|\lambda_h|^{1/2}(T-s)) \, d\langle h, W_s \rangle_H. \quad (2.8)$$

Itô's isometry hence shows for all $h \in \mathbb{H}$ that

$$\begin{aligned} \text{Var}(\langle h, X^{I,1} \rangle_{H_0}) &= \mathbb{E}[\langle h, X^{I,1} \rangle_{H_0}^2] \\ &= \mathbb{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \int_0^T |\sin(|\lambda_h|^{1/2}(T-s))|^2 \, ds \\ &= \mathbb{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \frac{1}{2} \left(T - \frac{\sin(2|\lambda_h|^{1/2}T)}{2|\lambda_h|^{1/2}} \right), \end{aligned} \quad (2.9)$$

$$\begin{aligned} \text{Var}(\langle |\lambda_h|^{1/2} h, X^{I,2} \rangle_{H_{-1/2}}) &= \mathbb{E}[\langle |\lambda_h|^{1/2} h, X^{I,2} \rangle_{H_{-1/2}}^2] \\ &= \mathbb{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \int_0^T |\cos(|\lambda_h|^{1/2}(T-s))|^2 \, ds \\ &= \mathbb{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \frac{1}{2} \left(T + \frac{\sin(2|\lambda_h|^{1/2}T)}{2|\lambda_h|^{1/2}} \right). \end{aligned} \quad (2.10)$$

Furthermore, observe that it holds for all $h \in \mathbb{H}$ that

$$\begin{aligned} \text{Cov}\left(\langle h, X^{I,1} \rangle_{H_0}, \langle |\lambda_h|^{1/2} h, X^{I,2} \rangle_{H_{-1/2}}\right) &= \mathbb{E}\left[\langle h, X^{I,1} \rangle_{H_0} \langle |\lambda_h|^{1/2} h, X^{I,2} \rangle_{H_{-1/2}}\right] \\ &= \mathbf{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \int_0^T \sin(|\lambda_h|^{1/2}(T-s)) \cos(|\lambda_h|^{1/2}(T-s)) \, ds \\ &= \mathbf{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \left(\frac{|\sin(|\lambda_h|^{1/2}T)|^2}{2|\lambda_h|^{1/2}} \right) \\ &= \mathbf{1}_I(h) \frac{|\mu_h|^2}{|\lambda_h|} \left(\frac{1 - \cos(2|\lambda_h|^{1/2}T)}{4|\lambda_h|^{1/2}} \right). \end{aligned} \tag{2.11}$$

The proof of Lemma 2.2 is thus completed. □

Lemma 2.3 *Assume the setting in Sect. 2.1 and let $I \in \mathcal{P}(\mathbb{H})$. Then it holds for all $i \in \{1, 2\}$ that*

$$\mathbb{E}[\|\mathbf{X}^I\|_{\mathbf{H}_0}^2] = T \sum_{h \in I} \frac{|\mu_h|^2}{|\lambda_h|} < \infty, \tag{2.12}$$

$$\mathbb{E}[\|\mathbf{X}^{I,i}\|_{H_{1/2-i/2}}^2] = \frac{1}{2} \sum_{h \in I} \frac{|\mu_h|^2}{|\lambda_h|} \left(T + \frac{\sin(2|\lambda_h|^{1/2}T)}{(-1)^i 2|\lambda_h|^{1/2}} \right) < \infty. \tag{2.13}$$

Proof of Lemma 2.3. Itô’s isometry and Lemma 2.6 in [4] imply that

$$\begin{aligned} \mathbb{E}[\|\mathbf{X}^I\|_{\mathbf{H}_0}^2] &= \mathbb{E}\left[\left\| \int_0^T e^{(T-s)\mathbf{A}} \mathbf{P}_I \mathbf{B} \, dW_s \right\|_{\mathbf{H}_0}^2\right] \\ &= T \|\mathbf{P}_I \mathbf{B}\|_{\text{HS}(H, \mathbf{H}_0)}^2 = T \sum_{h \in I} \frac{|\mu_h|^2}{|\lambda_h|} < \infty. \end{aligned} \tag{2.14}$$

