Chapter 6
Black Holes and Nilpotent Orbits

Deep into that darkness peering, long I stood there,
wondering, fearing, doubting,
dreaming dreams no mortal
ever dared to dream before.

Edgar Allan Poe

6.1 Historical Introduction

When on September 14th 2015 the gravitational wave signal emitted 1.5 billion year ago by two coalescing black stars was detected at LIGO I and LIGO II, we not only obtained a new spectacular confirmation of General Relativity but we actually saw the dynamical process of formation of the most intriguing objects populating the Universe, namely black holes (Fig. 6.1).

Black Holes are on one side physical objects capable of interacting with the emission of enormous quantities of energy, on the other side they are just pure geometries. Indeed a classical black-hole is nothing else but a solution of Einstein equations which are just geometrical statements on the curvature tensor.

6.1.1 Black Holes in Supergravity and Superstrings

A new season of research in Black Hole theory started in the middle nineties of the XXth century with the contributions of Sergio Ferrara, Renata Kallosh, Andrew Strominger and Cumrun Vafa, that are described in the following short summary:

1. In 1995 R. Kallosh, S. Ferrara and A. Strominger considered black holes in the context of $\mathcal{N} = 2$ supergravity and introduced the notion of attractors [1, 2].
2. In 1996 S. Ferrara (see Fig. 4.2) and R. Kallosh (see Fig. 6.2) formalized the attractor mechanism for supergravity black holes [1, 2].

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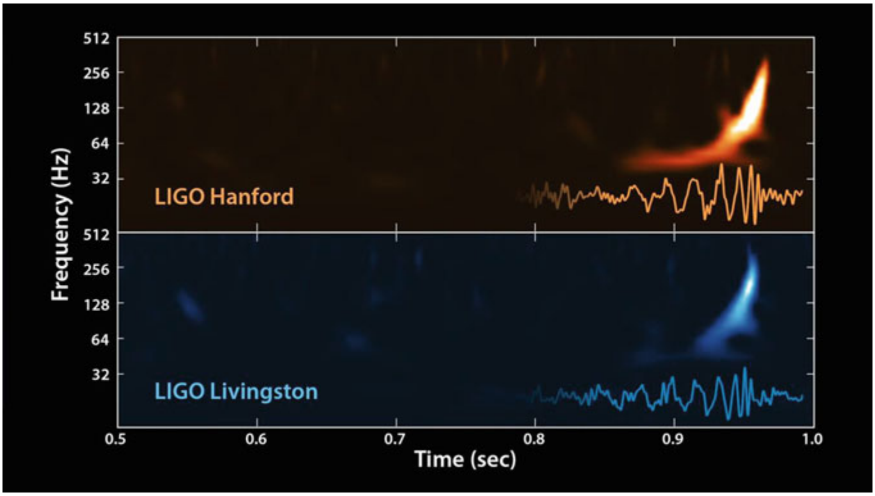

Fig. 6.1 The gravitational wave signal emitted in the coalescence of two black holes which occurred 1.5 billion of years ago was simultaneously detected September 14th 2015 by the two interferometers LIGO I and LIGO II

3. In 1996 A. Strominger (see Fig. 4.7) and C. Vafa (see Fig. 6.3) showed that an extremal BPS black hole in d $= 5$ has a horizon area that exactly counts the number of string microstates it corresponds to [3].[1]
4. In the years 1997–2000 the horizon area of BPS supergravity black holes was interpreted in terms of a symplectic invariant constructed with the black hole electromagnetic charges (for a review containing also an extensive bibliography see [11]).
5. In the years 2006–2009 new insights extended the attractor mechanism to non BPS black-holes [12–25].
6. Since 2010 new exact integration techniques for Sugra Black Holes were found by A. Sorin, P. Fré, M. Trigiante and their younger collaborators [26–33].

6.1.2 *Black Holes in This Chapter*

The intriguing relation between Geometry and Physics arises at several levels, the most profound and challenging being provided by the identification of the *horizon area* with the *statistical entropy* of the mysterious dynamical system which is encoded in a *classical black solution*.

[1] There followed a vast literature some items of which are are quoted in [4–10].

Fig. 6.2 Renata Kallosh (on the left) born in Moscow in 1943 completed her Bachelor's from Moscow State University in 1966 and obtained her Ph.D. from Lebedev Physical Institute, Moscow in 1968. She then held a position, as professor, at the same institute, before moving to CERN for a year in 1989. Kallosh joined Stanford University in 1990 and continues to work there. She is married with the famous cosmologist Andrei Linde. Renata Kallosh is renowned for her pioneering contributions with Ferrara to the attractor mechanism in supergravity black holes, for her studies in supergravity cosmology and for her early work with A. Van Proeyen on the AdS/CFT correspondence. Indeed Kallosh and Van Proeyen were the first to propose the interpretation of the anti de Sitter group as the conformal group on a brane boundary. Anna Ceresole (on the right), born 1961 in Torino, graduated from Torino University in 1984 with a thesis on Kaluza Klein supergravity written under the supervision of Hermann Nicolai and the author of this book. In 1989 she obtained her Ph.D. from Stony Brook University under the supervision of Peter van Nieuwenhuizen. Post doctoral fellow at Caltech for two years she was Assistant Professor at the Politecnico di Torino for several years. Then she became Senior Research Scientist of INFN and joined the Torino University String Group. Anna Ceresole has given many important contributions to the development of supergravity, in particular in relation with special Kähler Geometry and black hole charges, duality transformations, gaugings and inflaton potentials. She has worked both with younger students and post-doc and, in different combinations, with all the main actors in the development of supergravity theory

We are not going to touch upon the physics of black holes and on the exciting question of their interpretation in terms of *microstates*, yet we cannot avoid discussing their several nested geometrical aspects, glimpses of which were already provided in Chap. 5.

We emphasized there that in the context of supergravity a black hole solution of Einstein equations comes equipped with other associated geometrical data, namely those encoded in a set of electromagnetic fields that are connections on suitable bundles and those encoded in scalar fields that describe a map from 4-dimensional space-time $\mathscr{M}_4$ to *special manifolds* $\mathscr{S}\mathscr{K}_n$. We also stressed the remarkable picture of a black-hole solution as a map from a three-dimensional Euclidean manifold $\mathscr{M}_3$ to a Lorentzian pseudo-quaternionic manifold $\mathscr{Q}_r$ lying in the image of the $c^\star$-map.

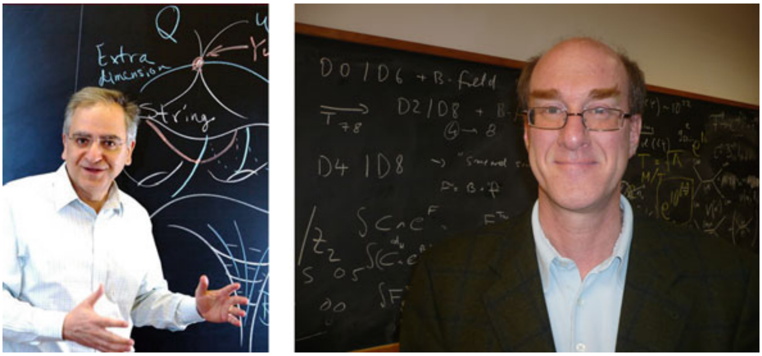

Fig. 6.3 Cumrun Vafa (on the left) was born in Tehran, Iran in 1960. He graduated from Alborz High School and went to the US in 1977. He got his undergraduate degree from the Massachusetts Institute of Technology with a double major in physics and mathematics. He received his Ph.D. from Princeton University in 1985 under the supervision of Edward Witten. He then became a junior fellow at Harvard, where he later got a junior faculty position. In 1989 he was offered a senior faculty position, and he has been there ever since. Currently, he is the Donner Professor of Science at Harvard University. Vafa's most relevant achievement is, together with Strominger, the first example of interpretation of the Bekenstein Hawking black hole entropy in terms of superstring microstates. He has also given pioneering contributions to topological strings, F-theory and to the general vision named *geometric engineering of quantum field theories*, which is a programme aimed at decoding quantum field theories in terms of algebraic geometry constructions. Dieter Luest (on the right) born 1956 in Chicago, graduated from the Ludvig Maximillian University in Muenchen in 1985. He was postdoctoral fellow in Caltech, Pasadena, in the Max Planck Institute in Muenchen and at CERN in Geneva. From 1993 to 2004 he was full professor of Quantum Field Theory at the von Humboldt University in Berlin. Since 2004 he made return to Muenchen where he is both full professor at the Ludwig Maximilan University and Research Director at the Max Planck Institute. Dieter Luest has given very important contributions in a large variety of topics connected with String Theory and Supergravity, in particular in relation with Black Hole solutions, D-brane engineering, Calabi Yau compactifications, double geometries, flux compactifications and string cosmology

This last viewpoint corresponds to the σ-model approach to black-hole solutions and it was developed in the last two decades.

If the special manifold $\mathcal{SK}_n = \frac{\mathrm{U_{D=4}}}{\mathrm{H_{D=4}}}$ is a symmetric coset manifold, then also the pseudo-quaternionic manifold $\mathcal{Q}_r = \frac{\mathrm{U_{D=3}}}{\mathrm{H_{D=3}}}$ is such and the classification of possible extremal black-hole solutions is turned into an algebraic problem that is the contemporary frontier of research in Lie algebra theory: *the classification of nilpotent orbits*.

In this chapter we analyze in detail the new very rich *geometric lore* which emerges from the issue of black–hole constructions within the σ-model approach. Here all the issues discussed in previous chapters enter the game in an essential way:

1. Special Kähler Geometry,
2. Lie Algebra invariants,
3. $c^{\star}$ map,
4. Tits Satake projection and its universality classes,

5. Weyl Group and its extensions,
6. Classification of nilpotent orbits.

In view of the deep relation between quantum physics and geometry encapsulated into black-holes it is to be expected that all the intriguing geometrical relations listed above are the tip of an iceberg of theoretical knowledge yet to be uncovered.

Hence let us resume the σ-model approach to black-holes.

6.2 The σ-Model Approach to Black-Hole Resumed

We start from Eq. (5.2.21) and from the golden splitting (1.7.12) which we rewrite as follows:

$$\mathrm{adj}(\mathbb{U}_{D=3}) = \mathrm{adj}(\mathbb{U}_{D=4}) \oplus \mathrm{adj}(\mathfrak{sl}(2,\mathbb{R})_E) \oplus W_{(2,\mathbf{W})} \tag{6.2.1}$$

where $\mathbf{W}$ is the **symplectic** representation of $\mathbb{U}_{D=4}$ to which the electric and magnetic field strengths are assigned.

Next we consider a gravity coupled three-dimensional Euclidean σ-model, whose fields

$$\Phi^A(x) \equiv \{U(x), a(x), \phi(x), Z(x)\}$$

describe mappings:

$$\Phi : \quad \mathscr{M}_3 \to \mathscr{Q} \tag{6.2.2}$$

from a three-dimensional manifold $\mathscr{M}_3$, whose metric we denote by $\gamma_{ij}(x)$, to the target space $\mathscr{Q}$. The action of this σ-model is the following:

$$\mathscr{A}^{[3]} = \int \sqrt{\det\gamma}\, \mathfrak{R}[\gamma]\, d^3x + \int \sqrt{\det\gamma}\, \mathscr{L}^{(3)}\, d^3x \tag{6.2.3}$$

$$\begin{aligned}\mathscr{L}^{(3)} = \Big(&\partial_i U\, \partial_j U + h_{rs}\, \partial_i\phi^r\, \partial_j\phi^s \\ &+ e^{-2U}\left(\partial_i a + \mathbf{Z}^T\mathbb{C}\partial_i\mathbf{Z}\right)\left(\partial_j a + \mathbf{Z}^T\mathbb{C}\partial_j\mathbf{Z}\right) + 2\, e^{-U}\, \partial_i\mathbf{Z}^T\, \mathscr{M}_4\, \partial_j\mathbf{Z}\Big)\, \gamma^{ij}\end{aligned} \tag{6.2.4}$$

where $\mathfrak{R}[\gamma]$ denotes the scalar curvature of the metric γ_{ij}.

The field equations of the σ-model are obtained by varying the action both in the metric γ_{ij} and in the fields $\Phi^A(x)$. The Einstein equation reads as usual:

$$\mathfrak{R}_{ij} - \tfrac{1}{2}\gamma_{ij}\, \mathfrak{R} = \mathfrak{T}_{ij} \tag{6.2.5}$$

where:

$$\mathfrak{T}_{ij} = \frac{\delta\mathscr{L}^{(3)}}{\delta\gamma^{ij}} - \gamma_{ij}\, \mathscr{L}^{(3)} \tag{6.2.6}$$

is the stress energy tensor, while the matter field equations assume the standard form:

$$\frac{1}{\sqrt{\det\gamma}}\,\gamma^{ij}\,\partial_i\left[\sqrt{\det\gamma}\,\frac{\delta\mathscr{L}^{(3)}}{\delta\,\partial^j\Phi^A}\right]-\frac{\delta\mathscr{L}^{(3)}}{\delta\Phi^A}\;=\;0 \tag{6.2.7}$$

As it is well known, in $D = 3$ there is no propagating graviton and the Riemann tensor is completely determined by the Ricci tensor, namely, via Einstein equations, by the stress-energy tensor of the matter fields.[2]

Extremal solutions of the σ-model are those for which the three-dimensional metric can be consistently chosen flat:

$$\gamma_{ij}\;=\;\delta_{ij} \tag{6.2.8}$$

corresponding to a vanishing stress-energy tensor:

$$\begin{aligned}&\partial_i U\,\partial_j U+\,h_{rs}\,\partial_i\phi^r\,\partial_j\phi^s+e^{-2U}\,\left(\partial_i a+\mathbf{Z}^T\mathbb{C}\partial_i\mathbf{Z}\right)\left(\partial_j a+\mathbf{Z}^T\mathbb{C}\partial_j\mathbf{Z}\right)\\&+2\,e^{-U}\,\partial_i\mathbf{Z}^T\,\mathscr{M}_4\,\partial_j\mathbf{Z}\;=\;0\end{aligned} \tag{6.2.9}$$

We will see in the sequel how the nilpotent orbits of the group $\mathrm{H}^\star$ in the $\mathbb{K}^\star$ representation can be systematically associated with general extremal solutions of the field equations.

6.2.1 Oxidation Rules for Extremal Multicenter Black Holes

Let us now describe the oxidation rules, namely the procedure by means of which to every configuration of the three-dimensional fields $\Phi(x) = \{U(x), a(x), \phi(x), Z(x)\}$, satisfying the field equations (6.2.7) and also the extremality condition (6.2.9), we can associate a well defined configuration of the four-dimensional fields satisfying the field equations of supergravity that follow from the lagrangian (5.2.3). We might write such oxidation rules for general solutions of the σ-model, also non extremal, yet given our present goal we confine ourselves to spell out such rule in the extremal case, which is somewhat simpler since it avoids the extra complications related with the three-dimensional metric γ_{ij}.

In order to write the $D = 4$ fields, the first necessary item we have to determine is the Kaluza–Klein vector field $\mathbf{A}^{[KK]} = A_i^{[KK]}\,dx^i$. This latter is worked out through the following dualization procedure:

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[2] *Clarification for mathematicians*: General Relativity in $D = 3 = 1 \oplus 2$ dimensions is a rather empty field theory. Einstein equations do not describe the propagation of any particle since there are no solutions of the wave-type and the only degree of freedom is the analogue of the Newton potential. Mathematically this follows from the fact that the Riemann tensor is fully determined by the Ricci tensor and the latter is identified by Einstein equations with the stress-energy tensor of matter fields.

$$\begin{aligned}
\mathbf{F}^{[KK]} &= d\mathbf{A}^{[KK]} \\
\mathbf{F}^{[KK]} &= -\varepsilon_{ijk}\, dx^i \wedge dx^j \left[\exp[-2U] \left(\partial^k a + Z\,\mathbb{C}\,\partial^k Z\right)\right]
\end{aligned} \tag{6.2.10}$$

Given the Kaluza–Klein vector we can write the four-dimensional metric which is the following:

$$ds^2 = -\exp[U] \left(dt + \mathbf{A}^{[KK]}\right)^2 + \exp[-U]\, dx^i \otimes dx^j\, \delta_{ij} \tag{6.2.11}$$

The vielbein description of the same metric is immediate. We just write:

$$\begin{aligned}
ds^2 &= -E^0 \otimes E^0 + E^i \otimes E^i \\
E^0 &= \exp[\tfrac{U}{2}] \left(dt + \mathbf{A}^{[KK]}\right) \\
E^i &= \exp[-\tfrac{U}{2}]\, dx^i
\end{aligned} \tag{6.2.12}$$

Next we can present the form of the electromagnetic field strengths:

$$\begin{aligned}
\mathbf{F}^{\Lambda} &= \mathbb{C}^{\Lambda M} \partial_i Z_M\, dx^i \wedge (dt + \mathbf{A}^{[KK]}) \\
&\quad + \varepsilon_{ijk} dx^i \wedge dx^j \left[\exp[-U] \left(\mathrm{Im}\mathcal{N}^{-1}\right)^{\Lambda\Sigma} \left(\partial^k Z_\Sigma + \mathrm{Re}\mathcal{N}_{\Sigma\Gamma} \partial^k Z^\Gamma\right)\right]
\end{aligned} \tag{6.2.13}$$

Next we define the electromagnetic charges and the Taub-NUT charges for multicenter solutions. Considering the metric (6.2.11) the black hole centers are defined by the zeros of the warp-factor $\exp[U(\mathbf{x})]$. In a composite m-black hole solution there are m three-vectors $\mathbf{r}_\alpha$ ($\alpha = 1, \ldots, m$), such that:

$$\lim_{\mathbf{x} \to \mathbf{r}_\alpha} \exp[U(\mathbf{x})] = 0 \tag{6.2.14}$$

Each of these zeros defines a non trivial homology two-cycle $\mathbb{S}^2_\alpha$ of the 4-dimensional space-time which surrounds the singularity $\mathbf{r}_\alpha$. The electromagnetic charges of the individual holes are obtained by integrating the field strengths and their duals on such homology cycles.

$$\begin{pmatrix} p^\Lambda \\ q_\Sigma \end{pmatrix}_\alpha = \frac{1}{4\pi\sqrt{2}} \begin{pmatrix} \int_{\mathbb{S}^2_\alpha} \mathbf{F}^\Lambda \\ \int_{\mathbb{S}^2_\alpha} \mathbf{G}_\Sigma \end{pmatrix} \equiv \frac{1}{4\pi} \int_{\mathbb{S}^2_\alpha} j^{EM} \tag{6.2.15}$$

Utilizing the form of the field strengths we obtain the explicit formula:

$$\begin{aligned}
\mathcal{Q}_\alpha \equiv \begin{pmatrix} p^\Lambda \\ q_\Sigma \end{pmatrix}_\alpha &= \frac{1}{4\pi\sqrt{2}} \int_{\mathbb{S}^2_\alpha} \varepsilon_{ijk} dx^i \wedge dx^j \left[\exp[-U]\, \mathcal{M}_4\, \partial^k Z \right. \\
&\quad \left. + \exp[-2U] \left(\partial^k a + Z\,\mathbb{C}\,\partial^k Z\right) \mathbb{C}\, Z\right]
\end{aligned} \tag{6.2.16}$$

which provides m-sets of electromagnetic charges associated with the solution. Similarly we have m Taub-NUT charges defined by:

$$\mathbf{n}_\alpha = -\frac{1}{4\pi}\int_{\mathbb{S}^2_\alpha} \varepsilon_{ijk} dx^i \wedge dx^j \exp[-2U]\left(\partial^k a + Z\,\mathbb{C}\,\partial^k Z\right) \equiv \frac{1}{4\pi}\int_{\mathbb{S}^2_\alpha} j^{TN} \tag{6.2.17}$$

6.2.1.1 Reduction to the Spherical Case

The spherical symmetric one-center solutions are retrieved from the general case by assuming that all the three-dimensional fields depend only on one radial coordinate:

$$\tau = -\frac{1}{r} \quad ; \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{6.2.18}$$

On functions only of τ we have the identity:

$$\partial_i f(\tau) = -x^i\,\tau^3 \frac{d}{d\tau} f(\tau) \tag{6.2.19}$$

and introducing polar coordinates:

$$\begin{aligned} x_1 &= \frac{1}{\tau}\cos\theta \\ x_2 &= \frac{1}{\tau}\sin\theta\,\sin\varphi \\ x_3 &= \frac{1}{\tau}\sin\theta\,\cos\varphi \end{aligned} \tag{6.2.20}$$

we obtain:

$$\tau^3 \varepsilon_{ijk}\, x^i dx^j \wedge dx^k = -2\sin\theta\, d\theta \wedge d\varphi \tag{6.2.21}$$

By using these identities and restricting one's attention to the extremal case, the action of the σ-model (6.2.3) reduces to:

$$\begin{aligned} \mathscr{A} &= \int d\tau\,\mathscr{L} \\ \mathscr{L} &= \dot{U}^2 + h_{rs}\,\dot{\varphi}^r\,\dot{\varphi}^s + e^{-2U}\,(\dot{a} + \mathbf{Z}^T\mathbb{C}\dot{\mathbf{Z}})^2 + 2\,e^{-U}\,\dot{\mathbf{Z}}^T\,\mathscr{M}_4\,\dot{\mathbf{Z}} \end{aligned} \tag{6.2.22}$$

where the dot denotes derivatives with respect to the τ variable. The σ-model field equations take the standard form of the Euler Lagrangian equations:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}\mathscr{L}}{\mathrm{d}\dot{\Phi}} = \frac{\mathrm{d}\mathscr{L}}{\mathrm{d}\Phi} \tag{6.2.23}$$

and the extremality conditions (6.2.9) reduces to:

$$\mathscr{L} = \dot{U}^2 + h_{rs}\,\dot{\varphi}^r\,\dot{\varphi}^s + e^{-2U}\,(\dot{a} + \mathbf{Z}^T\mathbb{C}\dot{\mathbf{Z}})^2 + 2\,e^{-U}\,\dot{\mathbf{Z}}^T\,\mathscr{M}_4\,\dot{\mathbf{Z}} = 0 \tag{6.2.24}$$

It appears from this that spherical extremal black holes are in one-to-one correspondence with light-like geodesics of the manifold $\mathscr{Q}$.

The Reduced Oxidation Rules

In the spherical case the above discussed oxidation rules reduce as follows. For the metric we have

$$ds^2_{(4)} = -\,e^{U(\tau)}\,(dt + 2\,\mathbf{n}\,\cos\theta\,d\varphi)^2 + e^{-U(\tau)}\left[\frac{1}{\tau^4}\,d\tau^2 + \frac{1}{\tau^2}\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \tag{6.2.25}$$

where $\mathbf{n}$ denotes the Taub-NUT charge obtained from the form of the Kaluza–Klein field strength:

$$\begin{aligned}\mathbf{F}^{KK} &= -2\,\mathbf{n}\,\sin\theta\,d\theta\,\wedge\,d\varphi\\ \mathbf{n} &= \left(\dot{a} + Z\,\mathbb{C}\,\dot{Z}\right)\end{aligned} \tag{6.2.26}$$

The electromagnetic field-strengths are instead the following ones:

$$F^\Lambda = 2\,p^\Lambda\,\sin\theta\,d\theta\wedge d\varphi + \dot{Z}_\Lambda d\tau\wedge(dt + 2\mathbf{n}\,\cos\theta\,d\varphi) \tag{6.2.27}$$

where the magnetic charges p^Λ are extracted from the reduction of the general formula (6.2.16), namely:

$$\mathscr{Q}^M = \begin{pmatrix} p^\Lambda \\ q_\Sigma \end{pmatrix} = \sqrt{2}\,\left[e^{-U}\,\mathscr{M}_4\,\dot{Z} - \mathbf{n}\,\mathbb{C}\,Z\right]^M \tag{6.2.28}$$

6.3 The $\mathfrak{g}_{2(2)}$ Lie Algebra and the S^3 Model

In Sect. 1.6 we discussed the structure of the smallest exceptional Lie algebra $\mathfrak{g}_2$ and we anticipated that it plays an important role in relation with the simplest example of special Kähler geometry and of its quaternionic images under the c and the $c^\star$ maps. Indeed the simplest example of special Kähler geometry occurs when we have only one complex scalar coordinate z which parameterizes the complex lower half-plane endowed with the standard Poincaré metric. In other words[3]:

$$g_{z\bar{z}} dz\, d\bar{z} \;=\; \frac{3}{4}\,\frac{1}{(\mathrm{Im} z)^2}\, dz\, d\bar{z} \tag{6.3.1}$$

From the point of view of geometry the lower half-plane is the symmetric coset manifold $\frac{\mathrm{SL(2,R)}}{\mathrm{SO(2)}} \sim \frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}}$.

According to the presented theory and to Table 5.2 the c-map and $c^\star$-map images of this special Kähler manifold are:

$$\begin{aligned} c\left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right] &= \frac{\mathrm{G}_{2(2)}}{\mathrm{SU}(2)\times\mathrm{SU}(2)} \\ c^\star\left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right] &= \frac{\mathrm{G}_{2(2)}}{\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)} \end{aligned} \tag{6.3.2}$$

and the architecture of the (pseudo)-quaternionic manifold is algebraically governed by the golden splitting (1.7.21) and analytically determined by the explicit form of the $\mathcal{N}$-matrix of special geometry appearing in Eqs. (5.2.17) and (5.2.18).

In our discussion of supergravity black-holes from the point of view of the $D = 3$ σ-model and of nilpotent orbits, the master model we will constantly utilize is the simplest one based on the above mentioned one dimensional special Kähler manifold traditionally dubbed the S^3 model. Hence we are interested in the explicit derivation of its special geometry items.

The manifold $\frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}}$ admits a standard solvable parametrization constructed as it follows. Let:

$$L_0 \;=\; \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad L_+ \;=\; \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad ; \quad L_- \;=\; \tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.3.3}$$

be the standard three generators of the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra satisfying the commutation relations $\left[L_0, L_\pm\right] = \pm L_\pm$ and $\left[L_+, L_-\right] = 2L_0$. The coset manifold $\frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}}$ is metrically equivalent with the solvable group manifold generated by L_0 and L_+. Correspondingly we can introduce the coset representative:

$$\mathbb{L}_4(\phi, y) \;=\; \exp[y\, L_1]\, \exp[\varphi\, L_0] \;=\; \begin{pmatrix} e^{\varphi/2} & e^{-\varphi/2} y \\ 0 & e^{-\varphi/2} \end{pmatrix} \tag{6.3.4}$$

Generic group elements of $\mathrm{SL}(2,\mathbb{R})$ are just 2×2 real matrices with determinant one:

$$\mathrm{SL}(2,\mathbb{R}) \;\ni\; \mathfrak{A} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ; \quad ad - bc = 1 \tag{6.3.5}$$

[3]The special overall normalization of the Poincaré metric is chosen in order to match the general definitions of special geometry applied to the present case.

and their action on the lower half-plane is defined by usual fractional linear transformations:

$$\mathfrak{A} \quad : \quad z \to \frac{a\,z\, + b}{c\,z\, + d} \tag{6.3.6}$$

The correspondence between the lower complex half-plane $\mathbb{C}_{-}$ and the solvable -parameterized coset (6.3.4) is easily established observing that the entire set of $\text{Im}z \, < \, 0$ complex numbers is just the orbit of the number i under the action of $\mathbb{L}(\phi, y)$:

$$\mathbb{L}_4(\phi, y) \quad : \quad \text{i} \to \frac{-e^{\varphi/2}\,\text{i} + e^{-\varphi/2}\,y}{e^{-\varphi/2}} = y - \text{i}e^{\varphi} \tag{6.3.7}$$

This simple argument shows that we can rewrite the coset representative $\mathbb{L}(\phi, y)$ in terms of the complex scalar field z as follows:

$$\mathbb{L}_4(z) = \begin{pmatrix} \sqrt{|\text{Im}z\,|} & \frac{\text{Re}z}{\sqrt{|\text{Im}z\,|}} \\ 0 & \frac{1}{\sqrt{|\text{Im}z\,|}} \end{pmatrix} \tag{6.3.8}$$

The issue of special Kähler geometry becomes clear at this stage. If we did not consider the symplectic vector bundle, the choice of the coset metric would be sufficient and nothing more would have to be said. The point is that we still have to define the $\mathscr{N}$–matrix associated with the flat symplectic bundle which enters the definition of special Kähler geometry. On the same base manifold $\text{SL}(2, \mathbb{R})/\text{SO}(2)$ we have different special structures which lead to different physical models and to different duality groups $\text{U}_{\text{D}=3}$ upon reduction to $D = 3$. The special structure is determined by the choice of the symplectic embedding $\text{SL}(2, \mathbb{R}) \to \text{Sp}(4, \mathbb{R})$. The symplectic embedding that defines our master model and which eventually leads to the duality group $\text{U}_{\text{D}=3} = \text{G}_{2(2)}$ is cubic and it was already described in Sect. 1.7.1.1. It is explicitly given by Eq. (1.7.28).

