

Chapter 9

A Brief Guide to Reversing and Extended Symmetries of Dynamical Systems



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9.1 Introduction

Symmetries of dynamical systems are important objects to study, as they help in understanding the orbit structure and many other properties. Moreover, the group of symmetries is a topological invariant that can be useful for distinguishing between different dynamical systems. Naturally, this invariant is generally weaker than other invariants (such as those from (co-)homology or homotopy theory), but often easier to access.

For both aspects, studying properties and defining invariants, one is clearly interested in effective generalisations or extensions of the symmetry group. Inspired by the time-reversal symmetry of many fundamental equations in physics, one obvious step in this direction is provided by the *reversing symmetry group* of a dynamical system, which—in the case of reversibility—is an index-2 extension of the symmetry group.

Traditionally, the majority of the studies has concentrated on concrete dynamical systems, where the space is usually simple, but the mapping(s) might be complicated. Even for toral automorphism, the answer is amazingly rich. There is a complementary picture, which arises through the coding of itineraries and leads to the analogous questions in symbolic dynamics [59]. Here, the mapping(s) are simple, but the space (usually a closed shift space) is complicated, and this is particularly so when going to higher-dimensional shifts.

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In this brief introductory review, we recall the basic definitions and notions, and present some results from the large body of literature that has accumulated. Clearly, the exposition cannot be complete in any way, whence the references will provide further directions.

After some examples from the classic theory of concrete dynamical systems, we shall stroll through some more recent results on the complementary picture from symbolic dynamics.

9.2 General Setting and Notions

A convenient starting point is a topological space \mathbb{X} , which is usually (but not always) assumed to be compact, and a mapping $T \in \text{Aut}(\mathbb{X})$, where the automorphism group is understood in the Smale sense, meaning that it is the group of *all* homeomorphisms of \mathbb{X} . The pair (\mathbb{X}, T) then defines a (topological) *dynamical system*, and the group $\langle T \rangle \subset \text{Aut}(\mathbb{X})$ is important. Now, we define the *symmetry group* of (\mathbb{X}, T) as

$$\mathcal{S}(\mathbb{X}, T) := \{G \in \text{Aut}(\mathbb{X}) : G \circ T = T \circ G\} = \text{cent}_{\text{Aut}(\mathbb{X})}(\langle T \rangle) = \text{Aut}(\mathbb{X}, T). \quad (9.1)$$

This group plays an important role in the analysis of (\mathbb{X}, T) , for instance in the context of periodic orbits and dynamical zeta functions. Its is also a useful tool in the classification of dynamical systems, because it is a topological invariant.

Remark 9.1 The group $\text{Aut}(\mathbb{X}, T)$ is often used as a starting point for algebraic considerations, and then simply called the automorphism group of the dynamical system, but this is—as we shall see later—a use of the word that is too restrictive, and effectively excludes many natural mappings from the consideration. We will thus not use this notation, and rather view $\mathcal{S}(\mathbb{X}, T)$ as a subgroup of $\text{Aut}(\mathbb{X})$ in the Smale sense.

In some cases, the group $\text{Aut}(\mathbb{X})$ might be too big a ‘universe’ to consider, and some subgroup of it is a more natural choice, for instance when some additional structure of \mathbb{X} should be preserved. This is particularly so if some general results are available that imply $\mathcal{S}(\mathbb{X}, T)$ and $\mathcal{R}(\mathbb{X}, T)$ to be subgroups of some group $\mathcal{U} \subset \text{Aut}(\mathbb{X})$. In this case, one can start with \mathcal{U} , and simplify the algebraic derivations considerably. The above point simply is that \mathcal{U} should generally *not* be chosen as $\text{Aut}(\mathbb{X}, T)$, as this is too restrictive.

Since we will not consider the case that T is not invertible, a natural extension of $\mathcal{S}(\mathbb{X}, T)$ is given by

$$\mathcal{R}(\mathbb{X}, T) := \{G \in \text{Aut}(\mathbb{X}) : G \circ T \circ G^{-1} = T^{\pm 1}\}, \quad (9.2)$$

which is motivated by the time-reversal symmetry of many fundamental equations of physics; see [53, 71] and references therein for background. From now on, we write GT instead of $G \circ T$ etc. for ease of notation. The relation between $\mathcal{S}(\mathbb{X}, T)$ and $\mathcal{R}(\mathbb{X}, T)$ can be summarised as follows; see [12] and references therein.

Theorem 9.2 *If (\mathbb{X}, T) is a topological dynamical system, $\mathcal{R}(\mathbb{X}, T) \subset \text{Aut}(\mathbb{X})$ is a group, with $\langle T \rangle$ and $\mathcal{S}(\mathbb{X}, T)$ as normal subgroups. Moreover, one either has $\mathcal{R}(\mathbb{X}, T) = \mathcal{S}(\mathbb{X}, T)$ or $[\mathcal{R}(\mathbb{X}, T) : \mathcal{S}(\mathbb{X}, T)] = 2$. In the latter case, the systems is reversible.*

Further, if $T^2 \neq \text{Id}$ and if there is an involution H with $HTH^{-1} = T^{-1}$, one has

$$\mathcal{R}(\mathbb{X}, T) = \mathcal{S}(\mathbb{X}, T) \rtimes \langle H \rangle \simeq \mathcal{S}(\mathbb{X}, T) \rtimes C_2,$$

which is the standard form of reversibility.

