Old Problems Revisited from New Perspectives in Implicit Theories of Fluids

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Abstract Three of the most studied problems in fluid dynamics are revisited within implicit theories of fluids. Specifically, the onset of convection, the determination of laminar flows and the motion of a fluid down an inclined plane are studied under the assumption that the Cauchy stress tensor and the rate-of-strain tensor are related through implicit constitutive equations. Particular attention is paid to fluids whose viscosities are pressure-dependent.

1 Introduction

The *principium reddendae rationis* is one of the most powerful tool that have been used in philosophical argumentations [28]. It has been used as principle by many philosophers (Spinoza, Liebniz, Descartes, Hamilton, to cite a few of them) and is very useful also in Science. Anaximander of Miletus suggestively used the *principium reddendae rationis* to argue that Earth was a round cylinder statically floating at the center of Universe without any support. In Anaximander's reasoning, since Earth was equidistant from all other bodies there was no reason why it should move in any one direction.

The *principium reddendae rationis* has been applied more rigorously in classical continuum mechanics, and, here, we shall appeal to it to sustain the suitability of introducing implicit constitutive relations rather than explicit models. The main aim of these notes is indeed the introduction, in the framework of implicit theories of

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fluids, of the class of fluids whose viscosities depend on pressure and that, for such a peculiarity, are called *piezo-viscous fluids* [10].

It is well known that in any continuum theory the motion of a real body $\mathscr{B}$ is governed by the balance laws of mass, linear and angular momenta, and energy as well as by the second law of Thermodynamics. In particular, if the Cauchy axiom holds, namely the internal actions in $\mathscr{B}$ can be represented only by a vector field, in the absence of body couples the balance of angular momentum requires that the Cauchy stress tensor T is symmetric, i.e. $\mathsf{T} = \mathsf{T}^T$, and the equations of mass, linear momentum and energy read

$$\dot{\rho} + \rho \mathrm{div} \boldsymbol{v} = 0, \tag{1}$$

$$\rho \dot{\boldsymbol{v}} = \mathrm{div} \mathsf{T} + \rho \boldsymbol{b}, \tag{2}$$

$$\rho \dot{e} + \mathrm{div} \boldsymbol{q} = \mathsf{T} \cdot \mathsf{D} + \rho r, \tag{3}$$

respectively. In (1)–(3) ρ denotes the mass density, $\boldsymbol{v}$ the velocity field, $\boldsymbol{b}$ the specific body force, e the specific internal energy, r the specific radiant heating, $\boldsymbol{q}$ the heat flux vector, and D the rate-of-strain tensor, i.e. the symmetric part of the velocity gradient $\mathsf{L} = \nabla \boldsymbol{v}$. The superimposed dot denotes the material time derivative.[1] The second law of Thermodynamics is instead usually written as the Clausius-Duhem inequality

$$\rho \dot{\eta} \geq \rho \frac{r}{\theta} - \mathrm{div} \left(\frac{\boldsymbol{q}}{\theta} \right), \tag{4}$$

with η denoting the specific entropy and θ the temperature.

Regarding the specific body force $\boldsymbol{b}$ as known, Eqs. (1)–(3) provide seven scalar equations for 13 scalar fields—the mass density ρ, the velocity components v_i ($i = 1, 2, 3$), and the stress tensor components T_{ij} ($i, j = 1, 2, 3$). The system of PDEs (1)–(3) is then not closed. The disparity in the number of equations and unknowns is however not surprising as (1)–(3) are valid for all the non-polar materials (i.e. materials satisfying the Cauchy axiom), but do not differentiate the special material the body is made of. Therefore, to obtain a system the number of equations of which matches the number of unknowns, one has to introduce some constitutive equations characterizing the thermomechanical response of the material.

Constitutive equations are very often referred to as constitutive relations. According to the Cambridge dictionary, a *relation* is *a connection or similarity between two things*. This is exactly what modelers well educated in mechanics usually do. They connect thermodynamical quantities through an equation which is specific to a particular material or substance.

[1]For the sake of self-consistency, if Σ is a smooth scalar, vector or tensor field defined on the trajectory of the body $\mathscr{B}$, $\dot{\Sigma} = \Sigma_t + (\boldsymbol{v} \cdot \nabla)\Sigma$.

For example, in classical fluid mechanics the stress tensor is related to the rate-of-strain tensor D, the density of the fluid and the temperature through an equation of the form

$$\mathsf{H}(\mathsf{T}, \mathsf{D}, \rho, \theta) = \mathsf{O}. \tag{5}$$

Such a relationship is sufficient to close the subsystem (1)–(3) and thus there is no clear reason (no *reddendae rationis*) to consider *a priori* an *explicit* representation of the form

$$\mathsf{T} = \hat{\mathsf{T}}(\mathsf{D}, \rho, \theta), \tag{6}$$

such as the Navier-Stokes constitutive equation:

$$\mathsf{T} = -p_{|E}(\rho, \theta)\mathsf{I} + \lambda(\rho, \theta)\mathrm{tr}(\mathsf{D})\mathsf{I} + 2\mu(\rho, \theta)\mathsf{D}, \tag{7}$$

where $p_{|E}$ is the pressure at thermodynamic equilibrium, and λ and μ are the bulk and shear viscosity, respectively. The reason of considering a representation of the form (6) stands exclusively in its mathematical ease.

The search for models of mathematical ease is common when dealing with problem of the motion of real bodies. A simple example comes from particle mechanics. It is well known that the motion of a free particle X is governed by Newton's second law

$$m\boldsymbol{a} = \boldsymbol{F}, \tag{8}$$

where m is the mass of the particle, $\boldsymbol{a}$ is the acceleration vector, and $\boldsymbol{F}$ is the resultant force acting on X. In direct problems the resultant force $\boldsymbol{F}$ is usually known, and the motion is to be determined by solving (8) under prescribed initial conditions for the position $\boldsymbol{x}$ and the velocity $\boldsymbol{v}$ of X. To solve uniquely the resulting Cauchy problem, one assumes that any *experimental* model for the force $\boldsymbol{F}$ depends on the motion of the particle through a relation of the form $\boldsymbol{F} = \widehat{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{v}, t)$, where $\widehat{\boldsymbol{F}}$ satisfies the smoothness assumptions of Cauchy's theorem for ordinary differential equations.

Behind the introduction of a constitutive relation of the form (6) there is then the expectation that, in the framework of a field theory like continuum mechanics, any initial and boundary value problem (IBVP) governing the motion of a real body admits a unique solution as the Cauchy problem governing the motion of a free particle in the framework of Newtonian's particle mechanics.

Jacques Hadamard [18] introduced the concept of well-posedness of IBVPs. An IBVP is well posed if it admits a unique solution which depends continuously on the data (that is on initial and boundary conditions). Well-posedness in the sense of Hadamard has always influenced strongly applied mathematicians and scholars involved in continuum mechanics research when modeling a real-world phenomenon. From a mathematical point of view it is clear that well-posedness is intriguing. From the point of view of continuum mechanics well-posedness is closely related to the concept of determinism. From this perspective, the principle of determinism for the stress stated in Truesdell and Noll's celebrated handbook [52],

> The stress in a body is determined by the history of the motion of that body,

can be regarded as a sort of 'pathway' to well-posedness.

John Ball and Richard James [5] commenting on the concept of well-posedness in the sense of Hadamard wrote:

> Any reader of that paper will see the unmistakable influence of Truesdell. At the end of the day, perhaps it would have been realized that Hadamard's notions of well-posedness are far too restrictive in the nonlinear setting, that non-uniqueness and even non-existence comprise acceptable behavior, and that there are probably no fundamental restrictions on the strain-energy function at all besides those arising from material symmetry and frame-indifference.

This means not only that the principle of determinism is highly questionable from a physical point of view, but also that *mathematical feasibility* is not a good argument to support explicit constitutive equations where stress is given in terms of motion.

Nature does not care if the relationship at the basis of its phenomena are graphs or functions. It is our idealization of natural phenomena that sometimes realizes the fact that relationships in the form of functions are more convenient. Moreover, an explicit approach rules out *a priori* the possibility to describe interesting phenomena like, for instance, constitutive branching. There is then no advantage to sacrifice the *principium reddendae rationis* on the altar of mathematical well-posedness and there are no *a priori* physical reason to support explicit constitutive equations.

It is well known that in an experiment it is possible to control stress or deformation. For example, when pulling a bar of steel it is possible to control the engineering stress and to record the engineering strain (the test is said to be performed in a *soft* device), or to control the engineering strain and to record the engineering stress (test performed in a *hard* device). Therefore, in the first case why can we not postulate that the motion in a body is determined by the history of the stress of that body? The fact that constitutive relationships are described by graphs and not functions is also confirmed by several experimental works (see [37] and references therein).

The aim of these lectures is to investigate classical problems in fluid mechanics (such as the onset of Rayleigh-Bénard convection, laminar flows and flows over an inclined plane) by employing implicit constitutive relations for the stress tensor. We do not claim that the use of such implicit constitutive models represents a scientific revolution. In the light of the previous discussion, we only claim that implicit constitutive equations are useful tools for investigating real-world phenomena and there is no a clear and neat reason to throw away *a priori* this class of relations.

These notes are not meant to be a detailed review of the literature concerning piezo-viscous fluids. The choices of the topics treated and the literature presented are based mainly on our personal tastes and pedagogical aims. The level of these lecture notes is basic and tailored for undergraduate students with an elementary knowledge of continuum mechanics.

2 Implicit Constitutive Models for the Cauchy Stress Tensor

Consider the constitutive relation for the Cauchy stress tensor T of a linearly viscous fluid given by

$$\mathsf{T} = -p\mathsf{I} + 2\mu\left[\mathsf{D} - \frac{1}{3}\mathrm{tr}(\mathsf{D})\mathsf{I}\right], \tag{9}$$

where

$$p = -\frac{1}{3}\mathrm{tr}\mathsf{T} \tag{10}$$

is the pressure and the positive parameter μ is the viscosity.

A quick look reveals that (9)–(10) differs from the classical constitutive model for the Cauchy stress

$$\mathsf{T} = -p\mathsf{I} + 2\mu\mathsf{D}. \tag{11}$$

In fact, contrarily to (11), in (9) and (10) it is explicitly stated what it is meant by "pressure": the negative of the mean normal stress. On using the terminology widely adopted in the literature, by "pressure" we mean the "mechanical pressure". It is extremely important point out this from the beginning not only because we are going to examine some aspects of the flows in fluids with material parameters depending on pressure, but also because, as observed by Rajagopal [36], the term "pressure" has been used in a plethora of different contexts and, as an unavoidable consequence, it has been often misused in the literature. Referring the interested reader to [36] for a detailed discussion on the issues related with the usage of the word "pressure", we limit to observe that, as argued by Huilgol [20], if one wishes to include pressure in the rheological material functions (as we intend to), defining the pressure through (10) is the only unambiguous way of introducing such a physical quantity. In addition, this definition can be used to interpret experimental data systematically both for compressible and incompressible fluids (the motions of which, as is well known, are subjected to the kinematic constraint $\mathrm{div}\boldsymbol{v} = \mathrm{tr}(\mathsf{D}) = 0$) [20].

It is widely accepted that the viscosity of fluids depends on temperature. In particular, experimental observations have shown that viscosity decreases with increasing temperature. On the other hand, there is also a vast literature on the dependence of the viscosity on pressure. For the sake of brevity, below, we report only some of the most important studies on this topic.

The first scholar to realize that the viscosity of a fluid may depend on pressure was Stokes. In fact, in his celebrated paper [49] on the constitutive response of fluids, Stokes stated

> Let us now consider in what cases it is allowable to suppose μ to be independent of the pressure. It has been concluded by Du Buat from his experiments on the motion of water in

> pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure… I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of the pressure.

Stokes's comment clearly implies that only in special circumstances the viscosity of a fluid is independent of pressure. While for flows in canals and pipes under normal conditions inclusion of the dependence of the viscosity on pressure does not affect the results of the experiments, there are several other situations where one needs to take this dependence into account. Eight decades later, Bridgman [10] gave a measure of the effect of pressure on the viscosity of water as well as of other forty-two pure liquids. In addition, Bridgman observed that, while it is true that all the physical quantities do vary with pressure, the variation in the viscosity with pressure may be far more dramatic than the variation of the other quantities with pressure. To this aim, Bridgman reported:

> It may be said in general that the effects of pressure on viscosity are greater than on any other physical property hitherto measured,[2] and vary very widely with the nature of the liquid. The increase of viscosity produced by 12000 kg varies from two or three fold to millions of fold for the liquids investigated here, whereas such properties as the volume decrease under 12000 kg seldom vary by as much as a factor of two from substance to substance.

As early as 1893, based on experiments on marine glue, Barus [6] proposed an empirical relation between the viscosity μ and the pressure p of the form

$$\mu(p, \theta) = \mu_{\mathrm{ref}} \exp[\beta(\theta)(p - p_{\mathrm{ref}})], \tag{12}$$

where μ_{ref} is the viscosity at the reference pressure p_{ref}, and the piezo-viscous coefficient β is temperature dependent. Later, Andrade [2] proposed a model expressing the viscosity in terms of the pressure, the mass density and the temperature, namely

$$\mu(p, \rho, \theta) = A\rho^{1/2} \exp[(p + \rho r^2)s/\theta], \tag{13}$$

where r, s and A are constants. References to much of the literature concerning the pressure dependence of the viscosity of fluids prior 1931 can be found in the book of Bridgman [11]. More recently, Laun has modeled the viscosity of polymer melts through

$$\mu(p, \theta) = \mu_{\mathrm{ref}} \exp[\beta(p - p_{\mathrm{ref}}) - \delta(\theta - \theta_{\mathrm{ref}})], \tag{14}$$

where μ_{ref} is the viscosity at the reference state (p_{ref}, θ_{ref}), and β and δ are positive constants. There have been numerous other experiments by Bair and coworkers that shows that the dependence of the viscosity on pressure is exponential (see the experiments of Bair and Kottke [4]). Mention must be made of the work of

[2] The other physical properties measured by Bridgman are the isothermal compressibility, the thermal expansion coefficient, the specific heat and the thermal conductivity.

Martín-Alfonso and co-workers [26] wherein an intricate relationship among the temperature, viscosity and pressure is provided for bitumen. In this context, it ought to be pointed out that the pressure dependence of the properties of bitumen was recognized very early. For instance, Saal and Koens [46] not only allowed for viscosity to depend on pressure (and hence on the mean normal stress), they also allowed it to depend on the shear stresses.