The 2×2 blocks A, B, C, D of the 4×4 symplectic matrix $\Lambda\,(\mathfrak{A})$ are easily readable from Eq. (1.7.28) so that, assuming that the matrix $\mathfrak{A}(z)$ is the coset representative of the manifold $\text{SU}(1, 1)/\text{U}(1)$, we can apply the Gaillard-Zumino formula (5.2.16) and obtain the explicit form of the kinetic matrix $\mathscr{N}_{\Lambda\Sigma}$:

$$\overline{\mathscr{N}} = \begin{pmatrix} -\frac{2ac-ibc+iad+2bd}{a^2+b^2} & -\frac{\sqrt{3}(c+id)(ac+bd)}{(a-ib)(a+ib)^2} \\ -\frac{\sqrt{3}(c+id)(ac+bd)}{(a-ib)(a+ib)^2} & -\frac{(c+id)^2(2ac+ibc-iad+2bd)}{(a-ib)(a+ib)^3} \end{pmatrix} \tag{6.3.9}$$

Inserting the specific values of the entries a, b, c, d corresponding to the coset representative (6.3.8), we get the explicit dependence of the $\mathscr{N}$-matrix on the complex coordinate z:

$$\overline{\mathscr{N}}_{\Lambda\Sigma}(z) = \begin{pmatrix} -\frac{3z+\bar{z}}{2z\bar{z}} & -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} \\ -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} & -\frac{z+3\bar{z}}{2z\bar{z}^3} \end{pmatrix} \tag{6.3.10}$$

This might conclude the determination of the quaternionic or pseudo-quaternionic metric of our master example, yet we have not yet seen the special Kähler structure induced by the cubic embedding. Let us present it.

The key point is the construction of the required holomorphic symplectic section $\Omega(z)$. As usual the transformation properties of a geometrical object indicate the way to build it explicitly. For consistency we should have that:

$$\Omega\left(\frac{a\,z + b}{c\,z + d}\right) = f(z)\,\Lambda(\mathfrak{A})\,\Omega(z) \tag{6.3.11}$$

where $\Lambda(\mathfrak{A})$ is the symplectic representation (1.7.28) of the considered $\mathrm{SL}(2,\mathbb{R})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $f(z)$ is the associated transition function for that line-bundle whose Chern-class is the Kähler class of the base-manifold. The identification of the symplectic fibres with the cubic symmetric representation provide the construction mechanism of Ω. Consider a vector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ that transforms in the fundamental doublet representation of $\mathrm{SL}(2,\mathbb{R})$. On one hand we can identify the complex coordinate z on the lower half-plane as $z = v_1/v_2$, on the other we can construct a symmetric three-index tensor taking the tensor products of three v_i, namely: $t_{ijk} = v_i\,v_j\,v_k$. Dividing the resulting tensor by v_2^3 we obtain a four vector:

$$\widehat{\Omega}(z) = \frac{1}{v_2^3}\begin{pmatrix} v_1^3 \\ v_1^2\,v_2 \\ v_1\,v_2^2 \\ v_2^3 \end{pmatrix} = \begin{pmatrix} z^3 \\ z^2 \\ z \\ 1 \end{pmatrix} \tag{6.3.12}$$

Next, recalling the change of basis (1.7.25), (1.7.26) required to put the cubic representation into a standard symplectic form we set:

$$\Omega(z) = S\,\widehat{\Omega}(z) = \begin{pmatrix} -\sqrt{3}z^2 \\ z^3 \\ \sqrt{3}z \\ 1 \end{pmatrix} \tag{6.3.13}$$

and we can easily verify that this object transforms in the appropriate way. Indeed we obtain:

$$\Omega\left(\frac{a\,z + b}{c\,z + d}\right) = (c\,z + d)^{-3}\,\Lambda(\mathfrak{A})\,\Omega(z) \tag{6.3.14}$$

The pre-factor $(c\,z + d)^{-3}$ is the correct one for the prescribed line-bundle. To see this let us first calculate the Kähler potential and the Kähler form. Inserting (6.3.13) into Eq. (4.2.15) we get:

$$\mathcal{K} = -\log\left(\mathrm{i}\langle\Omega \mid \bar{\Omega}\rangle\right) = -\log\left(-\mathrm{i}(z-\bar{z})^3\right)$$
$$\mathrm{K} = \frac{\mathrm{i}}{2\pi}\,\partial\,\bar{\partial}\,\mathcal{K} = \frac{\mathrm{i}}{2\pi}\,\frac{3}{(\mathrm{Im}z)^2}dz\wedge d\bar{z} \tag{6.3.15}$$

This shows that the constructed symplectic bundle leads indeed to the standard Poincaré metric and the exponential of the Kähler potential transforms with the prefactor $(c\,z + d)^3$ whose inverse appears in Eq. (6.3.14).

To conclude let us show that the special geometry definition of the period matrix $\mathcal{N}$ agrees with the Gaillard-Zumino definition holding true for all symplectically embedded cosets. To this effect we calculate the necessary ingredients:

$$\nabla_z V(z) = \exp\left[\frac{\mathcal{K}}{2}\right](\partial_z\Omega(z) + \partial_z\mathcal{K}\,\Omega(z)) = \begin{pmatrix} \frac{\sqrt{3}z(z+2\bar{z})}{(z-\bar{z})\sqrt{-i(z-\bar{z})^3}} \\ -\frac{3z^2\bar{z}}{(z-\bar{z})\sqrt{-i(z-\bar{z})^3}} \\ -\frac{\sqrt{3}(2z+\bar{z})}{(z-\bar{z})\sqrt{-i(z-\bar{z})^3}} \\ -\frac{3}{(z-\bar{z})\sqrt{-i(z-\bar{z})^3}} \end{pmatrix} \equiv \begin{pmatrix} f_z^{\Lambda} \\ h_{\Sigma z} \end{pmatrix} \tag{6.3.16}$$

Then according to Eq. (4.2.21) we obtain:

$$f_I^{\Lambda} = \begin{pmatrix} \frac{\sqrt{3}z(z+2\bar{z})}{(z-\bar{z})\sqrt{-i(z-\bar{z})^3}} & -\frac{2\sqrt{6}\bar{z}^2}{(-i(z-\bar{z}))^{3/2}} \\ -\frac{3z^2\bar{z}}{(z-\bar{z})\sqrt{-i(z-\bar{z})^3}} & \frac{2\sqrt{2}\bar{z}^3}{(-i(z-\bar{z}))^{3/2}} \end{pmatrix}$$
$$h_{\Lambda|I} = \begin{pmatrix} -\frac{\sqrt{3}(2z+\bar{z})}{(z-\bar{z})\sqrt{-i(z-\bar{z})^3}} & \frac{2\sqrt{6}\bar{z}}{(-i(z-\bar{z}))^{3/2}} \\ -\frac{3}{(z-\bar{z})\sqrt{-i(z-\bar{z})^3}} & \frac{2\sqrt{2}}{(-i(z-\bar{z}))^{3/2}} \end{pmatrix} \tag{6.3.17}$$

and applying definition (4.2.21) we exactly retrieve the same form of $\mathcal{N}_{\Lambda\Sigma}$ as given in Eq. (6.3.10).

For completeness and also for later use we calculate the remaining items pertaining to special geometry, in particular the symmetric C-tensor. From the general definition (4.2.18) applied to the present one-dimensional case we get:

$$\nabla_z U_z = \mathrm{i}\,C_{zzz}\,h^{zz^\star}\bar{U}_{z^\star} \quad\Rightarrow\quad C_{zzz} = -\frac{6\mathrm{i}}{(z-z^\star)^3} \tag{6.3.18}$$

As for the standard Levi-Civita connection we have:

$$\Gamma^z_{zz} = \frac{2}{z-z^\star} \quad ; \quad \Gamma^{z^\star}_{z^\star z^\star} = -\frac{2}{z-z^\star} \quad ; \quad \text{all other components vanish} \tag{6.3.19}$$

This concludes our illustration of the cubic special Kähler structure on $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$.

6.3.1 *The Quartic Invariant*

In the cubic spin $j = \frac{3}{2}$ representation of $\mathrm{SL}(2, \mathbb{R})$ there is a quartic invariant which plays an important role in the discussion of black-holes. As it happens for all the other supergravity models, the quartic invariant of the symplectic vector of magnetic and electric charges:

$$\mathscr{Q} = \begin{pmatrix} p^\Lambda \\ q_\Sigma \end{pmatrix} \tag{6.3.20}$$

is related to the entropy of the extremal black-holes, the latter being its square root. The origin of the quartic invariant is easily understood in terms of the symmetric tensor t_{ijk}. Using the $\mathrm{SL}(2, \mathbb{R})$-invariant antisymmetric symbol ε^{ij} we can construct an invariant order four polynomial in the tensor t_{ijk} by writing:

$$\mathfrak{I}_4 \propto \varepsilon^{ai}\, \varepsilon^{bj}\, \varepsilon^{pl}\, \varepsilon^{qm}\, \varepsilon^{kr}\, \varepsilon^{cn}\, t_{abc}\, t_{ijk}\, t_{pqr}\, t_{lmn} \tag{6.3.21}$$

If we use the standard basis $t_{111}, t_{112}, t_{122}, t_{222}$, we rotate it with the matrix (1.7.25) and we identify the components of the resultant vector with those of the charge vector $\mathscr{Q}$ the explicit form of the invariant quartic polynomial is the following one:

$$\mathfrak{I}_4 = \frac{1}{3\sqrt{3}} q_2 p_1^3 + \frac{1}{12} q_1^2 p_1^2 - \frac{1}{2} p_2 q_1 q_2 p_1 - \frac{1}{3\sqrt{3}} p_2 q_1^3 - \frac{1}{4} p_2^2 q_2^2 \tag{6.3.22}$$

where we have also chosen a specific overall normalization which turns out to be convenient in the sequel.

6.4 Attractor Mechanism, the Entropy and Other Special Geometry Invariants

One of the most important features of supergravity black-holes is the attractor mechanism discovered in the nineties by Ferrara and Kallosh for the case of BPS solutions[4] [1, 2] and in recent time extended to non-BPS cases [12–14, 21–25]. According to this mechanism, if we focus on spherical symmetric configurations, the evolving

[4] *Clarification for mathematicians:* the acronym BPS stands for Bogomolny, Prasad and Sommerfeld. It is a notion occuring in the theory of monopoles where one always derives a bound according to which the energy (or mass) of a quasi-particle corresponding to a localized solution of non linear propagation equations is always larger or equal than some kind of *charge* carried by the quasi-particle. BPS states are those that saturate the bound and typically correspond to shortened representations of the space-time group. In the case of supergravity black–holes the BPS bound relates the mass of the hole with the modulus of the central charge of the supersymmetry algebra. Because of the scope of this book we omit the original definition of the central charge in terms of superalgebras and we confine to give its expression in terms of special Kähler geometrical items (see Eq. (6.4.4)).

scalar fields $z^i(\tau)$ flow to fixed values at the horizon of the black-hole ($\tau = -\infty$), which do not depend from their initial values at infinite radius ($\tau = 0$) but only on the electromagnetic charges p, q.

In order to establish the relation of the quartic invariant $\mathfrak{I}_4$ defined in Eq. (6.3.22) with the black-hole entropy and review the attractor mechanism, we must briefly recall the essential items of black hole field equations in the *geodesic potential approach* [10]. In this framework we do not consider all the fields listed in Eq. (5.2.2). We introduce only the warp factor $U(\tau)$ and the original scalar fields of $D = 4$ supergravity. The information about vector gauge fields is encoded solely in the set of electric and magnetic charges $\mathcal{Q}$ defined by Eq. (6.3.20) which is retrieved in Eq. (6.2.28). Under these conditions the correct field equations for an $\mathcal{N} = 2$ black-hole are derived from the geodesic one dimensional field-theory described by the following lagrangian:

$$\begin{aligned} S_{eff} &\equiv \int \mathcal{L}_{eff}(\tau)\, d\tau \quad ; \quad \tau = -\frac{1}{r} \\ \mathcal{L}_{eff}(\tau) &= \tfrac{1}{4}\left(\frac{dU}{d\tau}\right)^2 + g_{ij^\star}\,\frac{dz^i}{d\tau}\,\frac{dz^{j^\star}}{d\tau} + e^U\, V_{BH}(z, \bar{z}, \mathcal{Q}) \end{aligned} \tag{6.4.1}$$

where, by definition, the *geodesic potential* $V(z, \bar{z}, \mathcal{Q})$ is given by the following formula in terms of the matrix $\mathcal{M}_4$ introduced in Eq. (4.3.4):

$$V_{BH}(z, \bar{z}, \mathcal{Q}) \;=\; \tfrac{1}{4}\, \mathcal{Q}^t\, \mathcal{M}_4^{-1}\,(\mathcal{N})\; \mathcal{Q} \tag{6.4.2}$$

The effective lagrangian (6.4.1) is derived from the σ-model lagrangian (6.2.24) upon substitution of the first integrals of motion corresponding to the electromagnetic charges (6.2.28) under the condition that the Taub-NUT charge, defined in (6.2.17), vanishes[5] ($\mathbf{n} = 0$). Indeed, when the Taub-NUT charge $\mathbf{n}$ vanishes, which will be our systematic choice, we can invert the above mentioned relations, expressing the derivatives of the Z^M fields in terms of the charge vector $\mathcal{Q}^M$ and the inverse of the matrix $\mathcal{M}_4$. Upon substitution in the $D = 3$ sigma model lagrangian (4.3.4) we obtain the effective lagrangian for the $D = 4$ scalar fields z^i and the warping factor U given by Eqs. (6.4.1)–(6.4.3).

The important thing is that, thanks to various identities of special geometry, the effective geodesic potential admits the following alternative representation:

$$V_{BH}(z, \bar{z}, \mathcal{Q}) = -\tfrac{1}{2}\,\left(|Z|^2 + |Z_i|^2\right) \equiv -\tfrac{1}{2}\,\left(Z\,\bar{Z} + Z_i\, g^{ij^\star}\,\bar{Z}_{j^\star}\right) \tag{6.4.3}$$

[5] As we are going to see later, each orbit of Lax operators always contains representatives such that the Taub-NUT charge is zero. Alternatively from a dynamical system point of view the Taub-NUT charge can be annihilated by setting a constraint which is consistent with the hamiltonian and which reduces the dimension of the system by one unit. The problem of black hole physics is therefore equivalent to the sigma model based on an appropriate codimension one hypersurface in the coset manifold $G/H^\star$.

where the symbol Z denotes the complex scalar field valued central charge of the supersymmetry algebra:

$$Z \equiv V^T \mathbb{C} \mathcal{Q} = M_\Sigma p^\Sigma - L^\Lambda q_\Lambda \tag{6.4.4}$$

and Z_i denote its covariant derivatives:

$$\begin{aligned} Z_i &= \nabla_i Z = U_i \mathbb{C} \mathcal{Q} \quad ; \quad Z^{j^\star} = g^{j^\star i} Z_i \\ \bar{Z}_{j^\star} &= \nabla_{j^\star} Z = \bar{U}_{j^\star} \mathbb{C} \mathcal{Q} \quad ; \quad \bar{Z}^i = g^{i j^\star} \bar{Z}_{j^\star} \end{aligned} \tag{6.4.5}$$

Equation (6.4.3) is a result in special geometry whose proof can be found in several articles and reviews of the late nineties.[6]

6.4.1 Critical Points of the Geodesic Potential and Attractors

The structure of the geodesic potential illustrated above allows for a detailed discussion of its critical points, which are relevant for the asymptotic behavior of the scalar fields.

By definition, critical points correspond to those values of z^i for which the first derivative of the potential vanishes: $\partial_i V_{BH} = 0$. Utilizing the fundamental identities of special geometry and Eq. (6.4.3), the vanishing derivative condition of the potential can be reformulated as follows:

$$0 = 2 Z_i \bar{Z} + \mathrm{i} C_{ijk} \bar{Z}^j \bar{Z}^k \tag{6.4.6}$$

From this equation it follows that there are three possible types of critical points:

$$\begin{array}{lll} Z_i = 0 \,;\ Z \neq 0 \,; & & \text{BPS attractor} \\ Z_i \neq 0 \,;\ Z = 0 \,;\ \mathrm{i}\, C_{ijk} \bar{Z}^j \bar{Z}^k = & 0 & \text{non BPS attractor I} \\ Z_i \neq 0 \,;\ Z \neq 0 \,;\ \mathrm{i}\, C_{ijk} \bar{Z}^j \bar{Z}^k = & -2 Z_i \bar{Z} & \text{non BPS attractor II} \end{array} \tag{6.4.7}$$

It should be noted that in the case of one-dimensional special geometries, like the S^3-model, only BPS attractors and non BPS attractors of type II are possible. Indeed non BPS attractors of type I are forbidden unless C_{zzz} vanishes identically.

In order to characterize the various type of attractors, the authors of [20] and [34] introduced a certain number of special geometry invariants that obey different and characterizing relations at attractor points of different type. They are defined as follows. Let us introduce the symbols:

$$N_3 \equiv C_{ijk} \bar{Z}^i \bar{Z}^j \bar{Z}^k \quad ; \quad \bar{N}_3 \equiv C_{i^\star j^\star k^\star} Z^{i^\star} Z^{j^\star} Z^{k^\star} \tag{6.4.8}$$

[6] See for instance the lecture notes [11].

and let us set:

$$
\begin{aligned}
i_1 &= Z\,\bar{Z} \quad ; \quad i_2 = Z_i\,\bar{Z}_{j^\star}\,g^{ij^\star} \\
i_3 &= \tfrac{1}{6}\left(Z\,N_3 + \bar{Z}\bar{N}_3\right) \quad ; \quad i_4 = \mathrm{i}\,\tfrac{1}{6}\left(Z\,N_3 - \bar{Z}\bar{N}_3\right) \\
i_5 &= C_{ijk}\,C_{\bar{\ell}\bar{m}\bar{n}}\,\bar{Z}^j\;\bar{Z}^k\,Z^{\bar{m}}\,Z^{\bar{n}}\,g^{i\bar{\ell}}\;;
\end{aligned}
\tag{6.4.9}
$$

An important identity satisfied by the above invariants, that depend both on the scalar fields z^i and the charges (p, q), is the following one:

$$
\mathfrak{I}_4(p,q) \;=\; \tfrac{1}{4}(i_1 - i_2)^2 + i_4 - \tfrac{1}{4}\,i_5 \tag{6.4.10}
$$

where $\mathfrak{I}_4(p,q)$ is the quartic symplectic invariant that depends only on the charges (see Eq. (6.3.22)). This means that in the above combination the dependence on the fields z^i cancels identically.

In the case of the one-dimensional S^3 model there are two additional identities [34] that read as follows:

$$
i_2^2 \;=\; \tfrac{3}{4}\,i_5\,;\;\; i_3^2 + i_4^2 = 4 i_1\left(\frac{i_2}{3}\right)^3 \quad ; \quad \text{for the } S^3 \text{ model} \tag{6.4.11}
$$

In [20] it was proposed that the three types of critical points can be characterized by the following relations among the above invariants holding at the attractor point:

At BPS Attractor Points

we have:

$$
i_1 \;\neq\; 0 \quad ; \quad i_2 = i_3 = i_4 = i_5 = 0 \quad ; \tag{6.4.12}
$$

At Non BPS Attractor Points of Type I

we have:

$$
i_2 \;\neq\; 0 \quad ; \quad i_1 = i_3 = i_4 = i_5 = 0 \tag{6.4.13}
$$

At Non BPS Attractor Points of Type II

we have:

$$
i_2 \;=\; 3 i_1 \quad ; \quad i_3 = 0 \quad ; \quad i_4 = -2\,i_1^2 \quad ; \quad i_5 \;=\; 12\,i_1^2 \tag{6.4.14}
$$

These relations follow from the definition of the critical point with the use of standard special geometry manipulations. Their values resides in that they inform us in a simple way about the nature of the black-hole solution we are considering. Indeed they provide a partial classification of solution orbits since, given a configuration of charges (p, q), whose structure depends, as we are going to see, from the choice of an $\mathrm{H}^\star$ orbit for the Lax operator, we can calculate the possible critical points of the corresponding geodesic potential and find out to which type they belong. We might expect several different critical points for each (p, q)-choice, yet it turns out that there is only one and it always belongs to the same type for all elements of the same H⋆ orbit. This fact, whose *a priori proof* has still to be given, implies that a classification of attractor points is also a partial classification of Lax operator orbits. We shall come back on this crucial issue later on. Yet it is appropriate to emphasize the word *partial classification*. Although the type of fixed point is the same for each element of the same orbit we should by no means assume that fixed point types select orbits. Indeed there are Lax operators belonging to different H⋆ orbits that have the same electromagnetic charges and therefore define the same fixed point. Furthermore the fact that a Lax operator defines certain charges and hence an associated fixed point does not imply that the solution generated by such Lax will necessarily reach that fixed point. The solution can break up at a finite value of τ, stopping before the fixed point is attained. Hence the classification of fixed points is not a classification of H⋆ orbits although the two classifications have partial relations to each other.

6.4.2 Fixed Scalars at BPS Attractor Points

In the case of BPS attractors we can find the explicit expression in terms of the (p,q)-charges for the scalar field fixed values at the critical point.

By means of standard special geometry manipulations the BPS critical point equation

$$\nabla_j Z = 0 \quad ; \quad \nabla_{j^\star} \bar{Z} = 0 \tag{6.4.15}$$

can be rewritten in the following celebrated form which, in the late nineties, appeared in numerous research and review papers (see for instance [11]):

$$p^\Lambda = \mathrm{i}\left(Z_{fix}\, \bar{L}^\Lambda_{fix} - \bar{Z}_{fix}\, L^\Lambda_{fix}\right) \tag{6.4.16}$$

$$q_\Sigma = \mathrm{i}\left(Z_{fix}\, \bar{M}^{fix}_\Sigma - \bar{Z}_{fix}\, M^{fix}_\Sigma\right) \tag{6.4.17}$$

Using the explicit form of the symplectic section $\Omega(z)$ given in Eq. (6.3.13), we can easily solve Eq. (6.4.17) for the S^3 model and obtain the following fixed scalars:

$$z_{fixed} = -\frac{p_1 q_1 + 3 p_2 q_2 + \mathrm{i}\, 6\sqrt{\mathfrak{I}_4(p,q)}}{2\left(q_1^2 + \sqrt{3} p_1 q_2\right)} \tag{6.4.18}$$

where $\mathfrak{I}_4(p,q)$ is the quartic invariant defined in Eq. (6.3.22). In fact, one can give the BPS solution in a closed form by replacing in the expression (6.4.18) z_{fixed} the quantized charges with harmonic functions

$$q_\Lambda \to H_\Lambda \equiv h_\Lambda - \sqrt{2}\, q_\Lambda\, \tau \;;\quad p^\Lambda \to H^\Lambda \equiv h^\Lambda - \sqrt{2}\, p^\Lambda\, \tau \tag{6.4.19}$$

The same substitution allows to describe the radial evolution of the warp factor:

$$e^{-U} = \frac{1}{2} \sqrt{\mathfrak{I}_4(H^\Lambda, H_\Lambda)} \tag{6.4.20}$$

The constants h^Λ, h_Λ in the harmonic functions are subject to two conditions: one originates from the requirement of asymptotic flatness ($\lim_{\tau\to 0^-} e^U = 1$), while the other reads $h^\Lambda q_\Lambda - h_\Lambda p^\Lambda = 0$. The remaining two free parameters are fixed by the choice of the value of z at radial infinity.

By replacing the fixed values (6.4.18) into the expression (6.4.3) for the potential we find:

$$V_{BH}\left(z_{fixed}, \bar{z}_{fixed}, \mathcal{Q}\right) = -\sqrt{\mathfrak{I}_4(p,q)} \tag{6.4.21}$$

The above result implies that the horizon area in the case of an extremal BPS black-hole is proportional to the square root of $\mathfrak{I}_4(p,q)$ and, as such, depends only on the charges[7] The argument goes as follows.

Consider the behavior of the warp factor $\exp[-U]$ in the vicinity of the horizon, when $\tau \to -\infty$. For regular black-holes the near horizon metric must factorize as follows:

$$ds^2_{\text{near hor.}} \approx \underbrace{-\frac{1}{r_H^2\,\tau^2}\,dt^2 + r_H^2 \left(\frac{d\tau}{\tau}\right)^2}_{\text{AdS}_2 \text{ metric}} + \underbrace{r_H^2 \left(d\theta^2 \sin^2\theta\, d\phi^2\right)}_{\text{S}^2 \text{ metric}} \tag{6.4.22}$$

where r_H is the Schwarzschild radius defining the horizon. This implies that the asymptotic behavior of the warp factor, for $\tau \to -\infty$ is the following one:

$$\exp[-U] \sim r_H^2\, \tau^2 \tag{6.4.23}$$

In the same limit the scalar fields go to their fixed values and their derivatives become essentially zero. Hence near the horizon we have:

$$\begin{aligned} \left(\dot{U}\right)^2 \approx \frac{4}{\tau^2} \quad &; \quad g_{ij^\star} \frac{dz^i}{d\tau} \frac{dz^{j^\star}}{d\tau} \approx 0 \\ e^U\, V_{BH}(z,\bar{z},\mathcal{Q}) &\approx \frac{1}{r_H^2\,\tau^2}\, V\left(z_{fixed}, \bar{z}_{fixed}, \mathcal{Q}\right) \end{aligned} \tag{6.4.24}$$

Since for extremal black-holes the sum of the above three terms vanishes (see Eq. (6.2.3)), we conclude that:

$$r_H^2 = -V_{BH}\left(z_{fixed}, \bar{z}_{fixed}, \mathcal{Q}\right) \tag{6.4.25}$$

which yields

$$\text{Area}_H = 4\pi\, r_H^2 = 4\pi \sqrt{\mathfrak{I}_4(p,q)} \tag{6.4.26}$$

[7] *Clarification for mathematicians*: for a short but comprehensive introduction to the theory of Black Holes we refer the interested reader to Chaps. 2 and 3 of Volume II of [35] by the present author.

6.5 A Counter Example: The Extremal Kerr Metric

In this section, in order to better clarify the notion of extremality provided by conditions (6.2.8)–(6.2.9) we consider the physically relevant counter-example of the extremal Kerr metric. Such static solution of Einstein equations is certainly encoded in the σ-model approach yet it is not extremal in the sense of Eqs. (6.2.8)–(6.2.9) and therefore it is not related to any nilpotent orbit. Indeed the extremal Kerr metric is a solution of pure gravity and as such its σ-model representation lies in the Euclidean submanifold:

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{O}(2)} \tag{6.5.1}$$

for which the coset tangent space $\mathbb{K}$ contains no nilpotent elements.

Instead the so named BPS Kerr–Newman metric, which is not extremal in the sense of General Relativity and actually displays a naked singularity, is extremal in the sense of Eqs. (6.2.8)–(6.2.9) and can be retrieved in one of the nilpotent orbits of the S^3-model. We will show that explicitly in Sect. 6.11.4.

As a preparation to such discussions let us recall the general form of the Kerr–Newman metric which we represent in polar coordinates as it follows:

$$ds^2_{KN} = -V^0 \otimes V^0 + \sum_{i=1}^{3} V^i \otimes V^i \tag{6.5.2}$$

$$V^0 = \frac{\delta(r)}{\sigma(r,\theta)} \left(dt - \alpha \sin^2\theta \, d\phi\right) \tag{6.5.3}$$

$$V^1 = \frac{\sigma(r,\theta)}{\delta(r)} \, dr \tag{6.5.4}$$

$$V^2 = \sigma(r,\theta) \, d\theta \tag{6.5.5}$$

$$V^3 = \frac{\sin(\theta)}{\sigma(r,\theta)} \left(\left(r^2+\alpha^2\right) d\phi - \alpha \, dt\right) \tag{6.5.6}$$

$$\delta(r) = \sqrt{q^2 + r^2 + \alpha^2 - 2mr} \tag{6.5.7}$$

$$\sigma(r,\theta) = \sqrt{r^2 + \alpha^2 \cos^2(\theta)} \tag{6.5.8}$$

Parameters of the Kerr–Newman solution are the mass m, the electric charge q and the angular momentum $J = m\,\alpha$ of the Black Hole. The two particular cases we shall consider in this paper correspond to:

(a) The extremal Kerr solution: $q = 0$ and $m = \alpha$.
(b) The BPS Kerr–Newman solution $q = m$, arbitrary α.

Let us then focus now on the extremal Kerr solution. With the choice $m = \alpha$, $q = 0$, the metric (6.5.2) can be rewritten in the following form:

$$ds^2_{EK} = -\exp[U]\left(dt + \mathbf{A}^{[KK]}\right)^2 + \exp[-U]\,\gamma_{ij}\,dy^i \otimes dy^j \tag{6.5.9}$$

where $y^i = \{r, \theta, \phi\}$ are the polar coordinates, the three dimensional metric γ_{ij} is the following one:

$$\gamma_{ij} = \begin{pmatrix} \frac{2r^2-\alpha^2+\alpha^2\cos(2\theta)}{2r^2} & 0 & 0 \\ 0 & r^2 - \frac{\alpha^2}{2} + \frac{1}{2}\alpha^2\cos(2\theta) & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{pmatrix} \tag{6.5.10}$$

the warp factor is:

$$U = \log\left[\frac{r^2 - \alpha^2\sin^2(\theta)}{(r+\alpha)^2 + \alpha^2\cos^2(\theta)}\right] \tag{6.5.11}$$

and the Kaluza Klein vector has the following appearance:

$$\mathbf{A}^{[KK]} = \frac{2\alpha^2(r+\alpha)\sin^2(\theta)}{r^2 - \alpha^2\sin^2(\theta)}\, d\phi \tag{6.5.12}$$

In presence of the metric γ_{ij} the duality relation between the Kaluza Klein vector field and the σ-model scalar field a reads as follows:

$$\mathbf{F}^{[KK]}_{ij} \equiv \partial_{[i}\mathbf{A}^{[KK]}_{j]} = \exp[-2U]\sqrt{\det\gamma}\,\varepsilon_{ijk}\,\gamma^{k\ell}\,\partial_\ell\, a \tag{6.5.13}$$

and it is solved by:

$$a = -\frac{2\alpha^2\cos(\theta)}{2r^2 + 4\alpha r + 3\alpha^2 + \alpha^2\cos(2\theta)} \tag{6.5.14}$$

In this way, by means of inverse engineering we have showed how the extremal Kerr metric is retrieved in the σ-model approach. The crucial point is that the metric γ_{ij} is not flat and hence such a configuration of the U, a fields does not correspond to an extremal solution of the σ-model field equations. Indeed calculating the curvature two-form of the three-dimensional metric (6.5.10) we find

$$\mathfrak{R}^{12} = \frac{4\alpha^2\left(2r^2+\alpha^2-\alpha^2\cos(2\theta)\right)}{\left(2r^2-\alpha^2+\alpha^2\cos(2\theta)\right)^3}\, e^1\wedge e^2 \tag{6.5.15}$$

$$\mathfrak{R}^{13} = \frac{4\alpha^2}{\left(2r^2-\alpha^2+\alpha^2\cos(2\theta)\right)^2}\, e^1\wedge e^3 \tag{6.5.16}$$

$$\mathfrak{R}^{23} = -\frac{4\alpha^2}{\left(2r^2-\alpha^2+\alpha^2\cos(2\theta)\right)^2}\, e^2\wedge e^3 \tag{6.5.17}$$

where

$$e^1 = \frac{dr\sqrt{\frac{\cos(2\theta)\alpha^2}{r^2} - \frac{\alpha^2}{r^2} + 2}}{\sqrt{2}} \tag{6.5.18}$$

$$e^2 = d\theta\sqrt{r^2 - \frac{\alpha^2}{2} + \frac{1}{2}\alpha^2\cos(2\theta)} \tag{6.5.19}$$

$$e^3 = d\phi\, r\sin(\theta) \tag{6.5.20}$$

is the *dreibein* corresponding to (6.5.10).

Hopefully this explicit calculation should have convinced the reader that the extremal Kerr solution and, by the same token, also the extremal Kerr–Newman solution are not extremal in the σ-model sense and are retrieved in regular rather than in nilpotent orbits[8] of $\mathrm{U}/\mathrm{H}^\star$.

6.6 The Standard Triple Classification of Nilpotent Orbits

The construction and classification of nilpotent orbits in semi-simple Lie algebras is a relatively new field of mathematics which has already generated a vast literature. Notwithstanding this, a well established set of results ready to use by physicists is not yet available mainly because existing classifications are concerned with orbits with respect to the full complex group $\mathrm{G}_\mathbb{C}$ or of one of its real forms $\mathrm{G}_\mathbb{R}$ [36], which is not exactly what the problem of supergravity black-holes requires (i.e. the classification of the nilpotent $\mathrm{H}^\star$-orbits in $\mathbb{K}$). Furthermore the complexity of the existing mathematical papers and books is rather formidable and their reading not too easy. Yet the main mathematical idea underlying all classification schemes is very simple and intuitive and can be rephrased in a language very familiar to physicists, namely that of angular momentum. Such rephrasing allows for what we named a *practitioner's approach* to the method of triples. In other words after decoding this method in terms of angular momentum we can derive case by case the needed results by using a relatively elementary algorithm supplemented with some hints borrowed from the mathematical literature.

[8] *Clarification for mathematicians*: Extremal in the GR sense means something different than extremal in the σ-model sense. As we mentioned above the extremal Kerr solution, according to General Relativity is the solution where $m = \alpha$. In the σ-model sense any extremal solution corresponds to a light-like geodesic of the of the $\mathrm{U}/\mathrm{H}^\star$ manifold. Light-like geodesics, on their turn are associated with $\mathrm{H}^\star$ orbits of nilpotent U Lie algebra elements. As shown above the extremal Kerr solution is obtained from a $\mathrm{U}/\mathrm{H}^\star$ geodesic that is not light-like so it is not extremal in the σ-model sense.

6.6.1 Presentation of the Method

In this section we shall denote the isometry group $\mathrm{U}_{D=3}$ by $\mathrm{G}_{\mathbb{R}}$ to emphasize that it is a real form of some complex semisimple Lie group.

We will present the practitioner's argument in the form of an ordered list.

1. The basic theorem proved by mathematicians (the Jacobson–Morozov theorem [36]) is that any nilpotent element of a Lie algebra $X \in \mathfrak{g}$ can be regarded as belonging ($X = x$) to a triple of elements $\{x, y, h\}$ that satisfy the standard commutation relations of the $\mathfrak{sl}(2)$ Lie algebra, namely:

$$[h\,,\,x] = x \quad ; \quad [h\,,\,y] = -\,y \quad ; \quad [x\,,\,y] = 2\,h \tag{6.6.1}$$

   Hence the classification of nilpotent orbits is just the classification of embeddings of an $\mathfrak{sl}(2)$ Lie algebra in the ambient one, modulo conjugation by the full group $\mathrm{G}_{\mathbb{R}}$ or by one of its subgroups. In our case the relevant subgroup is $\mathrm{H}^{\star} \subset \mathrm{G}_{\mathbb{R}}$.
2. The second relevant point in our decoding is that embeddings of subalgebras $\mathfrak{h} \subset \mathfrak{g}$ are characterized by the branching law of any representation of $\mathfrak{g}$ into irreducible representations of $\mathfrak{h}$. Clearly two embeddings might be conjugate only if their branching laws are identical. Embeddings with different branching laws necessarily belong to different orbits. In the case of the $\mathfrak{sl}(2) \sim \mathfrak{so}(1,2)$ Lie algebra, irreducible representations are uniquely identified by their spin j, so that the branching law is expressed by listing the angular momenta $\{j_1, j_2, \ldots j_n\}$ of the irreducible blocks into which any representation of the original algebra, for instance the fundamental, decomposes with respect to the embedded subalgebra. The dimensions of each irreducible module is $2j+1$ so that an a priori constraint on the labels $\{j_1, j_2, \ldots j_n\}$ characterizing an orbit is the summation rule:

$$\sum_{i=1}^{n}(2j_i + 1) \;=\; N \;=\; \text{dimension of the fundamental representation} \tag{6.6.2}$$

   Taking into account that j_i are integer or half integer numbers, the sum rule (6.6.2) is actually a partition of N into integers and this explains why mathematicians classify nilpotent orbits starting from partitions of N and use Young tableaux in the process.
3. The next observation is that the central element h of any triple is by definition a diagonalizable (semisimple) non-compact element of the Lie algebra and as such it can always be rotated into the Cartan subalgebra by means of a $\mathrm{G}_{\mathbb{R}}$ transformation. In the case of interest to us, the Cartan subalgebra $\mathscr{C}$ can be chosen, as we will do, inside the subalgebra $\mathbb{H}^{\star}$ and consequently we can argue that for any standard triple $\{x, y, h\}$ the central element is inside that subalgebra:

$$h \in \mathbb{H}^{\star} \tag{6.6.3}$$

Since we shall work with real representations of $G_{\mathbb{R}}$, we choose a basis in which h is a symmetric matrix. Indeed there are two possibilities: either $x \in \mathbb{H}^\star$ or $x \in \mathbb{K}$. In the first case we have $y \in \mathbb{H}^\star$, while in the second we have $y \in \mathbb{K}$. This follows from matrix transposition. Given x, the element y is just its transposed $y = x^T$ and transposition maps $\mathbb{H}^\star$ into $\mathbb{H}^\star$ and $\mathbb{K}$ into $\mathbb{K}$. Since it is already in $\mathbb{H}^\star$, in order to rotate the central element h into the Cartan subalgebra it suffices an $H^\star$ transformation. Therefore to classify $H^\star$ orbits of nilpotent $\mathbb{K}$ elements we can start by considering central elements h belonging to the Cartan subalgebra $\mathscr{C}$ chosen inside $\mathbb{H}^\star$.

4. The central element h of the standard triple, chosen inside the Cartan subalgebra, is identified by its eigenvalues and by their ordering with respect to a standard basis. Since h is the third component of the angular momentum, *i.e.* the operator J_3, its eigenvalues in a representation of spin j are $-j, -j+1, \ldots, j-1, j$. Hence if we choose a branching law $\{j_1, j_2, \ldots j_n\}$, we also decide the eigenvalues of h and consequently its components along a standard basis of simple roots. The only indeterminacy which remains to be resolved is the order of the available eigenvalues.
5. The question which remains to be answered is how much we can order the eigenvalues of Cartan elements by means of $H^\star$ group rotations. The answer is given in terms of the generalized Weyl group $\mathscr{GW}$ and the Weyl group $\mathscr{W}$.
6. The generalized Weyl group is the discrete group generated by all matrices of the form:
$$\mathscr{O}_\alpha = \exp\left[\theta_\alpha \left(E^\alpha - E^{-\alpha}\right)\right] \tag{6.6.4}$$
where $E^{\pm\alpha}$ are the step operators associated with the roots $\pm\alpha$ and the angle θ_α is chosen in such a way that it realizes the α-reflection on a Cartan subalgebra element $\beta \cdot \mathscr{H}$ associated with a vector β:
$$\begin{aligned} O_\alpha\, \beta \cdot \mathscr{H}\, O_\alpha^{-1} &= \sigma_\alpha(\beta) \cdot \mathscr{H} \\ \sigma_\alpha(\beta) &\equiv \beta - 2\,\frac{(\alpha\,,\,\beta)}{(\alpha\,,\,\alpha)}\,\alpha \end{aligned} \tag{6.6.5}$$
The generalized Weyl group has the property that for each of its elements $\gamma \in \mathscr{GW}$ and for each element $h \in \mathscr{C}$ of the Cartan subalgebra $\mathscr{C}$, we have:
$$\gamma\, h\, \gamma^{-1} = h' \in \mathscr{C} \tag{6.6.6}$$
7. The generalized Weyl group contains a normal subgroup $\mathscr{HW} \subset \mathscr{GW}$, named the Weyl stability group and defined by the property that for each element $\xi \in \mathscr{HW}$ and for each Cartan subalgebra element $h \in \mathscr{HW}$ we have:
$$\gamma\, h\, \gamma^{-1} = h \tag{6.6.7}$$

8. The proper Weyl group is defined as the quotient of the generalized Weyl group with respect to the Weyl stability subgroup:

$$\mathscr{W} \equiv \frac{\mathscr{G}\mathscr{W}}{\mathscr{H}\mathscr{W}} \tag{6.6.8}$$

9. The above definition of the Weyl group shows that we can distinguish among its elements those that can be realized by H⋆ transformations, namely those whose corresponding generalized Weyl group elements satisfy the condition $O^T \eta O = \eta$ and those that are outside of H⋆.
10. If we were to consider nilpotent orbits with respect to the whole group G we would just have to mod out all Weyl transformations. In the case of H⋆ orbits this is too much since the entire Weyl group is not contained in H⋆ as we just said. The rotations that have to be modded out are those of the intersection of the generalized Weyl group $\mathscr{G}\mathscr{W}_H$ with H⋆, namely:

$$\mathscr{G}\mathscr{W}_H \equiv \mathscr{G}\mathscr{W} \bigcap \mathrm{H}^\star \tag{6.6.9}$$

It should be noted that the Weyl stability subgroup is always contained in H⋆ so that, by definition, it is also a subgroup of $\mathscr{G}\mathscr{W}_H$:

$$\mathscr{H}\mathscr{W} \subset \mathscr{G}\mathscr{W}_H \tag{6.6.10}$$

which happens to be normal. Hence we can define the ratio

$$\mathscr{W}_H \equiv \frac{\mathscr{G}\mathscr{W}_H}{\mathscr{H}\mathscr{W}} \tag{6.6.11}$$

which is a subgroup of the Weyl group.
11. There is a simple method to find directly $\mathscr{W}_H$. The Weyl group is the symmetry group of the root system Δ. When we choose the Cartan subalgebra inside H⋆ the root system splits into two disjoint subsets:

$$\Delta = \Delta_H \bigoplus \Delta_K \tag{6.6.12}$$

respectively containing the roots represented in $\mathbb{H}^\star$ and those represented in $\mathbb{K}$. Clearly the looked for subgroup $\mathscr{W}_H \subset \mathscr{W}$ is composed by those Weyl elements which do not mix Δ_H with Δ_K and thus respect the splitting (6.6.12). According to this viewpoint, given a Cartan element h corresponding to a partition $\{j_1, j_2, \ldots j_n\}$, we consider its Weyl orbit and we split this Weyl orbit into m suborbits corresponding to the m cosets:

$$\frac{\mathscr{W}}{\mathscr{W}_H} \quad ; \quad m \equiv \frac{|\mathscr{W}|}{|\mathscr{W}_H|} \tag{6.6.13}$$

Each Weyl suborbit corresponds to an $H^\star$-orbit of the neutral elements h in the standard triples. We just have to separate those triples whose x and y elements lie in $\mathbb{K}$ from those whose x and y elements lie in $\mathbb{H}^\star$. By construction if the x and y elements of one triple lie in $\mathbb{K}$, the same is true for all the other triples in the same $\mathscr{W}_H$ orbit. Weyl transformations outside $\mathscr{W}_H$ mix instead $\mathbb{K}$-triples with $\mathbb{H}^\star$ ones.

12. The construction described in the above points fixes completely the choice of the central element h in a standard triple providing a standard representative of an $H^\star$ orbit. The work would be finished if the choice of h uniquely fixed also x and $y = x^T$ that are our main target. This is not so. Given h one can impose the commutation relations:

$$[h\,,\,x] = x \tag{6.6.14}$$
$$\left[x\,,\,x^T\right] = 2\,h \tag{6.6.15}$$

as a set of algebraic equations for x. Typically these equations admit more than one solution.[9] The next task is that of arranging such solutions in orbits with respect to the stability subgroup $\mathscr{S}_h \subset H^\star$ of the central element. Typically such a group is the product, direct or semidirect, of the discrete group $\mathscr{HW}$, which stabilizes any Cartan Lie algebra element, with a continuous subgroup of $H^\star$ which stabilizes only the considered central element h. The presence of such a continuous part of the stabilizer $\mathscr{S}_h$ manifests itself in the presence of continuous parameters in the solution of the second equation (6.6.15) at fixed h.

13. When there are no continuous parameters in the solution of Eq. (6.6.15) what we have to do is quite simple. We just need to verify which solutions are related to which by means of $\mathscr{HW}$ transformations and we immediately construct the $\mathscr{HW}$-orbits. Each $\mathscr{HW}$ orbit of x solutions corresponds to an independent $H^\star$ orbit of nilpotent operators.
14. When continuous parameters are left over in the solutions space, signaling the existence of a continuous part in the $\mathscr{S}_h$ stabilizer, the direct construction of $\mathscr{S}_h$ orbits is more involved and time consuming. An alternative method, however, is available to distribute the obtained solutions into distinct orbits which is based on invariants. Let us define the non-compact operator:

$$X_c \equiv \mathrm{i}\left(x \,-\, x^T\right) \tag{6.6.16}$$

and consider its adjoint action on the maximal compact subalgebra $\mathbb{H} \subset \mathbb{U}$ which, by construction, has the same dimension as $\mathbb{H}^\star$. We name β-labels the spectrum of eigenvalues of that adjoint matrix[10]:

[9] Such solutions actually correspond to different $G_\mathbb{R}$-orbits [36].

[10] In the literature, see [36], β-labels are defined as the value of the simple roots β^i of the complexification $\mathbb{H}_\mathbb{C}$ of $\mathbb{H}^\star$ on the non-compact element X_c, viewed as a Cartan element of $\mathbb{H}_\mathbb{C}$ in the Weyl chamber of (β^i). We find it more practical to work with the equivalent characterization (6.6.17).

$$\beta - \text{label} \;=\; \text{Spectrum}\left[\text{adj}_{\mathbb{H}}\left(X_c\right)\right] \tag{6.6.17}$$

Since the spectrum is an invariant property with respect to conjugation, x-solutions that have different β-labels belong to different $\mathrm{H}^\star$ orbits necessarily. Actually they even belong to different orbits with respect to the full group U. In fact there exists a one-to-one correspondence between nilpotent U orbits in $\mathbb{U}$ and β-labels, which directly follows from the celebrated Kostant-Sekiguchi theorem [36]. So we arrange the different solutions of Eq. (6.6.15) into orbits by grouping them according to their β-labels.

15. The set of possible β-labels at fixed choice of the partition $\{j_1, j_2, \ldots j_n\}$ is predetermined since it corresponds to the set of γ-labels [37]. Let us define these latter. Given the central element h of the triple, we consider its adjoint action on the subalgebra $\mathbb{H}^\star$ and we set:

$$\gamma - \text{label} \;=\; \text{Spectrum}\left[\text{adj}_{\mathbb{H}^\star}\left(h\right)\right] \tag{6.6.18}$$

Obviously all h-operators in the same $\mathscr{W}_H$-orbit have the same γ-label. Hence the set of possible γ-labels corresponding to the same partition $\{j_1, j_2, \ldots j_n\}$ contains at most as many elements as the order of lateral classes $\frac{\mathscr{W}}{\mathscr{W}_H}$. The actual number can be less when some $\mathscr{W}_H$-orbits of h-elements coincide.[11] Given the set of γ-labels pertaining to one $\{j_1, j_2, \ldots j_n\}$-partition the set of possible β-labels pertaining to the same partition is the same. We know a priori that the solutions to Eq. (6.6.15) will distribute in groups corresponding to the available β-labels. Typically all available β-labels will be populated, yet for some partition $\{j_1, j_2, \ldots j_n\}$ and for some chosen γ-label one or more β-labels might be empty.

16. The above discussion shows that by naming α-label the partition $\{j_1, j_2, \ldots j_n\}$ (branching rule of the fundamental representation of $\mathbb{U}$ with respect to the embedded $\mathfrak{sl}(2)$) the orbits can be classified and named with a triple of indices:

$$\mathscr{O}^{\alpha}_{\gamma\beta} \tag{6.6.19}$$

the set of $\gamma\beta$-labels available for each α-label being determined by means of the action of the Weyl group as we have thoroughly explained.

What we have described in the above list is a concrete algorithm to single out standard triple representatives of nilpotent $\mathrm{H}^\star$ orbits of $\mathbb{K}$ operators. In the next section we apply it to the example of the $\mathfrak{g}_{(2,2)}$ model in order to show how it works.

[11] Note that the action of certain Weyl group elements $g \in \mathscr{W}$ on specific h.s can be the identity: $g \cdot h = h$. When such stabilizing group elements g are inside $\mathscr{W}_H$ the number of different h.s inside each lateral classes is accordingly reduced. If there are stabilizing elements g that are not inside $\mathscr{W}_H$ than two or more $\mathscr{W}_H$ orbits coincide.

6.7 The Nilpotent Orbits of the $\mathfrak{g}_{(2,2)}$ Model

In the present section we consider the classification of nilpotent H*-orbits in $\mathfrak{g}_{(2,2)}$ by using the algorithm described in the previous section.

6.7.1 *The Weyl and the Generalized Weyl Groups for* $\mathfrak{g}_{(2,2)}$

According to our general discussion the most important tools for the orbit classification are the generalized Weyl groups and its subgroups.

We begin with the structure of the Weyl group for the $\mathfrak{g}_{(2,2)}$ root system Δ_{g2}. By definition this is the group of rotations in a two-dimensional plane generated by the reflections along all the roots contained in Δ_{g2}. Abstractly the structure of the group is given by the semidirect product of the permutation group of three object S_3 with a $\mathbb{Z}_2$ factor:

$$\mathscr{W} = S_3 \ltimes \mathbb{Z}_2 \tag{6.7.1}$$

Correspondingly the order of the group is:

$$|\mathscr{W}| = 12 \tag{6.7.2}$$

An explicit realization by means of 2×2 orthogonal matrices is the following one:

$$\begin{array}{llllll} \text{Id} = & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & ;\ \alpha_1 = & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & ;\ \alpha_2 = & \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \\ \alpha_3 = & \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} & ;\ \alpha_4 = & \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} & ;\ \alpha_5 = & \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \\ \alpha_6 = & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & ;\ \xi_1 = & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & ;\ \xi_2 = & \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ \xi_3 = & \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} & ;\ \xi_4 = & \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} & ;\ \xi_5 = & \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \end{array} \tag{6.7.3}$$

where Id is the identity element, α_i $(i = 1, \ldots, 6)$ denote the reflections along the corresponding roots and ξ_i $(i = 1, \ldots, 5)$ are the additional elements created by products of reflections. The multiplication table of this group is displayed below:

0	Id	α_1	α_2	α_3	α_4	α_5	α_6	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
Id	Id	α_1	α_2	α_3	α_4	α_5	α_6	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
α_1	α_1	Id	ξ_4	ξ_2	ξ_3	ξ_5	ξ_1	α_6	α_3	α_4	α_2	α_5
α_2	α_2	ξ_5	Id	ξ_4	ξ_1	ξ_3	ξ_2	α_4	α_6	α_5	α_3	α_1
α_3	α_3	ξ_3	ξ_5	Id	ξ_2	ξ_1	ξ_4	α_5	α_4	α_1	α_6	α_2
α_4	α_4	ξ_2	ξ_1	ξ_3	Id	ξ_4	ξ_5	α_2	α_1	α_3	α_5	α_6
α_5	α_5	ξ_4	ξ_2	ξ_1	ξ_5	Id	ξ_3	α_3	α_2	α_6	α_1	α_4
α_6	α_6	ξ_1	ξ_3	ξ_5	ξ_4	ξ_2	Id	α_1	α_5	α_2	α_4	α_3
ξ_1	ξ_1	α_6	α_4	α_5	α_2	α_3	α_1	Id	ξ_5	ξ_4	ξ_3	ξ_2
ξ_2	ξ_2	α_4	α_5	α_1	α_3	α_6	α_2	ξ_5	ξ_3	Id	ξ_1	ξ_4
ξ_3	ξ_3	α_3	α_6	α_4	α_1	α_2	α_5	ξ_4	Id	ξ_2	ξ_5	ξ_1
ξ_4	ξ_4	α_5	α_1	α_2	α_6	α_4	α_3	ξ_3	ξ_1	ξ_5	ξ_2	Id
ξ_5	ξ_5	α_2	α_3	α_6	α_5	α_1	α_4	ξ_2	ξ_4	ξ_1	Id	ξ_3

(6.7.4)

Next let us discuss the structure of the generalized Weyl group. In this case $\mathscr{GW}$ is composed by 48 elements and its stability subgroup $\mathscr{HW} \sim \mathbb{Z}_2 \times \mathbb{Z}_2$ is made by the following four 7×7 matrices belonging to the $\mathrm{G}_{(2,2)}$ group:

$$
hw_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} ; \; hw_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
hw_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \; \mathrm{Id} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.7.5}
$$

In order to complete the description of the generalized Weyl group it is now sufficient to write one representative for each equivalence class of the quotient:

$$
\frac{\mathscr{GW}}{\mathscr{HW}} \simeq \mathscr{W} \tag{6.7.6}
$$

We have:

$$\alpha_1 \sim \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} ; \alpha_2 \sim \begin{pmatrix} -\frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$$\alpha_3 \sim \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} ; \alpha_4 \sim \begin{pmatrix} -\frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$$\alpha_5 \sim \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} ; \alpha_6 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{6.7.7}$$

and

$$\xi_1 \sim \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} ; \xi_2 \sim \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$\xi_3 \sim \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} ; \xi_4 \sim \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$\xi_5 \sim \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 \end{pmatrix} ; \tag{6.7.8}$$

We can explicitly verify that all the elements of the $\mathscr{HW}$ subgroup are in $\mathrm{H}^\star = \mathfrak{su}(1,1) \times \mathfrak{su}(1,1)$ since they satisfy the condition:

$$hw_i^T \, \eta \, hw_i \; = \; \eta \tag{6.7.9}$$

where

$$\eta \; = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.7.10}$$

is the invariant metric which defines the $\mathrm{H}^\star$ subgroup. Note that here we use all the conventions and the definitions introduced in [32].

The next required ingredient of our construction is the subgroup $\mathscr{W}_H$. As it was shown in [32], when we diagonalize the adjoint action of a Cartan Subalgebra contained in the $\mathbb{H}^\star$ subalgebra, the root system of the $\mathfrak{g}_2$ Lie algebra (see Fig. 6.4), decomposes in two subsystems Δ_H and Δ_K such that the step operators corresponding to roots in Δ_H belong to $\mathbb{H}^\star$ while the step operators corresponding to roots in Δ_K belong to $\mathbb{K}$. The subsystem Δ_H is composed by the roots $\pm\alpha_3$, $\pm\alpha_5$, while Δ_K is made by the remaining ones. The subgroup $\mathscr{W}_H \subset \mathscr{W}$ can be easily derived. It is made by all those elements of the Weyl group which map Δ_H into itself and Δ_K into itself, as well. Referring to the previously introduced notation, we easily see that (Fig. 6.5):

$$\mathscr{W}_H \; = \; \{\mathrm{Id}, \alpha_3, \alpha_5, \xi_1\} \tag{6.7.11}$$

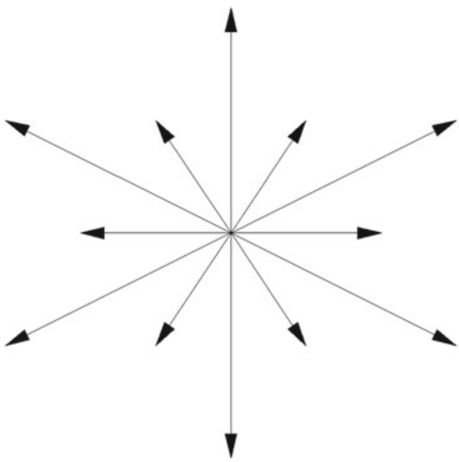

Fig. 6.4 The $\mathfrak{g}_2$ root system Δ_{g2} is made of six positive roots and of their negatives

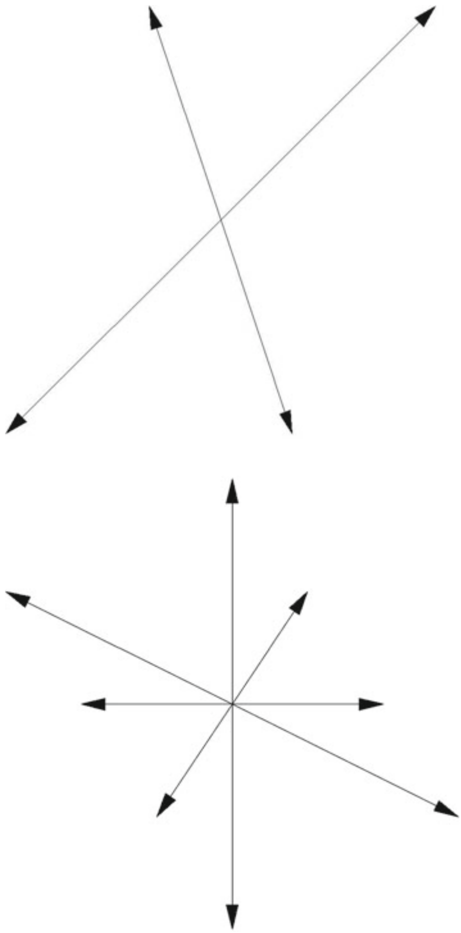

Fig. 6.5 The root system Δ_{g2} splits in two subsystems, the system Δ_H on the left, the system Δ_K on the right

Abstractly the structure of $\mathscr{W}_H$ is the following:

$$\mathscr{W}_H \sim \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{6.7.12}$$

since all of its elements square to the identity.

There are three lateral classes in $\mathscr{W}/\mathscr{W}_H$, respectively associated with the identity element and with the reflection along the two simple roots.

$$[\mathrm{Id}] = \{\mathrm{Id}, \alpha_3, \alpha_5, \xi_1\} \tag{6.7.13}$$

$$[\alpha_1] = \{\alpha_1, \alpha_6, \xi_3, \xi_4\} \tag{6.7.14}$$

$$[\alpha_2] = \{\alpha_2, \alpha_4, \xi_2, \xi_5\} \tag{6.7.15}$$

It follows that for each partition $\{j_1, j_2, \dots j_n\}$ (α-label) there are three possible γ-labels and three possible β-labels. It remains to be seen for which combinations of these γ and β-labels there exist an x-operator purely contained in $\mathbb{K}$ which completes the standard triple.

6.7.2 The Table of $\frac{\mathbf{G_{(2,2)}}}{\mathbf{SU(1,1)\times SU(1,1)}}$ Nilpotent Orbits

In order to derive the desired table of nilpotent orbits we begin from the first step namely from partitions or, said differently, from α-labels.

6.7.2.1 α-Labels

Taking into account the restriction (see [36]) that every half-integer spin j should appear an even number of times we easily conclude that the possible branching laws of the 7-dimensional fundamental representation of $\mathfrak{g}_{(2,2)}$ into irreducible representations of $\mathfrak{sl}(2)$ are the following ones:

$$\alpha_1 - \text{label} = [\text{j=3}] \tag{6.7.16}$$

$$\alpha_2 - \text{label} = [\text{j=1}] \times 2[j = 1/2] \tag{6.7.17}$$

$$\alpha_3 - \text{label} = 2[\text{j=1}] \times [j = 0] \tag{6.7.18}$$

$$\alpha_4 - \text{label} = 2[\text{j=1/2}] \times 3[j = 0] \tag{6.7.19}$$

6.7.2.2 γ-Labels

Analyzing the two Eqs. (6.6.14), (6.6.15) for the x-triple element at fixed h we find the following result:

α_1 In this sector there are x operators in $\mathbb{K}$ only for the second lateral class (6.7.14). This means that there is only one γ-label which has the following form:

$$\gamma_1 = \{\pm 8, \pm 4, 0, 0\} \equiv \{8_1, 4_1, 0_1\} \tag{6.7.20}$$

The notation introduced in Eq. (6.7.20) is based on the following observation. The dimension of $\mathbb{H}$ or $\mathbb{H}^\star$ is six and every eigenvalue appears together with its negative. Hence it suffices to mention the non-negative eigenvalues (including the zero) with their multiplicity (all zeros appear in pairs as well). It follows that the β-label is also unique so that in this sector there is only one nilpotent orbit.

α_2 For this partition the $\mathscr{W}_H$ orbits (6.7.13) and (6.7.14) coincide: within them we find x operators in $\mathbb{K}$. In the third $\mathscr{W}_H$ orbit there are no solutions for x in $\mathbb{K}$. So we have only one γ-label:

Table 6.1 Classification of nilpotent orbits of $\frac{\mathrm{G}_{(2,2)}}{\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)}$

N	d_n	α − label	$\gamma\beta$ − labels	Orbits	$\mathscr{W}_H$ − classes
1	7	[j=3]	$\gamma\beta_1 = \{8_1 4_1 0_1\}$	$\mathscr{O}_1^1$	$(\times, \gamma_1, \times)$
2	3	[j=1] $\times$ 2[$j = 1/2$]	$\gamma\beta_1 = \{3_1 1_1 0_1\}$	$\mathscr{O}_1^2$	$(\gamma_1, \gamma_1, \times)$
7	3	2[j=1] $\times$ [$j = 0$]	$\gamma\beta_1 = \{4_1 0_2\}$ $\gamma\beta_2 = \{2_2 0_1\}$	$\begin{array}{ccc} & \beta_1 & \beta_2 \\ \gamma_1 & \mathscr{O}^3_{1,1} & \mathscr{O}^3_{1,2} \\ \gamma_2 & \mathscr{O}^3_{2,1} & \mathscr{O}^3_{2,2} \end{array}$	$(\gamma_1, \gamma_2, \gamma_2)$
4	2	2[j=1/2] $\times$ 3[$j = 0$]	$\gamma\beta_1 = \{1_2 0_1\}$	$\mathscr{O}_1^4$	$(0, \gamma_1, \gamma_1)$

$$\gamma_1 = \{3_1, 1_1, 0_1\} \tag{6.7.21}$$

and consequently only one nilpotent orbit.

α_3 For this partition the $\mathscr{W}_H$ orbits (6.7.14) and (6.7.15) coincide while the first is distinct. We find solutions for x in $\mathbb{K}$ both for the first $\mathscr{W}_H$-orbit (6.7.13) and for the coinciding subsequent two. That means that we have two γ-labels

$$\gamma_1 = \{4_1, 0_2\} \tag{6.7.22}$$

$$\gamma_2 = \{2_2, 0_1\} \tag{6.7.23}$$

Considering the solutions for x both in the case of γ_1 and γ_2 they group in two non empty classes corresponding to β-labels β_1 and β_2. This means that we have a total of 4 nilpotent orbits from this sector.

α_4 For this partition the situation is similar to that of partition one and two. There are no $\mathbb{K}$ solutions for x in the first $\mathscr{W}_H$ orbit while there are such solutions in the second and third $\mathscr{W}_H$-orbits, which coincide. Hence there is only one γ-label:

$$\gamma_1 = \{1_2, 0_1\} \tag{6.7.24}$$

and one nilpotent orbit.

In Table 6.1 the results we have described are summarized.

6.8 Construction of Multicenter Solutions Associated with Nilpotent Orbits

In this section we summarize in purely mathematical terms the algorithm that associates extremal black hole solutions of supergravity to nilpotent orbits of the Lie algebra $\mathbb{U}$. As the reader will appreciate the algorithm is completely sequential and constructive so that it can be easily implemented by means of computer codes.

For spherically symmetric black holes the construction of solutions is associated with nilpotent orbits in the following way. A representative of the $H^\star$ orbit is a standard triple $\{h, X, Y\}$ and hence an embedding of an $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra:

$$[h, X] = 2X \quad ; \quad [h, Y] = -2Y \quad ; \quad [X, Y] = 2h \tag{6.8.1}$$

into $\mathbb{U}_{D=3}$ in such a way that $h \in \mathbb{H}^\star$ and $X, Y \in \mathbb{K}^\star$. The nilpotent operator X is identified with the Lax operator L_0 at Euclidean time $\tau = 0$ and the corresponding solution depending on τ is constructed by using the algorithm described in [27, 29, 32].

In the multicenter approach of [15–19, 38] one utilizes the standard triple to single out a nilpotent subalgebra $\mathbb{N}$, as follows. One diagonalizes the adjoint action of the central element h of the triple on the Lie Algebra $\mathbb{U}_{D=3}$:

$$\left[h\,,\,C_\mu\right] = \mu\,C_\mu \tag{6.8.2}$$

The set of all eigen-operators C_μ corresponding to positive gradings $\mu > 0$ spans a subalgebra $\mathbb{N} \subset \mathbb{U}_{D=3}$ which is necessarily nilpotent

$$\mathbb{N} = \text{span}\,[C_2\,,\,C_3\,,\,\ldots\,,\,C_{max}] \tag{6.8.3}$$

Such a nilpotent subalgebra has an intersection $\mathbb{N}\bigcap\mathbb{K}^\star$ with the space $\mathbb{K}^\star$ which is not empty since at least the operator $C_2 = X$ is present by definition of a standard triple. The next steps of the construction are as follows.

6.8.1 The Coset Representative in the Symmetric Gauge

Given a basis A^i of the space $\mathbb{N}_\mathbb{K} \equiv \mathbb{N}\bigcap\mathbb{K}^\star$, whose dimension we denote:

$$\ell \equiv \dim \mathbb{N}_\mathbb{K} \tag{6.8.4}$$

and a basis B^α of the subalgebra $\mathbb{N}_\mathbb{H} \equiv \mathbb{N}\bigcap\mathbb{H}^\star$, whose dimension we denote

$$\mathfrak{m} \equiv \dim \mathbb{N}_\mathbb{H} \tag{6.8.5}$$

we can construct a map:

$$\mathfrak{H} \;:\; \mathbb{R}^3 \to \mathbb{N}_\mathbb{K} \tag{6.8.6}$$

by writing:

$$\mathbb{N}_\mathbb{K} \ni \mathfrak{H}(\mathbf{x}) = \sum_{i=1}^{\ell} \mathfrak{h}_i(\mathbf{x})\,A^i \tag{6.8.7}$$

By construction, the point dependent Lie algebra element $\mathfrak{H}(\mathbf{x})$ is nilpotent of a certain maximal degree $\mathrm{d_n}$, so that its exponential map to the nilpotent group $\mathrm{N} \subset \mathrm{U_{D=3}}$ truncates to a finite sum:

$$\mathscr{Y}(x) = \exp[\mathfrak{H}(\mathbf{x})] = \mathbf{1} + \sum_{a=1}^{\mathrm{d_n}} \frac{1}{a!}\,\mathfrak{H}^a(\mathbf{x}) \tag{6.8.8}$$

The above constructed object realizes an explicit $\mathbf{x}$-dependent coset representative from which we can construct the Maurer Cartan left-invariant one form:

$$\Sigma = \mathscr{Y}^{-1}\partial_i\mathscr{Y}\,dx^i \tag{6.8.9}$$

Next let us decompose Σ along the $\mathbb{K}^\star$ subspace and the $\mathbb{H}^\star$ subalgebra, respectively. This is done by setting:

$$\mathbf{P} = \mathrm{Tr}(\Sigma\,K^A)K_A \quad ; \quad \Omega = \mathrm{Tr}(\Sigma\,H^m)H_m \tag{6.8.10}$$

where K_A and H_m denote a basis of generators for the two considered subspaces, K^A and H^m being their duals:

$$\mathrm{Tr}(K^A\,K_B) = \delta^A_B \quad ; \quad \mathrm{Tr}(H^m\,H_n) = \delta^m_n \quad ; \quad \mathrm{Tr}(K^A\,H_n) = 0 \tag{6.8.11}$$

Denoting:

$$^\star\mathbf{P} \equiv \tfrac{1}{2}\,\varepsilon_{ijk}\,\delta^{im}\,\mathbf{P}_m\,dx^j \wedge dx^k \tag{6.8.12}$$

the Hodge-dual of the coset vielbein

$$\mathbf{P} = \mathbf{P}_m\,dx^m \tag{6.8.13}$$

the field equations of the three dimensional σ-model reduce to the following one:

$$d^\star\mathbf{P} = \Omega \wedge {}^\star\mathbf{P} - {}^\star\mathbf{P} \wedge \Omega \tag{6.8.14}$$

Actually, since $\mathbb{N} \subset \mathbb{U}_{D=3}$ forms a nilpotent subalgebra the constructed object $\mathscr{Y}$ realizes a map from the three-dimensional space to the much smaller coset manifold:

$$\mathscr{Y} \; : \; \mathbb{R}^3 \to \frac{\mathrm{N}}{\mathrm{N_H}} \tag{6.8.15}$$

and due to the polynomial form of the coset representative the final equations of motion obtain a triangular solvable form that we describe here below. Since the algebra $\mathbb{N}$ is nilpotent, its derivative series terminates, namely we have:

$$\mathbb{N} \supset \mathscr{D}\mathbb{N} \supset \ldots \supset \mathscr{D}^n\mathbb{N} \supset \mathscr{D}^{n+1}\mathbb{N} = \mathbf{0} \tag{6.8.16}$$

where at each step $\mathscr{D}^i\mathbb{N}$ is a proper subspace of $\mathscr{D}^{i-1}\mathbb{N}$. Correspondingly let us define:

$$\mathscr{D}^i\mathbb{N}_{\mathbb{K}} = \mathscr{D}^i\mathbb{N}\bigcap\mathbb{K}^\star \tag{6.8.17}$$

the intersections of the derivative subalgebras with the $\mathbb{K}^\star$ subspace and let us introduce the complementary orthogonal subspaces:

$$\mathscr{D}^i\mathbb{N}_{\mathbb{K}} = \mathbb{N}^{(i)}_K \oplus \mathscr{D}^{i+1}\mathbb{N}_{\mathbb{K}} \tag{6.8.18}$$

This yields an orthogonal graded decomposition of the space $\mathbb{N}_{\mathbb{K}}$ of the following form:

$$\mathbb{N}_{\mathbb{K}} = \bigoplus_{a=0}^{n} \mathbb{N}^{(a)}_{\mathbb{K}} \tag{6.8.19}$$

The space $\mathbb{N}^{(0)}_{\mathbb{K}}$ contains those generators that cannot be produced by any commutator within the algebra, $\mathbb{N}^{(1)}_{\mathbb{K}}$ contains those generators that are produced in simple commutators, $\mathbb{N}^{(2)}_{\mathbb{K}}$ contains those that are produced in double commutators and so on. Let us name

$$\ell_a = \dim\mathbb{N}^{(a)}_{\mathbb{K}} \quad ; \quad \sum_a^n \ell_a = \ell \tag{6.8.20}$$

Correspondingly we can arrange the ℓ functions $\mathfrak{h}_i(\mathbf{x})$ according to the graded decomposition (6.8.19), by writing:

$$\mathfrak{H}(\mathbf{x}) = \sum_{\alpha=0}^{n} \underbrace{\sum_{i=1}^{\ell_\alpha} \mathfrak{h}^{(\alpha)}_i(\mathbf{x})\, A^i_\alpha}_{\in\mathbb{N}^{(\alpha)}_{\mathbb{K}}} \tag{6.8.21}$$

and Eq. (6.8.14) take the following triangular form:

$$\begin{aligned}
\nabla^2\mathfrak{h}^{(0)}_i &= 0\\
\nabla^2\mathfrak{h}^{(1)}_i &= \mathfrak{F}^{(1)}_i\left(\mathfrak{h}^{(0)}, \nabla\mathfrak{h}^{(0)}\right)\\
\nabla^2\mathfrak{h}^{(2)}_i &= \mathfrak{F}^{(2)}_i\left(\mathfrak{h}^{(0)}, \nabla\mathfrak{h}^{(0)}, \mathfrak{h}^{(1)}, \nabla\mathfrak{h}^{(1)}\right)\\
\dots &= \dots\\
\nabla^2\mathfrak{h}^{(n)}_i &= \mathfrak{F}^{(n)}_i\left(\mathfrak{h}^{(0)}, \nabla\mathfrak{h}^{(0)}, \mathfrak{h}^{(1)}, \nabla\mathfrak{h}^{(1)}, \dots, \mathfrak{h}^{(n-1)}, \nabla\mathfrak{h}^{(n-1)}\right),
\end{aligned} \tag{6.8.22}$$

where ∇^2 denotes the three-dimensional Laplacian and at each level α, by $\mathfrak{F}^{(\alpha)}_i(\dots)$ we denote an $\mathfrak{so}(3)$ invariant polynomial of all the functions h^β up to level $\alpha - 1$ and of their derivatives.

Therefore the first ℓ_0 functions $\mathfrak{h}_i^{(0)}$ are just harmonic functions, while the higher ones satisfy Laplace equation with a source that is provided by the previously determined functions.

6.8.2 *Transformation to the Solvable Gauge*

Given the symmetric coset representative $\mathcal{Y}(\mathbf{x})$, parameterized by functions $\mathfrak{h}_i^{(\alpha)}(\mathbf{x})$ which satisfy the field equations (6.8.22), in order to retrieve the corresponding supergravity fields satisfying supergravity field equations, we need to solve a technical, yet quite crucial problem. We need to construct a new upper triangular coset representative:

$$\mathbb{L}(\mathcal{Y}) = \begin{pmatrix} L_{1,1}(\mathcal{Y}) & L_{1,2}(\mathcal{Y}) & \cdots & L_{1,n-1}(\mathcal{Y}) & L_{1,n}(\mathcal{Y}) \\ 0 & L_{2,2}(\mathcal{Y}) & \cdots & L_{2,n-1}(\mathcal{Y}) & L_{2,n}(\mathcal{Y}) \\ 0 & 0 & L_{3,3}(\mathcal{Y}) & \cdots & L_{3,n}(\mathcal{Y}) \\ \vdots & \dots & 0 & \dots & \vdots \\ 0 & 0 & \cdots & 0 & L_{3,n}(\mathcal{Y}) \end{pmatrix} \quad (6.8.23)$$

which depends algebraically on the matrix entries of $\mathcal{Y}$ and satisfies the following equivalence condition

$$\mathbb{L}(\mathcal{Y})\,\mathcal{Q}(\mathcal{Y}) = \mathcal{Y} \quad ; \quad \mathcal{Q}(\mathcal{Y}) \in \mathrm{H}^\star \quad (6.8.24)$$

where, as specified above, $\mathcal{Q}(\mathcal{Y})$ is a suitable element of the subgroup $\mathrm{H}^\star$. It should be stressed that in the existing literature, this transition from the symmetric to the solvable gauge, which is compulsory in order to make the construction of the black hole solutions explicit, has been advocated, yet it has been left to *ad hoc* procedures to be invented case by case.

Actually a universal and very elegant solution of such a problem exists and was found, from a different perspective, by the author of the present book in collaboration with A. Sorin. It was presented in [27–30, 32]. Defining the following determinants:

$$\mathfrak{D}_i(\mathcal{Y}) := \mathrm{Det}\begin{pmatrix} \mathcal{Y}_{1,1} & \dots & \mathcal{Y}_{1,i} \\ \vdots & \vdots & \vdots \\ \mathcal{Y}_{i,1} & \dots & \mathcal{Y}_{i,i} \end{pmatrix}, \quad \mathfrak{D}_0(\mathcal{Y}) := 1 \quad (6.8.25)$$

the matrix elements of the inverse of the upper triangular coset representative satisfying both Eqs. (6.8.23) and (6.8.24) are given by the following expressions:

$$\left(\mathbb{L}(\mathscr{Y})^{-1}\right)_{ij} \equiv \frac{1}{\sqrt{\mathfrak{D}_i(\mathscr{Y})\mathfrak{D}_{i-1}(\mathscr{Y})}}\text{Det}\begin{pmatrix} \mathscr{Y}_{1,1} & \dots & \mathscr{Y}_{1,i-1} & \mathscr{Y}_{1,j} \\ \vdots & \vdots & \vdots & \vdots \\ \mathscr{Y}_{i,1} & \dots & \mathscr{Y}_{i,i-1} & \mathscr{Y}_{i,j} \end{pmatrix} \tag{6.8.26}$$

Equation (6.8.26) provides a universal non-trivial and very elegant solution to the gauge-change problem and makes the entire construction based on harmonic functions truly algorithmic from the start to the very end.

6.8.3 Extraction of the Three Dimensional Scalar Fields

The result of the procedure described in the previous section is a triangular coset representative $\mathbb{L}(\mathfrak{h}_i^{(\alpha)})$ whose entries are polynomial and square root of polynomials in the functions $\mathfrak{h}_i^{(\alpha)}(x)$. The extraction of the scalar fields $\{U(x), a(x), Z(x), \phi(x)\}$ can now be performed according to the rules already presented in [32], which we recall here in full.

The general form of the solvable coset representative in terms of the fields is the following one:

$$\mathbb{L}(\Phi) = \exp\left[-a\, L_+^E\right] \exp\left[\sqrt{2}\, Z^M\, \mathscr{W}_M\right] \mathbb{L}_4(\phi) \exp\left[U\, L_0^E\right] \tag{6.8.27}$$

where $L_0^E, L_\pm^E$ are the generators of the Ehlers group and $\mathscr{W}^M \equiv W^{1M}$ are the generators in the W-representation, according to the general structure (1.7.13) of the $\mathbb{U}_{D=3}$ Lie algebra; furthermore $\mathbb{L}_4(\phi)$ is the coset representative of the $D = 4$ scalar coset manifold immersed in the $\mathrm{U}_{\mathrm{D=3}}$ group. From this structure, identifying $\mathbb{L}(\Phi) = \mathbb{L}(\mathfrak{h}_i^{(\alpha)})$ we deduce the following iterative procedure for the extraction of the relevant fields:

First of all we can determine the warp factor U by means of the following simple formula:

$$U(\mathfrak{h}) = \log\left[\tfrac{1}{2}\,\text{Tr}\left(\mathbb{L}(\mathfrak{h})\, L_+^E\, \mathbb{L}^{-1}(\mathfrak{h})\, L_-^E\right)\right] \tag{6.8.28}$$

Secondly we obtain the fields ϕ_i as follows. Defining the functionals

$$\Xi_i(\mathfrak{h}) = \text{Tr}\left(\mathbb{L}^{-1}(\mathfrak{h})\, T_i\, \mathbb{L}(\tau)\right) \tag{6.8.29}$$

from the form of the coset representative (6.8.27) it follows that Ξ_i depend only on the $D = 4$ scalar fields and, according to the explicit form of the $D = 4$ coset, one can work out the scalar fields ϕ_i.

The knowledge of U, ϕ_i allows to define:

$$\Omega(\mathfrak{h}) = \mathbb{L}(\mathfrak{h}) \exp\left[-\,U\, L_0^E\right] \mathbb{L}_4(\phi)^{-1} \tag{6.8.30}$$

from which we extract the Z^M fields by means of the following formula:

$$Z^M(\mathfrak{h}) = \frac{1}{2\sqrt{2}} \operatorname{Tr}\left[\Omega(\mathfrak{h}) \mathscr{W}_M^T\right] \tag{6.8.31}$$

where T means transposed. Finally the knowledge of $Z^M(\mathfrak{h})$ allows to extract the a field by means of the following trace:

$$a(\mathfrak{h}) = -\tfrac{1}{2}\operatorname{Tr}\left[\Omega(\mathfrak{h}) \exp\left[-\sqrt{2}\, Z^M(\mathfrak{h}) \mathscr{W}_M\right] L_+^E\right] \tag{6.8.32}$$

6.9 General Properties of the Black Hole Solutions and Structure of Their Poles

Having discussed the structure of supergravity solutions in terms of black-boxes that are a set of harmonic functions and of their descendants generated through the solution of the hierarchical equations (6.8.22), it is appropriate to study the general form of the geometries one obtains in this way and the properties of the available harmonic functions.

First of all, naming:

$$\mathfrak{W} = \exp[U(x)] \tag{6.9.1}$$

the warp factor that defines the 4-dimensional metric (6.2.11), we would like to investigate the general properties of the corresponding geometries. For the case where the Kaluza–Klein monopole is zero $\mathbf{A}^{[KK]} = 0$ we can write the general form of the curvature two-form of such spaces and therefore the intrinsic form of the Riemann tensor. Using the vielbein formalism introduced in Eq. (6.2.12) we obtain:

$$\begin{aligned}
\mathfrak{R}^{0i} &= -\mathfrak{W}\,\nabla^i\nabla_k\mathfrak{W}\; E^0 \wedge E^k - 2\,\nabla^i\mathfrak{W}\nabla_k\mathfrak{W}\; E^0 \wedge E^k \\
\mathfrak{R}^{ij} &= -2\,\mathfrak{W}\,\nabla^{[i}\nabla_k\mathfrak{W}\; E^{j]} \wedge E^k + (\nabla\mathfrak{W}\cdot\nabla\mathfrak{W})\; \nabla_k\mathfrak{W}\; E^i \wedge E^j
\end{aligned} \tag{6.9.2}$$

where the derivatives used in the above equations are defined as follows. Let the flat metric in three dimension be described by a Euclidean *dreibein* e^i such that:

$$\begin{aligned}
ds^2_{flat} &= \sum_{i=1}^{3} e^i \otimes e^i \\
E^i &= \frac{1}{\mathfrak{W}}\, e^i
\end{aligned} \tag{6.9.3}$$

then the total differential of the warp factor expanded along e^i yields the derivatives $\nabla_k\mathfrak{W}$, namely:

$$d\,\mathfrak{W} = \nabla_k\mathfrak{W}\, e^k \tag{6.9.4}$$

Next let us consider the general form of harmonic functions. These latter form a linear space since any linear combination of harmonic functions is still harmonic. There are three types of building blocks that we can use:

a Real center pole:

$$\mathscr{H}_\alpha(\mathbf{x}) = \frac{1}{|\mathbf{x} - \mathbf{x}_\alpha|} \tag{6.9.5}$$

b Real part of an imaginary center pole:

$$\mathscr{R}_\alpha(\mathbf{x}) = \mathrm{Re}\left[\frac{1}{|\mathbf{x} - \mathrm{i}\,\mathbf{x}_\alpha|}\right] \tag{6.9.6}$$

c Imaginary part of an imaginary center pole:

$$\mathscr{I}_\alpha(\mathbf{x}) = \mathrm{Im}\left[\frac{1}{|\mathbf{x} - \mathrm{i}\,\mathbf{x}_\alpha|}\right] \tag{6.9.7}$$

Hence the most general harmonic function can be written as the following sum:

$$\mathrm{Harm}(\mathbf{x}) = h_\infty + \sum_\alpha \frac{p_\alpha}{|\mathbf{x} - \mathbf{x}_\alpha|} + \sum_\beta q_\beta\,\mathrm{Re}\left[\frac{1}{|\mathbf{x} - \mathrm{i}\,\mathbf{x}_\beta|}\right] + \sum_\gamma k_\gamma\,\mathrm{Im}\left[\frac{1}{|\mathbf{x} - \mathrm{i}\,\mathbf{x}_\gamma|}\right] \tag{6.9.8}$$

where the constant h_∞ is the boundary value of the harmonic function at infinity far from all the poles. In order to study the behavior of $\mathrm{Harm}(\mathbf{x})$ in the vicinity of a real pole ($|\mathbf{x} - \mathbf{x}_\alpha| << 1$) it is convenient to adopt local polar coordinates:

$$\begin{aligned} x^1 - x^1_\alpha &= r\cos\theta \\ x^2 - x^2_\alpha &= r\sin\theta\sin\phi \\ x^3 - x^3_\alpha &= r\sin\theta\cos\phi \end{aligned} \tag{6.9.9}$$

In this coordinates the harmonic function is approximated by:

$$\mathrm{Harm}(\mathbf{x}) \simeq h_\alpha + \frac{p_\alpha}{r} \tag{6.9.10}$$

where the effective constant h_α encodes the finite part of the function contributed by all the other poles. In polar coordinates the Laplacian operator on functions of r becomes:

$$\varDelta = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \tag{6.9.11}$$

The general outcome of the construction procedure outlined in the previous section is that the warp factor is the square root of a rational function of n harmonic functions, where $n = \dim \mathbb{N}_{\mathbb{K}}$

$$\mathfrak{W}(\mathbf{x}) = \sqrt{\frac{\mathbb{P}\left(\widehat{\mathrm{Harm}}_1(\mathbf{x}), \ldots, \widehat{\mathrm{Harm}}_n(\mathbf{x})\right)}{\mathbb{Q}\left(\widehat{\mathrm{Harm}}_1(\mathbf{x}), \ldots, \widehat{\mathrm{Harm}}_n(\mathbf{x})\right)}} \tag{6.9.12}$$

where $\mathbb{P}$ and $\mathbb{Q}$ are two polynomials. By $\widehat{\mathrm{Harm}}_1(\mathbf{x})$ we denote both harmonic functions and their descendants generated by the hierarchical system (6.8.22). For a given multicenter solution it is convenient to enumerate all the poles displayed by one or the other of the harmonic functions and in the vicinity of each of those poles we will have:

$$\widehat{\mathrm{Harm}}_i(\mathbf{x}) \simeq \frac{p_i}{r^{m_i}} \tag{6.9.13}$$

where $p_i \neq 0$ if the considered pole belongs to the considered function and it is zero otherwise. Furthermore if $\mathrm{Harm}_i(\mathbf{x})$ is one of the level one harmonic function the exponent $m_i = 1$. Otherwise it is bigger, but in any case $m_i \geq 1$. Taking this into account the effective behavior of the warp factor will always be of the following form:

$$\mathfrak{W}(\mathbf{x}) \simeq r^{\ell_\alpha} \sqrt{c_\alpha} \tag{6.9.14}$$

where ℓ is some integer or half integer power (positive or negative) and c_α is a constant. In order for the pole to be a regular point of the solution, two conditions have to be satisfied:

1. The constant $c_\alpha > 0$ must be positive so that the warp factor is real.
2. The power $\ell_\alpha \geq 1$ so that the Riemann tensor does not diverge at the pole.

The second condition follows from the form (6.9.2) of the Riemann tensor which implies that all of its components behave as:

$$\mathfrak{R}^{ab}_{cd} \simeq r^{2\ell_\alpha - 2} \times \text{const} \tag{6.9.15}$$

Near the pole the metric behaves as follows:

$$ds^2 \simeq -\sqrt{c_\alpha}\, r^{\ell_\alpha} dt^2 + \frac{1}{\sqrt{c_\alpha}} \frac{1}{r^{\ell_\alpha}} \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{6.9.16}$$

In order for the pole to be an event horizon of finite or of vanishing area, we must have $2 - \ell_\alpha > 0$, so that the volume of the two-sphere described by $\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$ does not diverge. Hence for regular black holes we have only three possibilities:

$$\underbrace{\ell_\alpha = 2}_{\text{Large Black Holes}} \quad ; \quad \underbrace{\ell_\alpha = \tfrac{3}{2}}_{\text{Small Black Holes}} \quad ; \quad \underbrace{\ell_\alpha = 1}_{\text{Very Small Black Holes}} \tag{6.9.17}$$

When we are in the case of Large Black Holes, the near horizon geometry is approximated by that:

$$\mathrm{AdS}_2 \times \mathbb{S}^2 \tag{6.9.18}$$

The case of the harmonic functions with an imaginary center requires a different treatment. Their near singularity behavior is best analyzed by using spheroidal coordinates.

These are easily introduced by setting:

$$\begin{aligned} x^1 &= \sqrt{r^2+\alpha^2}\,\sin\theta\,\sin\phi \\ x^2 &= \sqrt{r^2+\alpha^2}\,\sin\theta\,\cos\phi \\ x^3 &= r\cos\theta \end{aligned} \tag{6.9.19}$$

where r, θ, ϕ are the new coordinates and α is a deformation parameter which represents the position of the center in the complex plane. In terms of these coordinates the flat Euclidean three-dimensional metric takes the following form:

$$\begin{aligned} ds^2_{\mathbb{E}^3} = d\Omega^2_{spheroidal} \equiv& \frac{\left(r^2+\alpha^2\cos^2\theta\right)dr^2}{r^2+\alpha^2} + \left(r^2+\alpha^2\right)\sin^2\theta\,d\phi^2 \\ &+ \left(r^2+\alpha^2\cos^2\theta\right)\,d\theta^2 \end{aligned} \tag{6.9.20}$$

and the two harmonic functions that correspond to the real and imaginary part of a complex harmonic function with center on the imaginary z-axis at α-distance from zero are:

$$\mathscr{P}_\alpha(r,\theta) = \frac{r}{r^2+\alpha^2\cos^2\theta} \tag{6.9.21}$$

$$\mathscr{R}_\alpha(r,\theta) = \frac{\alpha\cos\theta}{r^2+\alpha^2\cos^2\theta} \tag{6.9.22}$$

and the Hodge duals of their gradients, in spheroidal coordinates have the following form:

$$\begin{aligned} \star\,\nabla\mathscr{P}_\alpha =& \frac{\sin\theta}{\left(r^2+\alpha^2\cos^2\theta\right)^2}\,\big[2\alpha^2\,r\cos\theta\sin\theta\,dr\wedge d\phi \\ &+ \left(r^2+\alpha^2\right)\left(r^2-\alpha^2\cos^2\theta\right)d\theta\wedge d\phi\big] \end{aligned} \tag{6.9.23}$$

$$\begin{aligned} \star\nabla\mathscr{R}_\alpha =& \frac{\alpha\sin\theta}{\left(r^2+\alpha^2\cos^2\theta\right)^2}\,\big[\left(\alpha^2\cos^2\theta-r^2\right)\sin\theta dr\wedge d\phi \\ &+2r\left(r^2+\alpha^2\right)\cos\theta d\theta\wedge d\phi\big] \end{aligned} \tag{6.9.24}$$

These are the building blocks we can use to construct Kerr–Newman like solutions and we shall outline a pair of examples in the sequel.

6.10 The Example of the S^3 Model: Classification of the Nilpotent Orbits

As an illustration of the general procedure we explore the case of the S^3 model, leading to the $\mathrm{G}_{2,2}$ group in $D = 3$. The detailed classification of the nilpotent orbits pertaining to this case was derived in Sect. 6.7. According to it, for the case of the coset manifold[12]:

$$\frac{\mathrm{U}_{\mathrm{D}=3}}{\mathrm{H}^\star} = \frac{\mathrm{G}_{(2,2)}}{\widehat{\mathrm{SL}(2,\mathbb{R})} \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{h}^\star}} \tag{6.10.1}$$

there just seven distinct nilpotent orbits of the $\mathrm{H}^\star = \widehat{\mathrm{SL}(2,\mathbb{R})} \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{h}^\star}$ subgroup in the $\mathbb{K}^\star$ representation $\left(2, \frac{3}{2}\right)$, which are enumerated by the three set of labels $\alpha\beta\gamma$ and are denoted $\mathscr{O}^{\alpha}_{\beta\gamma}$ as described in Table 6.1. An explicit choice of a representative for each of the seven orbits is provided below.

$$\mathscr{O}^1_{11} = \begin{pmatrix} \sqrt{\frac{3}{2}} & \frac{\sqrt{\frac{5}{2}}}{2} & \sqrt{\frac{3}{2}} & \frac{\sqrt{5}}{2} & 0 & \frac{\sqrt{\frac{5}{2}}}{2} & 0 \\ \frac{\sqrt{\frac{5}{2}}}{2} & \sqrt{6} & -\frac{\sqrt{\frac{5}{2}}}{2} & -\sqrt{3} & -\frac{\sqrt{\frac{5}{2}}}{2} & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \\ -\sqrt{\frac{3}{2}} & \frac{\sqrt{\frac{5}{2}}}{2} & -\sqrt{\frac{3}{2}} & \frac{\sqrt{5}}{2} & 0 & \frac{\sqrt{\frac{5}{2}}}{2} & 0 \\ -\frac{\sqrt{5}}{2} & \sqrt{3} & \frac{\sqrt{5}}{2} & 0 & \frac{\sqrt{5}}{2} & -\sqrt{3} & -\frac{\sqrt{5}}{2} \\ 0 & \frac{\sqrt{\frac{5}{2}}}{2} & 0 & \frac{\sqrt{5}}{2} & \sqrt{\frac{3}{2}} & \frac{\sqrt{\frac{5}{2}}}{2} & \sqrt{\frac{3}{2}} \\ \frac{\sqrt{\frac{5}{2}}}{2} & 0 & -\frac{\sqrt{\frac{5}{2}}}{2} & \sqrt{3} & -\frac{\sqrt{\frac{5}{2}}}{2} & -\sqrt{6} & \frac{\sqrt{\frac{5}{2}}}{2} \\ 0 & \frac{\sqrt{\frac{5}{2}}}{2} & 0 & \frac{\sqrt{5}}{2} & -\sqrt{\frac{3}{2}} & \frac{\sqrt{\frac{5}{2}}}{2} & -\sqrt{\frac{3}{2}} \end{pmatrix} \tag{6.10.2}$$

$$\mathscr{O}^4_{11} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{6.10.3}$$

$$\mathscr{O}^2_{11} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \tag{6.10.4}$$

[12] For the rationale of our notation we refer the reader to previous Sect. 5.8.

$$\mathscr{O}^3_{11} = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -1 \end{pmatrix} \tag{6.10.5}$$

$$\mathscr{O}^3_{22} = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -1 \end{pmatrix} \tag{6.10.6}$$

$$\mathscr{O}^3_{21} = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 1 \end{pmatrix} \tag{6.10.7}$$

$$\mathscr{O}^3_{12} = \begin{pmatrix} -1 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 0 & -\frac{1}{2} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 1 \end{pmatrix} \tag{6.10.8}$$

Each orbit representative $\mathscr{O}^{\alpha}_{\beta\gamma}$ identifies a standard triple $\{h, X, Y\}$ and hence an embedding of an $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra:

$$[h, X] = 2\,X \quad ; \quad [h, Y] = -2\,Y \quad ; \quad [X, Y] = 2\,h \tag{6.10.9}$$

into $\mathfrak{g}_{(2,2)}$ in such a way that $h \in \mathbb{H}^{\star}$ and $X, Y \in \mathbb{K}^{\star}$. The triple is obtained by setting:

$$X_{\alpha|\beta\gamma} \equiv \mathscr{O}^{\alpha}_{\beta\gamma} \quad ; \quad Y_{\alpha|\beta\gamma} \equiv X^{T}_{\alpha|\beta\gamma} \quad ; \quad h_{\alpha|\beta\gamma} \equiv \left[X_{\alpha|\beta\gamma}, Y_{\alpha|\beta\gamma}\right] \tag{6.10.10}$$

The relevant item in the construction of solutions based on the integration of equations in the symmetric gauge is provided by the central element of the triple $h_{\alpha|\beta\gamma}$ which defines the gradings. In the present example of the S^3 model, it turns out the orbits having the same α and γ labels but different β-labels have the same central element, namely:

$$h_{\alpha|\beta\gamma} = h_{\alpha|\beta'\gamma} \tag{6.10.11}$$

so that the solutions pertaining both to orbit $\mathscr{O}^{\alpha}_{\beta\gamma}$ and to orbit $\mathscr{O}^{\alpha}_{\beta'\gamma}$ are obtained from the same construction and are distinguished only by different choices in the space of the available harmonic functions parameterizing the general solution.

The explicit form of the central elements are the following ones:

Large Orbit $\mathscr{O}^{1}_{11}$: Central Element

$$h_{1|11} = \begin{pmatrix} 0 & 0 & -1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\text{Eigenvalues}\left[\tfrac{1}{2}\, h_{1|11}\right] = \{-3, 3, -2, 2, -1, 1, 0\} \tag{6.10.12}$$

Very Small Orbit $\mathscr{O}^{4}_{11}$: Central Element

$$h_{4|11} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Eigenvalues}\left[\tfrac{1}{2}\, h_{4|11}\right] = \left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right\} \tag{6.10.13}$$

Small Orbit $\mathcal{O}^2_{11}$: Central Element

$$h_{2|11} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\text{Eigenvalues}\left[\tfrac{1}{2}h_{2|11}\right] = \left\{-1, 1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right\} \tag{6.10.14}$$

Large BPS Orbit $\mathcal{O}^3_{11}$: Central Element

$$h_{3|11} = h_{3|21} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Eigenvalues}\left[\tfrac{1}{2}h_{3|11}\right] = \{-1, -1, 1, 1, 0, 0, 0\} \tag{6.10.15}$$

Large Non BPS Orbit $\mathcal{O}^3_{22}$: Central Element

$$h_{3|12} = h_{3|22} = \begin{pmatrix} 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Eigenvalues}\left[\tfrac{1}{2}h_{3|22}\right] = \{-1, -1, 1, 1, 0, 0, 0\} \tag{6.10.16}$$

6.11 Explicit Construction of the Multicenter Black Holes Solutions of the S^3 Model

Having enumerated the central elements for the independent orbits we proceed to the construction and discussion of the corresponding black hole solutions, whose properties are summarized in Table 6.2.

Table 6.2 Properties of the $\mathfrak{g}_{(2,2)}$ orbits in the S^3 model. The structure of the electromagnetic charge vector is that obtained for solutions with vanishing Taub-NUT current. The symbol $\rhd$ is meant to denote semidirect product. $\mathscr{S}_{\mathbf{W}}$ denotes the subgroup of the $D = 4$ duality group which leaves the charge vector invariant, while $\mathfrak{S}_{\mathrm{H}^\star}$ denotes the subgroup of the $\mathrm{H}^\star$ isotropy group of the $D = 3$ sigma-model which leaves invariant the X element of the standard triple. This latter is the Lax operator in the one-dimensional spherical symmetric approach

Name of orbit	pq charges	Quart. Inv. $\mathfrak{I}_4$	**W** – stab. group $\mathscr{S}_{\mathbf{W}} \subset \mathfrak{sl}(2,\mathbb{R})$	$\mathrm{H}^\star$ – stab. group $\mathfrak{S}_{\mathrm{H}^\star} \subset \widehat{\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathrm{h}^\star}}$	dim $\mathbb{N}$	dim $\mathbb{N} \bigcap \mathbb{K}^\star$
$\mathscr{O}^4_{11}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ q \end{pmatrix}$	0	$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$	$\underbrace{\mathrm{ISO}(1,1)}_{3\ \text{gen.}}$	3	3
$\mathscr{O}^2_{11}$	$\begin{pmatrix} \sqrt{3}\,p \\ 0 \\ 0 \\ 0 \end{pmatrix}$	0	$\mathbf{1}$	$\underbrace{\mathrm{SO}(1,1) \rhd \mathbb{R}}_{2\ \text{gen.}}$	4	3
$\mathscr{O}^3_{11}$	$\begin{pmatrix} 0 \\ p \\ -\sqrt{3}q \\ 0 \end{pmatrix}$	$9\,p\,q^3 > 0$	$\mathbb{Z}_3$	$\underbrace{\mathbb{R}}_{1\ \text{gen.}} \quad A^2 = 0$	5	4
$\mathscr{O}^3_{22}$	$\begin{pmatrix} 0 \\ p \\ \sqrt{3}q \\ 0 \end{pmatrix}$	$-9\,p\,q^3 < 0$	$\mathbf{1}$	$\underbrace{\mathbb{R}}_{1\ \text{gen.}} \quad A^3 = 0$	3	3
$\mathscr{O}^1_{11}$	$\begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{2}}\,p \\ 0 \\ \frac{7}{6}p \\ \sqrt{2}q \end{pmatrix}$	$\frac{1}{128}p^3 \times (49p + 72q)$	$\mathbf{1}$	$\mathbf{1}$	6	4

6.11.1 The Very Small Black Holes of $\mathscr{O}^4_{11}$

We begin with the smallest orbits which, in a sense that will become clear further on, represent the elementary blocks in terms of which bigger black holes are constructed.

Focusing on any orbit $\mathscr{O}^\alpha_{\beta\gamma}$ and considering the nilpotent element of the corresponding triple $X_{\alpha|\beta\gamma} \in \mathbb{K}^\star$ as a Lax operator L_0, we easily workout the electromagnetic charges by calculating the traces displayed below (see Sect. 5.9, for more explanations)

$$\mathscr{Q}^{\mathbf{w}} = \mathrm{Tr}(X_{\alpha|\beta\gamma}\mathscr{T}^{\mathbf{w}}) \tag{6.11.1}$$

W-Representation

In the case of the orbit $\mathscr{O}^4_{11}$ we obtain:

$$\mathscr{Q}^{\mathbf{w}}_{4|11} = (0, 0, 0, 1) \tag{6.11.2}$$

Substituting such a result in the expression for the quartic symplectic invariant (see [32]):

$$\mathfrak{I}_4 = \frac{1}{4}\left(4\sqrt{3}Q_4Q_1^3 + 3Q_3^2Q_1^2 - 18Q_2Q_3Q_4Q_1 - Q_2\left(4\sqrt{3}Q_3^3 + 9Q_2Q_4^2\right)\right) \tag{6.11.3}$$

of the **W** representation which happens to be the spin $\frac{3}{2}$ of $\mathfrak{sl}(2, \mathbb{R})$ we find:

$$\mathfrak{I}_4 = 0 \tag{6.11.4}$$

The result is meaningful since, by calculating the trace $\mathrm{Tr}(X_{4|11}L_+^E) = 0$, we can also check that the Taub-NUT charge vanishes. We can also address the question whether there are subgroups of the original duality group in four-dimensions SL(2, R) that leave the charge vector (6.11.2) invariant. Using the explicit form of the $j = \frac{3}{2}$ representation displayed in Eq. (3.13) of [32], we realize that indeed such group exists and it is the parabolic subgroup described below:

$$\forall c \in \mathbb{R} \quad : \quad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in \mathscr{S}_{4|11} \subset \mathrm{SL}(2, \mathbb{R}) \tag{6.11.5}$$

This stability subgroup together with the vanishing of the quartic invariant are the intrinsic definition of the **W**-orbit pertaining to very small black holes.

H*-Stability Subgroup

In a parallel way we can pose the question what is the stability subgroup of the nilpotent element $X_{4|11}$ in $\mathrm{H}^\star = \widehat{\mathfrak{sl}(2, \mathbb{R})} \oplus \mathfrak{sl}(2, \mathbb{R})_{\mathrm{h}^\star}$ (For further explanations on H* and its structure see Sect. 5.8). The answer is the following:

$$\mathfrak{S}_{4|11} = \mathrm{ISO}(1, 1) \tag{6.11.6}$$

A generic element of the corresponding Lie algebra is a linear combination of three generators J, T_1, T_2, satisfying the commutation relations:

$$\begin{aligned} [J\,,\, T_1] &= \frac{1}{\sqrt{2}}T_1 + \frac{3}{2\sqrt{6}}T_2 \\ [J\,,\, T_2] &= \frac{3}{2\sqrt{2}}T_1 \quad ; \quad [T_1\,,\, T_2] = 0 \end{aligned} \tag{6.11.7}$$

It is explicitly given by the following matrix:

$$\omega\, J \;+\; x\, T_1 \;+\; y\, T_2 \;=\; \begin{pmatrix} 0 & -\frac{x}{2\sqrt{2}} & \frac{\omega}{2\sqrt{2}} & -\frac{x}{2} & 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}y & 0 \\ \frac{x}{2\sqrt{2}} & 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}y & -\frac{\omega}{2} & \frac{x}{2\sqrt{2}} & 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}y \\ \frac{\omega}{2\sqrt{2}} & -\frac{1}{2}\sqrt{\frac{3}{2}}y & 0 & -\frac{x}{2} & 0 & -\frac{x}{2\sqrt{2}} & 0 \\ -\frac{x}{2} & -\frac{\omega}{2} & \frac{x}{2} & 0 & -\frac{x}{2} & -\frac{\omega}{2} & \frac{x}{2} \\ 0 & \frac{x}{2\sqrt{2}} & 0 & \frac{x}{2} & 0 & \frac{1}{2}\sqrt{\frac{3}{2}}y & \frac{\omega}{2\sqrt{2}} \\ \frac{1}{2}\sqrt{\frac{3}{2}}y & 0 & -\frac{x}{2\sqrt{2}} & -\frac{\omega}{2} & \frac{1}{2}\sqrt{\frac{3}{2}}y & 0 & -\frac{x}{2\sqrt{2}} \\ 0 & \frac{1}{2}\sqrt{\frac{3}{2}}y & 0 & \frac{x}{2} & \frac{\omega}{2\sqrt{2}} & \frac{x}{2\sqrt{2}} & 0 \end{pmatrix} \tag{6.11.8}$$

Nilpotent Algebra $\mathbb{N}_{4|11}$

Considering next the adjoint action of the central element $h_{4|11}$ on the subspace $\mathbb{K}^\star$ we find that its eigenvalues are the following ones:

$$\text{Eigenvalues}^{\mathbb{K}^\star}_{4|11} \;=\; \{-2, 2, -1, -1, 1, 1, 0, 0\} \tag{6.11.9}$$

Therefore the three eigenoperators A_1, A_2, A_3 corresponding to the positive eigenvalues 2, 1, 1, respectively, form the restriction to $\mathbb{K}^\star$ of a nilpotent algebra $\mathbb{N}_{4|11}$. In this case A_i commute among themselves so that $\mathbb{N}_{4|11} = \mathbb{N}_{4|11} \bigcap \mathbb{K}^\star$ and it is abelian. This structure of the nilpotent algebra implies that for the orbit $\mathscr{O}^4_{11}$ we have only three functions $\mathfrak{h}^0_i$ which will be harmonic and independent.

Explicitly we set:

$$\mathfrak{H}(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3) \;=\; \sum_{i=1}^{3} \mathfrak{h}_i\, A_i \;=\; \begin{pmatrix} -\mathfrak{h}_1 & \mathfrak{h}_3 & 0 & -\sqrt{2}\mathfrak{h}_3 & -\mathfrak{h}_1 & -\mathfrak{h}_2 & 0 \\ \mathfrak{h}_3 & 0 & -\mathfrak{h}_2 & 0 & \mathfrak{h}_3 & 0 & -\mathfrak{h}_2 \\ 0 & \mathfrak{h}_2 & -\mathfrak{h}_1 & \sqrt{2}\mathfrak{h}_3 & 0 & -\mathfrak{h}_3 & -\mathfrak{h}_1 \\ \sqrt{2}\mathfrak{h}_3 & 0 & \sqrt{2}\mathfrak{h}_3 & 0 & \sqrt{2}\mathfrak{h}_3 & 0 & \sqrt{2}\mathfrak{h}_3 \\ \mathfrak{h}_1 & -\mathfrak{h}_3 & 0 & \sqrt{2}\mathfrak{h}_3 & \mathfrak{h}_1 & \mathfrak{h}_2 & 0 \\ -\mathfrak{h}_2 & 0 & \mathfrak{h}_3 & 0 & -\mathfrak{h}_2 & 0 & \mathfrak{h}_3 \\ 0 & -\mathfrak{h}_2 & \mathfrak{h}_1 & -\sqrt{2}\mathfrak{h}_3 & 0 & \mathfrak{h}_3 & \mathfrak{h}_1 \end{pmatrix} \tag{6.11.10}$$

Considering $\mathfrak{H}(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$ as a Lax operator and calculating its Taub-NUT charge and electromagnetic charges we find:

$$\mathbf{n}_{TN} \;=\; -2\,\mathfrak{h}_2 \quad ; \quad \mathscr{Q} \;=\; \Big(0\,,\; 2\mathfrak{h}_2\,,\; -2\sqrt{3}\mathfrak{h}_3\,,\; -2\mathfrak{h}_1\Big) \tag{6.11.11}$$

This implies that constructing the multi-centre solution with harmonic functions the condition $\mathfrak{h}_2 \;=\; 0$ should be sufficient to annihilate the Taub-NUT current.

For later convenience let us change the normalization in the basis of harmonic functions as follows:

$$\mathfrak{h}^{(0)}_1 \;=\; \frac{1}{\sqrt{2}}\mathscr{H}_1 \quad ; \quad \mathfrak{h}^{(0)}_2 \;=\; \frac{1}{2}\,(1 \;-\; \mathscr{H}_2) \quad ; \quad \mathfrak{h}^{(0)}_3 \;=\; \frac{1}{\sqrt{2}}\mathscr{H}_3 \tag{6.11.12}$$

Implementing the symmetric coset construction with:

$$\mathscr{Y}\left(\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3\right) \equiv \exp\left[\mathfrak{H}\left(\frac{1}{\sqrt{2}}\mathscr{H}_1, \frac{1}{2}\left(1 - \mathscr{H}_2\right), \frac{1}{\sqrt{2}}\mathscr{H}_3\right)\right] \tag{6.11.13}$$

and calculating the upper triangular coset representative $\mathbb{L}(\mathscr{Y})$ according to Eq. (6.8.26) we find a relatively simple expression which, however, is still too large to be displayed. Yet the extraction of the σ-model scalar fields produces a quite compact answer which we list below:

$$\exp[-U] = \sqrt{\mathscr{H}_2^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1} \tag{6.11.14}$$

$$\mathrm{Im}\, z = \frac{\sqrt{\mathscr{H}_2^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1}}{\mathscr{H}_2^2 - \mathscr{H}_3^2 + \mathscr{H}_1} \tag{6.11.15}$$

$$\mathrm{Re}\, z = -\frac{\sqrt{2}\mathscr{H}_3}{\mathscr{H}_2^2 - \mathscr{H}_3^2 + \mathscr{H}_1} \tag{6.11.16}$$

$$Z^M = \begin{pmatrix} \frac{\sqrt{6}\mathscr{H}_3^2}{\mathscr{H}_2^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1} \\ \frac{(\mathscr{H}_2 - 2\mathscr{H}_3)(\mathscr{H}_2 + \mathscr{H}_3)^2 + \mathscr{H}_1\mathscr{H}_2}{\sqrt{\left(\mathscr{H}_2^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1\right)^2}} \\ -\frac{\sqrt{3}\mathscr{H}_3}{\mathscr{H}_2^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1} \\ \frac{\mathscr{H}_2^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1 - 1}{\sqrt{2}\left(\mathscr{H}_2^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1\right)} \end{pmatrix} \tag{6.11.17}$$

$$a = \frac{\mathscr{H}_2^3 + \left(-3\mathscr{H}_3^2 + \mathscr{H}_1 + 1\right)\mathscr{H}_2 - 2\mathscr{H}_3^3}{\sqrt{2}\left(\mathscr{H}_2^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1\right)} \tag{6.11.18}$$

The Taub-NUT Current

Given this explicit result we can turn to the explicit oxidation formulae described in Sect. 6.2.1 and calculate the Taub-NUT current which is the integrand of Eq. (6.2.17). We find:

$$j^{TN} = \sqrt{2}\,{}^{\star}\nabla\,\mathscr{H}_2 \tag{6.11.19}$$

Hence the vanishing of the Taub-NUT current is guaranteed by the very simple condition:

$$\mathscr{H}_2 = \alpha \quad ; \quad \nabla\mathscr{H}_2 = 0 \tag{6.11.20}$$

where α is just a constant. This confirms the preliminary analysis obtained from the Lax operator which requires a vanishing component of the Lax along the second generator A_2 of the nilpotent algebra.

General Form of the Solution

Imposing this condition we arrive at the following form of the solution depending on two harmonic functions $\mathscr{H}_1$, $\mathscr{H}_3$:

$$\exp[-U] = \sqrt{\alpha^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1} \tag{6.11.21}$$

$$z = \mathrm{i}\,\frac{1}{\sqrt{\alpha^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1}} - \frac{\sqrt{2}\,\mathscr{H}_3}{\alpha^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1} \tag{6.11.22}$$

$$j^{TN} = 0 \tag{6.11.23}$$

$$j^{EM} = \star\nabla \begin{pmatrix} 0 \\ 0 \\ \sqrt{3}\,\mathscr{H}_3 \\ -\frac{1}{\sqrt{2}}\,\mathscr{H}_1 \end{pmatrix} \tag{6.11.24}$$

Obviously the physical range of the solution is determined by the condition $(\alpha^2 - 3\mathscr{H}_3^2 + \mathscr{H}_1) > 0$ which can always be arranged, by tuning the parameters contained in the harmonic functions.

To this effect let us discuss the nature of the black holes encompassed by this solution, that, by definition, are located at the poles of the harmonic functions $\mathscr{H}_1$, $\mathscr{H}_3$.

According to the argument developed in Sect. 6.9, in the vicinity of each pole $|\mathbf{x} - \mathbf{x}_I| = r < \varepsilon$ we can choose polar coordinates centered at $\mathbf{x}_\alpha$ and the behavior of the harmonic functions, for $\varepsilon \to 0$ is the following one:

$$\mathscr{H}_1 \sim a_1 + \frac{b_1}{r} \tag{6.11.25}$$

$$\mathscr{H}_3 \sim a_3 + \frac{b_3}{r} \tag{6.11.26}$$

which corresponds to the following behavior of the warp factor:

$$\exp[-U] \sim \sqrt{\alpha^2 - 3a_3^2 - \frac{3b_3^2}{r^2} + a_1 + \frac{b_1}{r} - \frac{6a_3b_3}{r}} \tag{6.11.27}$$

In order for the warp factor to be real for all values of $r \to 0$ we necessarily find

$$\begin{aligned} b_3 &= 0 \\ b_1 &> 0 \\ \alpha^2 - 3a_3^2 + a_1 &> 0 \end{aligned} \tag{6.11.28}$$

Since conditions (6.11.28) hold true for each available pole, it means the harmonic function $\mathscr{H}_3$ has actually no pole and is therefore equal to some constant. The boundary condition of asymptotic flatness fixes the value of such a constant:

$$\lim_{r\to\infty} \exp[-U] = 1 \quad \Leftrightarrow \quad \mathscr{H}_3 = \frac{\sqrt{\alpha^2 + \mathscr{H}_1(\infty) - 1}}{\sqrt{3}} \tag{6.11.29}$$

Under such conditions in the vicinity of each pole $\mathbf{x}_\alpha$, the warp factor has the following behavior:

$$|\mathbf{x} - \mathbf{x}_\alpha|^2 \exp[-U] \overset{\mathbf{x}\to\mathbf{x}_\alpha}{\sim} \sqrt{b_1}\,|\mathbf{x} - \mathbf{x}_\alpha|^{3/2} + \mathscr{O}\left(|\mathbf{x} - \mathbf{x}_\alpha|^{5/2}\right) \tag{6.11.30}$$

leading to a vanishing horizon area:

$$\text{Area}_{\,\mathrm{H}_\alpha} = \lim_{\mathbf{x}\to\mathbf{x}_\alpha} |\mathbf{x} - \mathbf{x}_\alpha|^2 \exp[-U] = 0 \tag{6.11.31}$$

At the same time using the form of the electromagnetic current in Eq. (6.11.24) and the behavior of the harmonic function in the vicinity of the poles we obtain the charge vector of each black hole encompassed by the solution:

$$\mathscr{Q}_\alpha = \int_{\mathbb{S}^2_\alpha} j^{EM} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}}\, q_\alpha \end{pmatrix} \quad ; \quad \text{where } q_\alpha = b_1 \text{ for pole } \mathbf{x}_\alpha \tag{6.11.32}$$

Summarizing

For the regular multicenter solutions associated with the orbit 4|11 all blacks holes localized at each pole are of the same type, namely they are very small black holes with vanishing horizon area and a charge vector $\mathscr{Q}$ belonging to **W**-orbit which is characterized by both a vanishing quartic invariant and the existence of a continuous parabolic stability subgroup of $\mathrm{SL}(2, \mathbb{R})$. Every black hole is a repetition in a different place of the spherical symmetric black hole which gives its name to the orbit.

6.11.2 The Small Black Holes of $\mathscr{O}^2_{11}$

Next let us consider the orbit $\mathscr{O}^2_{11}$.

W-Representation

Applying the same strategy as in the previous case, from the general formula we obtain

$$\mathscr{Q}^{\mathbf{w}}_{2|11} = \mathrm{Tr}(X_{2|11}\mathscr{T}^{\mathbf{w}}) = \left(\sqrt{3}, 0, 0, 0\right) \tag{6.11.33}$$

Substituting such a result in the expression for the quartic symplectic invariant (see Eq. (6.11.3) we find:

$$\mathfrak{I}_4 = 0 \tag{6.11.34}$$

Just as before we stress that this result is meaningful since, by calculating the trace $\mathrm{Tr}(X_{2|11}L_+^E) = 0$, we can also check that the Taub-NUT charge vanishes. Addressing the question whether there are subgroups of the original duality group in four-dimensions SL(2, R) that leave the charge vector (6.11.33) invariant we realize that such a group contains only the identity

$$\mathrm{SL}(2,\mathbb{R}) \supset \mathscr{S}_{2|11} = \mathbf{1} \tag{6.11.35}$$

Hence we clearly establish the intrinsic difference between the two type of small black holes at the level of the **W**-representation. Both have vanishing quartic invariant, yet only the orbit 4|11 has a residual symmetry.

H*-Stability Subgroup

Considering next the stability subgroup of the nilpotent element $X_{2|11}$ in $\mathrm{H}^\star = \widehat{\mathfrak{sl}(2,\mathbb{R})} \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathrm{h}^\star}$ we obtain:

$$\mathfrak{S}_{2|11} = \mathrm{SO}(1,1) \triangleright \mathbb{R} \tag{6.11.36}$$

A generic element of the corresponding Lie algebra is a linear combination of two generators J, T, satisfying the commutation relations:

$$[J\,,\,T] = \frac{3}{2\sqrt{6}}\,T \tag{6.11.37}$$

We do not give its explicit form which we do not use in the sequel.

Nilpotent Algebra $\mathbb{N}_{4|11}$

Considering next the adjoint action of the central element $h_{2|11}$ on the subspace $\mathbb{K}^\star$ we find that its eigenvalues are the following ones:

$$\mathrm{Eigenvalues}^{\mathbb{K}^\star}_{4|11} = \{-3, 3, -2, 2, -1, 1, 0, 0\} \tag{6.11.38}$$

Therefore the three eigenoperators A_3, A_2, A_1 corresponding to the positive eigenvalues 3, 2, 1, respectively, form the restriction to $\mathbb{K}^\star$ of a nilpotent algebra $\mathbb{N}_{2|11}$. In this case A_i do not all commute among themselves so that, differently from the previous case we have $\mathbb{N}_{4|11} \neq \mathbb{N}_{4|11} \bigcap \mathbb{K}^\star$. In particular we find a new generator:

$$B \in \mathbb{H}^\star \tag{6.11.39}$$

which completes a four-dimensional algebra with the following commutation relations:

$$
\begin{aligned}
0 &= [A_3\,,\ A_2] \;=\; [A_1\,,\ A_3] && (6.11.40)\\
B &= [A_2\,,\ A_1] \\
0 &= [B\,,\ A_1] \\
0 &= [B\,,\ A_2] \\
0 &= [B\,,\ A_3] && (6.11.41)
\end{aligned}
$$

As in the previous case, the structure of the nilpotent algebra implies that for the orbit $\mathscr{O}^2_{11}$ we have only three functions $\mathfrak{h}^0_i$ which will be harmonic and independent. This is so because $\mathscr{D}^2\mathbb{N}_{2|11} \;=\; 0$ and $\mathscr{D}\mathbb{N}_{2|11} \bigcap \mathbb{K}^\star \;=\; 0$.

Explicitly we set:

$$\mathfrak{H}(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3) \;=\; \sum_{i=1}^3 \mathfrak{h}_i\, A_i \;=$$

$$
\begin{pmatrix}
-\mathfrak{h}_2 & \mathfrak{h}_1-\mathfrak{h}_3 & \mathfrak{h}_2 & -\sqrt{2}\mathfrak{h}_1-\sqrt{2}\mathfrak{h}_3 & 0 & -3\mathfrak{h}_1-\mathfrak{h}_3 & 0\\
\mathfrak{h}_1-\mathfrak{h}_3 & -2\mathfrak{h}_2 & \mathfrak{h}_3-3\mathfrak{h}_1 & -\sqrt{2}\mathfrak{h}_2 & \mathfrak{h}_1+\mathfrak{h}_3 & 0 & -3\mathfrak{h}_1-\mathfrak{h}_3\\
-\mathfrak{h}_2 & 3\mathfrak{h}_1-\mathfrak{h}_3 & \mathfrak{h}_2 & \sqrt{2}\mathfrak{h}_1-\sqrt{2}\mathfrak{h}_3 & 0 & -\mathfrak{h}_1-\mathfrak{h}_3 & 0\\
\sqrt{2}\mathfrak{h}_1+\sqrt{2}\mathfrak{h}_3 & \sqrt{2}\mathfrak{h}_2 & \sqrt{2}\mathfrak{h}_1-\sqrt{2}\mathfrak{h}_3 & 0 & \sqrt{2}\mathfrak{h}_1-\sqrt{2}\mathfrak{h}_3 & -\sqrt{2}\mathfrak{h}_2 & \sqrt{2}\mathfrak{h}_1+\sqrt{2}\mathfrak{h}_3\\
0 & -\mathfrak{h}_1-\mathfrak{h}_3 & 0 & \sqrt{2}\mathfrak{h}_1-\sqrt{2}\mathfrak{h}_3 & -\mathfrak{h}_2 & 3\mathfrak{h}_1-\mathfrak{h}_3 & \mathfrak{h}_2\\
-3\mathfrak{h}_1-\mathfrak{h}_3 & 0 & \mathfrak{h}_1+\mathfrak{h}_3 & \sqrt{2}\mathfrak{h}_2 & \mathfrak{h}_3-3\mathfrak{h}_1 & 2\mathfrak{h}_2 & \mathfrak{h}_1-\mathfrak{h}_3\\
0 & -3\mathfrak{h}_1-\mathfrak{h}_3 & 0 & -\sqrt{2}\mathfrak{h}_1-\sqrt{2}\mathfrak{h}_3 & -\mathfrak{h}_2 & \mathfrak{h}_1-\mathfrak{h}_3 & \mathfrak{h}_2
\end{pmatrix}
\tag{6.11.42}
$$

Considering $\mathfrak{H}(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$ as a Lax operator and calculating its Taub-NUT charge and electromagnetic charges we find:

$$\mathbf{n}_{TN} \;=\; -2\,(3\mathfrak{h}_1+\mathfrak{h}_3) \quad ; \quad \mathscr{Q} \;=\; \left\{-2\sqrt{3}\mathfrak{h}_2, 6\mathfrak{h}_1-2\mathfrak{h}_3, -2\sqrt{3}\,(\mathfrak{h}_1+\mathfrak{h}_3)\,, 0\right\} \tag{6.11.43}$$

This implies that constructing the multi-centre solution with harmonic functions the condition $\mathfrak{h}_3 \;=\; -\,3\,\mathfrak{h}_1$ might be sufficient to annihilate the Taub-NUT current. We shall demonstrate that in this case the condition is slightly more complicated.

For later convenience let us change the normalization in the basis of harmonic functions as follows:

$$\mathfrak{h}_1^{(0)} \;=\; \tfrac{1}{4}\mathscr{H}_3 \quad ; \quad \mathfrak{h}_2^{(0)} \;=\; \tfrac{1}{2}\,(1\;-\;\mathscr{H}_2) \quad ; \quad \mathfrak{h}_3^{(0)} \;=\; \tfrac{1}{4}\mathscr{H}_1 \tag{6.11.44}$$

Implementing the symmetric coset construction with:

$$\mathscr{Y}\,(\mathscr{H}_3, \mathscr{H}_2, \mathscr{H}_1) \;\equiv\; \exp\left[\mathfrak{H}\left(\tfrac{1}{4}\mathscr{H}_3, \tfrac{1}{2}\,(1\;-\;\mathscr{H}_2)\,, \tfrac{1}{4}\mathscr{H}_1\right)\right] \tag{6.11.45}$$

calculating the upper triangular coset representative $\mathbb{L}(\mathscr{Y})$ according to equations (6.8.26) and extracting the σ-model scalar fields we obtain the answer which we list below:

$$\exp\left[-U\right] = \frac{1}{2}\sqrt{-\mathscr{H}_3^2 + \left(4\mathscr{H}_1^3 + 6\mathscr{H}_2\mathscr{H}_1\right)\mathscr{H}_3 + \mathscr{H}_2^2\left(3\mathscr{H}_1^2 + 4\mathscr{H}_2\right)} \tag{6.11.46}$$

$$\mathrm{Im}\, z = \frac{\sqrt{-\mathscr{H}_3^2 + \left(4\mathscr{H}_1^3 + 6\mathscr{H}_2\mathscr{H}_1\right)\mathscr{H}_3 + \mathscr{H}_2^2\left(3\mathscr{H}_1^2 + 4\mathscr{H}_2\right)}}{2\left(\mathscr{H}_1^2 + \mathscr{H}_2\right)} \tag{6.11.47}$$

$$\mathrm{Re}\, z = \frac{\mathscr{H}_3 - \mathscr{H}_2\mathscr{H}_1}{2\left(\mathscr{H}_1^2 + \mathscr{H}_2\right)} \tag{6.11.48}$$

$$Z^M = \begin{pmatrix} \frac{\sqrt{\frac{3}{2}}\left(\mathscr{H}_3{}^2 - 2\mathscr{H}_1\left(2\mathscr{H}_1{}^2 + 3\mathscr{H}_2 - 1\right)\mathscr{H}_3 + \mathscr{H}_2\left(-4\mathscr{H}_2{}^2 + \left(4 - 3\mathscr{H}_1{}^2\right)\mathscr{H}_2 + 2\mathscr{H}_1{}^2\right)\right)}{\mathscr{H}_3{}^2 - 2\left(2\mathscr{H}_1{}^3 + 3\mathscr{H}_2\mathscr{H}_1\right)\mathscr{H}_3 - \mathscr{H}_2{}^2\left(3\mathscr{H}_1{}^2 + 4\mathscr{H}_2\right)} \\ \frac{\sqrt{2}\left(2\mathscr{H}_1{}^3 + 3\mathscr{H}_2\mathscr{H}_1 - \mathscr{H}_3\right)}{-\mathscr{H}_3{}^2 + \left(4\mathscr{H}_1{}^3 + 6\mathscr{H}_2\mathscr{H}_1\right)\mathscr{H}_3 + \mathscr{H}_2{}^2\left(3\mathscr{H}_1{}^2 + 4\mathscr{H}_2\right)} \\ \frac{\sqrt{6}\left(\mathscr{H}_1\mathscr{H}_2{}^2 + \mathscr{H}_3\left(2\mathscr{H}_1{}^2 + \mathscr{H}_2\right)\right)}{\mathscr{H}_3{}^2 - 2\left(2\mathscr{H}_1{}^3 + 3\mathscr{H}_2\mathscr{H}_1\right)\mathscr{H}_3 - \mathscr{H}_2{}^2\left(3\mathscr{H}_1{}^2 + 4\mathscr{H}_2\right)} \\ \frac{4\mathscr{H}_3\mathscr{H}_1{}^3 + 3\mathscr{H}_2{}^2\mathscr{H}_1{}^2 + \mathscr{H}_3{}^2}{\sqrt{2}\left(-\mathscr{H}_3{}^2 + \left(4\mathscr{H}_1{}^3 + 6\mathscr{H}_2\mathscr{H}_1\right)\mathscr{H}_3 + \mathscr{H}_2{}^2\left(3\mathscr{H}_1{}^2 + 4\mathscr{H}_2\right)\right)} \end{pmatrix} \tag{6.11.49}$$

$$a = \frac{\mathscr{H}_3\left(-6\mathscr{H}_1^2 - 3\mathscr{H}_2 + 1\right) - \mathscr{H}_1\left(3\mathscr{H}_2^2 + 3\mathscr{H}_2 + 2\mathscr{H}_1^2\right)}{\mathscr{H}_3^2 - 2\left(2\mathscr{H}_1^3 + 3\mathscr{H}_2\mathscr{H}_1\right)\mathscr{H}_3 - \mathscr{H}_2^2\left(3\mathscr{H}_1^2 + 4\mathscr{H}_2\right)} \tag{6.11.50}$$

The Taub-NUT Current

Given this explicit result we can turn to the explicit oxidation formulae described in Sect. 6.2.1 and calculate the Taub-NUT current which is the integrand of Eq. (6.2.17). We find:

$$j^{TN} = \tfrac{1}{2}\left({}^\star\nabla\mathscr{H}_3 + 3\left(\mathscr{H}_2{}^\star\nabla\mathscr{H}_1 - \mathscr{H}_1{}^\star\nabla\mathscr{H}_2\right)\right) \tag{6.11.51}$$

Analyzing Eq. (6.11.51) we see that there are just two possible solutions to the condition $j^{TN} = 0$:

(case a) $\mathscr{H}_3 = \beta = \text{const}$; $\mathscr{H}_1 = 0$. With this condition we obtain:

$$\exp[-U] = \frac{1}{2}\sqrt{4\mathscr{H}_2^3 - \beta^2} \tag{6.11.52}$$

$$z = \frac{\beta + \mathrm{i}\sqrt{4\mathscr{H}_2^3 - \beta^2}}{2\mathscr{H}_2} \tag{6.11.53}$$

$$j^{EM} = \star\nabla\begin{pmatrix} -\sqrt{\frac{3}{2}}\mathscr{H}_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6.11.54}$$

(case b) $\mathscr{H}_3 = \beta = \text{const}$; $\mathscr{H}_2 = 0$

$$\exp[-U] = \frac{1}{2}\sqrt{\beta\left(4\mathscr{H}_1^3 - \beta\right)} \tag{6.11.55}$$

$$z = \frac{\beta + \mathrm{i}\sqrt{\beta\left(4\mathscr{H}_1^3 - \beta\right)}}{2\mathscr{H}_3^2} \tag{6.11.56}$$

$$j^{EM} = \begin{pmatrix} 0 \\ 0 \\ -\sqrt{\frac{3}{2}}\mathscr{H}_1 \\ 0 \end{pmatrix} \tag{6.11.57}$$

It might seem that these two solutions correspond to different types of black holes but this is not the case, as we now show. From the asymptotic flatness boundary condition we find that the value of β is fixed in terms of the value at infinity of the corresponding harmonic function $\mathscr{H}_{1,2}$, which of course must satisfy the necessary condition for reality of the solution $\mathscr{H}_{1,2}(\infty) \geq 1$:

$$\begin{cases} \beta = \quad 2\sqrt{[\mathscr{H}_2(\infty)]^3 - 1} & \text{case a} \\ \beta = 2\left([\mathscr{H}_1(\infty)]^3 + \sqrt{[\mathscr{H}_1(\infty)]^6 - 1}\right) & \text{case b} \end{cases} \tag{6.11.58}$$

In the vicinity of a pole by means of the usual argument we obtain the following behavior of the warp factor:

$$|\mathbf{x} - \mathbf{x}_\alpha|^2 \exp[-U] \overset{\mathbf{x}\to\mathbf{x}_\alpha}{\sim} \begin{cases} \sqrt{b_2^3}\sqrt{|\mathbf{x} - \mathbf{x}_\alpha|} + \mathscr{O}\left(|\mathbf{x} - \mathbf{x}_\alpha|^{3/2}\right) & : \text{case a} \\ \sqrt{\beta\, b_1^3\sqrt{|\mathbf{x} - \mathbf{x}_\alpha|}} + \mathscr{O}\left(|\mathbf{x} - \mathbf{x}_\alpha|^{3/2}\right) & : \text{case b} \end{cases} \tag{6.11.59}$$

Hence in both cases the horizon area vanishes at all poles $\mathbf{x}_\alpha$ and the reality conditions are satisfied choosing the appropriate sign of $b_{1,2}$. The charge vector has the same structure for all black holes encompassed in the first or in the second solution, namely:

$$\mathscr{Q}_\alpha = \begin{cases} \left\{-\sqrt{\frac{3}{2}}\, p_\alpha\,,\, 0\,,\, 0\,,\, 0\right\} : p_\alpha = b_2 & \text{for pole } \alpha \\ \left\{0\,,\, 0\,,\, -\sqrt{\frac{3}{2}}\, q_\alpha\,,\, 0\right\} : q_\alpha = b_1 & \text{for pole } \alpha \end{cases} \tag{6.11.60}$$

In both cases the quartic invariant $\mathfrak{I}_4$ is zero for all black holes in the solutions, yet one might still doubt whether the **W**-orbit for the two cases might be different. It is not so, since a direct calculation shows that the image in the $j = \frac{3}{2}$ representation $\Lambda[\mathfrak{A}]$[13] of the following $\mathrm{SL}(2,\mathbb{R})$ element:

$$\mathfrak{A} = \begin{pmatrix} 0 & \frac{p}{q} \\ -\frac{q}{p} & 0 \end{pmatrix} \tag{6.11.61}$$

[13] See [32] for details, in particular Eq. (3.13) of that reference for the explicit form of the spin $\frac{3}{2}$ matrices.

maps the charge vector $\mathscr{Q}_{[q]} = \{0\,,\,0\,,\,-q\,,\,0\}$, into the charge vector $\mathscr{Q}_{[p]} = \{p\,,\,0\,,\,0\,,\,0\}$, namely we have $\Lambda[\mathfrak{A}]\,\mathscr{Q}_{[q]} = \mathscr{Q}_{[p]}$. Hence the two solutions we have here discussed simply give different representatives of the same **W**-orbit.

SUMMARY

Just as in the previous case for a multicenter solution associated with the $\mathscr{O}^2_{11}$ orbit all the black holes included in one solution are of the same type, namely small black holes with the same identical properties.

6.11.3 The Large BPS Black Holes of $\mathscr{O}^3_{11}$

Next let us consider the orbit $\mathscr{O}^3_{11}$, which in the spherical symmetric case leads to BPS Black holes with a finite horizon area.

W-Representation

In order to better appreciate the structure of these solutions, let us slightly generalize our orbit representative, writing the following nilpotent matrix that depends on two parameters (p, q) to be interpreted later as the magnetic and the electric charge of the hole:

$$X_{3|11}(p,q) = \begin{pmatrix} q & 0 & 0 & -\frac{q}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{p+q}{2} & -\frac{p}{2} & 0 & \frac{q}{2} & 0 & 0 \\ 0 & \frac{p}{2} & \frac{q-p}{2} & 0 & 0 & -\frac{q}{2} & 0 \\ \frac{q}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{q}{\sqrt{2}} \\ 0 & -\frac{q}{2} & 0 & 0 & \frac{p-q}{2} & \frac{p}{2} & 0 \\ 0 & 0 & \frac{q}{2} & 0 & -\frac{p}{2} & \frac{1}{2}(-p-q) & 0 \\ 0 & 0 & 0 & -\frac{q}{\sqrt{2}} & 0 & 0 & -q \end{pmatrix} \quad (6.11.62)$$

The standard triple representative mentioned in Eq. (6.10.5) is just the particular case $X_{3|11}(1, 1)$. Applying the same strategy as in the previous case, from the general formula we obtain

$$\mathscr{Q}^{\mathbf{w}}_{3|11} = \mathrm{Tr}(X_{3|11}(p,q)\mathscr{T}^{\mathbf{w}}) = \left(0,\, p,\, -\sqrt{3}q,\, 0\right) \quad (6.11.63)$$

Substituting such a result in the expression for the quartic symplectic invariant (see Eq. (6.11.3)) we find:

$$\mathfrak{I}_4 = 9\,p\,q^3 > 0 \quad \text{if } p \text{ and } q \text{ have the same sign} \quad (6.11.64)$$

Just as before we stress that this result is meaningful since, by calculating the trace $\mathrm{Tr}(X_{3|11}L^E_+) = 0$, we can also check that the Taub-NUT charge vanishes. Furthermore we note that the condition that p and q have the same sign was singled out in [32] as the defining condition of the orbit O^3_{11} which, in the spherical symmetry approach leads to regular BPS solutions. The choice of opposite signs was proved in [32] to correspond to a different H* orbit, the non diagonal O^3_{21} which instead contains only singular solutions. Here we will show another important and intrinsically four dimensional reason to separate the two cases.

Addressing the question whether there are subgroups of the original duality group in four-dimensions SL(2, R) that leave the charge vector (6.11.63) invariant we realize that such a subgroup exists and is the finite cyclic group of order three:

$$\mathrm{SL}(2,\mathbb{R}) \supset \mathscr{S}_{3|11} = \mathbb{Z}_3 \tag{6.11.65}$$

$\mathscr{S}_{3|11}$ is made by the following three elements:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.11.66}$$

$$\mathfrak{B} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2}\sqrt{\frac{p}{q}} \\ \frac{\sqrt{3}}{2}\sqrt{\frac{q}{p}} & -\frac{1}{2} \end{pmatrix} \tag{6.11.67}$$

$$\mathfrak{B}^2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}\sqrt{\frac{p}{q}} \\ -\frac{\sqrt{3}}{2}\sqrt{\frac{q}{p}} & -\frac{1}{2} \end{pmatrix} \quad ; \quad \mathfrak{B}^3 = \mathbf{1} \tag{6.11.68}$$

It is evident that such a $\mathbb{Z}_3$ subgroup exists if and only if the two charges p, q have the same sign. Otherwise the corresponding matrices develop imaginary elements and migrate to SL(2, $\mathbb{C}$). The existence of this isotropy group $\mathbb{Z}_3$ can be considered the very definition of the **W**-orbit corresponding to BPS black holes. Indeed let us name $\lambda = \sqrt{\frac{p}{q}}$ and consider the algebraic condition imposed on a generic charge vector: $\mathscr{Q} = \{Q_1, Q_2, Q_3, Q_4\}$ by the request that it should admit the above described $\mathbb{Z}_3$ stability group:

$$\Lambda[\mathfrak{B}]\mathscr{Q} = \mathscr{Q} \quad \Leftrightarrow \quad \mathscr{Q} = \left(\sqrt{3}\lambda^2 Q_4, -\frac{\lambda^2 Q_3}{\sqrt{3}}, Q_3, Q_4\right) \tag{6.11.69}$$

It is evident from the above explicit result that the charge vectors having this symmetry depend only on three parameters (λ^2, Q_3, Q_4). The very relevant fact is that substituting this restricted charge vector in the general formula (6.11.3) for the quartic invariant we obtain:

$$\mathfrak{I}_4 = \lambda^2 \left(Q_3^2 + 3\lambda^2 Q_4^2\right)^2 > 0 \tag{6.11.70}$$

Hence the $\mathbb{Z}_3$ guarantees that the quartic invariant is a perfect square and hence positive. It is an intrinsic restriction characterizing the $\mathbb{W}$-orbit.

H*-Stability Subgroup

Considering next the stability subgroup of the nilpotent element $X_{3|11}(1,1)$ in H* $=$ $\widehat{\mathfrak{sl}(2,\mathbb{R})} \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathrm{h}^\star}$ we obtain:

$$\mathfrak{S}_{3|11} \;=\; \mathbb{R} \tag{6.11.71}$$

the group being generated by a matrix $\mathbb{A}_{3|11}$ of nilpotency degree 2:

$$\mathbb{A}_{3|11}^2 \;=\; \mathbf{0} \tag{6.11.72}$$

We do not give its explicit form which we do not use in the sequel.

Nilpotent Algebra $\mathbb{N}_{3|11}$

Considering next the adjoint action of the central element $h_{3|11}$ on the subspace $\mathbb{K}^\star$ we find that its eigenvalues are the following ones:

$$\text{Eigenvalues}_{3|11}^{\mathbb{K}^\star} \;=\; \{-2,-2,-2,-2,2,2,2,2\} \tag{6.11.73}$$

Therefore the four eigenoperators A_1, A_2, A_3, A_4 corresponding to the four positive eigenvalues 2, respectively, form the restriction to $\mathbb{K}^\star$ of a nilpotent algebra $\mathbb{N}_{3|11}$. Also in this case the A_i do not all commute among themselves so that, we have $\mathbb{N}_{3|11} \neq \mathbb{N}_{3|11} \bigcap \mathbb{K}^\star$. In particular we find a new generator:

$$B \;\in\; \mathbb{H}^\star \tag{6.11.74}$$

which completes a five-dimensional algebra with the following commutation relations:

$$\begin{aligned}
\left[A_i\,,\;A_j\right] &= \Omega_{ij}\,B \\
[B\,,\;A_i] &= 0 \\
B &= \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{6.11.75}$$

The structure of the nilpotent algebra implies that for the orbit $\mathscr{O}^3_{11}$ we have only four functions $\mathfrak{h}_i^0$ which will be harmonic and independent. This is so because $\mathscr{D}^2\mathbb{N}_{3|11} = 0$ and $\mathscr{D}\mathbb{N}_{3|11} \bigcap \mathbb{K}^\star \;=\; 0$.

Explicitly we set:

$$\mathfrak{H}(\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3,\mathfrak{h}_4) \;=\; \sum_{i=1}^4 \mathfrak{h}_i\,A_i \;=$$

$$\begin{pmatrix} 2\mathfrak{h}_3 & \mathfrak{h}_1-2\mathfrak{h}_2 & 2\mathfrak{h}_1-\mathfrak{h}_2 & -\sqrt{2}\mathfrak{h}_3 & -3\mathfrak{h}_2 & -3\mathfrak{h}_1 & 0 \\ \mathfrak{h}_1-2\mathfrak{h}_2 & \mathfrak{h}_3-\mathfrak{h}_4 & \mathfrak{h}_4 & \sqrt{2}\mathfrak{h}_2-2\sqrt{2}\mathfrak{h}_1 & \mathfrak{h}_3 & 0 & -3\mathfrak{h}_1 \\ \mathfrak{h}_2-2\mathfrak{h}_1 & -\mathfrak{h}_4 & \mathfrak{h}_3+\mathfrak{h}_4 & \sqrt{2}\mathfrak{h}_1-2\sqrt{2}\mathfrak{h}_2 & 0 & -\mathfrak{h}_3 & -3\mathfrak{h}_2 \\ \sqrt{2}\mathfrak{h}_3 & 2\sqrt{2}\mathfrak{h}_1-\sqrt{2}\mathfrak{h}_2 & \sqrt{2}\mathfrak{h}_1-2\sqrt{2}\mathfrak{h}_2 & 0 & \sqrt{2}\mathfrak{h}_1-2\sqrt{2}\mathfrak{h}_2 & \sqrt{2}\mathfrak{h}_2-2\sqrt{2}\mathfrak{h}_1 & \sqrt{2}\mathfrak{h}_3 \\ 3\mathfrak{h}_2 & -\mathfrak{h}_3 & 0 & \sqrt{2}\mathfrak{h}_1-2\sqrt{2}\mathfrak{h}_2 & -\mathfrak{h}_3-\mathfrak{h}_4 & -\mathfrak{h}_4 & 2\mathfrak{h}_1-\mathfrak{h}_2 \\ -3\mathfrak{h}_1 & 0 & \mathfrak{h}_3 & 2\sqrt{2}\mathfrak{h}_1-\sqrt{2}\mathfrak{h}_2 & \mathfrak{h}_4 & \mathfrak{h}_4-\mathfrak{h}_3 & \mathfrak{h}_1-2\mathfrak{h}_2 \\ 0 & -3\mathfrak{h}_1 & 3\mathfrak{h}_2 & -\sqrt{2}\mathfrak{h}_3 & \mathfrak{h}_2-2\mathfrak{h}_1 & \mathfrak{h}_1-2\mathfrak{h}_2 & -2\mathfrak{h}_3 \end{pmatrix} \tag{6.11.76}$$

Considering $\mathfrak{H}(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4)$ as a Lax operator and calculating its Taub-NUT charge and electromagnetic charges we find:

$$\mathbf{n}_{TN} = -6\mathfrak{h}_1 \quad ; \quad \mathscr{Q} = \left\{ 2\sqrt{3}\,(\mathfrak{h}_2 - 2\mathfrak{h}_1)\,, -2\mathfrak{h}_4, -2\sqrt{3}\mathfrak{h}_3, -6\mathfrak{h}_2 \right\} \tag{6.11.77}$$

This implies that constructing the multi-centre solution with harmonic functions the condition $\mathfrak{h}_1 = 0$ might be sufficient to annihilate the Taub-NUT current. We shall demonstrate that also in this case the condition is slightly more complicated. This emphasizes the difference between the Lax operator one-dimensional approach and the multicenter construction based on harmonic functions.

For later convenience let us change the normalization in the basis of harmonic functions as follows:

$$\mathfrak{h}_1^{(0)} = \frac{1}{\sqrt{12}}\mathscr{H}_1 \quad ; \quad \mathfrak{h}_2^{(0)} = \frac{1}{\sqrt{12}}\mathscr{H}_2 \quad ; \quad \mathfrak{h}_3^{(0)} = \frac{1}{2}\,(\mathscr{H}_3 - 1) \quad ; \quad \mathfrak{h}_4^{(0)} = \frac{1}{2}\,(\mathscr{H}_4 + 1) \tag{6.11.78}$$

Implementing the symmetric coset construction with:

$$\mathscr{Y}\,(\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3, \mathscr{H}_4) \equiv \exp\left[\mathfrak{H}\left(\frac{1}{\sqrt{12}}\mathscr{H}_1, \frac{1}{\sqrt{12}}\mathscr{H}_2, \frac{1}{2}\,(\mathscr{H}_3 - 1)\,, \frac{1}{2}\,(\mathscr{H}_4 + 1) \right) \right] \tag{6.11.79}$$

calculating the upper triangular coset representative $\mathbb{L}(\mathscr{Y})$ according to Eq. (6.8.26) and extracting the σ-model scalar fields we obtain an explicit but rather messy answer which we omit. In particular we obtain the Taub-NUT current in the following form:

$$j^{TN} = \sum_{i=1}^{4} \mathfrak{R}_i\,(\mathscr{H})\;\nabla\mathscr{H}_i \tag{6.11.80}$$

where $\mathfrak{R}_i\,(\mathscr{H})$ are rational functions of the four harmonic functions, the maximal degree of involved polynomials being 16. A priori, imposing the vanishing of the Taub-NUT current is a problem without guaranteed solutions. In the 4-dimensional linear space of the harmonic functions we can introduce r-linear relations of the form:

$$0 = V_\alpha^i\,\mathscr{H}_i \quad ; \quad \alpha = 1, \ldots, r \tag{6.11.81}$$

Let U^i_a be a set of $4 - r$ linear independent 4-vectors orthogonal to the vectors V^i_α. Then it must happen that on the locus defined by Eqs. (6.11.81), the following rational functions should also vanish

$$0 = \mathfrak{P}_a(\mathscr{H}) \equiv U^i_a \, \mathfrak{R}_i(\mathscr{H}) \quad ; \quad (a = 1, \ldots, r-4) \tag{6.11.82}$$

For generic rational functions this will never happen, yet we know that for our system such solutions should exist and in want of a clear cut algorithm it is a matter of ingenuity to find them. We do not find any solution with $r = 1$ but we find two nice solutions with $r = 2$. They are the following ones:

(a) $\mathscr{H}_1 = \mathscr{H}_2 = 0$. The complete form of the supergravity solution corresponding to this choice is:

$$\exp[-U] = \sqrt{-\mathscr{H}_3^3 \mathscr{H}_4} \tag{6.11.83}$$

$$z = \mathrm{i} \, \frac{\sqrt{-\mathscr{H}_3^3 \mathscr{H}_4}}{\mathscr{H}_3^2} \tag{6.11.84}$$

$$j^{TN} = 0 \tag{6.11.85}$$

$$j^{EM} = \star\nabla \begin{pmatrix} 0 \\ \frac{\mathscr{H}_4}{\sqrt{2}} \\ \sqrt{\frac{3}{2}} \mathscr{H}_3 \\ 0 \end{pmatrix} \tag{6.11.86}$$

(b) $\mathscr{H}_1 = 0$, $\mathscr{H}_3 = -\mathscr{H}_4$. The complete form of the supergravity solution corresponding to this choice is:

$$\exp[-U] = \sqrt{-\frac{\mathscr{H}_2^4}{3} - 2\mathscr{H}_4^2 \mathscr{H}_2^2 + \mathscr{H}_4^4} \tag{6.11.87}$$

$$z = \frac{2\mathscr{H}_2 \mathscr{H}_4 - \mathrm{i}\sqrt{-\mathscr{H}_2^4 - 6\mathscr{H}_4^2 \mathscr{H}_2^2 + 3\mathscr{H}_4^4}}{\sqrt{3}\left(\mathscr{H}_2^2 - \mathscr{H}_4^2\right)} \tag{6.11.88}$$

$$j^{TN} = 0 \tag{6.11.89}$$

$$j^{EM} = \star\nabla \begin{pmatrix} -\frac{\mathscr{H}_2}{\sqrt{2}} \\ \frac{\mathscr{H}_4}{\sqrt{2}} \\ -\sqrt{\frac{3}{2}} \mathscr{H}_4 \\ \sqrt{\frac{3}{2}} \mathscr{H}_2 \end{pmatrix} \tag{6.11.90}$$

We can now make some comments about the two solutions. First of all both in case a) and in case b) we have to fix the asymptotic value of the harmonic functions at spatial infinity $r = \infty$, in such a way as to obtain asymptotic flatness. This is quite easy and we do not dwell on it. Secondly we have to fix the parameters of the harmonic functions in such a way that the warp factor is always real on the whole physical range. These conditions are also easily spelled out:

$$\begin{array}{llr} a) & -\mathscr{H}_3\mathscr{H}_4 & > 0 \\ b) & -\frac{\mathscr{H}_2^4}{3} - 2\mathscr{H}_4^2\mathscr{H}_2^2 + \mathscr{H}_4^4 & > 0 \end{array} \tag{6.11.91}$$

and in a multicenter solution can be easily arranged adjusting the coefficients of each pole. Thirdly we can comment about the structure of the charge vector that we obtain at each pole:

$$\mathscr{H}_i \sim a_i + \frac{Q_i}{|x - x_\alpha|} \tag{6.11.92}$$

In case (a) and (b) we respectively obtain:

$$\mathscr{Q}_\alpha = \begin{pmatrix} 0 \\ \frac{Q_4}{\sqrt{2}} \\ \sqrt{\frac{3}{2}}Q_3 \\ 0 \end{pmatrix} \tag{6.11.93}$$

$$\mathscr{Q}_\alpha = \begin{pmatrix} -\frac{Q_2}{\sqrt{2}} \\ \frac{Q_4}{\sqrt{2}} \\ -\sqrt{\frac{3}{2}}Q_4 \\ \sqrt{\frac{3}{2}}Q_2 \end{pmatrix} \tag{6.11.94}$$

Comparing with Eqs. (6.11.69), (6.11.70) we see that in both cases the structure of these charges is that imposed by the $\mathbb{Z}_3$ invariance which characterizes BPS black holes. The necessary choice of signs in the case (a)

$$\frac{Q_4}{Q_3} < 0 \tag{6.11.95}$$

is the same which is required by the reality of the warp factor. Hence in case (b) all the black holes encompassed by the solution at each pole are finite area BPS black holes. In case (a) the same is true for all the poles common to the harmonic function $\mathscr{H}_3$ and $\mathscr{H}_4$: they are finite area BPS black holes. Yet we can envisage the situation where some poles of $\mathscr{H}_3$ are not shared by $\mathscr{H}_4$ and viceversa. In this case the pole of $\mathscr{H}_4$ defines a very small black hole, while the pole of $\mathscr{H}_3$ defines a small black hole. This is confirmed by the fact that a charge vector of type $\{0, p, 0, 0\}$ is mapped into $\{0, 0, 0, p\}$ by $\Lambda\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]$ and as such admits a parabolic subgroup of stability $\Lambda\left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right]$.

Summary

For a multicenter solution associated with the $\mathscr{O}^3_{11}$ orbit there are two possibilities either all the black holes included in one solution are regular, finite area, BPS black holes, either we have a mixture of very small and small black holes. A finite area BPS black hole emerges when the center of a very small black hole coincides with the center of a small one. This provides the challenging suggestion that a BPS black hole can be considered quantum mechanically as a composite object where the "quarks" are small and very small black holes.

6.11.4 BPS Kerr–Newman Solution

Next we want to show how this orbit encompasses also the BPS Kerr–Newman solution that was found by Luest et al. in [39].

To this effect we go back to the general formulae for the scalar fields in this orbit and we make the following reduction from four to two independent harmonic functions:

$$\mathscr{H}_2 = 0 \quad ; \quad \mathscr{H}_4 = -\tfrac{1}{3}\,\mathscr{H}_3 \tag{6.11.96}$$

With such a choice the expressions for all the scalar fields dramatically simplify and we obtain:

$$\mathfrak{W} = \frac{\sqrt{3}}{\mathscr{H}_1^2 + \mathscr{H}_3^2} \tag{6.11.97}$$

$$z = \mathrm{i}\,\frac{1}{\sqrt{3}} \tag{6.11.98}$$

$$Z = \begin{pmatrix} -\frac{3\mathscr{H}_1}{\sqrt{2}\left(\mathscr{H}_1^2+\mathscr{H}_3^2\right)} \\ \frac{\mathscr{H}_1^2+(\mathscr{H}_3-3)\mathscr{H}_3}{\sqrt{2}\left(\mathscr{H}_1^2+\mathscr{H}_3^2\right)} \\ -\frac{\sqrt{\frac{3}{2}}\left(\mathscr{H}_1^2+(\mathscr{H}_3-1)\mathscr{H}_3\right)}{\mathscr{H}_1^2+\mathscr{H}_3^2} \\ -\frac{\mathscr{H}_1}{\sqrt{6}\left(\mathscr{H}_1^2+\mathscr{H}_3^2\right)} \end{pmatrix} \tag{6.11.99}$$

$$a = \frac{5\mathscr{H}_1}{\sqrt{3}\left(\mathscr{H}_1^2 + \mathscr{H}_3^2\right)} \tag{6.11.100}$$

Utilizing the above expressions in the final oxidation formulae we obtain the following result for the Taub-Nut current and for the electromagnetic currents:

$$j^{TN} = \frac{2\left(\star\nabla\mathscr{H}_1\mathscr{H}_3 - \star\nabla\mathscr{H}_3\mathscr{H}_1\right)}{\sqrt{3}} \tag{6.11.101}$$

$$j^{EM} = \begin{pmatrix} \frac{2\star\nabla\mathscr{H}_3\mathscr{H}_1\left(\mathscr{H}_1^2+(\mathscr{H}_3-2)\mathscr{H}_3\right)-\star\nabla\mathscr{H}_1\left((2\mathscr{H}_3+1)\mathscr{H}_1^2+\mathscr{H}_3^2(2\mathscr{H}_3-3)\right)}{\sqrt{2}\left(\mathscr{H}_1^2+\mathscr{H}_3^2\right)} \\ \frac{\star\nabla\mathscr{H}_3\left(3\mathscr{H}_1^2-\mathscr{H}_3^2\right)-4\star\nabla\mathscr{H}_1\mathscr{H}_1\mathscr{H}_3}{3\sqrt{2}\left(\mathscr{H}_1^2+\mathscr{H}_3^2\right)} \\ \frac{\sqrt{\frac{3}{2}}\left(4\star\nabla\mathscr{H}_1\mathscr{H}_1\mathscr{H}_3+\star\nabla\mathscr{H}_3\left(\mathscr{H}_3^2-3\mathscr{H}_1^2\right)\right)}{\mathscr{H}_1^2+\mathscr{H}_3^2} \\ \frac{2\star\nabla\mathscr{H}_3\mathscr{H}_1\left(\mathscr{H}_1^2+(\mathscr{H}_3-6)\mathscr{H}_3\right)-\star\nabla\mathscr{H}_1\left((2\mathscr{H}_3+3)\mathscr{H}_1^2+\mathscr{H}_3^2(2\mathscr{H}_3-9)\right)}{\sqrt{6}\left(\mathscr{H}_1^2+\mathscr{H}_3^2\right)} \end{pmatrix} \tag{6.11.102}$$

Next identifying the two harmonic functions with those introduced in Eqs. (6.9.21)–(6.9.24), according to:

$$\mathscr{H}_1 = 3^{\frac{1}{4}} \, (1 + m\,\mathscr{P}) \quad ; \quad \mathscr{H}_3 = 3^{\frac{1}{4}} \, m\,\mathscr{R} \tag{6.11.103}$$

we obtain the following result for the warp-factor:

$$\exp[U] = \frac{(m+r)^2 + \alpha^2 \cos^2(\theta)}{r^2 + \alpha^2 \cos^2(\theta)} \tag{6.11.104}$$

and for the Kaluza–Klein vector:

$$\mathbf{A}^{[KK]} = \omega \equiv \frac{m(m+2r)\alpha \sin^2(\theta)}{r^2 + \alpha^2 \cos^2(\theta)} \, d\phi \tag{6.11.105}$$

Indeed one can easily check that, in the spheroidal coordinates (6.9.19) with flat metric Eq. (6.9.20) we have:

$$2m \, (\star\nabla\mathscr{P}\,\mathscr{R} - \mathscr{P} \star\nabla\mathscr{R}) = \mathrm{d}\omega \tag{6.11.106}$$

where $\star\nabla$ denotes the Hodge dual of the exterior derivative d. Writing the corresponding final form of the metric:

$$ds^2_{BPSKN} = -\exp[U]\,(dt + \omega)^2 + \exp[-U]\,d\Omega^2_{spheroidal} \tag{6.11.107}$$

we can easily check that it is just the Kerr–Newman metric (6.5.2) with $q = m$. The only necessary step, in order to verify such an identity is a redefinition of the coordinate r. If in the metric (6.5.2) one replaces $r \to r + m$, then (6.5.2) becomes identical to (6.11.107).

It is interesting to consider the expressions for the vector field strengths that solve the Maxwell-Einstein system together with the BPS Kerr–Newman metric. For the first two field strengths (magnetic), from Eq. (6.11.102) we find:

$$F^1 = -\frac{1}{\sqrt{2}\left(r^2+\alpha^2\cos^2\theta\right)^2\left((m+r)^2+\alpha^2\cos^2\theta\right)}\left(\sqrt[4]{3}m\alpha\sin\theta\left(\left(\left(-3 \right.\right.\right.\right.$$
$$\left. +2\sqrt[4]{3}\right)\alpha^4\cos^4\theta$$
$$+m\left(2\sqrt[4]{3}m+m+2\left(1+\sqrt[4]{3}\right)r\right)\alpha^2\cos^2\theta$$
$$-r(m+r)^2\left(2\sqrt[4]{3}m+\left(-3+2\sqrt[4]{3}\right)r\right)\Big)\sin\theta dr\wedge d\phi$$
$$+2\left(r^2+\alpha^2\right)\cos\theta\left(\left(\left(\left(-2+\sqrt[4]{3}\right)m\right.\right.\right.$$
$$+\left(-3+2\sqrt[4]{3}\right)r\Big)\alpha^2\cos^2\theta+(m+r)\left(\sqrt[4]{3}m^2\right.$$
$$\left.+\left(-1+3\sqrt[4]{3}\right)rm+\left(-3+2\sqrt[4]{3}\right)r^2\right)\Big)d\theta\wedge d\phi\Big)\Big) \tag{6.11.108}$$

$$F^2 = \frac{1}{\sqrt{2}3^{3/4}\left(r^2+\alpha^2\cos^2\theta\right)^2\left((m+r)^2+\alpha^2\cos^2\theta\right)}\left(m\sin\theta\left(\alpha^2\left(-2\cos\theta\sin\theta r^3\right.\right.\right.$$
$$\left.+m^2\sin 2\theta r-2(2m+r)\alpha^2\cos^3\theta\sin\theta\right)dr\wedge d\phi$$
$$-\frac{1}{8}\left(r^2+\alpha^2\right)\left(8r^4+16mr^3+8m^2r^2+\alpha^4\right.$$
$$\left.-8\alpha^2\left(-3m^2-6rm+\alpha^2\right)\cos^2\theta-\alpha^4\cos(4\theta)\right)d\theta\wedge d\phi\Big)\Big) \tag{6.11.109}$$

while for the second two we get:

$$G^3 = \frac{1}{\sqrt{2}\left(r^2+\alpha^2\cos^2\theta\right)^2\left((m+r)^2+\alpha^2\cos^2\theta\right)}\left(3^{3/4}m\sin\theta\left(\left(\sin 2\theta r^3\right.\right.\right.$$
$$-2m^2\cos\theta\sin\theta r$$
$$\left.+2(2m+r)\alpha^2\cos^3\theta\sin\theta\right)dr\wedge d\phi\alpha^2$$
$$+\frac{1}{8}\left(r^2+\alpha^2\right)\left(8r^4+16mr^3+8m^2r^2+\alpha^4\right.$$
$$\left.-8\alpha^2\left(-3m^2-6rm+\alpha^2\right)\cos^2\theta-\alpha^4\cos(4\theta)\right)d\theta\wedge d\phi\Big)\Big) \tag{6.11.110}$$

$$G^4 = -\frac{1}{\sqrt{2}\left(r^2+\alpha^2\cos^2\theta\right)^2\left((m+r)^2+\alpha^2\cos^2\theta\right)}\left(m\alpha\sin\theta\left(\left(\right.\right.\right.$$
$$-\left(-2+33^{3/4}\right)\alpha^4\cos^4\theta$$
$$+m\left(\left(2+3^{3/4}\right)m+2\left(1+3^{3/4}\right)r\right)\alpha^2\cos^2\theta$$
$$+r(m+r)^2\left(\left(-2+33^{3/4}\right)r-2m\right)\Big)\sin\theta dr\wedge d\phi$$
$$-2\left(r^2+\alpha^2\right)\cos\theta\left(-m^3+\left(-4+3^{3/4}\right)rm^2+\left(-5+43^{3/4}\right)r^2m\right.$$
$$+\left(-2+33^{3/4}\right)r^3$$
$$\left.+\left(\left(-1+23^{3/4}\right)m+\left(-2+33^{3/4}\right)r\right)\alpha^2\cos^2\theta\right)d\theta\wedge d\phi\Big)\Big) \tag{6.11.111}$$

The above expressions are rather formidable, yet considering them in some limit their meaning can be decoded. First of all we recall that in the limit $\alpha \to 0$ the metric (6.11.107) becomes the Reissner–Nordstrom metric. Correspondingly in the same limit the above four-vector of field strengths degenerates into:

$$\begin{pmatrix} F^1 \\ F^2 \\ G^3 \\ G^4 \end{pmatrix} \overset{\alpha \to 0}{\Longrightarrow} \begin{pmatrix} 0 \\ -\frac{m \sin(\theta) d\theta \wedge d\phi}{\sqrt{2} 3^{3/4}} \\ \frac{3^{3/4} m \sin(\theta) d\theta \wedge d\phi}{\sqrt{2}} \\ 0 \end{pmatrix} \tag{6.11.112}$$

showing that the black hole charges $\left(0, -\frac{m}{\sqrt{2}\, 3^{1/4}}, \frac{m\, 3^{1/4}}{\sqrt{2}}, 0\right)$ have the correct form for a BPS black hole and are endowed with the characteristic $\mathbb{Z}_3$ symmetry.

Also in the $\alpha \neq 0$ we can easily determine the black hole charges by integrating the field strengths on a two-sphere of very large radius $r \to \infty$. For this purpose it is important to evaluate the asymptotic expansion of the field strengths for large radius. We find:

$$\begin{pmatrix} F^1 \\ F^2 \\ G^3 \\ G^4 \end{pmatrix} \overset{r \to \infty}{\simeq} \begin{pmatrix} -\frac{\sqrt{2} \sqrt[4]{3} \left(-3+2 \sqrt[4]{3}\right) m \alpha \cos\theta \sin\theta d\theta \wedge d\phi}{r} + \mathscr{O}\left(\frac{1}{r^2}\right) \\ -\frac{m \sin\theta d\theta \wedge d\phi}{\sqrt{2} 3^{3/4}} + \mathscr{O}\left(\frac{1}{r^2}\right) \\ \frac{3^{3/4} m \sin\theta d\theta \wedge d\phi}{\sqrt{2}} + \mathscr{O}\left(\frac{1}{r^2}\right) \\ \frac{\sqrt{2}\left(-2+3 3^{3/4}\right) m \alpha \cos\theta \sin\theta d\theta \wedge d\phi}{r} + \mathscr{O}\left(\frac{1}{r^2}\right) \end{pmatrix} \tag{6.11.113}$$

and the integration on the angular variables produces the same result as for the corresponding Reissner–Nordstrom black hole:

$$\mathscr{Q}_{BPSKN} = \left(0, -\frac{m}{\sqrt{2}\, 3^{1/4}}, \frac{m\, 3^{1/4}}{\sqrt{2}}, 0\right) \tag{6.11.114}$$

In conclusion the BPS Kerr–Newman solution is a deformation of the Reissner–Nordstrom BPS black hole. It is extremal in the σ-model sense and for this reason could be retrieved from the nilpotent orbit construction. However it is not extremal in the sense of General Relativity since the mass is less than $\sqrt{q^2 + \alpha^2}$ being equal to m. For this reason we are below the limit of the cosmic censorship, there is no horizon and we have instead a naked singularity.

The important message is that, notwithstanding the deformation and the presence of a Kaluza–Klein vector, the structure of the charges is that pertaining to the orbit where the solution has been constructed, namely the BPS orbit $\mathscr{O}^3_{11}$.

6.11.5 The Large Non BPS Black Holes of $\mathscr{O}^3_{22}$

Next let us consider the orbit $\mathscr{O}^3_{22}$, which in the spherical symmetric case leads to non BPS Black holes with a finite horizon area.

W-Representation

As in the previous case, in order to better appreciate the structure of these solutions, let us slightly generalize our orbit representative, writing the following nilpotent matrix that depends on two parameters (p, q)

$$X_{3|22}(p,q) = \begin{pmatrix} q & 0 & 0 & \frac{q}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{p+q}{2} & -\frac{p}{2} & 0 & -\frac{q}{2} & 0 & 0 \\ 0 & \frac{p}{2} & \frac{q-p}{2} & 0 & 0 & \frac{q}{2} & 0 \\ -\frac{q}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{q}{\sqrt{2}} \\ 0 & \frac{q}{2} & 0 & 0 & \frac{p-q}{2} & \frac{p}{2} & 0 \\ 0 & 0 & -\frac{q}{2} & 0 & -\frac{p}{2} & \frac{1}{2}(-p-q) & 0 \\ 0 & 0 & 0 & \frac{q}{\sqrt{2}} & 0 & 0 & -q \end{pmatrix} \tag{6.11.115}$$

The standard triple representative mentioned in Eq. (6.10.6) is just the particular case $X_{3|22}(1, 1)$. Applying the usual strategy from the general formula we obtain

$$\mathscr{Q}^{\mathbf{w}}_{3|22} = \mathrm{Tr}(X_{3|22}(p,q)\mathscr{T}^{\mathbf{w}}) = \left(0, p, \sqrt{3}q, 0\right) \tag{6.11.116}$$

Substituting such a result in the expression for the quartic symplectic invariant (see Eq. (6.11.3) we find:

$$\mathfrak{I}_4 = -9\,p\,q^3 < 0 \quad \text{if } p \text{ and } q \text{ have the same sign} \tag{6.11.117}$$

This result is meaningful since, by calculating the trace $\mathrm{Tr}(X_{3|22}L^E_+) = 0$, we find that the Taub-NUT charge vanishes. Furthermore we note that the condition that p and q have the same sign was singled out in [32] as the defining condition of the orbit O^3_{22} which, in the spherical symmetry approach leads to regular non BPS solutions. The choice of opposite signs was proved in [32] to correspond to a different $\mathrm{H}^\star$ orbit, the non diagonal O^3_{12} which instead contains only singular solutions.

Addressing the question of stability subgroups of the original duality group in four-dimensions SL(2, R), we realize that for the charge vector (6.11.116) this subgroup is just trivial:

$$\mathrm{SL}(2,\mathbb{R}) \supset \mathscr{S}_{3|22} = \mathbf{1} \tag{6.11.118}$$

$\mathrm{H}^\star$-Stability Subgroup

Considering next the stability subgroup of the nilpotent element $X_{3|22}(1, 1)$ in $\mathrm{H}^\star = \widehat{\mathfrak{sl}(2,\mathbb{R})} \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathrm{h}^\star}$ we obtain:

$$\mathfrak{S}_{3|22} = \mathbb{R} \tag{6.11.119}$$

the group being generated by a matrix $\mathbb{A}_{3|22}$ of nilpotency degree 2:

$$\mathbb{A}^3_{3|22} = \mathbf{0} \tag{6.11.120}$$

We do not give its explicit form which we do not use in the sequel.

Nilpotent Algebra $\mathbb{N}_{3|22}$

Considering next the adjoint action of the central element $h_{3|22}$ on the subspace $\mathbb{K}^\star$ we find that its eigenvalues are the following ones:

$$\text{Eigenvalues}^{\mathbb{K}^\star}_{3|22} = \{-4, 4, -2, -2, 2, 2, 0, 0\} \tag{6.11.121}$$

Therefore the three eigenoperators A_1, A_2, A_3 corresponding to the three positive eigenvalues 4, 2, 2, respectively, form the restriction to $\mathbb{K}^\star$ of a nilpotent algebra $\mathbb{N}_{3|22}$. In this case the A_i do all commute among themselves so that we have $\mathbb{N}_{3|22} = \mathbb{N}_{3|22} \bigcap \mathbb{K}^\star$ and it is abelian. The abelian structure of the nilpotent algebra implies that for the orbit $\mathscr{O}^3_{22}$ we have only three functions $\mathfrak{h}^0_i$ which will be harmonic and independent. This is so because $\mathscr{D}\mathbb{N}_{3|22} = 0$

Explicitly we set:

$$\mathfrak{H}(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3) = \sum_{i=1}^3 \mathfrak{h}_i\, A_i =$$

$$\begin{pmatrix}
2\mathfrak{h}_3 & \mathfrak{h}_1 - 2\mathfrak{h}_2 & 2\mathfrak{h}_1 - \mathfrak{h}_2 & -\sqrt{2}\mathfrak{h}_3 & -3\mathfrak{h}_2 & -3\mathfrak{h}_1 & 0 \\
\mathfrak{h}_1 - 2\mathfrak{h}_2 & \mathfrak{h}_3 & 0 & \sqrt{2}\mathfrak{h}_2 - 2\sqrt{2}\mathfrak{h}_1 & \mathfrak{h}_3 & 0 & -3\mathfrak{h}_1 \\
\mathfrak{h}_2 - 2\mathfrak{h}_1 & 0 & \mathfrak{h}_3 & \sqrt{2}\mathfrak{h}_1 - 2\sqrt{2}\mathfrak{h}_2 & 0 & -\mathfrak{h}_3 & -3\mathfrak{h}_2 \\
\sqrt{2}\mathfrak{h}_3 & 2\sqrt{2}\mathfrak{h}_1 - \sqrt{2}\mathfrak{h}_2 & \sqrt{2}\mathfrak{h}_1 - 2\sqrt{2}\mathfrak{h}_2 & 0 & \sqrt{2}\mathfrak{h}_1 - 2\sqrt{2}\mathfrak{h}_2 & \sqrt{2}\mathfrak{h}_2 - 2\sqrt{2}\mathfrak{h}_1 & \sqrt{2}\mathfrak{h}_3 \\
3\mathfrak{h}_2 & -\mathfrak{h}_3 & 0 & \sqrt{2}\mathfrak{h}_1 - 2\sqrt{2}\mathfrak{h}_2 & -\mathfrak{h}_3 & 0 & 2\mathfrak{h}_1 - \mathfrak{h}_2 \\
-3\mathfrak{h}_1 & 0 & \mathfrak{h}_3 & 2\sqrt{2}\mathfrak{h}_1 - \sqrt{2}\mathfrak{h}_2 & 0 & -\mathfrak{h}_3 & \mathfrak{h}_1 - 2\mathfrak{h}_2 \\
0 & -3\mathfrak{h}_1 & 3\mathfrak{h}_2 & -\sqrt{2}\mathfrak{h}_3 & \mathfrak{h}_2 - 2\mathfrak{h}_1 & \mathfrak{h}_1 - 2\mathfrak{h}_2 & -2\mathfrak{h}_3
\end{pmatrix} \tag{6.11.122}$$

Considering $\mathfrak{H}(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$ as a Lax operator and calculating its Taub-NUT charge and electromagnetic charges we find:

$$\mathbf{n}_{TN} = -6\mathfrak{h}_1 \quad ; \quad \mathscr{Q} = \left\{2\sqrt{3}\,(\mathfrak{h}_2 - 2\mathfrak{h}_1)\,, 0, -2\sqrt{3}\mathfrak{h}_3, -6\mathfrak{h}_2\right\} \tag{6.11.123}$$

This implies that constructing the multi-centre solution with harmonic functions the condition $\mathfrak{h}_1 = 0$ might be sufficient to annihilate the Taub-NUT current. In this case we will be lucky and such a condition suffices.

