On the Linear Essential Spectrum Operator

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Abstract. This paper presents: Let A be a unital C^*-algebra of real rank zero and B be a unital semisimple complex Banach algebra. We characterize linear maps from A onto B that compress different essential spectral sets such as the (left, right) essential spectrum, the semi-Fredholm spectrum, and the Weyl spectrum. Essentially spectrally bounded linear mapping from A onto B are also characterized.

Keywords: Fredholm elements · Essential spectrum
Essential spectral radius

1 Introduction

Linear preserver problems is an active research area in matrix and operator theory. These problems involve linear or additive maps that leave invariant certain relations, or subsets, or functions. Over the past decades much work has been done on linear preserver problems on matrix or operator spaces. Often, the characterization of such linear preservers reveal the algebraic structures, in many cases, they are in fact Jordan homomorphisms; see surveys papers [3] and the references therein.

Throughout, A and B will denote infinite dimensional unital semisimple Banach algebras over the field $\mathbb{C}$ of complex number, unless specified otherwise. The unit is denoted by $\mathbf{1}$. A linear mapping $\varphi : A \to B$ is said to be Jordan homomorphism if $\varphi(a^2) = \varphi(a)^2$ for all $a \in A$, or equivalently

$$\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$$

for all $a, b \in A$. Clearly, every homomorphism and every anti-homomorphism is a Jordan homomorphism. For further properties of Jordan homomorphisms, we refer the reader to [3,9–11]. The map φ is said to be essentially spectrally bounded if there exists a positive constant M such that

$$r_e(\varphi(a)) \leq M r_e(a)$$

for all $a \in A$, where $r_e(.)$ stands for the essential spectral radius. In [6], the authors characterized linear maps on the algebra $\mathcal{L}(\mathcal{H})$ of all bounded

M. Ben Ahmed and A. A. Boudhir (Eds.): SCAMS 2017, LNNS 37, pp. 961–965, 2018.
https://doi.org/10.1007/978-3-319-74500-8_85

linear operators on an infinite dimensional Hilbert space $\mathcal{H}$ that are essentially spectrally bounded, extending some recent results obtained in [5] concerning linear essential spectral radius (essential spectrum) preservers. They proved that a linear surjective up to compact operators map from $\mathcal{L}(\mathcal{H})$ into itself is essentially spectrally bounded if and only if it preserves the ideal of compact operators and the induced linear map on the Calkin algebra is either a continuous epimorphism or a continuous anti-epimorphism multiplied by a nonzero scalar. Recently, in [7], as a local version, the authors studied linear maps on the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$ that compress the local spectrum and the ones that are locally spectrally bounded.

The object of this note is to study essential spectrum compressors and essentially spectrally bounded linear maps between Banach algebras.

2 Essentially Spectrally Bounded Linear Maps

First, let us recall the following useful facts about Fredholm theory in semisimple Banach algebras that will be often used in the sequel.

Let A be a semisimple Banach algebra. The socle of A, $\mathrm{Soc}\,(A)$, is defined to be the sum of all minimal left (or right) ideals of A. The ideal of inessential elements of A is given by

$$I(A) := \bigcap\{P : P \in \Pi_A : \mathrm{Soc}\,(A) \subseteq P\},$$

where Π_A denotes the set of all primitive ideals of A. It is a closed ideal of A. We call $\mathcal{C}(A) := A/I(A)$ the generalized Calkin algebra of A. It should be noted that a semisimple Banach algebra is finite dimensional if and only if it coincides with its socle; see for instance [2, Theorem 5.4.2]. Since our algebras are always supposed to be infinite dimensional, the generalized Calkin algebra introduced above is not trivial.

An element $a \in A$ is called left semi-Fredholm (resp. right semi-Fredholm) if it is left (resp. right) invertible modulo $\mathrm{Soc}\,(A)$, and is called Fredholm if it is invertible modulo $\mathrm{Soc}\,(A)$. The element a is said to be Atkinson if it is left or right semi-Fredholm. It is known that left (resp. right) invertible modulo $\mathrm{Soc}\,(A)$ is equivalent to left (resp. right) invertible modulo $I(A)$.

