Chapter 3
Complex and Quaternionic Geometry

Mathematics, however, is, as it were, its own explanation; this, although it may seem hard to accept, is nevertheless true, for the recognition that a fact is so is the cause upon which we base the proof.

Girolamo Cardano

3.1 Imaginary Units and Geometry

Considering the possible types of numbers we have $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. This is a message for geometry. Keeping the fundamental idea that a geometrical space should be viewed as a manifold, constructed by means of an atlas of open charts, the local coordinates could be chosen not only as real numbers but also as complex, quaternionic or even octonionic numbers. Yet an important lesson is immediately learnt from the story told in my other book [1], twin of the present one: the possible numbers are, anyhow, division algebras over the reals, whose classification is due to Frobenius, so that the real structure remains the basis for everything.

This must be the same also in geometry. Manifolds of complex, quaternionic or octonionic type, if they exist, are, first of all, real manifolds. Their characterization as complex, quaternionic or octonionic must reside in some additional richer structure they are able to support. It is evident that this additional structure are the imaginary units, the same that provide the extensions of the field $\mathbb{R}$ to $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

Hence the conceptual path we have to follow starts revealing itself. We have to imagine what the imaginary units might be in the context of differential geometry. The catch is the relation $\mathbf{J}^2 = -\mathbf{1}$. How to reinterpret such a relation? It is rather natural to consider $\mathbf{J}$ as a map, in particular a linear map, and $\mathbf{1}$ as the identity map which always exists. We are almost there, the remaining question is *on which space does* $\mathbf{J}$ *act*? The answer is obvious since for linear maps we need vector spaces and if

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we want to do things locally, point by point on the manifold, we need *vector bundles*. The universal vector-bundle that it is intrinsically associated with any manifold $\mathscr{M}$ is the tangent bundle $T\mathscr{M} \rightarrow \mathscr{M}$. Hence the imaginary units, that from now on we will name *complex structures*, are linear maps operating on sections of the tangent bundle that square to minus one.

Complex and quaternionic or hyper-complex geometries arise when a manifold admits one or more complex structures satisfying appropriate algebraic relations. This mixture of algebra and geometry leads to new classes of very interesting spaces:

(a) Complex Manifolds
(b) Complex Kähler Manifolds
(c) HyperKähler Manifolds
(d) Quaternionic Kähler Manifolds

that is the mission of the present chapter to define and illustrate.

Furthermore when we come to discuss the symmetries of such manifolds, namely their isometries, which is the main interest of this book, we discover that the presence of the complex-structures entrains a new very much challenging viewpoint on continuous symmetries. To the Killing vectors, thanks to the symplectic structures implied by the complex-structures we are able to associate *hamiltonian functions*, named *moment maps*. These moment maps open a vast playing ground for new constructions of high relevance both in Physics and Mathematics.

3.1.1 The Precognitions of Supersymmetry

Supersymmetric field-theories and in particular Supergravity have the remarkable property of an intrinsic precognition of geometric and algebraic structures. All classes of existing geometries found, in due time, their proper role within the frame of supersymmetric field theories. For instance Kähler Manifolds describe the most general coupling of scalar multiplets in $\mathscr{N} = 1$ rigid supersymetry, while HyperKähler Manifolds do the same for the rigid $\mathscr{N} = 2$ case (see [2] which will be extensively discussed in Chap. 8). Quaternionic Kähler Manifolds are the obligatory structure for the coupling of hypermultiplets to $\mathscr{N} = 2$ supergravity [3–5]. In these cases the precognition resides in algebraic relations that come from supersymmetry and, once duely interpreted, were shown to imply the mentioned geometry. In other, even more spectacular cases, the geometric structures required by supersymmetry were not yet available in the mathematical supermarkets when the corresponding supermultiplets were studied. They were just discovered by the physicists working in supergravity and now constitute new chapters of mathematics. These are the *Special Geometries* to which Chap. 4 is devoted.

Let us now turn to complex structures and their heritage.

3.2 Complex Structures on 2n-Dimensional Manifolds

Let $\mathscr{M}$ be a 2n-dimensional manifold, $T\mathscr{M}$ its tangent space and $T^*\mathscr{M}$ its cotangent space. Denoting by $\{\phi^\alpha\}$ $(\alpha = 1, \ldots, 2n)$ the $2n$ coordinates in a patch, a section $\mathbf{t} \in \Gamma(T\mathscr{M}, \mathscr{M})$ is represented by a linear differential operator:

$$\mathbf{t} = t^\alpha \boldsymbol{\partial}_\alpha \tag{3.2.1}$$

while a section in $T^*\mathscr{M}$ is a differential 1-form

$$\omega = d\phi^\alpha \omega_\alpha(\phi) \tag{3.2.2}$$

The contraction is an operation that to each vector field $\mathbf{t} \in \Gamma(T\mathscr{M}, \mathscr{M})$ associates a map

$$i_\mathbf{t} \; : \; T^*\mathscr{M} \longrightarrow \mathbb{C}^\infty(\mathscr{M}) \tag{3.2.3}$$

of 1-forms into 0-forms locally given by the following expression:

$$i_\mathbf{t}\, \omega = t^\alpha(\phi)\omega_\alpha(\phi) \tag{3.2.4}$$

In particular, if $\omega = df$ we have

$$i_\mathbf{t}\, df = t^\alpha \partial_\alpha f = \mathbf{t} f \tag{3.2.5}$$

The contraction is also canonically extended to higher forms:

$$\forall \mathbf{t} \in \Gamma(T\mathscr{M}, \mathscr{M}) \quad : \quad \begin{cases} i_\mathbf{t} \; : \; \Omega^p(\mathscr{M}) \longrightarrow \Omega^{p-1}(\mathscr{M}) \\ i_\mathbf{t}\, \omega = t^\alpha(\phi)\omega_{\alpha\beta_1\ldots\beta_{p-1}}(\phi)\, d\phi^{\beta_1} \wedge \cdots \wedge d\phi^{\beta_{p-1}} \end{cases} \tag{3.2.6}$$

Now we can consider a linear operator L acting on the tangent bundle $T\mathscr{M}$, or more precisely acting on $\Gamma(T\mathscr{M}, M)$:

$$\begin{aligned} &L : \Gamma(T\mathscr{M}, \mathscr{M}) \to \Gamma(T\mathscr{M}, \mathscr{M}) \\ &\forall \mathbf{t} \in \Gamma(T\mathscr{M}, \mathscr{M}) : \quad L\mathbf{t} \in \Gamma(T\mathscr{M}, \mathscr{M}) \\ &\forall \alpha, \beta \in \mathbb{C}, \;\; \forall \mathbf{t}_1, \mathbf{t}_2 \in \Gamma(T\mathscr{M}, \mathscr{M}) : \quad L(\alpha \mathbf{t}_1 + \beta \mathbf{t}_2) = \alpha L\mathbf{t}_1 + \beta L\mathbf{t}_2 \end{aligned} \tag{3.2.7}$$

In every local chart L is represented by a mixed tensor $L_\alpha^\beta(\phi)$ with one covariant index and one controvariant index such that

$$L\mathbf{t} = t^\alpha(\phi) L_\alpha^\beta(\phi) \boldsymbol{\partial}_\beta \tag{3.2.8}$$

Moreover the action of L is naturally pulled back on the cotangent space:

$$L : \Gamma(T\mathscr{M}^*, \mathscr{M}) \to \Gamma(T\mathscr{M}^*, \mathscr{M}) \tag{3.2.9}$$

by defining

$$i_{\mathbf{t}} L\omega = i_{L\mathbf{t}}\omega \tag{3.2.10}$$

which in a local chart yields

$$L\omega = d\phi^\alpha L_\alpha^\beta(\phi)\omega_\beta \tag{3.2.11}$$

Definition 3.2.1 A 2n-dimensional manifold $\mathscr{M}$ is called almost complex if it has an almost complex structure. An almost complex structure is a linear operator J : $\Gamma(T\mathscr{M}, \mathscr{M}) \to \Gamma(T\mathscr{M}, \mathscr{M})$ which satisfies the following property:

$$J^2 = -\mathbb{1} \tag{3.2.12}$$

In every local chart the operator J is represented by a tensor $J_\beta^\alpha(\phi)$ such that

$$J_\alpha^\beta(\phi) J_\beta^\gamma(\phi) = -\delta_\alpha^\gamma \tag{3.2.13}$$

and by a suitable change of basis at every point $p \in \mathscr{M}$ we can reduce J_α^β to the form

$$\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

where $\mathbb{1}$ is the $n \times n$ unity matrix. A local frame where J takes the form (3.2.14) is called a "well-adapted" frame to the almost complex structure. Naming

$$\mathbf{e}_\alpha = \boldsymbol{\partial}_\alpha = \frac{\partial}{\partial\phi_\alpha} \tag{3.2.14}$$

the basis of the well-adapted frame we have

$$\begin{aligned} J\mathbf{e}_\alpha &= -\mathbf{e}_{\alpha+n} \quad if \quad \alpha \leq n \\ J\mathbf{e}_\alpha &= \mathbf{e}_{\alpha-n} \quad if \quad \alpha > n \end{aligned} \tag{3.2.15}$$

At this point, introducing the index i with range $i = 1, \ldots, n$ we can define the complex vectors:

$$\begin{aligned} \mathbf{E}_i &= \mathbf{e}_i - i\mathbf{e}_{i+n} \\ \mathbf{E}_{i^*} &= \mathbf{e}_i + i\mathbf{e}_{i+n} \end{aligned} \tag{3.2.16}$$

and we obtain the following result:

$$\begin{aligned} J\mathbf{E}_i &= i\mathbf{E}_i \\ J\mathbf{E}_{i^*} &= -i\mathbf{E}_{i^*} \end{aligned} \tag{3.2.17}$$

The tangent vectors $\mathbf{E}_i$ are the partial derivatives along the complex coordinates:

$$z^i = \phi^i + i\phi^{i+n} \tag{3.2.18}$$

while $\mathbf{E}_{i^*}$ are the partial derivatives along the complex conjugate coordinates $\overline{z}^{i^*} = \phi^i - i\phi^{i+n}$:

$$\mathbf{E}_i = \boldsymbol{\partial}_i = \frac{\partial}{\partial z^i} \quad \mathbf{E}_{i^*} = \boldsymbol{\partial}_{i^*} = \frac{\partial}{\partial z^{i^*}} \tag{3.2.19}$$

This construction is the reason why J is called an almost complex structure: the existence of this latter guarantees that at every point $p \in \mathscr{M}$ we can replace the $2n$ real coordinates by n complex coordinates, corresponding to a well-adapted frame. Moreover every two well-adapted frames are related to each other by a coordinate transformation which is a holomorphic function of the corresponding complex coordinates. Indeed let

$$\phi^\alpha \to \phi^\alpha + \zeta^\alpha(\phi) \tag{3.2.20}$$

be an infinitesimal coordinate transformation connecting two well adapted frames. By definition this means

$$\partial_\alpha \zeta^\beta J^\gamma_\beta = J^\beta_\alpha \partial_\beta \zeta^\gamma \tag{3.2.21}$$

which is nothing but the Cauchy–Riemann equation for the real and imaginary parts of a holomorphic function. Hence Eq. (3.2.20) can be replaced by

$$z^i \to z^i + \zeta^i(z) \tag{3.2.22}$$

where $\zeta^i(z)$ is a holomorphic function of z^j. Conversely if $\mathscr{M}$ is a complex analytic manifold,[1] in every local chart $\{z^i\}$ we can set

$$\phi^\alpha = \text{Re} z^i \;\; (\alpha \leq n) \quad \phi^\alpha = \text{Im} z^i \;\; (\alpha > n) \tag{3.2.23}$$

and we can define an almost complex structure J. Now let J act on $T^*(\mathscr{M})$. In a well-adapted frame we have

$$\begin{aligned} J dz^i &= i dz^i \\ J dz^{i^*} &= -i dz^{i^*} \end{aligned} \tag{3.2.24}$$

[1]Complex analytic manifold means a manifold whose transition functions in the intersection of two charts are holomorphic functions of the local coordinates.

Equation (3.2.24) characterize the holomorphic coordinates. More generally let $\{x^\alpha\}$ be a generic coordinate system (not necessarily well-adapted) and let $w(x)$ be a complex-valued function on the manifold $\mathscr{M}$: we say that w is holomorphic if it satisfies the equation:

$$J dw = i dw \tag{3.2.25}$$

which in the generic coordinate system $\{x^\alpha\}$ reads as follows:

$$J_\alpha^\beta \partial_\beta w(x) = i \partial_\alpha w(x) \tag{3.2.26}$$

As we have seen, at every point $p \in \mathscr{M}$, J can be reduced to the canonical form (3.2.14) by a suitable coordinate transformation: what is not guaranteed is whether J can be reduced to this canonical form in a whole open neighbourhood $\mathscr{U}_p$. This amounts to asking the question whether Eq. (3.2.26) admits n $\mathbb{C}$-linearly independent solutions in some open subset $\mathscr{U} \in \mathscr{U}_X$, where $\mathscr{U}_X$ is the domain of the considered local chart $\{x^\alpha\}$. If these solutions $w^i(x)$ exist we can consider them as the holomorphic coordinates in the neighbourhood $\mathscr{U}$, that is we can set

$$z^i = w^i(z) \tag{3.2.27}$$

In view of what we discussed before, the transition function between any two such coordinate systems is holomorphic. Hence if Eq. (3.2.25) is integrable, then a holomorphic coordinate system exists and any function ϕ on the manifold can be viewed as a function of z^i and z^{i^*}: $\phi = \phi(z, z^{i^*})$. In this case we have

$$\begin{aligned} d\phi &= \partial_i \phi dz^i + \partial_{i^*} \phi d\overline{z}^{i^*} \\ J d\phi &= i(\partial_i \phi dz^i - \partial_{i^*} \phi dz^{i^*}) \end{aligned} \tag{3.2.28}$$

By taking the exterior derivative of Eq. (3.2.28) we obtain

$$dJ \wedge d\phi = -2i \partial_i \partial_{i^*} \phi dz^i \wedge dz^{i^*} \tag{3.2.29}$$

and we can verify the equation

$$(1 - J) dJ \wedge d\phi = 0 \tag{3.2.30}$$

which follows from

$$J dJ \wedge d\phi = -2i \partial_i \partial_{j^*} \phi J dz^i \wedge J dz^{j^*} = -2i \partial_i \partial_{j^*} \phi dz^i \wedge dz^{j^*} = dJ \wedge d\phi \tag{3.2.31}$$

Equation (3.2.30) is true in a holomorphic coordinate system and, being an exterior algebra statement, must be true in every coordinate system. In the real coordinate system Eq. (3.2.30) reads

$$T^\alpha_{\beta\gamma} \partial_\alpha \phi dx^\beta \wedge dx^\gamma = 0 \tag{3.2.32}$$

where the tensor

$$T^{\alpha}_{\beta\gamma} = \partial_{[\beta} J^{\alpha}_{\gamma]} - J^{\mu}_{\beta} J^{\nu}_{\gamma} \partial_{[\mu} J^{\alpha}_{\nu]} \tag{3.2.33}$$

is called the "torsion", or the Nienhuis tensor of the almost complex structure J^{α}_{β}. The vanishing of $T^{\alpha}_{\beta\gamma}$ is a necessary condition for the integrability of Eq. (3.2.26) and hence for the existence of a complex structure. It can be shown that it is also sufficient provided $T^{\alpha}_{\beta\gamma}$ is real analytic with respect to some real coordinate system.

3.3 Metric and Connections on Holomorphic Vector Bundles

In the previous section we considered the structure of complex manifolds. When both the base space and the standard fibre are complex manifolds we can refine the notion of fibre bundle by requiring that the transition function be locally holomorphic functions. In particular a very relevant concept, which plays a major role in our subsequent developments, is that of holomorphic vector bundle. For convenience we recall the complete definition that follows from the general definition of fibre-bundle.

Definition 3.3.1 Let $\mathscr{M}$ be a complex manifold and E be another complex manifold. A holomorphic vector bundle with total space E and base manifold $\mathscr{M}$ is given by a projection map:

$$\pi \; : \; E \longrightarrow \mathscr{M} \tag{3.3.1}$$

such that

(a) π is a holomorphic map of E onto $\mathscr{M}$

(b) Let $p \in \mathscr{M}$, then the fibre over p

$$E_p = \pi^{-1}(p) \tag{3.3.2}$$

is a complex vector space of dimension r. (The number r is called the rank of the vector bundle.)

(c) For each $p \in \mathscr{M}$ there is a neighbourhood U of p and a holomorphic homeomorphism

$$h \; : \; \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^r \tag{3.3.3}$$

such that

$$h\left(\pi^{-1}(p)\right) = \{\, p \,\} \times \mathbb{C}^r \tag{3.3.4}$$

(The pair (U, h) is called a local trivialization.)

(d) The transition functions between two local trivializations (U_α, h_α) and (U_β, h_β):

$$h_\alpha \circ h_\beta^{-1} \; : \; (U_\alpha \cap U_\beta) \otimes \mathbb{C}^r \longrightarrow (U_\alpha \cap U_\beta) \otimes \mathbb{C}^r \tag{3.3.5}$$

induce holomorphic maps

$$g_{\alpha\beta} \,:\, (U_\alpha \,\cap\, U_\beta\,) \,\longrightarrow\, \mathrm{GL}\,(r,\mathbb{C}) \tag{3.3.6}$$

Let $E \longrightarrow \mathscr{M}$ be a holomorphic vector bundle of rank r and $U \subset \mathscr{M}$ an open subset of the base manifold. A frame over U is a set of r holomorphic sections $\{s_1, \ldots, s_r\}$ such that $\{s_1(z), \ldots, s_r(z)\}$ is a basis for $\pi^{-1}(z)$ for any $z \in U$. Let $f \equiv \{e_I(z)\}$ be a frame of holomorphic sections. Any other holomorphic section ξ is described by

$$\xi \;=\; \xi^I(z)\, e_I \tag{3.3.7}$$

where

$$\overline{\partial}\, \xi^I \;=\; d\overline{z}^{j^\star}\, \overline{\partial}_{j^\star}\, \xi^I \;=\; 0 \tag{3.3.8}$$

Given a holomorphic bundle with a frame of sections we can discuss metrics connections and curvatures, as we already did for the general case of bundles.

In general a connection θ is defined by introducing the covariant derivative of any section ξ

$$D\,\xi \;=\; d\,\xi \,+\, \theta\,\xi \tag{3.3.9}$$

where $\theta \;=\; \theta^I{}_J$, the connection coefficient, is an $r \,\times\, r$ matrix-valued 1-form. On a complex manifold this 1-form can be decomposed into its parts of holomorphic type (1, 0) and (0, 1), respectively:

$$\begin{aligned} \theta &= \theta^{(1,0)} \,+\, \theta^{(0,1)} \\ \theta^{(1,0)} &= dz^i\, \theta_i \\ \theta^{(0,1)} &= d\overline{z}^{i^\star}\, \theta_{i^\star} \end{aligned} \tag{3.3.10}$$

Let now a *fiber hermitian metric h* be defined on the holomorphic vector bundle. This is a sesquilinear form that yields the scalar product of any two holomorphic sections ξ and η at each point of the base manifold:

$$\langle\, \xi\,,\, \eta\,\rangle_h \;\equiv\; \overline{\xi}^{I^\star}(\overline{z})\, \eta^J(z)\, h_{I^\star J}(z,\overline{z}) \;=\; \xi^\dagger\, h\, \eta \tag{3.3.11}$$

As it is evident from the above formula, the metric h is defined by means of the point-dependent hermitian matrix $h_{I^\star J}(z,\overline{z})$, which is requested to transform, from one local trivialization to another, with the inverses of the transition functions $g_{\alpha\beta}$ defined in Eq. (3.3.6). This is so because the scalar product $\langle\, \xi\,,\, \eta\,\rangle_h$ is by definition an invariant (namely a scalar function globally defined on the manifold).

Definition 3.3.2 A hermitian metric for a complex manifold $\mathscr{M}$ is a hermitian fibre metric on the canonical tangent bundle $T\mathscr{M}$. In this case the transition functions $g_{\alpha\beta}$ are given by the jacobians of the coordinate transformations.

