Chapter 4
Special Geometries

La géométrie…est une science née à propos de l'expérience…nous avons créé l'espace qu'elle etudie, mais en l'adaptant au monde où nous vivons. Nous avons choisie l'espace le plus commode…

Henri Poincaré.

4.1 The Evolution of Geometry in the Second Half of the XXth Century

Relying for a complete historical account on the tale told in the twin book [1], let us summarize the steps that led, in the 1990's to Special Geometries.

4.1.1 Complex Geometry Rises to Prominence

On the purely mathematical front in the years from 1953 to 1955, Pierre Dolbeault introduced a new very important mathematical instrument: the $\overline{\partial}$-cohomology of the differential forms defined on complex analytic manifolds, namely the holomorphic analogue of de Rham cohomology defined on real manifolds. The essence of Dolbeault cohomology (described in Sect. 3.3) is the topic of Dolbeault's thesis, prepared by him under the direction of Henri Cartan, Élie's son and one of the closest friends of André Weil. The thesis was defended in Paris in 1955.

Complex Geometry and, within it Kähler Geometry, arose to high prominence in the three decades from 1950 to 1980. The language of fibre-bundles and characteristic classes was combined with the notion of holomorphicity and line-bundles, namely Principal Bundles whose structural group is the group of non vanishing complex numbers $\mathbb{C}^\star$, became ubiquitous in the discussion of complex manifolds.

A new innovative conception developed in this context, namely that of characterizing the geometry of base manifolds $\mathscr{M}$ by means of statements on the characteristic classes of bundles defined over them.

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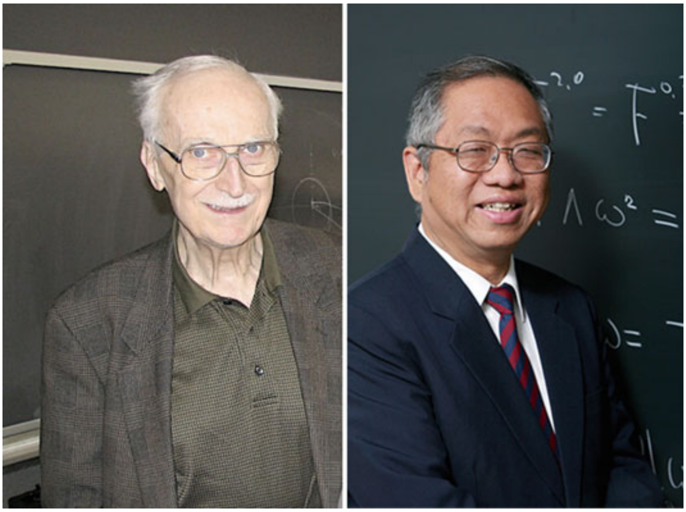

Fig. 4.1 On the left Eugenio Calabi (Milano, Italy 1923). On the right Shing–Tung Yau (Shantou, China 1949). Born Italian, Calabi is an American citizen. He graduated in 1946 from MIT and obtained his Ph.D from Princeton in 1950. He held temporary positions in Minnesota and in Princeton, then since 1967 to retirement he was Full Professor of Mathematics at the University of Pennsylvania, successor of Hans Rademacher. He came to the definition of Calabi–Yau n-folds while exploring the geometry of complex manifolds that support harmonic spinors. Born in China, Yau studied first at Hong Kong University, then he went to the USA where he got his Ph.D. in 1971 from Berkeley under the supervision of Chern. Post-doctoral fellow in Princeton and in Stony Brook, he became Professor in Stanford. Since 1987 he is Professor of Mathematics at Harvard University. Yau's proof of Calabi 1964 conjecture was published in 1977

The first example, which plays an important role in the sequel, is that of *Hodge–Kähler manifolds* that are Kähler manifolds $\mathscr{M}$ characterized by the existence of a line bundle $\mathscr{L} \to \mathscr{M}$, such that its first Chern Class coincides with the cohomology class of the Kähler 2-form: $c_1(\mathscr{L}) = [K]$.

Another important example is provided by Calabi–Yau n-folds. These latter were introduced by Eugenio Calabi (see Fig. 4.1) in 1964 with the definition of complex n-dimensional algebraic varieties $\mathscr{M}_n$, the first Chern class of whose tangent bundle vanishes: $c_1\,(T\mathscr{M}_n) = 0$. Later, the American-Chinese mathematician Shin–Tung Yau (see Fig. 4.1) proved the theorem that for Calabi–Yau n-folds, every $(1, 1)$ Dolbeault cohomology class contains a representative that can be identified with the Kähler 2-form of a Ricci flat Kähler metric: the Calabi–Yau metric.

4.1.2 On the Way to Special Geometries

Other notable examples of this way of thinking, applying both to complex and to real geometry are the *manifolds of restricted holonomy*. One considers Riemannian manifolds $\mathscr{M}_n$ in dimension n and their *spin bundles*, namely the principal bundles on which their spin connections ω^{ab} are defined as Ehresman connections. Generically such bundles have, as structural group, Spin(n), which is the double covering of SO(n), yet it may happen that ω^{ab} is Lie algebra–valued in a proper subalgebra $\mathbb{G} \subset \mathfrak{so}(n)$. Choosing algebras $\mathbb{G}$ for which this might happen and imposing that it should happen is a strong constraint on the geometry of the manifold $\mathscr{M}_n$.

Research on manifolds of restricted holonomy went on in the 1980s and 1990s in the mathematical community but, not too surprisingly, it was heavily stimulated by issues in theoretical physics and particularly in Superstring/Supergravity theory.

It is easy to understand why. The main input in Superstring/Supergravity is Supersymmetry, a generalization of Lie algebras where spinor representations and vector representations of groups SO(n) are transformed one into the other by new symmetry operators Q^α, dubbed the *supercharges*, that are themselves spinors. At the level of field theories we work with fibre-bundles and the fields we consider are sections of such bundles. Field theories can be supersymmetric if the supercharges Q^α find a field-theoretic realization which is a symmetry of the action, leaving the door open for its desired spontaneous breaking. It is quite intuitive that such a realization of the supercharges requires special restrictions on the bundles and this reflects into heavy constraints on the geometry of the base manifolds.

The above simple reasoning reveals what, in the opinion of this author, is the main conceptual contribution of Supergravity theories to the development of geometrical thought and, eventually, of physical thought, provisionally assuming that geometry and physics are, once properly interpreted, the same thing. Supersymmetry tackles with one of the most fundamental and so far unexplained pillars of physics, namely the separation of the physical world into bosons and fermions and the spin-statistics theorem. The distinction between vector and spinor representations is at the basis of all that and it is a distinctive property of the $\mathfrak{so}(n)$ Lie algebras, unexisting for the other simple Lie algebras. On the other hand the reduction of the tangent-bundle to an $\mathfrak{so}(n)$-bundle is the same thing as the existence of a metric and can be interpreted as gravity. Special Geometries arise because of supersymmetry, in order to allow the mixing of boson and fermions. It is the mathematical investigation of *Space* from this new viewpoint the new quality of geometrical studies inspired by supergravity. Before telling such a story we need to recall another mathematical conception, that was developed independently from Superstring/Supergravity yet found its most ample and fertile applications in the supersymmetric context.

4.1.3 The Geometry of Geometries

Let us recall Hermann Weyl's discussion of the ellipses, used by him to introduce his conception of mathematical thinking and reported by us in the twin book [1]. The coefficients a, b, c of the quadratic form quoted by Weyl are the first example of *moduli* and the portion of $\mathbb{R}^3$ where they are allowed to take values is the first example of a *moduli-space*. In complex algebraic geometry one considers loci of some projective space $\mathbb{P}_n(\mathbb{C})$ cut out by some homogeneous polynomial constraint of degree m:

$$0 = \mathscr{W}(a, X) = \sum_{i_1 \ldots i_m} a_{i_1 \ldots i_m} X^{i_1} \ldots X^{i_m} \tag{4.1.1}$$

imposed on the $n+1$ homogeneous coordinates X^i $(i = 1, \ldots, n+1)$. The complex coefficients $a_{i_1 \ldots i_m}$ are also *moduli* and fill some complex manifold $\mathscr{M}$. If we consider the following constraint imposed on the metric tensor of some Riemannian manifold $\mathscr{M}_n$:

$$R_{\mu\nu}[g] = \lambda\, g_{\mu\nu} \tag{4.1.2}$$

where $R_{\mu\nu}[g]$ is the Ricci tensor and λ some constant, we actually write a set of differential equations for the metric tensor $g_{\mu\nu}$, which, on the manifold $\mathscr{M}_n$, generically admit a solution depending on a set of parameters $\{p_1, \ldots p_r\}$, among which λ is included. Also these are moduli and they fill a space named *the moduli space of Einstein metrics* on $\mathscr{M}_n$.

Several other examples can be made of manifolds $\mathscr{M}_{mod}$ whose points correspond to the specification of a particular geometry within a class, for instance the moduli ρ^i of an instanton parameterize the solution of the self duality constraint[1]:

$$F^{\Lambda}_{\mu\nu}(\rho, x) = \tfrac{1}{2}\, \varepsilon_{\mu\nu\lambda\sigma}\, F^{\Lambda}_{\lambda\sigma}(\rho, x) \tag{4.1.3}$$

imposed on the field strength of a connection on a principal fibre bundle $P(\mathrm{G}, \mathscr{M}_4)$.

A new mathematical idea that is of outmost relevance both for physics and for mathematics is encoded in the following almost obvious argument. Being a manifold, the moduli space $\mathscr{M}_{mod}$ can support such geometrical structures like a metric, like a complex structure, or a fibration. We call this the *geometry of geometries*. There are several mathematical constructions, dictated by the mathematical nature of the objects of which we consider the moduli, that single out a canonical determination of the *geometry of geometries*, yet it is precisely at this level that the interaction between physics and mathematics becomes most profound and fertile. Indeed the geometry of geometries is typically what enters the supergravity lagrangians under the form of sigma-models for scalar fields that on one side are the spin zero members of supersymmetry multiplets,[2] while on the other side they are *moduli* of some

[1] *Clarification for readers with a mostly mathematical background*: in the physical literature *instantons* play a very important role. They are field configurations that in the Wick-rotated space-time with Euclidean signature satisfy first-order equations more restrictive than the second order Euler Lagrangian equations (the latter are implied by the former). In the *path integral* formulation of *quantum field theory*, instanton correspond to the absolute minimal of the action functional and provide the dominant contribution to quantum correlators. Depending on the type of considered fields instantons have different definitions. For gauge fields, instantons are the connections on the underlying principal fibre-bundle whose field strengths are self dual, namely satisfy Eq. (4.1.3).