In virtue of the experimental evidences reported above, it is then reasonable to assume that the viscosity of a fluid depends on pressure and temperature. Consequently, since we have defined the pressure as the negative mean normal stress, the constitutive model (9) with $\mu = \mu(p, \theta)$ prescribes the Cauchy stress tensor in terms of the strain-rate tensor and temperature through the implicit relation

$$\mathsf{T} - \frac{1}{3}\mathrm{tr}(\mathsf{T})\mathsf{I} - 2\mu\left(-\frac{1}{3}\mathrm{tr}(\mathsf{T}), \theta\right)\mathsf{D} = \mathsf{O}. \tag{15}$$

In the following sections, we shall mainly use the implicit model (15) or its variants. However, for the sake of generality, we now determine the most general implicit model for the Cauchy stress of an isotropic fluid. We start with an implicit relation of the form

$$\mathsf{G}(\mathsf{T}, \mathsf{D}, \theta) = \mathsf{O}. \tag{16}$$

Since the fluid is isotropic, G is an isotropic tensor function of the two second-order tensors T and D, i.e. G satisfies the property

$$\mathsf{G}(\mathsf{Q}\mathsf{T}\mathsf{Q}^T, \mathsf{Q}\mathsf{D}\mathsf{Q}^T, \theta) = \mathsf{Q}\mathsf{G}(\mathsf{T}, \mathsf{D}, \theta)\mathsf{Q}^T, \tag{17}$$

for all proper orthogonal tensors Q. Next, following Spencer [48], the most general implicit model for the Cauchy stress tensor of an isotropic fluid can be written as

$$\begin{aligned} &\alpha_0\mathsf{I} + \alpha_1\mathsf{T} + \alpha_2\mathsf{D} + \alpha_3\mathsf{T}^2 + \alpha_4\mathsf{D}^2 + \alpha_5(\mathsf{T}\mathsf{D} + \mathsf{D}\mathsf{T}) \\ &\quad + \alpha_6(\mathsf{T}^2\mathsf{D} + \mathsf{D}\mathsf{T}^2) + \alpha_7(\mathsf{T}\mathsf{D}^2 + \mathsf{D}^2\mathsf{T}) + \alpha_8(\mathsf{T}^2\mathsf{D}^2 + \mathsf{D}^2\mathsf{T}^2) = \mathsf{O}, \end{aligned} \tag{18}$$

where the coefficients α_i, $i = 0, 1, \ldots, 8$, depend on θ and the integrity basis of the two tensors T and D. The integrity basis consists of the invariants of any combination of tensor products up to second order. For the current problem, these are given by

$$\begin{gathered} \mathrm{tr}(\mathsf{T}),\ \mathrm{tr}(\mathsf{D}),\ \mathrm{tr}(\mathsf{T}^2),\ \mathrm{tr}(\mathsf{D}^2),\ \mathrm{tr}(\mathsf{T}^3),\ \mathrm{tr}(\mathsf{D}^3), \\ \mathrm{tr}(\mathsf{T}\mathsf{D}),\ \mathrm{tr}(\mathsf{T}^2\mathsf{D}),\ \mathrm{tr}(\mathsf{T}\mathsf{D}^2),\ \mathrm{tr}(\mathsf{T}^2\mathsf{D}^2). \end{gathered} \tag{19}$$

This is a minimal set of invariants since the trace of the product of two second-order Cartesian tensors is equal to the trace of the tensor product with the factors written in reverse.

When we consider fluid models of the form (18)–(19), with

$$\alpha_0 = -\frac{1}{3}\mathrm{tr}(\mathsf{T}) + 2\mu\left(-\frac{1}{3}\mathrm{tr}(\mathsf{T}), \theta\right)\mathrm{tr}(\mathsf{D}),$$
$$\alpha_1 = 1, \quad \alpha_2 = -2\mu\left(-\frac{1}{3}\mathrm{tr}(\mathsf{T}), \theta\right), \tag{20}$$

and all the remaining α_i equal to zero, we recover the model (15) for a piezo-viscous fluid.

It might seem that the previous discussion on implicit constitutive models for the Cauchy stress tensor is valid only for compressible isotropic fluids. The reader might be led to such a conclusion by the usual practice in continuum mechanics to associate a Lagrange multiplier with the constraint of incompressibility, split the stress tensor into the sum of the constraint stress T_C and the extra stress T_E, and assume that the constraint stress is workless and independent of the state variables, and the extra stress is independent of T_C [52]. This conclusion is not right and the discussion above can be easily adapted to incompressible isotropic fluids. In fact, Rajagopal [35] showed that, when dealing with incompressible fluids, appropriate choices of the material parameters α_i in (18) guarantee the incompressibility without the introduction of a Lagrange multiplier and any split of the stress tensor. In these notes we shall not introduce special models for incompressible fluids because, as we shall show in the following section, incompressibility is an approximation that is valid under specific flow regimes. Therefore, when dealing with these flow regimes, there is no need at all to introduce *a priori* appropriate models which automatically meet the kinematic restriction of invariability of volume elements during motion.

3 Isochoric Motions of Fluids as Approximations Under Different Flow Regimes

All real bodies are compressible. In fact, if a sufficiently high pressure is employed, the body undergoes a reduction in volume. On the contrary, for most liquids in Nature, experience teaches that volume increases with increasing temperature. However, it is possible that some bodies do not undergo a significant change in volume over a sufficiently large ranges of pressures or temperatures and can hence be approximated as being incompressible in those ranges. When the ranges of pressures and temperatures are what is considered ‘normal’, in view of day to day applications, the body is considered to be incompressible. Of course, what is deemed to be a ‘significant change in volume’ is quite arbitrary and it boils down to whether neglecting the volume change and modeling the body as an incompressible body yet captures the essential features of the response of the body when subject to external stimuli. Most liquids can be approximated as incompressible liquids provided the pressures to which they are subject to are not very high and temperature changes are small enough. On the other hand, if the ranges of pressures and temperatures to which the liquid is subject are large, then volume changes do take place and, moreover, all the properties that characterize a fluid must be considered pressure- and temperature-dependent.

Müller [29] defined a body to be incompressible if the density and the internal energy depend only on temperature and do not depend on pressure. Appealing to the material frame indifference and the entropy principle Müller [29] showed that a Navier-Stokes-Fourier fluid cannot undergo changes in volume due to temperature changes. Such a behavior is clearly contradicted by experiments which show that volume changes do take place with temperature. Motivated by the fact that experimental evidence clearly contradicts Müller's conclusion (often referred to as the *Müller paradox*), Gouin et al. [16] studied a class of Navier-Stokes-Fourier fluids for which the internal energy, shear and bulk viscosities, and thermal conductivity depend on pressure and temperature, while the density depends only on temperature. They referred to such materials as quasi-thermally compressible fluids and found a critical value of the pressure, denoted by p_{cr}, below which a quasi-thermally compressible fluid behaves like a perfectly compressible fluid in Müller's sense. Since the value of p_{cr} is large with respect to the normal pressure conditions (for instance, for water at 20 °C, $p_{cr} \simeq 2 \times 10^5$ atm), Gouin et al. [16] concluded that a quasi thermal-incompressible fluid is experimentally similar to a perfectly incompressible fluid, removing in a such a way the Müller paradox.

The analysis of Gouin et al. [16] is based on the assumption that the density depends only on the temperature. Clearly, for a homogeneous fluid this assumption is equivalent to assuming that the deformation gradient F depends only on the temperature, i.e.

$$\det \mathsf{F} = \varphi(\theta). \tag{21}$$

Recently, Rajagopal et al. [42] proved that assumption (21) leads to three physically unrealistic deductions:

- the specific heat at constant volume is zero,
- thermodynamic instability (the specific entropy fails to be a concave function of the pressure and specific volume),
- imaginary speed of sound.

To overcome these drawbacks Rajagopal et al. [42] modified the assumption (21) by postulating that

$$\det \mathsf{F} = \varphi(p, \theta), \tag{22}$$

and showed that, for several classes of flow regimes of interest in the applications, the velocity field of a fluid with pressure and temperature dependent material properties is, to a first approximation, solenoidal. Therefore, in comparison to the findings by Gouin et al. [16], instead of determining pressure ranges in which a real fluid behaves like an idealized incompressible fluid, Rajagopal et al. [42]

determined the flow regimes in which the motions of a fluid can be regarded, to a first approximation, as isochoric.

3.1 Equations Governing the Flows in a Piezo-Viscous Fluid

To rigorously derive the sets of approximated equations we shall employ in the following sections, we follow the same procedure as in [42]. We start by assuming that the fluid is slightly compressible due to variations in the pressure and temperature and thus assume that (22) holds.

We also assume that the motion of the fluid is sufficiently smooth so that the derivatives that are taken are meaningful. Then, differentiating the determinant of the deformation gradient with respect to time yields

$$\mathrm{div}\boldsymbol{v} = -k_T(p,\theta)\dot{p} + \alpha(p,\theta)\dot{\theta}, \tag{23}$$

where

$$k_T = -\frac{1}{\varphi}\frac{\partial \varphi}{\partial p}, \quad \alpha = \frac{1}{\varphi}\frac{\partial \varphi}{\partial \theta} \tag{24}$$

are the isothermal compressibility and the coefficient of thermal expansion, respectively. Clearly, k_T and α are related through the integrability condition

$$\frac{\partial k_T}{\partial \theta} = -\frac{\partial \alpha}{\partial p}. \tag{25}$$

From (1) and (23) we deduce that

$$\frac{\dot{\rho}}{\rho} = k_T\dot{p} - \alpha\dot{\theta}. \tag{26}$$

Hence

$$\mathrm{d}\rho = \rho(k_T\mathrm{d}p - \alpha\mathrm{d}\theta), \tag{27}$$

and, denoting $v = 1/\rho$ the specific volume,

$$\mathrm{d}v = -\frac{k_T}{\rho}\mathrm{d}p + \frac{\alpha}{\rho}\mathrm{d}\theta. \tag{28}$$

Next, we introduce the enthalpy

$$h = e + pv, \tag{29}$$

and the Gibbs free enthalpy

$$g = h - \theta\eta, \tag{30}$$

and combine these two thermodynamic potentials with the balance equations (1)–(3) and the Clausius-Duhem inequality (4) to get

$$\rho(\dot{g} + \eta\dot{\theta}) - \dot{p} - p\,\mathrm{div}\boldsymbol{v} - \mathsf{T}\cdot\mathsf{D} + \frac{\boldsymbol{q}}{\theta}\cdot\nabla\theta \le 0. \tag{31}$$

In our analysis we shall regard the mechanical pressure and the temperature as independent variables on which the material parameters of the fluid depend, and, since we are interested in fluids of grade 1 (see Truesdell and Noll [52]), the requirement of material frame indifference and the representation theorems for isotropic functions lead us to consider the Cauchy stress tensor to be constitutively prescribed by the implicit relation (15), the response functions of the specific internal energy and entropy to be of the form

$$e = \hat{e}(p, \theta, \mathrm{tr}(\mathsf{D})), \quad \eta = \hat{\eta}(p, \theta, \mathrm{tr}(\mathsf{D})), \tag{32}$$

and the heat flux vector given by the Fourier law

$$\boldsymbol{q} = -k(p, \theta)\nabla\theta, \tag{33}$$

with k being the thermal conductivity. Finally, we introduce the specific heats at constant pressure and at constant volume through

$$c_p = \left(\frac{\partial h}{\partial \theta}\right)_p, \quad c_v = \left(\frac{\partial e}{\partial \theta}\right)_v, \tag{34}$$

respectively, and the specific heat ratio $\gamma = c_p/c_v$ [15].

Inserting (9), (23) and (33) into (31) yields the inequality

$$\left(\rho\frac{\partial g}{\partial p} - 1\right)\dot{p} + \rho\left(\frac{\partial g}{\partial \theta} + \eta\right)\dot{\theta} + \rho\frac{\partial g}{\partial \mathrm{tr}(\mathsf{D})}\overline{\mathrm{tr}(\mathsf{D})}^{\boldsymbol{\cdot}} - 2\mu\left\{\|\mathsf{D}\|^2 - \frac{1}{3}[\mathrm{tr}(\mathsf{D})]^2\right\} - \frac{k}{\theta}\|\nabla\theta\|^2 \le 0, \tag{35}$$

that holds true for any thermodynamical processes, i.e. for any fields ρ, $\boldsymbol{v}$ and θ satisfying the balance equations (1)–(3). Therefore, by using standard arguments in continuum thermodynamics, we deduce that

$$\frac{\partial g}{\partial p} = \frac{1}{\rho}, \quad \frac{\partial g}{\partial \theta} = -\eta, \quad \frac{\partial g}{\partial \mathrm{tr}(\mathsf{D})} = 0, \tag{36}$$

and the constitutive functions for the viscosity of the fluid μ and the thermal conductivity k are non-negative.

From (36) we deduce that the differential of the Gibbs free enthalpy is

$$\mathrm{d}g = \frac{1}{\rho}\mathrm{d}p - \eta \mathrm{d}\theta \tag{37}$$

and the constitutive functions for the specific internal energy and entropy are, respectively, of the form $e = \hat{e}(p, \theta)$ and $\eta = \hat{\eta}(p, \theta)$ (namely, both the specific internal energy and the specific entropy do not depend on $\mathrm{tr}(\mathsf{D})$), with

$$\left(\frac{\partial \eta}{\partial p}\right)_\theta = -\frac{\alpha}{\rho}. \tag{38}$$

Next, combining (29) and (30) with (37) yields

$$\theta \mathrm{d}\eta = \mathrm{d}e + p\mathrm{d}v \tag{39}$$

which in turn leads to

$$\theta \left(\frac{\partial \hat{\eta}}{\partial \theta}\right)_v = \left(\frac{\partial \hat{e}}{\partial \theta}\right)_v = c_v, \tag{40}$$

and then by virtue of (27) and (28) we obtain

$$\theta \left(\frac{\partial \hat{\eta}}{\partial \theta}\right)_p = \left(\frac{\partial \hat{e}}{\partial \theta}\right)_p + \frac{\alpha}{\rho}p = \left(\frac{\partial h}{\partial \theta}\right)_p = c_p. \tag{41}$$

As far the specific internal energy is concerned, from $(36)_1$, (38) and (41) we have

$$\left(\frac{\partial \hat{e}}{\partial p}\right)_\theta = \frac{k_T p - \alpha\theta}{\rho}, \quad \left(\frac{\partial \hat{e}}{\partial \theta}\right)_p = c_p - \frac{\alpha}{\rho}p. \tag{42}$$

Finally, by using (9), (23) and (42) the equations of balance of linear momentum (2) and energy (3) can be expressed as

$$\rho \dot{\boldsymbol{v}} = -\nabla p + 2\mathrm{div}\left\{\mu\left[\mathsf{D} - \frac{1}{3}(\mathrm{div}\boldsymbol{v})\mathsf{I}\right]\right\} + \rho \boldsymbol{b} \tag{43}$$

and

$$-\alpha\theta\dot{p} + \rho c_p \dot{\theta} = \mathrm{div}(k\nabla\theta) + 2\mu\left[\|\mathsf{D}\|^2 - \frac{1}{3}(\mathrm{div}\boldsymbol{v})^2\right] + \rho r, \tag{44}$$

respectively. Equations (26), (23), (43) and (44) constitute a system of partial differential equations for determining the thermodynamic fields ρ, $\boldsymbol{v}$, p and θ.