In general h is just a metric on the fibres and the transition functions are different objects from the Jacobian of the coordinate transformations. In any case, given a fibre metric on a holomorphic vector bundle we can introduce a canonical connection θ associated with it. It is defined by requiring that

$$\begin{array}{lll} (A) & d\,\langle\,\xi\,,\,\eta\,\rangle_h & = \langle\,D\,\xi\,,\,\eta\,\rangle_h \,+\, \langle\,\xi\,,\,D\,\eta\,\rangle_h \\ (B) & D^{(0,1)}\xi \,\equiv\, \left[\,\overline{\partial}\,+\,\theta^{(0,1)}\,\right]\xi = 0 & \end{array} \tag{3.3.12}$$

namely by demanding that the scalar product be invariant with respect to the parallel transport defined by θ and by requiring that the holomorphic sections be transported into holomorphic sections. Let f be a holomorphic frame. In this frame the canonical connection is given by

$$\theta(f) \;=\; h(f)^{-1}\,\partial\,h(f) \tag{3.3.13}$$

or, in other words, by

$$\theta^I{}_J \;=\; dz^i\,h^{IJ^\star}\,\partial_i\,h_{K^\star J} \tag{3.3.14}$$

In the particular case of a manifold metric (see Definition 3.3.2), where h is a fibre metric on the tangent bundle $T\mathscr{M}$, the general formula (3.3.14) provides the definition of the Levi-Civita connection:

$$dz^k\,\Gamma^i_{kj} \;=\; -\,g^{il^\star}\,\partial\,g_{l^\star j} \tag{3.3.15}$$

Given a connection we can compute its curvature by means of the standard formula $\Theta \;=\; d\theta + \theta\,\wedge\theta$. In the case of the above-defined canonical connection we obtain

$$\Theta\;(f\,) \;=\; \partial\,\theta \,+\, \overline{\partial}\,\theta \,+\, \theta\,\wedge\,\theta \,=\, \overline{\partial}\,\theta \tag{3.3.16}$$

This identity follows from $\partial\,\theta \;+\; \theta\,\wedge\,\theta \;=\; 0$, which is identically true for the canonical connection (3.3.13). Component-wise the curvature 2-form is given by

$$\Theta^I{}_J \;=\; \overline{\partial}_i\,\left(h^{IK^\star}\,\partial_j\,h_{K^\star J}\right)\,d\overline{z}^i\,\wedge\,dz^j \tag{3.3.17}$$

For the case of the Levi-Civita connection defined in Eq. (3.3.15) we find

$$\begin{aligned} \Gamma^i_j &= \Gamma^i_{kj} dz^k \\ \Gamma^i_{kj} &= -g^{i\ell^*}(\partial_j g_{k\ell^*}) \\ \Gamma^{i^*}_{j^*} &= \Gamma^{i^*}_{k^* j^*} d\overline{z}^{k^*} \\ \Gamma^{i^*}_{k^* j^*} &= -g^{i^*\ell}(\partial_{j^*} g_{k^*\ell}) \end{aligned} \tag{3.3.18}$$

for the connection coefficients and

$$\begin{aligned}
\mathscr{R}^{i}_{j} &= \mathscr{R}^{i}_{jk^*\ell} d\overline{z}^{k^*} \wedge dz^{\ell} \\
\mathscr{R}^{i}_{jk^*\ell} &= \partial_{k^*} \Gamma^{i}_{j\ell} \\
\mathscr{R}^{i^*}_{j^*} &= \mathscr{R}^{i^*}_{j^*k\ell^*} dz^{k} \wedge d\overline{z}^{\ell^*} \\
\mathscr{R}^{i^*}_{j^*k\ell^*} &= \partial_{k} \Gamma^{i^*}_{j^*\ell^*}
\end{aligned} \tag{3.3.19}$$

for the curvature 2-form. The Ricci tensor has a remarkable simple expression:

$$\mathscr{R}^{n}_{m^*} = \mathscr{R}^{i}_{m^*n\,i} = \partial_{m^*} \Gamma^{i}_{ni} = \partial_{m^*} \partial_{n}\, ln\,(\sqrt{g}) \tag{3.3.20}$$

where $g = det|g_{\alpha\beta}| = (det|g_{i\,j^*}|)^2$.

3.4 Characteristic Classes and Elliptic Complexes

The cohomology[2] of differential forms on differentiable manifolds is named *de Rham cohomology*.[3] There are more general constructions of the same type. They are named *elliptic complexes*.

Elliptic complexes are associated with fibre-bundles and their general definition is provided below. To each elliptic complex we can associate a topological number that is named its *index*. On its turn the index of a complex can be calculated as the integral of certain polynomials in the curvature 2-forms of the connection that can be introduced on the corresponding principle bundle. These polynomials are named characteristic classes.

More precisely characteristic classes are maps from the ring $I^\star(\mathbb{G})$ of invariant polynomials on the Lie algebra $\mathbb{G}$ of the structural group of the bundle to the de Rham cohomology ring $H^\star(\mathscr{M})$ of its base manifold. They provide an intrinsic way of measuring the twisting, or deviation from triviality, of a fibre bundle. They are also an essential ingredient of the index theorems that express the difference of zero modes of an elliptic operator minus its adjoint precisely in terms of integrals of characteristic classes. Index theorems play a fundamental role in many physical problems. Characteristic classes are also needed in the definition of special geometries that we later consider. For this reason we devote the present section to their general discussion.

We begin by recalling the notion of de Rham cohomology groups. The differential forms of degree r on a k-dimensional manifold $\mathscr{M}$ are sections of a vector bundle, namely of the completely antisymmetrized tensor product $\Lambda^r\,(T^\star\mathscr{M})$ of the cotangent bundle $T^\star\mathscr{M}$, r times with itself. We name $\Omega^r = \Gamma\,(\mathscr{M},\,\Lambda^r\,(T^\star\mathscr{M}))$ the

[2]For a pedagogical short introduction to cohomology theory I refer the reader to my book [6], Vol 1, Chap. 2.

[3]The development of de Rham cohomology and of characteristic classes is historically reviewed in the twin book to this one [1], within the general frame of the evolution of geometry in the XXth century.

space of sections of this bundle, namely the space of r-forms. The exterior derivative d provides a sequence of maps d_i:

$$\Omega^0(\mathscr{M}) \xrightarrow{d_0} \Omega^1(\mathscr{M}) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-2}} \Omega^{k-1}(\mathscr{M}) \xrightarrow{d_{k-1}} \Omega^k(\mathscr{M}) \xrightarrow{d_k} 0 \quad (3.4.1)$$

where d_r is the exterior derivative acting on r-forms and producing $r + 1$-forms as a result. The property of the exterior derivative $d^2 = 0$ implies that

$$d_i \, d_{i+1} = 0 \qquad \forall i = 0, \ldots, k \quad (3.4.2)$$

What we have just described is named the **de Rham complex** and provides the first and most prominent example of an elliptic complex. More generally we have

Definition 3.4.1 An elliptic complex $(E^\star, D)$ is a sequence of vector bundles $E_i \xrightarrow{\pi_i} \mathscr{M}$ constructed over the same base manifold and a sequence of Fredholm operators D_i mapping the sections of the ith bundle into those of the (i+1)th bundle:

$$\Gamma(\mathscr{M}, E_0) \xrightarrow{D_0} \Gamma(\mathscr{M}, E_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{k-2}} \Gamma(\mathscr{M}, E_{k-1}) \xrightarrow{D_{k-1}} \Gamma(\mathscr{M}, E_k) \xrightarrow{D_k} 0 \quad (3.4.3)$$

such that

$$D_i \, D_{i+1} = 0 \qquad \forall i = 0, \ldots, k \quad (3.4.4)$$

A Fredholm operator is a differential operator of elliptic type with finite kernel and cokernel, as we discuss below. To each elliptic complex and to the de Rham complex in particular we can attach the notion of cohomology groups. The ith cohomology group is defined as follows:

$$H^i\left(E^\star, \mathscr{M}\right) = \frac{\ker D_i}{\mathrm{Im}\, D_{i-1}} \quad (3.4.5)$$

It is the space of sections of the ith bundle E_i satisfying $D_i \, s = 0$, modulo those of the form $s = D_{i-1} \, s'$. In the de Rham complex $H^r\,(\Omega^\star(\mathscr{M}))$ is the space of closed r-forms modulo exact forms. For any Fredholm operator D_i appearing in the elliptic complex (3.4.3) we denote $D_i^\dagger$ its adjoint, which is defined by

$$\begin{aligned} D_i^\dagger &: \Gamma(\mathscr{M}, E_{i+1}) \to \Gamma(\mathscr{M}, E_i) \\ (s', D_i s)_{E_{i+1}} &= (D_i^\dagger s', s)_{E_i} \end{aligned} \quad (3.4.6)$$

where $s \in \Gamma(\mathscr{M}, E_i)$, $s' \in \Gamma(\mathscr{M}, E_{i+1})$ and $(\ ,\ )_E$ denotes the fibre metric in the specified fibre. The laplacian operator is defined by

$$\begin{aligned} \Delta_i &: \Gamma(\mathscr{M}, E_i) \to \Gamma(\mathscr{M}, E_i) \\ \Delta_i &\equiv D_{i-1} D_{i-1}^\dagger + D_i^\dagger D_i \end{aligned} \quad (3.4.7)$$

The cohomology group $H^i(E^*, \mathscr{M})$ is isomorphic to the kernel of the operator Δ_i, so that we have

$$\dim H^i(E^*, D) = \dim \mathrm{Harm}^i(E^*, D) \tag{3.4.8}$$

where by $\mathrm{Harm}^i(E^*, D)$ we denote the vector space spanned by sections $h_i \in \Gamma(\mathscr{M}, E_i)$ which satisfy

$$\Delta_i h_i = 0\,. \tag{3.4.9}$$

Given a section $s_i \in \Gamma(\mathscr{M}, E_i)$ we can write the Hodge decomposition:

$$s_i = D_i s_{i-1} + D_i^{\dagger} s_{i+1} + h_i \tag{3.4.10}$$

where $s_{i\pm 1} \in \Gamma(\mathscr{M}, E_i)$.

Definition 3.4.2 Given an elliptic complex (E^*, D) we define the index of this complex by

$$\mathrm{ind}\,(E^*, D) = \sum (-)^i \dim H^i(E^*, D) = \sum (-)^i \dim \ker \Delta_i \tag{3.4.11}$$

Equation (3.4.11), when specialized to the de Rham complex, gives the Euler characteristic of the base manifold:

$$\mathrm{ind}\, d = \sum (-)^i \dim H^i(E^*, d) \equiv \chi(\mathscr{M}) = \sum (-)^i b^i \tag{3.4.12}$$

where b^i is the ith Betti number, equal, by definition, to the number of linearly independent harmonic i-forms. For a generic Fredholm operator $D : \Gamma(\mathscr{M}, E) \to \Gamma(\mathscr{M}, F)$ we can define the *analytical index* of D as

$$\mathrm{ind} D = \dim \ker D \;-\; \dim \mathrm{coker} D \tag{3.4.13}$$

To show the relation between Eqs. (3.4.11) and (3.4.13), we have to resume our discussion on Fredholm operators. Let $D : \Gamma(\mathscr{M}, E) \to \Gamma(\mathscr{M}, F)$ be an elliptic operator. The kernel of D is the following set of sections:

$$\ker D = \{s \in \Gamma(\mathscr{M}, E) | Ds = 0\}\,. \tag{3.4.14}$$

We define the cokernel of D by

$$\mathrm{coker} D = \frac{\Gamma(\mathscr{M}, F)}{\mathrm{Im} D} \tag{3.4.15}$$

We now state without proof the following theorem:

Theorem 3.4.1 *Let $D : \Gamma(\mathscr{M}, E) \to \Gamma(\mathscr{M}, F)$ be a Fredholm operator. Then*

$$\mathrm{coker} D \sim \ker D^{\dagger} \tag{3.4.16}$$

Using Theorem 3.4.1 we immediately rewrite Eq. (3.4.11) as

$$\mathrm{ind} D = \dim\ker D - \dim\ker D^{\dagger} \tag{3.4.17}$$

Consider now the one-operator complex $\Gamma(\mathscr{M}, E) \xrightarrow{D} \Gamma(\mathscr{M}, F)$, which can also be written as

$$0 \xrightarrow{i} \Gamma(\mathscr{M}, E) \xrightarrow{D} \Gamma(\mathscr{M}, F) \xrightarrow{\phi} 0 \tag{3.4.18}$$

where i is the inclusion map (defined by $i(0) = 0$), and ϕ is a map from a generic section in $\Gamma(\mathscr{M}, F)$ into 0. Using Eq. (3.4.11) for the complex (3.4.18) we find

$$\dim\ker D - [\dim \Gamma(\mathscr{M}, F) - \dim \mathrm{Im} D] = \dim\ker D \; - \; \dim\mathrm{coker}\, D \tag{3.4.19}$$

The above equation shows the simple relation between the analytical index (3.4.13) and the index of the elliptic complex (3.4.11). Equation (3.4.13) provides an easy formula that is always recalled in physical literature. Moreover, given an elliptic complex, it is always possible to construct a Fredholm operator whose analytical index coincides with the index of the complex (E^*, D). Indeed if we define

$$E_+ = \oplus_i E_{2i}, \qquad E_- = \oplus_i E_{2i+1} \tag{3.4.20}$$

which are respectively called the even and the odd bundles and we consider the operators

$$D \equiv \oplus_i (D_{2i} + D^{\dagger}_{2i-1}) \quad D^{\dagger} \equiv \oplus_i (D_{2i+1} + D^{\dagger}_{2i}) \tag{3.4.21}$$

we easily verify that

$$\begin{aligned} D &: \Gamma(\mathscr{M}, E_+) \to \Gamma(\mathscr{M}, E_-) \\ D &: \Gamma(\mathscr{M}, E_-) \to \Gamma(\mathscr{M}, E_+) \end{aligned} \tag{3.4.22}$$

Next, if we define

$$\Delta_+ \equiv D^{\dagger} D = \oplus_i \Delta_{2i} \quad \Delta_- \equiv D D^{\dagger} = \oplus_i \Delta_{2i+1} \tag{3.4.23}$$

then we have

$$\mathrm{ind}(E_{\pm}, D) = \dim\ker \Delta_+ - \dim\ker \Delta_- = \sum (-)^i \dim\ker \Delta_i = \mathrm{ind}(E^*, D) \tag{3.4.24}$$

In general the index of an elliptic complex can be expressed by an integral over $\mathscr{M}$ of suitable characteristic classes. At the beginning of the present section we have defined characteristic classes as maps from the ring of invariant polynomials on the Lie algebra of the structural group to the de Rham cohomology group ring of the base manifold. Let us now go a little deeper on the meaning of this definition. Let

$\mathscr{M}(k, \mathbb{C})$ be the set of complex $k \times k$ matrices. We denote by $S^r(\mathscr{M}(k, \mathbb{C}))$ the vector space of symmetric r-linear $\mathbb{C}$-valued functions on $\mathscr{M}(k, \mathbb{C})$. A map

$$\hat{P} : \otimes_r \mathscr{M}(k, \mathbb{C}) \to \mathbb{C} \tag{3.4.25}$$

belongs to $S^r(\mathscr{M}(k, \mathbb{C}))$ if it satisfies, in addition to linearity in each entry, the symmetry

$$\hat{P}(a_1, \ldots, a_i, \ldots, a_j, \ldots .a_r) = \hat{P}(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_r) \qquad \forall i, j \leq r \tag{3.4.26}$$

Consider now the formal sum

$$S^*(\mathscr{M}(k, \mathbb{C})) = \oplus_0^\infty S^r(\mathscr{M}(k, \mathbb{C})) \tag{3.4.27}$$

and define a product of $\hat{P} \in S^p(\mathscr{M}(k, \mathbb{C}))$ and $\hat{Q} \in S^q(\mathscr{M}(k, \mathbb{C}))$ by

$$\hat{P} \cdot \hat{Q}(a_1, \ldots, a_{p+q}) = \frac{1}{(p+q)!} \sum_P \hat{P}(a_{P(1)}, \ldots, a_{P(p)}) \hat{Q}(a_{P(p+1)}, \ldots, a_{P(p+q)}) \tag{3.4.28}$$

where P denotes the permutation of the set $(1, \ldots, p+q)$. $S^*(\mathscr{M}(k, \mathbb{C}))$ equipped with the product (3.4.28) is an algebra. If we now consider a Lie algebra $\mathbb{G} \in \mathscr{M}(k, \mathbb{C})$, and the corresponding simply connected Lie group $\mathscr{G} = \exp[\mathbb{G}]$, in full analogy with Eqs. (3.4.27) and (3.4.26), we can define the sum $S^*(\mathbb{G}) = \oplus_{r \geq 0} S^r(\mathbb{G})$. An element $\hat{P}(h_1, \ldots, h_r) \in S^r(\mathbb{G})$ ($h_i \in \mathbb{G}$) is said to be invariant if, for any $g \in G$, it satisfies

$$\hat{P}(g^{-1} h_1 g, \ldots, g^{-1} h_r g) = \hat{P}(h_1, \ldots, h_r) \tag{3.4.29}$$

The set of invariant elements of $S^r(\mathbb{G})$ is denoted by $I^r(\mathbb{G})$. The product defined in (3.4.28) induces a natural multiplication

$$\cdot\ :\ I^p(\mathbb{G}) \otimes I^q(\mathbb{G}) \to I^{p+q}(\mathbb{G}) \tag{3.4.30}$$

The sum $I^* = \oplus_{r \geq 0} I^r(\mathbb{G})$ equipped with the product (3.4.30) is an algebra. The diagonal combination $P(h) = P(h, \ldots, h)$ containing r-times the element $h \in \mathbb{G}$ is a polynomial of degree r, which is said to be an *invariant polynomial*. Let now $P(\mathscr{M}, G)$ be a principal bundle that has as structural group a Lie group $\mathscr{G}$ with Lie algebra $\mathbb{G}$. We extend the domain of invariant polynomials from $\mathbb{G}$ to $\mathbb{G}$-valued p-forms on $\mathscr{M}$. We define

$$\hat{P}(h_1 \omega_1, \ldots, h_r \omega_r) \equiv \omega_1 \wedge \cdots \wedge \omega_r \hat{P}(h_1, \ldots, h_r) \tag{3.4.31}$$

where $h_i \in \mathbb{G}$, $\omega_i \in \Omega^{p_i}(\mathscr{M})$ $(i = 1 \ldots r)$. The diagonal combination is now given by

$$P(h\omega) = \omega \wedge \cdots \wedge \omega\, P(h) \tag{3.4.32}$$

where the wedge product of $\omega \in \Omega^p(\mathscr{M})$ is repeated r-times in (3.4.32). Consider now the curvature 2-form Θ associated with a connection in a complex fibre bundle. In the following we are particularly interested in invariant polynomials of the form $P(\Theta)$. We can state the following theorem (*Chern–Weil theorem*).

Theorem 3.4.2 *Let $P(\Theta)$ be an invariant polynomial in the curvature 2-form; then*
(i) $dP(\Theta) = 0$
(ii) Let Θ, Θ' be curvature 2-forms corresponding to different connections θ, θ' on the fibre bundle. Then the difference $P(\Theta) - P(\Theta')$ is exact.

This theorem proves that an invariant polynomial $P(\Theta)$ is closed and in general non-trivial. We can then associate to $P(\Theta)$ a cohomology class of $\mathscr{M}$. Moreover Theorem 3.4.2 ensures that this cohomology class is independent of the chosen connection. The cohomology class defined by $P(\Theta)$ is called a *characteristic class*. The characteristic class defined by an invariant polynomial P is denoted by $\chi_E(P)$, where E is the fibre bundle on which curvatures and connections are defined.

Theorem 3.4.3 *Let P be an invariant polynomial in $I^*(\mathbb{G})$ and E be a fibre bundle over $\mathscr{M}$, whose structural group $\mathscr{G}$ has $\mathbb{G}$ as Lie algebra. The map*

$$\chi_E : I^*(\mathbb{G}) \to H^*(\mathscr{M}) \tag{3.4.33}$$

defined by $P \to \chi_E(P)$ is a homomorphism.

Theorem 3.4.3 establishes a homomorphism, called the Chern–Weil homomorphism,[4] between the ring $I^*(\mathbb{G})$ and the de Rham cohomology ring $H^*(\mathscr{M})$, defined by

$$H^*(\mathscr{M}) = \oplus_r H^r(\mathscr{M}) \tag{3.4.34}$$

where H^r is the rth cohomolgy group. The Chern–Weil homomorphism is the fundamental instrument that allows one to relate the index of an elliptic complex with the integral of particular characteristic classes, through the so called *index theorem* (stated below in Eq. (3.4.56)). Before giving the statement of this theorem, due to Atiyah and Singer, we list some specific examples of characteristic classes, which will be useful in the following.

Definition 3.4.3 Given a complex vector bundle E equipped with a connection θ, whose fibre is $\mathbb{C}^r$, we can define its total Chern class $c(E, \Theta)$ as the following formal determinant:

$$c(E, \Theta) = det\left(\mathbf{1} + \frac{i}{2\pi}\Theta\right) \tag{3.4.35}$$

where Θ is the matrix-valued curvature 2-form.

The determinant is calculated with respect to the matrix indices. As it is well known, the determinant $det(\mathbf{1} + A)$ is a polynomial in the matrix elements of A and can

[4]The interesting history of the Chern–Weil homomorphism, independently discovered by the two great mathematicians in the years of World War II, is reported in the twin book [1].

be expanded in powers of A. Such an expansion of the total Chern class yields the definition of the individual Chern classes $c_k(E, \Theta)$. In particular, if we call $x_1, \dots x_r$ the (formal) eigenvaules[5] 2-forms of the matrix $\frac{i}{2\pi}\Theta$ we easily find

$$\det\left(1+\frac{i}{2\pi}\Theta\right)=\prod_{1}^{r}(1+x_j) \;=\; 1+(x_1+\cdots+x_r)+ \\ (x_1x_2+\cdots+x_{r-1}x_r)+\cdots+(x_1x_2\cdots x_r) \quad (3.4.36)$$

so that, by writing

$$c(E,\Theta) \;=\; \sum_{k=0}^{r} c_k(E,\Theta) \quad (3.4.37)$$

we get

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \frac{i}{2\pi}\,\mathrm{tr}\,(\Theta)\,, \\ c_2 &= \frac{1}{8\pi^2}\left[\mathrm{tr}\left(\Theta^2\right) \;-\; (\mathrm{tr}\,\Theta)^2\right] \\ &\vdots \; \vdots \; \vdots \\ c_r &= \det\frac{i\Theta}{2\pi} \end{aligned} \quad (3.4.38)$$

where, for a generic form Ω, by Ω^n we mean the nth wedge product $\wedge^n\Omega$. A remarkable property of the Chern class is the following: given two complex vector bundles $E \xrightarrow{\pi} \mathscr{M}$, $F \xrightarrow{\pi'} \mathscr{M}$ we have

$$c(E\oplus F)=c(E)\wedge c(F) \quad (3.4.39)$$

Definition 3.4.4 Given a rank r vector bundle $E \xrightarrow{\pi} \mathscr{M}$ we define the total Chern character by

$$\mathrm{ch}(E,\Theta)=\mathrm{tr}\,exp\left(\frac{i\Theta}{2\pi}\right)=\sum_{l=1}\frac{1}{l!}\mathrm{tr}\left(\frac{i\Theta}{2\pi}\right)^j \quad (3.4.40)$$

and the jth Chern character by

[5] We stress the word "formal eigenvalues" because the correct framework to understand these eigenvalues is the "splitting principle", which, for convenience, is mentioned after the Eq. (2.7.59).

$$\mathrm{ch}_j(E,\Theta) = \frac{1}{j!}\mathrm{tr}\left(\frac{i\Theta}{2\pi}\right)^j \tag{3.4.41}$$

From now on, for notational convenience we refer to $\mathrm{ch}(E,\Theta)$ as $\mathrm{ch}\,E$ or $\mathrm{ch}\,\Theta$ indifferently (and similarly for the Chern class $c(E,\Theta)$). In terms of the eigenvectors x_j we get

$$\mathrm{ch}(\Theta) = \sum_{j=1}^{r}\left(1 + x_j + \frac{1}{2}x_j^2 + \cdots\right) \tag{3.4.42}$$

so that we can write

$$\begin{aligned}\mathrm{ch}_0(\Theta) &= r\\ \mathrm{ch}_1(\Theta) &= c_1(\Theta)\\ \mathrm{ch}_2(\Theta) &= \frac{1}{2}[c_1^2(\Theta) - 2c_2(\Theta)]\end{aligned} \tag{3.4.43}$$

Theorem 3.4.4 *Let E and F be two vector bundles over a manifold $\mathscr{M}$. The Chern character of $E \otimes F$ and $E \oplus F$ are given by*

$$\begin{aligned}ch(E\otimes F) &= ch(E)\wedge ch(F)\\ ch(E\oplus F) &= ch(E) + ch(F)\end{aligned} \tag{3.4.44}$$

Another useful characteristic class associated with a complex vector bundle is the **Todd class** defined by

$$\mathrm{Td}(\Theta) = \prod_{j=1}^{r}\frac{x_j}{1-e^{-x_j}} \tag{3.4.45}$$

where x_j are the eigenvalues of the curvature 2-form $\frac{i}{2\pi}\Theta$. We obtain

$$\begin{aligned}\mathrm{Td}(\Theta) &= 1 + \frac{1}{2}\sum_j x_j + \frac{1}{12}x_j^2 + \cdots\\ &= \prod_j\left(1 + \tfrac{1}{2}x_j + \sum_{k\geq 1}(-)^{k-1}\frac{B_k}{2k!}x_j^{2k}\right)\\ &= 1 + \frac{1}{2}c_1(\Theta) + \frac{1}{12}[c_1^2(\Theta) + c_2(\Theta)] + \cdots\end{aligned} \tag{3.4.46}$$

where the numbers B_k appearing in Eq. (3.4.46) are the Bernoulli numbers.

Finally we define the **Euler class**. The characteristic classes previously introduced are naturally defined for complex vector bundles. On the other hand the Euler class can be defined for real vector bundles over an orientable Riemann manifold $\mathscr{M}$. In particular it is consistently defined for even rank real bundles, while it is zero for odd rank bundles. Given a rank k real bundle E it is useful to construct a complex vector bundle from E by a *complexification* procedure. The complexification of E is the bundle over $\mathscr{M}$ obtained by replacing the fibres $\mathbb{R}^k$ by $\mathbb{C}^k = (\mathbb{R} \oplus i\mathbb{R})^k$. We denote the complexification of E by $E^{\mathbb{C}}$. We can think of $E^{\mathbb{C}}$ as the following product

$$E^{\mathbb{C}} = E \otimes (\mathbb{R} \oplus i\mathbb{R}) \tag{3.4.47}$$

Complex vector bundles can also be complexified by converting them into real vector bundles and then complexifying the result. If the starting complex bundle has rank r, its complexification has rank $2r$. Notice that, given a complex vector bundle E, and denoting by $E_{\mathbb{R}}$ the underlying real bundle, we have

$$E_{\mathbb{R}}^{\mathbb{C}} = E_{\mathbb{R}} \otimes (\mathbb{R} + i\mathbb{R}) \sim E \oplus \overline{E} \tag{3.4.48}$$

where $\overline{E}$ denotes the conjugate complex bundle, defined by applying complex conjugation to the coordinates of the fibres $\mathbb{C}^r$ of E. Having outlined the complexification procedure for a real vector bundle, we define the Euler class through another typical characteristic class defined in real bundles: the Pontrjagin class. Let E be a real vector bundle of rank r over $\mathscr{M}$, the ith Pontrjagin class is defined as

$$p_i(E) = (-)^i c_{2i}(E^{\mathbb{C}}) \tag{3.4.49}$$

where $c_{2i}(E^{\mathbb{C}})$ is the $2i$th Chern class of the complexified bundle. The total Pontrjagin class is defined as

$$P(E) = 1 + p_1(E) + \cdots + p_{[r/2]} \tag{3.4.50}$$

where $[r/2]$ is the largest integer not greater than r. Consider now real vector bundles E of *even rank* over an orientable manifold $\mathscr{M}$. The Euler class is defined by

$$e^2(V) = p_{[r/2]} \tag{3.4.51}$$

The Euler class of a Whitney sum $E \oplus V$ is

$$e(E \oplus V) = e(E)e(V) \tag{3.4.52}$$

where we denote $c(E)c(V) = c(E) \wedge c(V)$. For a complex vector bundle the Pontrjagin and the Euler class are the Pontrjagin and the Euler class of the underlying real bundle. Since the eigenvalues of the curvature 2-form in the conjugate bundle are given by $-x_i$, we have

$$c(E^{\mathbb{C}}) = c(E \oplus \overline{E}) = c(E)c(\overline{E}) = \prod_{i=1}^{r}(1 + x_i)(1 - x_i) = \prod_{i=1}^{r}(1 - x_i^2) \tag{3.4.53}$$

so that

$$c_r(E^{\mathbb{C}}) = (-)^r x_1^2 \cdots x_r^2 \tag{3.4.54}$$

and (recalling that $E^{\mathbb{C}}$ has rank $2r$)

$$\begin{aligned} p_r(E) &= x_1^2 \cdots x_r^2 \\ e(E) &= x_1 x_2 \cdots x_r = c_r(E) \end{aligned} \tag{3.4.55}$$

We are now able to state the *Atiyah–Singer index theorem* in its full generality:

Theorem 3.4.5 *Given an elliptic complex* (E^*, D) *over an* m*-dimensional* ($\dim_{\mathbb{R}} \mathscr{M} = m$) *compact manifold* $\mathscr{M}$ *without a boundary, then*

$$ind(E^*, D) = (-)^{\frac{m(m+1)}{2}} \int_{\mathscr{M}} ch\left(\oplus_j (-)^j E_j\right) \frac{Td(T\mathscr{M}^{\mathbb{C}})}{e(T\mathscr{M})} \tag{3.4.56}$$

where $T\mathscr{M}$ *is the tangent bundle over* $\mathscr{M}$.

Let us now consider the application of the index theorem to some particular elliptic complexes. Consider an m-dimensional compact orientable manifold without boundaries and the elliptic de Rham complex:

$$\cdots \xrightarrow{d} \Omega^{r-1}(\mathscr{M})^{\mathbb{C}} \xrightarrow{d} \Omega^r(\mathscr{M})^{\mathbb{C}} \xrightarrow{d} \Omega^{r+1}(\mathscr{M})^{\mathbb{C}} \xrightarrow{d} \cdots \tag{3.4.57}$$

with $\Omega^r(\mathscr{M})^{\mathbb{C}} = \Gamma(\mathscr{M}, \wedge^r T^*\mathscr{M}^{\mathbb{C}})$, where we have complexified the forms to apply the Atiyah–Singer theorem. The analytical index is given by

$$\text{ind}\, d = \sum_{r=0}^{m} (-)^r \dim_{\mathbb{C}} H^r(\mathscr{M}, \mathbb{C}) = \sum_{r=0}^{m} (-)^r \dim_{\mathbb{R}} H^r(\mathscr{M}, \mathbb{R}) = \chi(\mathscr{M}) \tag{3.4.58}$$

where $\chi(\mathscr{M})$ is the Euler characteristic of $\mathscr{M}$. Suppose $\mathscr{M}$ is even dimensional $m = 2l$. Equation 3.4.56 gives the following result for the de Rham index:

$$\text{ind}\, d = (-)^{l(2l+1)} \int_{\mathscr{M}} \text{ch}\left(\oplus_r^{2l} (-)^r \wedge^r T^*\mathscr{M}^{\mathbb{C}}\right) \frac{\text{Td}T\mathscr{M}^{\mathbb{C}}}{e(T\mathscr{M})} \tag{3.4.59}$$

To compute $\text{ch}\left(\oplus_r^m (-)^r \wedge^r T^*\mathscr{M}^{\mathbb{C}}\right)$ we employ the splitting principle. The splitting principle uses the fact that in order to prove an identity for characteristic classes, it is sufficient to prove it only for bundles which decompose into a sum of line bundles. Suppose that a fibre bundle F is a Whitney sum of n line bundles L_i; then

$$\wedge^p F = \oplus_{1 \le i_1 \cdots i_p \le n} \left(L_{i_1} \otimes \cdots \otimes L_{i_p}\right) \tag{3.4.60}$$

This means that

$$\mathrm{ch}(\wedge^p F) = \sum_{1\leq i_1\cdots i_p\leq n} \mathrm{ch}(L_{i_1})\mathrm{ch}(L_{i_2})\cdots\mathrm{ch}(L_{i_p}) \tag{3.4.61}$$

Since for any line bundle appearing in the Whitney sum $\mathrm{ch}(L_i) = e^{x_i}$, we finally get

$$\mathrm{ch}(\wedge^p F) = \sum_{1\leq i_1\cdots i_p\leq n} e^{x_{i_i}+\cdots+x_{i_p}} \tag{3.4.62}$$

Applying this result to $\oplus_r^m (-)^r \wedge^r T^*\mathscr{M}^{\mathbb{C}}$, and using the fact that taking the dual bundle merely changes the sign of x_i we get

$$\mathrm{ch} \oplus_r^m (-)^r \wedge^r T^*\mathscr{M}^{\mathbb{C}} = \prod_{i=1}^m (1 - e^{-x_i})(T\mathscr{M}^{\mathbb{C}}) \tag{3.4.63}$$

Moreover we can write

$$\mathrm{Td}(T\mathscr{M}^{\mathbb{C}}) = \prod_{i=1}^m \frac{x_i}{1 - e^{-x_i}}(T\mathscr{M}^{\mathbb{C}}) \tag{3.4.64}$$

Then the index of the de Rham complex is given by

$$\mathrm{ind}\, d = (-)^l \int_{\mathscr{M}} \frac{\prod_{i=1}^m x_i(T\mathscr{M}^{\mathbb{C}})}{e(T\mathscr{M})} = (-)^l \int_{\mathscr{M}} \frac{c_m(T\mathscr{M}^{\mathbb{C}})}{e(T\mathscr{M})} = \int_{\mathscr{M}} e(T\mathscr{M}) \tag{3.4.65}$$

where we have used

$$c_m(T\mathscr{M}^{\mathbb{C}}) = (-)^{m/2} e(T\mathscr{M} \oplus T\mathscr{M}) = (-)^l x_1^2 \cdots x_m^2 = (-)^l e^2(T\mathscr{M})$$

By combining the results for the analytical index and for the Atiyah–Singer index (often referred to as the topological index), we get the Gauss–Bonnet theorem

$$\int_{\mathscr{M}} e(T\mathscr{M}) = \chi(\mathscr{M}) \tag{3.4.66}$$

For m odd, the de Rham index is zero. Let us consider now the application of the index theorem to the Dolbeault complex, which we are going to define below. Consider a complex manifold $\mathscr{M}$ with $dim_{\mathbb{C}}\mathscr{M} = m$. We denote by $T^{(1,0)}\mathscr{M}$ the tangent bundle spanned by the vectors $\{\partial/\partial z^\mu\}$ and by $T^{(0,1)}\mathscr{M}$ its complex conjugate. The space dual to $T^{(1,0)}\mathscr{M}$ is spanned by the 1-forms $\{dz^\mu\}$. We denote it by $T^{*(1,0)}\mathscr{M}$. The space $\Omega^r(\mathscr{M})^{\mathbb{C}}$ of complexified r-forms is decomposed as

$$\Omega^r(\mathscr{M})^{\mathbb{C}} = \oplus_{p+q=r} \Omega^{p,q}(\mathscr{M}) \tag{3.4.67}$$

where by $\Omega^{p,q}(\mathscr{M})$ we denote the space of (p, q) forms. The exterior derivative can be written as

$$d = dz^{\mu} \wedge \frac{\partial}{\partial z^{\mu}} + d\overline{z}^{\mu} \wedge \frac{\partial}{\partial \overline{z}^{\mu}} \tag{3.4.68}$$

It is immediate to verify that ∂, $\overline{\partial}$ satisfy the following relations:

$$\partial\overline{\partial} - \overline{\partial}\partial = \partial^2 = \overline{\partial}^2 = 0 \tag{3.4.69}$$

Moreover ∂ maps (p, q)-forms into $(p + 1, q)$-forms and $\overline{\partial}$ maps (p, q) forms into $(p, q + 1)$ forms. Let us consider the sequence

$$\cdots \xrightarrow{\overline{\partial}} \Omega^{(0,q)}(\mathscr{M}) \xrightarrow{\overline{\partial}} \Omega^{(0,q+1)}(\mathscr{M}) \xrightarrow{\overline{\partial}} \cdots \tag{3.4.70}$$

This sequence is called the **Dolbeault complex**. It can be shown that (3.4.70) defines an elliptic complex. The index theorem in this case gives

$$\mathrm{ind}\,\overline{\partial} = \int_{\mathscr{M}} \mathrm{ch}\left(\oplus_r (-)^r \wedge^r T^{*\,(0,1)}\mathscr{M}\right) \frac{\mathrm{Td}T\mathscr{M}^{\mathbb{C}}}{e(T\mathscr{M})} \tag{3.4.71}$$

The left hand side of the above equation can be computed using the Eq. (3.4.13), so that

$$\mathrm{ind}\,\overline{\partial} = \sum_{r=0}^{n} (-)^r h^{(0,r)} \tag{3.4.72}$$

where

$$h^{(0,r)} = \dim_{\mathbb{C}} H^{(0,r)}(\mathscr{M}) = \dim_{\mathbb{C}} \frac{\ker\overline{\partial}_r}{\mathrm{im}\overline{\partial}_{r-1}} \tag{3.4.73}$$

is the complex dimension of the cohomology group $H^{(0,r)}$. The application of theorem (3.4.56) to this case is analogous to the one presented for the de Rham complex and gives

$$\sum_{r=0}^{n} (-)^r b^{(0,r)} = \int_{\mathscr{M}} \mathrm{Td}(T^{(1,0)}\mathscr{M}) \tag{3.4.74}$$

In the Dolbeault complex the space $\Omega^{(0,r)}$ can be replaced by a tensor product bundle $\Omega^{(0,r)} \otimes V$, where V is a holomorphic vector bundle. In this case we define the following elliptic complex, named the **twisted Dolbeault complex**:

$$\cdots \xrightarrow{\overline{\partial}_V} \Omega^{(0,q)}(\mathscr{M}) \otimes V \xrightarrow{\overline{\partial}_V} \Omega^{(0,q+1)}(\mathscr{M}) \otimes V \xrightarrow{\overline{\partial}_V} \cdots \tag{3.4.75}$$

The Atiyah–Singer theorem for this particular complex reduces to the Hirzebruch–Riemann–Roch theorem:

$$\text{ind}\,\overline{\partial}_V = \int_{\mathscr{M}} \text{Td}\,(T^{(1,0)}\mathscr{M})\text{ch}(V) \tag{3.4.76}$$

In the case of complex dimension one, namely $dim_{\mathbb{C}}\mathscr{M} = 1$, we get

$$\text{ind}\,\overline{\partial}_V = \frac{1}{2}\text{dim}V \int_{\mathscr{M}} c_1(T^{(1,0)}\mathscr{M}) + \int_{\mathscr{M}} c_1(\mathscr{M}) \tag{3.4.77}$$

Since it can be shown that

$$\int_{\mathscr{M}} c_1(T^{(1,0)}\mathscr{M}) = \int_{\mathscr{M}} e(T\mathscr{M}) = 2(1-g) \tag{3.4.78}$$

where g is the genus of the base manifold, which in complex dimension one is nothing but a Riemann surface Σ_g, in this case we get

$$\text{ind}\,\overline{\partial}_V = \text{dim}V(1-g) + \int_{\Sigma_g} \frac{i\Theta}{2\pi} \tag{3.4.79}$$

In the general case of a complex manifold $\mathscr{M}$ of complex dimension n, the dimensions

$$h^{(p,q)} \stackrel{\text{def}}{=} \text{dim}_{\mathbb{C}}\, H^{(p,q)}\,(\mathscr{M}) \tag{3.4.80}$$

of the Dolbeault cohomology groups are named *Hodge numbers*.

3.5 Kähler Metrics

In the previous sections we have discussed the general notion of hermitian fibre metrics on holomorphic vector bundles and in particular of hermitian manifold metrics defined on the tangent bundle. In this section we introduce the more restricted concept of Kählerian metrics that plays a fundamental role in many applications.[6] The definition of the previous section Definition 3.3.2 can also be restated in the following way: a manifold metric g is a symmetric bilinear scalar valued functional on $\Gamma(T\mathscr{M},\mathscr{M}) \otimes \Gamma(T\mathscr{M},\mathscr{M})$

$$g\;:\;\Gamma(T\mathscr{M},\mathscr{M}) \otimes \Gamma(T\mathscr{M},\mathscr{M}) \;\rightarrow\; C^{\infty}(\mathscr{M}) \tag{3.5.1}$$

In every coordinate system it is represented by the familiar symmetric tensor $g_{\alpha\beta}(x)$. Indeed we have

$$g(\mathbf{u},\mathbf{w}) = g_{\alpha\beta}u^{\alpha}w^{\beta} \tag{3.5.2}$$

[6]For Kähler's life, his relations with Chern and other outstanding mathematicians and for the conceptual development of Kähler metrics we refer the reader to the twin book [1].

where u^α, w^β are the components of the vector fields **u** and **w**, respectively. In this language the hermiticity of the manifold metric g can be rephrased in the following way:

Definition 3.5.1 Let $\mathscr{M}$ be a 2n-dimensional manifold with an almost complex structure J. A metric g on $\mathscr{M}$ is called hermitian with respect to J if

$$g(J\mathbf{u}, J\mathbf{w}) = g(\mathbf{u}, \mathbf{w}) \tag{3.5.3}$$

Given a metric g and an almost complex structure J let us introduce the following differential 2-form K:

$$K(\mathbf{u}, \mathbf{w}) = \frac{1}{2\pi} g(J\mathbf{u}, \mathbf{w}) \tag{3.5.4}$$

The components $K_{\alpha\beta}$ of K are given by

$$K_{\alpha\beta} = g_{\gamma\beta} J^\gamma_\alpha \tag{3.5.5}$$

and by direct computation we can easily verify that:

Theorem 3.5.1 *g is hermitian if and only if K is anti-symmetric.*

Definition 3.5.2 A hermitian almost complex manifold is an almost complex manifold endowed with a hermitian metric g.

In a well-adapted basis we can write

$$g(u, w) = g_{ij} u^i w^j + g_{i^* j^*} u^{i^*} w^{j^*} + g_{\imath j^*} u^i w^{j^*} + g_{i^* j} u^i w^{j^*} \tag{3.5.6}$$

Reality of $g(u, w)$ implies

$$\begin{aligned} g_{ij} &= (g_{i^* j^*})^* \\ g_{i^* j} &= \left(g_{i j^*}\right)^\star \end{aligned} \tag{3.5.7}$$

symmetry ($g(u, w) = g(w, u)$) yields

$$\begin{aligned} g_{ij} &= g_{ji} \\ g_{j^* i} &= g_{i j^*} \end{aligned} \tag{3.5.8}$$

while the hermiticity condition gives

$$g_{ij} = g_{i^* j^*} = 0 \tag{3.5.9}$$

Finally in the well-adapted basis the 2-form K associated to the hermitian metric g can be written as

$$K = \frac{i}{2\pi} g_{i j^\star} dz^i \wedge d\overline{z}^{j^*} \tag{3.5.10}$$

Definition 3.5.3 A hermitian metric on a complex manifold $\mathscr{M}$ is called a Kähler metric if the associated 2-form K is closed:

$$dK = 0 \tag{3.5.11}$$

A hermitian complex manifold endowed with a Kähler metric is called a Kähler manifold.

Equation (3.5.11) is a differential equation for g_{ij^*} whose general solution in any local chart is given by the following expression:

$$g_{ij^*} = \partial_i \partial_{j^*} \mathscr{K} \tag{3.5.12}$$

where $\mathscr{K} = \mathscr{K}^* = \mathscr{K}(z, z^*)$ is a real function of z^i, z^{i^*}. The function $\mathscr{K}$ is called the Kähler potential and it is defined only up to the real part of a holomorphic function $f(z)$. Indeed one sees that

$$\mathscr{K}'(z, z^{i^*}) = \mathscr{K}(z, z^{i^*}) + f(z) + f^*(z^*) \tag{3.5.13}$$

give rise to the same metric g_{ij^*} as $\mathscr{K}$. The transformation (3.5.13) is called a Kähler transformation. The differential geometry of a Kähler manifold is described by Eqs. (3.3.18) and (3.3.19) with g_{ij^*} given by (3.5.12). Kähler geometry is that implied by $\mathscr{N} = 1$ supersymmetry for the scalar multiplets [7].

3.6 Hypergeometry

Next we turn our attention to the geometry that emerges when the manifold admits three complex structures satisfying the quaternionic algebra first discovered by Hamilton. To this effect the prerequisite is that the dimension of the manifold should be a multiple of 4. This is precisely what happens in supersymmetry when we consider the so called $\mathscr{N} = 2$ hypermultiplets. Each of them contains 4 real scalar fields and, at least locally, they can be regarded as the four components of a quaternion. The locality caveat is, in this case, very substantial because global quaternionic coordinates can be constructed only occasionally even on those manifolds that are denominated quaternionic in the mathematical literature [2, 3]. Anyhow, what is important is that, in the hypermultiplet sector, the scalar manifold $\mathscr{QM}$ has dimension multiple of four:

$$\dim_{\mathbf{R}} \mathscr{QM} = 4\,m \equiv 4\,\#\,\text{of hypermultiplets} \tag{3.6.1}$$

and, in some appropriate sense, it has a quaternionic structure.

We name *Hypergeometry* that pertaining to the hypermultiplet sector, irrespectively whether we deal with global or local $\mathcal{N} = 2$ theories. Yet there are two kinds of hypergeometries. Supersymmetry requires the existence of a principal SU(2)-bundle

$$\mathscr{SU} \longrightarrow \mathscr{QM} \tag{3.6.2}$$

The bundle $\mathscr{SU}$ is **flat** in the *rigid supersymmetry case* while its curvature is proportional to the Kähler forms in the *local case*.

These two versions of hypergeometry were already known in mathematics prior to their use [2–5, 8–10] in the context of $\mathcal{N} = 2$ supersymmetry and are identified as:

$$\begin{aligned} &\text{rigid hypergeometry} \equiv \text{HyperKähler geometry.} \\ &\text{local hypergeometry} \equiv \text{Quaternionic Kähler geometry} \end{aligned} \tag{3.6.3}$$

3.6.1 *Quaternionic Kähler, Versus HyperKähler Manifolds*

Both a Quaternionic Kähler or a HyperKähler manifold $\mathscr{QM}$ is a $4m$-dimensional real manifold endowed with a metric h:

$$ds^2 = h_{uv}(q) dq^u \otimes dq^v \quad ; \quad u, v = 1, \ldots, 4m \tag{3.6.4}$$

and three complex structures

$$(J^x) \ : \ T(\mathscr{QM}) \longrightarrow T(\mathscr{QM}) \qquad (x = 1, 2, 3) \tag{3.6.5}$$

that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} \, \mathbb{1} + \varepsilon^{xyz} J^z \tag{3.6.6}$$

and respect to which the metric is hermitian:

$$\forall \mathbf{X}, \mathbf{Y} \in T\mathscr{QM} \ : \quad h\left(J^x \mathbf{X}, J^x \mathbf{Y}\right) = h\left(\mathbf{X}, \mathbf{Y}\right) \qquad (x = 1, 2, 3) \tag{3.6.7}$$

From Eq. (3.6.7) it follows that one can introduce a triplet of 2-forms

$$K^x = K^x_{uv} dq^u \wedge dq^v \ ; \ K^x_{uv} = h_{uw} (J^x)^w_v \tag{3.6.8}$$

that provides the generalization of the concept of Kähler form occurring in the complex case. The triplet K^x is named the *HyperKähler* form. It is an SU(2) Lie-algebra valued 2-form in the same way as the Kähler form is a U(1) Lie-algebra valued 2-form. In the complex case the definition of Kähler manifold involves the statement that the Kähler 2-form is closed. At the same time in Hodge–Kähler manifolds the Kähler 2-form can be identified with the curvature of a line-bundle which in the case of rigid supersymmetry is flat. Similar steps can be taken also here and lead to two possibilities: either HyperKähler or Quaternionic Kähler manifolds.

Let us introduce a principal SU(2)-bundle $\mathscr{SU}$ as defined in Eq. (3.6.2). Let ω^x denote a connection on such a bundle. To obtain either a HyperKähler or a Quaternionic Kähler manifold we must impose the condition that the HyperKähler 2-form is covariantly closed with respect to the connection ω^x:

$$\nabla K^x \equiv dK^x + \varepsilon^{xyz}\omega^y \wedge K^z = 0 \tag{3.6.9}$$

The only difference between the two kinds of geometries resides in the structure of the $\mathscr{SU}$-bundle.

Definition 3.6.1 A HyperKähler manifold is a $4m$-dimensional manifold with the structure described above and such that the $\mathscr{SU}$-bundle is **flat**

Defining the $\mathscr{SU}$-curvature by:

$$\Omega^x \equiv d\omega^x + \frac{1}{2}\varepsilon^{xyz}\omega^y \wedge \omega^z \tag{3.6.10}$$

in the HyperKähler case we have:

$$\Omega^x = 0 \tag{3.6.11}$$

Viceversa

Definition 3.6.2 A Quaternionic Kähler manifold is a $4m$-dimensional manifold with the structure described above and such that the curvature of the $\mathscr{SU}$-bundle is proportional to the HyperKähler 2-form

Hence, in the quaternionic case we can write:

$$\Omega^x = \lambda K^x \tag{3.6.12}$$

where λ is a non vanishing real number.

As a consequence of the above structure the manifold $\mathscr{QM}$ has a holonomy group of the following type:

$$\begin{aligned} \mathrm{Hol}(\mathscr{QM}) &= \mathrm{SU}(2) \otimes \mathrm{H} \quad \text{(Quaternionic Kähler)} \\ \mathrm{Hol}(\mathscr{QM}) &= \mathbb{1} \otimes \mathrm{H} \quad \text{(HyperKähler)} \\ \mathrm{H} &\subset \mathrm{Sp}(2\mathrm{m}, \mathbb{R}) \end{aligned} \tag{3.6.13}$$

In both cases, introducing flat indices $\{A, B, C = 1, 2\}\{\alpha, \beta, \gamma = 1, \ldots, 2m\}$ that run, respectively, in the fundamental representation of SU(2) and of $\mathrm{Sp}(2m, \mathbb{R})$, we can find a vielbein 1-form

$$\mathcal{U}^{A\alpha} = \mathcal{U}_u^{A\alpha}(q)dq^u \tag{3.6.14}$$

such that

$$h_{uv} = \mathcal{U}_u^{A\alpha}\mathcal{U}_v^{B\beta}\mathbb{C}_{\alpha\beta}\varepsilon_{AB} \tag{3.6.15}$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\varepsilon_{AB} = -\varepsilon_{BA}$ are, respectively, the flat Sp$(2m)$ and Sp$(2) \sim$ SU(2) invariant metrics. The vielbein $\mathcal{U}^{A\alpha}$ is covariantly closed with respect to the SU(2)-connection ω^z and to some Sp$(2m, \mathbb{R})$-Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$\begin{aligned}\nabla\mathcal{U}^{A\alpha} &\equiv d\mathcal{U}^{A\alpha} + \frac{i}{2}\omega^x(\varepsilon\sigma_x\varepsilon^{-1})^A{}_B \wedge \mathcal{U}^{B\alpha} \\ &+ \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma}\mathbb{C}_{\beta\gamma} = 0\end{aligned} \tag{3.6.16}$$

where $(\sigma^x)_A{}^B$ are the standard Pauli matrices. Furthermore $\mathcal{U}^{A\alpha}$ satisfies the reality condition:

$$\mathcal{U}_{A\alpha} \equiv (\mathcal{U}^{A\alpha})^* = \varepsilon_{AB}\mathbb{C}_{\alpha\beta}\mathcal{U}^{B\beta} \tag{3.6.17}$$

Equation (3.6.17) defines the rule to lower the symplectic indices by means of the flat symplectic metrics ε_{AB} and $\mathbb{C}_{\alpha\beta}$. More specifically we can write a stronger version of Eq. (3.6.15) [7]:

$$(\mathcal{U}_u^{A\alpha}\mathcal{U}_v^{B\beta} + \mathcal{U}_v^{A\alpha}\mathcal{U}_u^{B\beta})\mathbb{C}_{\alpha\beta} = h_{uv}\varepsilon^{AB} \tag{3.6.18}$$

We have also the inverse vielbein $\mathcal{U}^u_{A\alpha}$ defined by the equation

$$\mathcal{U}^u_{A\alpha}\mathcal{U}_v^{A\alpha} = \delta^u_v \tag{3.6.19}$$

Flattening a pair of indices of the Riemann tensor $\mathcal{R}^{uv}{}_{ts}$ we obtain

$$\mathcal{R}^{uv}{}_{ts}\mathcal{U}_u^{\alpha A}\mathcal{U}_v^{\beta B} = -\frac{\mathrm{i}}{2}\Omega^x_{ts}\varepsilon^{AC}(\sigma_x)_C{}^B\mathbb{C}^{\alpha\beta} + \mathbb{R}^{\alpha\beta}_{ts}\varepsilon^{AB} \tag{3.6.20}$$

where $\mathbb{R}^{\alpha\beta}_{ts}$ is the field strength of the Sp$(2m)$ connection:

$$d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma} \wedge \Delta^{\delta\beta}\mathbb{C}_{\gamma\delta} \equiv \mathbb{R}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}_{ts}dq^t \wedge dq^s \tag{3.6.21}$$

Equation (3.6.20) is the explicit statement that the Levi Civita connection associated with the metric h has a holonomy group contained in SU$(2) \otimes$ Sp$(2m)$. Consider now Eqs. (3.6.6), (3.6.8) and (3.6.12). We easily deduce the following relation:

$$h^{st}K^x_{us}K^y_{tw} = -\delta^{xy}h_{uw} + \varepsilon^{xyz}K^z_{uw} \tag{3.6.22}$$

that holds true both in the HyperKähler and in the quaternionic case. In the latter case, using Eqs. (3.6.12), (3.6.22) can be rewritten as follows:

$$h^{st}\Omega^x_{us}\Omega^y_{tw} = -\lambda^2\delta^{xy}h_{uw} + \lambda\varepsilon^{xyz}\Omega^z_{uw} \tag{3.6.23}$$

Equation (3.6.23) implies that the intrinsic components of the curvature 2-form Ω^x yield a representation of the quaternion algebra. In the HyperKähler case such a representation is provided only by the HyperKähler form. In the quaternionic case we can write:

$$\Omega^x_{A\alpha,B\beta} \equiv \Omega^x_{uv}\mathcal{U}^u_{A\alpha}\mathcal{U}^v_{B\beta} = -i\lambda\mathbb{C}_{\alpha\beta}(\sigma_x)_A{}^C\varepsilon_{CB} \tag{3.6.24}$$

Alternatively Eq. (3.6.24) can be rewritten in an intrinsic form as

$$\Omega^x = -\mathrm{i}\,\lambda\mathbb{C}_{\alpha\beta}(\sigma_x)_A{}^C\varepsilon_{CB}\mathcal{U}^{\alpha A}\wedge\mathcal{U}^{\beta B} \tag{3.6.25}$$

whence we also get:

$$\frac{i}{2}\Omega^x(\sigma_x)_A{}^B = \lambda\mathcal{U}_{A\alpha}\wedge\mathcal{U}^{B\alpha} \tag{3.6.26}$$

3.7 Moment Maps

The conception of moment maps has its root in Hamiltonian mechanics where the time-derivative of any dynamical variable can be represented by the Poisson bracket of that variable with the hamiltonian. More generally the action of any vector field $\mathbf{t}$ on functions defined over the phase-space $\mathcal{M}$ can be represented as the Poisson bracket of that function with a generalized hamiltonian $\mathcal{H}_\mathfrak{t}$ which is associated with the vector field:

$$\begin{aligned}\mathbf{t} &\equiv t^i(p,q)\frac{\partial}{\partial q^i} + t_i(p,q)\frac{\partial}{\partial p_i}\\ \mathbf{t}\,\mathfrak{f}(p,q) &= \{\mathfrak{f},\,\mathcal{H}_\mathfrak{t}\}\end{aligned} \tag{3.7.1}$$

The moment map is the map:

$$\begin{aligned}\mu \,:\, \Gamma[T\mathcal{M},\mathcal{M}] &\to \mathbb{C}[\mathcal{M}]\\ \mu[\mathbf{t}] &= \mathcal{H}_\mathfrak{t}\end{aligned} \tag{3.7.2}$$

which to every vector field associates its proper hamiltonian.

In the present geometrical context, conceptually very much different from that of dynamical systems which are of no concern to us in this book, the focus is on the moment-maps of Killing vectors, associated with isometries of the manifold $\mathcal{M}$. The symplectic structure which allows for the definition of Poisson-like brackets is provided by the presence of the complex-structure leading to closed or covariantly closed 2-forms, the Kähler or the HyperKähler ones. Our generalized hamiltonians or simply *moment-maps* have another important role to play. On one hand they appear as constructive items in supergravity lagrangians with gauge-symmetries, on the other, purely mathematical side, they are the building blocks in a general procedure, the *Kähler or HyperKähler quotient* which allows to construct non trivial Kähler or HyperKähler manifolds starting from simple trivial ones.

In Chap. 8 we plan to exemplify such constructions with the derivation of ALE-manifolds by means of HyperKḧaler quotients. Here we just begin with the general definitions of holomorphic and tri-holomorphic moment maps.

3.7.1 The Holomorphic Moment Map on Kähler Manifolds

The concept of holomorphic moment map applies to all Kähler manifolds, not necessarily special. Indeed it can be constructed just in terms of the Kähler potential without advocating any further structure. In this subsection we review its properties and definition, as usual in order to fix conventions, normalizations and notations.

Let $g_{ij^\star}$ be the Kähler metric of a Kähler manifold $\mathscr{M}$ and let us assume that $g_{ij^\star}$ admits a non trivial group of continuous isometries $\mathscr{G}$ generated by Killing vectors $k^i_{\mathbf{I}}$ ($\mathbf{I} = 1, \ldots, \dim \mathscr{G}$) that define the infinitesimal variation of the complex coordinates z^i under the group action:

$$z^i \to z^i + \varepsilon^{\mathbf{I}} k^i_{\mathbf{I}}(z) \tag{3.7.3}$$

Let $k^i_{\mathbf{I}}(z)$ be a basis of holomorphic Killing vectors for the metric $g_{ij^\star}$. Holomorphicity means the following differential constraint:

$$\partial_{j^*} k^i_{\mathbf{I}}(z) = 0 \leftrightarrow \partial_j k^{i^*}_{\mathbf{I}}(\overline{z}) = 0 \tag{3.7.4}$$

while the generic Killing equation (suppressing the gauge index $\mathbf{I}$):

$$\nabla_\mu k_\nu + \nabla_\mu k_\nu = 0 \tag{3.7.5}$$

in holomorphic indices reads as follows:

$$\nabla_i k_j + \nabla_j k_i = 0 \; ; \; \nabla_{i^*} k_j + \nabla_j k_{i^*} = 0 \tag{3.7.6}$$

where the covariant components are defined as $k_j = g_{ji^*} k^{i^*}$ (and similarly for k_{i^*}).

The vectors $k^i_{\mathbf{I}}$ are generators of infinitesimal holomorphic coordinate transformations $\delta z^i = \varepsilon^{\mathbf{I}} k^i_{\mathbf{I}}(z)$ which leave the metric invariant. In the same way as the metric is the derivative of a more fundamental object, the Killing vectors in a Kähler manifold are the derivatives of suitable prepotentials. Indeed the first of Eq. (3.7.6) is automatically satisfied by holomorphic vectors and the second equation reduces to the following one:

$$k^i_{\mathbf{I}} = i g^{ij^*} \partial_{j^*} \mathscr{P}_{\mathbf{I}}, \quad \mathscr{P}^*_{\mathbf{I}} = \mathscr{P}_{\mathbf{I}} \tag{3.7.7}$$

In other words if we can find a real function $\mathscr{P}^{\mathbf{I}}$ such that the expression $i g^{ij^*} \partial_{j^*} \mathscr{P}_{(\mathbf{I})}$ is holomorphic, then Eq. (3.7.7) defines a Killing vector.

The construction of the Killing prepotential can be stated in a more precise geometrical fashion through the notion of *moment map*. Let us review this construction.

Consider a Kählerian manifold $\mathscr{M}$ of real dimension $2n$. Consider an isometry group $\mathscr{G}$ acting on $\mathscr{M}$ by means of Killing vector fields $\overrightarrow{X}$ which are holomorphic with respect to the complex structure J of $\mathscr{M}$; then these vector fields preserve also the Kähler 2-form

$$\left.\begin{array}{l} \mathscr{L}_{\overrightarrow{X}} g = 0 \leftrightarrow \nabla_{(\mu} X_{\nu)} = 0 \\ \mathscr{L}_{\overrightarrow{X}} J = 0 \end{array}\right\} \Rightarrow 0 = \mathscr{L}_{\overrightarrow{X}} K = i_{\overrightarrow{X}} dK + d(i_{\overrightarrow{X}} K) = d(i_{\overrightarrow{X}} K) \tag{3.7.8}$$

Here $\mathscr{L}_{\overrightarrow{X}}$ and $i_{\overrightarrow{X}}$ denote respectively the Lie derivative along the vector field $\overrightarrow{X}$ and the contraction (of forms) with it.

If $\mathscr{M}$ is simply connected, $d(i_{\overrightarrow{X}} K) = 0$ implies the existence of a function $\mathscr{P}_{\overrightarrow{X}}$ such that

$$-\frac{1}{2} d\mathscr{P}_{\overrightarrow{X}} = i_{\overrightarrow{X}} K \tag{3.7.9}$$

The function $\mathscr{P}_{\overrightarrow{X}}$ is defined up to a constant, which can be arranged so as to make it equivariant:

$$\overrightarrow{X} \mathscr{P}_{\overrightarrow{Y}} = \mathscr{P}_{[\overrightarrow{X}, \overrightarrow{Y}]} \tag{3.7.10}$$

$\mathscr{P}_{\overrightarrow{X}}$ constitutes then a *moment map*. This can be regarded as a map

$$\mathscr{P} : \mathscr{M} \longrightarrow \mathbb{R} \otimes \mathbb{G}^* \tag{3.7.11}$$

where $\mathbb{G}^*$ denotes the dual of the Lie algebra $\mathbb{G}$ of the group $\mathscr{G}$. Indeed let $x \in \mathbb{G}$ be the Lie algebra element corresponding to the Killing vector $\overrightarrow{X}$; then, for a given $m \in \mathscr{M}$

$$\mu(m) \, : \, x \longrightarrow \mathscr{P}_{\overrightarrow{X}}(m) \in \mathbb{R} \tag{3.7.12}$$

is a linear functional on $\mathbb{G}$. If we expand $\overrightarrow{X} = a^{\mathbf{I}} k_{\mathbf{I}}$ in a basis of Killing vectors $k_{\mathbf{I}}$ such that

$$[k_{\mathbf{I}}, k_{\mathbf{L}}] = f_{\mathbf{IL}}{}^{\mathbf{K}} k_{\mathbf{K}} \tag{3.7.13}$$

we have also

$$\mathscr{P}_{\overrightarrow{X}} = a^{\mathbf{I}} \mathscr{P}_{\mathbf{I}} \tag{3.7.14}$$

In the following we use the shorthand notation $\mathscr{L}_{\mathbf{I}}, i_{\mathbf{I}}$ for the Lie derivative and the contraction along the chosen basis of Killing vectors $k_{\mathbf{I}}$.