[2] *Clarification for mathematicians:* the wording *supermultiplets* is universally used in the context of supersymmetric field theories to denote a finite set of standard fields of various spins that form a *unitary irreducible representation* of the supersymmetry algebra extending the Poincaré Lie algebra.

manifold, for a example a Calabi–Yau threefold, on which the superstring has been compactified.

This evenience produces a double check on the geometry of geometries. Its use in supersymmetric lagrangians, imposes strong constraints on the geometry of the scalar fields that, in many cases, have a recognizable solution in terms of known geometrical categories, in other cases it leads to the definition of new types of restricted geometries, generically dubbed *special geometries*. It is particularly rewarding that the *special geometries* selected by supersymmetry are just those apt to accomodate *the moduli spaces* of such mathematical structures as *the complex structures* or the *Kähler structures* of a compactification manifold like a Calabi–Yau threefold.

Altogether, a really new chapter has been written in the two decades from 1990 to 2010 in the history of geometry, where the distinction between physics and mathematics has become somewhat obsolete, ideas from one field compenetrating the other in an essential way.

4.1.4 The Advent of Special Geometries

The first instance of a special geometry was found by brute force, immediately after the discovery in 1976 by Sergio Ferrara, Daniel Freedman and Peter van Nieuwenhuizen of $\mathcal{N} = 1$, $d = 4$ supergravity (see Fig. 4.2). The next year, considering the coupling of a scalar multiplet to the newly found gravitational theory, the three supergravity founders, together with Breitenlohner, Gliozzi and Scherk, constructed a rather impressive and cumbersome lagrangian, depending on an arbitrary real function $G(A, B)$ of a scalar A and a pseudoscalar B and on all its derivatives up to the fourth one [2]. It was Bruno Zumino (see Fig. 4.3) who, in 1979, decoded the meaning of this monster, showing that $G(A, B)$ is just the Kähler potential of a Kähler metric, all of the introduced derivatives obtaining their adequate interpretation as metric, connection and curvature of the Kählerian manifold [3]. In this way the generalization to several scalar multiplets was singled out: it suffices to utilize an n-dimensional Kähler manifold.

Shortly after, the so named holomorphic superpotential introduced by physicists to describe fermion–scalar interactions and to produce a scalar potential consistent with supersymmetry, was also interpreted geometrically. The superpotential is just a holomorphic section of the Hodge line-bundle over the Kähler manifold.

In this way the firstly found special geometry was a known one, namely Hodge-Kähler geometry. This is not so for the next case.

At the beginning of the 1980's the next obvious case was the coupling of vector multiplets to $\mathcal{N} = 2$, $d = 4$ supergravity. Each multiplet contains a complex scalar field and the question was what is the geometry of the scalar manifold $\mathcal{M}_{scalar}$ in the case of several such multiplets. Certainly $\mathcal{M}_{scalar}$ had to be Kähler, since $\mathcal{N} = 2$ is in particular $\mathcal{N} = 1$. Yet the stronger supersymmetry imposes additional constraints so that $\mathcal{M}_{scalar}$ had to be a *special Kähler manifold*. A pioneering work on this problem was conducted in several different combinations by a group of French, Belgian,

Fig. 4.2 From left to right the three founders of Supergravity Theory, Daniel Freedman (1939), Sergio Ferrara (1945), Peter van Nieuwenhuizen (1938). Dan Freedman was born in the USA, graduated from Wisconsin University. He has been professor at Stony Brook University and he is currently full-professor at MIT. Sergio Ferrara born in Rome in 1945 graduated from la Sapienza University under the supervision of Raoul Gatto. Permanent Member of the CERN Theoretical Division for many years he is also professor of physics at UCLA. Peter van Nieuwenhuizen born in Holland in 1938, graduated in Utrecht under the supervision of Veltman, held various positions in the United States and since the middle 1980s he is full-professor of physics at Stony Brook University. The paper containing the lagrangian and the transformation rules of $\mathcal{N} = 1, d = 4$ supergravity was published by the three founders of the theory in 1976. Since then all the three have contributed extensively and in various different directions to the development of supergravity. Sergio Ferrara among the three has largely contributed to the development of special geometries

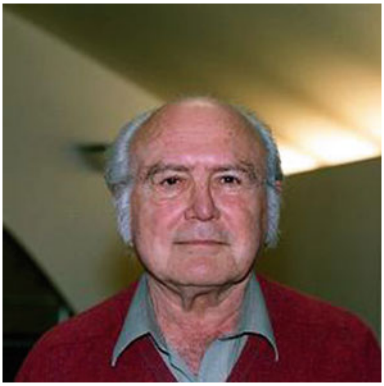

Fig. 4.3 Bruno Zumino (1923–2014). Born in 1923 in Rome, he graduated from the University La Sapienza in 1945. He died in 2014 in California, where he was emeritus professor of Berkeley University. For many years he was permanent member of the Theoretical Division at CERN. Zumino has given many important contributions to Theoretical Physics in several directions: supersymmetry, anomalies, conformal field theories, quantum groups

Fig. 4.4 On the left Antoine Van Proeyen (1953 Belgium), on the right Eugene Cremmer (Paris 1942). Antoine Van Proeyen graduated from KU Leuven and worked in several Laboratories and Universities, among which the École Normale of Paris, CERN Theoretical Division and Torino University, before becoming full-professor in Leuven. He is currently the Head of the Theoretical Physics Section at the K.U. Leuven. Since 1979, he has been involved in the construction of various supergravity theories, the resulting special geometries and their applications to phenomenology and cosmology. Cremmer is *directeur de recherche* of the CNRS working at the École Normale Supérieure of Paris. In 1978, together with Bernard Julia and Joël Scherk, he derived the space-time formulation of 11 dimensional supergravity theory, regarded today as the low energy limit of the so far mysterious M-theory. In the following few years, Cremmer, together with Bernard Julia, constructed the dimensional reductions of $d = 11$ supergravity, arriving in $d = 4$ at the maximal extended $\mathcal{N} = 8$ theory, whose structure is completely determined by the non-compact coset $\frac{\mathrm{E}_{7(7)}}{\mathrm{SU(8)}}$ accomodating the 70 scalars of the gravitational multiplet. Active research is going on at the present time to demonstrate that $\mathcal{N} = 8$ supergravity is a finite theory

Dutch, Swiss and Italian theoretical physicists in the papers mentioned in [4–6]. Using a special set of complex coordinates, the special Kähler manifolds that can accomodate the scalar fields of $\mathcal{N} = 2$ vector multiplets were described as those where the Kähler potential is obtained from a holomorphic prepotential according to a specific formula.

Once this was established, a natural question arose whether among so defined *special Kähler manifolds* there were symmetric spaces G/H. The answer to this question was given in Paris in 1985 by Eugene Cremmer and Antoine Van Proeyen (see Fig. 4.4) who, in a beautiful paper absolutely worth of Cartan's tradition [7], provided the exhaustive classification shown in the first column of Table 4.1. As one sees, exceptional Lie groups make their appearance in such a list through peculiar real forms. This was no longer a surprise for supergravity researchers since, four years before, the same Eugene Cremmer, in collaboration with Bernard Julia (see Fig. 4.5), had shown that the dimensional reduction of maximally extended supergravity from $D = 11$ down to $D = 10$, $D = 9, \ldots, D = 4$, $D = 3$ produces, as scalar manifolds, the following maximally split symmetric spaces:

$$\mathrm{M}_D = \frac{\mathrm{E}_{11-\mathrm{D}(11-\mathrm{D})}}{\mathrm{H_c}} \tag{4.1.4}$$

Table 4.1 List of special Kähler symmetric spaces with their Quaternionic Kähler c-map images. The number n denotes the complex dimension of the Special Kähler preimage. On the other hand $4n + 4$ is the real dimension of the Quaternionic Kähler c-map image

$\mathscr{SK}_n$ Special Kähler manifold	$\mathscr{QM}_{4n+4}$ Quaternionic Kähler manifold	$\dim \mathscr{SK}_n = n$
$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$	$\frac{\mathrm{G}_{2(2)}}{\mathrm{SU}(2)\times\mathrm{SU}(2)}$	$n = 1$
$\frac{\mathrm{Sp}(6,\mathrm{R})}{\mathrm{SU}(3)\times\mathrm{U}(1)}$	$\frac{\mathrm{F}_{4(4)}}{\mathrm{USp}(6)\times\mathrm{SU}(2)}$	$n = 6$
$\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1)}$	$\frac{\mathrm{E}_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)}$	$n = 9$
$\frac{\mathrm{SO}^\star(12)}{\mathrm{SU}(6)\times\mathrm{U}(1)}$	$\frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SU}(2)}$	$n = 15$
$\frac{\mathrm{E}_{7(-25)}}{\mathrm{E}_{6(-78)}\times\mathrm{U}(1)}$	$\frac{\mathrm{E}_{8(-24)}}{\mathrm{E}_{7(-133)}\times\mathrm{SU}(2)}$	$n = 27$
$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(2,2+p)}{\mathrm{SO}(2)\times\mathrm{SO}(2+p)}$	$\frac{\mathrm{SO}(4,4+p)}{\mathrm{SO}(4)\times\mathrm{SO}(4+p)}$	$n = 3 + p$
$\frac{\mathrm{SU}(p+1,1)}{\mathrm{SU}(p+1)\times\mathrm{U}(1)}$	$\frac{\mathrm{SU}(p+2,2)}{\mathrm{SU}(p+2)\times\mathrm{SU}(2)}$	$n = p + 1$

Fig. 4.5 Bernard Julia (Paris 1952). He graduated from Université de Paris-Sud in 1978, and he is *directeur de recherche* of the CNRS working at the *École Normale Suṕrieure*. In 1978, together with Eugne Cremmer and Joël Scherk, he constructed 11-dimensional supergravity. Shortly afterwards, Cremmer and Julia constructed the classical Lagrangian of four-dimensional $\mathscr{N} = 8$ supergravity by dimensional reduction from the 11-dimensional theory

where:

$$\begin{aligned}
E_{5(5)} &\simeq D_{5(5)} \simeq \mathrm{SO}(5,5) \\
E_{4(4)} &\simeq A_{4(4)} \simeq \mathrm{SL}(5,\mathbb{R}) \\
E_{3(3)} &\simeq A_{1(1)} \times A_{2(2)} \simeq \mathrm{SL}(2,\mathbb{R}) \otimes \mathrm{SL}(3,\mathbb{R}) \\
E_{2(2)} &\simeq A_{1(1)} \times A_{1(1)} \simeq \mathrm{SL}(2,\mathbb{R}) \otimes \mathrm{SL}(2,\mathbb{R})
\end{aligned} \tag{4.1.5}$$