For every $a \in A$ we set

$$\sigma_e(a) := \{\lambda \in \mathbb{C} : a - \lambda\, isnotFredholm\},$$
$$\sigma_{le}(a) := \{\lambda \in \mathbb{C} : a - \lambda\, isnotleftsemi - Fredholm\},$$
$$\sigma_{re}(a) := \{\lambda \in \mathbb{C} : a - \lambda\, isnotrightsemi - Fredholm\},$$

and

$$\sigma_{SF}(a) := \{\lambda \in \mathbb{C} : a - \lambda\, isnotAtkinson\}.$$

These are called respectively the essential spectrum, the left essential spectrum, the right essential spectrum, and the semi-Fredholm spectrum of a.

For an element $a \in A$, the essential spectral radius is defined by

$$r_e(a) := \max\{|\lambda| : \lambda \in \sigma_e(a)\}.$$

It coincides with the limit of the convergent sequence $(\|a^n\|_e^{\frac{1}{n}})_{n\geq 1}$, where $\|a\|_e := \|\pi(a)\|$ is the essential norm of a and π is the canonical projection from A onto $\mathcal{C}(A)$. We refer the reader to [12,13] and the monographs [1,4] for basic facts concerning Atkinson and Fredholm theory in Banach algebras.

A linear map $\varphi : A \to B$ is said to be surjective up to inessential elements if $B = \varphi(A) + I(B)$. It is called spectrally bounded if there exists a positive constant M such that $r(\varphi(a)) \leq Mr(a))$ for all $a \in A$, where $r(.)$ denotes the classical spectral radius. The following, quoted in [6], is needed in what follows.

Let A be a unital purely infinite C*-algebra with real rank zero and let B be a unital semi-prime Banach algebra. If $\varphi : A \to B$ be a surjective spectrally bounded linear map, then there exist a central invertible element c, viz., $\varphi(\mathbf{1})$, and a Jordan epimorphism $J : A \to B$ such that $\varphi(x) = cJ(x)$ for all $x \in A$.

Proof. See [6, Lemma 1] □

The problem of characterizing essentially spectrally bounded it seems to be difficult even when A and B are supposed to be C^*-algebras of real rank zero. Recall that a C^*-algebra A is of real rank zero if the set of all finite real linear combinations of orthogonal projections is dense in the set of all self adjoint elements of A; see [8]. However, We give a positive answer to this problem when A is a purely infinite C^*-algebra of real rank zero. A C^*-algebras A is purely infinite if it has no characters and if, for every pair of positive elements a, b in $\mathcal{A}$ with $a \in \overline{\mathcal{A}b\mathcal{A}}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $a = \lim_n x_n^* b x_n$; see [14].

The main result of this section is the following. It characterizes essentially spectrally bounded linear maps.

Theorem 2.1. *Let φ be a linear surjective up to compact operators map from a purely infinite C^*-algebras with real rank zero A into semisimple a Banach algebra B. If φ is essentially spectrally bounded, then*

$$\varphi(I(A)) \subseteq I(B)$$

and the induced mapping $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ defined by

$$\widehat{\varphi}(\pi(a)) := \pi(\varphi(a)),\ (a \in A),$$

is a continuous Jordan epimorphism multiplied by an invertible central element of $\mathcal{C}(B)$.

Proof. Assume that there is a positive constant M such that $r_e(\varphi(x)) \leq Mr_e(x)$ for all $x \in A$. We first show that φ maps $I(A)$ into $I(B)$. So pick an inessential element $a \in I(A)$, and let us prove that $\varphi(a)$ is inessential as well. Let y be an arbitrary element in B and note that, since φ is surjective up to inessential elements, there exist $x \in A$ and $y_0 \in I(B)$ such that $y = \varphi(x) + y_0$. For every $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} r(\lambda\pi(\varphi(a)) + \pi(y)) &= r(\pi(\lambda\varphi(a) + y)) = r_e(\lambda\varphi(a) + y) \\ &= r_e(\varphi(\lambda a + x) + y_0) \\ &= r_e(\varphi(\lambda a + x)) \\ &\leq Mr_e(\lambda a + x) = Mr_e(x) = Mr(\pi(x)). \end{aligned}$$