From a geometrical point of view the prepotential, or moment map, $\mathscr{P}_{\mathbf{I}}$ is the Hamiltonian function providing the Poissonian realization of the Lie algebra on the Kähler manifold. This is just another way of stating the already mentioned *equivariance*. Indeed the very existence of the closed 2-form K guarantees that every Kähler space is a symplectic manifold and that we can define a Poisson bracket.

Consider Eq. (3.7.7). To every generator of the abstract Lie algebra $\mathbb{G}$ we have associated a function $\mathscr{P}_{\mathbf{I}}$ on $\mathscr{M}$; the Poisson bracket of $\mathscr{P}_{\mathbf{I}}$ with $\mathscr{P}_{\mathbf{J}}$ is defined as follows:

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\} \equiv 4\pi K(\mathbf{I}, \mathbf{J}) \tag{3.7.15}$$

where $K(\mathbf{I}, \mathbf{J}) \equiv K(\mathbf{k}_{\mathbf{I}}, \mathbf{k}_{\mathbf{J}})$ is the value of K along the pair of Killing vectors.

In Ref. [4] the following lemma was proved:

Lemma 3.1 *The following identity is true:*

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\} = f_{\mathbf{IJ}}{}^{\mathbf{L}} \mathscr{P}_{\mathbf{L}} + C_{\mathbf{IJ}} \tag{3.7.16}$$

where $C_{\mathbf{IJ}}$ is a constant fulfilling the cocycle condition

$$f_{\mathbf{IM}}{}^{\mathbf{L}} C_{\mathbf{LJ}} + f_{\mathbf{MJ}}{}^{\mathbf{L}} C_{\mathbf{LI}} + f_{\mathbf{JI}}{}^{\mathbf{L}} C_{\mathbf{LM}} = 0 \tag{3.7.17}$$

If the Lie algebra $\mathbb{G}$ has a trivial second cohomology group $H^2(\mathbb{G}) = 0$, then the cocycle $C_{\mathbf{IJ}}$ is a coboundary; namely we have

$$C_{\mathbf{IJ}} = f_{\mathbf{IJ}}{}^{\mathbf{L}} C_{\mathbf{L}} \tag{3.7.18}$$

where $C_{\mathbf{L}}$ are suitable constants. Hence, assuming $H^2(\mathbb{G}) = 0$ we can reabsorb $C_{\mathbf{L}}$ in the definition of $\mathscr{P}_{\mathbf{I}}$:

$$\mathscr{P}_{\mathbf{I}} \to \mathscr{P}_{\mathbf{I}} + C_{\mathbf{I}} \tag{3.7.19}$$

and we obtain the stronger equation

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\} = f_{\mathbf{IJ}}{}^{\mathbf{L}} \mathscr{P}_{\mathbf{L}} \tag{3.7.20}$$

Note that $H^2(\mathbb{G}) = 0$ is true for all semi-simple Lie algebras. Using Eqs. (3.7.16), (3.7.20) can be rewritten in components as follows:

$$\frac{i}{2} g_{ij^*} (k^i_{\mathbf{I}} k^{j^*}_{\mathbf{J}} - k^i_{\mathbf{J}} k^{j^*}_{\mathbf{I}}) = \frac{1}{2} f_{\mathbf{IJ}}{}^{\mathbf{L}} \mathscr{P}_{\mathbf{L}} \tag{3.7.21}$$

Equation (3.7.21) is identical with the equivariance condition in Eq. (3.7.10).

Finally let us recall the explicit general way of solving Eq. (3.7.9) obtaining the real valued function $\mathscr{P}_{\mathbf{I}}$ which satisfies Eq. (3.7.7). In terms of the Kähler potential $\mathscr{K}$ we have:

$$\mathscr{P}_{\mathbf{I}}{}^{x} = -\frac{i}{2} \left(k^i_{\mathbf{I}} \partial_i \mathscr{K} - k^{\bar{\imath}}_{\mathbf{I}} \partial_{\bar{\imath}} \mathscr{K} \right) + \mathrm{Im}(f_{\mathbf{I}}) \,, \tag{3.7.22}$$

where $f_{\mathbf{I}} = f_{\mathbf{I}}(z)$ is a holomorphic transformation on the line-bundle, defining a compensating Kähler transformation:

$$k_{\mathbf{I}}^{i}\partial_{i}\mathcal{K} + k_{\mathbf{I}}^{\bar{\imath}}\partial_{\bar{\imath}}\mathcal{K} = -f_{\mathbf{I}}(z) - \overline{f}_{\mathbf{I}}(\bar{z})\,. \tag{3.7.23}$$

3.7.2 The Triholomorphic Moment Map on Quaternionic Manifolds

Next, following closely the original derivation of [4, 11] let us turn to a discussion of the triholomorphic isometries of the manifold $\mathcal{QM}$ associated with hypermultiplets. In $D = 4$ supergravity the manifold of hypermultiplet scalars $\mathcal{QM}$ is a Quaternionic Kähler manifold and we can gauge only those of its isometries that are triholomorphic and that either generate an abelian group $\mathcal{G}$ or are *suitably realized* as isometries also on the special manifold $\widehat{\mathcal{SK}}_n$. This means that on $\mathcal{QM}$ we have Killing vectors:

$$\mathbf{k}_{\mathbf{I}} = k_{\mathbf{I}}^{u}\frac{\boldsymbol{\partial}}{\partial q^{u}} \tag{3.7.24}$$

satisfying the same Lie algebra as the corresponding Killing vectors on $\widehat{\mathcal{SK}}_n$. In other words

$$\mathfrak{K}_{\mathbf{I}} = \hat{k}_{\mathbf{I}}^{i}\boldsymbol{\partial}_{i} + \hat{k}_{\mathbf{I}}^{i^*}\boldsymbol{\partial}_{i^*} + k_{\mathbf{I}}^{u}\boldsymbol{\partial}_{u} \tag{3.7.25}$$

is a Killing vector of the block diagonal metric:

$$\mathfrak{g} = \begin{pmatrix} \widehat{g}_{ij^\star} & 0 \\ 0 & h_{uv} \end{pmatrix} \tag{3.7.26}$$

defined on the product manifold[7] $\widehat{\mathcal{SK}} \otimes \mathcal{QM}$.

Let us first focus on the manifold $\mathcal{QM}$. Triholomorphicity means that the Killing vector fields leave the HyperKähler structure invariant up to SU(2) rotations in the SU(2)-bundle defined by Eq. (3.6.2). Namely:

$$\mathcal{L}_{\mathbf{I}}K^{x} = \varepsilon^{xyz}K^{y}W_{\mathbf{I}}^{z}\ ;\ \mathcal{L}_{\mathbf{I}}\omega^{x} = \nabla W_{\mathbf{I}}^{x} \tag{3.7.27}$$

[7] Special Kähler geometry will be discussed in Chap. 4, yet we anticipate here that it is the geometrical structure imposed by $\mathcal{N} = 2$ supersymmetry on the scalars belonging to vector multiplets (the scalar partners of the gauge vectors). In our notations the Special Kähler manifold which describes the interaction of vector multiplets is denoted $\widehat{\mathcal{SK}}$ and all the Special Geometry Structures are endowed with a hat in order to distinguish this Special Kähler manifold from the other one which is incapsulated into the Quaternionic Kähler manifold $\mathcal{QM}$ describing the hypermultiplets when this latter happens to be in the image of the c-map. For all these concepts we refer the reader to Chap. 4. They are not necessary to understand the present constructions, yet they were essential part for their establishment in the original papers mentioned here above.

where $W^x_{\mathbf{I}}$ is an SU(2) compensator associated with the Killing vector $k^u_{\mathbf{I}}$. The compensator $W^x_{\mathbf{I}}$ necessarily fulfills the cocycle condition:

$$\mathscr{L}_{\mathbf{I}} W^x_{\mathbf{J}} - \mathscr{L}_{\mathbf{J}} W^x_{\mathbf{I}} + \varepsilon^{xyz} W^y_{\mathbf{I}} W^z_{\mathbf{J}} = f_{\mathbf{IJ}}^{\cdot\cdot\mathbf{L}} W^x_{\mathbf{L}} \tag{3.7.28}$$

In the HyperKähler case the SU(2)-bundle is flat and the compensator can be reabsorbed into the definition of the HyperKähler forms. In other words we can always find a map

$$\mathscr{Q}\mathscr{M} \longrightarrow L^x{}_y(q) \in \mathrm{SO}(3) \tag{3.7.29}$$

that trivializes the $\mathscr{S}\mathscr{U}$-bundle globally. Redefining:

$$K^{x\prime} = L^x{}_y(q)\, K^y \tag{3.7.30}$$

the new HyperKähler form obeys the stronger equation:

$$\mathscr{L}_{\mathbf{I}} K^{x\prime} = 0 \tag{3.7.31}$$

On the other hand, in the quaternionic case, the non-triviality of the $\mathscr{S}\mathscr{U}$-bundle forbids to eliminate the W-compensator completely. Due to the identification between HyperKähler forms and SU(2) curvatures Eq. (3.7.27) is rewritten as:

$$\mathscr{L}_{\mathbf{I}} \Omega^x = \varepsilon^{xyz} \Omega^y W^z_{\mathbf{I}} \; ; \; \mathscr{L}_{\mathbf{I}} \omega^x = \nabla W^x_{\mathbf{I}} \tag{3.7.32}$$

In both cases, anyhow, and in full analogy with the case of Kähler manifolds, to each Killing vector we can associate a triplet $\mathscr{P}^x_{\mathbf{I}}(q)$ of 0-form prepotentials. Indeed we can set:

$$\mathbf{i}_{\mathbf{I}} K^x = -\nabla \mathscr{P}^x_{\mathbf{I}} \equiv -(d\mathscr{P}^x_{\mathbf{I}} + \varepsilon^{xyz} \omega^y \mathscr{P}^z_{\mathbf{I}}) \tag{3.7.33}$$

where ∇ denotes the SU(2) covariant exterior derivative.

As in the Kähler case Eq. (3.7.33) defines a moment map:

$$\mathscr{P} : \mathscr{M} \longrightarrow \mathbb{R}^3 \otimes \mathbb{G}^* \tag{3.7.34}$$

where $\mathbb{G}^*$ denotes the dual of the Lie algebra $\mathbb{G}$ of the group $\mathscr{G}$. Indeed let $x \in \mathbb{G}$ be the Lie algebra element corresponding to the Killing vector $\overrightarrow{X}$; then, for a given $m \in \mathscr{M}$

$$\mu(m) \; : \; x \longrightarrow \mathscr{P}_{\overrightarrow{X}}(m) \in \mathbb{R}^3 \tag{3.7.35}$$

is a linear functional on $\mathscr{G}$. If we expand $\overrightarrow{X} = a^{\mathbf{I}} k_{\mathbf{I}}$ on a basis of Killing vectors $k_{\mathbf{I}}$ such that

$$[k_{\mathbf{I}}, k_{\mathbf{L}}] = f_{\mathbf{IL}}{}^{\mathbf{K}} k_{\mathbf{K}} \tag{3.7.36}$$

and we also choose a basis $\mathbf{i}_x$ $(x = 1, 2, 3)$ for $\mathbb{R}^3$ we get:

$$\mathscr{P}_{\vec{X}} = a^{\mathbf{I}} \mathscr{P}^x_{\mathbf{I}} \, \mathbf{i}_x \tag{3.7.37}$$

Furthermore we need a generalization of the equivariance defined by Eq. (3.7.10)

$$\vec{X} \circ \mathscr{P}_{\vec{Y}} = \mathscr{P}_{[\vec{X}, \vec{Y}]} \tag{3.7.38}$$

In the HyperKähler case, the left-hand side of Eq. (3.7.38) is defined as the usual action of a vector field on a 0-form:

$$\vec{X} \circ \mathscr{P}_{\vec{Y}} = \mathbf{i}_{\vec{X}} \, d\mathscr{P}_{\vec{Y}} = X^u \frac{\partial}{\partial q^u} \mathscr{P}_{\vec{Y}} \tag{3.7.39}$$

The equivariance condition implies that we can introduce a triholomorphic Poisson bracket defined as follows:

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\}^x \equiv 2K^x(\mathbf{I}, \mathbf{J}) \tag{3.7.40}$$

leading to the triholomorphic Poissonian realization of the Lie algebra:

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\}^x = f^{\mathbf{K}}{}_{\mathbf{IJ}} \, \mathscr{P}^x_{\mathbf{K}} \tag{3.7.41}$$

which in components reads:

$$K^x_{uv} \, k^u_{\mathbf{I}} \, k^v_{\mathbf{J}} = \frac{1}{2} f^{\mathbf{K}}{}_{\mathbf{IJ}} \, \mathscr{P}^x_{\mathbf{K}} \tag{3.7.42}$$

In the quaternionic case, instead, the left-hand side of Eq. (3.7.38) is interpreted as follows:

$$\vec{X} \circ \mathscr{P}_{\vec{Y}} = \mathbf{i}_{\vec{X}} \, \nabla \mathscr{P}_{\vec{Y}} = X^u \, \nabla_u \, \mathscr{P}_{\vec{Y}} \tag{3.7.43}$$

where ∇ is the SU(2)-covariant differential. Correspondingly, the triholomorphic Poisson bracket is defined as follows:

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\}^x \equiv 2K^x(\mathbf{I}, \mathbf{J}) - \lambda \, \varepsilon^{xyz} \, \mathscr{P}^y_{\mathbf{I}} \, \mathscr{P}^z_{\mathbf{J}} \tag{3.7.44}$$

and leads to the Poissonian realization of the Lie algebra

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\}^x = f^{\mathbf{K}}{}_{\mathbf{IJ}} \, \mathscr{P}^x_{\mathbf{K}} \tag{3.7.45}$$

which in components reads:

$$K^x_{uv} \, k^u_{\mathbf{I}} \, k^v_{\mathbf{J}} - \frac{\lambda}{2} \varepsilon^{xyz} \, \mathscr{P}^y_{\mathbf{I}} \, \mathscr{P}^z_{\mathbf{J}} = \frac{1}{2} f^{\mathbf{K}}{}_{\mathbf{IJ}} \, \mathscr{P}^x_{\mathbf{K}} \tag{3.7.46}$$

Equation (3.7.46), which is the most convenient way of expressing equivariance in a coordinate basis was originally written in [4] and has played a fundamental role in the construction of supersymmetric actions for gauged $\mathcal{N} = 2$ supergravity both in $D = 4$ [4, 5] and in $D = 5$ [12].

3.8 Kähler Surfaces with One Continuous Isometry

As an illustration of the concepts introduced in the previous sections we consider here a class of very simple manifolds for which a lot of explicit calculations can be explicitly done, quite non trivial conceptual questions can be addressed and answered. These are 2-dimensional surfaces endowed with a one-dimensional continuous group of isometries $\mathcal{G}_{iso}$. As we advocate below the geometry of such manifolds is completely encoded in a single positive real function $V(\phi)$ of a single real coordinate ϕ. We name such a function the *potential*.[8] The main point is that any two-dimensional Euclidean manifold is actually complex and Kähler. This offers us the possibility of exemplifying all the structures we have discussed. We have to find the complex structure, the Kähler form and the Kähler potential. Furthermore since we have a Killing vector we can construct its moment map. Finally we can calculate the curvature. All these objects are functions of a single coordinate related with the initial potential $V(\phi)$ and its derivatives. Last but not least we have to decide the topological nature of the isometry group.

Within this class of manifolds we are able to construct several interesting examples that hopefully should clarify the non trivial aspects of the geometrical apparatus developed in previous sections. In particular, since we are dealing with 2-dimensional surfaces we can visualize them by means of their embedding in three-dimensional space.

With the above motivations let us consider Riemannian 2-dimensional manifolds Σ whose metric is the following one:

$$ds^2_{\Sigma} = p(U)\, dU^2 + q(U)\, dB^2 \tag{3.8.1}$$

$p(U), q(U)$ being two positive definite functions of their argument. The isometry group of the manifold Σ is generated by the Killing vector $\mathbf{k}_{[B]} = \partial_B$.

A fundamental geometrical question is whether $\mathbf{k}_{[B]}$ generates a *compact rotation symmetry*, or a *non compact symmetry either parabolic or hyperbolic*. We plan to discuss this issue in detail in the sequel.

Actually when $\Sigma = \Sigma_{max}$ is a constant curvature surface namely the coset manifold $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \sim \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{O}(2)}$, there is also a third possibility. In such a situation the Killing vector $\mathbf{k}_{[B]}$ can be the generator of a *dilatation*, namely it can correspond to

[8] This name is related with the use of this class of surfaces in supergravity inflationary models as described in [13–15], yet this is not relevant to us here. In this book our view point is just geometrical. Most of the material presented in this section was originally worked out in [13–15].

a non-compact but semi-simple element $\mathbf{d} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of the Lie algebra $SL(2, \mathbb{R})$ rather then to a nilpotent one $\mathbf{t} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

As all other two-dimensional surfaces, Σ has an underlying complex Kählerian structure that we can systematically uncover with the methods described in this chapter. The first step is to determine the complex structure with respect to which the metric (3.8.1) is hermitian. By definition an almost complex structure is a tensor $\mathfrak{J}_\alpha^\beta$ which squares to minus the identity:

$$\mathfrak{J}_\alpha^\beta \, \mathfrak{J}_\beta^\gamma = -\delta_\alpha^\gamma \tag{3.8.2}$$

The almost complex structure $\mathfrak{J}_\alpha^\beta$ becomes a true complex structure if its Nienhuis tensor vanishes:

$$N_{\alpha\beta}^\gamma \equiv \partial_{[\alpha} \, \mathfrak{J}_{\beta]}^\gamma - \mathfrak{J}_\alpha^\mu \, \mathfrak{J}_\beta^\nu \, \partial_{[\mu} \, \mathfrak{J}_{\nu]}^\gamma = 0 \tag{3.8.3}$$

Given a complex structure, a metric $g_{\alpha\beta}$ is hermitian with respect to it if the following identity is true:

$$g_{\alpha\beta} = \mathfrak{J}_\alpha^\gamma \, \mathfrak{J}_\beta^\delta \, g_{\gamma\delta} \tag{3.8.4}$$

Given the metric (3.8.1) there is a unique tensor $\mathfrak{J}_\alpha^\beta$, which simulatenously satisfies Eqs. (3.8.2), (3.8.3), (3.8.4) and it is the following:

$$\mathfrak{J} = \begin{pmatrix} 0 & \mathfrak{J}_B^U \\ \mathfrak{J}_U^B & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\frac{p(U)}{q(U)}} \\ -\sqrt{\frac{q(U)}{p(U)}} & 0 \end{pmatrix} \tag{3.8.5}$$

Next, according to theory, the Kähler 2-form is defined by:

$$\begin{aligned} \mathrm{K} = \mathrm{K}_{\alpha\beta} \, dx^\alpha \wedge dx^\beta &= g_{\alpha\gamma} \, \mathfrak{J}_\beta^\gamma \, dx^\alpha \wedge dx^\beta \\ &= -\sqrt{p(U)\, q(U)} \, dU \wedge dB \end{aligned} \tag{3.8.6}$$

and it is clearly closed. Hence the metric (3.8.1) is Kählerian and necessarily admits a representation in terms of a complex coordinate ζ and a Kähler potential $\mathscr{K}(\zeta, \overline{\zeta})$. In terms of the complex coordinate:

$$\zeta = \zeta(U, B) \tag{3.8.7}$$

the Kähler 2-form K in Eq. (3.8.6) should be rewritten as:

$$\mathrm{K} = \partial \, \overline{\partial} \, \mathscr{K} = \partial_\zeta \, \partial_{\overline{\zeta}} \, \mathscr{K} \, d\zeta \wedge d\overline{\zeta} \tag{3.8.8}$$

Next one aims at reproducing the Kählerian metric (3.8.1) in terms of a complex coordinate $\mathfrak{z} = \mathfrak{z}(U, B)$ and a Kähler potential $\mathscr{K}(\mathfrak{z}, \overline{\mathfrak{z}}) = \mathscr{K}^\star(\mathfrak{z}, \overline{\mathfrak{z}})$ such that:

$$\mathrm{K} = \mathrm{i}\,\partial\,\overline{\partial}\,\mathscr{K} = \mathrm{i}\partial_{\mathfrak{z}}\,\partial_{\bar{\mathfrak{z}}}\,\mathscr{K}\,d\mathfrak{z}\wedge d\bar{\mathfrak{z}} \quad ; \quad ds^2_{\Sigma} = \partial_{\mathfrak{z}}\,\partial_{\bar{\mathfrak{z}}}\,\mathscr{K}\,d\mathfrak{z}\otimes d\bar{\mathfrak{z}} \tag{3.8.9}$$

The complex coordinate $\mathfrak{z}$ is necessarily a solution of the complex structure equation:

$$\mathfrak{J}_\alpha^{\beta}\,\partial_\beta\,\mathfrak{z} = \mathrm{i}\partial_\alpha\,\mathfrak{z} \quad \Rightarrow \quad \sqrt{\frac{p(U)}{q(U)}}\,\partial_B\,\mathfrak{z}(U,B) = \mathrm{i}\,\partial_U\,\mathfrak{z}(U,B) \tag{3.8.10}$$

The general solution of such an equation is easily found. Define the linear combination[9]:

$$w \equiv \mathrm{i}\,C(U) - B \quad ; \quad C(U) = \int\sqrt{\frac{p(U)}{q(U)}}\,dU \tag{3.8.11}$$

and consider any holomorphic function $f(w)$. As one can immediately verify, the position $\mathfrak{z}(U,B) = f(w)$ solves Eq. (3.8.10). What is the appropriate choice of the holomorphic function $f(w)$? Locally (in an open neighborhood) this is an empty question, since the holomorphic function $f(w)$ simply corresponds to a change of coordinates and gives rise to the same Kähler metric in a different basis. Globally, however, there are significant restrictions that concern the range of the variables B and $C(U)$, namely the global topology of the manifold Σ. By definition B is the coordinate that, within Σ, parameterizes points along the $\mathscr{G}_\Sigma$-orbits, having denoted by $\mathscr{G}_\Sigma$ the isometry group. If $\mathscr{G}_\Sigma$ is compact, then B is a coordinate on the circle and it must be defined up to identifications $B \simeq B + 2\,n\,\pi$, where n is an integer. On the other hand if B is non compact its range extends on the full real line $\mathbb{R}$.