Fig. 4.6 On the left Leonardo Castellani (born 1953 in Freiburg, Switzerland). On the right Riccardo D'Auria (born 1940 in Rome). Leonardo Castellani studied physics at the University of Florence in Italy and obtained his Ph.D from Stony Brook University in the US, with a thesis written under the supervision of van Nieuwenhuizen. He had post-doctoral positions at Caltech and at CERN, then he became permanent Researcher in the Torino section of the National Institute of Nuclear Research (INFN) and in 1993 he was appointed full-professor of Theoretical Physics at the University of Eastern Piedmont, position that he holds at the present time. He is especially known for his contributions, together with D'Auria and Fré to the rheonomic formulation of supersymmetric theories, for his derivation together with Larry Romans of the list of G/H compactifications of $d = 11$ supergravity and more recently for developments in quantum group theories and, together with P.A. Grassi and R. Catenacci for the extension of Hodge theory to supermanifolds. Riccardo D'Auria studied at the University of Torino and graduated there with a thesis written under the supervision of Tullio Regge. He was for several years Associate Professor at the University of Torino, in 1987 he was appointed full-professor of Theoretical Physics at the University of Padua. Few years later he was offered a full professor chair at the Politecnico of Torino where he concluded his academic career becoming emeritus professor in 2011. D'Auria, together with Fré has been the founder of the rheonomic formulation of supergravity and also with Fré he introduced the notion of super Free Differential Algebras, that were singled out as the algebraic basis of all supergravity theories in dimension higher than four. In particular in 1982, D'Auria and Fré obtained the FDA formulation of $d = 11$ supergravity. D'Auria has given many more contributions to supergravity theory in particular in connection with special geometries, with the classification of black-hole solutions, with duality rotations, with the various formulations of the $d = 6$ theories and with several other aspects of the superworld

So exceptional Lie groups that had been regarded for long time as mathematical curiosities were brought to prominence by supergravity and in parallel also by superstring theory.

The fact that all such results were obtained in the *École Normale Supérieure de Paris* demonstrates the far reaching influence of Élie Cartan's tradition.

At the end of the eighties the intrinsic definition of *special Kähler geometry*, free from the use of special coordinates, was independently obtained with two different strategies by Andrew Strominger (see Fig. 4.7) and by Leonardo Castellani, Riccardo D'Auria and Sergio Ferrara (see Fig. 4.6).

While Strominger derived his definition from the properties of Calabi–Yau moduli spaces [8], Castellani, D'Auria and Ferrara [9, 10] (and later D'Auria Ferrara and Fré [11]) derived their own definition from the constraints imposed by supersymmetry on

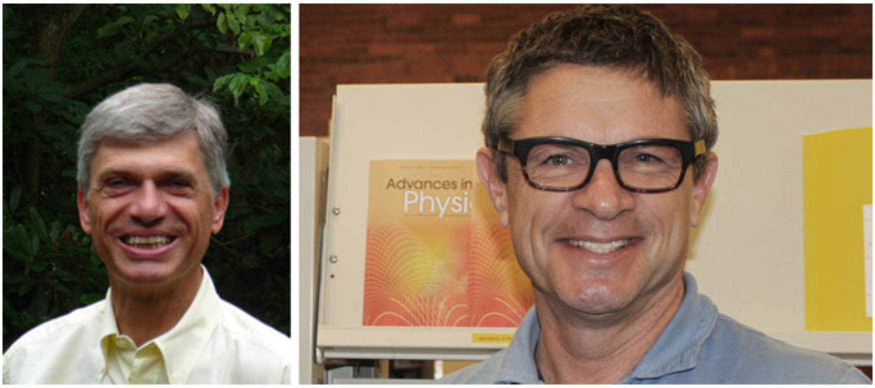

Fig. 4.7 On the left Bernard Quirinus Petrus Joseph de Wit (born 1945 in the Netherlands). On the right Andrew Eben Strominger (born 1955 in the USA). Bernard de Wit studied theoretical physics at Utrecht University, where he got his PhD under the supervision of the Nobel Prize laureate Martinus Veltman in 1973. He held postdoc positions in Stony Brook, Utrecht and Leiden. He became a staff member at the National Institute for Nuclear and High Energy Physics (NIKHEF) in 1978, where he became head of the theory group in 1981. In 1984 he was appointed professor of theoretical physics at Utrecht University where he has stayed for the rest of his career. Bernard de Wit has given important contributions to the development of supergravity theory building, in collaboration mainly with Van Proeyen, the so named *conformal tensor calculus*. Together with Herman Nicolai he constructed the $\mathfrak{so}(8)$-gauged version of $\mathcal{N} = 8$ supergravity that has provided the paradigmatic example for all supergravity gaugings. Andrew Strominger completed his undergraduate studies at Harvard in 1977 before attending the University of California, Berkeley for his Master diploma. He received his PhD from MIT in 1982 under the supervision of Roman Jackiw. Prior to joining Harvard as a professor in 1997, he held a faculty position at the University of California, Santa Barbara. Strominger is especially known for introducing, together with Cumrun Vafa the string theory explanations of the microscopic origin of black hole entropy, originally calculated thermodynamically by Stephen Hawking and Jacob Bekenstein. Strominger, together with Philippe Candelas, Gary Horowitz and Edward Witten was the first proposer of Calabi–Yau threefolds as compactification manifolds for superstrings and supergravities in $d = 10$

the curvature tensor of the Kählerian manifold. With some labour they also showed the full equivalence of the two definitions.

In the same years, Antoine Van Proeyen and Bernard de Wit (see Fig. 4.7), in some publications together with a younger collaborator, established a full classification of *homogeneous special geometries*, namely of special manifolds that admit a solvable transitive group of isometries [12–14]. They also explored the relation [12, 13] between *special Kähler geometries* and quaternionic geometries that can be obtained from them by means of a very interesting map, originally discovered by Cecotti [15] and further developed by Ferrara et al. in [16, 17]. So doing they came in touch with the classification of quaternionic manifolds with a transitive solvable group of motion that had been performed several years before by Alekseevsky [18, 19].

The map mentioned above is named the *c*-map and can be given a modern compact definition exhibited in [20]. Furthermore the *c*-map has a non Euclidean analogue, the $c^{\star}$-map that plays an important role in the discussion of supergravity based black-holes, another instance of geometry that will occupy us in later chapters.

4.1.5 A Survey of the Topics in This Chapter

In the sequel the special geometries motivated by supergravity will be thoroughly discussed and the properties of the c-map will be analyzed in detail. In that we closely follow the recent paper [20].[3] Indeed, coming to these topics our history of *Symmetry and Geometry* has reached the front of current research. Here physics and mathematics are fully entangled.

4.2 Special Kähler Geometry

In this section we present Special Kähler Geometry in a full-fledged rigorous mathematical form. Let us begin by summarizing some relevant concepts and definitions that are propaedeutical to the main definition.

4.2.1 Hodge–Kähler Manifolds

Consider a *line bundle* $\mathscr{L} \stackrel{\pi}{\longrightarrow} \mathscr{M}$ over a Kähler manifold $\mathscr{M}$. By definition this is a holomorphic vector bundle of rank $r = 1$. For such bundles the only available Chern class is the first:

$$c_1(\mathscr{L}) = \frac{i}{2}\,\overline{\partial}\,\left(h^{-1}\,\partial\,h\right) = \frac{i}{2}\,\overline{\partial}\,\partial\log h \tag{4.2.1}$$

where the 1-component real function $h(z, \overline{z})$ is some hermitian fibre metric on $\mathscr{L}$. Let $\xi(z)$ be a holomorphic section of the line bundle $\mathscr{L}$: noting that under the action of the operator $\overline{\partial}\,\partial$ the term $\log\left(\overline{\xi}(\overline{z})\,\xi(z)\right)$ yields a vanishing contribution, we conclude that the formula in Eq. (4.2.1) for the first Chern class can be re-expressed as follows:

$$c_1(\mathscr{L}) = \frac{i}{2}\,\overline{\partial}\,\partial\log\,\parallel\,\xi(z)\,\parallel^2 \tag{4.2.2}$$

where $\parallel\,\xi(z)\,\parallel^2\; = h(z, \overline{z})\,\overline{\xi}(\overline{z})\,\xi(z)$ denotes the norm of the holomorphic section $\xi(z)$.

Equation (4.2.2) is the starting point for the definition of Hodge–Kähler manifolds. A Kähler manifold $\mathscr{M}$ is a Hodge manifold if and only if there exists a line bundle

[3] An early review of Special Kähler Geometry was written by this author in 1996 in [21].

$\mathscr{L} \xrightarrow{\pi} \mathscr{M}$ such that its first Chern class equals the cohomology class of the Kähler two-form K:

$$c_1(\mathscr{L}) = [\,\mathrm{K}\,] \qquad (4.2.3)$$

In local terms this means that there is a holomorphic section $\xi(z)$ such that we can write

$$\mathrm{K} = \frac{i}{2}\, g_{ij^\star}\, dz^i \wedge d\bar{z}^{j^\star} = \frac{i}{2}\, \bar{\partial}\, \partial \log \parallel \xi(z) \parallel^2 \qquad (4.2.4)$$

Recalling the local expression of the Kähler metric in terms of the Kähler potential $g_{ij^\star} = \partial_i\, \partial_{j^\star} \mathscr{K}(z, \bar{z})$, it follows from Eq. (4.2.4) that if the manifold $\mathscr{M}$ is a Hodge manifold, then the exponential of the Kähler potential can be interpreted as the metric $h(z, \bar{z}) = \exp(\mathscr{K}(z, \bar{z}))$ on an appropriate line bundle $\mathscr{L}$.

4.2.2 Connection on the Line Bundle

On any complex line bundle $\mathscr{L}$ there is a canonical hermitian connection defined as:

$$\theta \equiv h^{-1}\, \partial\, h = \tfrac{1}{h}\, \partial_i h\, dz^i \;;\; \bar{\theta} \equiv h^{-1}\, \bar{\partial}\, h = \tfrac{1}{h}\, \partial_{i^\star} h\, d\bar{z}^{i^\star} \qquad (4.2.5)$$

For the line-bundle advocated by the Hodge-Kähler structure we have

$$\left[\,\bar{\partial}\,\theta\,\right] = c_1(\mathscr{L}) = [\mathrm{K}] \qquad (4.2.6)$$

and since the fibre metric h can be identified with the exponential of the Kähler potential we obtain:

$$\theta = \partial\, \mathscr{K} = \partial_i \mathscr{K}\, dz^i \;;\; \bar{\theta} = \bar{\partial}\, \mathscr{K} = \partial_{i^\star} \mathscr{K}\, d\bar{z}^{i^\star} \qquad (4.2.7)$$

To define special Kähler geometry, in addition to the afore-mentioned line–bundle $\mathscr{L}$ we need a flat holomorphic vector bundle $\mathscr{SV} \longrightarrow \mathscr{M}$ whose sections play an important role in the construction of the supergravity Lagrangians. For reasons intrinsic to such constructions the rank of the vector bundle $\mathscr{SV}$ must be $2\, n_V$ where n_V is the total number of vector fields in the theory. If we have n-vector multiplets the total number of vectors is $n_V = n + 1$ since, in addition to the vectors of the vector multiplets, we always have the graviphoton sitting in the graviton multiplet. On the other hand the total number of scalars is $2n$. Suitably paired into n-complex fields z^i, these scalars span the n complex dimensions of the base manifold $\mathscr{M}$ to the rank $2n + 2$ bundle $\mathscr{SV} \longrightarrow \mathscr{M}$.