An element that conjugates T into its inverse (where we assume $T^2 \neq \text{Id}$) is called a *reversor*. An elementary observation is the fact that a reversor cannot be of odd order, so it is either of even or of infinite order. When the order is finite, hence of the form $2^\ell(2m + 1)$ for some $\ell \geq 1$, there exists another reversor of order 2^ℓ . When T possesses an involutory reversor, R say, one has $T = TR^2 = (TR)R$, where $(TR)^2 = TRTR = TT^{-1} = \text{Id}$, so T is the product of two involutions. This is a frequently used approach in the older literature, before the group-theoretic setting showed [38, 52] that the more general approach is natural and helpful; see [12, 53] and references therein for details.

As is implicit from our formulation so far, reversibility is not an interesting concept when T itself is an involution. More generally, when T has finite order, the structure of $\mathcal{R}(\mathbb{X}, T)$ is a group-theoretic problem, and of independent interest; see [60] for a concise exposition. However, in the context of dynamical systems, one is mainly interested in the case that $\langle T \rangle \simeq \mathbb{Z}$. Then, one can slightly change the point of view by considering T as defining a continuous group action of \mathbb{Z} on \mathbb{X} , which is often reflected by the modified notation (\mathbb{X}, \mathbb{Z}) for the topological dynamical system. From now on, unless explicitly stated otherwise, we shall adopt this point of view here, too. The following result is elementary.

Fact 9.3 *When T is not of finite order, one has $\mathcal{R}(\mathbb{X}, T) = \text{norm}_{\text{Aut}(\mathbb{X})}(\langle T \rangle)$.*

It is thus the interplay between the (topological) centraliser and normaliser that is added in the extension from $\mathcal{S}(\mathbb{X}, T)$ to $\mathcal{R}(\mathbb{X}, T)$. One simple (but frequently useful) instance of this is given by the following result, where C_∞ and $D_\infty = C_\infty \rtimes C_2$ denote the infinite cyclic and dihedral group, respectively.

Theorem 9.4 ([12, Thm. 1 and Cor. 1]) *Let $T \in \text{Aut}(\mathbb{X})$ be of infinite order. If one has $\mathcal{S}(\mathbb{X}, T) \simeq C_\infty$ and if T is reversible, one has $\mathcal{R}(\mathbb{X}, T) = \mathcal{S}(\mathbb{X}, T) \rtimes C_2 \simeq D_\infty$, and all reversors of T are involutions.*

Conversely, if all reversors of T are involutions, the symmetry group $\mathcal{S}(\mathbb{X}, T)$ is Abelian.

Clearly, in the setting of dynamical systems, one could equally well consider the analogous questions for the measure-theoretic centraliser and normaliser, and this is indeed frequently done in the literature; compare [35, 39, 69] and references therein. Since, in many relevant cases, the measure-theoretic symmetry groups turn out to be topological (see [69] for results in this direction), we concentrate on the latter situation in this overview.

In what follows, we shall meet two rather different general situations, as briefly indicated in the introduction. On the one hand, there are many systems from nonlinear dynamics where the space is simple, but the map is complicated. In this case, we will write $\mathcal{S}(T)$ instead of $\mathcal{S}(\mathbb{X}, T)$ to emphasise the mapping. Likewise, when we are in the complementary situation (of symbolic dynamics, say) with a simple map acting on a more complicated space, we will use $\mathcal{S}(\mathbb{X})$ instead to highlight the difference. This also matches the widely used conventions in these two directions.

9.3 Concrete Systems from Nonlinear Dynamics

In this section, we will describe, in a somewhat informal manner, how symmetries and reversing symmetries arise in three particular families of dynamical systems, namely trace maps, toral automorphisms, and polynomial automorphisms of the plane. Clearly, there are many other relevant examples, some of which can be found in [53, 60, 71] and references therein.

9.3.1 Trace Maps

This class of dynamical system arises in the study of one-dimensional Schrödinger operators with aperiodic potentials of substitutive origin, compare [31] and references therein, and provide a powerful tool for the study of their spectra and transport properties. The paradigmatic *Fibonacci trace map* in 3-space is given by

$$(x, y, z) \mapsto (y, z, 2yz - x)$$

and is reversible, with involutory reversor $(x, y, z) \mapsto (z, y, x)$; see [65] and references given there. The group-theoretic ‘universe’ to consider here is given by the group of 3-dimensional invertible polynomial mappings that preserve the Fricke–Vogt invariant

$$I(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$$

and fix the point $(1, 1, 1)$; see [9, 14, 65] and references therein for more. This group of mappings is isomorphic with $\mathrm{PGL}(2, \mathbb{Z})$, and can thus be analysed by classic methods, including the theory of binary quadratic forms.

In other words, the analysis of (reversing) symmetries of trace maps is equivalent to the determination of $\mathcal{S}(M)$ and $\mathcal{R}(M)$ for matrices $M \in \mathrm{PGL}(2, \mathbb{Z})$. Since

$$\mathrm{PGL}(2, \mathbb{Z}) \simeq \mathrm{GL}(2, \mathbb{Z})/\{\pm 1\},$$

the following result is obvious.

Fact 9.5 *Let $M \in \mathrm{PGL}(2, \mathbb{Z})$ and M' be either of the two corresponding matrices in $\mathrm{GL}(2, \mathbb{Z})$. Then, the symmetry group $\mathcal{S}(M)$ is given by*

$$\mathcal{S}(M) = \mathrm{cent}_{\mathrm{PGL}(2, \mathbb{Z})}(\langle M \rangle) = \mathrm{cent}_{\mathrm{GL}(2, \mathbb{Z})}(\langle M' \rangle)/\{\pm 1\}.$$

The symmetry groups can thus be derived from the analysis of general (two-dimensional) toral automorphisms, which we will review in Sect. 9.3.2. For the reversing symmetry group, the role of $\{\pm 1\}$ changes. Let $M \in \mathrm{PGL}(2, \mathbb{Z})$ be given, and view it as a $\mathrm{GL}(2, \mathbb{Z})$ -matrix. Then, we have to find all solutions H to

$$HMH^{-1} = \pm M^{-1},$$

where the calculation modulo ± 1 means that we get *more* cases with reversibility than in $\mathrm{GL}(2, \mathbb{Z})$. For instance, $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is reversible in $\mathrm{PGL}(2, \mathbb{Z})$, with an involutory reversor, but not within $\mathrm{GL}(2, \mathbb{Z})$, while M^2 (known as Arnold's cat map [5, Ex. 1.15]) is reversible in both groups. Within $\mathrm{GL}(2, \mathbb{Z})$, this phenomenon is called 2-reversibility; see [9] for details.