In addition, Lemma 2.2 shows for all $i \in \{1, 2\}$ that

$$\begin{aligned} \mathbb{E}[\|\mathbf{X}^{I,i}\|_{H_{1/2-i/2}}^2] &= \sum_{h \in \mathbb{H}} \mathbb{E}\left[\left| \langle |\lambda_h|^{i/2-1/2} h, X^{I,i} \rangle_{H_{1/2-i/2}} \right|^2\right] \\ &= \frac{1}{2} \sum_{h \in I} \frac{|\mu_h|^2}{|\lambda_h|} \left(T + \frac{\sin(2|\lambda_h|^{1/2}T)}{(-1)^i 2|\lambda_h|^{1/2}} \right) < \infty. \end{aligned} \tag{2.15}$$

The proof of Lemma 2.3 is thus completed. □

Corollary 2.4 *Assume the setting in Sect. 2.1 and let $I \in \mathcal{P}(\mathbb{H})$. Then it holds for all $(v, w) \in \mathbf{P}_I(\mathbf{H}_0)$ that*

$$\begin{aligned}
\text{CovOp}(\mathbf{X}^I) \begin{pmatrix} v \\ w \end{pmatrix} &= \frac{1}{2} \sum_{h \in I} \frac{|\mu_h|^2}{|\lambda_h|} \left[\left(T - \frac{\sin(2|\lambda_h|^{1/2}T)}{2|\lambda_h|^{1/2}} \right) \langle h, v \rangle_{H_0} \begin{pmatrix} h \\ 0 \end{pmatrix} \right. \\
&\quad + \left(\frac{1 - \cos(2|\lambda_h|^{1/2}T)}{2|\lambda_h|^{1/2}} \right) \langle |\lambda_h|^{1/2}h, w \rangle_{H_{-1/2}} \begin{pmatrix} h \\ 0 \end{pmatrix} \\
&\quad + \left(\frac{1 - \cos(2|\lambda_h|^{1/2}T)}{2|\lambda_h|^{1/2}} \right) \langle h, v \rangle_{H_0} \begin{pmatrix} 0 \\ |\lambda_h|^{1/2}h \end{pmatrix} \\
&\quad \left. + \left(T + \frac{\sin(2|\lambda_h|^{1/2}T)}{2|\lambda_h|^{1/2}} \right) \langle |\lambda_h|^{1/2}h, w \rangle_{H_{-1/2}} \begin{pmatrix} 0 \\ |\lambda_h|^{1/2}h \end{pmatrix} \right] \in \mathbf{P}_I(\mathbf{H}_0).
\end{aligned} \tag{2.16}$$

Proof of Corollary 2.4. Lemma 2.1, and Lemma 2.2 prove for all $x_1 = (v_1, w_1)$, $x_2 = (v_2, w_2) \in \mathbf{P}_I(\mathbf{H}_0)$ that