Since $\lambda \mapsto r(\lambda\pi(\varphi(a)) + \pi(y))$ is a subharmonic function on $\mathbb{C}$, Liouville's Theorem implies that

$$r(\pi(\varphi(a)) + \pi(y)) = r(\pi(y)).$$

As y is arbitrary in B, it follows from semi-simplicity of $\mathcal{C}(B)$ and the Zemánek's characterization of the radical [2, Theorem 5.3.1] that $\pi(\varphi(a)) = 0$ and $\varphi(a) \in I(B)$.

Therefore $\varphi(I(A)) \subseteq I(B)$, and φ induces a linear map $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ defined by $\widehat{\varphi}(\pi(x)) := \pi(\varphi(x))$ for all $x \in A$. The map $\widehat{\varphi}$ is obviously surjective and spectrally bounded. Note that, by [14, Proposition 4.3] the quotient of a C^*-algebra of real rank zero by a closed ideal is a C^*-algebra of real rank zero and the quotient of a purely infinite C^*-algebra by a closed ideal is a purely infinite C^*-algebra. Thus, the desired conclusion follows by applying Lemma 1. □

For an infinite-dimensional complex Hilbert space $\mathcal{H}$, $\mathrm{Soc}\,(\mathcal{L}(\mathcal{H})) = \mathcal{F}(\mathcal{H})$ is the ideal of all finite rank operators on $\mathcal{H}$, $I(\mathcal{L}(\mathcal{H})) = \mathcal{K}(\mathcal{H})$ is the closed ideal of all compact operators on $\mathcal{H}$, and the generalized Calkin algebra $\mathcal{C}(\mathcal{L}(\mathcal{H}))$ coincides with the usual Calkin algebra $\mathcal{C}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and it is prime. Thus, a Jordan homomorphism $\widehat{\varphi} : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ is either a homomorphism or an anti-homomorphism by [9,10].

More generally, if A is a C^*-algebra, then $\mathrm{Soc}\,(A)$ is the set of all finite rank element in A, and $I(A) = \overline{\mathrm{Soc}\,(A)} = \mathcal{K}(A)$, the set of all compact elements in A; see [4]. Recall that an element x of A is said to be finite (resp. compact) in A if the wedge operator $x \wedge x : A \to A$, given by $x \wedge x(a) = xax$, is a finite rank (resp. compact) operator on A. Note that even if the C^*-algebra A is prime, the generalized Calkin algebra $\mathcal{C}(A) = A/\mathcal{K}(A)$ is not necessary prime. For example, consider A the C^*-algebra generated by $\mathcal{K}(\mathcal{H})$ and two orthogonal infinite dimensional projections on a Hilbert space $\mathcal{H}$, and note that $\mathcal{C}(A) \cong \mathbb{C}^2$ is not prime. However, when A is factor, the ideal $\mathcal{K}(A)$ is the largest ideal of type I, and $\mathcal{C}(A)$ is a prime C^*-algebra.

Let φ be a surjective up to inessential elements linear map from a purely infinite C^*-algebra A with rank real zero into a factor B. Then the following assertions are equivalent.

- (i) φ is essentially spectrally bounded.
- (ii) $\varphi(I(A)) \subseteq I(B)$ and the induced mapping $\widehat{\varphi} : \mathcal{C}(A) \to \mathcal{C}(B)$ is either a continuous epimorphism or a continuous anti-epimorphism up to a nonzero complex scalar.

Proof. We only need to proof that (i) ⇒ (ii) as (ii) ⇒ (i) follows easily. If φ is essentially spectrally bounded then, by Theorem 2.1 and the fact that the center of $\mathcal{C}(B)$ is trivial, we infer that $\widehat{\varphi}$ is a continuous Jordan epimorphism multiplied by a nonzero complex number c. As the algebra $\mathcal{C}(B)$ is prime, then by [11] the map $\widehat{\varphi}$ is, in fact, either an epimorphism or an anti-epimorphism multiplied by c. □

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