3.2 Approximations

In order to introduce the most appropriate non-dimensionalization, it is necessary to record before some thermodynamic identities. Rajagopal et al. [42] observed that the specific heats ratio $\gamma > 1$, the isothermal compressibility is related to the speed of sound C in the fluid through

$$k_T = \frac{\gamma}{\rho C^2}, \tag{45}$$

and the square of the coefficient of thermal expansion can be written as

$$\alpha^2 = \frac{c_p(\gamma - 1)}{C^2\theta}. \tag{46}$$

Let $Oxyz$ be a Cartesian frame of reference with orthonormal basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. Let $\Omega_d = \mathbb{R}^2 \times [0, d]$ be a horizontal fluid layer of thickness d and assume that gravity is the only body force acting on the fluid, namely $\boldsymbol{b} = -g\boldsymbol{k}$, where g is the acceleration due to gravity.[3] We assume also that no heat is supplied, i.e., $r = 0$. To non-dimensionalize the equations governing the fluid motion, we choose a convenient reference state $(p_{\text{ref}}, \theta_{\text{ref}})$ and introduce the following scales

$$\begin{gathered} \boldsymbol{x}^* = \frac{\boldsymbol{x}}{d}, \quad \boldsymbol{v}^* = \frac{\boldsymbol{v}}{V}, \quad \rho^* = \frac{\rho}{\rho_{\text{ref}}}, \quad t^* = \frac{V}{d}t, \\ p^* = \frac{p - p_{\text{ref}}}{\rho_{\text{ref}} g d}, \quad \theta^* = \frac{\theta - \theta_{\text{ref}}}{\theta_M - \theta_m}, \quad \alpha^* = \frac{\alpha}{\alpha_{\text{ref}}}, \quad C^* = \frac{C}{C_{\text{ref}}}, \\ \mu^* = \frac{\mu}{\mu_{\text{ref}}}, \quad c_p^* = \frac{c_p}{c_{p\text{ref}}}, \quad k_T^* = \frac{\rho_{\text{ref}} C_{\text{ref}}^2}{\gamma_{\text{ref}}} k_T. \end{gathered} \tag{47}$$

In (47) the subscript 'ref' indicates that the corresponding material parameters are evaluated at the reference state $(p_{\text{ref}}, \theta_{\text{ref}})$, V is the reference velocity, $\theta_M = \max_\Omega \theta$, $\theta_m = \min_\Omega \theta$ and the isothermal compressibility has been scaled by taking into account (45). Hereinafter, we choose θ_M as the reference temperature, *viz* $\theta_{\text{ref}} = \theta_M$.

Substituting (47) into (23), (26), (43) and (44) yields the dimensionless equations (omitting the asterisks for convenience)

$$\text{div}\boldsymbol{v} = -\gamma_{\text{ref}} \frac{\text{Ma}^2}{\text{Fr}^2} \rho k_T \dot{p} + \alpha_{\text{ref}}(\theta_M - \theta_m)\alpha\dot{\theta}, \tag{48}$$

$$\dot{\rho} = \gamma_{\text{ref}} \frac{\text{Ma}^2}{\text{Fr}^2} \rho k_T \dot{p} - \alpha_{\text{ref}}(\theta_M - \theta_m)\rho\alpha\dot{\theta}, \tag{49}$$

[3] Assuming that Ω is a horizontal layer is convenient for deriving the set of approximations we shall adopt in this paper. However, the analysis we are going to perform can be adapted, by means of slight changes, to the case in which Ω is bounded in one direction provided that such a direction is non-horizontal.

$$\mathrm{Fr}^2\rho\dot{\boldsymbol{v}} = -\nabla p + 2\frac{\mathrm{Fr}^2}{\mathrm{Re}}\mathrm{div}\left\{\mu\left[\mathsf{D} - \frac{1}{3}(\mathrm{div}\boldsymbol{v})\mathsf{I}\right]\right\} - \rho\boldsymbol{k}, \tag{50}$$

$$\begin{aligned} - \alpha_{\mathrm{ref}}(\theta_M - \theta_m)\frac{\mathrm{ReBr}}{\mathrm{Fr}^2}\alpha\left(\theta + \frac{1}{\mathrm{Ca}}\right)\dot{p} + \mathrm{Pe}\rho c_p\dot{\theta} &= \mathrm{div}(k\nabla\theta) \\ &+ 2\mathrm{Br}\mu\left[\|\mathsf{D}\|^2 - \frac{1}{3}(\mathrm{div}\boldsymbol{v})^2\right], \end{aligned} \tag{51}$$

where

$$\begin{gathered} \mathrm{Ma} = \frac{V}{C_{\mathrm{ref}}}, \quad \mathrm{Fr}^2 = \frac{V^2}{gd}, \quad \mathrm{Re} = \frac{\rho_{\mathrm{ref}}Vd}{\mu_{\mathrm{ref}}}, \\ \mathrm{Br} = \frac{\mu_{\mathrm{ref}}V^2}{k_{\mathrm{ref}}(\theta_M - \theta_m)}, \quad \mathrm{Pe} = \frac{\rho_{\mathrm{ref}}c_{p_{\mathrm{ref}}}Vd}{k_{\mathrm{ref}}}, \quad \mathrm{Ca} = \frac{\theta_M - \theta_m}{\theta_M} \end{gathered} \tag{52}$$

are the Mach, second Froude, Reynolds, Brinkman, Péclet and Carnot numbers, respectively.

We now assume that the material parameters α, k_T, c_p and k are analytic functions and limit our analysis to the departures of the pressure and temperature from the reference state $(p_{\mathrm{ref}}, \theta_M)$ for which we can write

$$\alpha(p,\theta) = \sum_{j_1,j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{j_1+j_2}\alpha}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2}, \tag{53}$$

$$k_T(p,\theta) = \sum_{j_1,j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{j_1+j_2}k_T}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2}, \tag{54}$$

$$c_p(p,\theta) = \sum_{j_1,j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{j_1+j_2}c_p}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2}, \tag{55}$$

and

$$k(p,\theta) = \sum_{j_1,j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{j_1+j_2}k}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2}. \tag{56}$$

From the integrability condition (25), the expansions (53) and (54) and the scales (47) we deduce that

$$\gamma_{\mathrm{ref}}\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\frac{\partial^{j_1+j_2}k_T}{\partial p^{j_1-1}\partial\theta^{j_2+1}}(0,0) = -\alpha_{\mathrm{ref}}(\theta_M - \theta_m)\frac{\partial^{j_1+j_2}\alpha}{\partial p^{j_1}\partial\theta^{j_2}}(0,0), \tag{57}$$

for all $(j_1, j_2) \in \mathbb{N} \times \mathbb{N}_0$. In virtue of (57) we can integrate equation (49) to obtain

$$\rho = \exp\Bigg[-\alpha_{\text{ref}}(\theta_M - \theta_m) \sum_{j_1,j_2=0}^{+\infty} \frac{1}{j_1!(j_2+1)!} \frac{\partial^{j_1+j_2}\alpha}{\partial p^{j_1} \partial \theta^{j_2}}(0,0) p^{j_1} \theta^{j_2+1} \tag{58}$$
$$+ \gamma_{\text{ref}} \frac{\text{Ma}^2}{\text{Fr}^2} \sum_{j_1=0}^{+\infty} \frac{1}{(j_1+1)!} \frac{\partial^{j_1} k_T}{\partial p^{j_1}}(0,0) p^{j_1+1} \Bigg].$$

Evaluating the identity (46) at the reference state $(p_{\text{ref}}, \theta_M)$ yields the following relation

$$\alpha_{\text{ref}}^2(\theta_M - \theta_m)^2 = \frac{\text{Ma}^2}{\text{E}} \text{Ca}(\gamma_{\text{ref}} - 1), \tag{59}$$

where

$$\text{E} = \frac{V^2}{c_{p_{\text{ref}}}(\theta_M - \theta_m)} \tag{60}$$

is the Eckert number. We henceforth consider thermodynamic processes for which

$$\alpha_{\text{ref}}(\theta_M - \theta_m) \equiv \varepsilon \ll 1. \tag{61}$$

Therefore, as long as $\text{E}/[\text{Fr}^2\text{Ca}(\gamma_{\text{ref}} - 1)]$ is of order $O(1)$ or smaller, from (59) we deduce that Ma^2/Fr^2 is of order $O(\varepsilon^2)$ or smaller.

We are now in position to carry out a perturbation analysis with respect to the small parameter ε. Let

$$\boldsymbol{v} = \sum_{n=0}^{+\infty} \varepsilon^2 \boldsymbol{v}_n, \quad p = \sum_{n=0}^{+\infty} \varepsilon^n p_n, \quad \theta = \sum_{n=0}^{+\infty} \varepsilon^n \theta_n \tag{62}$$

be the power series in ε of the thermodynamic fields $\boldsymbol{v}$, p and θ. As far as the power series expansion of the fluid density is concerned, it may be derived from (58) and $(62)_{2,3}$ by taking into account the fact that Ma^2/Fr^2 is of order $O(\varepsilon^2)$. However, the expression is quite complicated and of no interest to our analysis. In our analytical scheme it suffices to know that

$$\rho = 1 - \varepsilon \sum_{j_1,j_2=0}^{+\infty} \frac{1}{j_1!(j_2+1)!} \frac{\partial^{j_1+j_2}\alpha}{\partial p^{j_1} \partial \theta^{j_2}}(0,0) p_0^{j_1} \theta_0^{j_2+1} + o(\varepsilon) \tag{63}$$
$$= 1 - \varepsilon \int_0^{\theta_0} \alpha(p_0, \theta_0) \mathrm{d}\theta_0 + o(\varepsilon),$$

where $o(\varepsilon)$ accounts for terms of order $O(\varepsilon^2)$ and higher. Therefore, by inserting (62) and (63) into (48), (50) and (51) we deduce a system of equations from which different approximations can be derived:

$$\sum_{n=0}^{+\infty} \varepsilon^n \mathrm{div}\boldsymbol{v}_n = -\gamma_{\mathrm{ref}} \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \sum_{n=0}^{+\infty} \varepsilon^n \left[k_T(p,\theta) \left(\frac{\partial p}{\partial t} + \boldsymbol{v} \cdot \nabla p \right) \right]_n + \sum_{n=0}^{+\infty} \varepsilon^{n+1} \left[\alpha(p,\theta) \left(\frac{\partial \theta}{\partial t} + \boldsymbol{v} \cdot \nabla \theta \right) \right]_n , \tag{64}$$

$$\begin{aligned} &\mathrm{Fr}^2 \sum_{n=0}^{+\infty} \varepsilon^n \left\{ \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] \right\}_n \\ &= -\sum_{n=0}^{+\infty} \varepsilon^n \nabla p_n + 2\frac{\mathrm{Fr}^2}{\mathrm{Re}} \sum_{n=0}^{+\infty} \varepsilon^n \left\{ \mathrm{div} \left[\mu(p,\theta) \left(\mathsf{D} - \frac{1}{3}(\mathrm{div}\boldsymbol{v})\mathsf{I} \right) \right] \right\}_n \\ &\quad - \left[1 - \varepsilon \int_0^{\theta_0} \alpha(p_0,\theta_0)\mathrm{d}\theta_0 + o(\varepsilon) \right] \boldsymbol{k} \end{aligned} \tag{65}$$

and

$$\begin{aligned} &-\frac{\mathrm{ReBr}}{\mathrm{Fr}^2} \sum_{n=0}^{+\infty} \varepsilon^{n+1} \left[\alpha(p,\theta) \left(\theta + \frac{1}{\mathrm{Ca}} \right) \left(\frac{\partial p}{\partial t} + \boldsymbol{v} \cdot \nabla p \right) \right]_n \\ &+\mathrm{Pe} \sum_{n=0}^{+\infty} \varepsilon^n \left[\rho c_p(p,\theta) \left(\frac{\partial \theta}{\partial t} + \boldsymbol{v} \cdot \nabla \theta \right) \right]_n \\ &= \sum_{n=0}^{+\infty} \varepsilon^n \{ \mathrm{div}[k(p,\theta)\nabla\theta] \}_n + 2\mathrm{Br} \sum_{n=0}^{+\infty} \varepsilon^n \left\{ \mu(p,\theta) \left[\|\mathsf{D}\|^2 - \frac{1}{3}(\mathrm{div}\boldsymbol{v})^2 \right] \right\}_n . \end{aligned} \tag{66}$$

Since $\gamma_{\mathrm{ref}}\mathrm{Ma}^2/\mathrm{Fr}^2$ is of order $O(\varepsilon^2)$ or smaller, collecting terms of order $O(1)$ in (64) yields

$$\mathrm{div}\boldsymbol{v}_0 = 0, \tag{67}$$

whence the fluid motions can be regarded as isochoric to a first approximation.

According to the magnitude of the dimensionless numbers occurring in (64) and (65), we can derive different sets of approximate equations such as, just to mention a few of them, those which have been employed in the last few years to study the flows at low Reynolds and Froude numbers [42, 53], the effects of viscous dissipation in a piezo-viscous fluid [40], viscous stratified flows [17, 56] turbulence in forced convection [55], and heat transfer in turbulent mixed convection [54]. Here, we instead derive the flow regimes for which (63)–(66) approximate to generalizations of the celebrated Oberbeck-Boussinesq approximation and the Navier-Stokes-Fourier equations for fluids with variable material properties.

3.2.1 Generalized Oberbeck-Boussinesq Approximation

If the second Froude number is of order $O(\varepsilon)$, the Reynolds number of order of unity, the Brinkman number of order $O(\varepsilon)$ or smaller and the Péclet number of order of unity or greater, then at the leading order equations (65) and (66) are

$$\nabla p_0 + \boldsymbol{k} = \mathbf{0} \tag{68}$$

and

$$\mathrm{Pe} c_p(p_0, \theta_0)\dot{\theta}_0 = \mathrm{div}[k(p_0, \theta_0)\nabla\theta_0]. \tag{69}$$

Obviously, Eq. (68) can be integrated and, taking the atmospheric pressure as the reference pressure, we deduce that p_0 coincides with the hydrostatic pressure $p_h = 1 - z$. We now notice that Eqs. (67) and (69) are not sufficient to determine all the thermodynamic fields at $O(1)$. Therefore, in order to attain the closure, we collect the terms of order $O(\varepsilon)$ in (65) and, in view of (67), get

$$\mathrm{Fr}^2\dot{\boldsymbol{v}}_0 = \varepsilon\nabla p_1 + 2\frac{\mathrm{Fr}^2}{\mathrm{Re}}\mathrm{div}[\mu(p_h, \theta_0)\mathsf{D}_0] + \varepsilon\left[\int_0^{\theta_0} \alpha(p_h, \theta_0)\mathrm{d}\theta_0\right]\boldsymbol{k}. \tag{70}$$

Now equations (67), (69) and (70) form a closed system, in which p_1 can be regarded as the hydrodynamic pressure. Finally, setting $P = \varepsilon p_1$, re-dimensionalizing (67), (69) and (70), and omitting the subscript '0' yield the generalized Oberbeck-Boussinesq approximation derived by Rajagopal et al. [39]

$$\begin{cases} \rho_{\mathrm{ref}}\dot{\boldsymbol{v}} = -\nabla P + 2\mathrm{div}[\mu(p_h, \theta)\mathsf{D}] + \rho_{\mathrm{ref}}g\left[\int_{\theta_{\mathrm{ref}}}^{\theta} \alpha(p_h, \theta)\mathrm{d}\theta\right]\boldsymbol{k}, \\ \mathrm{div}\boldsymbol{v} = 0, \\ \rho_{\mathrm{ref}}c_p(p_h, \theta)\dot{\theta} = \mathrm{div}[k(p_h, \theta)\nabla\theta], \end{cases} \tag{71}$$

where the dimensionalized hydrostatic pressure is given by the well-known Stevin's law $p_h = \rho_{\mathrm{ref}}g(d - z)$.