9.3.2 Toral Automorphisms

These systems, which are also known as 'cat maps', are much studied examples in chaotic dynamics and ergodic theory. Here, in order to preserve the linear structure of \mathbb{T}^d , the d -dimensional torus, one usually works within $\mathcal{U} = \mathrm{GL}(d, \mathbb{Z}) \subset \mathrm{Aut}(\mathbb{T}^d)$; see [1, 2, 5, 62] for background.

In the planar case ($d = 2$), one thus has to deal with the group $\mathrm{GL}(2, \mathbb{Z})$. Here, if M is an element of infinite order, one always finds $\mathcal{S}(M) \simeq C_2 \times C_\infty$, where $C_2 = \{\pm 1\}$. This follows for any parabolic element by a simple calculation, and, for the hyperbolic elements, is a consequence of Dirichlet's unit theorem for real quadratic number fields; see [24] for background.

Remark 9.6 Reversible cases among elements of infinite order are of three possible types: When all reversors are involutions, one has $\mathcal{R}(M) \simeq C_2 \times D_\infty$, again with $D_\infty = C_\infty \rtimes C_2$; when all reversors are of fourth order, one has $\mathcal{R}(M) \simeq C_\infty \rtimes C_4$;

finally, when reversors both of order 2 and 4 exist, one has $\mathcal{R}(M) \simeq (C_2 \times C_\infty) \rtimes C_2$. All three types occur; see [12, Thm. 2 and Ex. 4] and references given there for more.

In this context, it is certainly a valid and interesting question how the concepts can be extended to cover toral endomorphisms, or what happens when one restricts to rational sublattices. This is connected with looking at the related questions over finite fields and residue class rings; see [16, 18] and references therein for some results.

The situation becomes more complex, and also more interesting, in higher dimensions. In a first step, one has to analyse the symmetry group of a toral automorphism, $M \in \text{GL}(d, \mathbb{Z})$ say, within this matrix group. In the generic case, where M is simple (meaning that its eigenvalues are distinct) one can employ Dirichlet’s unit theorem again. Let us first look at the case that the characteristic polynomial $P(x) = \det(M - x1)$ of M is irreducible over \mathbb{Z} , and hence also over \mathbb{Q} . Then, if λ is any of the d eigenvalues of M , it is an algebraic integer of degree $d = n_1 + 2n_2$, where n_1 is the number of real algebraic conjugates of λ and n_2 the number of complex conjugate pairs among the algebraic conjugates.

Now, if \mathcal{O} is the maximal order in the algebraic number field $\mathbb{Q}(\lambda)$, Dirichlet’s unit theorem states that the unit group \mathcal{O}^\times is of the form

$$\mathcal{O}^\times \simeq T \times \mathbb{Z}^{n_1+n_2-1} \tag{9.3}$$

with $T = \mathcal{O} \cap \{\text{roots of unity}\}$ being a finite cyclic group. The latter is known as the *torsion subgroup* of \mathcal{O}^\times . Due to the isomorphism of $\mathbb{Z}[\lambda]$ with the ring $\mathbb{Z}[M]$ under our irreducibility assumption on P , one then has the following result [10, Prop. 1 and Cor. 1].

Theorem 9.7 *Let $M \in \text{GL}(d, \mathbb{Z})$ have an irreducible characteristic polynomial, $P(x)$, of degree $d = n_1 + 2n_2$, with n_1 and n_2 as above. Then, $\mathcal{S}(M)$ is isomorphic with a subgroup of \mathcal{O}^\times of maximal rank, so*

$$\mathcal{S}(M) \simeq T' \times \mathbb{Z}^{n_1+n_2-1},$$

where T' is a subgroup of the torsion group T from Eq. (9.3).

Moreover, whenever $P(x)$ has a real root, which includes all cases with d odd, one simply has $T' = \{\pm 1\} \simeq C_2$.

For our previous example, $M = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, one finds $\mathcal{S}(M) = \{\pm 1\} \times \langle M \rangle \simeq C_2 \times \mathbb{Z}$. Note that, in general, the generators of the free part of $\mathcal{S}(M)$ can correspond to powers of fundamental units, which is related with the question of the existence of matrix roots within $\text{GL}(d, \mathbb{Z})$; see [10] for more. One can quite easily extend Theorem 9.7 to the case that M is simple. This is done by factoring P over \mathbb{Z} and treating the factors separately [9, Thm. 1].

Let us look at the reversibility of a matrix $M \in \text{GL}(2, \mathbb{Z})$. A necessary condition clearly is that M and M^{-1} have the same spectrum (including multiplicities). In other words, if P is the characteristic polynomial of M with integer coefficients, it

must satisfy the self-reciprocity condition

$$P(x) = \frac{(-1)^d x^d}{\det(M)} P\left(\frac{1}{x}\right). \tag{9.4}$$

Now, if d is odd or if $\det(M) = -1$, this relation implies that 1 or -1 is a root, and P is reducible over \mathbb{Z} . In particular, d odd and P irreducible immediately excludes reversibility. This means that, generically, reversible cases can only occur when d is even and $\det(M) = 1$.