For later convenience let us change the normalization in the basis of harmonic functions as follows:

$$\mathfrak{h}^{(0)}_1 = \mathscr{H}_1 \quad ; \quad \mathfrak{h}^{(0)}_2 = \tfrac{1}{2}\,(1 - \mathscr{H}_2) \quad ; \quad \mathfrak{h}^{(0)}_3 = \tfrac{1}{2}\,(1 - \mathscr{H}_3) \tag{6.11.124}$$

Implementing the symmetric coset construction with:

$$\mathscr{Y}\left(\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3\right) \equiv \exp\left[\mathfrak{H}\left(\mathscr{H}_1, \tfrac{1}{2}\left(1-\mathscr{H}_2\right), \tfrac{1}{2}\left(1-\mathscr{H}_3\right)\right)\right] \qquad (6.11.125)$$

calculating the upper triangular coset representative $\mathbb{L}(\mathscr{Y})$ according to Eq. (6.8.26) and extracting the σ-model scalar fields we obtain an explicit expression which is sufficiently simple to be displayed:

$$\exp\left[-U\right] = \sqrt{\mathscr{H}_2\mathscr{H}_3^3 - 4\mathscr{H}_1^2} \qquad (6.11.126)$$

$$\operatorname{Im} z = \frac{\sqrt{\mathscr{H}_2\mathscr{H}_3^3 - 4\mathscr{H}_1^2}}{\mathscr{H}_3^2} \qquad (6.11.127)$$

$$\operatorname{Re} z = -\frac{2\mathscr{H}_1}{\mathscr{H}_3^2} \qquad (6.11.128)$$

$$Z^M = \begin{pmatrix} -\frac{\sqrt{6}\mathscr{H}_1\mathscr{H}_3}{4\mathscr{H}_1^2 - \mathscr{H}_2\mathscr{H}_3^3} \\ \frac{4\mathscr{H}_1^2 - (\mathscr{H}_2 - 1)\mathscr{H}_3^3}{\sqrt{2}\left(4\mathscr{H}_1^2 - \mathscr{H}_2\mathscr{H}_3^3\right)} \\ \frac{\sqrt{\frac{3}{2}}\left(4\mathscr{H}_1^2 - \mathscr{H}_2(\mathscr{H}_3 - 1)\mathscr{H}_3^2\right)}{4\mathscr{H}_1^2 - \mathscr{H}_2\mathscr{H}_3^3} \\ \frac{\sqrt{2}\mathscr{H}_1\mathscr{H}_2}{4\mathscr{H}_1^2 - \mathscr{H}_2\mathscr{H}_3^3} \end{pmatrix} \qquad (6.11.129)$$

$$a = -\frac{\mathscr{H}_1\left(\mathscr{H}_2 + 3\mathscr{H}_3 - 2\right)}{4\mathscr{H}_1^2 - \mathscr{H}_2\mathscr{H}_3^3} \qquad (6.11.130)$$

Using these results we easily obtain the Taub-NUT current in the following form:

$$j^{TN} = 2 \star \nabla \mathscr{H}_1 \qquad (6.11.131)$$

In this case the predicted condition $\mathscr{H}_1 = 0$ is sufficient to annihilate the Taub-NUT current and we obtain an extremely simple result.[14] The complete form of the supergravity solution corresponding to this choice is:

$$\exp[-U] = \sqrt{\mathscr{H}_3^3\mathscr{H}_2} \qquad (6.11.132)$$

$$z = \mathrm{i}\,\frac{\sqrt{\mathscr{H}_3^3\mathscr{H}_2}}{\mathscr{H}_3^2} \qquad (6.11.133)$$

$$j^{TN} = 0 \qquad (6.11.134)$$

[14] Actually even the condition $\mathscr{H}_1 = \text{const}$ suffices to annihilate the Taub-NUT charge allowing for a non trivial real part of the z-field. However in this section we analyze the case $\mathscr{H}_1 = 0$ for its remarkable simplicity.

$$j^{EM} = \star\nabla \begin{pmatrix} 0 \\ -\frac{\mathscr{H}_2}{\sqrt{2}} \\ -\sqrt{\frac{3}{2}}\mathscr{H}_3 \\ 0 \end{pmatrix} \tag{6.11.135}$$

Comparing with the case of the large BPS orbit we see that the only difference is the relative sign of the harmonic functions in the electromagnetic current. What we said for the BPS black holes extends to the non BPS ones in the same way.

Summary

For a multicenter solution associated with the $\mathscr{O}^3_{22}$ orbit we have a mixture of very small and small black holes as in the case of the orbit $\mathscr{O}^3_{22}$. Also here a finite area non BPS black hole emerges when the center of a very small black comes to coincides with the center of a small one. The only difference is the relative sign of the two charges. With equal signs we construct a non BPS state, while with opposite charges we construct a BPS one. This reinforces the conjecture that at the quantum level finite black holes can be interpreted as composite states.

This conjecture is also supported by an angular momentum analysis. Looking at the representations in Table 6.1, we see that the representation $2(j = 1) + (j = 0)$ that corresponds to BPS and non BPS large black holes can be obtained by summing the representation $(j = 1) + 2(j = \frac{1}{2})$ that corresponds to small black holes with the representation $3(j = 0) + 2(j = \frac{1}{2})$ that corresponds to very small black holes. Consider the following table:

1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
1	1	0	0	0	-1	-1

the numbers in the first line are the eigenvalues of the central element h in the triplet (h, X, Y) characterizing the orbit $\mathscr{O}^4_{11}$. The second line contains the eigenvalues for the central element of the triplet of the orbit $\mathscr{O}^4_{11}$. In the last line we have the eigenvalues for the h in the triplet characterizing the orbit $\mathscr{O}^3_{i,j}$. We realize that the coincidence of centres correspond to the identification of a new SL(2, R) subgroup which is the direct sum of the original two associated with the two small black holes.

6.11.6 *The Largest Orbit* $\mathscr{O}^1_{11}$

Next let us consider the orbit $\mathscr{O}^1_{11}$, which in the spherical symmetric case leads only to singular solutions.

W-Representation

Applying the usual strategy from the general formula we obtain a charge vector

$$\mathcal{Q}^{\mathbf{w}}_{1|11} = \mathrm{Tr}(X_{1|11}(p,q)\mathcal{T}^{\mathbf{w}}) \tag{6.11.136}$$

which has no invariance:

$$\mathrm{SL}(2,\mathbb{R}) \supset \mathcal{S}_{1|11} = \mathbf{1} \tag{6.11.137}$$

and yields a quartic invariant generically different from zero:

$$\mathfrak{I}_4 \neq 0 \tag{6.11.138}$$

Because of our simplified choice of the representative the Taub-NUT charge is not zero and only later we will enforce the vanishing of the Taub-NUT current on the harmonic function parameterized solution.

$\mathrm{H}^\star$-Stability Subgroup

Considering next the stability subgroup of the nilpotent element $X_{1|11}$ in $\mathrm{H}^\star = \widehat{\mathfrak{sl}(2,\mathbb{R})} \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathrm{h}^\star}$ we obtain that it is trivial:

$$\mathfrak{S}_{1|11} = \mathbf{1} \tag{6.11.139}$$

Nilpotent Algebra $\mathbb{N}_{1|11}$

Considering next the adjoint action of the central element $h_{1|11}$ on the subspace $\mathbb{K}^\star$ we find that its eigenvalues are the following ones:

$$\mathrm{Eigenvalues}^{\mathbb{K}^\star}_{3|22} = \{-5, 5, -3, 3, -1, -1, 1, 1\} \tag{6.11.140}$$

Therefore the four eigenoperators A_1, A_2, A_3, A_4 corresponding to the four positive eigenvalues 5, 3, 1, 1, respectively, form the restriction to $\mathbb{K}^\star$ of a nilpotent algebra $\mathbb{N}_{1|11}$. In this case the A_i do not all commute among themselves so that we have $\mathbb{N}_{1|11} \neq \mathbb{N}_{1|11} \bigcap \mathbb{K}^\star$. The full algebra involves also two operators $B_1, B_2 \in \mathbb{H}^\star$ and the full set of commutation relations is the following one:

$$\begin{aligned}
0 &= [A_1\,,\ A_2] = [A_1\,,\ A_3] = [A_1\,,\ A_4] \\
0 &= [A_2\,,\ A_3] \\
0 &= [B_1\,,\ B_2] = [B_1\,,\ A_1] = [B_1\,,\ A_2] \\
0 &= [B_1\,,\ A_4] = [B_2\,,\ A_1] = [B_2\,,\ A_3] \\
B_1 &= [A_2\,,\ A_4] \\
B_2 &= [A_3\,,\ A_4] \\
-16\,A_1 &= [B_1\,,\ A_3] \\
-16\,A_1 &= [B_2\,,\ A_1]
\end{aligned}$$

$$24\, A_2 = [B_2\,,\; A_4] \tag{6.11.141}$$

By inspection of Eq. (6.11.141) we easily see that:

$$\mathscr{D}\mathbb{N}_{1|11} = \text{span}\,\{B_1,\, B_2,\, A_1,\, A_2\} \quad ; \quad \mathscr{D}\mathbb{N}_{1|11} \bigcap \mathbb{K}^\star \;=\; \text{span}\,\{A_1,\, A_2\} \tag{6.11.142}$$

$$\mathscr{D}^2\mathbb{N}_{1|11} = \text{span}\,\{A_1\} \;=\; \mathscr{D}^2\mathbb{N}_{1|11} \bigcap \mathbb{K}^\star \tag{6.11.143}$$

This structure of the nilpotent algebra implies that for the orbit $\mathscr{O}^1_{11}$ we have only two functions $\mathfrak{h}^0_3$, $\mathfrak{h}^0_4$ which are harmonic and independent. The other two functions $\mathfrak{h}^2_1$, $\mathfrak{h}^1_2$, obey instead equations in which the previous two play the role of sources. Not surprisingly $\mathfrak{h}^2_1$, $\mathfrak{h}^1_2$ correspond to the higher gradings 5 and 3, while $\mathfrak{h}^0_3$, $\mathfrak{h}^0_4$ correspond to the gradings 1, 1. More precisely $\mathfrak{h}^1_2$ receives source contributions only from $\mathfrak{h}^0_3$, $\mathfrak{h}^0_4$, while $\mathfrak{h}^2_1$ receives source contributions from $\mathfrak{h}^1_2$, $\mathfrak{h}^0_3$, $\mathfrak{h}^0_4$

Explicitly we set:

$$\mathfrak{H}(\mathfrak{h}_1, \ldots, \mathfrak{h}_4) \;=\; \textstyle\sum_{i=1}^4\, \mathfrak{h}_i\, A_i \;=$$

$$\left(\begin{array}{ccccccc}
\mathfrak{h}_1+\mathfrak{h}_4 & \frac{\mathfrak{h}_2}{3}-\mathfrak{h}_3 & \mathfrak{h}_4 & \frac{\sqrt{2}\mathfrak{h}_2}{3}-\sqrt{2}\mathfrak{h}_3 & -\mathfrak{h}_1 & -\mathfrak{h}_2-\mathfrak{h}_3 & 0 \\
\frac{\mathfrak{h}_2}{3}-\mathfrak{h}_3 & 2\mathfrak{h}_4 & \mathfrak{h}_2+\mathfrak{h}_3 & -\sqrt{2}\mathfrak{h}_4 & \mathfrak{h}_3-\frac{\mathfrak{h}_2}{3} & 0 & -\mathfrak{h}_2-\mathfrak{h}_3 \\
-\mathfrak{h}_4 & -\mathfrak{h}_2-\mathfrak{h}_3 & \mathfrak{h}_1-\mathfrak{h}_4 & \frac{\sqrt{2}\mathfrak{h}_2}{3}-\sqrt{2}\mathfrak{h}_3 & 0 & \frac{\mathfrak{h}_2}{3}-\mathfrak{h}_3 & -\mathfrak{h}_1 \\
\sqrt{2}\mathfrak{h}_3-\frac{\sqrt{2}\mathfrak{h}_2}{3} & \sqrt{2}\mathfrak{h}_4 & \frac{\sqrt{2}\mathfrak{h}_2}{3}-\sqrt{2}\mathfrak{h}_3 & 0 & \frac{\sqrt{2}\mathfrak{h}_2}{3}-\sqrt{2}\mathfrak{h}_3 & -\sqrt{2}\mathfrak{h}_4 & \sqrt{2}\mathfrak{h}_3-\frac{\sqrt{2}\mathfrak{h}_2}{3} \\
\mathfrak{h}_1 & \frac{\mathfrak{h}_2}{3}-\mathfrak{h}_3 & 0 & \frac{\sqrt{2}\mathfrak{h}_2}{3}-\sqrt{2}\mathfrak{h}_3 & \mathfrak{h}_4-\mathfrak{h}_1 & -\mathfrak{h}_2-\mathfrak{h}_3 & \mathfrak{h}_4 \\
-\mathfrak{h}_2-\mathfrak{h}_3 & 0 & \mathfrak{h}_3-\frac{\mathfrak{h}_2}{3} & \sqrt{2}\mathfrak{h}_4 & \mathfrak{h}_2+\mathfrak{h}_3 & -2\mathfrak{h}_4 & \frac{\mathfrak{h}_2}{3}-\mathfrak{h}_3 \\
0 & -\mathfrak{h}_2-\mathfrak{h}_3 & \mathfrak{h}_1 & \frac{\sqrt{2}\mathfrak{h}_2}{3}-\sqrt{2}\mathfrak{h}_3 & -\mathfrak{h}_4 & \frac{\mathfrak{h}_2}{3}-\mathfrak{h}_3 & -\mathfrak{h}_1-\mathfrak{h}_4
\end{array}\right) \tag{6.11.144}$$

Considering $\mathfrak{H}(\mathfrak{h}_1, \ldots, \mathfrak{h}_4)$ as a Lax operator and calculating its Taub-NUT charge and electromagnetic charges we find:

$$\mathbf{n}_{TN} \;=\; -2(\mathfrak{h}_2+\mathfrak{h}_3) \quad ; \quad \mathscr{Q} \;=\; \left\{-2\sqrt{3}\mathfrak{h}_4,\, -2\,(\mathfrak{h}_2+\mathfrak{h}_3)\,,\, \frac{2\,(\mathfrak{h}_2-3\mathfrak{h}_3)}{\sqrt{3}},\, -2\mathfrak{h}_1\right\} \tag{6.11.145}$$

This implies that constructing the multi-centre solution with harmonic functions the condition $\mathfrak{h}_2 \;=\; -\,\mathfrak{h}_3$ might be sufficient to annihilate the Taub-NUT current.

Implementing the symmetric coset construction with:

$$\mathscr{Y}\,(\mathfrak{h}_1, \ldots, \mathfrak{h}_4) \;\equiv\; \exp\left[\mathfrak{H}\,(\mathfrak{h}_1, \ldots, \mathfrak{h}_4)\right] \tag{6.11.146}$$

and imposing the field equations (6.8.14) we obtain the following conditions:

$$
\begin{aligned}
0 = & \frac{224}{5}\nabla\mathfrak{h}_3 \circ \nabla\mathfrak{h}_3\, \mathfrak{h}_4^3 - \frac{16}{5}\,\mathfrak{h}_3\Delta\,\mathfrak{h}_3\,\mathfrak{h}_4^3 - \frac{416}{5}\nabla\,\mathfrak{h}_3 \circ \nabla\mathfrak{h}_4\,\mathfrak{h}_3\,\mathfrak{h}_4^2 + \frac{16}{5}\,\mathfrak{h}_3^2\Delta\,\mathfrak{h}_4\,\mathfrak{h}_4^2 \\
& + \frac{192}{5}\nabla\,\mathfrak{h}_4 \circ \nabla\mathfrak{h}_4\,\mathfrak{h}_3^2\,\mathfrak{h}_4 + \frac{32}{3}\nabla\mathfrak{h}_2 \circ \nabla\mathfrak{h}_3\,\mathfrak{h}_4 - \frac{8}{3}\,\mathfrak{h}_3\Delta\mathfrak{h}_2\,\mathfrak{h}_4 - \frac{8}{3}\,\mathfrak{h}_2\Delta\,\mathfrak{h}_3\,\mathfrak{h}_4 \\
& - \frac{16}{3}\nabla\,\mathfrak{h}_3 \circ \nabla\mathfrak{h}_4\,\mathfrak{h}_2 - \frac{16}{3}\nabla\mathfrak{h}_2 \circ \nabla\mathfrak{h}_4\,\mathfrak{h}_3 + \Delta\,\mathfrak{h}_1 + \frac{16}{3}\,\mathfrak{h}_2\,\mathfrak{h}_3\Delta\mathfrak{h}_4 \\
0 = & 4\Delta\mathfrak{h}_3\,\mathfrak{h}_4^2 - 8\nabla\,\mathfrak{h}_3 \circ \nabla\mathfrak{h}_4\,\mathfrak{h}_4 - 4\,\mathfrak{h}_3\Delta\mathfrak{h}_4\,\mathfrak{h}_4 + 8\nabla\mathfrak{h}_4 \circ \nabla\mathfrak{h}_4\,\mathfrak{h}_3 + \Delta\mathfrak{h}_2 \\
0 = & \Delta\mathfrak{h}_3 \\
0 = & \Delta\mathfrak{h}_4
\end{aligned}
\tag{6.11.147}
$$

Solutions of the above system can be quite complicated and can encompass many different types of behaviors, yet what is generically true is that the contributions from the source term introduces in $\mathfrak{h}_1$ and $\mathfrak{h}_2$ poles $1/r^p$ stronger than $p = 1$, while $\mathfrak{h}_3$ and $\mathfrak{h}_4$ have only simple poles. Hence if the structure of the polynomials in the functions $\mathfrak{h}_{1,2,3,4}$ is such that at simple poles the divergence of the inverse warp factor is already too strong or the coefficient already becomes imaginary, introducing stronger poles can only make the situation worse. For this reason we confine ourselves to analyze solutions encompassed in this orbit in which the source terms vanish identically upon the implementation of some identifications.

There are few different reductions with such a property and we choose just one that has also the additional feature of annihilating the Taub-NUT current. It is the following one:

$$
\mathfrak{h}_3 \;=\; \mathfrak{h}_4 \;=\; -\,\mathfrak{h}_2 \;\equiv\; \mathfrak{h}
\tag{6.11.148}
$$

The reader can easily check that with the choice (6.11.148) the system of equations (6.11.147) reduces to:

$$
\Delta\mathfrak{h} \;=\; \Delta\mathfrak{h}_1 \;=\; 0
\tag{6.11.149}
$$

For later convenience let us change the normalization in the basis of harmonic functions as follows:

$$
\mathfrak{h}_4 \;=\; \tfrac{1}{4}\,\mathscr{H} \quad ; \quad \mathfrak{h}_3 \;=\; \tfrac{1}{4}\,\mathscr{H} \quad ; \quad \mathfrak{h}_2 \;=\; -\,\tfrac{1}{4}\,\mathscr{H} \quad ; \quad \mathfrak{h}_1 \;=\; -\,\tfrac{1}{4} + \mathscr{W}
\tag{6.11.150}
$$

calculating the upper triangular coset representative $\mathbb{L}(\mathscr{Y})$ according to Eq. (6.8.26) and extracting the σ-model scalar fields we obtain explicit expressions which are sufficiently simple to be displayed:

$$
\exp[\,U\,] \;=\; \frac{8\sqrt{15}}{\sqrt{-(\mathscr{H}+2)^3\left(\mathscr{H}^5 + 10\mathscr{H}^4 + 40\mathscr{H}^3 + 80\mathscr{H}^2 - 60(4\mathscr{W}+1)\right)}}
\tag{6.11.151}
$$

$$
\mathrm{Im}\, z \;=\; \frac{3\sqrt{15}(\mathscr{H}+2)}{\sqrt{-\frac{\mathscr{H}+2}{\mathscr{H}^2(\mathscr{H}(\mathscr{H}(\mathscr{H}+10)+40)+80)-60(4\mathscr{W}+1)}}\left((\mathscr{H}(\mathscr{H}(\mathscr{H}+10)+20)-40)\mathscr{H}^2 + 90(4\mathscr{W}+1)\right)}
\tag{6.11.152}
$$

$$
\mathrm{Re}\, z \;=\; \frac{15\mathscr{H}(\mathscr{H}+2)(\mathscr{H}+4)}{\mathscr{H}^5 + 10\mathscr{H}^4 + 20\mathscr{H}^3 - 40\mathscr{H}^2 + 360\mathscr{W} + 90}
\tag{6.11.153}
$$

We skip the form of the Z fields and of a but we mention their consequences, namely the Taub-NUT current

$$j^{TN} = 0 \tag{6.11.154}$$

and the electromagnetic currents

$$j^{EM} = \star\nabla \left\{ \frac{1}{2}\sqrt{\frac{3}{2}}\mathscr{H}, 0, \frac{7\mathscr{H}}{6}, \sqrt{2}\mathscr{W} \right\} \tag{6.11.155}$$

This shows that a black hole belonging to this orbit has a charge vector $\mathscr{Q} = \left\{ \frac{1}{2}\sqrt{\frac{3}{2}}p, 0, \frac{7p}{6}, \sqrt{2}q \right\}$, whose quartic invariant is:

$$\mathfrak{I}_4 = \frac{1}{128}p^3(49p + 72q) \tag{6.11.156}$$

This latter can be positive or negative depending on the choices for p and q. The problem, however, is that this solution is always singular around all poles of $\mathscr{H}$. Indeed setting:

$$\mathscr{H} \sim \frac{p}{r} \quad ; \quad \mathscr{W} \sim \frac{q}{r} \tag{6.11.157}$$

we find that for $r \to 0$ the inverse warp factor behaves as follows:

$$\exp[-U] \sim \frac{\sqrt{-p^8}}{8\sqrt{15}r^4} + \frac{\sqrt{-p^8}}{\sqrt{15}pr^3} + \frac{\sqrt{\frac{3}{5}}\sqrt{-p^8}}{p^2r^2} + \frac{4\sqrt{-p^8}}{\sqrt{15}p^3r} + \frac{\sqrt{\frac{3}{5}}p^3(p+5q)}{\sqrt{-p^8}} + \mathscr{O}(r) \tag{6.11.158}$$

The coefficient $\sqrt{-p^8}$ indicates that approaching the pole the warp factor becomes imaginary at a finite distance from it and the would be horizon $r = 0$ is never reached. If it were reached, the divergence $\frac{1}{r^4}$ would imply an infinite area of the horizon. As we know from our general discussion the Riemann tensor diverges if the warp factor goes to zero faster than r^2 so that the would be horizon would actually be a singularity. Yet since the warp factor becomes imaginary at a finite distance from the pole it remains open the question if solutions of this type can be prolonged by suitably changing the coordinate system. In that case they might acquire a physical meaning. So far such a question has not been tackled but it deserves to be.

6.12 Conclusions on the Episteme Contained in This Chapter

In this very long chapter we have tackled quite advanced issues of current or of quite recent research. Although all the inspiring motivations come from *Supergravity*, the material here presented is of genuine algebraic and geometrical character; indeed it might be understood and treated within the scope of pure Mathematics. As usual, the role of supersymmetry was just that of directing our choices, leading us to focus on *special manifolds* endowed with *special geometries*.

Actually the methods and the constructions considered in this chapter are general and might be dealt with no knowledge of supermultiplets and supercharges. Additional inspiration coming from *Supergravity* is encoded in the strategic attention paid to the *Tits–Satake projection* and to *Tits–Satake universality classes*, which, however, are purely mathematical phenomena, self-contained in Lie algebra theory.

Even the very final physical motivation of constructing *extremal black-hole solutions* might be forgotten once, in the spirit of *the geometry of geometries*, a physical–geometrical problem has been mapped into another purely geometrical one.

Thus let us summarize into a list of points the mathematical logic of what we have been discussing in the present chapter.

(A) The problem of constructing extremal black-hole solutions is reduced to the construction and classification of mappings:

$$\Phi \quad : \quad \mathbb{R}^3 \Longrightarrow \mathscr{M}_s \tag{6.12.1}$$

where $(\mathscr{M}_s, g)$ is a pseudo-Riemmannian manifold and the map Φ satisfies both the σ-model equations of motion and the stress-tensor vanishing condition:

$$\partial_i \left(\frac{\partial \Phi^\mu}{\partial x^i} \nabla_\mu \Phi^\nu \right) = 0 \quad ; \quad g_{\mu\nu}(\Phi)\, \partial_i \Phi^\mu \, \partial_j \Phi^\nu = 0 \tag{6.12.2}$$

(B) The geometrical problem posed in (A) can be considered for any Lorentzian-manifold $\mathscr{M}_s$ but, instructed by supersymmetry, we localize it on the homogeneous manifolds:

$$\mathscr{M}_s = \frac{\mathrm{U}_{\mathrm{D}=3}}{\mathrm{H}^\star} \tag{6.12.3}$$

listed in Table 5.4 that are in the image of the $c^\star$-map and have a structure fitting the golden splitting (1.7.12)

(C) For the reasons discussed at length in previous sections and chapters we are actually interested only in those maps of the type (6.12.1) where:

$$\Phi\left[\mathbb{R}^3\right] \subset \frac{\mathrm{U}^{\mathrm{TS}}_{\mathrm{D}=3}}{\mathrm{H}^\star_{\mathrm{TS}}} \subset \frac{\mathrm{U}_{\mathrm{D}=3}}{\mathrm{H}^\star} \tag{6.12.4}$$

namely where the image of the three-dimensional space $\mathbb{R}^3$ lies entirely inside the Tits-Satake submanifold.

(D) These $\mathrm{H}^\star$–orbits of solutions can be classified and explicitly constructed thanks to an algorithm, thoroughly explained in Sect. 6.8, that associates such solutions to each $\mathrm{H}^\star$–orbit of nilpotent operators $X \in \mathbb{K}$, where $\mathbb{K}$ is the orthogonal complement of the subalgebra $\mathbb{H}^\star \subset \mathbb{U}$. The classification of U-nilpotent orbits is a frontier topic in Mathematics and, further specialized to $\mathrm{H}^\star \subset \mathrm{U}$ orbits, involves items and techniques generically not yet available in the mathematical supermarket, like the generalized Weyl group $\mathscr{GW}$ and the H-Weyl subgroup $\mathscr{W}_{\mathrm{H}}$.

(E) Within the class of manifolds in the image of the $c^{\star}$-map, the problem of $\mathrm{H}^{\star}$ nilpotent orbits acquires very special features because of the special nature of the subgroup $\mathrm{H}^{\star}$. These special features are ultimately related with the golden splitting structure (1.7.12) which is on its turn a land-mark of special geometries. The complicated mechanisms here at work relate the classification of $\mathrm{H}^{\star}$–orbits with the classification of $\mathrm{U}_{\mathrm{D}=4}$–orbits in the **W**-representation.

(F) The association of the considered mathematical problem with extremal black-holes provides the features pointed out in (E) with physical interpretations in terms of electromagnetic charges, horizon areas and fixed scalars. Yet we might complete ignore such interpretations and ask ourself the question of what is the abstract, purely mathematical meaning of such relations as that between $\mathrm{U}_{\mathrm{D}=4}$–orbits in the **W**-representation and $\mathrm{H}^{\star}$ nilpotent orbits. Such a study has not yet been performed but might be the source of new precious insights.

Generally speaking the problem considered in this chapter unveils new very profound aspects of Special Geometries pertaining both to the scope of Geometry and of Lie Algebra Theory. As we tried to emphasize in point (F) of the above list a mathematical reformulation of all the mechanisms spotted in this context might be of great moment. We might find clues to some generalization of the golden splitting that goes beyond both supersymmetry and even homogeneous spaces and opens some new direction in differential and algebraic geometry. Inspiring clues come probably from a careful analysis of Weyl subgroups and the characterization among them of those that can be regarded as H-subgroups.

In this context an inspiring observation appears to be the one highlighted in previous pages that regular finite horizon black-holes can be regarded as bound-states of small or very small black-holes. An in depth investigation of the proper mathematics lurking behind this feature is potentially capable of revealing new exciting perspectives both in geometry and physics.

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