3.2.2 Generalized Navier-Stokes-Fourier Equations

Suppose that Fr^2, $\mathrm{Fr}^2/\mathrm{Re}$ and Pe are of order of unity or greater, and the Brinkman number is of order $O(\varepsilon)$ or smaller. Then collecting the terms of order $O(1)$ in Eqs. (65) and (66) and re-dimensionalizing lead to the Navier-Stokes-Fourier equations for a fluid with material properties depending on pressure and temperature:

$$\begin{cases} \rho_{\mathrm{ref}}\dot{\boldsymbol{v}} = -\nabla p + 2\mathrm{div}[\mu(p,\theta)\mathsf{D}] - \rho_{\mathrm{ref}} g \boldsymbol{k}, \\ \mathrm{div}\boldsymbol{v} = 0, \\ \rho_{\mathrm{ref}} c_p \dot{\theta} = \mathrm{div}[k(p,\theta)\nabla\theta]. \end{cases} \tag{72}$$

Obviously, in isothermal conditions (72) reduces to the Navier-Stokes equations for piezo-viscous fluids

$$\begin{cases} \rho_{\mathrm{ref}}\dot{\boldsymbol{v}} = -\nabla p + 2\mathrm{div}[\mu(p)\mathsf{D}] - \rho_{\mathrm{ref}} g \boldsymbol{k}, \\ \mathrm{div}\boldsymbol{v} = 0. \end{cases} \tag{73}$$

4 Rayleigh-Bénard Problem for Fluids with Pressure- and Temperature Dependent Viscosities

Problems involving thermal convection are amongst those that have been studied most assiduously in mechanics in virtue of their relevance to a plethora of problems in astrophysics and geophysics. Understanding thermal-convection is at the heart of explaining weather patterns, solar winds, flows in the interior of stars, thermal currents in oceans, as well as numerous important industrial applications. The prototypical theoretical model as well as experimental set up, within which one can systematically investigate the effect of thermal-convection, is the flow that occurs in a fluid layer due to a thermal gradient that is present across the layer. The earliest experiments of thermal-convection in a fluid layer, heated from below, were carried out by Bénard [7]. He found a pattern of polygonal cells, predominantly hexagonal, though a few rectangular, pentagonal and septagonal cellular structures were also present. Bénard also found that these cellular structures were also quite stable modes under certain circumstances. Lord Rayleigh [43] studied the stability of the flow in a fluid layer heated from below, when the upper layer was stress-free.

There have been numerous studies concerning the stability/instability of 'Rayleigh-Bénard flows'. Until a critical difference in temperature is reached, the main process for the transfer of heat is conduction and upon reaching the critical temperature gradient convective rolls set in. Depending on the nature of boundary conditions (flow between solid boundaries, flow when one boundary is free of stress, etc.) one finds various types of flows are possible. A detailed discussion of the literature pertinent to various aspects of Bénard convection can be found in [8, 12, 14, 23, 25, 30]. An elegant introduction to the problem can be found in the treatise by Chandrasekhar [13].

The governing equations for the study of Bénard convection are obtained by appealing to an approximation that was independently established by Oberbeck [31, 32] and Boussinesq [9]. Such an approximation was established for fluids with constant material parameters. Thus, since here we aim at studying the problem of the onset of convection in a fluid whose viscosity varies with pressure and temperature, we cannot appeal to the classical Oberbeck-Boussinesq approximation but we have to employ its generalization derived in Sect. 3.2.1. We shall next find a necessary and sufficient condition for the linear stability of the conduction solution and compare the critical thresholds for the onset of convection in fluids with pressure and temperature dependent viscosities with the classical results for fluids whose viscosity is constant.

4.1 Conduction Solution: Evolution Equations of Perturbations

Assume that the viscosity of the horizontal fluid layer Ω_d (see Sect. 3.2.1) is an analytic function of pressure and temperature, while the coefficient of thermal expansion, the specific heat at constant pressure and the thermal conductivity are constant. It is worth noting that this assumption is coherent with the experimental evidences by Bridgman [10] reported in Sect. 2 and permits to appreciate the effects of a variable viscosity on the critical threshold for the onset of convection. In this framework, the generalized Oberbeck-Boussinesq approximation (71) becomes

$$\begin{cases} \rho_{\text{ref}}\dot{\boldsymbol{v}} = -\nabla P + 2\text{div}[\mu(p_h,\theta)\mathsf{D}] + \rho_{\text{ref}}g\alpha(\theta-\theta_{\text{ref}})\boldsymbol{k}, \\ \text{div}\boldsymbol{v} = 0, \\ \rho_{\text{ref}}c_p\dot{\theta} = k\Delta\theta. \end{cases} \tag{74}$$

The appropriate boundary conditions for the temperature and hydrodynamic pressure to add to system (74) are

$$\begin{cases} \theta(x,y,0,t) = \theta_L, \quad \theta(x,y,d,t) = \theta_U, \\ P(x,y,d,t) = 0, \end{cases} \tag{75}$$

with $\theta_L > \theta_U$. Our aim is the study of stability of the steady static conduction solution m_0 to (74)–(75):

$$\begin{cases} \tilde{\boldsymbol{v}} = \boldsymbol{0}, \\ \tilde{\theta} = \theta_L - \dfrac{\theta_L-\theta_U}{d}z, \\ \tilde{P} = \rho_{\text{ref}}g\alpha(\theta_L-\theta_U)z\left(1-\dfrac{z}{2d}\right). \end{cases} \tag{76}$$

In order to study the stability of the conduction solution m_0 we introduce the perturbations $\boldsymbol{u} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$, ϑ and Π to $\bar{\boldsymbol{v}}$, $\bar{\theta}$ and $\bar{P}$, respectively, i.e.

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} + \boldsymbol{u}, \quad \theta = \tilde{\theta} + \vartheta, \quad P = \tilde{P} + \Pi. \tag{77}$$

Then, inserting (77) into (74) gives the evolution equations of perturbations

$$\begin{cases} \rho_{\text{ref}}(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\nabla\Pi + \mu(p_h, \tilde{\theta}+\vartheta)\Delta\boldsymbol{u} \\ \qquad\qquad +[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]\nabla\mu(p_h, \tilde{\theta}+\vartheta) + \rho_{\text{ref}}g\alpha\vartheta\boldsymbol{k}, \\ \text{div}\boldsymbol{u} = 0, \\ \rho_{\text{ref}}c_p\left(\vartheta_t + \boldsymbol{u}\cdot\nabla\vartheta - \dfrac{\theta_L-\theta_U}{d}w\right) = k\Delta\vartheta, \end{cases} \tag{78}$$

that are valid for all $(x, y, z, t) \in \mathbb{R}^2 \times [0, d] \times [0, +\infty[$. To (78) we append the initial conditions

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \vartheta(\boldsymbol{x}, 0) = \vartheta_0(\boldsymbol{x}), \tag{79}$$

and the boundary conditions

$$\Pi(x, y, d, t) = 0, \quad \vartheta(x, y, 0, t) = \vartheta(x, y, d, t) = 0, \tag{80}$$

and

$$\boldsymbol{u}(x, y, 0, t) = \boldsymbol{u}(x, y, d, t) = \boldsymbol{0} \tag{81}$$

for rigid boundaries, or

$$u_z = v_z = 0 \text{ and } w = 0 \quad \text{on } z = 0, d \tag{82}$$

for stress-free bounding surfaces. We refer to [13] for the derivation of the boundary conditions (81) and (82). In (79) $\boldsymbol{u}_0$ and ϑ_0 are regular fields, with $\boldsymbol{u}_0$ being divergence-free.

4.2 Linear Stability Analysis

Since the viscosity is an analytic function of the temperature and pressure, for sufficiently small disturbances we can approximate the two terms containing μ in $(78)_1$ as:

$$\mu(p_h, \tilde{\theta}+\vartheta)\Delta\boldsymbol{u} = \left[\sum_{n=0}^{+\infty}\frac{1}{n!}\frac{\partial^n\mu}{\partial\theta^n}(p_h, \tilde{\theta})\vartheta^n\right]\Delta\boldsymbol{u} \approx \mu(p_h, \tilde{\theta})\Delta\boldsymbol{u} \equiv \hat{\mu}(z)\Delta\boldsymbol{u}, \tag{83}$$

and

$$[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]\nabla\mu(p_h, \tilde{\theta} + \vartheta) = [\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]\left\{\sum_{n=0}^{+\infty}\frac{1}{n!}\nabla\left[\frac{\partial^n \mu}{\partial \theta^n}(p_h, \tilde{\theta})\vartheta^n\right]\right\}$$
$$\approx \hat{\mu}'(z)\left[(u_z + w_x)\boldsymbol{i} + (v_z + w_y)\boldsymbol{j} + 2w_z\boldsymbol{k}\right], \quad (84)$$

where, henceforth, a prime denotes the derivative of a function which depends only on one variable.

Thanks to (83) and (84) we can linearize the evolution equations of perturbations (78) to obtain

$$\begin{cases} \rho_{\text{ref}}\boldsymbol{u}_t = -\nabla \Pi + \hat{\mu}(z)\Delta \boldsymbol{u} \\ \qquad\qquad +\hat{\mu}'(z)\left[(u_z + w_x)\boldsymbol{i} + (v_z + w_y)\boldsymbol{j} + 2w_z\boldsymbol{k}\right] + \rho_{\text{ref}} g\alpha\vartheta\boldsymbol{k}, \\ \text{div}\boldsymbol{u} = 0, \\ \rho_{\text{ref}} c_p\left(\vartheta_t - \dfrac{\theta_L - \theta_U}{d}w\right) = k\Delta\vartheta. \end{cases} \quad (85)$$

It is now convenient to non-dimensionalize (85) and the boundary conditions (80)–(82) by introducing the following scales:

$$\begin{gathered} \boldsymbol{x}^* = \frac{\boldsymbol{x}}{d}, \quad t^* = \frac{\mu_{\text{ref}}}{\rho_{\text{ref}} d^2}t, \quad \boldsymbol{u}^* = \frac{\rho_{\text{ref}} d}{\mu_{\text{ref}}}\boldsymbol{u}, \quad \mu^* = \frac{\mu}{\mu_{\text{ref}}}, \\ p_h^* = \frac{p_h}{\rho_{\text{ref}} g d} = 1 - z^*, \quad \bar{\theta}^* = \frac{\theta - \theta_U}{\theta_L - \theta_U} = 1 - z^*, \\ \Pi^* = \frac{\rho_{\text{ref}} d^2}{\mu_{\text{ref}}^2}\Pi, \quad \vartheta^* = \frac{\vartheta}{\theta_L - \theta_U}. \end{gathered} \quad (86)$$

Inserting the dimensionless quantities (86) into (85) and (80)–(82) yields the non-dimensional equations (omitting the asterisks)

$$\begin{cases} \boldsymbol{u}_t = -\nabla \Pi + \hat{\mu}(z)\Delta \boldsymbol{u} \\ \qquad\qquad +\hat{\mu}'(z)\left[(u_z + w_x)\boldsymbol{i} + (v_z + w_y)\boldsymbol{j} + 2w_z\boldsymbol{k}\right] + \dfrac{\mathcal{R}}{\text{Pr}}\vartheta\boldsymbol{k}, \\ \text{div}\boldsymbol{u} = 0, \\ \text{Pr}\,(\vartheta_t - w) = \Delta\vartheta, \end{cases} \quad (87)$$

where

$$\mathcal{R} = \frac{\rho_{\text{ref}}^2 g d^3 c_p \alpha(\theta_L - \theta_U)}{\mu_{\text{ref}} k} \quad \text{and} \quad \text{Pr} = \frac{c_p \mu_{\text{ref}}}{k} \quad (88)$$

are the Rayleigh and Prandtl numbers, respectively, and the dimensionless boundary conditions

$$\Pi(x, y, 1, t) = 0, \quad u = v = w = \vartheta = 0 \quad \text{on } z = 0, 1 \quad (89)$$

for rigid boundaries, and

$$\varPi(x, y, 1, t) = 0, \quad u_z = v_z = w = \vartheta = 0 \quad \text{on } z = 0, 1 \tag{90}$$

for bounding surfaces free of stress.

As common praxis in the linear stability analysis of isochoric flows, we take the third component of the curlcurl of (87) to eliminate the disturbance $\varPi$ and obtain the following coupled system in w and ϑ:

$$\begin{cases} \dfrac{\partial}{\partial t}\varDelta w = 2\hat{\mu}'(z)\dfrac{\partial}{\partial z}\varDelta w + \hat{\mu}(z)\varDelta\varDelta w + \hat{\mu}''(z)\dfrac{\partial^2 w}{\partial z^2} \\ \qquad -\hat{\mu}''(z)\varDelta_s w + \dfrac{\mathcal{R}}{\mathrm{Pr}}\varDelta_s\vartheta, \\ \mathrm{Pr}(\theta_t - w) = \varDelta\vartheta, \end{cases} \tag{91}$$

where $\varDelta_s = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the so-called horizontal Laplacian. Since the coefficients in equations (91) depend only on z, the equations admit solutions which depend on x, y and t exponentially. We therefore look for solutions of the form:

$$\begin{cases} w(x, y, z, t) = \dfrac{W(z)}{\mathrm{Pr}} \exp[i(a_x x + a_y y) + \sigma t], \\ \vartheta(x, y, z, t) = \dfrac{\varTheta(z)}{\sqrt{\mathcal{R}}} \exp[i(a_x x + a_y y) + \sigma t], \end{cases} \tag{92}$$

in which it is understood that the real parts of these expressions must be taken into consideration to obtain physical quantities. The wave speed σ may be complex, say $\sigma = \sigma_r + i\sigma_i$. Thus, expressions (92) represent waves which travel in the x and y co-ordinate directions with phase speed σ_i/a, where $a = \sqrt{a_x^2 + a_y^2}$ is the two-dimensional wave number, and whose growth or decay in time is given by $\exp(\sigma_r t)$. A wave of the form (92) is then stable if $\sigma_r \leq 0$ (marginally stable if $\sigma_r = 0$), and unstable if $\sigma_r > 0$.