Note that, even if Eq. (9.4) is satisfied, the reversibility still depends on the underlying integer matrix M , and the class number of $\mathbb{Z}[\lambda]$ enters. It is then clear that deciding on reversibility is a problem that increases with growing d ; we refer to the discussion in [10] for more. However, for any given characteristic polynomial that is self-reciprocal according to the condition of Eq. (9.4), there is at least one reversible class of matrices, and this can be represented by the Frobenius companion matrix [10, Thm. 3].

A natural extension of symmetries can be considered in the setting of matrix rings rather than groups, such as $\text{Mat}(d, K)$ instead of $\text{GL}(d, K)$, where K can itself be a ring (such as \mathbb{Z}) or a field (such as \mathbb{Q}). Then, one can define

$$\mathcal{S}(M) = \{G \in \text{Mat}(d, K) : [M, G] = 0\}.$$

Concretely, if M is an integer matrix with irreducible characteristic polynomial, and λ is any of its roots, one finds $\mathcal{S}(M)$ to be isomorphic with an order \mathcal{O} in the number field $\mathbb{Q}(\lambda)$ that satisfies $\mathbb{Z}[\lambda] \subseteq \mathcal{O} \subseteq \mathcal{O}_{\max}$, where \mathcal{O}_{\max} denotes the maximal order in $\mathbb{Q}(\lambda)$; see [41, Ch. III] as well as [10, Sec. 3.3] and references given there for more.

9.3.3 Polynomial Automorphisms of the Plane

Let K be a field and consider the group $\mathcal{U}_K = \text{GA}_2(K)$ of polynomial automorphisms of the affine plane over K . Consequently, we have $\mathbb{X} = K^2$ in this case, which need not be compact. \mathcal{U}_K consists of all mappings of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix}$$

with $P, Q \in K[x, y]$, subject to the condition that the inverse exists and is also polynomial. Note that, over general fields, different polynomials might actually define the same mapping on K^2 , but we will distinguish them on the level of the polynomials.

In nonlinear dynamics, where $\text{GA}_2(\mathbb{R})$ and $\text{GA}_2(\mathbb{C})$ have received considerable attention, a common alternative notation is

$$x' = P(x, y), \quad y' = Q(x, y).$$

Frequently studied examples include the Hénon quadratic map family, defined by $P(x, y) = y$ and $Q(x, y) = -\delta x + y^2 + c$ with constants $c, \delta \in \mathbb{C}$ and $\delta \neq 0$. Quite often, for instance in the context of area-preserving mappings, the starting point is a polynomial automorphism in *generalised standard form*,

$$x' = x + P_1(y), \quad y' = y + P_2(x'),$$

with single-variable polynomials P_1 and P_2 ; compare [37, 66] and references therein. Here, the inverse is simply given by $y = y' - P_2(x')$ together with $x = x' - P_1(y)$.

In a certain sense, such particular normal forms are important, but do not exhaust the full power of the algebraic setting. Let us explain this a little in the context of combinatorial group theory. We begin by defining three subgroups of $\text{GA}_2(K)$ as follows. First,

$$\mathcal{A} := \{(\underline{a}, M) : \underline{a} \in K^2, M \in \text{GL}(2, K)\}$$

is the group of *affine* transformations, where (\underline{a}, M) encodes the mapping $\underline{x} \mapsto M\underline{x} + \underline{a}$. We write \underline{a} for a column vector, and tacitly identify the elements of \mathcal{A} with the canonically corresponding elements of $\text{GA}_2(K)$. Multiplication is defined by

$$(\underline{a}, A)(\underline{b}, B) = (\underline{a} + A\underline{b}, AB),$$

whence \mathcal{A} is a semi-direct product, namely $\mathcal{A} = K^2 \rtimes \text{GL}(2, K)$. The inverse of an element is $(\underline{a}, A)^{-1} = (-A^{-1}\underline{a}, A^{-1})$.

The second group, \mathcal{E} , is known as the group of *elementary* transformations. It consists of all mappings of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha x + P(y) \\ \beta y + v \end{pmatrix}$$

with P a single-variable polynomial and $\alpha, \beta, v \in K$ subject to the condition $\alpha\beta \neq 0$. It is easy to check that the inverse exists and it of the same form. Transformations of this kind map lines with constant y -coordinate to lines of the same type. It is a well-known fact that the group $\text{GA}_2(K)$ is generated by \mathcal{A} and \mathcal{E} ; see [42, 72] as well as [73, Sec. 1.5].

Finally, our third group, \mathcal{B} , is defined as the intersection $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$, with obvious meaning as subgroups of $\text{GA}_2(K)$. The elements of \mathcal{B} are called *basic* transformations, and are mappings of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}$$

with $\alpha, \beta, \gamma, u, v \in K$ and $\alpha\beta \neq 0$. Clearly, also \mathcal{B} is a semi-direct product, namely $\mathcal{B} = K^2 \rtimes \mathcal{T}$, where \mathcal{T} denotes the subgroups of $\text{GL}(2, K)$ that consists of all invertible upper triangular matrices over K .

Now, the following result [68, 73] is fundamental to the classification of (reversing) symmetries of polynomial automorphisms.

