Chapter 4
Euler's Gamma Function

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For proofs of the following results, see for example Chapters 12 and 13 of [2].

Definition 4.1 For all $z \in \mathbb{C}$ with $\Re(z) > 0$, we define

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Theorem 4.2 (Weierstrass Product) *For all* $z \in \mathbb{C}$,

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where $\gamma = 0.5772156649\ldots$ *is the Euler constant.*

Thus the function Γ *is meromorphic on* $\mathbb{C}$, *has no zero, and admits simple poles at* $z = 0, -1, -2, \ldots$.

Proposition 4.3 *For* $z \in \mathbb{C} \setminus (-\mathbb{N})$, *we have*

$$\Gamma(z+1) = z\,\Gamma(z).$$

Corollary 4.4 *For* $n \in \mathbb{N}$, *we have* $\Gamma(n+1) = n!$.

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Proposition 4.5 *For* $z \in \mathbb{C} \setminus \mathbb{Z}$*,*

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Corollary 4.6 *We have* $\Gamma(\frac{1}{2}) = \sqrt{\pi}$*.*

Proposition 4.7 (Legendre Duplication Formula) *For* $z \in \mathbb{C} \setminus (-\frac{1}{2}\mathbb{N})$*,*

$$\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) = \pi^{1/2}\,2^{1-2z}\,\Gamma(2z).$$

Corollary 4.8 *For* $z \in \mathbb{C} \setminus (2\mathbb{Z})$*, we have*

$$\Gamma(\tfrac{z}{2})/\Gamma(\tfrac{1-z}{2}) = \pi^{-1/2}\,2^{1-z}\,\cos(\tfrac{\pi z}{2})\,\Gamma(z).$$

Theorem 4.9 (Stirling Formula) *For fixed* $0 \leqslant \vartheta < \pi$ *and* $|arg(z)| < \vartheta$*, we have*

$$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z - \tfrac{1}{2} \log 2\pi + O(|z|^{-1}), \quad |z| \to +\infty.$$

Proposition 4.10 *For fixed* $0 \leqslant \vartheta < \pi$ *and* $|arg(z)| < \vartheta$*, we have*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z + O(|z|^{-1}), \quad |z| \to +\infty.$$

Proof See [1, footnote, p.57].

References

1. A.E. Ingham, *The Distribution of Prime Numbers*. Cambridge Mathematical Library (Cambridge University Press, Cambridge, 1990); Reprint of the 1932 original, With a foreword by R. C. Vaughan
2. E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*. Cambridge Mathematical Library (Cambridge University Press, Cambridge, 1996); An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition