A Note on Diffusion Processes with Jumps

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Abstract. We focus on stochastic diffusion processes with jumps occurring at random times. After each jump the process is reset to a fixed state from which it restarts with a different dynamics. We analyze the transition probability density function, its moments and the first passage time density. The obtained results are used to study the lognormal diffusion process with jumps which is of interest in the applications.

1 Introduction and Description of the Model

Stochastic processes with jumps play a relevant role in many fields of applications. For example, in [3,7,10], diffusion processes with jumps are studied in order to model an intermittent treatment for tumor diseases, in [4] birth-and-death processes with jumps are analyzed as queuing models with catastrophes, in [5] a non-homogeneous Ornstein-Uhlenbeck with jumps is considered in relation to neuronal activity. In these contexts, a jump is random event that changes the state of the process leading it to another random state from which the dynamics restarts with the same or different law.

We consider diffusion processes assuming that the jumps occur at random times chosen with a fixed probability density function (pdf). After each jump the process is reset to a fixed state from which it restarts with a different dynamics.

Let $\{\widetilde{X}_k(t), t \geq t_0 \geq 0\}$ $(k = 0, 1, \ldots)$ be a stochastic diffusion process. Following [6], we construct the stochastic process $X(t)$ with random jumps. Starting from the initial state $\rho_0 = X(t_0)$, the process $X(t)$ evolves according to $\widetilde{X}_0(t)$ until a jump occurs that shifts the process to a state ρ_1. From here, $X(t)$ restarts according to $\widetilde{X}_1(t)$ until another jump occurs resetting the process to ρ_2 and so on. The effect of the k-th jump $(k = 1, 2, \ldots)$ is to shift the state of $X(t)$ in ρ_k. Then, the process evolves like $\widetilde{X}_k(t)$, until a new jump occurs.

$X(t)$ consists of cycles, whose durations, I_1, $I_2, \ldots$, representing the time intervals between two consecutive jumps, are independent random variables distributed with pdf $\psi_k(\cdot)$. We denote by $\Theta_1, \Theta_2, \ldots$ the times in which the jumps occur. We set $\Theta_0 = t_0$ that corresponds the initial time and for $k = 1, 2, \ldots$, let $\gamma_k(\tau)$ be the pdf of the random variable Θ_k. The variables I_k and Θ_k are

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related, indeed $\Theta_1 = I_1$ and for $k > 1$ it results $\Theta_k = I_1 + I_2 + \ldots I_k$. Hence, the pdf $\gamma_k(\cdot)$ of Θ_k and the pdf $\psi_k(\cdot)$ of I_k are related, indeed $\gamma_1(t) = \psi_1(t)$ and $\gamma_k(t) = \psi_1(t) * \psi_2(t) * \cdots * \psi_k(t)$, where "$*$" denotes the convolution operator.

In the paper we study $X(t)$ by analyzing the transition pdf, its moments and the first passage time of $X(t)$ through a constant boundary. We consider some particular cases when the inter-jumps I_k are deterministic or exponentially distributed. Finally, the lognormal diffusion process with jumps is studied.

2 Some Probabilistic Features of the Process

Let $f(x,t|y,\tau) = \frac{\partial}{\partial x} P[X(t) \le x | X(\tau) = y]$, $\widetilde{f}_k(x,t|y,\tau) = \frac{\partial}{\partial x} P[\widetilde{X}(t) \le x | \widetilde{X}(\tau) = y]$ be the transition pdf's of $X(t)$ and $\widetilde{X}_k(t)$, respectively. The densities f and $\widetilde{f}_k$ are related. Indeed, considering the age of the process with jumps, we have the following expression of the transition pdf of the process $X(t)$

$$f(x,t|\rho_0,t_0) = \left(1 - \int_0^{t-t_0} \psi_1(s)\,ds\right) \widetilde{f}_0(x,t|\rho_0,t_0) + \sum_{k=1}^{\infty} \int_{t_0}^{t} \left(1 - \int_0^{t-\tau} \psi_k(s)ds\right) \widetilde{f}_k(x,t|\rho_k,\tau)\,\gamma_k(\tau)\,d\tau. \qquad (1)$$

We analyze the right hand side of (1). The first term represents the case in which there are not jumps in the interval (t_0, t), so that $X(t)$ evolves as $\widetilde{X}_0(t)$. The factor $1 - \int_0^{t-t_0} \psi_1(s)\,ds$ represents the probability that the first jump occurs after the time t. The sum in (1) concerns the circumstance that one or more jumps occur in (t_0, t). In this case, the last jump, the k-th one, occurs at the time $\tau \in (t_0, t)$ with probability $1 - \int_0^{t-\tau} \psi_k(s)\,ds$; then $X(t)$ evolves according to $\widetilde{X}_k(t)$ to reach x at time t, starting from ρ_k.

Denoting by $m^{(n)}(x,t|y,\tau) = E[X^n(t)|X(\tau) = y]$ and $\widetilde{m}_k^{(n)}(x,t|y,\tau) = E[\widetilde{X}^n(t)|\widetilde{X}(\tau) = y]$ the conditional moments of $X(t)$ and $\widetilde{X}_k(t)$, respectively, from (1) it follows

$$m^{(n)}(t|\rho_0,t_0) = \left(1 - \int_0^{t-t_0} \psi_1(s)\,ds\right) \widetilde{m}_0^{(n)}(t|\rho_0,t_0) + \sum_{k=1}^{\infty} \int_{t_0}^{t} \left(1 - \int_0^{t-\tau} \psi_k(s)ds\right) \widetilde{m}_k^{(n)}(t|\rho_k,\tau)\,\gamma_k(\tau)\,d\tau. \qquad (2)$$

To analyze the first passage time (FPT) of $X(t)$, we consider a state $S > \rho_k$ ($k = 0, 1, 2 \ldots$). For $X(t_0) < S$ we denote by $T_{\rho_0}(t_0) = \inf\{t \ge t_0 : X(t) > S\}$ the random variable FPT of $X(t)$ through S and with $g(S,t|\rho_0,t_0) = \frac{\partial}{\partial t} P\{T_{\rho_0}(t_0) < t\}$. Similarly let $\widetilde{T}_{\rho_0}(t_0) = \inf\{t \ge t_0 : \widetilde{X}(t) > S\}$ with $\widetilde{X}(t_0) < S$ be the FPT for $\widetilde{X}(t)$ through S and $g(S,t|\rho_0,t_0) = \frac{\partial}{\partial t} P\{T_{\rho_0}(t_0) < t\}$. Recalling that $X(t)$

consists of independent cycles and that $\widetilde{X}_k(t)$ evolves in I_k, the following relation can be obtained

$$g(S,t|\rho_0,t_0) = \left[1 - \int_0^{t-t_0} \psi(s)\,ds\right] \widetilde{g}_0(S,t|\rho_0,t_0) \tag{3}$$
$$+\sum_{k=1}^{\infty} \int_{t_0}^{t} \left[1 - \int_0^{t-\tau} \psi(s)ds\right] \widetilde{g}_k(S,t|\rho_k,\tau)\gamma_k(\tau)\,d\tau \left\{\prod_{j=0}^{k-1} \left[1 - P(\widetilde{T_j}(\Theta_j) < \Theta_{j+1})\right]\right\},$$

where the product $\prod_{j=0}^{k-1}\left[1 - P(\widetilde{T_j}(\Theta_j) < \Theta_{j+1})\right]$ represents the probability that none of the processes $\widetilde{X}_0(t), \widetilde{X}_1(t), \ldots, \widetilde{X}_{k-1}(t)$ crosses S before τ.

3 Deterministic Inter-jumps

We assume that the jumps occur at fixed times denoted by $\tau_1, \tau_2, \ldots, \tau_N$. Therefore, $X(t)$ consists of a combination of processes $\widetilde{X}_k(t)$ with $\widetilde{X}_k(\tau_k) = X(\tau_k) = \rho_k$. Assuming that $\tau_0 = t_0$, $\tau_{N+1} = \infty$, one has

$$X(t) = \sum_{k=0}^{N} \widetilde{X}_k(t)\mathbf{1}_{(\tau_k,\tau_{k+1})}(t) \quad \text{with} \quad \mathbf{1}_{(\tau_k,\tau_{k+1})}(t) = \begin{cases} 1, & t \in (\tau_k, \tau_{k+1}) \\ 0, & t \notin (\tau_k, \tau_{k+1})\,. \end{cases}$$

After the time τ_N, the process $X(t)$ evolves as $\widetilde{X}_N(t)$. For $k = 0, 1, \ldots, N$, $\Theta_k = \tau_k$ a.s. and I_k are degenerate random variables; in particular, denoting by δ the delta Dirac function, the pdf 's of Θ_k and of I_k are

$$\gamma_k(t) = \delta(t - \tau_k), \qquad \psi_k(t) = \delta\left[t - (\tau_k - \tau_{k-1})\right],$$

respectively. We note that

$$\int_a^b \delta(s - \tau_k)ds = H(b - a - \tau_k), \qquad H(x) = \int_{-\infty}^{x} \delta(u)\,du = \begin{cases} 0, & x < 0 \\ 1, & x > 0, \end{cases} \tag{4}$$

where $H(\cdot)$ is the Heaviside unit step function. Hence, from (1) one has:

$$f(x,t|\rho_0,t_0) = \left[1 - H\left(t - \tau_1\right)\right] \widetilde{f}_0(x,t|\rho_0,t_0)$$
$$+ \sum_{k=1}^{\infty} H\left(t - \tau_k\right)\left[1 - H\left(t - \tau_k - (\tau_k - \tau_{k-1})\right)\right] \widetilde{f}_k\left(x,t|\rho_k,\tau_k\right),$$

from which, recalling (4), it follows:

$$f(x,t|\rho_0,t_0) = \sum_{k=0}^{N} \widetilde{f}_k(x,t|\rho_k,\tau_k)\mathbf{1}_{(\tau_k,\tau_{k+1})}(t). \tag{5}$$

Similarly, from (2) the conditional moments of $X(t)$ can be obtained.

Concerning the FPT problem, we note that since $\Theta_k = \tau_k$ *a.s.*, one has

$$1 - P[\widetilde{T}_k(\tau_k) < \tau_{k+1}] = 1 - \int_{\tau_k}^{\tau_{k+1}} \widetilde{g}_k(S, \tau|\rho_k, \tau_k) d\tau;$$

so, following the procedure used to obtain (5), one has:

$$g(S, t|\rho_0, t_0) = \begin{cases} \widetilde{g}_0(S, t|\rho_0, t_0), & t \in \mathcal{I}_1 \\ \displaystyle\prod_{j=0}^{k-1} \left[1 - \int_{\tau_j}^{\tau_{j+1}} \widetilde{g}_j(S, \tau|\rho_i, \tau_j) d\tau\right] \widetilde{g}_k(S, t|\rho_k, \tau_k), & t \in \mathcal{I}_k \\ & (k = 2, 3, \ldots). \end{cases} \quad (6)$$

4 Exponentially Distributed Inter-jumps

We assume that, for $k \geq 1$, $\rho_k = \rho$ and I_k are identically distributed with pdf $\psi_k(s) = \psi(s) = \xi e^{-\xi s}$ for $s > 0$. In this case the pdf of Θ_k is an Erlang distribution with parameters (k, ξ), i.e. $\gamma_k(t) = \xi^k t^{k-1} e^{-\xi t}/(k-1)!$ for $t > 0$. From (1) and (2) the transition pdf and the conditional moments of $X(t)$ result:

$$f(x, t|\rho, t_0) = e^{-\xi(t-t_0)} \widetilde{f}_0(x, t|\rho, t_0) + e^{-\xi t} \sum_{k=1}^{\infty} \int_{t_0}^{t} \frac{\xi^k \tau^{k-1}}{(k-1)!} \widetilde{f}_k(x, t|\rho, \tau) d\tau, \quad (7)$$

$$m^{(n)}(t|\rho, t_0) = e^{-\xi(t-t_0)} \widetilde{m}_0^{(n)}(t|\rho, t_0) + e^{-\xi t} \sum_{k=1}^{\infty} \int_{t_0}^{t} \frac{\xi^k \tau^{k-1}}{(k-1)!} \widetilde{m}_k^{(n)}(t|\rho, \tau) d\tau, \quad (8)$$

respectively. Moreover, concerning the FPT pdf, from (3) one has:

$$g(S, t|\rho, t_0) = e^{-\xi(t-t_0)} \widetilde{g}(S, t|\rho, t_0) + \sum_{k=1}^{\infty} \int_{t_0}^{t} \frac{(\xi\tau)^{k-1} e^{-\xi\tau}}{(k-1)!} \xi e^{-\xi(t-\tau)} \widetilde{g}_k(S, t|\rho, \tau) d\tau \left\{ \prod_{j=0}^{k-1} \left[1 - P(\widetilde{T}_j(\Theta_j) < \Theta_{j+1})\right] \right\}. \quad (9)$$

We assume that each $\widetilde{X}_k(t)$ evolves as $\widetilde{X}(t)$, from (7) and (8) one obtains:

$$f(x, t|\rho, t_0) = e^{-\xi(t-t_0)} \widetilde{f}(x, t|\rho, t_0) + \xi \int_{t_0}^{t} e^{-\xi(t-\tau)} \widetilde{f}(x, t|\rho, \tau) d\tau; \quad (10)$$

$$m^{(n)}(t|\rho, t_0) = e^{-\xi(t-t_0)} \widetilde{m}^{(n)}(t|\rho, t_0) + \xi \int_{t_0}^{t} e^{-\xi(t-\tau)} \widetilde{m}^{(n)}(t|\rho, \tau) d\tau, \quad (11)$$

in agreement with the analogue results in [1,2]. Moreover, if the involved processes are time homogeneous, one has that $P(\widetilde{T_j}(\Theta_j) < \Theta_{j+1}) = P(\widetilde{T}(0) <$

$I_{j+1}) = P(\widetilde{T}(0) < I)$, where $\widetilde{T}(0)$ is the FPT of $\widetilde{X}_0(t)$ through the threshold S and $I_k \stackrel{d}{=} I$. Therefore, Eq. (9) becomes:

$$g(S, t - t_0|\rho) = e^{-\xi(t-t_0)}\widetilde{g}(S, t - t_0|\rho) + \sum_{k=1}^{\infty} \int_0^{t-t_0} \frac{\left(\xi\tau \left[1 - P(\widetilde{T}(0) < I)\right]\right)^{k-1}}{(k-1)!}$$
$$\times\ xie^{-\xi t}\widetilde{g}(S, t - \tau|\rho)d\tau \left[1 - P(\widetilde{T}(0) < I)\right]$$
$$= e^{-\xi(t-t_0)}\widetilde{g}(x, t - t_0|\rho)$$
$$+\ \xi \left[1 - P(\widetilde{T}(0) < I)\right] e^{-\xi t} \int_0^{t-t_0} e^{\xi\tau\left[1-P(\widetilde{T}(0)<I)\right]}\widetilde{g}(S, t - \tau|\rho)d\tau. \quad (12)$$

5 The Lognormal Process with Jumps

We construct a new process with jumps on the lognormal process. This is an interesting process to study because it and its transformations are largely used in the applications. For example, in [8] a gamma-type diffusion process is transformed in a lognormal process to model the trend of the total stock of the private car-petrol. So, the study is performed on a lognormal process to provide a statistical methodology by which it can be fitted real data and obtain forecasts that, in statistical term, are quite accurate. In this context, a process with jumps can take into consideration the possibility of stock collapsed and a threshold can represent a control value. Also such process with stock collapses can be studied to give forecasts and, eventually, prevent problems. More recently, in [9], a gamma diffusion process with exogenous factors is transformed also in a lognormal process to describe the electric power consumption during a period of economic crisis. The transformation in the lognormal process allows to infer on parameters to give forecasts and, moreover, an application on the total consumption in Spain is considered. In this context, we can construct a process with jumps to take into consideration the possibility of a breakdown. Regarding this process with breakdowns, a threshold can represent a control value that gives an alarm in some cases which can be of interest for the authority.

Let $\widetilde{X}_k(t)$ be the lognormal time homogeneous diffusion processes with drift $A_1^k(x) = a_k x$ and infinitesimal variance $A_2^k(x) = \sigma_k^2 x^2$. For $\widetilde{X}_k(t)$ one has

$$\widetilde{f}_k(x, t|\rho_k, t_k) = \frac{1}{x\sigma_k\sqrt{2\pi(t - \tau_k)}} \exp\left\{-\frac{[\log(\frac{x}{\rho_k}) - (a_k - \frac{\sigma_k^2}{2})(t - \tau_k)]^2}{2\sigma_k^2(t - \tau_k)}\right\}, \quad (13)$$

$$\widetilde{m}_k^{(n)}(t|\rho_k, \tau_k) = \exp\left\{n\left[\log\rho_k + \left(a_k - \frac{\sigma_k^2}{2}\right)(t - \tau_k)\right] + \frac{n^2}{2}\sigma_k^2(t - \tau_k)\right\}, \quad (14)$$

$$\widetilde{g}_k(S, t|\rho_k, \tau_k) = \frac{|\log(\frac{S}{\rho_k})|}{\sqrt{2\pi\sigma_k^2(t - \tau_k)^3}} exp\left\{-\frac{[\log(\frac{S}{\rho_k}) - (a_k - \frac{\sigma_k^2}{2})(t - \tau_k)]^2}{2\sigma_k^2(t - \tau_k)}\right\}. \quad (15)$$

5.1 Lognormal Process with Deterministic Jumps

Let $\tau_1, \tau_2, \ldots, \tau_N$ be the instants in which jumps occur. From (5), recalling (13) one obtains the transition pdf of $X(t)$:

$$f(x,t|\rho_0,t_0) = \sum_{k=0}^{N} \frac{\mathbf{1}_{(\tau_k,\tau_{k+1})}(t)}{x\sqrt{2\pi\sigma_k^2(t-\tau_k)}} \exp\left\{-\frac{[\log(\frac{x}{\rho_k}) - (a_k - \frac{\sigma_k^2}{2})(t-\tau_k)]^2}{2{\sigma_k}^2(t-\tau_k)}\right\}$$

and, making use of (14), the conditional moments of $X(t)$ can be obtained from (2). Moreover, the FPT pdf is obtainable by (6) by remarking that from (15) one has:

$$\int_{\tau_j}^{\tau_{j+1}} \widetilde{g}_j(S,\tau|\rho_j,\tau_j)\tau = \frac{1}{2}\mathrm{Erfc}\left[\frac{\log(\frac{S}{\rho_j}) + (a_j - \frac{\sigma_j^2}{2})(t_{j+1}-t_j)}{\sqrt{2\sigma_j^2(t_{j+1}-t_j)}}\right]$$
$$+ \frac{1}{2}\exp\left\{-\frac{2(a_j - \frac{\sigma_j^2}{2})\log(\frac{S}{\rho_j})}{\sigma_j^2}\right\}\mathrm{Erfc}\left[\frac{\log(\frac{S}{\rho_j}) - (a_j - \frac{\sigma_j^2}{2})(t_{j+1}-t_j)}{\sqrt{2\sigma_j^2(t_{j+1}-t_j)}}\right],$$

where $\mathrm{Erfc}(x) = (2/\sqrt{\pi})\int_x^\infty e^{-t^2}dt$ denotes the complementary error function.

5.2 Lognormal Process with Exponentially Distributed Jumps

Let I_k be identically distributed random variables with $\psi_k(s) \equiv \psi(s) = \xi e^{-\xi s}$, so that the expression (7) holds, with $\widetilde{f}_k(x,t|\rho,\tau_k)$ defined in (13). Moreover, making use of the moments of the single process $\widetilde{X}_k(t)$, also the moments of $X(t)$ can be evaluated via (8). Similarly, recalling (15), from (9) the FPT pdf can be written.

Now we consider the special case in which $\rho_k = \rho$ and $\widetilde{X}_k(t) \stackrel{d}{=} \widetilde{X}(t)$ with $A_1^{(k)}(x) = ax$ and $A_2^{(k)}(x) = \sigma^2 x^2$. In this case, from (10) and (13) one has:

$$f(x,t|\rho,t_0) = \frac{e^{-\xi(t-t_0)}}{x\sqrt{2\pi\sigma^2(t-t_0)}} \exp\left\{-\frac{[\log(\frac{x}{\rho}) - (a-\frac{\sigma^2}{2})(t-t_0)]^2}{2\sigma^2(t-t_0)}\right\}$$
$$+ \xi\int_{t_0}^{t} \frac{e^{-\xi(t-\tau)}}{x\sqrt{2\pi\sigma^2(t-\tau)}} \exp\left\{-\frac{[\log(\frac{x}{\rho}) - (a-\frac{\sigma^2}{2})(t-\tau)]^2}{2\sigma^2(t-\tau)}\right\}d\tau,$$

where

$$\int_{t_0}^{t} \frac{e^{-\xi(t-\tau)}}{x\sqrt{2\pi\sigma^2(t-\tau)}} \exp\left\{-\frac{[\log(\frac{x}{\rho}) - (a-\frac{\sigma^2}{2})(t-\tau)]^2}{2\sigma^2(t-\tau)}\right\}d\tau$$
$$= \frac{e^{\log(\frac{x}{\rho})\left(a-\frac{\sigma^2}{2}-\sqrt{(a-\frac{\sigma^2}{2})^2+2\sigma^2\xi}\right)}}{2x\sqrt{\mu^2+2\sigma^2\xi}}\left[\mathrm{Erfc}\left(\frac{\log(\frac{x}{\rho}) - (t-t_0)\sqrt{(a-\frac{\sigma^2}{2})^2+2\sigma^2\xi}}{\sqrt{2(t-t_0)\sigma^2}}\right)\right.$$
$$\left. - e^{\frac{2\log(\frac{x}{\rho})\sqrt{(a-\frac{\sigma^2}{2})^2+2\sigma^2\xi}}{\sigma^2}}\mathrm{Erfc}\left(\frac{\log(\frac{x}{\rho}) + (t-t_0)\sqrt{(a-\frac{\sigma^2}{2})^2+2\sigma^2\xi}}{\sqrt{2(t-t_0)\sigma^2}}\right)\right].$$

Moreover, the conditional moments of $X(t)$ can be evaluated from (11) with $\widetilde{m}^{(n)}(t|\rho,t_0)$ given in (14); so it follows:

$$m^{(n)}(t|\rho,t_0) = e^{-\xi(t-t_0)}\widetilde{m}^{(n)}(t|\rho,t_0) + \frac{\xi\rho^n}{n(a-\frac{\sigma^2}{2})+n^2\frac{\sigma^2}{2}-\xi} \times \Big[\exp\Big\{[n\Big(a-\frac{\sigma^2}{2}\Big)+n^2\frac{\sigma^2}{2}-\xi](t-t_0)\Big\}-1\Big]. \qquad (16)$$

Concerning the FPT pdf, recalling that $\widetilde{g}_j(S,\tau|\rho,t_j) = \widetilde{g}(S,\tau|\rho,t_j)$ is defined in (15), from (12) one has:

$$g(S,t|\rho,t_0) = e^{-\xi(t-t_0)}\frac{\log(\frac{S}{\rho})}{\sqrt{2\pi\sigma^2(t-t_0)^3}}\exp\left\{-\frac{[\log(\frac{S}{\rho})-(a-\frac{\sigma^2}{2})(t-t_0)]^2}{2\sigma^2(t-t_0)}\right\} + \xi e^{-\xi(t-t_0)}\big[1-P(\widetilde{T}(0)<I)\big]\int_{t_0}^{t} e^{\xi\tau\left[1-P(\widetilde{T}(0)<I)\right]}\frac{\log(\frac{S}{\rho})}{\sqrt{2\pi\sigma^2(t-\tau)^3}} \times \exp\left\{-\frac{[\log(\frac{S}{\rho})-(a-\frac{\sigma^2}{2})(t-\tau)]^2}{2\sigma^2(t-\tau)}\right\}d\tau,$$

with

$$P(\widetilde{T}(0)<I) = -\frac{1}{2}\xi\left\{L\left[\mathrm{Erfc}\left(\frac{-\log(\frac{S}{\rho})+(a-\frac{\sigma^2}{2})\theta}{\sqrt{2\sigma^2\theta}}\right)\right] + L\left[\mathrm{Erfc}\left(\frac{-\log(\frac{S}{\rho})-(a-\frac{\sigma^2}{2})\theta}{\sqrt{2\sigma^2\theta}}\right)\right]\right\},$$

where L is the Laplace Transform.

In Fig. 1 the mean of $X(t)$ (full line) and the mean of $\widetilde{X}_0(t)$ (dashed line) are plotted for the deterministic jumps (on the left) and for exponential jumps (on the right).

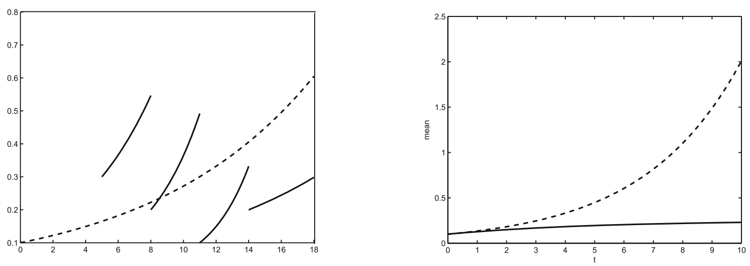

Fig. 1. The mean of $X(t)$ (full line) and the mean of $\widetilde{X}_0(t)$ (dashed line) are plotted for the deterministic jumps (on the left) and for exponential jumps (on the right). For the deterministic case $\rho_k = 0.1, 0.3, 0.2, 0.1, 0.2$, $a_k = 0.1, 0.2, 0.3, 0.4, 0.1$, $\tau_k = 0, 5, 8, 11, 14$ for $k = 0, 1, 2, 3, 4$. For the exponential case the parameters are $\rho = 0.1$, $a = 0.3$ and $\xi = 0.2$. In both cases $\sigma = 1$.

6 Conclusion and Future Developments

Stochastic diffusion processes with jumps occurring at random times have been studied by analyzing the transition pdf and its moments, the FPT density in the presence of constant and exponential distributed jumps. Particular attention has been payed on the lognormal process with jumps.

As future develops one could insert a dead time after a jump representing a delay period after that the process re-starts. This period can be represented by a random variable and the expressions for the transition pdf, the conditional moments and the FPT density can be obtained. Moreover, one can consider other probability distributions for the inter-jump intervals. In general, one can construct other processes with jumps, unknown in literature, on diffusion processes that are of interest in the applications. Finally, a general methodology to infer on parameters could be interesting to fit real data and provide forecasts in application context.

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