Setting $D = \mathrm{d}/\mathrm{d}z$ and inserting (92) into (91) gives the system of ordinary differential equations

$$\begin{cases} \sigma(D^2 - a^2)W = \hat{\mu}(z)(D^2 - a^2)^2 W + 2\hat{\mu}'(z)D(D^2 - a^2)W \\ \qquad +\hat{\mu}''(z)(D^2 + a^2)W - \sqrt{\mathcal{R}}a^2\varTheta, \\ \sigma\mathrm{Pr}\varTheta - \sqrt{\mathcal{R}}W = (D^2 - a^2)\varTheta, \end{cases} \tag{93}$$

to which we add the boundary conditions

$$W = DW = \varTheta = 0 \quad \text{at } z = 0, 1 \tag{94}$$

for rigid boundaries, or

$$W = D^2W = \Theta = 0 \quad \text{at } z = 0, 1 \tag{95}$$

for stress-free boundaries.

Denoting by the superscript $\star$ the complex conjugate, multiplying $(93)_1$ by $W^\star$, $(93)_2$ by $a^2\Theta^\star$, summing and integrating over the interval [0, 1] taking into account the boundary conditions (94) or (95), we obtain

$$\begin{aligned} \sigma \int_0^1 [|DW|^2 + a^2(|W|^2 + \mathrm{Pr}|\Theta|^2)]\mathrm{d}z &= a^2\sqrt{R}\int_0^1 (W\Theta^\star + W^\star\Theta)\mathrm{d}z \\ -\int_0^1 \hat{\mu}(z)[|(D^2+a^2)W|^2 + 4a^2|DW|^2]\mathrm{d}z &- a^2\int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)\mathrm{d}z. \end{aligned} \tag{96}$$

Considering the imaginary part of (96) yields that $\sigma_i = 0$, that is the wave speed of the perturbation is real. Therefore the linearized equations of Bénard convection (85) satisfy the strong form of principle of exchange of stabilities [50] also in the case of fluids with pressure- and temperature-dependent viscosity. In addition, rewriting (96) as

$$\sigma\mathcal{L}(W, \Theta; a^2) = \left[\sqrt{\mathcal{R}}\frac{\mathcal{I}(W, \Theta; a^2)}{\mathcal{D}(W, \Theta; a^2)} - 1\right]\mathcal{D}(W, \Theta; a^2), \tag{97}$$

with

$$\mathcal{L}(W, \Theta; a^2) = \int_0^1 [|DW|^2 + a^2(|W|^2 + \mathrm{Pr}|\Theta|^2)]\mathrm{d}z, \tag{98}$$

$$\mathcal{I}(W, \Theta; a^2) = a^2\int_0^1 (W\Theta^\star + W^\star\Theta)\mathrm{d}z \tag{99}$$

and

$$\begin{aligned} \mathcal{D}(W, \Theta; a^2) = \int_0^1 &\hat{\mu}(z)[|(D^2+a^2)W|^2 + 4a^2|DW|^2]\mathrm{d}z \\ &+ a^2\int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)\mathrm{d}z, \end{aligned} \tag{100}$$

we deduce that the modes (92) with two-dimensional wave number a are linearly stable if and only if

$$\mathcal{R} \le \mathcal{R}_L(a) \equiv \left[\max_{(W,\Theta)\in\mathscr{H}} \frac{\mathcal{I}(W, \Theta; a^2)}{\mathcal{D}(W, \Theta; a^2)}\right]^{-2}, \tag{101}$$

where $\mathscr{H}$ is the space of kinematically admissible disturbances:

$$\mathscr{H} = \left\{(W, \Theta) \in \mathrm{H}^2(0,1) \times \mathrm{H}^1(0,1) : W = DW = \Theta = 0 \text{ at } z = 0, 1\right\} \tag{102}$$

for rigid boundaries, or

$$\mathscr{H} = \left\{(W, \Theta) \in \mathrm{H}^2(0,1) \times \mathrm{H}^1(0,1) : W = D^2W = \Theta = 0 \text{ at } z = 0, 1\right\} \tag{103}$$

for stress-free bounding surfaces. The existence of the maximum of the functional $\mathcal{I}/\mathcal{D}$ in $\mathscr{H}$ can be proved by following similar arguments as in [45].

It is easy to check that the Euler-Lagrange equations associated with the variational problem (101) coincide with (93) with $\sigma = 0$ giving the marginally stable states. Following [13], (93) with $\sigma = 0$ can be simplified further to the following sixth-order ordinary differential equation

$$\hat{\mu}(z)(D^2-a^2)^3\Theta + 2\hat{\mu}'(z)D(D^2-a^2)^2\Theta + \hat{\mu}''(z)(D^4-a^4)\Theta + \mathcal{R}a^2\Theta = 0 \tag{104}$$

to which we add the boundary conditions

$$\Theta = \mathrm{D}^2\Theta = D(D^2 - a^2)\Theta = 0 \quad \text{at } z = 0, 1 \tag{105}$$

for rigid boundaries, or

$$\Theta = D^2\Theta = D^4\Theta = 0 \quad \text{at } z = 0, 1 \tag{106}$$

for stress-free boundaries. The square of the maximum of the functional $\mathcal{I}/\mathcal{D}$ is then the reciprocal of the least eigenvalue of the characteristic-value problem (104) with boundary conditions (105) or (106) and thus the marginal stability curve has equation $\mathcal{R} = \mathcal{R}_L(a)$. Finally, we introduce the so-called critical Rayleigh number

$$\mathcal{R}_{cr} = \min_{a>0} \mathcal{R}_L(a), \tag{107}$$

and note that if $\mathcal{R} \leq \mathcal{R}_{cr}$ then all modes are stable, while if $\mathcal{R} > \mathcal{R}_{cr}$ there exists at least one unstable mode. Thus, we may conclude that the conduction solution m_0 is linearly stable if and only if

$$\mathcal{R} \leq \mathcal{R}_{cr}. \tag{108}$$

To appreciate the departures from the classical results for fluids with constant viscosities, in Table 1 we display the critical thresholds for the Rayleigh number for rigid and stress-free boundaries when the viscosity depends exponentially on pressure and temperature according to (14). In this case, the nondimensional

Table 1 Approximations of the critical Rayleigh and two-dimensional wave numbers against the dimensionless parameter Γ for (a) rigid and (b) stress-free boundaries

(a)

Γ	$\mathcal{R}_{cr}$	a_{cr}
0.5	2200.315	3.115
0.3	1986.687	3.116
0.2	1888.573	3.116
0.1	1795.744	3.116
0	1707.937	3.116
−0.1	1624.857	3.116
−0.2	1546.233	3.116
−0.3	1471.774	3.116
−0.5	1334.559	3.115

(b)

Γ	$\mathcal{R}_{cr}$	a_{cr}
0.5	850.079	2.216
0.3	765.847	2.219
0.2	727.500	2.221
0.1	691.451	2.221
0	657.548	2.221
−0.1	625.651	2.221
−0.2	595.627	2.221
−0.3	567.353	2.219
−0.5	515.599	2.216

viscosity function $\hat{\mu}$ reads

$$\hat{\mu} = \exp[\Gamma(1-z)] \quad \text{with} \quad \Gamma = \beta\rho_{\text{ref}} g d - \delta(\theta_L - \theta_U). \tag{109}$$

The critical thresholds $\mathcal{R}_{cr}$ have been found for different values of the dimensionless parameter Γ by solving the eigenvalue problems (104) and (105) or (106) with the aid of the MATLAB www.bvp4c solver. Observe that, both in the rigid and stress-free case the critical threshold $\mathcal{R}_{cr}$ increases with increasing Γ and equals the critical thresholds for fluids with constant viscosities when $\Gamma = 0$. This result is physically reasonable as the viscosity, and hence the resistance to motion from the rest state, increases as Γ increases. From the definition of Γ we can then assert that the pressure dependence of viscosity has a stabilizing effect on the onset of convection, in the sense that the critical threshold for the Rayleigh number is greater than the one that can be predicted starting from the assumption that viscosity is constant or dependent only on temperature.

The nonlinear stability of the conduction solution m_0 in a fluid whose viscosity is an analytic function of the pressure and temperature has been studied by Rajagopal et al. [38]. They proved that, under appropriate conditions on the initial disturbance of the temperature field ϑ_0, the conduction solution is nonlinearly stable with respect to the energy of the perturbations if inequality (108) holds. In this way, Rajagopal et al. proved that (108) is a necessary and sufficient condition for the local nonlinear stability of m_0.

5 Parallel Shear Flows of Piezo-Viscous Fluids

In the last two sections we shall consider isothermal flows in fluids with variable viscosities. In particular, in this section the viscosity will be assumed to depend only on pressure, while in the next section, to include shear-thickening/thinning effects, viscosity will be expressed in terms of pressure and shear.

The main goal of this section is the determination of some classes of steady unidirectional shear flows that are possible in a piezo-viscous fluid. We shall show that, under the assumption that gravity is the only body force acting on the fluid, Couette flows are possible for any constitutive model of the viscosity, whereas Poiseuille flows are possible only in fluids with constant viscosity. To the best of our knowledge, this result is novel as results on the existence of Poiseuille flows in piezo-viscous fluids are available only in the absence of body forces. Assuming that body forces are negligible compared to viscous forces and pressure gradients, Bair et al. [3] claimed to have proven that Poiseuille flows in piezo-viscous fluids are not possible, a secondary flow being necessary. This claim is not true in general. In fact, Hron et al. [19] showed that steady unidirectional flows are possible if the dependence of the viscosity on pressure is linear, and explicit exact continuous solutions can be established even if shear-thinning effects are included. On the other hand, for other forms of the viscosity, with polynomial and exponential dependence on the pressure, Hron et al. [19] reconfirmed the results of Bair et al. [3]. Two years later, Renardy [44] gave an elegant proof on the existence/nonexistence of Poiseuille flows in piezo-viscous fluids. He proved that, in the absence of body forces, Poiseuille flows are possible only if viscosity depends linearly on pressure. In what follows we shall consider also the case in which body forces are negligible and give an alternative proof of the result by Renardy [44].

5.1 Governing Equations

When gravity is the only force acting on the fluid, the equations governing the isothermal flows in the horizontal fluid layer Ω_d are the generalized Navier-Stokes equation (73).

We are here interested in two types of unidirectional flow: Couette flow, when one plate is fixed ($z = 0$) and the other one ($z = d$) moves with a prescribed velocity; and Poiseuille flow, when homogeneous Dirichlet boundary conditions at the plates $z = 0, d$ are considered. We then look for solutions to (73) of the form

$$\boldsymbol{v} = u(z)\boldsymbol{i}, \quad p = p(x, y, z), \tag{110}$$

which satisfy the following boundary conditions

$$u(0) = 0, \quad u(d) = V, \qquad \text{(Couette flow)}, \tag{111}$$

or

$$u(0) = u(d) = 0, \qquad \text{(Poiseuille flow)}. \tag{112}$$

Inserting the ansatz (110) into (73) and non-dimensionalizing the resulting system of pdes by means of the scales

$$\begin{aligned} \boldsymbol{x}^* &= \frac{\boldsymbol{x}}{d}, \quad u^* = \frac{u}{U}, \quad p^* = \frac{p - p_{\text{ref}}}{\rho_{\text{ref}} g d}, \\ \mu^* &= \frac{\mu}{\mu_{\text{ref}}}, \quad U = \frac{\rho_{\text{ref}} g d^2}{\mu_{\text{ref}}}, \quad \mu_{\text{ref}} = \mu(p_{\text{ref}}), \end{aligned} \tag{113}$$

yield (omitting the asterisks for simplicity of notation) the dimensionless equations

$$\begin{cases} p_x = \tau_z, \\ p_y = 0, \\ p_z = \tau_x - 1, \end{cases} \tag{114}$$

where

$$\tau = \mu(p) u_z \tag{115}$$

is the shear stress. To (114) we add the dimensionless boundary conditions

$$u(0) = 0, \quad u(1) = V^* \equiv \frac{V}{U}, \qquad \text{(Couette flow)}, \tag{116}$$

or

$$u(0) = u(1) = 0 \qquad \text{(Poiseuille flow)}. \tag{117}$$

From (114) we easily deduce that $p = p(x, z)$ and that both the shear stress and the pressure are solutions of the wave equation $\psi_{xx} - \psi_{zz} = 0$. This leads to the representations

$$\begin{cases} p = \Xi(x + z) + \Psi(x - z) - z, \\ \tau = \Xi(x + z) - \Psi(x - z). \end{cases} \tag{118}$$

5.2 *Couette Flows*

Let us first consider the Couette flows. Assuming that the pressure at the upper boundary is constant and equal to the reference pressure, namely $p(1) = 0$, from $(118)_1$ the functions Ξ and Ψ are such that $\Xi(x + 1) + \Psi(x - 1) = 1$ for all $x \in \mathbb{R}$, by which one deduces that Ξ and Ψ are of the form

$$\Xi(x + z) = a(x + z) + b, \quad \Psi(x - z) = -a(x - z) + 1 - 2a - b, \tag{119}$$

with a and b constants. Hence, inserting (119) into (118) we deduce that the pressure, and thus the shear stress, do not depend on x, whence $a = 0$ and

$$\begin{cases} p = 1 - z, \\ \tau = 2b - 1. \end{cases} \tag{120}$$

Finally, from (115), (116) and $(120)_2$ the velocity component u is found to be

$$u(z) = V^* \frac{\displaystyle\int_0^z \frac{\mathrm{d}z}{\mu(1-z)}}{\displaystyle\int_0^1 \frac{\mathrm{d}z}{\mu(1-z)}}. \tag{121}$$

Couette flows are then possible in a piezo-viscous fluid with a very general response function of the viscosity. As examples, if the (dimensionless) viscosity is given by a power law of the form

$$\mu(p) = 1 + \varpi p^n \quad (\varpi \geq 0,\ n > 0), \tag{122}$$

then

$$u(z) = 1 - \frac{\mathrm{Lerch}\Phi\left(-\varpi(1-z)^n, 1, \frac{1}{n}\right)}{\mathrm{Lerch}\Phi\left(-\varpi, 1, \frac{1}{n}\right)}(1-z), \tag{123}$$

where LerchΦ is the Lerch Phi function; while if viscosity depends exponentially on the pressure according to the Barus law

$$\mu(p) = \mathrm{e}^{\varpi p} \quad (\varpi \geq 0), \tag{124}$$

then

$$u(z) = \frac{\mathrm{e}^{\varpi z} - 1}{\mathrm{e}^{\varpi} - 1}. \tag{125}$$

Figure 1 displays the velocity profiles of the Couette flow for different models of the viscosity.