Lemma 9.8 *The group $\text{GA}_2(K)$ is the free product of the groups \mathcal{A} and \mathcal{E} , amalgamated along their intersection, \mathcal{B} , which is abbreviated as $\text{GA}_2(K) = \mathcal{A}^*_\mathcal{B}\mathcal{E}$.*

Through this result, the problem has been reset in a purely algebraic way, and one can now explore the subgroup structure [43] of the amalgamated free product. In particular, one can classify the Abelian subgroups of $\text{GA}_2(K)$, which has trivial centre. Naturally, $\mathcal{S}(T)$ for a given $T \in \text{GA}_2(K)$ is more complex, and need no longer be Abelian. When K has characteristic 0, one can derive restrictions on the order of other symmetries, which gives access to the finite subgroups of $\mathcal{S}(T)$; for details, the reader is referred to [11].

For an important subclass of transformations known as CR elements, one can say a lot more. In particular, if K is a field of characteristic 0, all reversors must be of finite order. If, in addition, the roots of unity in K are just $\{\pm 1\}$, any reversor is an involution or an element of order 4, which makes their detection feasible. The possible reversing symmetry groups in this case are then the same three types we saw earlier, in Remark 9.6, for 2-dimensional toral automorphisms of infinite order. Since further details in this setting of combinatorial group theory tend to be a bit technical, we refer to [11] and references therein for more.

9.4 Shift Spaces with Faithful \mathbb{Z} -action

All examples in the previous section shared the feature that the space \mathbb{X} is simple, but the map T on it is not. This is the standard situation in most dynamical systems that arise from concrete problems, for instance in nonlinear dynamics. However, it has long been known [59] that there is a complementary picture, which arises by coding orbits in such systems by symbolic sequences, for instance via itineraries. The latter keep track of a coarse-grained structure in such a way that the full dynamics can be recovered from them—at least almost surely in some suitable measure-theoretic sense.

This leads to *symbolic dynamics*, where the space \mathbb{X} is ‘replaced’ by a closed shift space \mathbb{Y} (often over a finite alphabet), and T by the action of the left shift, S . More precisely, one constructs a conjugacy, a semi-conjugacy, or (typically) a measure-theoretic isomorphism that makes the diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{T} & \mathbb{X} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{Y} & \xrightarrow{S} & \mathbb{Y} \end{array}$$

commutative and ϕ as ‘invertible as possible’. This motivates to also consider symmetries and reversing symmetries of shift spaces, where we shall always assume that the action of \mathbb{Z} on the shift space is *faithful* in order to exclude degenerate situations. We refer to [50, 56] for general background, and to [49, 55] for the study of topological Markov chains in this context.

One immediate problem that arises is the fact that the symmetry group of a shift space (now called \mathbb{X} again) is generally huge, in the sense that it contains a copy of the free group of two generators—and is thus not amenable [56]. This turns a potential classification into a wild problem, and not much has been done in this direction. On the other hand, as has long been known, it is also possible that one simply gets $\mathcal{S}(\mathbb{X}) = \langle S \rangle \simeq \mathbb{Z}$, in which case one speaks of a *trivial centraliser*, or of a *minimal symmetry group*. This is a form of *rigidity*, for which different mechanisms are possible. Interestingly, rigidity is not a rare phenomenon [23], but actually generic in some sense [40], which makes it rather relevant also in practice.

To explore the possibilities a little, let us assume that \mathcal{A} is a finite set, called the *alphabet*, and that $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}}$ is a closed and shift-invariant set, which is then automatically compact. Such a space is called a *shift space*, or *subshift* for short.

A special role has the ‘canonical’ reversor R defined by

$$(Rx)_n := x_{-n} \tag{9.5}$$

or any combination of R with a power of the shift S . It is clear that R conjugates S into its inverse on the full shift, $\mathbb{X} = \mathcal{A}^{\mathbb{Z}}$. More generally, one has the following property.

Lemma 9.9 *Let \mathbb{X} be a shift space with faithful shift action. If \mathbb{X} is reflection-invariant, which means $R(\mathbb{X}) = \mathbb{X}$ with the mapping R from Eq. (9.5), the system is reversible, with $\mathcal{R}(\mathbb{X}) = \mathcal{S}(\mathbb{X}) \rtimes C_2$, where $C_2 = \langle R \rangle$.*

Let us collect a few examples of reversible subshifts, in an informal manner; see [20] and references therein for precise statements and proofs, and [3, 7, 26, 63, 64]

for general background on substitution generated subshifts. Among these examples are

1. the *full* shift [50, 56], $\mathbb{X} = \mathcal{A}^{\mathbb{Z}}$, where $\mathcal{S}(\mathbb{X})$ is huge (and not amenable);
2. any *Sturmian* shift [27], which is always palindromic [33] and hence reversible, with symmetry group $\mathcal{S}(\mathbb{X}) \simeq \mathbb{Z}$;
3. the *period doubling* shift, defined by the primitive substitution rule $a \mapsto ab$, $b \mapsto aa$, again with $\mathcal{S}(\mathbb{X}) \simeq \mathbb{Z}$;
4. the *Thue–Morse* (TM) shift, defined by $a \mapsto ab$, $b \mapsto ba$, here with $\mathcal{S}(\mathbb{X}) \simeq \mathbb{Z} \times C_2$, where the extra symmetry is the letter exchange map defined by $a \leftrightarrow b$;
5. the *square-free* shift, obtained as the orbit closure of the characteristic function of the square-free integers, also with $\mathcal{S}(\mathbb{X}) \simeq \mathbb{Z}$.

In fact, the last example is quite remarkable, as its rigidity mechanism relies on the heredity of the shift, as was recently shown by Mentzen [57]. Note that the square-free shift has positive topological entropy, but nevertheless possesses minimal centraliser. Though this is not surprising in view of known results from Toeplitz sequences [25], it does show that rigidity as a result of low complexity, as studied in [28–30, 32], is only one of *several* mechanisms. We shall see more in Sect. 9.5. The square-free shift is a prominent example from the class of \mathcal{B} -free shifts, see [22, 34] and references therein, and also of interest in the context of Sarnak’s conjecture on the statistical independence of the Möbius function from deterministic sequences (as discussed at length in other contributions to this volume).