$$\begin{aligned}
\langle x_1, \text{CovOp}(\mathbf{X}^I)x_2 \rangle_{\mathbf{H}_0} &= \text{Cov}(\langle x_1, \mathbf{X}^I \rangle_{\mathbf{H}_0}, \langle x_2, \mathbf{X}^I \rangle_{\mathbf{H}_0}) = \mathbb{E}[\langle x_1, \mathbf{X}^I \rangle_{\mathbf{H}_0} \langle x_2, \mathbf{X}^I \rangle_{\mathbf{H}_0}] \\
&= \mathbb{E}[\langle (v_1, X^{I,1})_{H_0} + (w_1, X^{I,2})_{H_{-1/2}} \rangle \langle (v_2, X^{I,1})_{H_0} + (w_2, X^{I,2})_{H_{-1/2}} \rangle] \\
&= \sum_{h \in \mathbb{H}} \text{Var}(\langle h, X^{I,1} \rangle_{H_0}) \langle h, v_1 \rangle_{H_0} \langle h, v_2 \rangle_{H_0} \\
&\quad + \sum_{h \in \mathbb{H}} \text{Cov}(\langle h, X^{I,1} \rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2} \rangle_{H_{-1/2}}) \langle h, v_1 \rangle_{H_0} \langle |\lambda_h|^{1/2}h, w_2 \rangle_{H_{-1/2}} \\
&\quad + \sum_{h \in \mathbb{H}} \text{Cov}(\langle h, X^{I,1} \rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2} \rangle_{H_{-1/2}}) \langle h, v_2 \rangle_{H_0} \langle |\lambda_h|^{1/2}h, w_1 \rangle_{H_{-1/2}} \\
&\quad + \sum_{h \in \mathbb{H}} \text{Var}(\langle |\lambda_h|^{1/2}h, X^{I,2} \rangle_{H_{-1/2}}) \langle |\lambda_h|^{1/2}h, w_1 \rangle_{H_{-1/2}} \langle |\lambda_h|^{1/2}h, w_2 \rangle_{H_{-1/2}} \\
&= \left\langle v_1, \sum_{h \in \mathbb{H}} \left[\text{Var}(\langle h, X^{I,1} \rangle_{H_0}) \langle h, v_2 \rangle_{H_0} \right. \right. \\
&\quad \left. \left. + \text{Cov}(\langle h, X^{I,1} \rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2} \rangle_{H_{-1/2}}) \langle |\lambda_h|^{1/2}h, w_2 \rangle_{H_{-1/2}} \right] h \right\rangle_{H_0} \tag{2.17} \\
&\quad + \left\langle w_1, \sum_{h \in \mathbb{H}} \left[\text{Cov}(\langle h, X^{I,1} \rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2} \rangle_{H_{-1/2}}) \langle h, v_2 \rangle_{H_0} \right. \right. \\
&\quad \left. \left. + \text{Var}(\langle |\lambda_h|^{1/2}h, X^{I,2} \rangle_{H_{-1/2}}) \langle |\lambda_h|^{1/2}h, w_2 \rangle_{H_{-1/2}} \right] |\lambda_h|^{1/2}h \right\rangle_{H_{-1/2}} \\
&= \left\langle x_1, \sum_{h \in \mathbb{H}} \left[\text{Var}(\langle h, X^{I,1} \rangle_{H_0}) \langle h, v_2 \rangle_{H_0} \right. \right. \\
&\quad \left. \left. + \text{Cov}(\langle h, X^{I,1} \rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2} \rangle_{H_{-1/2}}) \langle |\lambda_h|^{1/2}h, w_2 \rangle_{H_{-1/2}} \right] \begin{pmatrix} h \\ 0 \end{pmatrix} \right\rangle_{\mathbf{H}_0} \\
&\quad + \left\langle x_1, \sum_{h \in \mathbb{H}} \left[\text{Cov}(\langle h, X^{I,1} \rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2} \rangle_{H_{-1/2}}) \langle h, v_2 \rangle_{H_0} \right. \right.
\end{aligned}$$

$$+ \text{Var}\left(\langle |\lambda_h|^{1/2}h, X^{I,2} \rangle_{H_{-1/2}} \langle |\lambda_h|^{1/2}h, w_2 \rangle_{H_{-1/2}} \right) \left(\begin{matrix} 0 \\ |\lambda_h|^{1/2}h \end{matrix} \right)_{\mathbf{H}_0} \Bigg\}.$$

This and again Lemma 2.2 complete the proof of Corollary 2.4. □

Proposition 2.5 *Assume the setting in Sect. 2.1. Then it holds for all $I \in \mathcal{P}(\mathbb{H})$ that*

$$\mathbb{E}[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2] - \mathbb{E}[\|\mathbf{X}^I\|_{\mathbf{H}_0}^2] = \mathbb{E}[\|\mathbf{X}^{\mathbb{H} \setminus I}\|_{\mathbf{H}_0}^2] \geq T \inf_{h \in \mathbb{H}} |\mu_h|^2 \sum_{h \in \mathbb{H} \setminus I} \frac{1}{|\lambda_h|}. \tag{2.18}$$