5.3 Poiseuille Flows

We now consider the Poiseuille flows and observe that, since the velocity field satisfies the boundary conditions (117), the assumption that either Ξ or Ψ is

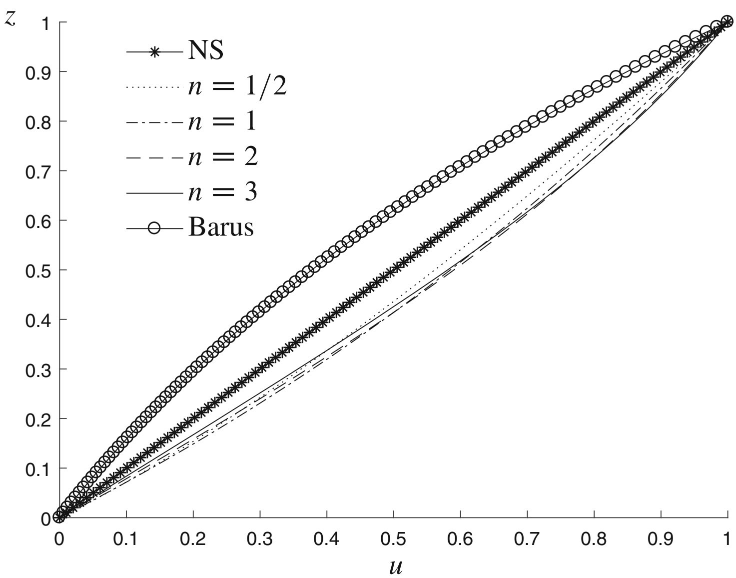


Fig. 1 Velocity profiles of the Couette flow when the dimensionless viscosity is constant (i.e. the classical Navier-Stokes (NS) model), or of the form (122) or (124). The dimensionless piezo-viscous coefficient ϖ is taken to be equal to unity

constant leads to the trivial flow $u \equiv 0$ (no motion). Thus, we have

$$\Xi'(x+z) \neq 0 \quad \text{and} \quad \Psi'(x-z) \neq 0. \tag{126}$$

Moreover, it can be easily proven that also the assumption $\mu'(p)u_z = \mu'(p)\tau/\mu(p)$ =constant leads to the trivial motionless flow. Hence,

$$\nabla\left(\frac{\mu'(p)}{\mu(p)}\tau\right) \neq \mathbf{0}. \tag{127}$$

Next, from (115) we deduce that

$$0 = \frac{\partial u_z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\tau}{\mu(p)}\right) = \frac{\tau_x}{\mu(p)} - \frac{\mu'(p)}{\mu^2(p)}\tau p_x, \tag{128}$$

whence, on using $(114)_1$,

$$\mu(p)\tau_x - \mu'(p)\tau\tau_z = 0. \tag{129}$$

In the light of (126) and (127), on using the representations of p and τ (118), (129) can be rewritten as

$$\frac{\Xi'(x+z)}{\Psi'(x-z)} = \frac{\mu(p) + \mu'(p)[\Xi(x+z) - \Psi(x-z)]}{\mu(p) - \mu'(p)[\Xi(x+z) - \Psi(x-z)]}, \tag{130}$$

with p as in $(118)_1$. Therefore, because of the explicit dependence of the pressure on the vertical variable z (and not only through $\Xi(x+z)$ and $\Psi(x-z)$), (130) holds if and only if the viscosity of the fluid is constant. Consequently, when gravity is the only body force acting on the fluid, Poiseuille flows are possible only if the viscosity is constant, in which case

$$\begin{cases} p = A_0 x - z + \kappa, \\ u = \dfrac{A_0}{2} z(z-1), \end{cases} \tag{131}$$

where A_0 is the constant pressure gradient that induces the flow, and κ is an integration constant that, once A_0 is known, can be determined by measuring the pressure at a single point on the boundary.

Assume now that body forces are negligible compared to the viscous forces and pressure gradients. Then, following the same arguments as in the case in which the effects due gravity are taken into account we arrive at (130) with p given by

$$p = \Xi(x+z) + \Psi(x-z). \tag{132}$$

Since in the absence of body forces the pressure does not depend explicitly on z but only through $\Xi(x+z)$ and $\Psi(x-z)$, (130) with p as in (132) holds if and only if μ is a linear function of pressure. In other words, the dimensionless viscosity must be of the form

$$\mu(p) = 1 + \varpi p \quad (\varpi \geq 0). \tag{133}$$

For such a dependency on the pressure, a simple manipulation of (114) and (115) gives

$$\begin{cases} \dfrac{p_x}{1+\varpi p} = \dfrac{u_{zz}}{1-\varpi^2 u_z^2}, \\ \dfrac{p_z}{1+\varpi p} = \dfrac{\varpi u_z u_{zz}}{1-\varpi^2 u_z^2}. \end{cases} \tag{134}$$

Integrating $(134)_2$ yields

$$p = \frac{1}{\varpi}\left[\frac{\phi(x)}{\sqrt{1-\varpi^2 u_z^2}} - 1\right], \tag{135}$$

where $\phi(x)$ is an arbitrary function to be determined with the aid of $(134)_1$. Indeed, inserting (135) into $(134)_1$ we obtain

$$\frac{1}{\varpi}\frac{\phi'(x)}{\phi(x)} = \frac{u_{zz}}{1-\varpi^2 u_z^2}, \tag{136}$$

which holds if and only if

$$\frac{1}{\varpi}\frac{\phi'(x)}{\phi(x)} = A_0 \quad \text{and} \quad \frac{u_{zz}}{1-\varpi^2 u_z^2} = A_0, \tag{137}$$

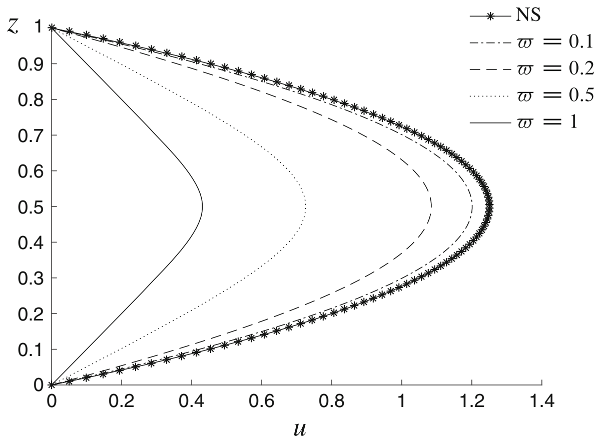
with A_0 being constant. Next, integrating (137) and taking into account the boundary conditions (117), we derive the Poiseuille flow

$$\begin{cases} p = \dfrac{1}{\varpi}\left\{\kappa \exp(\varpi A_0 x)\cosh\left[\varpi A_0\left(z-\dfrac{1}{2}\right)\right]-1\right\}, \\ u = \dfrac{1}{\varpi^2 A_0}\ln\dfrac{\cosh\left[\varpi A_0\left(z-\dfrac{1}{2}\right)\right]}{\cosh\dfrac{\varpi A_0}{2}}, \end{cases} \tag{138}$$

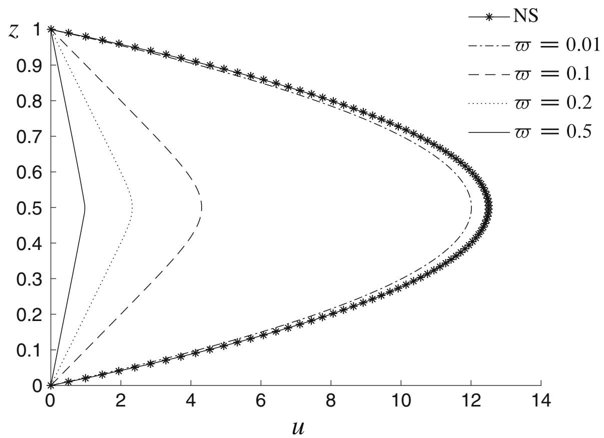
where κ is an integration constant that, as indicated before, can be determined once A_0 is known. Finally, from $(138)_2$ it follows that the maximum speed in a Poiseuille flow decreases with increasing ϖ, and, in the limit as $\varpi \to 0$, the velocity profile tends to the one which has been determined for a fluid with constant viscosity (see Fig. 2).

6 Flow of Fluids with Pressure and Shear Dependent Viscosity Down an Inclined Plane

In this final section, we carry out an analysis of the flow of a fluid with a pressure and shear dependent viscosity down an inclined plane within the context of the lubrication approximation. It is legitimate to ask where the pressure dependence of viscosity could become important within the context of the lubrication approximation. Thin film flows are ubiquitous in engineering, geophysics, biology and elsewhere, and low aspect ratios are often the basis for simplified fluid dynamical models. An important relevant application in geophysics is the flow of glaciers and ice sheets as well as rock glaciers. For instance, while the ice sheet covering Antarctica is several kilometers thick, it however has a horizontal extent of several thousand kilometres, yielding a length scale ratio epsilon of order 10^{-3} [47]. These glaciers clearly exhibit non-Newtonian characteristics in that their viscosities depend on the shear rate so that their flows are modelled using a shallow-ice approximation and Glen's flow law [33]: in other words, as gravity currents with



(a)



(b)

Fig. 2 Poiseuille flow. Velocity profiles for different values of the piezo-viscous coefficient ϖ and (**a**) $A_0 = -10$ and (**b**) $A_0 = -100$

non-Newtonian (power law) rheology. On the other hand, there are several papers that investigate the possibility of normal stress effects in the creep of polycrystalline ice (see, for instance, [27] and [24]). In particular, Jones and Chew [22] have shown that hydrostatic pressure decreases the creep of polycristalline ice slightly and, then, above 15 MPa, a minimum creep rate is reached followed by an increase in rate with increasing hydrostatic pressure. Therefore, in view of the depths of glaciers we would expect that the pressure would also influence the viscosity. As the viscosity depends on both the shear rate as well as the pressure, it is possible that these two effects could either compete against each other thereby mitigating their effects, or join forces to enhance the qualitative and quantitative differences. As the fluid can shear-thin or shear-thicken, both possibilities may come to pass.

6.1 Basic Equations

We consider a fluid moving on an inclined plane, whose angle of inclination is ι. Let now $Oxyz$ be a Cartesian frame of reference with fundamental unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, where the coordinate z is perpendicular to the plane, the x and y coordinates lie in the plane, y is horizontal and x increasing downward. We denote the components of the velocity $\boldsymbol{v}$ of the fluid in the directions x, y and z as u, v and w, respectively. The generalized Navier-Stokes equations read then

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0, \\ \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right) = -\dfrac{\partial p}{\partial x} + \dfrac{\partial S_{xx}}{\partial x} + \dfrac{\partial S_{xy}}{\partial y} + \dfrac{\partial S_{xz}}{\partial z} + \rho g \sin\iota, \\ \rho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right) = -\dfrac{\partial p}{\partial y} + \dfrac{\partial S_{yx}}{\partial x} + \dfrac{\partial S_{yy}}{\partial y} + \dfrac{\partial S_{yz}}{\partial z}, \\ \rho\left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right) = -\dfrac{\partial p}{\partial z} + \dfrac{\partial S_{zx}}{\partial x} + \dfrac{\partial S_{zy}}{\partial y} + \dfrac{\partial S_{zz}}{\partial z} - \rho g \cos\iota, \end{cases} \tag{139}$$

where the Cauchy stress is now given by the implicit relation

$$\mathsf{T} - \frac{1}{3}\mathrm{tr}(\mathsf{T}) - \mu\left(-\frac{1}{3}\mathrm{tr}(\mathsf{T}), \mathrm{tr}(\mathsf{D}^2)\right)\mathsf{D} = \mathsf{O}, \tag{140}$$

or, equivalently, since $\|\mathsf{D}\| = [\mathrm{tr}(\mathsf{D}^2)]^{1/2}$,

$$\mathsf{T} = -p\mathsf{I} + 2\mu(p, \|\mathsf{D}\|)\mathsf{D} \equiv -p\mathsf{I} + \mathsf{S}. \tag{141}$$

The viscosity is taken of the form

$$\mu(p, \|\mathsf{D}\|) = \varsigma(p - p_{\mathrm{ref}})\|\mathsf{D}\|^{(1-\chi)/\chi}, \tag{142}$$

with $\chi > 0$, p_{ref} the reference pressure and, as is reasonable to expect since the fluid viscosity increases as the pressure increases, ς is a positive function whose value increases with increasing pressure. Model (141), with a viscosity of the type (142), allows for a fluid that is capable of shear thinning, when $\chi > 1$, or shear thickening, when $\chi \in (0, 1)$. Here, for the sake of definiteness, we shall consider the following forms for ς:

$$\text{(exponential model)} \qquad \varsigma(p) = \varsigma_{\mathrm{ref}} e^{\beta(p - p_{\mathrm{ref}})}, \tag{143}$$

$$\text{(power law model)} \qquad \varsigma(p) = \varsigma_{\mathrm{ref}} + \beta(p - p_{\mathrm{ref}})^n, \tag{144}$$

where $\varsigma_{\mathrm{ref}} > 0$, $\beta \geq 0$ and $n \geq 0$ are constants. In general, the material parameters that appear in (143) and (144) can be obtained by corroboration with experimental data. Here, in order to illustrate the effects due to the pressure dependence of viscosity, we merely carry out a parametric study.

We prescribe the following boundary conditions for the velocity and pressure fields

$$\begin{cases} u = v = w = 0 \text{ on } z = 0, \\ \mathsf{T}\boldsymbol{n} = -p_{\mathrm{ref}}\boldsymbol{n} \quad \text{ on } z = h(x, y, t), \end{cases} \tag{145}$$

where

$$\boldsymbol{n} = \frac{1}{\sqrt{1 + \left(\dfrac{\partial h}{\partial x}\right)^2 + \left(\dfrac{\partial h}{\partial y}\right)^2}} \left(-\frac{\partial h}{\partial x}\boldsymbol{i} - \frac{\partial h}{\partial y}\boldsymbol{j} + \boldsymbol{k}\right) \tag{146}$$

is the unit normal to the free surface of the current $z = h(x, y, t)$.

Let H and L denote the characteristic thickness and characteristic length along the plane of the current free surface $z = h(x, y, t)$, respectively. The main assumption in lubrication approximation is that the lengthscale ratio H/L is small [51]. Here, as we are interested in fluids whose viscosity depends on the pressure, we assume that the ratio H/L is small though H is large enough to have a significant dependence of the viscosity on the pressure.

As a consequence of the smallness of the lengthscale ratio H/L, the component of the velocity parallel to the plane is much larger than the normal component, so that

$$\sqrt{u^2 + v^2} \gg |w|. \tag{147}$$

We call U, V and W the characteristic velocities along x, y and z directions, respectively. Hence, $U_\parallel = \sqrt{U^2 + V^2}$ and W are the characteristic velocities parallel and perpendicular to the inclined plane, respectively. From Eqs. $(139)_1$ and (147) we find that $W = HU_\parallel/L$.

There are many ways of transforming the governing equation (139) and boundary conditions (145) into dimensionless expressions. Here we introduce a scaling which is similar to that introduced in [1]:

$$\begin{cases} \boldsymbol{x}^* = \dfrac{1}{L}(x\boldsymbol{i} + y\boldsymbol{j}) + \dfrac{z}{H}\boldsymbol{k}, \quad \boldsymbol{v}^* = \dfrac{1}{U_\parallel}(u\boldsymbol{i} + v\boldsymbol{j}) + \dfrac{w}{W}\boldsymbol{k}, \quad h^* = \dfrac{h}{H}, \\ W = \dfrac{H}{L}U_\parallel, \quad t^* = \dfrac{U_\parallel}{L}t, \quad p^* = \dfrac{p - p_{\mathrm{ref}}}{\rho g \cos \iota H}, \quad \varsigma^* = \dfrac{\varsigma}{\varsigma_{\mathrm{ref}}}. \end{cases} \tag{148}$$

Substituting the dimensionless quantities (148) into equations (139), (141) and (146) and into the boundary conditions (145) leads to (omitting all asterisks)

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0, \\ \epsilon \mathrm{Re} \left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} \right) = \epsilon \dfrac{\mathrm{Re}}{\mathrm{Fr}^2} \left(\dfrac{\tan \iota}{\epsilon} - \dfrac{\partial p}{\partial x} \right) \\ \qquad\qquad + \epsilon \dfrac{\partial S_{xx}}{\partial x} + \epsilon \dfrac{\partial S_{xy}}{\partial y} + \dfrac{\partial S_{xz}}{\partial z}, \\ \epsilon \mathrm{Re} \left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} \right) = -\epsilon \dfrac{\mathrm{Re}}{\mathrm{Fr}^2} \dfrac{\partial p}{\partial y} \\ \qquad\qquad + \epsilon \dfrac{\partial S_{yx}}{\partial x} + \epsilon \dfrac{\partial S_{yy}}{\partial y} + \dfrac{\partial S_{yz}}{\partial z}, \\ \epsilon^2 \mathrm{Re} \left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} \right) = -\dfrac{\mathrm{Re}}{\mathrm{Fr}^2} \left(1 + \dfrac{\partial p}{\partial z} \right) \\ \qquad\qquad + \epsilon \dfrac{\partial S_{zx}}{\partial x} + \epsilon \dfrac{\partial S_{zy}}{\partial y} + \dfrac{\partial S_{zz}}{\partial z}, \end{cases} \tag{149}$$

$$\begin{cases} u = v = w = 0 & \text{on } z = 0, \\ (-p\mathsf{I} + \mathsf{S}) \left(-\epsilon \dfrac{\partial h}{\partial x} \boldsymbol{i} - \epsilon \dfrac{\partial h}{\partial y} \boldsymbol{j} + \boldsymbol{k} \right) = \boldsymbol{0} & \text{on } z = h(x, y, t), \end{cases} \tag{150}$$

where $\epsilon = H/L \ll 1$, and

$$\begin{aligned} \mathsf{S} = \varsigma(p) \Bigg[& \epsilon^2 \left(\frac{\partial u}{\partial x} \right)^2 + \epsilon^2 \left(\frac{\partial v}{\partial y} \right)^2 + \epsilon^2 \left(\frac{\partial w}{\partial z} \right)^2 + \frac{\epsilon^2}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial u}{\partial z} + \epsilon^2 \frac{\partial w}{\partial x} \right)^2 \\ & + \frac{1}{2} \left(\frac{\partial v}{\partial z} + \epsilon^2 \frac{\partial w}{\partial y} \right)^2 \Bigg]^{\frac{1-\chi}{2\chi}} \times \Bigg[2\epsilon \left(\frac{\partial u}{\partial x} \boldsymbol{i} \otimes \boldsymbol{i} + \frac{\partial v}{\partial y} \boldsymbol{j} \otimes \boldsymbol{j} + \frac{\partial w}{\partial z} \boldsymbol{k} \otimes \boldsymbol{k} \right) \\ & + \epsilon \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) (\boldsymbol{i} \otimes \boldsymbol{j} + \boldsymbol{j} \otimes \boldsymbol{i}) + \left(\frac{\partial u}{\partial z} + \epsilon^2 \frac{\partial w}{\partial x} \right) (\boldsymbol{i} \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \boldsymbol{i}) \\ & + \left(\frac{\partial v}{\partial z} + \epsilon^2 \frac{\partial w}{\partial y} \right) (\boldsymbol{j} \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \boldsymbol{j}) \Bigg]. \end{aligned} \tag{151}$$

In this framework the dimensionless version of ς is an increasing function such that $\varsigma(0) = 1$. In particular, (143) and (144) become, respectively,

$$\varsigma(p) = \mathrm{e}^{\omega p} \quad \text{with} \quad \omega = \beta \rho g \cos \iota H, \tag{152}$$

and

$$\varsigma(p) = 1 + \omega p^n \quad \text{with} \quad \omega = \beta (\rho g \cos \iota H)^n. \tag{153}$$

The dimensionless quantities

$$\mathrm{Re} = \frac{\rho U_{\parallel}^{(2\chi-1)/\chi} H^{1/\chi}}{\varsigma_{\mathrm{ref}}} \quad \text{and} \quad \mathrm{Fr} = \frac{U_{\parallel}}{\sqrt{g \cos \iota H}} \tag{154}$$

are, respectively, the Reynolds and Froude numbers for a fluid film moving over an inclined plane.

Depending on the values considered for the characteristic scales, different types of flow regime occur. Here we shall focus on the following two types of flow regimes:

1. The *nearly steady uniform regime*, where the viscous contribution is comparable to the gravitational effect. In this case, we have

$$U_{\parallel} = \left[\frac{\rho g \sin \iota H^{(\chi+1)/\chi}}{\varsigma_{\mathrm{ref}}} \right]^{\chi} \tag{155}$$

and $\mathrm{Fr}^2 = O(\mathrm{Re})$. Inertial terms and pressure gradient terms must be negligible, which means $\epsilon\mathrm{Re} \ll 1$. Therefore, from (149) and (151) the approximate equations are found to be given by

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial}{\partial z} \left\{ \varsigma(p) \left[\left(\dfrac{\partial u}{\partial z} \right)^2 + \left(\dfrac{\partial v}{\partial z} \right)^2 \right]^{(1-\chi)/(2\chi)} \dfrac{\partial u}{\partial z} \right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad + 2^{(1-\chi)/(2\chi)} = 0, \\ \dfrac{\partial}{\partial z} \left\{ \varsigma(p) \left[\left(\dfrac{\partial u}{\partial z} \right)^2 + \left(\dfrac{\partial v}{\partial z} \right)^2 \right]^{(1-\chi)/(2\chi)} \dfrac{\partial v}{\partial z} \right\} = 0, \\ \dfrac{\partial p}{\partial z} + 1 = 0. \end{cases} \tag{156}$$

2. The *viscous regime*, where the effect of the pressure gradient is balanced by stresses induced due to the viscosity within the bulk. In this case, we have

$$U_{\parallel} = \left[\frac{\rho g \cos \iota H^{(2\chi+1)/\chi}}{\varsigma_{\mathrm{ref}} L} \right]^{\chi} \tag{157}$$

and consequently $\mathrm{Fr}^2 = \epsilon\mathrm{Re}$. Inertial terms must be small compared to the effect of the pressure gradient and the slope must be gentle ($\tan \iota = O(\epsilon)$). This imposes the constraint $\epsilon\mathrm{Re} \ll 1$. In such a way, from (149) and (151) we deduce the approximate equations

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial}{\partial z}\left\{\varsigma(p)\left[\left(\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial v}{\partial z}\right)^2\right]^{(1-\chi)/(2\chi)} \dfrac{\partial u}{\partial z}\right\} \\ \qquad\qquad +2^{(1-\chi)/(2\chi)}\left(\dfrac{\tan\iota}{\epsilon} - \dfrac{\partial p}{\partial x}\right) = 0, \\ \dfrac{\partial}{\partial z}\left\{\varsigma(p)\left[\left(\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial v}{\partial z}\right)^2\right]^{(1-\chi)/(2\chi)} \dfrac{\partial v}{\partial z}\right\} \\ \qquad\qquad -2^{(1-\chi)/(2\chi)}\dfrac{\partial p}{\partial y} = 0, \\ \dfrac{\partial p}{\partial z} + 1 = 0. \end{cases} \tag{158}$$

Moreover, from (150) and (151), by virtue of the smallness of ϵ, the boundary conditions (145) approximate to

$$\begin{cases} u = v = w = 0 \text{ on } z = 0, \\ p = 0 \qquad\qquad\ \text{on } z = h(x, y, t), \\ \dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0 \ \text{ on } z = h(x, y, t). \end{cases} \tag{159}$$

Finally, we derive the evolution equation for the free surface $z = h(x, y, t)$. We first integrate the constraint of incompressibility over the flow depth to obtain, by means of boundary condition $(159)_1$,

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \mathrm{d}z = \frac{\partial}{\partial x}\int_0^h u\mathrm{d}z + \frac{\partial}{\partial y}\int_0^h v\mathrm{d}z - u_{|z=h}\frac{\partial h}{\partial x} - v_{|z=h}\frac{\partial h}{\partial y} - w_{|z=h}. \tag{160}$$

But, obviously,

$$w_{|z=h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial t} + u_{|z=h}\frac{\partial h}{\partial x} + v_{|z=h}\frac{\partial h}{\partial y}. \tag{161}$$

Therefore, combining (160) and (161) gives the required equation for h:

$$h_t + \frac{\partial(h\bar{u})}{\partial x} + \frac{\partial(h\bar{v})}{\partial y} = 0, \tag{162}$$

where we have introduced the depth-averaged variables defined as

$$\bar{\varphi}(x, y, t) = \frac{1}{h(x, y, t)} \int_0^{h(x,y,t)} \varphi(x, y, z, t)\mathrm{d}z. \tag{163}$$

6.2 Nearly Steady Uniform Regime

It is easy to verify that system (156) with boundary conditions (159) admits the solution

$$\begin{cases} u = 2^{(1-\chi)/2} \displaystyle\int_0^z \left[\frac{h-\zeta}{\varsigma(h-\zeta)}\right]^\chi \mathrm{d}\zeta, \\ v = 0, \\ w = -2^{(1-\chi)/2} \dfrac{\partial}{\partial x} \displaystyle\int_0^z \left\{ \int_0^{\zeta_1} \left[\frac{h-\zeta}{\varsigma(h-\zeta)}\right]^\chi \mathrm{d}\zeta_2 \right\} \mathrm{d}\zeta_1, \\ p = h - z. \end{cases} \tag{164}$$

Therefore

$$h\bar{u} = 2^{(1-\chi)/2} \int_0^h \xi \left[\frac{\xi}{\varsigma(\xi)}\right]^\chi \mathrm{d}\xi \equiv F(h) \tag{165}$$

and (162) becomes

$$\frac{\partial h}{\partial t} + F'(h)\frac{\partial h}{\partial x} = 0. \tag{166}$$

Equation (166) is a quasilinear first order partial differential equation whose general solution can be found by the method of characteristics. If $f(\xi)$ is an initial profile, then the corresponding solution is given by

$$h = f(x - F'(h)t). \tag{167}$$

The wave (167) breaks, i.e. its profile becomes multivalued, at time $t_B = -\left[F''(f(\xi_B))\dfrac{\mathrm{d}f}{\mathrm{d}\xi}(\xi_B)\right]^{-1}$ at the point $x_B = \xi_B + F'(f(\xi_B))t_B$, provided that ξ_B satisfies the following two conditions

$$\begin{cases} F''(f(\xi_B))\dfrac{\mathrm{d}f}{\mathrm{d}\xi}(\xi_B) < 0 \\ \left| F''(f(\xi_B))\dfrac{\mathrm{d}f}{\mathrm{d}\xi}(\xi_B) \right| = \max \left| \dfrac{\mathrm{d}F'(f(\xi))}{\mathrm{d}\xi} \right|. \end{cases} \tag{168}$$

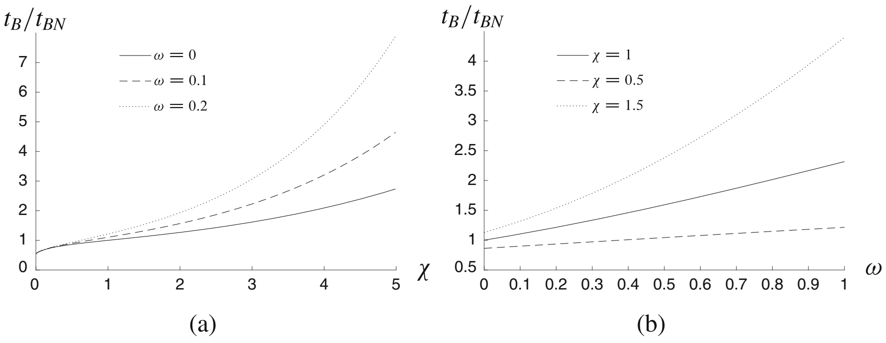


Fig. 3 Ratio t_B/t_{BN} as a function of (**a**) χ and (**b**) the piezo-viscous coefficient ω. The pressure-dependent parameter ς is assumed to be of the form (143) and the initial profile considered is $h(x,0) = 1 - x^2$

Since ς is a positive increasing function, from (168) we deduce that the pressure dependence of the viscosity has the effect of delaying the time at which the wave could break. To quantify this delaying effect we consider ς of the form (152) and assume that $h(x,0) = f(x) = 1 - x^2$. If the fluid is Newtonian with a constant viscosity μ_0 (i.e., $\chi = 1$ and $\varsigma(p - p_{\text{ref}}) = \mu_{\text{ref}}$ in (142)), it is easy to show that the wave breaks at time $t_{BN} = 3\sqrt{3}/8$. In order to make the differences between the non-Newtonian case that is being considered and the classical Newtonian case more evident, we have plotted the ratio between the breaking time t_B in the non-Newtonian case and t_{BN} as a function of χ (Fig. 3a) and as a function of the non-dimensional piezo-viscous coefficient ω (Fig. 3b). Furthermore, the solutions to the wave equation (166) with $\chi = 0.5$ (Fig. 4a) and $\chi = 1.5$ (Fig. 4b) are plotted at different times together with the profiles of the free surface $z = h(x,t)$ in the classical Newtonian case. We find that the solutions are qualitatively similar, though quantitatively different.