Of course, things are generally more subtle than in these examples. First of all, a subshift can be irreversible, as happens for the one defined by the binary substitution $a \mapsto aba$, $b \mapsto baa$, where $\mathcal{R}(\mathbb{X}) = \mathcal{S}(\mathbb{X}) \simeq \mathbb{Z}$. Next, consider the subshift $\mathbb{X}_{k,\ell}$ defined by the primitive substitution

$$a \mapsto a^k b^\ell, \quad b \mapsto b^k a^\ell$$

with $k, \ell \in \mathbb{N}$, which is reversible if and only if $k = \ell$. This is an extension of the TM shift (which is the case $k = \ell = 1$), in the spirit of [17, 45]. The symmetry group is $\mathcal{S}(\mathbb{X}_{k,\ell}) \simeq \mathbb{Z} \times C_2$ in all cases, where C_2 is once again the group generated by the letter exchange map.

Going to larger alphabets, $\mathcal{A} = \{a_0, a_1, \dots, a_{N-1}\}$ say, one can look at a cyclic extension of the TM shift, as defined by the substitution $a_i \mapsto a_i a_{i+1}$ with the index taken modulo N . This shift is reflection invariant only for $N = 2$, but nevertheless reversible for any N , and even with an involutory reversor. The symmetry group is $\mathbb{Z} \times C_N$.

The quaternary Rudin–Shapiro shift shows another phenomenon. Its symmetry group is $\mathbb{Z} \times C_2$, and it is reversible, but no reversor is an involution. Instead, there is a reversor of order 4 (and all reversors have this order), and the reversing symmetry group is $\mathbb{Z} \rtimes C_4$, where the square of the generating element of the cyclic group C_4 is the extra (involutory) symmetry; see [20] for details on this and the previous examples.

9.5 Shift Spaces with Faithful \mathbb{Z}^d -action

It is more than natural to also consider higher-dimensional shift actions. Here, given some alphabet \mathcal{A} , a subshift is any closed subspace $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ that is invariant under the shift in each of the d directions. With $\underline{n} = (n_1, \dots, n_d)^T \in \mathbb{Z}^d$ as well as $x_{\underline{n}} = (x_{n_1}, \dots, x_{n_d})$, one defines the shift in direction i by

$$(S_i x)_{\underline{n}} := x_{\underline{n} + \underline{e}_i},$$

where \underline{e}_i is the standard unit vector in direction i . The individual shifts commute with one another, $S_i S_j = S_j S_i$, for all $1 \leq i, j \leq d$. Now, we define the *symmetry group* of \mathbb{X} as

$$\mathcal{S}(\mathbb{X}) = \text{cent}_{\text{Aut}(\mathbb{X})}(\mathcal{G}),$$

where $\mathcal{G} := \langle S_1, \dots, S_d \rangle$ is a subgroup of $\text{Aut}(\mathbb{X})$.

As before, we are only interested in subshifts with faithful shift action, which means $\mathcal{G} = \langle S_1 \rangle \times \dots \times \langle S_d \rangle \simeq \mathbb{Z}^d$, where the direct product structure is a consequence of the commutativity of the individual shifts. In this case, we define the *group of extended symmetries* as

$$\mathcal{R}(\mathbb{X}) = \text{norm}_{\text{Aut}(\mathbb{X})}(\mathcal{G}),$$

which is the obvious extension of the one-dimensional case. As we shall see shortly, many of the obvious ‘symmetries’ of \mathbb{X} are only captured by this extension step.

Unlike before, the structure of the normaliser is generally much richer now, which also means that $\mathcal{R}(\mathbb{X})$ is a considerably better topological invariant than $\mathcal{S}(\mathbb{X})$. Indeed, the normaliser can even be an *infinite* extension of the centraliser when $d > 1$, as can be seen from the full shift as follows; see [20, Lemma 4].

Fact 9.10 *Let $d \in \mathbb{N}$ and let $\mathbb{X} = \mathcal{A}^{\mathbb{Z}^d}$ be the full d -dimensional shift over the (finite or infinite) alphabet \mathcal{A} . Then, the group of extended symmetries is given by $\mathcal{R}(\mathbb{X}) = \mathcal{S}(\mathbb{X}) \rtimes \text{GL}(d, \mathbb{Z})$.*

The reasoning behind this observation is simple. Each element of $\mathcal{R}(\mathbb{X})$ must map generators of $\mathcal{G} = \langle S_1, \dots, S_d \rangle \simeq \mathbb{Z}^d$ onto generators of \mathcal{G} , and thus induces a mapping into $\text{GL}(d, \mathbb{Z})$, which is the automorphism group of the free Abelian group of rank d . Now, one checks that, for any $M \in \text{GL}(d, \mathbb{Z})$, the mapping h_M defined by

$$(h_M x)_{\underline{n}} = x_{M^{-1}\underline{n}},$$

with \underline{n} considered as a column vector, defines an automorphism of the full shift. This leads to the semi-direct product structure as stated.