Proof of Proposition 2.5. Orthogonality and Lemma 2.1 imply for all $I \in \mathcal{P}(\mathbb{H})$ that

$$\begin{aligned} \mathbb{E}[\|\mathbf{X}^I\|_{\mathbf{H}_0}^2] + \mathbb{E}[\|\mathbf{X}^{\mathbb{H} \setminus I}\|_{\mathbf{H}_0}^2] &= \mathbb{E}[\|\mathbf{P}_I \mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2] + \mathbb{E}[\|\mathbf{P}_{\mathbb{H} \setminus I} \mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2] \\ &= \mathbb{E}[\|(\mathbf{P}_I + \mathbf{P}_{\mathbb{H} \setminus I}) \mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2] = \mathbb{E}[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2]. \end{aligned} \tag{2.19}$$

This and Lemma 2.3 show for all $I \in \mathcal{P}(\mathbb{H})$ that

$$\begin{aligned} \mathbb{E}[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2] - \mathbb{E}[\|\mathbf{X}^I\|_{\mathbf{H}_0}^2] &= \mathbb{E}[\|\mathbf{X}^{\mathbb{H} \setminus I}\|_{\mathbf{H}_0}^2] \\ &= T \sum_{h \in \mathbb{H} \setminus I} \frac{|\mu_h|^2}{|\lambda_h|} \geq T \inf_{h \in \mathbb{H}} |\mu_h|^2 \sum_{h \in \mathbb{H} \setminus I} \frac{1}{|\lambda_h|}. \end{aligned} \tag{2.20}$$

The proof of Proposition 2.5 is thus completed. □

In Corollary 2.7 and Corollary 2.8 below lower bounds on the weak approximation error with the squared norm as test function are presented. Our proofs of Corollary 2.7 and Corollary 2.8 use the following elementary and well-known lemma (cf., e.g., Proposition 7.4 in Conus et al. [1]).

Lemma 2.6 *Let $p \in (0, \infty)$, $\delta \in (-\infty, 1/2 - 1/(2p))$. Then it holds for all $N \in \mathbb{N}$ that*

$$\sum_{n=N+1}^{\infty} n^{p(2\delta-1)} \geq \frac{N^{p(2\delta-1)+1}}{[p(1-2\delta) - 1]2^{p(1-2\delta)-1}}. \tag{2.21}$$

Proof of Lemma 2.6. Observe that the assumption that $\delta \in (-\infty, 1/2 - 1/(2p))$ ensures that $p(2\delta - 1) \in (-\infty, -1)$. This implies for all $N \in \mathbb{N}$ that

$$\begin{aligned} \sum_{n=N+1}^{\infty} n^{p(2\delta-1)} &= \sum_{n=N+1}^{\infty} \int_n^{n+1} n^{p(2\delta-1)} dx \geq \sum_{n=N+1}^{\infty} \int_n^{n+1} x^{p(2\delta-1)} dx \\ &= \int_{N+1}^{\infty} x^{p(2\delta-1)} dx = -\frac{(N+1)^{p(2\delta-1)+1}}{p(2\delta-1)+1} \\ &\geq \frac{N^{p(2\delta-1)+1}}{[p(1-2\delta) - 1]2^{p(1-2\delta)-1}}. \end{aligned} \tag{2.22}$$

This completes the proof of Lemma 2.6. \square

Corollary 2.7 *Assume the setting in Sect. 2.1, let $c \in (0, \infty)$, $p \in (1, \infty)$, let $e: \mathbb{N} \rightarrow \mathbb{H}$ be a bijection which satisfies for all $n \in \mathbb{N}$ that $\lambda_{e_n} = -cn^p$, and let $I_N \in \mathcal{P}(\mathbb{H})$, $N \in \mathbb{N}$, be the sets which satisfy for all $N \in \mathbb{N}$ that $I_N = \{e_1, e_2, \dots, e_N\} \subseteq \mathbb{H}$. Then it holds for all $N \in \mathbb{N}$ that*

$$\mathbb{E}[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2] - \mathbb{E}[\|\mathbf{X}^{I_N}\|_{\mathbf{H}_0}^2] \geq \frac{T \inf_{h \in \mathbb{H}} |\mu_h|^2 N^{1-p}}{c(p-1)2^{p-1}}. \quad (2.23)$$