In the sequel we make extensive use of covariant derivatives with respect to the canonical connection of the line–bundle $\mathscr{L}$. Let us review its normalization. As it is well known there exists a correspondence between line–bundles and U(1)–bundles. If $\exp[f_{\alpha\beta}(z)]$ is the transition function between two local trivializations of the line–

bundle $\mathscr{L} \xrightarrow{\pi} \mathscr{M}$, the transition function in the corresponding principal $\mathrm{U}(1)$–bundle $\mathscr{U} \longrightarrow \mathscr{M}$ is just $\exp[\mathrm{i}\mathrm{Im} f_{\alpha\beta}(z)]$ and the Kähler potentials in two different charts are related by: $\mathscr{K}_\beta = \mathscr{K}_\alpha + f_{\alpha\beta} + \overline{f}_{\alpha\beta}$. At the level of connections this correspondence is formulated by setting: $\mathrm{U}(1)$–connection $\equiv \mathscr{Q} = \mathrm{Im}\theta = -\frac{\mathrm{i}}{2}\left(\theta - \overline{\theta}\right)$. If we apply this formula to the case of the $\mathrm{U}(1)$–bundle $\mathscr{U} \longrightarrow \mathscr{M}$ associated with the line–bundle $\mathscr{L}$ whose first Chern class equals the Kähler class, we get:

$$\mathscr{Q} = \frac{\mathrm{i}}{2}\left(\partial_i \mathscr{K} dz^i - \partial_{i^\star} \mathscr{K} d\overline{z}^{i^\star}\right) \tag{4.2.8}$$

Let now $\Phi(z, \overline{z})$ be a section of $\mathscr{U}^p$. By definition its covariant derivative is $\nabla\Phi = (d - ip\mathscr{Q})\Phi$ or, in components,

$$\nabla_i \Phi = (\partial_i + \tfrac{1}{2} p \partial_i \mathscr{K})\Phi \ ; \ \nabla_{i^*}\Phi = (\partial_{i^*} - \tfrac{1}{2} p \partial_{i^*}\mathscr{K})\Phi \tag{4.2.9}$$

A covariantly holomorphic section of $\mathscr{U}$ is defined by the equation: $\nabla_{i^*}\Phi = 0$. We can easily map each section $\Phi(z, \overline{z})$ of $\mathscr{U}^p$ into a section of the line–bundle $\mathscr{L}$ by setting:

$$\tilde{\Phi} = e^{-p\mathscr{K}/2}\Phi \, . \tag{4.2.10}$$

With this position we obtain:

$$\nabla_i \tilde{\Phi} = (\partial_i + p\partial_i \mathscr{K})\tilde{\Phi} \ ; \ \nabla_{i^*}\tilde{\Phi} = \partial_{i^*}\tilde{\Phi} \tag{4.2.11}$$

Under the map of Eq. (4.2.10) covariantly holomorphic sections of $\mathscr{U}$ flow into holomorphic sections of $\mathscr{L}$ and viceversa.

4.2.3 Special Kähler Manifolds

We are now ready to give the first of two equivalent definitions of special Kähler manifolds:

Definition 4.2.1 A Hodge Kähler manifold is **Special Kähler (of the local type)** if there exists a completely symmetric holomorphic 3-index section W_{ijk} of $(T^\star\mathscr{M})^3 \otimes \mathscr{L}^2$ (and its antiholomorphic conjugate $W_{i^* j^* k^*}$) such that the following identity is satisfied by the Riemann tensor of the Levi–Civita connection:

$$\begin{aligned}
\partial_{m^*} W_{ijk} &= 0 \quad \partial_m W_{i^* j^* k^*} = 0 \\
\nabla_{[m} W_{i]jk} &= 0 \quad \nabla_{[m} W_{i^*]j^* k^*} = 0 \\
\mathscr{R}_{i^* j \ell^* k} &= g_{\ell^* j} g_{k i^*} + g_{\ell^* k} g_{j i^*} - e^{2\mathscr{K}} W_{i^* \ell^* s^*} W_{tkj} g^{s^* t}
\end{aligned} \tag{4.2.12}$$

In the above equations ∇ denotes the covariant derivative with respect to both the Levi–Civita and the U(1) holomorphic connection of Eq. (4.2.8). In the case of W_{ijk}, the U(1) weight is $p = 2$.

Out of the W_{ijk} we can construct covariantly holomorphic sections of weight 2 and - 2 by setting:

$$C_{ijk} = W_{ijk}\, e^{\mathscr{K}} \quad ; \quad C_{i^\star j^\star k^\star} = W_{i^\star j^\star k^\star}\, e^{\mathscr{K}} \tag{4.2.13}$$

The flat bundle mentioned in the previous subsection apparently does not appear in this definition of special geometry. Yet it is there. It is indeed the essential ingredient in the second definition whose equivalence to the first we shall shortly provide.

Let $\mathscr{L}\xrightarrow{\pi}\mathscr{M}$ denote the complex line bundle whose first Chern class equals the cohomology class of the Kähler form K of an n-dimensional Hodge–Kähler manifold $\mathscr{M}$. Let $\mathscr{SV} \longrightarrow \mathscr{M}$ denote a holomorphic flat vector bundle of rank $2n + 2$ with structural group $\mathrm{Sp}(2n + 2, \mathbb{R})$. Consider tensor bundles of the type $\mathscr{H} = \mathscr{SV} \otimes \mathscr{L}$. A typical holomorphic section of such a bundle will be denoted by Ω and will have the following structure:

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \quad \Lambda, \Sigma = 0, 1, \ldots, n$$

By definition the transition functions between two local trivializations $U_i \subset \mathscr{M}$ and $U_j \subset \mathscr{M}$ of the bundle $\mathscr{H}$ have the following form:

$$\begin{pmatrix} X \\ F \end{pmatrix}_i = e^{f_{ij}} M_{ij} \begin{pmatrix} X \\ F \end{pmatrix}_j$$

where f_{ij} are holomorphic maps $U_i \cap U_j \to \mathbb{C}$ while M_{ij} is a constant $\mathrm{Sp}(2n + 2, \mathbb{R})$ matrix. For a consistent definition of the bundle the transition functions are obviously subject to the cocycle condition on a triple overlap: $e^{f_{ij}+f_{jk}+f_{ki}} = 1$ and $M_{ij} M_{jk} M_{ki} = 1$.

Let $\mathrm{i}\langle\ |\ \rangle$ be the compatible hermitian metric on $\mathscr{H}$

$$\mathrm{i}\langle \Omega \,|\, \overline{\Omega} \rangle \equiv -\mathrm{i}\Omega^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \overline{\Omega}$$

Definition 4.2.2 We say that a Hodge–Kähler manifold $\mathscr{M}$ is **special Kähler** if there exists a bundle $\mathscr{H}$ of the type described above such that for some section $\Omega \in \Gamma(\mathscr{H}, \mathscr{M})$ the Kähler two form is given by:

$$\mathrm{K} = \frac{\mathrm{i}}{2} \partial \overline{\partial} \log \left(\mathrm{i} \langle \Omega \,|\, \overline{\Omega} \rangle \right) = \frac{i}{2}\, g_{ij^*}\, dz^i \wedge d\overline{z}^{j^*} \tag{4.2.14}$$

From the point of view of local properties, Eq. (4.2.14) implies that we have an expression for the Kähler potential in terms of the holomorphic section Ω:

$$\mathscr{K} = -\log\left(\mathrm{i}\langle\Omega\,|\,\overline{\Omega}\rangle\right) = -\log\left[\mathrm{i}\left(\overline{X}^{\Lambda}F_{\Lambda} - \overline{F}_{\Sigma}X^{\Sigma}\right)\right] \tag{4.2.15}$$

The relation between the two definitions of special manifolds is obtained by introducing a non–holomorphic section of the bundle $\mathscr{H}$ according to:

$$V = \begin{pmatrix} L^{\Lambda} \\ M_{\Sigma} \end{pmatrix} \equiv e^{\mathscr{K}/2}\Omega = e^{\mathscr{K}/2}\begin{pmatrix} X^{\Lambda} \\ F_{\Sigma} \end{pmatrix} \tag{4.2.16}$$

so that Eq. (4.2.15) becomes:

$$1 = \mathrm{i}\langle V\,|\,\overline{V}\rangle = \mathrm{i}\left(\overline{L}^{\Lambda}M_{\Lambda} - \overline{M}_{\Sigma}L^{\Sigma}\right) \tag{4.2.17}$$

Since V is related to a holomorphic section by Eq. (4.2.16) it immediately follows that:

$$\nabla_{i^\star}V = \left(\partial_{i^\star} - \frac{1}{2}\partial_{i^\star}\mathscr{K}\right)V = 0 \tag{4.2.18}$$

On the other hand, from Eq. (4.2.16), defining:

$$U_i = \nabla_i V = \left(\partial_i + \frac{1}{2}\partial_i\mathscr{K}\right)V \equiv \begin{pmatrix} f_i^{\Lambda} \\ h_{\Sigma|i} \end{pmatrix}$$
$$\overline{U}_{i^\star} = \nabla_{i^\star}\overline{V} = \left(\partial_{i^\star} + \frac{1}{2}\partial_{i^\star}\mathscr{K}\right)\overline{V} \equiv \begin{pmatrix} \overline{f}_{i^\star}^{\Lambda} \\ \overline{h}_{\Sigma|i^\star} \end{pmatrix}$$

it follows that:

$$\nabla_i U_j = \mathrm{i}C_{ijk}\,g^{k\ell^\star}\,\overline{U}_{\ell^\star} \tag{4.2.19}$$

where ∇_i denotes the covariant derivative containing both the Levi–Civita connection on the bundle $\mathscr{TM}$ and the canonical connection θ on the line bundle $\mathscr{L}$. In Eq. (4.2.19) the symbol C_{ijk} denotes a covariantly holomorphic ($\nabla_{\ell^\star}C_{ijk} = 0$) section of the bundle $\mathscr{TM}^3 \otimes \mathscr{L}^2$ that is totally symmetric in its indices. This tensor can be identified with the tensor of Eq. (4.2.13) appearing in Eq. (4.2.12). Alternatively, the set of differential equations:

$$\begin{aligned}
&\nabla_i V = U_i \\
&\nabla_i U_j = \mathrm{i}C_{ijk}g^{k\ell^\star}U_{\ell^\star} \\
&\nabla_{i^\star}U_j = g_{i^\star j}V \\
&\nabla_{i^\star}V = 0
\end{aligned} \tag{4.2.20}$$

with V satisfying equation (4.2.17) give yet another definition of special geometry. In particular it is easy to find Eq. (4.2.12) as integrability conditions of (4.2.20).[4]