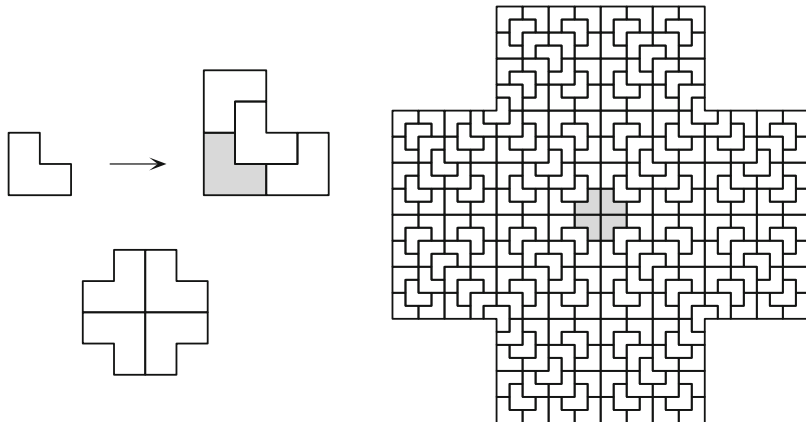


Fig. 9.1 The chair inflation rule (upper left panel; rotated tiles are inflated to rotated patches), a legal patch with full D_4 symmetry (lower left) and a level-3 inflation patch generated from this legal seed (shaded; right panel). Note that this patch still has the full D_4 point symmetry (with respect to its centre), as will the infinite inflation tiling fixed point emerging from it

9.5.1 Tiling Dynamical Systems as Subshifts

Substitution tilings of constant block size are a generalisation of substitutions of constant length, and admit an alternative description as subshifts, for instance via a suitable symbolic coding. Classic examples include the chair and the table tiling [67], but many more are known [8, 36].

Here, we take a look at the chair tiling, which is illustrated in Fig. 9.1; see [7] for more. Its geometric realisation makes it particularly obvious that any reasonable notion of a group of full symmetries must somehow contain the elementary symmetries of the square, simply because the inflation tiling (whose orbit closure under the translation action of \mathbb{Z}^2 defines the tiling dynamical system, with compact space \mathbb{X}) is invariant under a fourfold rotation and a reflection in the horizontal axis. These two operations generate a group that is isomorphic with the dihedral group D_4 , a maximal finite subgroup of $GL(2, \mathbb{Z})$.

Now, none of these orthogonal transformations occur in the centraliser of the shift group, which was shown to be minimal in [61]. This is a rigidity phenomenon of *topological* origin, due to the fibre structure of \mathbb{X} over its maximal equicontinuous factor (MEF). Consequently, this example provides ample evidence that one also needs to consider the normaliser. The general result reads as follows; see [20] for the details.

Theorem 9.11 *Let \mathbb{X} be the hull of the chair tiling, and $(\mathbb{X}, \mathbb{Z}^2)$ the corresponding dynamical system. It is topologically conjugate to a subshift of $\{0, 1, 2, 3\}^{\mathbb{Z}^2}$ with faithful shift action. Moreover, one has $\mathcal{S}(\mathbb{X}) \simeq \mathbb{Z}^2$ and $\mathcal{R}(\mathbb{X}) \simeq \mathbb{Z}^2 \rtimes D_4$, where D_4 is the symmetry group of the square, and a maximal finite subgroup of $GL(2, \mathbb{Z})$.*

Proof (Sketch) It is well known that \mathbb{X} is a.e. one-to-one over its MEF, which is a two-dimensional odometer here. The orbits of non-singleton fibres over the MEF create the topological rigidity that enforce the centraliser to agree with the group generated by the lattice translations.

The extension by D_4 is constructive, via the symmetries of the inflation fixed point. Any further extension would require the inclusion of a $\text{GL}(2, \mathbb{Z})$ -element of infinite order (because D_4 is a maximal finite subgroup of $\text{GL}(2, \mathbb{Z})$), which is impossible by the geometric structure (and rigidity) of the prototiles.

Similar results will occur for other tiling dynamical system, also in higher dimensions. For instance, it is clear that the d -dimensional chair (with $d \geq 2$; see [7]) will have $\mathcal{S} = \mathbb{Z}^d$ and $\mathcal{R} = \mathbb{Z}^d \rtimes W_d$, where W_d is the symmetry group of the d -dimensional cube, also known as the hyperoctahedral group [6].

Let us note that there is no general reason why the extended symmetry group should be a semi-direct product (though this will be the most frequent case to encounter in the applications). In fact, in (periodic) crystallography, the classification of space groups in dimensions $d \geq 2$ contains so-called *non-symmorphic* cases that do not show a semi-direct product structure between translations and linear isometries [70]. It will be an interesting question to identify or construct planar shift spaces that show the planar wallpaper groups as their extended symmetry groups. This and similar results would emphasise once more that and how the extension from $\mathcal{S}(\mathbb{X})$ to $\mathcal{R}(\mathbb{X})$ is relevant to capture the full symmetry of faithful shift actions.

9.5.2 Shifts of Algebraic Origin

There is a particularly interesting and important class of subshifts that has attracted a lot of attention. They are known as subshifts of algebraic origin; see [69] and references therein. The important point here is that such a subshift is also an Abelian group under pointwise addition, and thus carries the corresponding Haar measure as a canonical invariant measure.

Here, we take a look at one of the paradigmatic examples from this class, the Ledrappier shift [54]. This is the subshift $\mathbb{X}_L \subset \{0, 1\}^{\mathbb{Z}^2}$ defined as

$$\mathbb{X}_L = \ker(1 + S_1 + S_2) = \{x \in \{0, 1\}^{\mathbb{Z}^2} : x_{\underline{n}} + x_{\underline{n}+\underline{e}_1} + x_{\underline{n}+\underline{e}_2} = 0 \text{ for all } \underline{n} \in \mathbb{Z}^2\}, \quad (9.6)$$

where the sums are pointwise, and to be taken modulo 2. This definition highlights the special role of elementary lattice triangles, whose vertices are supporting the local variables that need to sum to 0; see Fig. 9.2 for an illustration. The symmetry group is known to be minimal, which can be seen as a rigidity phenomenon of *algebraic* type. More generally, one has the following result.