Proof of Corollary 2.7. Proposition 2.5 and Lemma 2.6 prove for all $N \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E}[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2] - \mathbb{E}[\|\mathbf{X}^{I_N}\|_{\mathbf{H}_0}^2] &\geq T \inf_{h \in \mathbb{H}} |\mu_h|^2 \sum_{h \in \mathbb{H} \setminus I_N} \frac{1}{|\lambda_h|} = c^{-1} T \inf_{h \in \mathbb{H}} |\mu_h|^2 \sum_{n=N+1}^{\infty} \frac{1}{n^p} \\ &\geq \frac{T \inf_{h \in \mathbb{H}} |\mu_h|^2 N^{1-p}}{c(p-1)2^{p-1}}. \end{aligned} \quad (2.24)$$

The proof of Corollary 2.7 is thus completed. \square

Corollary 2.8 *Assume the setting in Sect. 2.1, let $c, p \in (0, \infty)$, $\delta \in (-\infty, 1/2 - 1/(2p))$, let $e: \mathbb{N} \rightarrow \mathbb{H}$ be a bijection which satisfies for all $n \in \mathbb{N}$ that $\lambda_{e_n} = -cn^p$, let $I_N \in \mathcal{P}(\mathbb{H})$, $N \in \mathbb{N}$, be the sets which satisfy for all $N \in \mathbb{N}$ that $I_N = \{e_1, e_2, \dots, e_N\} \subseteq \mathbb{H}$, and assume for all $h \in \mathbb{H}$ that $|\mu_h| = |\lambda_h|^\delta$. Then it holds for all $N \in \mathbb{N}$ that*

$$\mathbb{E}[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2] - \mathbb{E}[\|\mathbf{X}^{I_N}\|_{\mathbf{H}_0}^2] \geq \frac{T c^{2\delta-1} N^{p(2\delta-1)+1}}{[p(1-2\delta) - 1]2^{p(1-2\delta)-1}}. \quad (2.25)$$

Proof of Corollary 2.8. Proposition 2.5, Lemma 2.3, and Lemma 2.6 show for all $N \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E}[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2] - \mathbb{E}[\|\mathbf{X}^{I_N}\|_{\mathbf{H}_0}^2] &= T \sum_{h \in \mathbb{H} \setminus I_N} \frac{|\mu_h|^2}{|\lambda_h|} = T \sum_{h \in \mathbb{H} \setminus I_N} |\lambda_h|^{2\delta-1} \\ &= T c^{2\delta-1} \sum_{n=N+1}^{\infty} n^{p(2\delta-1)} \geq \frac{T c^{2\delta-1} N^{p(2\delta-1)+1}}{[p(1-2\delta) - 1]2^{p(1-2\delta)-1}}. \end{aligned} \quad (2.26)$$

This completes the proof of Corollary 2.8. \square

2.3 Lower Bounds for the Weak Error of a Particular Regular Test Function

The next result, Proposition 2.9 below, follows directly from Lemma 2.2, and Lemma 2.3 above and Lemma 9.5 in Jentzen and Kurniawan [5].