4.2.4 The Vector Kinetic Matrix $\mathcal{N}_{\Lambda\Sigma}$ in Special Geometry

In the construction of supergravity actions another essential item is the complex symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$ whose real and imaginary parts are necessary in order to write the kinetic terms of the vector fields. From the physicist's viewpoint the matrix $\mathcal{N}_{\Lambda\Sigma}$ is an essential item since the Lagrangian cannot be written without it. From the mathematical viewpoint it is very much significant that the same $\mathcal{N}_{\Lambda\Sigma}$ constitutes an integral part of the Special Geometry set up. We provide its general definition in the following lines. Explicitly $\mathcal{N}_{\Lambda\Sigma}$ which, in relation to its interpretation in the case of Calabi–Yau threefolds, is named the *period matrix*, is defined by means of the following relations:

$$\overline{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\overline{L}^\Sigma \quad ; \quad h_{\Sigma|i} = \mathcal{N}_{\Lambda\Sigma} f_i^\Sigma \tag{4.2.21}$$

which can be solved introducing the two $(n+1) \times (n+1)$ vectors

$$f_I^\Lambda = \begin{pmatrix} f_i^\Lambda \\ \overline{L}^\Lambda \end{pmatrix} \quad ; \quad h_{\Lambda|I} = \begin{pmatrix} h_{\Lambda|i} \\ \overline{M}_\Lambda \end{pmatrix}$$

and setting:

$$\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda|I} \circ \left(f^{-1}\right)^I_\Sigma \tag{4.2.22}$$

Let us now consider the case where the Special Kähler manifold $\mathcal{SK}_n$ of complex dimension n has some isometry group $\mathrm{U}_{\mathcal{SK}}$. Compatibility with the Special Geometry structure requires the existence of a $2n+2$-dimensional symplectic representation of such a group that we name the **W** representation. In other words that there necessarily exists a symplectic embedding of the isometry group $\mathcal{SK}_n$

$$\mathrm{U}_{\mathcal{SK}} \mapsto \mathrm{Sp(2n+2,\mathbb{R})} \tag{4.2.23}$$

such that for each element $\xi \in \mathrm{U}_{\mathcal{SK}}$ we have its representation by means of a suitable real symplectic matrix:

$$\xi \mapsto \Lambda_\xi \equiv \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \tag{4.2.24}$$

[4]We omit the detailed proof that from Eq. (4.2.20) one obtains Eq. (4.2.12). The essential link between the two formulations resides in the second of Eq. (4.2.20) which identifies the tensor C_{ijk} with the expression of the derivative of U_i in terms of the same objects U_k.

satisfying the defining relation (in terms of the symplectic antisymmetric metric $\mathbb{C}$):

$$\Lambda_\xi^T \underbrace{\begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}}_{\equiv\mathbb{C}} \Lambda_\xi = \underbrace{\begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}}_{\mathbb{C}} \tag{4.2.25}$$

which implies the following relations on the $n \times n$ blocks:

$$\begin{aligned} A_\xi^T\, C_\xi - C_\xi^T\, A_\xi &= 0 \\ A_\xi^T\, D_\xi - C_\xi^T\, B_\xi &= \mathbf{1} \\ B_\xi^T\, C_\xi - D_\xi^T\, A_\xi &= -\mathbf{1} \\ B_\xi^T\, D_\xi - D_\xi^T\, B_\xi &= 0 \end{aligned} \tag{4.2.26}$$

Under an element of the isometry group the symplectic section Ω of Special Geometry transforms as follows:

$$\Omega\,(\xi\,\cdot\,z) \;=\; \Lambda_\xi\,\Omega\,(z) \tag{4.2.27}$$

As a consequence of its definition, under the same isometry the matrix $\mathcal{N}$ transforms by means of a generalized linear fractional transformation:

$$\mathcal{N}\,(\xi\cdot z, \xi\cdot\overline{z}) = \left(C_\xi + D_\xi\,\mathcal{N}(z,\overline{z})\right)\,\left(A_\xi + B_\xi\,\mathcal{N}(z,\overline{z})\right)^{-1} \tag{4.2.28}$$

4.3 The Quaternionic Kähler Geometry in the Image of the *c*-Map

The main object of study in the present section are those Quaternionic Kähler manifolds that are in the image of the *c*-map.[5] This latter

$$\text{c-map} \;:\; \mathcal{SK}_n \Longrightarrow \mathcal{QM}_{4n+4} \tag{4.3.1}$$

is a universal construction that starting from an arbitrary Special Kähler manifold $\mathcal{SK}_n$ of complex dimension n, irrespectively whether it is homogeneous or not, leads to a unique Quaternionic Kähler manifold $\mathcal{QM}_{4n+4}$ of real dimension $4n+4$ which contains $\mathcal{SK}_n$ as a submanifold. The precise modern definition of the *c*-map, originally introduced in [16, 17], is provided below.

[5]Not all non-compact, homogeneous Quaternionic Kähler manifolds which are relevant to supergravity (which are *normal*, i.e. exhibiting a solvable group of isometries having a free and transitive action on it) are in the image of the c-map, the only exception being the quaternionic projective spaces [14, 15].

Definition 4.3.1 Let $\mathcal{SK}_n$ be a special Kähler manifold whose complex coordinates we denote by z^i and whose Kähler metric we denote by $g_{ij^\star}$. Let moreover $\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})$ be the symmetric period matrix defined by Eq. (4.2.22), introduce the following set of $4n+4$ coordinates:

$$\left\{q^u\right\} \equiv \underbrace{\{U,a\}}_{\text{2 real}} \bigcup \underbrace{\underbrace{\{z^i\}}_{\text{n complex}}}_{\text{2n real}} \bigcup \underbrace{\mathbf{Z} = \{Z^\Lambda\,,\,Z_\Sigma\}}_{\text{(2n+2) real}} \tag{4.3.2}$$

Let us further introduce the following $(2n+2)\times(2n+2)$ matrix $\mathcal{M}_4^{-1}$:

$$\mathcal{M}_4^{-1} = \left(\begin{array}{c|c} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\,\mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}\,\mathrm{Im}\,\mathcal{N}^{-1} \\ \hline -\,\mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} & \mathrm{Im}\mathcal{N}^{-1} \end{array}\right) \tag{4.3.3}$$

which depends only on the coordinate of the Special Kähler manifold. The c-map image of $\mathcal{SK}_n$ is the unique Quaternionic Kähler manifold $\mathcal{QM}_{4n+4}$ whose coordinates are the q^u defined in (4.3.2) and whose metric is given by the following universal formula

$$ds^2_{\mathcal{QM}} = \frac{1}{4}\left(dU^2 + 4g_{ij^\star}\,dz^j\,d\bar{z}^{j^\star} + e^{-2U}\,(da + \mathbf{Z}^T\,\mathbb{C}d\mathbf{Z})^2 \,-\, 2\,e^{-U}\,d\mathbf{Z}^T\,\mathcal{M}_4^{-1}\,d\mathbf{Z}\right) \tag{4.3.4}$$

The metric (4.3.4) has the following positive definite signature

$$\mathrm{sign}\left[ds^2_{\mathcal{QM}}\right] = \left(\underbrace{+,\cdots,+}_{4+4n}\right) \tag{4.3.5}$$

since the matrix $\mathcal{M}_4^{-1}$ is negative definite.

In the case the Special Kähler pre-image is a symmetric space $\mathrm{U}_{\mathcal{SK}}/\mathrm{H}_{\mathcal{SK}}$, the manifold $\mathcal{QM}$ turns out to be symmetric spaces, $\mathrm{U}_Q/\mathrm{H}_Q$. We will come back to the issue of symmetric homogeneous Quaternionic Kähler manifolds in Sect. 4.3.4

4.3.1 The HyperKähler Two-Forms and the $\mathfrak{su}(2)$-Connection

The reason why we state that $\mathcal{QM}_{4n+4}$ is Quaternionic Kähler is that, by utilizing only the identities of Special Kähler Geometry we can construct the three complex structures $J_u^{x|v}$ satisfying the quaternionic algebra (3.6.6) the corresponding Hyper-

Kähler two-forms K^x and the $\mathfrak{su}(2)$ connection ω^x with respect to which they are covariantly constant.

The construction is extremely beautiful, it was found in [20] and it is the following one.

Consider the Kähler connection $\mathcal{Q}$ defined by Eq. (4.2.8) and furthermore introduce the following differential form:

$$\Phi \; = \; da + \mathbf{Z}^T \, \mathbb{C} \, \mathrm{d}\mathbf{Z} \tag{4.3.6}$$

Next define the two dimensional representation of both the $\mathfrak{su}(2)$ connection and of the HyperKähler 2-forms as it follows:

$$\omega = \frac{\mathrm{i}}{\sqrt{2}} \sum_{x=1}^{3} \omega^x \, \gamma_x \tag{4.3.7}$$

$$\mathbf{K} = \frac{\mathrm{i}}{\sqrt{2}} \sum_{x=1}^{3} K^x \, \sigma_x \tag{4.3.8}$$

where γ_x denotes a basis of 2×2 Euclidean γ-matrices for which we utilize the following basis which is convenient in the explicit calculations we perform in later chapters[6]:

$$\gamma_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$
$$\gamma_2 = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$
$$\gamma_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{4.3.9}$$

These γ-matrices satisfy the following Clifford algebra:

$$\left\{ \gamma_x \, , \, \gamma_y \right\} \; = \; \delta^{xy} \, \mathbf{1}_{2\times 2} \tag{4.3.10}$$

and $\frac{\mathrm{i}}{2} \, \gamma_x$ provide a basis of generators of the $\mathfrak{su}(2)$ algebra.