Furthermore, it is convenient to choose a canonical variable ϕ and codify the geometry of the surface in terms of a single positive potential function $V(\phi)$ rewriting it in the following way:

$$ds^2_g = d\phi^2 + \underbrace{\left(\frac{\mathrm{d}\sqrt{V(\phi)}}{\mathrm{d}\phi}\right)^2}_{f^2(\phi)} dB^2 \tag{3.8.12}$$

Hence we aim at a Kähler potential $\mathscr{K}(\mathfrak{z},\bar{\mathfrak{z}})$ that in terms of the variables $C(U)$ and B should actually depend only on C, being constant on the $\mathscr{G}$-orbits. Starting from the metric (3.8.1) we can always choose a canonical variable ϕ defined by the position:

$$\phi = \phi(U) = \int\sqrt{p(U)}\,dU \quad ; \quad d\phi = \sqrt{p(U)}\,dU \tag{3.8.13}$$

[9] As it follows from the present discussion the coordinate $C(U)$ has an intrinsic geometric characterization as that one which solves the differential equation of the complex structure. For the historical reasons explained in [13–15] we name C the Van Proeyen coordinate, abbreviated VP-coordinate.

and assuming that $\phi(U)$ can be inverted $U = U(\phi)$ we can rewrite (3.8.1) in the following canonical form:

$$ds^2_{can} = d\phi^2 + \left(\mathscr{P}'(\phi)\right)^2 dB^2 \quad ; \quad \mathscr{P}'(\phi) = \sqrt{q\,(U(\phi))} \quad ; \quad \underbrace{\sqrt{p(U(\phi))}\,\frac{dU}{d\phi} = 1}_{\text{by construction}} \tag{3.8.14}$$

The reason to call the square root of $q\,(U(\phi))$ with the name $\mathscr{P}'(\phi)$ is the interpretation of such a function as the derivative with respect to the canonical variable ϕ of the moment map of the Killing vector $\mathbf{k}_{[B]}$.

By using the canonical variable ϕ, the coordinate C defined in Eq. (3.8.11) becomes:

$$C(\phi) = C\,(U(\phi)) = \int \frac{d\phi}{\mathscr{P}'(\phi)} \tag{3.8.15}$$

and the metric $ds^2_{\Sigma} = ds^2_{can}$ of the Kähler surface Σ can be rewritten as:

$$ds^2_{\Sigma} = \tfrac{1}{2}\,\frac{\mathrm{d}^2 J}{\mathrm{d}C^2}\left(\mathrm{d}C^2 + \mathrm{d}B^2\right) \tag{3.8.16}$$

where the function $J(C)$ is defined as follows:

$$\mathscr{J}(\phi) \equiv 2\int \frac{\mathscr{P}(\phi)}{\mathscr{P}'(\phi)}\,d\phi \quad ; \quad J(C) \equiv \mathscr{J}\,(\phi(C)) \tag{3.8.17}$$

It appears from the above formula that the crucial step in working out the analytic form of the function $J(C)$ is the ability of inverting the relation between the coordinate C, defined by the integral (3.8.15), and the canonical one ϕ, a task which, in the general case, is quite hard in both directions. The indefinite integral (3.8.15) can be expressed in terms of special functions only in certain cases and even less frequently one has at his own disposal inverse functions. In any case the problem is reduced to quadratures and one can proceed further. Having already established in Eq. (3.8.11) the general solution of the complex structure equations, there are three possibilities that correspond, in the case of constant curvature manifolds Σ_{max}, to the three conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ elements (elliptic, hyperbolic and parabolic). In the three cases $J(C)$ is identified with the Kähler potential $\mathscr{K}(\mathfrak{z},\bar{\mathfrak{z}})$, but it remains to be decided whether the coordinate C is to be identified with the imaginary part of the complex coordinate, namely $C = \mathrm{Im}\,\mathfrak{z}$, with the logarithm of its modulus $C = \frac{1}{2}\log|\mathfrak{z}|^2$, or with a third combination of $\mathfrak{z}$ and $\bar{\mathfrak{z}}$, namely whether we choose the first the second or the third of the options listed below:

$$\mathfrak{z} = \left\{\begin{array}{lll} \zeta \equiv & \exp[-\mathrm{i}\,w] & = \underbrace{\exp[C(\phi)]}_{\rho(\phi)}\exp[\mathrm{i}B] \\ t \equiv & w & = \mathrm{i}\,C(\phi) - B \\ \hat{\zeta} \equiv \mathrm{i}\tanh\left(-\tfrac{1}{2}w\right) & & = \mathrm{i}\tanh\left(-\tfrac{1}{2}\left(\mathrm{i}\,C(\phi) - B\right)\right) \end{array}\right. \quad \Bigg| \quad C(\phi) \equiv \int \frac{1}{\mathscr{P}'(\phi)}\,d\phi \tag{3.8.18}$$

If we choose the first solution $\mathfrak{z} = \zeta$, that we name name of the *Disk-type*, we obtain that the basic isometry generated by the Killing vector $\mathbf{k}_{[B]}$ is a compact rotation symmetry. Choosing the second solution $\mathfrak{z} = t$, that we name of *Plane-type*, is appropriate instead to the case of a non compact shift symmetry. The third possibility mentioned above certainly occurs in the case of constant curvature surfaces Σ_{max} and leads to the interpretation of the B-shift as an SO(1, 1)-hyperbolic transformation.

In Sect. 3.8.5 we recall that the classification of a one dimensional isometry group as elliptic, parabolic or hyperbolic exists also for non maximally symmetric manifolds and it can be unambiguously formulated for *Hadamard manifolds* that are, by definition, simply connected, smooth Riemannian manifolds with a non positive definite curvature, *i.e.* $R(x) \leq 0, \forall x \in \Sigma$, having denoted by $R(x)$ the scalar curvature at the point x.

In the three cases mentioned in Eq. (3.8.18) the analytic form of the holomorphic Killing vector $\mathbf{k}_{[B]}$ is quite different:

$$\mathbf{k}_{[B]} = \begin{cases} \mathrm{i}\zeta\,\partial_\zeta & \equiv k^{\mathfrak{z}}\partial_{\mathfrak{z}} \Rightarrow k^{\mathfrak{z}} = \mathrm{i}\,\mathfrak{z} & ;\ \text{Disk-type, compact rotation} \\ \partial_t & \equiv k^{\mathfrak{z}}\partial_{\mathfrak{z}} \Rightarrow k^{\mathfrak{z}} = 1 & ;\ \text{Plane-type, non-compact shift} \\ \mathrm{i}\left(1+\hat{\zeta}^2\right)\partial_{\hat{\zeta}} & \equiv k^{\mathfrak{z}}\partial_{\mathfrak{z}} \Rightarrow k^{\mathfrak{z}} = \mathrm{i}\,\left(1+\mathfrak{z}^2\right) & ;\ \text{Disk-type, hyperbolic boost} \end{cases} \tag{3.8.19}$$

Choosing the complex structure amounts to the same as introducing one half of the missing information on the global structure of Σ, namely the range of the coordinate B. The other half is the range of the coordinate U or C.

Actually, by means of the constant curvature examples, a criterion able to discriminate the relevant topologies is encoded in the asymptotic behavior of the function $\partial_C^2 J(C)$ for large and small values of its argument, namely in the center of the bulk and on the boundary of the surface Σ. The main conclusions that we can reach by considering the case of constant curvature surfaces are those summarized below and are also encoded in Table 3.1:

(I) The global topology of the group $\mathscr{G}_\Sigma$ reflects into a different asymptotic behavior of the function $\partial_C^2 J(C)$ in the region that we can call the origin of the manifold. In the compact case the complex coordinate $\mathfrak{z}$ is charged with respect to U(1) and, for consistency, this symmetry should exist at all orders in an expansion of the line element ds_Σ^2 for small coordinates. Hence for $\mathfrak{z} \to 0$ the line element should approach the canonical one of a flat complex-manifold:

$$ds_\Sigma^2 \propto \mathrm{d}\mathfrak{z}\,\mathrm{d}\bar{\mathfrak{z}} \tag{3.8.20}$$

Assuming, as it is necessary for the U(1) interpretation of the B-shift symmetry, that $\mathfrak{z} = \zeta = \exp\left[\delta(C + \mathrm{i}\,B)\right]$, where δ is some real coefficient, Eq. (3.8.20) can be satisfied if and only if we have:

$$\lim_{C \to -\infty} \exp\left[-2\,\delta\,C\right]\,\partial_C^2 J(C) = \text{const.} \tag{3.8.21}$$

Table 3.1 Summary of the functions $V(\phi)$ defining the line element (3.8.12) which are obtained from constant curvature Kähler manifolds

Curv.	Isometry group	$V(\phi)$	$V(C)$	$V(\mathfrak{z})$	Comp. Struct.
$-\hat{\nu}^2$	U(1)	$\left(\cosh\left(\hat{\nu}\,\hat{\phi}\right) + \mu\right)^2$	$\left(\mu + \frac{2\,e^{4\,C\,\hat{\nu}^2}}{1-e^{4\,C\,\hat{\nu}^2}} + 1\right)^2$	$\frac{1}{\nu^4}\left(\frac{\mu+1-\mu\,\zeta\,\bar{\zeta}}{1-\zeta\,\bar{\zeta}}\right)^2$	$\zeta = e^{C-\mathrm{i}B}$
$-\hat{\nu}^2$	SO(1, 1)	$\left(\sinh\left(\hat{\nu}\,\hat{\phi}\right) + \mu\right)^2$	$\left(\mu + \tan\left(2\,C\,\hat{\nu}^2\right)\right)^2$	$\left(\frac{\bar{\zeta}(\zeta+\bar{\zeta})\zeta+\zeta+\bar{\zeta}+2\mu(\zeta\,\bar{\zeta}-1)}{4\zeta\,\bar{\zeta}-4}\right)^2$	$\zeta =$ $\mathrm{i}\,\tanh\left(\frac{1}{2}(B-\mathrm{i}C)\,\nu^2\right)$
$-\hat{\nu}^2$	parabolic	$\left(\exp\left(\hat{\nu}\,\hat{\phi}\right) + \mu\right)^2$	$\left(\mu + \frac{1}{2\,\hat{\nu}^2\,C}\right)^2$	$\left(\frac{1}{2}\,\mu + \frac{\mathrm{i}}{2\,\hat{\nu}^2}\,(t-\bar{t})^{-1}\right)^2$	$t = \mathrm{i}\,C - B$
0	U(1)	$M^4\left[\left(\frac{\phi}{\phi_0}\right)^2 \pm 1\right]^2$	$M^4\left[\frac{e^{2a_2\,C}}{\phi_0^2} \pm 1\right]^2$	$\frac{1}{4}\left(\mathfrak{z}\,\bar{\mathfrak{z}} - \frac{2\,a_0}{a_2}\right)^2$	$\mathfrak{z} = \exp\left[a_2\,(C - \mathrm{i}B)\right]$
0	parabolic	$(a_0 + a_1\,\phi)^2$	$(a_1\,C + \beta)^2$	$\frac{1}{2}\,(a_1 \mathrm{Im}\mathfrak{z} + \beta)^2$	$\mathfrak{z} = \mathrm{i}C - B$

or more precisely:

$$\partial_C^2 J(C) \overset{C \to -\infty}{\approx} \text{const} \times \exp[2\,\delta\,C] + \text{subleading}$$
$$J(C) \overset{C \to -\infty}{\approx} \text{const} \times \exp[2\,\delta\,C] + \text{subleading} \tag{3.8.22}$$

The above stated is an intrinsic clue to establish the global topology of the Kähler surface Σ. In Sect. 3.8.5 we present some rigorous mathematical results that justify the above criterion to establish the compact nature of the gauged isometry. Indeed what, in heuristic jargon we call the origin of the manifold is, in rigorous mathematical language, the fixed point for all $\Gamma \in \mathscr{G}_\Sigma$, located in the interior of the manifold, whose existence is a necessary defining feature of an elliptic[10]isometry group $\mathscr{G}$.

(II) The above properties are general and apply to all surfaces of type (3.8.1)–(3.8.12). In the particular case of constant curvature Kähler surfaces there are five ways of writing the line-element (3.8.12), two associated with a flat Kähler manifold and three with the unique negative curvature two-dimensional symmetric space SL(2, R)/O(2).

(III) Global topology amounts, at the end of the day, to giving the precise range of the coordinates C and B labeling the points of Σ. In the five constant curvature cases these ranges are as follows. In the elliptic and parabolic case C is in the range $[-\infty, 0]$, while it is in the range $[-\infty, +\infty]$ for the flat case and it is periodic in the hyperbolic case. The cooordinate B instead is periodic in the elliptic case, while it is unrestricted in the hyperbolic and parabolic cases. The manifold Σ in the flat case with B periodic is just a strip. It is instead the full plane in the flat parabolic case.

Our goal is to extend the above results to examples where the curvature of the Kähler surface Σ is not constant. In such examples we will verify the criterion that singles out the interpretation of the B-shift isometry as a parabolic shift-symmetry. In all such cases the range of the C coordinate is $[-\infty, 0]$[11] or $[-\infty, \infty]$. The limit $C \to 0$ always correspond to a boundary of the Kähler manifold Σ irrespectively whether the isometry group $\mathscr{G}_\Sigma$ is elliptic or parabolic. If the curvature is negative we always have:

$$\partial_C^2 J(C) \overset{C \to 0}{\approx} \text{const} \times \frac{1}{C^2} + \text{subleading}$$
$$J(C) \overset{C \to 0}{\approx} \text{const} \times \log[C] + \text{subleading} \tag{3.8.23}$$

[10]Let us stress that this is true for Hadamard manifolds and possibly for CAT(k) manifolds, in any case for simple connected manifolds. In the presence of a non trivial fundamental group the presence of a fixed point is not necessary in order to establish the compact nature of the isometry group.

[11]Note that $[-\infty, 0]$ as range of the C-coordinate is conventional. Were it to be $[\infty, 0]$, we could just replace $C \to -C$ which is always possible since the Kähler metric is given by Eq. 3.8.16.

In case the curvature at $C = 0$ is zero, the gauge group is necessarily parabolic, since we cannot organize an exponential behavior of $J(C)$ for $C \to 0$. Such exponential behavior is instead requested by an elliptic isometry, so the only conclusion is that a limiting zero curvature at a boundary $C = 0$ can occur only in parabolic models and there we have:

$$\begin{aligned} \partial_C^2 J(C) &\overset{C\to 0}{\approx} \text{const} + \text{subleading} \\ J(C) &\overset{C\to 0}{\approx} \text{const} \times C^2 + \text{subleading} \end{aligned} \tag{3.8.24}$$

In the case of a parabolic structure of the isometry group $\mathscr{G}_\Sigma$, the locus $C = -\infty$ is always a boundary and not an interior fixed point which does not exist. Differently from Eq. (3.8.22) the asymptotic behavior of the metric and of the J-function is either:

$$\begin{aligned} \partial_C^2 J(C) &\overset{C\to -\infty}{\approx} \text{const} \times \frac{1}{C^2} + \text{subleading} \\ J(C) &\overset{C\to -\infty}{\approx} \frac{1}{R_\infty} \times \log[C] + \text{subleading} \end{aligned} \tag{3.8.25}$$

or

$$\begin{aligned} \partial_C^2 J(C) &\overset{C\to -\infty}{\approx} \text{const} + \text{subleading} \\ J(C) &\overset{C\to -\infty}{\approx} \text{const} \times C^2 + \text{subleading} \end{aligned} \tag{3.8.26}$$

The asymptotic behavior (3.8.25) obtains when the limit of the curvature for $C \to -\infty$ is $R_\infty < 0$. On the other hand, the exceptional asymptotic behavior (3.8.26) occurs when the limit of the curvature for $C \to -\infty$ is $R_\infty = 0$. As we did for the compact case, also for the parabolic case, in Sect. 3.8.5 we present rigorous mathematical arguments that sustain the heuristic criteria (3.8.25) and (3.8.26). Hence in the case where we deal with a parabolic isometry group, the Kähler potential has typically two logarithmic divergences one at $C = 0$, and one at $C = -\infty$, the two boundaries of the manifold Σ. One logarithm can be replaced by C^2 in case the limiting curvature on the corresponding boundary is zero. In other regions the behavior of J is different from logarithmic because of the non constant curvature.

Finally we can wonder what is the criterion to single out a hyperbolic characterization of the isometry group $\mathscr{G}_\Sigma$. A very simple answer arises from the example in the second line of Table 3.1. The hallmark of such isometries is a periodic coordinate C or anyhow a C that takes values in a finite range $[C_{min}, C_{max}]$. We will present an example of a non constant curvature Kähler surface with a hyperbolic isometry in Sect. 3.8.3.

There is still one subtle case of which we briefly discuss an example in Sect. 3.8.2. As we know there are two versions of flat manifolds, one where the selected isometry is a compact U(1) and one where it is a parabolic translation. In both cases the curvature is zero but in the former case the $J(C)$ function is:

$$J(C) \propto \exp[\delta C] \quad ; \quad \text{elliptic case} \tag{3.8.27}$$

while in the latter case we have

$$J(C) \propto C^2 \quad ; \quad \text{parabolic case} \tag{3.8.28}$$

Hence the following question arises. For Σ surfaces with a parabolic isometry group we foresaw the possibility, realized for instance in the example discussed in Sect. 3.8.4, that the limiting curvature might be zero on one of the boundaries so that the asymptotic behavior (3.8.25) is replaced by (3.8.26). In a similar way we might expect that there are elliptic models where the asymptotic behavior at $C \to \pm\infty$ is:

$$J(C) \overset{C\to\pm\infty}{\approx} \exp\left[\delta_{\pm} C\right] \tag{3.8.29}$$

one of the limits being interpreted as the symmetric fixed point in the interior of the manifold, the other being interpreted as the boundary on which the curvature should be zero. In Sect. 3.8.2 we will briefly sketch a model that realizes the above forseen situation. The corresponding manifold Σ has the topology of the disk. In the same section, as a counterexample, we consider a case where the same asymptotic (8.3.56) is realized in presence of an elliptic symmetry, yet $C \to -\infty$ no longer corresponds to an interior point, rather to a boundary. This is due to the non trivial homotopy group $\pi_1(\Sigma)$ of the surface which realizes such an asymptotic behavior. Being non-simply connected such Kähler surface is not a Hadamard manifold and presents new pathologies from the mathematical stand-point.

So let us turn to the analysis of the curvature.