Theorem 9.12 *The symmetry group of Ledrappier’s shift \mathbb{X}_L from Eq. (9.6) is*

$$S(\mathbb{X}_L) = \langle S_1, S_2 \rangle \simeq \mathbb{Z}^2,$$

while the group of extended symmetries is given by

$$\mathcal{R}(\mathbb{X}) = \langle S_1, S_2 \rangle \rtimes H \simeq \mathbb{Z}^2 \rtimes D_3,$$

where H is the finite group generated by the autormorphisms h_A and h_B , with $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This group is isomorphic with the dihedral group $D_3 \subset GL(2, \mathbb{Z})$ that is generated by the corresponding matrices, A and B .

Proof (Sketch) The triviality of the centraliser is a consequence of the group structure, which heavily restricts the homeomorphisms between irreducible subshifts that commute with the translations [23, 51, 69].

For the extension to the normaliser, the presence of D_3 is again constructive, and evident from Fig. 9.2. One then excludes any element of order 6 that would complete D_3 to D_6 , and finally any element of infinite order that could extend the group D_3 . Both types of extensions are impossible because any such additional element would change the defining condition by deforming the elementary triangles.

This example is of interest for a number of reasons. First of all, it shows the phenomenon of rank-1 entropy, which is to say that the number of circular configurations grows exponentially in the *radius* of the patch, but not in the area. While this means that the topological entropy still vanishes, Ledrappier’s shift is not an example of low complexity. Second, the spectral structure displays a

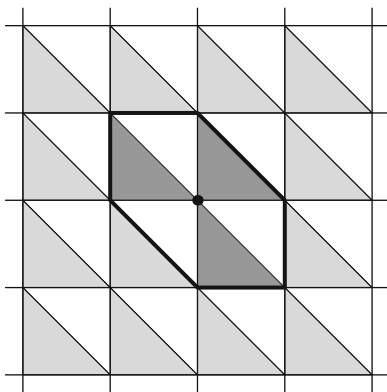


Fig. 9.2 Central configurational patch for Ledrappier’s shift condition, indicating the relevance of the triangular lattice. Equation (9.6) implies a condition for the values at the three vertices of all elementary L -triangles (shaded). The overall pattern of these triangles is preserved by all (extended) symmetries. The group D_3 from Theorem 9.12 can now be viewed as the colour-preserving symmetry group of the ‘distorted’ hexagon as indicated around the origin

mixture of trivial point spectrum with further absolutely continuous (Lebesgue) components [13], which highlights the fact that the inverse problem of structure determination, in the presence of mixed spectra, is really a lot more complex than in the case of pure point spectra. Once again, capturing the full extended symmetry group is an important first step in this analysis, as is well-known from classical crystallography [70].

Let us consider the planar point set

$$V := \{(x, y) \in \mathbb{Z}^2 : \gcd(x, y) = 1\} \subset \mathbb{Z}^2,$$

which is known as the set of *visible* (or primitive) *lattice points*; see the cover page of [4] for an illustration. The set V has numerous fascinating properties, both algebraically and geometrically. In particular, it fails to be a Delone set, because it has holes of arbitrary size that even repeat lattice-periodically. Nevertheless, the natural density exists and equals $6/\pi^2 = 1/\zeta(2)$. Moreover, the set V is invariant under the group $\mathrm{GL}(2, \mathbb{Z})$, which acts transitively on V ; see [15] and references therein.

The corresponding subshift \mathbb{X}_V is defined as the orbit closure of the characteristic function 1_V under the shift action of \mathbb{Z}^2 , which turns $(\mathbb{X}_V, \mathbb{Z}^2)$ into a topological dynamical system with faithful shift action and positive topological entropy. This system, like the square-free shift from above, is *hereditary*, which implies rigidity for the symmetry group. On the other hand, due to the way that $\mathrm{GL}(2, \mathbb{Z})$ -matrices act on it, the normaliser is the maximal extension of the centraliser in this case [21]. In fact, there is no reason to restrict to the planar case here, as the visible lattice points can be defined for \mathbb{Z}^d with any $d \geq 2$ (the case $d = 1$ gives a finite set that is not of interest). Thus, one has the following result.

Theorem 9.13 *Let \mathbb{X}_V be the subshift defined by the visible lattice points of \mathbb{Z}^d , where $d \geq 2$. Then, \mathbb{X}_V has faithful shift action with minimal symmetry group, $\mathcal{S}(\mathbb{X}_V) = \mathbb{Z}^d$, while the extended symmetry group emerges as the maximal extension of it, $\mathcal{R}(\mathbb{X}_V) = \mathbb{Z}^2 \rtimes \mathrm{GL}(2, \mathbb{Z})$.*

Proof (Sketch) Here, the triviality of the centraliser, as in the earlier example of the square-free shift, is a consequence of the *heredity* of the subshift [21], and really follows from a mild generalisation of Mentzen's approach [57]. The extension to the normaliser, as explained above, is by all of $\mathrm{GL}(d, \mathbb{Z})$, where the semi-direct product structure is the same as for the full shift in Fact 9.10.

More generally, one can study systems of this kind as defined from primitive lattice systems, for instance in the spirit of [19]. This also covers subshifts that are generated from rings of integers in general algebraic number fields subject to certain freeness conditions. This gives a huge class of examples that can be viewed as multi-dimensional generalisations of \mathcal{B} -free systems. Interestingly, they are also examples of weak model sets [19], which gives access to a whole new range of tools from the interplay of dynamical systems and algebraic number theory [44, 46–48], in the spirit of the original approach by Meyer [58].

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