Having fixed these conventions the expression of the quaternionic $\mathfrak{su}(2)$-connection in terms of Special Geometry structures is encoded in the following expression for the 2×2-matrix valued 1-form ω. Explicitly we have:

[6]The chosen γ-matrices are a permutation of the standard pauli matrices divided by $\sqrt{2}$ and multiplied by $\frac{\mathrm{i}}{2}$ can be used as a basis of anti-hermitian generators for the $\mathfrak{su}(2)$ algebra in the fundamental defining representation.

$$\omega = \begin{pmatrix} -\frac{\mathrm{i}}{2}\,\mathcal{Q} - \frac{\mathrm{i}}{4}\,e^{-U}\,\Phi & e^{-\frac{U}{2}}\,V^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} \\ -\,e^{-\frac{U}{2}}\,\overline{V}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} & \frac{\mathrm{i}}{2}\,\mathcal{Q} + \frac{\mathrm{i}}{4}\,e^{-U}\,\Phi \end{pmatrix} \tag{4.3.11}$$

where V and $\overline{V}$ denote the covariantly holomorphic sections of Special geometry defined in Eq. (4.2.16). The curvature of this connection is obtained from a straightforward calculation:

$$\begin{aligned} \mathbf{K} &\equiv d\omega + \omega \wedge \omega \\ &= \begin{pmatrix} \mathfrak{u} & \mathfrak{v} \\ -\,\overline{\mathfrak{v}} & -\,\mathfrak{u} \end{pmatrix} \end{aligned} \tag{4.3.12}$$

the independent 2-form matrix elements being given by the following explicit formulae:

$$\begin{aligned} \mathfrak{u} &= -\mathrm{i}\frac{1}{2}\,K - \frac{1}{8}dS \wedge d\overline{S} - e^{-U}\,V^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} \wedge \overline{V}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} - \frac{1}{4}\,e^{-U}\,\mathrm{d}\mathbf{Z}^T \wedge \mathbb{C}\,\mathrm{d}\mathbf{Z} \\ \mathfrak{v} &= e^{-\frac{U}{2}}\left(DV^T \wedge \mathbb{C}\,\mathrm{d}\mathbf{Z} - \frac{1}{2}dS \wedge V^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z}\right) \\ \overline{\mathfrak{v}} &= e^{-\frac{U}{2}}\left(D\overline{V}^T \wedge \mathbb{C}\,\mathrm{d}\mathbf{Z} - \frac{1}{2}d\overline{S} \wedge \overline{V}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z}\right) \end{aligned} \tag{4.3.13}$$

where

$$K = \frac{\mathrm{i}}{2}\,g_{ij^\star}\,dz^i \wedge d\overline{z}^{j^\star} \tag{4.3.14}$$

is the Kähler 2-form of the Special Kähler submanifold and where we have used the following short hand notations:

$$dS = dU + \mathrm{i}\,e^{-U}\left(da + \mathbf{Z}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z}\right) \tag{4.3.15}$$

$$d\overline{S} = dU - \mathrm{i}\,e^{-U}\left(da + \mathbf{Z}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z}\right) \tag{4.3.16}$$

$$DV = dz^i\,\nabla_i V \tag{4.3.17}$$

$$D\overline{V} = d\overline{z}^{i^\star}\,\nabla_{i^\star} V \tag{4.3.18}$$

The three HyperKähler forms[7] K^x are easily extracted from Eqs. (4.3.12)–(4.3.13) by collecting the coefficients of the γ-matrix expansion and we need not to write their form which is immediately deduced. The relevant thing is that the components of K^x with an index raised through multiplication with the inverse of the quaternionic metric h^{uv} exactly satisfy the algebra of quaternionic complex structures (3.6.6). Explicitly we have:

[7] See Sect. 3.6 for notations.

$$
\begin{aligned}
K^x &= -\mathrm{i}\,4\sqrt{2}\,\mathrm{Tr}\,\left(\gamma^x\,\mathbf{K}\right) \equiv K^x_{uv}\,dq^u \wedge dq^v \\
J^{x|s}_u &= K^x_{uv}\,h^{vs} \\
J^{x|s}_u\,J^{y|v}_s &= -\delta^{xy}\,\delta^v_u + \varepsilon^{xyz}\,J^{z|v}_u
\end{aligned}
\tag{4.3.19}
$$

The above formulae are not only the general proof that the Riemanniann manifold $\mathcal{QM}$ defined by the metric (4.3.4) is indeed a Quaternionic Kähler manifold, but, what is most relevant, they also provide an algorithm to write in terms of Special Geometry structures the tri-holomorphic moment map of the principal isometries possessed by $\mathcal{QM}$.

4.3.2 The Holomorphic Moment Map in Special Kähler Manifolds

In any Kähler manifold

$$
\mathscr{P}_{\mathbf{I}}{}^x = -\frac{i}{2}\left(k^i_{\mathbf{I}}\partial_i\mathscr{K} - k^{\bar{\imath}}_{\mathbf{I}}\partial_{\bar{\imath}}\mathscr{K}\right) + \mathrm{Im}(f_{\mathbf{I}})\,, \tag{4.3.20}
$$

where $f_{\mathbf{I}} = f_{\mathbf{I}}(z)$ is a holomorphic transformation on the line-bundle, defining a compensating Kähler transformation:

$$
k^i_{\mathbf{I}}\partial_i\mathscr{K} + k^{\bar{\imath}}_{\mathbf{I}}\partial_{\bar{\imath}}\mathscr{K} = -f_{\mathbf{I}}(z) - \overline{f}_{\mathbf{I}}(\overline{z})\,. \tag{4.3.21}
$$

We also have:

$$
\mathfrak{T}_{\mathbf{I}}\cdot\Omega = \mathfrak{T}_{\mathbf{I}}\cdot\Omega + f_{\mathbf{I}}\,\Omega\,, \tag{4.3.22}
$$

$$
\mathfrak{T}_{\mathbf{I}}\cdot V + i\,\mathrm{Im}(f_{\mathbf{I}})\,V = k^i_{\mathbf{I}}\partial_i V + k^{\bar{\imath}}_{\mathbf{I}}\partial_{\bar{\imath}} V\,, \tag{4.3.23}
$$

where $\mathfrak{T}_{\mathbf{I}}\cdot\Omega$ denotes the symplectic action of the isometry on the section V. If $\mathfrak{T}_{\mathbf{I}}$ is represented by the symplectic matrix $(\mathfrak{T}_{\mathbf{I}})_\alpha{}^\beta = -(\mathfrak{T}_{\mathbf{I}})^\beta{}_\alpha$, $\alpha,\ \beta = 1, \ldots, 2n+2$:

$$
\mathfrak{T}^T_{\mathbf{I}}\,\mathbb{C} + \mathbb{C}\,\mathfrak{T}_{\mathbf{I}} = 0 \tag{4.3.24}
$$

we have $(\mathfrak{T}_{\mathbf{I}}\cdot V)^\alpha = -\mathfrak{T}_{\mathbf{I}\beta}{}^\alpha\,V^\beta = \mathfrak{T}^\alpha_{\mathbf{I}\,\beta}\,V^\beta$. From (4.3.23) and (3.7.22) we derive the following useful symplectic-invariant expression for the moment maps:

$$
\mathscr{P}_{\mathbf{I}}{}^x = -\overline{V}^\alpha\,\mathfrak{T}_{\mathbf{I}\alpha}{}^\beta\mathbb{C}_{\beta\gamma}\,V^\gamma\,. \tag{4.3.25}
$$

Equations (3.7.22), (3.7.23), (4.3.23) generalize the corresponding formulae given in Sects. 7.1 and 7.2 of [22], where the condition $f_{\mathbf{I}} = 0$ was imposed, to gaugings of non-compact isometries which are associated with non-trivial compensating Kähler transformations and/or to gauged (non-compact) isometries whose symplectic action is not diagonal.

4.3.3 Isometries of $\mathscr{Q}\mathscr{M}$ in the Image of the c-Map and Their Tri-Holomorphic Moment Maps

Let us now consider the isometries of the metric (4.3.4). There are three type of isometries:

(a) The isometries of the $(2n + 3)$–dimensional Heisenberg algebra $\mathbb{H}\mathrm{eis}$ which is always present and is universal for any $(4n + 4)$–dimensional Quaternionic Kähler manifold in the image of the c-map. We describe it below.
(b) All the isometries of the pre-image Special Kähler manifold $\mathscr{S}\mathscr{K}_n$ that are promoted to isometries of the image manifold in a way described below.
(c) The additional $2n + 4$ isometries that occur only when $\mathscr{S}\mathscr{K}_n$ is a symmetric space and such, as a consequence, is also the c-map image $\mathscr{Q}\mathscr{M}_{4n+4}$. We will discuss these isometries in Sect. 4.3.4.

For the first two types of isometries (a) and (b) we are able to write general expressions for the tri-holomorphic moment maps that utilize only the structures of Special Geometry. In the case that the additional isometries (c) do exist we have another universal formula which can be used for all generators of the isometry algebra $\mathbb{U}_{\mathscr{Q}}$ and which relies on the identification of the generators of the $\mathfrak{su}(2) \subset \mathbb{H}$ subalgebra with the three complex structures. We will illustrate the details of such an identification while discussing the example of the S^3-model.

First of all let us fix the notation writing the general form of a Killing vector. This a tangent vector:

$$
\begin{aligned}
\mathbf{k} &= k^u(q)\,\partial_u \\
&= k^\diamond\,\frac{\partial}{\partial U} + k^i\,\frac{\partial}{\partial z^i} + k^{i^\star}\,\frac{\partial}{\partial \bar{z}^{i^\star}} + k^\bullet\,\frac{\partial}{\partial a} + k^\alpha\,\frac{\partial}{\partial \mathbf{Z}^\alpha} \\
&\equiv k^\diamond\,\partial_\diamond + k^i\,\partial_i + k^{i^\star}\,\partial_{i^\star} + k^\bullet\,\partial_\bullet + k^\alpha\,\partial_\alpha
\end{aligned}
\tag{4.3.26}
$$

with respect to which the Lie derivative of the metric element (4.3.4) vanishes:

$$
\ell_{\mathbf{k}}\, ds^2_{\mathscr{Q}\mathscr{M}} = 0 \tag{4.3.27}
$$

4.3.3.1 Tri-Holomorphic Moment Maps for the Heisenberg Algebra Translations

First let us consider the isometries associated with the Heisenberg algebra. The transformation:

$$Z^\alpha \mapsto Z^\alpha + \Lambda^\alpha \quad ; \quad a \mapsto a - \Lambda^T \mathbb{C}\,\mathbf{Z} \tag{4.3.28}$$

where Λ^α is an arbitrary set of $2n+2$ real infinitesimal parameters is an infinitesimal isometry for the metric $ds^2_{\mathcal{QM}}$ in (4.3.4). It corresponds to the following Killing vector:

$$\begin{aligned} \overrightarrow{\mathbf{k}}_{[\Lambda]} &= \Lambda^\alpha \overrightarrow{\mathbf{k}}_\alpha \\ &= \Lambda^\alpha \partial_\alpha - \Lambda^T \mathbb{C}\,\mathbf{Z}\,\partial_\bullet \end{aligned} \tag{4.3.29}$$

whose components are immediately deduced by comparison of Eq. (4.3.29) with Eq. (4.3.26).