3.8.1 The Curvature and the Kähler Potential of the Surface Σ

The curvature of a two-dimensional Kähler manifold with a one-dimensional isometry group can be written in two different ways in terms of the canonical coordinate ϕ or the coordinate C. In terms of the coordinate C we have the following formula:

$$\begin{aligned} R = R(C) &= -\tfrac{1}{2}\,\frac{J''''(C) - J'''(C)^2}{J''(C)^3} \\ &= -\tfrac{1}{2}\,\partial_C^2 \log\left[\partial_C^2 J(C)\right]\,\frac{1}{\partial_C^2 J(C)} \end{aligned} \tag{3.8.30}$$

which can be derived from the standard structural equations of the manifold [12]:

$$
\begin{aligned}
0 &= \mathrm{d}E^1 + \omega \wedge E^2 \\
0 &= \mathrm{d}E^2 - \omega \wedge E^1 \\
\mathfrak{R} \equiv \mathrm{d}\omega &\equiv 2\,R\,E^1 \wedge E^2
\end{aligned}
\tag{3.8.31}
$$

by inserting into them the appropriate form of the zweibein:

$$
E^1 = \sqrt{\frac{J''(C)}{2}}\,dC \quad ; \quad E^2 = \sqrt{\frac{J''(C)}{2}}\,dB \quad \Rightarrow \quad ds^2 = \tfrac{1}{2}\,J''(C)\left(dC^2 + dB^2\right)
\tag{3.8.32}
$$

Alternatively we can write the curvature in terms of the moment map $\mathscr{P}(\phi)$ or of the function $V(\phi) \propto \mathscr{P}^2(\phi)$ if we use the canonical coordinate ϕ and the corresponding appropriate zweibein:

$$
E^1 = d\phi \quad ; \quad E^2 = \mathscr{P}'(\phi)\,dB \quad \Rightarrow \quad ds^2 = \left(d\phi^2 + \left(\mathscr{P}'(\phi)\right)^2 dB^2\right)
\tag{3.8.33}
$$

Upon insertion of Eq. (3.8.33) into (3.8.31) we get:

$$
R(\phi) = -\tfrac{1}{2}\,\frac{\mathscr{P}'''(\phi)}{\mathscr{P}'(\phi)} = -\tfrac{1}{2}\left(\frac{V'''}{V'} - \tfrac{3}{2}\,\frac{V''}{V} - \tfrac{3}{4}\left(\frac{V'}{V}\right)^2\right)
\tag{3.8.34}
$$

The zero curvature and constant curvature cases can be easily analyzed. The general solution of the equation:

$$
R(\phi) = -\tfrac{1}{2}\,\nu^2 \equiv -\hat{\nu}^2
\tag{3.8.35}
$$

can be presented in terms of the moment map $\mathscr{P}(\phi)$ and of the canonical variable ϕ. We have:

$$
\mathscr{P}(\phi) = a\,\exp(\nu\,\phi) + b\,\exp(-\,\nu\,\phi) + c \quad ; \quad a, b, c \in \mathbb{R}
\tag{3.8.36}
$$

In order to convert this solution in terms of the Jordan function $J(C)$ of the coordinate C, it is convenient to remark that, up to constant shift redefinitions and sign flips of the canonical variable $\phi \to \pm\phi + \kappa$, which leave the $d\phi^2$ part of the line-element invariant there are only three relevant cases:

(A) $a \neq 0,\ b \neq 0$ and $a/b > 0$. In this case, up to an overall constant, we can just set:

$$
\mathscr{P}(\phi) = \cosh(\nu\,\phi) + \gamma \quad \Rightarrow \quad V(\phi) \propto (\cosh(\nu\,\phi) + \gamma)^2
\tag{3.8.37}
$$

[12] The factor 2 introduced in this equation is chosen in order to have a normalization of what we name curvature that agrees with the normalization used in several papers of the physical literature.

(B) $a \neq 0,\ b \neq 0$ and $a/b < 0$. In this case we can just set:

$$\mathscr{P}(\phi) = \sinh(\nu\,\phi) + \gamma \quad \Rightarrow \quad V(\phi) \propto (\sinh(\nu\,\phi) + \gamma)^2 \tag{3.8.38}$$

(C) $a \neq 0,\ b = 0$. In this case we can just set:

$$\mathscr{P}(\phi) = \exp(\nu\,\phi) + \gamma \quad \Rightarrow \quad V(\phi) \propto (\exp(\nu\,\phi) + \gamma)^2 \tag{3.8.39}$$

Since our main goal is to understand the topology of the Kähler surface Σ and possibly to generalize the above three-fold classification of isometries to the non constant curvature case, it is very useful to recall how, in the above three cases, the corresponding (Euclidean) metric ds^2_ϕ is realized as the pull-back on the hyperboloid surface

$$X_1^2 + X_2^2 - X_3^2 = -1 \tag{3.8.40}$$

of the flat Lorentz metric in the three-dimensional Minkowski space of coordinates $\{X_1, X_2, X_3\}$. The manifold is always the same but the three different parameterizations single out different gaussian curves on the same surface. It is indeed an excellent exercise in differential geometry to see how the same space can be described in apparently very much different coordinate systems. Furthermore the gaussian curves being integral curves of different Killing vectors give visual appreciation of the different global character of elliptic, parabolic and hyperbolic isometries.

3.8.1.1 Embedding of Case (A)

Let us consider the case of the moment map of Eq. (3.8.37). The corresponding two-dimensional metric is:

$$ds^2_\phi = d\phi^2 + \sinh^2(\nu\,\phi)\,dB^2 \tag{3.8.41}$$

It is the pull-back of the (2, 1)-Lorentz metric onto the hyperboloid surface (3.8.40). Indeed setting:

$$\begin{aligned} X_1 &= \sinh(\nu\phi)\cos(B\nu) \\ X_2 &= \sinh(\nu\phi)\sin(B\nu) \\ X_3 &= \pm\cosh(\nu\phi) \end{aligned} \tag{3.8.42}$$

we obtain a parametric covering of the algebraic locus (3.8.40) and we can verify that:

$$\frac{1}{\nu^2}\left(dX_1^2 + dX_2^2 - dX_3^2\right) = d\phi^2 + \sinh^2(\nu\,\phi)\,dB^2 = ds^2_\phi \tag{3.8.43}$$

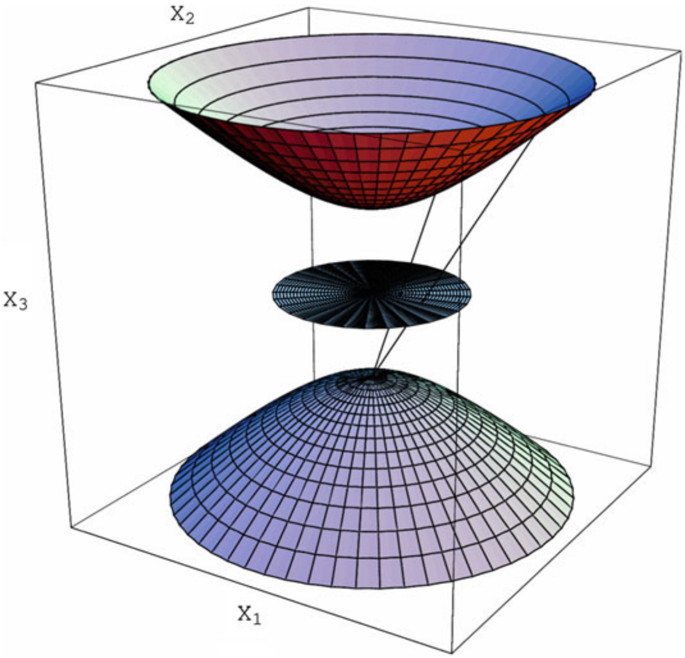


Fig. 3.1 In this figure we show the hyperboloid ruled by lines of constant ϕ that are circles and of constant B that are hyperbolae. In this figure we also show the stereographic projection of points of the hyperboloid onto points of the unit disk

A picture of the hyperboloid ruled by lines of constant ϕ and constant B according to the parametrization (3.8.42) is depicted in Fig. 3.1. In case of non constant curvature with a moment map which gives rise to a consistent U(1) interpretation of the isometry, the surface Σ is also a revolution surface but of a different curve than the hyperbola.

Setting:

$$f(\phi) = \mathscr{P}'(\phi) \tag{3.8.44}$$

we consider the parametric surface:

$$\begin{aligned} X_1 &= f(\phi) \cos B \\ X_2 &= f(\phi) \sin B \\ X_3 &= \pm g(\phi) \end{aligned} \tag{3.8.45}$$

where $g(\phi)$ is a function that satisfies the differential equation:

$$g'(\phi) = \sqrt{(f'(\phi))^2 - 1} \quad \Rightarrow \quad g(\phi) = \int d\phi \sqrt{(f'(\phi))^2 - 1} \tag{3.8.46}$$

The pull back on the parametric surface (3.8.45) of the flat Minkowski metric:

$$ds_M^2 \;=\; dX_1^2 + dX_2^2 - dX_3^2 \tag{3.8.47}$$

reproduces the metric of the surface Σ under analysis:

$$ds_\Sigma^2 \;=\; \mathrm{d}\phi^2 \;+\; f^2(\phi)\,\mathrm{d}B^2 \tag{3.8.48}$$

Hence the revolution surface (3.8.45) is generically an explicit geometrical model of the Kähler manifolds Σ where the considered isometry is elliptic, namely a compact $\mathrm{U}(1)$. Note that the last integral in Eq. (3.8.46) can be performed and yields a real function only for those functions $f(\phi)$ that satisfy the condition $\left(f'(\phi)\right)^2 > 1$. Hence the condition:

$$\left(\mathscr{P}''(\phi)\right)^2 > 1 \tag{3.8.49}$$

is a necessary requirement for the $\mathrm{U}(1)$ interpretation of the gauged isometry which has to be true together with the asymptotic expansion criterion (3.8.22).

Applying to the present constant curvature case the general rule given in Eq. (3.8.15) that defines the coordinate C we get:

$$C(\phi) \;=\; \int \frac{d\phi}{\mathscr{P}'(\phi)} \;=\; \frac{\log\left(\tanh\left(\frac{\nu\phi}{2}\right)\right)}{\nu^2} \quad \Leftrightarrow \quad \phi = \frac{2\mathrm{Arctanh}\left(e^{C\nu^2}\right)}{\nu} \tag{3.8.50}$$

from which we deduce that the allowed range of the flat variable C, in which the canonical variable ϕ is real and goes from 0 to ∞, is the following one:

$$C \;\in\; [-\infty\,,\,0] \tag{3.8.51}$$

The Kähler potential function is easily calculated and we get:

$$J(C) \;=\; 2\,(\gamma+1)\,C \;-\; 2\,\frac{\log\left(1-e^{2C\nu^2}\right)}{\nu^2} \;+\; 2\,\frac{\log(2)}{\nu^2} \tag{3.8.52}$$

In this case the appropriate relation between ζ in the unit circle and the real variables C, B is the following:

$$\zeta \;=\; e^{\nu^2(\mathrm{i}B+C)} \tag{3.8.53}$$

3.8.1.2 Embedding of Case (B)

Consider the case of Eq. (3.8.38). The corresponding two-dimensional metric is:

$$ds_\phi^2 \;=\; \left(d\phi^2 \;+\; \cosh^2(\nu\,\phi)\;dB^2\right) \tag{3.8.54}$$

which can be shown to be another form of the pull-back of the Lorentz metric onto a hyperboloid surface. Indeed setting:

$$\begin{aligned} X_1 &= \cosh(\nu\phi)\sinh(B\nu) \\ X_2 &= \sinh(\nu\phi) \\ X_3 &= \pm\cosh(B\nu)\cosh(\nu\phi) \end{aligned} \tag{3.8.55}$$

we obtain a parametric covering of the algebraic locus (3.8.40) and we can verify that:

$$\frac{1}{\nu^2}\left(dX_1^2 + dX_2^2 - dX_3^2\right) = \left(d\phi^2 + \cosh^2(\nu\,\phi)\, dB^2\right) = ds_\phi^2 \tag{3.8.56}$$

A three-dimensional picture of the hyperboloid ruled by lines of constant ϕ and constant B is displayed in Fig. 3.2. For other surfaces Σ (if they exist and are regular) possessing a hyperbolic isometry we can realize their geometrical model considering the following parametric surface:

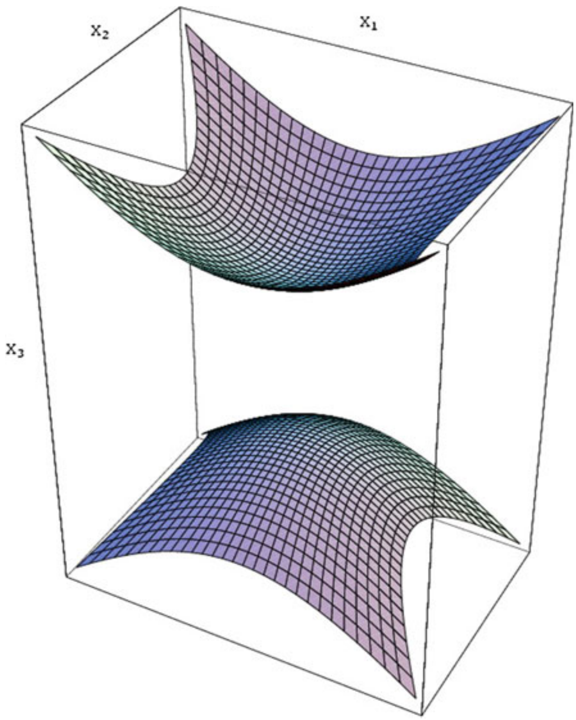


Fig. 3.2 The hyperboloid surface displayed in the parametrization (3.8.55). The lines drawn on the hyperboloid surface are those of constant B and constant ϕ respectively. Both of them are hyperbolae, in this case

$$\begin{aligned} X_1 &= f(\phi)\ \sinh B \\ X_2 &= g(\phi) \\ X_3 &= \pm\, f(\phi)\ \cosh B \end{aligned} \tag{3.8.57}$$

where:

$$f(\phi) \;=\; \mathscr{P}'(\phi) \tag{3.8.58}$$

and where $g(\phi)$ is a function that satisfies the following differential equation:

$$g'(\phi) \;=\; \sqrt{1 \,+\, (f'(\phi))^2} \quad \Rightarrow \quad g(\phi) \;=\; \int \mathrm{d}\phi \sqrt{1 \,+\, (f'(\phi))^2} \tag{3.8.59}$$

Once again the pull-back of the flat Minkowski metric (3.8.47) on the parametric surface (3.8.57) reproduces the looked for metric of the Σ-surface:

$$ds^2_\Sigma \;=\; \mathrm{d}\phi^2 \,+\, f^2(\phi)\,\mathrm{d}B^2 \tag{3.8.60}$$

Which is the appropriate interpretation is dictated by the asymptotic behavior of the $J(C)$ function and of its second derivative, or alternatively by the equivalent mathematical criteria discussed in Sect. 3.8.5.

Applying to the present constant curvature case the general rule given in Eq. (3.8.15) that defines the coordinate C we get:

$$C(\phi) \;=\; \int \frac{d\phi}{\mathscr{P}'(\phi)} \;=\; \frac{2\mathrm{Arctan}\left(\tanh\left(\frac{\nu\phi}{2}\right)\right)}{\nu^2} \quad \Leftrightarrow \quad \phi = \frac{2\mathrm{Arctanh}\left(\tan\left(\frac{C\nu^2}{2}\right)\right)}{\nu} \tag{3.8.61}$$

from which we deduce that the allowed range of the flat variable C, in which the canonical variable ϕ is real and goes from $-\infty$ to ∞, is the following one:

$$C \;\in\; \left[-\frac{\pi}{2\,\nu^2}\,,\;\frac{\pi}{2\,\nu^2}\right] \tag{3.8.62}$$

The Kähler function $J(\phi)$ is easily calculated and we obtain:

$$J(C) \;=\; 2\,\gamma\,C \;-\; \frac{2}{\nu^2}\,\log\left(\cos\left(C\nu^2\right)\right) \tag{3.8.63}$$

In this case the appropriate relation between ζ in the unit circle and the real variables C, B is different, it is:

$$\zeta \;=\; \mathrm{i}\tanh\left(\frac{1}{2}(B - \mathrm{i}\,C)\nu^2\right) \tag{3.8.64}$$

3.8.1.3 Embedding of Case (C)

In the case the moment map is given by Eq. (3.8.39) the parameterization of the hyperboloid is the following one:

$$\begin{aligned} X_1 &= \frac{1}{2}\left(-e^{\nu\phi}B^2 + e^{\nu\phi} - \frac{e^{-\nu\phi}}{\nu^2}\right)\nu \\ X_2 &= B e^{\nu\phi}\nu \\ X_3 &= \frac{1}{2}\left(e^{\nu\phi}B^2 + e^{\nu\phi} + \frac{e^{-\nu\phi}}{\nu^2}\right)\nu \end{aligned} \tag{3.8.65}$$

Indeed upon insertion of Eq. (3.8.65) into (3.8.40) we see that for all values of B and ϕ the constraint defining the algebraic locus is satisfied. At the same time by means of an immediate calculation one finds:

$$\frac{1}{\nu^2}\left(dX_1^2 + dX_2^2 - dX_3^2\right) = d\phi^2 + e^{2\nu\phi}\,dB^2 = ds_\phi^2 \tag{3.8.66}$$

so that the considered metric is the pull-back of the three-dimensional Lorentz metric on the surface Σ parameterized as in Eq. (3.8.65). The integration of Eq. (3.8.15) is immediate and the coordinate $C(\phi)$ takes the following very simple invertible form:

$$C(\phi) = -\frac{e^{-\nu\phi}}{\nu^2} \Leftrightarrow \phi(C) = -\frac{\log\left(-C\nu^2\right)}{\nu} \tag{3.8.67}$$

The range of definition of C is:

$$C \in [-\infty, 0] \tag{3.8.68}$$

A three-dimensional picture of the hyperboloid ruled by lines of constant ϕ and constant B, according to Eq. (3.8.65) is displayed in Fig. 3.3.

The integration of Eq. (3.8.17) for the Kähler potential is equally immediate and using the inverse function $\phi(C)$ we obtain:

$$J(C) = 2\gamma\,C - \frac{2}{\nu^2}\log(-C) + \text{const} \tag{3.8.69}$$

From the form of Eq. (3.8.69) we conclude that in this case the appropriate solution of the complex structure equation is:

$$\mathfrak{z} = t = -\mathrm{i}C + B \tag{3.8.70}$$

so that the Kähler metric becomes proportional to the Poincaré metric in the upper complex plane (note that C is negative definite for the whole range of the canonical variable ϕ):

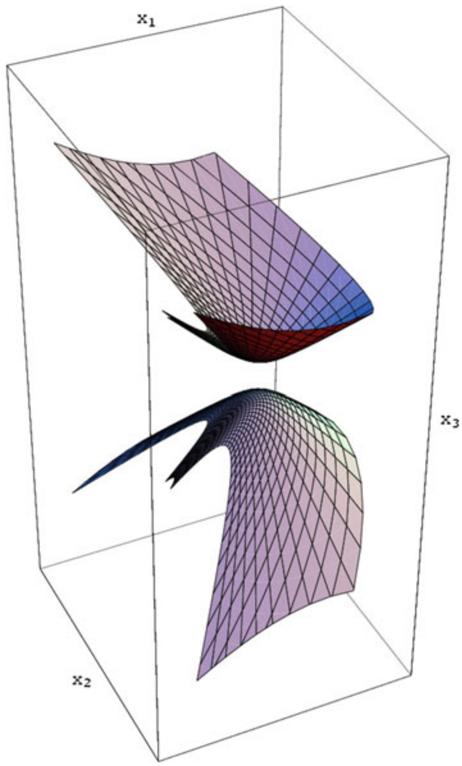


Fig. 3.3 The hyperboloid surface displayed in the parametrization (3.8.65). The lines drawn on the hyperboloid surface are those of constant B and constant ϕ respectively. The constant ϕ curves are parabolae and they are the orbits of the translation group

$$ds^2 = \tfrac{1}{2}\,\frac{\mathrm{d}^2 J}{\mathrm{d}C^2}\,\left(dC^2 + dB^2\right) = \frac{1}{4\,\nu^2}\,\frac{d\bar{t}\,dt}{(\mathrm{Im}t)^2} \tag{3.8.71}$$

As a consequence of Eq. (3.8.70), we see that the B-translation happens to be, in this case, a non-compact parabolic symmetry.

More generally for any surface Σ where the isometry of the metric:

$$\mathrm{d}s^2_\Sigma = \mathrm{d}\phi^2 + f^2(\phi)\,\mathrm{d}B^2 \tag{3.8.72}$$

is interpreted as a parabolic shift-symmetry we can construct a geometric model of Σ in three-dimensional Minkowski space by considering the following parametric surface:

$$\begin{aligned} X_1 &= \frac{1}{2}\left(-f(\phi)B^2 + f(\phi) + g(\phi)\right) \\ X_2 &= Bf(\phi) \\ X_3 &= \frac{1}{2}\left(f(\phi)B^2 + f(\phi) - g(\phi)\right) \end{aligned} \tag{3.8.73}$$

where $g(\phi)$ is a function that satisfies the differential equation:

$$f'(\phi)\, g'(\phi) = 1 \quad \Rightarrow \quad g(\phi) = \int \frac{1}{f'(\phi)}\, d\phi \tag{3.8.74}$$

The pull-back of the flat metric (3.8.47) onto the surface (3.8.73) is indeed the desired metric (3.8.72).

3.8.2 Asymptotically Flat Kähler Surfaces with an Elliptic Isometry Group

As announced above in this section we consider the problem of constructing a Kähler surface Σ with an elliptic isometry whose limiting curvature at the boundary vanishes $R_{\pm\infty} = 0$. In this case we can predict the asymptotic behavior of the function $J(C)$ for $C \to \pm\infty$. Indeed we know that for flat Kähler manifolds with an elliptic isometry, we have $J(C) \propto \exp[\delta\, C]$ for some value of $\delta \in \mathbb{R}$. Hence we expect that the function $J(C)$ for surfaces Σ with an elliptic isometry and a vanishing limiting curvature should behave has follows:

$$J(C) \overset{C\to\pm\infty}{\approx} \exp\left[\, \delta_{\pm}\, C\right] + \text{subleading terms} \tag{3.8.75}$$

There is however a fundamental subtlety that has to be immediately emphasized. If the topology of the surface Σ is the disk topology and Σ is simply connected $\pi_1(\Sigma) = 1$, then one of the two limits $C \to \infty$ has to be interpreted as the interior fixed point, required by Gromov criteria, for elliptic isometries in Hadamard manifolds (and possibly in CAT(k) manifolds). The other limit corresponds to the unique boundary of disk topology. On the other hand if $\pi_1(\Sigma) = \mathbb{Z}$ and the Kähler surface has the corona topology then there are two boundaries and the limiting curvature can be zero on both boundaries. We will illustrate this with two examples, respectively corresponding to the latter and to the former case.