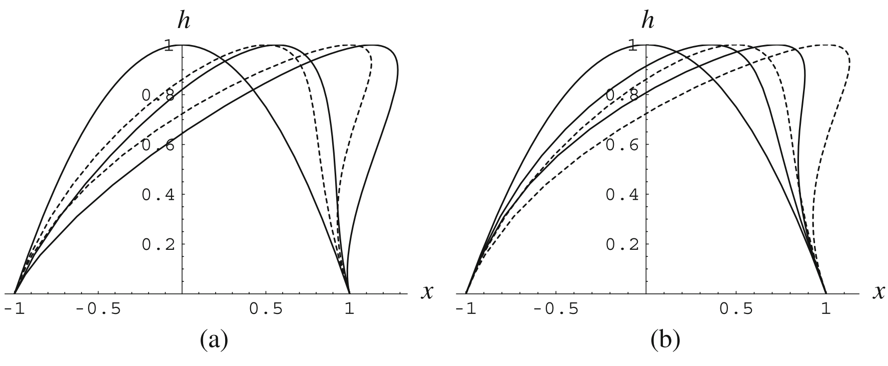


Fig. 4 Solutions of (166) with an initial profile $f(x) = 1 - x^2$ at $t = 0$, $t = 0.5$, $t = 1$. The dashed line represents the solution in the classical Newtonian case, whereas the solid line represents the solution in the case in which the dimensionless parameter ς depends on the pressure according to the law $\varsigma(p) = e^{0.1p}$ and (**a**) $\chi = 0.5$ and (**b**) $\chi = 1.5$

Finally, in order to look for self-similar solutions of (166), we need to know whether F' is invertible. The invertibility of F' is linked with the equation

$$\chi h \varsigma'(h) - (\chi + 1)\varsigma(h) = 0. \tag{169}$$

Indeed, if (169) admits positive roots, the least of which we denote by $\hat{h}$, then F' is invertible in $[0, \hat{h}[$. On the contrary, if (169) does not admit positive roots, then F' is invertible in $[0, +\infty[$. In any case F' is continuous and increasing. It is interesting to show that some time after the initiation of the current, no matter what the initial shape, the solution tends to the unique self-similar solution of Eq. (166), i.e.

$$h(x,t) \to F'^{-1}\left(\frac{x}{t}\right) \quad \text{as} \quad t \to +\infty. \tag{170}$$

In order to prove (170), from (166) we deduce that h is constant along the characteristics given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F'(h). \tag{171}$$

Thus, if initially $h(x, 0) = f(x)$, the characteristics are straight lines

$$x = x_0 + F'[f(x_0)]t, \tag{172}$$

with x_0 being the initial value of the characteristic. The solution of (166) is then

$$h(x,t) = F'^{-1}\left(\frac{x - x_0}{t}\right) \to F'^{-1}\left(\frac{x}{t}\right) \quad \text{as} \quad t \to +\infty. \tag{173}$$

If the viscosity does not depend on the pressure, Eq. (170) reduces to the self-similar solution found by Perazzo and Gratton [34] that in turn is the non-Newtonian counterpart of the self-similar solution derived by Huppert [21] for Newtonian fluids.

6.3 Viscous Regime

A lengthy but straightforward algebraic manipulation allows us to obtain the solution to the boundary-value problem (158)–(159):

$$
\begin{cases}
u = 2^{(1-\chi)/2}\left(\dfrac{\tan\iota}{\epsilon} - \dfrac{\partial h}{\partial x}\right)\left[\left(\dfrac{\partial h}{\partial x} - \dfrac{\tan\iota}{\epsilon}\right)^2 + \left(\dfrac{\partial h}{\partial y}\right)^2\right]^{(\chi-1)/2} \displaystyle\int_0^z \left[\dfrac{h-\zeta}{\varsigma\,(h-\zeta)}\right]^\chi \mathrm{d}\zeta, \\
v = -2^{(1-\chi)/2}\dfrac{\partial h}{\partial y}\left[\left(\dfrac{\partial h}{\partial x} - \dfrac{\tan\iota}{\epsilon}\right)^2 + \left(\dfrac{\partial h}{\partial y}\right)^2\right]^{(\chi-1)/2} \displaystyle\int_0^z \left[\dfrac{h-\zeta}{\varsigma\,(h-\zeta)}\right]^\chi \mathrm{d}\zeta, \\
w = -2^{(1-\chi)/2}\nabla_s \cdot \left\{\left|\dfrac{\tan\iota}{\epsilon}\boldsymbol{i} - \nabla_s h\right|^{\chi-1}\left(\dfrac{\tan\iota}{\epsilon}\boldsymbol{i} - \nabla_s h\right)\displaystyle\int_0^z \left[\int_0^{\zeta_1}\left(\dfrac{h-\zeta_2}{\varsigma(h-\zeta_2)}\right)^\chi \mathrm{d}\zeta_2\right]\mathrm{d}\zeta_1\right\}, \\
p = h - z,
\end{cases}
\tag{174}
$$

where ∇_s is the two-dimensional gradient:

$$
\nabla_s \varphi = \frac{\partial \varphi}{\partial x}\boldsymbol{i} + \frac{\partial \varphi}{\partial y}\boldsymbol{j}. \tag{175}
$$

Then

$$
h\bar{u} = F(h)\left(\frac{\tan\iota}{\epsilon} - \frac{\partial h}{\partial x}\right)\left[\left(\frac{\partial h}{\partial x} - \frac{\tan\iota}{\epsilon}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right]^{(\chi-1)/2}, \tag{176}
$$

$$
h\bar{v} = -F(h)\frac{\partial h}{\partial y}\left[\left(\frac{\partial h}{\partial x} - \frac{\tan\iota}{\epsilon}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right]^{(\chi-1)/2}, \tag{177}
$$

so that (162) becomes

$$
\frac{\partial h}{\partial t} + \nabla_s \cdot \left\{F(h)\left|\frac{\tan\iota}{\epsilon}\boldsymbol{i} - \nabla_s h\right|^{\chi-1}\left(\frac{\tan\iota}{\epsilon}\boldsymbol{i} - \nabla_s h\right)\right\} = 0. \tag{178}
$$

Now let us make the further assumption that the flow depends only on the x coordinate. Then $\dfrac{\partial h}{\partial y} = 0$ (so that $v = 0$) and (178) reduces to

$$
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left\{F(h)\left|\frac{\tan\iota}{\epsilon} - \frac{\partial h}{\partial x}\right|^{\chi-1}\left(\frac{\tan\iota}{\epsilon} - \frac{\partial h}{\partial x}\right)\right\} = 0. \tag{179}
$$

To find traveling wave solutions we assume that h depends on the single variable $s \equiv x - ct$, where c is a constant which represents the wave speed. Since in what follows we shall assume that the inclined plane is infinite and limit our analysis to waves propagating downwards, we take $c > 0$. We refer the interested reader to [41] for a more detailed study of downslope and upslope traveling wave solutions. Inserting the ansatz $h = h(s)$ into (179) and integrating once, one obtains

$$
\left|\frac{\tan\iota}{\epsilon} - \frac{\mathrm{d}h}{\mathrm{d}s}\right|^{\chi-1}\left(\frac{\tan\iota}{\epsilon} - \frac{\mathrm{d}h}{\mathrm{d}s}\right) = \frac{c_1 + ch}{F(h)}, \tag{180}
$$

c_1 being an integration constant. Let $c_1 = 0$ in (180). Then, (180) may be written as

$$\frac{\mathrm{d}h}{\mathrm{d}s} = \frac{\tan\iota}{\epsilon} - \left[\frac{ch}{F(h)}\right]^{1/\chi}. \tag{181}$$

In order to discuss the integrability of Eq. (181) with $c > 0$, we have to find the positive roots of the following equation

$$\left(\frac{\tan\iota}{\epsilon}\right)^{\chi} F(h) - ch = 0. \tag{182}$$

The roots of (182) may be found numerically. Nevertheless, we can deduce the number of positive roots of (182) by studying the function $\mathscr{F}(h) \equiv (\tan\iota/\epsilon)^{\chi} F(h)/h$. $\mathscr{F}$ is a continuous differentiable function that tends to zero as $h \to 0$, whose derivative may be written as

$$\begin{aligned} \mathscr{F}'(h) &= \left(\frac{\tan\iota}{\epsilon}\right)^{\chi} \frac{1}{h^2} \int_0^h \xi F''(\xi)\mathrm{d}\xi \\ &= 2^{(1-\chi)/2} \left(\frac{\tan\iota}{\epsilon}\right)^{\chi} \frac{1}{h^2} \int_0^h \left(\frac{\xi}{\varsigma(\xi)}\right)^{\chi+1} \left[(\chi+1)\varsigma(\xi) - \chi\xi\varsigma'(\xi)\right] \mathrm{d}\xi. \end{aligned} \tag{183}$$

From (183) it follows that $\mathscr{F}'$ is positive in a neighbourhood of $h = 0$, but it might change sign away from zero if (169) admits positive roots. Here, for the sake of simplicity, we shall limit our analysis to the constitutive functions for which (169) admits at most one positive root. It is easy to recognize that models (152) and (153) meet this requirement.

We are now able to say how many positive roots (182) admits. In fact:

1. if $[\varsigma(h)/h]^{\chi}$ has linear growth as $h \to +\infty$, then $\mathscr{F}$ is increasing and tends to $l > 0$ as $h \to +\infty$ so that (182) with $c \in]0, l[$ admits only one positive root, whereas it does not admit a positive root for $c \geq l$;
2. if $[\varsigma(h)/h]^{\chi}$ has sublinear growth as $h \to +\infty$, then $\mathscr{F}$ is increasing and tends to $+\infty$ as $h \to +\infty$ so that, for any $c > 0$, (182) admits a unique positive root;
3. if $[\varsigma(h)/h]^{\chi}$ has superlinear growth as $h \to +\infty$, then $\mathscr{F}$ attains its absolute maximum at $h = h_* > 0$ and tends to zero as $h \to +\infty$ so that (182) admits two positive roots if $c \in]0, \mathscr{F}(h_*)[$, only one positive root if $c = \mathscr{F}(h_*)$, and no positive root for $c > \mathscr{F}(h_*)$.

According to the number of positive roots of (182), one, two or three families of solutions to (180) may arise.

If (182) does not admit a positive root, then Eq. (180) may be numerically integrated over the range $(0, \bar{h})$ for all $\bar{h} > 0$. In this case the general solution is a decreasing function defined over the interval $(-\infty, c_2)$, c_2 being an integration constant and tends to $+\infty$ as $s \to -\infty$. Therefore, we do not consider these solutions as they do not satisfy the lubrication approximation.

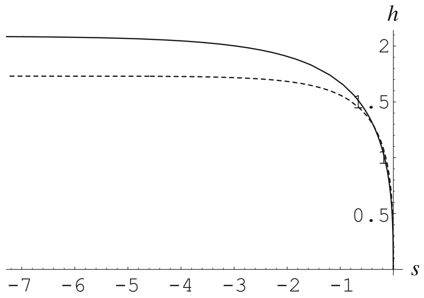


Fig. 5 Profiles of downslope traveling waves behind a front. The solid line represents the traveling wave solution when $\chi = 1.5$ and $\varsigma(p) = 1 + 0.2p$, whereas the dashed line represents the traveling wave solution in the classical Newtonian case. We have considered $(\tan\iota)/\epsilon = 1$ and $c = 1$

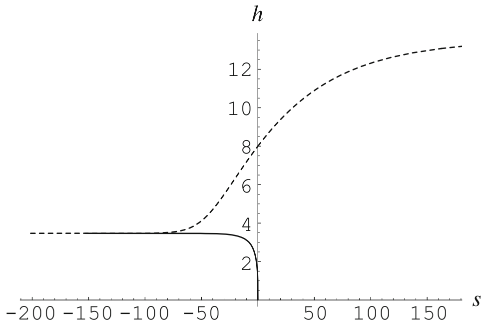


Fig. 6 Profiles of downslope traveling waves solutions for $\chi = 1.1$, $\varsigma(p) = 1 + 0.05p^3$, $\tan\iota/\epsilon = 0.5$ and $c = 1$. In the case we are considering two families of downslope traveling wave solutions that satisfy the lubrication approximation arise (see the text): downslope traveling waves behind a front (solid line) and compressive shock waves (dashed line)

If (182) admits only one positive root h_m, then two families of solutions to (180) arise. The first is formed by bounded decreasing functions defined over the range $(-\infty, c_2)$ satisfying the inequality $0 \leq h \leq h_m$. For these solutions we have $h \to h_m$ as $s \to -\infty$. Then they represent traveling waves behind a front running downslope that, far behind the front ($s \to -\infty$), tend to the steady downslope flow $h = h_m$ (see Fig. 5). The other family is formed by increasing functions bounded from below for which $h \geq h_m$. These solutions represents downslope traveling waves with no front for which $h \to h_m$ as $s \to -\infty$ and $h \to +\infty$ as $s \to +\infty$. Therefore, they do not satisfy the lubrication approximation.

If (182) admits two positive roots, $h_m < h_M$, then, as well as the downslope traveling waves behind a front, two other families of solutions to (180) arise, representing downslope traveling waves with no front (Fig. 6). The former is constituted by bounded increasing functions satisfying the inequality $h_m \leq h \leq h_M$ and for which we have $h \to h_m$ as $s \to -\infty$ and $h \to h_M$ as $s \to +\infty$. The latter is formed by decreasing functions that are bounded from below as they satisfy the inequality $h \geq h_M$ and for which we have $h \to +\infty$ as $s \to -\infty$ and $h \to h_M$ as $s \to +\infty$. We disregard the traveling wave solutions belonging to this family as the length-scale ratio fails to be small as $s \to -\infty$. On the contrary, the former class of downslope travelling waves with no front satisfies the lubrication approximation. Furthermore, as shown by Rajagopal et al. [41], the waves belonging to this family are compressive shock waves which can also be viewed as heteroclinic orbits connecting the two equilibria $h = h_m$ and $h = h_M$ of (181) (see Fig. 6).

We finally observe that

$$F(h) \simeq 2^{(\chi-1)/2} \frac{h^{\chi+2}}{\chi+2} \quad \text{as} \quad h \to 0. \tag{184}$$

Therefore near the wave front, where the effects of pressure can be neglected, the solution to Eq. (180) is approximated by that found by Perazzo and Gratton [34], namely

$$\begin{aligned} &\frac{\tan \iota}{\epsilon}(s - c_2) \\ &= h \left\{ 1 - {}_2F_1 \left[\frac{\chi}{\chi+1}, 1, \frac{2\chi+1}{\chi+1}, \frac{\tan \iota}{\epsilon} \left(\frac{c(\chi+2)}{2^{(1-\chi)/2}} \right)^{-1/\chi} h^{1+1/\chi} \right] \right\}, \end{aligned} \tag{185}$$

with ${}_2F_1(a, b, c, d)$ being the hypergeometric function. From (185) we deduce that h' tends to infinity as $s \to c_2$. Hence, near the wave front, the component of the fluid velocity normal to the incline is not small with respect to the parallel component and thus the solution does not satisfy the lubrication approximation.

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