We are interested in determining the expression of the tri-holomorphic moment map $\mathfrak{P}_{[\Lambda]}$ which satisfies the defining equation:

$$\mathbf{i}_{[\Lambda]}\,\mathbf{K} \equiv \begin{pmatrix} \mathbf{i}_{[\Lambda]}\,\mathfrak{u} & \mathbf{i}_{[\Lambda]}\,\mathfrak{v} \\ -\,\mathbf{i}_{[\Lambda]}\,\overline{\mathfrak{v}} & -\,\mathbf{i}_{[\Lambda]}\,\mathfrak{u} \end{pmatrix} = \mathrm{d}\mathfrak{P}_{[\Lambda]} + \left[\omega\,,\,\mathfrak{P}_{[\Lambda]}\right] \tag{4.3.30}$$

The general solution to this problem is

$$\mathfrak{P}_{[\Lambda]} = \begin{pmatrix} -\frac{\mathrm{i}}{4}\,e^{-U}\,\Lambda^T\,\mathbb{C}\,\mathbf{Z} & \frac{1}{2}\,e^{-\frac{U}{2}}\,\Lambda^T\,C\,V \\ -\frac{1}{2}\,e^{-\frac{U}{2}}\,\Lambda^T\,C\,\overline{V} & \frac{\mathrm{i}}{4}\,e^{-U}\,\Lambda^T\,\mathbb{C}\,\mathbf{Z} \end{pmatrix} \tag{4.3.31}$$

4.3.3.2 Tri-Holomorphic Moment Map for the Heisenberg Algebra Central Charge

Consider next the isometry associated with the Heisenberg algebra central charge. The transformation:

$$a \mapsto a + \varepsilon \tag{4.3.32}$$

where ε is an arbitrary real small parameter is an infinitesimal isometry for the metric $ds^2_{\mathcal{QM}}$ in (4.3.4). It corresponds to the following Killing vector:

$$\varepsilon\,\overrightarrow{\mathbf{k}}_{[\bullet]} = \varepsilon\,\partial_\bullet \tag{4.3.33}$$

whose components are immediately deduced by comparison of Eq. (4.3.33) with Eq. (4.3.26).

We are interested in determining the expression of the tri-holomorphic moment map $\mathfrak{P}_{[\bullet]}$ which satisfies the defining equation analogous to Eq. (4.3.30):

$$\mathbf{i}_{[\bullet]}\,\mathbf{K} = \mathrm{d}\mathfrak{P}_{[\bullet]} + \left[\omega\,,\,\mathfrak{P}_{[\bullet]}\right] \tag{4.3.34}$$

The solution of this problem is even simpler than in the previous case. Explicitly we obtain:

$$\mathfrak{P}_{[\bullet]} = \begin{pmatrix} -\frac{\mathrm{i}}{8}\, e^{-U} & 0 \\ 0 & \frac{\mathrm{i}}{8}\, e^{-U} \end{pmatrix} \tag{4.3.35}$$

The explicit expression of the moment maps and Killing vectors associated with the Heisenberg isometries was used in the gauging of abelian subalgebras of the Heisenberg algebra, which is relevant to the description of compactifications of Type II superstring on a generalized Calabi–Yau manifold.

4.3.3.3 Tri-Holomorphic Moment Map for the Extension of $\mathscr{SK}_n$ Holomorphic Isometries

Next we consider the question how to write the moment map associated with those isometries that where already present in the original Special Kähler manifold $\mathscr{SK}_n$ which we c-mapped to a Quaternionic Kähler manifold.

Suppose that $\mathscr{SK}_n$ has a certain number of holomorphic Killing vectors $k^i_{\mathbf{I}}(z)$ satisfying equations (3.7.6), (3.7.7), (8.4.85) necessarily closing some Lie algebra $\mathfrak{g}_{\mathscr{SK}}$ among themselves.[8] Their holomorphic momentum-map is provided by Eq. (3.7.22). Necessarily every isometry of a special Kähler manifold has a linear symplectic $(2n+2)$-dimensional realization on the holomorphic section $\Omega(z)$ up to an overall holomorphic factor. This means that for each holomorphic Killing vector we have (see Eq. (4.3.22)):

$$k^i_{\mathbf{I}}(z)\,\partial_i\,\Omega(z) \;=\; \exp\left[f_{\mathbf{I}}(z)\right]\,\mathfrak{T}_{\mathbf{I}}\,\Omega(z)\,. \tag{4.3.36}$$

where $f_{\mathbf{I}}(z)$ the holomorphic Kähler compensator. Then it can be easily checked that the transformation:

$$z^i \;\mapsto\; z^i \,+\, k^i_{\mathbf{I}}(z) \quad ; \quad \mathbf{Z} \mapsto \mathbf{Z} \,+\, \mathfrak{T}_{\mathbf{I}}\,\mathbf{Z} \tag{4.3.37}$$

is an infinitesimal isometry of the metric (4.3.4) corresponding to the Killing vector:

$$\mathbf{k}_{\mathbf{I}} \;=\; k^i_{\mathbf{I}}(z)\,\partial_i \,+\, k^{i^\star}_{\mathbf{I}}(\bar{z})\,\partial_{i^\star} \,+\, (\mathfrak{T}_{\mathbf{I}})^\alpha{}_\beta\,\mathbf{Z}^\beta\,\partial_\alpha \tag{4.3.38}$$

Also in this case we are interested in determining the expression of the tri-holomorphic moment map $\mathfrak{P}_{[\mathbf{I}]}$ satisfying the defining equation:

$$\mathbf{i}_{\mathbf{k}_{\mathbf{I}}}\,\mathbf{K} = \mathrm{d}\mathfrak{P}_{[\mathbf{I}]} \,+\, \left[\omega\,,\,\mathfrak{P}_{[\mathbf{I}]}\right] \tag{4.3.39}$$

[8] *Clarification for mathematicians*: in the jargon ubiquitously utilized in the physical literature one says that a set of operators closes a Lie algebra when any of the commutators thereof belongs to the linear span of the same operators.

The solution is given by the expression below:

$$\mathfrak{P}_{[\mathbf{I}]} = \begin{pmatrix} \frac{\mathrm{i}}{4}\left(\mathscr{P}_{\mathbf{I}} + \frac{1}{2}e^{-U}\,\mathbf{Z}^T\,\mathbb{C}\,\mathfrak{T}_{\mathbf{I}}\,\mathbf{Z}\right) & -\frac{1}{2}\,e^{-U/2}\,V^T\,\mathbb{C}\,\mathfrak{T}_{\mathbf{I}}\,\mathbf{Z} \\ \frac{1}{2}\,e^{-U/2}\,\overline{V}^T\,\mathbb{C}\,\mathfrak{T}_{\mathbf{I}}\,\mathbf{Z} & -\frac{\mathrm{i}}{4}\left(\mathscr{P}_{\mathbf{I}} + \frac{1}{2}e^{-U}\,\mathbf{Z}^T\,\mathbb{C}\,\mathfrak{T}_{\mathbf{I}}\,\mathbf{Z}\right) \end{pmatrix} \tag{4.3.40}$$

where $\mathscr{P}_{\mathbf{I}}$ is the moment map of the same Killing vector in pure Special Geometry.

4.3.4 Homogeneous Symmetric Special Quaternionic Kähler Manifolds

When the Special Kähler manifold $\mathscr{SK}_n$ is a symmetric coset space, it turns out that the metric (4.3.4) is actually the symmetric metric on an enlarged symmetric coset manifold

$$\mathscr{QM}_{4n+4} = \frac{\mathrm{U}_Q}{\mathrm{H}_Q} \supset \frac{\mathrm{U}_{\mathscr{SK}}}{\mathrm{H}_{\mathscr{SK}}} \tag{4.3.41}$$

Naming $\Lambda[\mathfrak{g}]$ the $\mathbf{W}$-representation of any finite element of the $\mathfrak{g} \in \mathrm{U}_{\mathscr{SK}}$ group, we have that the matrix $\mathscr{M}_4(z, \overline{z})$ transforms as follows:

$$\mathscr{M}_4\,(\mathfrak{g}\cdot z, \mathfrak{g}\cdot\overline{z}) = \Lambda[\mathfrak{g}]\,\mathscr{M}_4\,(z, \overline{z})]\,\Lambda^T[\mathfrak{g}] \tag{4.3.42}$$

where $\mathfrak{g}\cdot z$ denotes the non linear action of $\mathrm{U}_{\mathscr{SK}}$ on the scalar fields. Since the space $\frac{\mathrm{U}_{\mathscr{SK}}}{\mathrm{H}_{\mathscr{SK}}}$ is homogeneous, choosing any reference point z_0 all the others can be reached by a suitable group element $\mathfrak{g}_z$ such that $\mathfrak{g}_z \cdot z_0 = z$ and we can write:

$$\mathscr{M}_4^{-1}(z, \overline{z}) = \Lambda^T[\mathfrak{g}_z^{-1}]\,\mathscr{M}_4^{-1}(z_0, \overline{z}_0)]\,\Lambda[\mathfrak{g}_z^{-1}] \tag{4.3.43}$$

This allows to introduce a set of $4n + 4$ vielbein defined in the following way:

$$E^I_{\mathscr{QM}} = \frac{1}{2}\left\{dU\,,\,\underbrace{e^i(z)}_{2n}\,,\,e^{-U}\,\left(da + \mathbf{Z}^T\mathbb{C}d\mathbf{Z}\right)\,,\,\underbrace{e^{-\frac{U}{2}}\,\Lambda[\mathfrak{g}_z^{-1}]\,\mathrm{d}\mathbf{Z}}_{2n+2}\right\} \tag{4.3.44}$$

and rewrite the metric (4.3.4) as it follows:

$$ds^2_{\mathscr{QM}} = E^I_{\mathscr{QM}}\,\mathfrak{q}_{IJ}\,E^J_{\mathscr{QM}} \tag{4.3.45}$$

where the quadratic symmetric constant tensor $\mathfrak{q}_{IJ}$ has the following form:

$$\mathfrak{q}_{IJ} = \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & \delta_{ij} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -2\,\mathscr{M}_4^{-1}(z_0,\bar{z}_0) \end{array}\right) \tag{4.3.46}$$