Proposition 2.9 *Assume the setting in Sect. 2.1 and let $\varphi_i : \mathbf{H}_0 \rightarrow \mathbb{R}$, $i \in \{1, 2\}$, be the functions which satisfy for all $i \in \{1, 2\}$, $(v_1, v_2) \in \mathbf{H}_0$ that $\varphi_i(v_1, v_2) = \exp(-\|v_i\|_{H_{1/2-i/2}}^2)$. Then it holds for all $i \in \{1, 2\}$, $I \in \mathcal{P}(\mathbb{H})$ that $\varphi_i \in C_b^2(\mathbf{H}_0, \mathbb{R})$ and*

$$\mathbb{E}[\varphi_i(\mathbf{X}^I)] - \mathbb{E}[\varphi_i(\mathbf{X}^{\mathbb{H}})] \geq \frac{\mathbb{E}[\|X^{\mathbb{H},i}\|_{H_{1/2-i/2}}^2] - \mathbb{E}[\|X^{I,i}\|_{H_{1/2-i/2}}^2]}{\exp\left(6\mathbb{E}[\|X^{\mathbb{H},i}\|_{H_{1/2-i/2}}^2]\right)}. \tag{2.27}$$

Corollary 2.10 *Assume the setting in Sect. 2.1, let $c, p \in (0, \infty)$, $\delta \in (-\infty, 1/2 - 1/(2p))$, let $e : \mathbb{N} \rightarrow \mathbb{H}$ be a bijection which satisfies for all $n \in \mathbb{N}$ that $\lambda_{e_n} = -cn^p$, let $I_N \in \mathcal{P}(\mathbb{H})$, $N \in \mathbb{N}$, be the sets which satisfy for all $N \in \mathbb{N}$ that $I_N = \{e_1, e_2, \dots, e_N\} \subseteq \mathbb{H}$, assume for all $h \in \mathbb{H}$ that $|\mu_h| = |\lambda_h|^\delta$, and let $\varphi_i : \mathbf{H}_0 \rightarrow \mathbb{R}$, $i \in \{1, 2\}$, be the functions which satisfy for all $i \in \{1, 2\}$, $(v_1, v_2) \in \mathbf{H}_0$ that $\varphi_i(v_1, v_2) = \exp(-\|v_i\|_{H_{1/2-i/2}}^2)$. Then it holds for all $i \in \{1, 2\}$, $N \in \mathbb{N}$ that $\varphi_i \in C_b^2(\mathbf{H}_0, \mathbb{R})$ and*

$$\begin{aligned} & \mathbb{E}[\varphi_i(\mathbf{X}^{I_N})] - \mathbb{E}[\varphi_i(\mathbf{X}^{\mathbb{H}})] \\ & \geq \left[1 + \inf_{x \in [2c^{1/2}T, \infty)} \frac{\sin(x)}{(-1)^i x} \right] \frac{Tc^{2\delta-1}2^{p(2\delta-1)}N^{p(2\delta-1)+1}}{[p(1-2\delta) - 1] \exp\left(\frac{6p(2\delta-1)Tc^{2\delta-1}}{p(2\delta-1)+1}\right)} > 0. \end{aligned} \tag{2.28}$$

Proof of Corollary 2.10. Lemma 2.3, and Lemma 2.6, and the fact that $\forall x \in (0, \infty) : \left| \frac{\sin(x)}{x} \right| < 1$ prove for all $i \in \{1, 2\}$, $N \in \mathbb{N}$ that

$$\begin{aligned} & \mathbb{E}[\|X^{\mathbb{H},i}\|_{H_{1/2-i/2}}^2] - \mathbb{E}[\|X^{I_N,i}\|_{H_{1/2-i/2}}^2] = \frac{1}{2} \sum_{h \in \mathbb{H} \setminus I_N} \frac{|\mu_h|^2}{|\lambda_h|} \left(T + \frac{\sin(2|\lambda_h|^{1/2}T)}{(-1)^i 2|\lambda_h|^{1/2}} \right) \\ & \geq \left(1 + \inf_{h \in \mathbb{H}} \frac{\sin(2|\lambda_h|^{1/2}T)}{(-1)^i 2|\lambda_h|^{1/2}T} \right) \frac{T}{2} \sum_{h \in \mathbb{H} \setminus I_N} |\lambda_h|^{2\delta-1} \\ & \geq \left(1 + \inf_{x \in [2c^{1/2}T, \infty)} \frac{\sin(x)}{(-1)^i x} \right) \frac{Tc^{2\delta-1}}{2} \sum_{n=N+1}^{\infty} n^{p(2\delta-1)} \\ & \geq \left(1 + \inf_{x \in [2c^{1/2}T, \infty)} \frac{\sin(x)}{(-1)^i x} \right) \frac{Tc^{2\delta-1}2^{p(2\delta-1)}N^{p(2\delta-1)+1}}{[p(1-2\delta) - 1]} > 0. \end{aligned} \tag{2.29}$$

Furthermore, note that the fact that $p(2\delta - 1) \in (-\infty, -1)$ ensures that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{p(2\delta-1)} &\leq 1 + \sum_{n=1}^{\infty} \int_n^{n+1} x^{p(2\delta-1)} dx = 1 + \int_1^{\infty} x^{p(2\delta-1)} dx \\ &= 1 - \frac{1}{p(2\delta - 1) + 1} = \frac{p(2\delta - 1)}{p(2\delta - 1) + 1}. \end{aligned} \tag{2.30}$$

Lemma 2.3 hence implies for all $i \in \{1, 2\}$ that

$$\begin{aligned} \exp\left(-6 \mathbb{E}\left[\|X^{\mathbb{H},i}\|_{H_{1/2-j/2}}^2\right]\right) &\geq \exp\left(-6 \mathbb{E}\left[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2\right]\right) = \exp\left(-6T c^{2\delta-1} \sum_{n=1}^{\infty} n^{p(2\delta-1)}\right) \\ &\geq \exp\left(-\frac{6p(2\delta - 1)Tc^{2\delta-1}}{p(2\delta - 1) + 1}\right) > 0. \end{aligned} \tag{2.31}$$

Combining this and (2.29) with Proposition 2.9 concludes the proof of Corollary 2.10. \square

Roughly speaking, Corollary 2.11 below specifies Corollary 2.10 to the case where the linear operator $A : D(A) \subseteq H \rightarrow H$ in the setting in Sect. 2.1 is the Laplacian with Dirichlet boundary conditions on $H = L^2((0, 1); \mathbb{R})$. Corollary 2.11 is an immediate consequence of Corollary 2.10.

Corollary 2.11 *Assume the setting in Sect. 2.1, let $\delta \in (-\infty, 1/4)$, let $e : \mathbb{N} \rightarrow \mathbb{H}$ be a bijection which satisfies for all $n \in \mathbb{N}$ that $\lambda_{e_n} = -\pi^2 n^2$, let $I_N \in \mathcal{P}(\mathbb{H})$, $N \in \mathbb{N}$, be the sets which satisfy for all $N \in \mathbb{N}$ that $I_N = \{e_1, e_2, \dots, e_N\} \subseteq \mathbb{H}$, assume for all $h \in \mathbb{H}$ that $|\mu_h| = |\lambda_h|^\delta$, and let $\varphi_i : \mathbf{H}_0 \rightarrow \mathbb{R}$, $i \in \{1, 2\}$, be the functions which satisfy for all $i \in \{1, 2\}$, $(v_1, v_2) \in \mathbf{H}_0$ that $\varphi_i(v_1, v_2) = \exp(-\|v_i\|_{H_{1/2-j/2}}^2)$. Then it holds for all $i \in \{1, 2\}$, $N \in \mathbb{N}$ that $\varphi_i \in C_b^2(\mathbf{H}_0, \mathbb{R})$ and*

$$\begin{aligned} &\mathbb{E}[\varphi_i(\mathbf{X}^{I_N})] - \mathbb{E}[\varphi_i(\mathbf{X}^{\mathbb{H}})] \\ &\geq \left[1 + \inf_{x \in [2\pi T, \infty)} \frac{\sin(x)}{(-1)^i x}\right] \frac{T(4\pi^2)^{2\delta-1} N^{4\delta-1}}{[1 - 4\delta] \exp\left(\frac{12(2\delta-1)T\pi^{4\delta-2}}{4\delta-1}\right)} > 0. \end{aligned} \tag{2.32}$$

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