The above defined vielbein are endowed with a very special property namely they identically satisfy a set of Maurer Cartan equations:

$$dE^I_{\mathscr{Q}\mathscr{M}} - \frac{1}{2} f^I{}_{JK}\, E^J_{\mathscr{Q}\mathscr{M}} \wedge E^K_{\mathscr{Q}\mathscr{M}} = 0 \tag{4.3.47}$$

where f^I_{JK} are the structure constants of a solvable Lie algebra $\mathfrak{A}$ which can be identified as follows:

$$\mathfrak{A} = Solv\left(\frac{\mathrm{U}_{\mathscr{Q}}}{\mathrm{H}_{\mathscr{Q}}}\right) \tag{4.3.48}$$

In the above equation $Solv\left(\frac{\mathrm{U}_{\mathscr{Q}}}{\mathrm{H}_{\mathscr{Q}}}\right)$ denotes the Lie algebra of the solvable group manifold metrically equivalent to the non-comapact coset manifold $\frac{\mathrm{U}_{\mathscr{Q}}}{\mathrm{H}_{\mathscr{Q}}}$ according to what we explained in Sect. 2.5. In the case $\mathrm{U}_{\mathscr{S}\mathscr{K}}$ is a *maximally split* real form of a complex Lie algebra, then also $\mathrm{U}_{\mathscr{Q}}$ is maximally split and we have:

$$Solv\left(\frac{\mathrm{U}_{\mathscr{Q}}}{\mathrm{H}_{\mathscr{Q}}}\right) = \mathrm{Bor}\,(\mathbb{U}_{\mathscr{Q}}) \tag{4.3.49}$$

where $\mathrm{Bor}\,(\mathbb{U}_{\mathscr{Q}})$ denotes the *Borel subalgebra* of the semi-simple Lie algebra $\mathbb{G}$, generated by its Cartan generators and by the step operators associated with all positive roots.

According to the mathematical theory summarized in Sect. 2.5 above, the very fact that the vielbein (4.3.44) satisfies the Maurer Cartan equations of the Lie algebra $Solv\left(\frac{\mathrm{U}_{\mathscr{Q}}}{\mathrm{H}_{\mathscr{Q}}}\right)$ implies that the metric (4.3.45) is the symmetric metric on the coset manifold $\frac{\mathrm{U}_{\mathscr{Q}}}{\mathrm{H}_{\mathscr{Q}}}$ which therefore admits continuous isometries associated with all the generators of the Lie algebra $\mathbb{U}_{\mathscr{Q}}$. For reader's convenience the list of Symmetric Special manifolds and of their Quaternionic Kähler counterparts in the image of the c-map is recalled in Table 4.1 which reproduces the results of [7], according to which there is a short list of Symmetric Homogeneous Special manifolds comprising five discrete cases and two infinite series.

Inspecting Eq. (1.7.19) we immediately realize that the Lie Algebra $\mathbb{U}_Q$ contains two universal Heisenberg subalgebras of dimension $(2n+3)$, namely:

$$\begin{aligned} \mathbb{U}_{\mathscr{Q}} \supset \mathbb{H}\mathrm{eis}_1 &= \mathrm{span}_{\mathbb{R}} \left\{ \mathbf{W}^{1\alpha} , \mathbb{Z}_1 \right\} \quad ; \quad \mathbb{Z}_1 = L_+ \equiv L^1 + L^2 \\ & \left[\mathbf{W}^{1\alpha} , \mathbf{W}^{1\beta} \right] = -\frac{1}{2} \mathbb{C}^{\alpha\beta} \mathbb{Z}_1 \quad ; \quad \left[\mathbb{Z}_1 , \mathbf{W}^{1\beta} \right] = 0 \end{aligned} \tag{4.3.50}$$

$$\begin{aligned} \mathbb{U}_{\mathscr{Q}} \supset \mathbb{H}\mathrm{eis}_2 &= \mathrm{span}_{\mathbb{R}} \left\{ \mathbf{W}^{2\alpha} , \mathbb{Z}_2 \right\} \quad ; \quad \mathbb{Z}_2 = L_- \equiv L^1 - L^2 \\ & \left[\mathbf{W}^{2\alpha} , \mathbf{W}^{2\beta} \right] = -\frac{1}{2} \mathbb{C}^{\alpha\beta} \mathbb{Z}_2 \quad ; \quad \left[\mathbb{Z}_2 , \mathbf{W}^{2\beta} \right] = 0 \end{aligned} \tag{4.3.51}$$

The first of these Heisenberg subalgebras of isometries is the universal one that exists for all Quaternionic Kähler manifolds $\mathscr{QM}_{4n+4}$ lying in the image of the c-map, irrespectively whether the pre-image Special Kähler manifold $\mathscr{SK}_n$ is a symmetric space or not. The tri-holomorphic moment map of these isometries was presented in Eqs. (4.3.31) and (4.3.35). The second Heisenberg algebra exists only in the case when the Quaternionic Kähler manifold $\mathscr{QM}_{4n+4}$ is a symmetric space.

From this discussion we also realize that the central charge $\mathbb{Z}_1$ is just the L_+ generator of a universal $\mathfrak{sl}(2, \mathbb{R})_E$ Lie algebra that exists only in the symmetric space case and which was named the Ehlers algebra in Sect. 1.7 where we presented the golden splitting (1.7.12). When $\mathfrak{sl}(2, \mathbb{R})_E$ does exist we can introduce the universal compact generator:

$$\mathfrak{S} \equiv L_+ - L_- = 2\,\lambda^2 \tag{4.3.52}$$

which rotates the two sets of Heisenberg translations one into the other:

$$\left[\mathfrak{S} , \mathbf{W}^{i\alpha} \right] = \varepsilon^{ij}\, \mathbf{W}^{j\alpha} \tag{4.3.53}$$

The gauging of this generator is a rather essential ingredient in the inclusion of one-field cosmological models into gauged $\mathscr{N} = 2$ supergravity as it was explained in [20].

4.3.4.1 The Tri-Holomorphic Moment Map in Homogeneous Symmetric Quaternionic Kähler Manifolds

In the case the Quaternionic Kähler manifold $\mathscr{QM}_{4n+4}$ is a homogeneous symmetric space $\frac{\mathrm{U}_{\mathscr{Q}}}{\mathrm{H}_{\mathscr{Q}}}$, the tri-holomorphic moment map associated with any generator of $\mathfrak{t} \in \mathbb{U}_{\mathscr{Q}}$ of the isometry Lie algebra can be easily constructed by means of the formula:

$$\mathscr{P}^x_{\mathfrak{t}} = \mathrm{Tr}_{[\mathbf{fun}]} \left(J^x\, \mathbb{L}^{-1}_{Solv}\, \mathfrak{t}\, \mathbb{L}_{Solv} \right) \tag{4.3.54}$$

where:

(a) J^x are the three generators of the $\mathfrak{su}(2)$ factor in the isotropy subalgebra $\mathbb{H} = \mathfrak{su}(2) \oplus \mathbb{H}'$, satisfying the quaternionic algebra (4.3.19). They should be normalized in such a way as to realize the following condition. Naming:

$$\Xi = \mathbb{L}^{-1}_{Solv}(q)\,\mathrm{d}\mathbb{L}_{Solv}(q) \tag{4.3.55}$$

the Maurer Cartan differential one-form, its projection on J^x should precisely yield the $\mathfrak{su}(2)$ one-form defined in Eq. (4.3.11):

$$\omega = -\frac{\mathrm{i}}{\sqrt{2}N_f}\sum_{x=1}^{3}\mathrm{Tr}_{[\mathbf{fun}]}\left(J^x\,\Xi\right)\gamma_x = \begin{pmatrix} -\frac{\mathrm{i}}{2}\,\mathcal{Q} - \frac{\mathrm{i}}{4}\,e^{-U}\,\Phi & e^{-\frac{U}{2}}\,V^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} \\ -e^{-\frac{U}{2}}\,\overline{V}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} & \frac{\mathrm{i}}{2}\,\mathcal{Q} + \frac{\mathrm{i}}{4}\,e^{-U}\,\Phi \end{pmatrix} \tag{4.3.56}$$

In the above equation, which provides the precise link between the c-map description and the coset manifold description of the same geometry, $N_f = \dim \mathbf{fun}$ denotes the dimension of the fundamental representation of $\mathbb{U}_{\mathcal{Q}}$.

(b) The solvable coset representative $\mathbb{L}_{Solv}(q)$ is obtained by exponentiation of the Solvable Lie algebra:

$$\mathbb{L}_{Solv}(q) \simeq \exp\left[q \cdot Solv\left(\frac{\mathrm{U}_{\mathcal{Q}}}{\mathcal{H}_{\mathcal{Q}}}\right)\right] \tag{4.3.57}$$

but the detailed exponentiation rule has to be determined in such a way that projecting the same Maurer Cartan form (4.3.55) along an appropriate basis of generators $T_{I|Solv}$ of the solvable Lie algebra $Solv\left(\frac{\mathrm{U}_{\mathcal{Q}}}{\mathcal{H}_{\mathcal{Q}}}\right)$ we precisely obtain the vielbein E^I_{QM} defined in Eq. (4.3.44). This is summarized in the following general equations:

$$\begin{aligned} E^I_{\mathcal{Q}\mathcal{M}} &= \mathrm{Tr}_{[\mathbf{fun}]}\left(T^I_{Solv}\,\Xi\right) \\ \delta^I_J &= \mathrm{Tr}_{[\mathbf{fun}]}\left(T^I_{Solv}\,T_{I|Solv}\right) \\ \Xi &= E^I_{\mathcal{Q}\mathcal{M}}\,T_{I|Solv} \end{aligned} \tag{4.3.58}$$

In Eq. (4.3.58) by T^I_{Solv} we have denoted the conjugate (with respect to the trace) of the solvable Lie algebra generators.

A general comment is in order. The precise calibration of the basis of the solvable generators T^I_{Solv} and of their exponentiation outlined in Eq. (4.3.57) which allows the identification (4.3.58) is a necessary and quite laborious task in order to establish the bridge between the general c-map description of the quaternionic geometry and its actual realization in each symmetric coset model. This is also an unavoidable step in order to give a precise meaning to the very handy formula (4.3.54) for the tri-holomorphic map. It should also be noted that although (4.3.54) covers all the cases, the result of such a purely algebraic calculation is difficult to be guessed a priori. Hence educated guesses on the choice of generators whose gauging produces a priori determined features are difficult to be inferred from (4.3.54). The analytic structure of the tri-holomorphic moment map instead is much clearer in the c-map framework of formulae (4.3.31), (4.3.35), (4.3.40). The use of both languages and the construction of the precise bridge between them in each model is therefore an essential ingredient to understand the nature and the properties of candidate gaugings in whatever physical application.

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