3.8.2.1 The Catenoid Case with $\pi_1(\Sigma) = \mathbb{Z}$

We begin by considering explicit functions $J(C)$ that have the required asymptotic behavior and we try to work our way backward towards the canonical coordinate ϕ and the moment map $\mathscr{P}(\phi)$. In particular we want to make sure that the considered function $J(C)$ does indeed correspond to a compact isometry. This will certainly be the case if the corresponding metric is the pull-back of the flat three-dimensional Euclidean metric on a smooth surface of revolution.

To carry out such a program we consider the following one-parameter family of $J(C)$ functions:

$$J_{[\mu]}(C) \;=\; \frac{1}{8}\left(\mu\, C^2 + \cosh[2\,C]\right) \tag{3.8.76}$$

which fulfills condition (3.8.75), by construction. Many other examples can be obviously put forward, but this rather simple one is sufficient to single out the main subtlety that makes many asymptotically flat elliptic models pathological from the point of view of Gromov et al. classification of isometries. Using Eqs. (3.8.16) and (3.8.30) we write the metric and the curvature following from the $J(C)$ function of Eq. (3.8.76), obtaining

$$ds^2_{\Sigma} = \frac{1}{16}\,(\,2\mu + 4\cosh[2C])\,\left(dC^2 \,+\, dB^2\right) \tag{3.8.77}$$

$$R(C) = -\frac{4\mu\cosh(C)+1}{(4\,\mu + \cosh[C])^3} \tag{3.8.78}$$

From these formulae we draw an important conclusion. In order for Σ to be a smooth manifold the curvature should not develop a pole neither in the interior nor on the boundary. This means that $4\mu + \cosh[C] > 0$ in the whole range of C. This is guaranteed if and only if $\mu > -\frac{1}{4}$. On the other hand, according to our previous discussions, in the case of an elliptic isometry, there should be, for a finite value of C, a zero of the metric coefficient. Such a zero is the fixed point that characterizes elliptic isometries of Hadamard manifolds. Looking at Eq. (3.8.77) we see that such a zero exists, if and only if $\mu < -\frac{1}{2}$. It follows that, at least in this family of models, there are no smooth manifolds that are asymptotically flat in the elliptic sense and fulfill the physical condition for U(1)-symmetry which corresponds to the Gromov et al. identification of elliptic isometries of Hadamard manifolds. At first sight one should draw the conclusion that, in the case of the $J(C)$ functions of Eq. (3.8.76), the isometry is not elliptic. Yet this is somehow strange, since at the boundary, where the curvature goes to zero, the form of $J(C)$ is precisely that which corresponds to elliptic isometries. Furthermore we will shortly show that for every value of μ the metric in Eq. (3.8.77) is just the metric of a smooth revolution surface. Actually for $\mu = 2$ such a revolution surface is the well-known **catenoid**, constructed by Bernoulli in 1744 as the first example of a minimal surface. Hence we arrive at a puzzle with Gromov et al. criteria, whose only resolution can be that the manifolds associated with the $J(C)$ functions of Eq. (3.8.76) are not Hadamard manifolds. From Eq. (3.8.78) we see that, provided $\mu > -\frac{1}{4}$, the curvature is negative definite and attains its maximal value $R = 0$ only on the boundary. Hence in relation with the curvature there is no violation of the properties defining a Hadamard manifold. The violation must be in another item of the definition. Considering the Definition 3.8.1 of Hadamard manifolds provided in Sect. 3.8.5 we realize that the only way out from the puzzle is that the surfaces corresponding to the $J(C)$ functions of Eq. (3.8.76) have to be **non simply connected**. That this is the case becomes visually obvious when we consider the plot of the surface in three-dimensional space-time (see Fig. 3.4), yet it is quite clear also analytically. For constant C the orbits of the isometry group spanned by $B \in [0, 2\pi]$ are circles of radius:

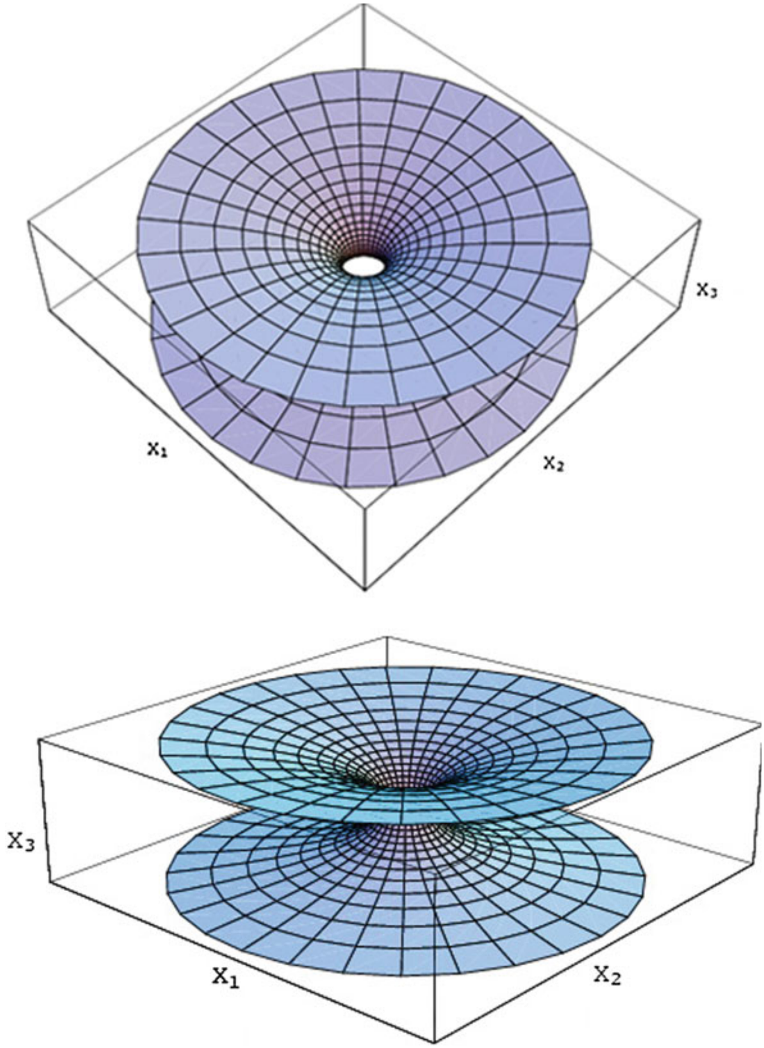


Fig. 3.4 In this picture we present two views of the **catenoid**, the revolution surface corresponding to $J_{[2]}(C) = \frac{1}{8}\left(2\,C^2 + \cosh[2\,C]\right)$. For large positive or negative values of C one is either in the superior or in inferior plane which is clearly flat with zero curvature. The center of the picture correspond instead to $C \to 0$ and is a sort of strongly negatively curved wormhole that connects the two asymptotic planes. Non simple connectedness is visually spotted. The circles on the surface winding around the throat cannot be contracted to zero and their homotopy class forms the non trivial element of the first homotopy group $\pi_1(\Sigma) = \mathbb{Z}$

$$r(C) = \frac{1}{4}\sqrt{2\mu + 4\cosh[2C]} \tag{3.8.79}$$

The fact that this radius has a minimum different from zero

$$r_{min} = \frac{1}{4}\sqrt{2\mu + 4} > 0 \tag{3.8.80}$$

is what spoils simple connectedness and prevents the existence of a fixed point for U(1). In this way the puzzle is resolved mathematically.

Having anticipated this conceptual discussion of their meaning let us work out the details of the models encoded in Eq. (3.8.76). Comparing Eqs. (3.8.16) and (3.8.14) we derive the relation between the canonical coordinate ϕ and C:

$$\phi = \sqrt{2}\int \sqrt{J''_{[\mu]}(C)}\, dC = \Phi_{[\mu]}(C) \equiv -\frac{1}{2}\,\mathrm{i}\,\sqrt{\mu + 2}\, E\left(\mathrm{i}\, C \,\middle|\, \frac{4}{\mu + 2}\right) \tag{3.8.81}$$

where $E\,(x\,|m\,)$ denotes the elliptic integral of its arguments. In the case $\mu = 2$ which turns out to be that of the **catenoid**, the function $\Phi_{[\mu]}(C)$ simplifies and it can be easily inverted in terms of elementary functions

$$\Phi_{[2]}(C) = \sinh(C) \Rightarrow \quad C(\phi) = \mathrm{ArcSinh}(C) \tag{3.8.82}$$

Substituting into the metric (3.8.77) one finds:

$$\mu = 2 \quad : \quad ds^2_\Sigma = \frac{\cosh^2(C)}{2}\left(dC^2 + dB^2\right) = \tfrac{1}{2}\left[d\phi^2 + \left(\phi^2 + 1\right) dB^2\right] \tag{3.8.83}$$

This implies that the derivative of the moment map is $\mathscr{P}'(\phi) = \sqrt{\phi^2 + 1}$ so that the moment map and the scalar potential are the following ones:

$$\mu = 2 : \mathscr{P}(\phi) = \frac{1}{2}\left(\sqrt{\phi^2 + 1}\,\phi + \mathrm{ArcSinh}[\phi\,]\right) \quad \Rightarrow$$
$$V(\phi) \propto \left(\sqrt{\phi^2 + 1}\,\phi + \mathrm{ArcSinh}[\phi\,]\right)^2 \tag{3.8.84}$$

The metric (3.8.83) can be easily recognized to be the pull-back of the flat three-dimensional Euclidean metric:

$$ds^2_{\mathbb{E}^3} = dX_1^2 + dX_2^2 + dX_3^2 \tag{3.8.85}$$

on the following parametric surface:

$$
\begin{aligned}
X_1 &= \frac{\cos(B)\cosh(C)}{\sqrt{2}} \\
X_2 &= \frac{\cosh(C)\sin(B)}{\sqrt{2}} \\
X_3 &= \frac{C}{\sqrt{2}}
\end{aligned} \tag{3.8.86}
$$

which is the classical catenoid. For other values of μ a similar parametric surface of revolution can be written in terms of appropriate functions of C. As we have already anticipated, although the catenoid is a rotation surface and its isometry is elliptic, its metric does not satisfy Gromov et al. criterion that requires the existence of a symmetric point. The reason for this pathology is the non trivial fundamental group $\pi_1(\Sigma)$.

Finally let us appreciate the nature of the same problem from the point of view of complex coordinates. If we introduce the complex coordinate:

$$
\zeta = \exp[C - \mathrm{i}\, B] \quad ; \quad \overline{\zeta} = \exp[C + \mathrm{i}\, B] \tag{3.8.87}
$$

and we insert it into the expression of (3.8.76) of the $J(C)$ function we easily obtain the Kähler potential:

$$
\mathscr{K}(\zeta, \overline{\zeta}) = 2\, J(C) = \frac{1}{16}\mu \log^2(\zeta\, \overline{\zeta}) + \frac{\zeta\, \overline{\zeta}}{8} + \frac{1}{8\, \zeta\, \overline{\zeta}} \tag{3.8.88}
$$

from which we obtain the metric:

$$
ds^2_\Sigma = \frac{d\zeta\, d\overline{\zeta}\left(\zeta\overline{\zeta}\left(\mu + \zeta\overline{\zeta}\right) + 1\right)}{8\left(\zeta\overline{\zeta}\right)^2} \overset{\mu \to 2}{\Longrightarrow} \frac{d\zeta\, d\overline{\zeta}\left(\zeta\overline{\zeta} + 1\right)^2}{8\left(\zeta\overline{\zeta}\right)^2} \tag{3.8.89}
$$

Examining Eq. (3.8.89) we see that the metric diverges at the symmetry restoration point $\zeta = 0$ which now is the boundary of the manifold rather than its interior.

3.8.2.2 An Asymptotically Flat Kähler Surface with an Elliptic Isometry and $\pi_1(\Sigma) = 1$

Let us consider the following moment map written in terms of the canonical variable ϕ:

$$
\mathscr{P}(\phi) = \phi^2 - \frac{1}{2}\mathrm{ArcTan}\left(\phi^2\right) \tag{3.8.90}
$$

Using the standard formulae (3.8.15) for the calculation of the coordinate C we obtain:

$$C(\phi) = \log\left(\frac{\phi}{\sqrt[8]{2\phi^4+1}}\right) \quad \Leftrightarrow \quad \phi = \begin{cases} \pm\sqrt[4]{\sqrt{e^{8C}+e^{16C}}+e^{8C}} \\ \pm\mathrm{i}\sqrt[4]{\sqrt{e^{8C}+e^{16C}}+e^{8C}} \\ \pm\sqrt[4]{\sqrt{e^{8C}-e^{16C}}+e^{8C}} \\ \pm\mathrm{i}\sqrt[4]{\sqrt{e^{8C}-e^{16C}}+e^{8C}} \end{cases} \tag{3.8.91}$$

The eighth-root implies the existence of eight branches of the inverse function, that have to considered carefully. Indeed we can accept only those branches where ϕ turns out to be everywhere real. Six branches have to be rejected because of that reason and the only acceptable ones are the first two which are equivalent under the always possible sign revers of ϕ. In conclusion we have:

$$\phi = \sqrt[4]{\sqrt{e^{8C}+e^{16C}}+e^{8C}} \tag{3.8.92}$$

Using this branch the infinite interval $[-\infty\,,\,\infty]$ of the variable C is mapped into the semi-infinite interval $[0\,,\,\infty]$ of the variable ϕ. Indeed we have $C(0) = -\infty$, $C(\infty) = \infty$. In the canonical coordinate the form of the metric is:

$$ds^2_\Sigma = d\phi^2 + f^2(\phi)\,dB^2 \quad ; \quad f^2(\phi) = \left(\frac{\phi^5}{\phi^4+1}+\phi\right)^2 \tag{3.8.93}$$

and using Eq. (3.8.92) we can easily convert it to the C variable:

$$\begin{aligned} ds^2_\Sigma &= \tfrac{1}{2}\,\frac{d^2J}{dC^2}\,\left(dC^2 + dB^2\right) \\ &= \frac{\sqrt{\sqrt{e^{8C}+e^{16C}}+e^{8C}}\left(2\sqrt{e^{8C}+e^{16C}}+2e^{8C}+1\right)^2}{\left(\sqrt{e^{8C}+e^{16C}}+e^{8C}+1\right)^2}\,\left(dC^2 + dB^2\right) \end{aligned} \tag{3.8.94}$$

For $C \to -\infty$ the behavior of the metric coefficient is:

$$\tfrac{1}{2}\,\frac{d^2J}{dC^2} \overset{C\to-\infty}{\approx} e^{2C} + \frac{5e^{6C}}{2} + \mathcal{O}\left(e^{10C}\right) \quad \Rightarrow \quad J(C) \overset{C\to-\infty}{\approx} \tfrac{1}{2}\,e^{2C} \tag{3.8.95}$$

while for $C \to \infty$ it is the following:

$$\tfrac{1}{2}\,\frac{d^2J}{dC^2} \overset{C\to\infty}{\approx} 4\sqrt{2}e^{4C} - \frac{3e^{-4C}}{\sqrt{2}} + \mathcal{O}\left(e^{-12C}\right) \quad \Rightarrow \quad J(C) \overset{C\to\infty}{\approx} \tfrac{1}{2}\,\tfrac{1}{\sqrt{2}}\,e^{4C} \tag{3.8.96}$$

From previous considerations we see that $C \to -\infty$ corresponds to $\phi = 0$ and hence to the fixed point in the interior of the manifold, so that the exponential behavior of $J(C)$ is the expected one for an elliptic isometry. At the same time the exponential

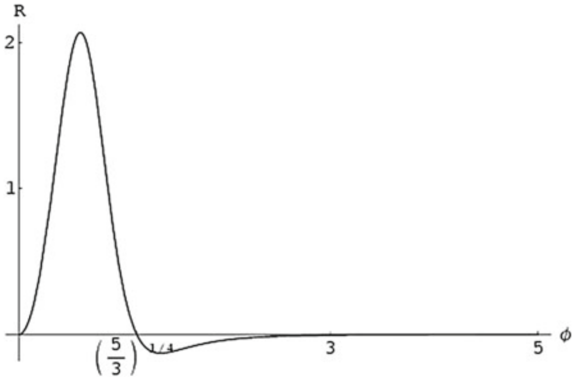


Fig. 3.5 In this picture we present the plot of the curvature for the elliptic model of Eq. (3.8.90). It is limited from above and has three zeros, one at the interior fixed point $\phi = 0$, a second one at $\phi = \left(\frac{5}{3}\right)^{1/4}$ and one on the boundary at $\phi = \infty$

behavior on the unique boundary implies that the limiting curvature on the boundary should be zero. Indeed from the standard formula (3.8.34) for the curvature we obtain:

$$R(\phi) = -\frac{2\phi^2\left(3\phi^4 - 5\right)}{\left(\phi^4 + 1\right)^2\left(2\phi^4 + 1\right)} \quad ; \quad R(0) = 0 \quad ; \quad R(\infty) = 0 \tag{3.8.97}$$

whose plot is displayed in Fig. 3.5. The vanishing of the limiting curvature is visually evident. Finally let us make sure that the isometry of this model is indeed elliptic. This we verify by showing that the metric (3.8.93) can be retrieved as the pull-back of the flat Lorentz metric in Minkowsian three-dimensional space (3.8.47) on the parametric revolution surface (3.8.45) defined by:

$$f(\phi) = \frac{\phi^5}{\phi^4 + 1} + \phi \quad ; \quad g(\phi) \equiv \int_0^\phi \sqrt{\frac{\sigma^4\left(\sigma^4 + 5\right)\left(3\sigma^8 + 9\sigma^4 + 2\right)}{\left(\sigma^4 + 1\right)^4}}\, d\sigma \tag{3.8.98}$$

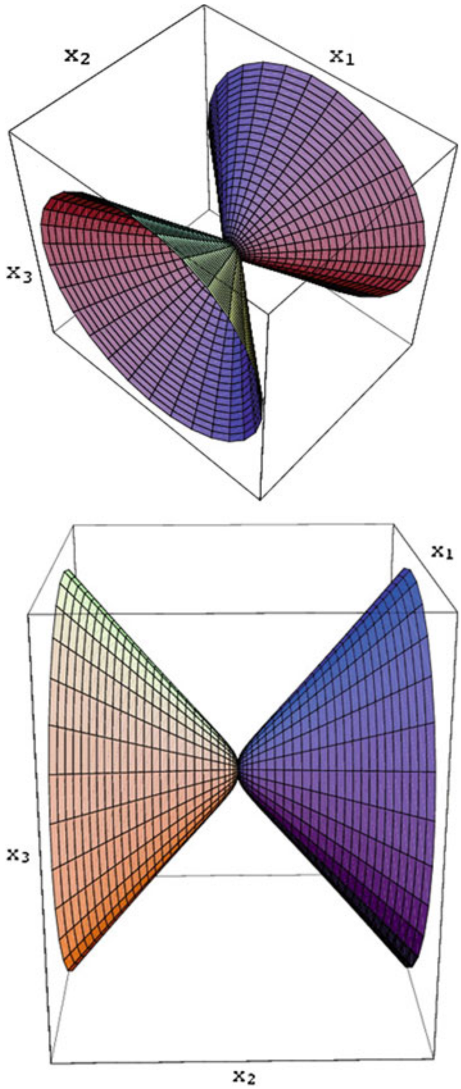
Two views of this surface are presented in Fig. 3.6. It is evident from the picture that this surface is simply connected and that there is in the interior of the manifold a fixed point. It is given by $X_1 = X_2 = X_3 = 0$ which lies on the surface and where the radius of the U(1) orbit shrinks to zero.

3.8.3 *An Example of a Non Maximally Symmetric Kähler Surface with an Isometry Group of the Hyperbolic Type*

In order to exhibit an example of a surface with non constant curvature that has a hyperbolic isometry we consider the following moment map and potential:

$$V(\phi) = [\mathscr{P}(\phi)]^2 \quad ; \quad \mathscr{P}(\phi) = \phi + \sinh(\phi) \tag{3.8.99}$$

Fig. 3.6 In this picture we present two views of the revolution surface Σ associated with the elliptic model of Eq. (3.8.90). It is clearly regular and smooth everywhere



which yields:

$$\mathscr{P}'(\phi) = 1 + \cosh(\phi) \quad ; \quad ds^2_\Sigma = d\phi^2 + (1 + \cosh(\phi))^2\, dB^2 \qquad (3.8.100)$$

According to the mathematical classification discussed in Sect. 3.8.5 the metric (3.8.100) has a hyperbolic type of isometry due to the two fixed points on the boundary of the manifold corresponding to the two singularities $\phi = \pm\infty$. The curvature of this manifold is finite but not constant. Indeed, applying Eq. (3.8.34) we obtain:

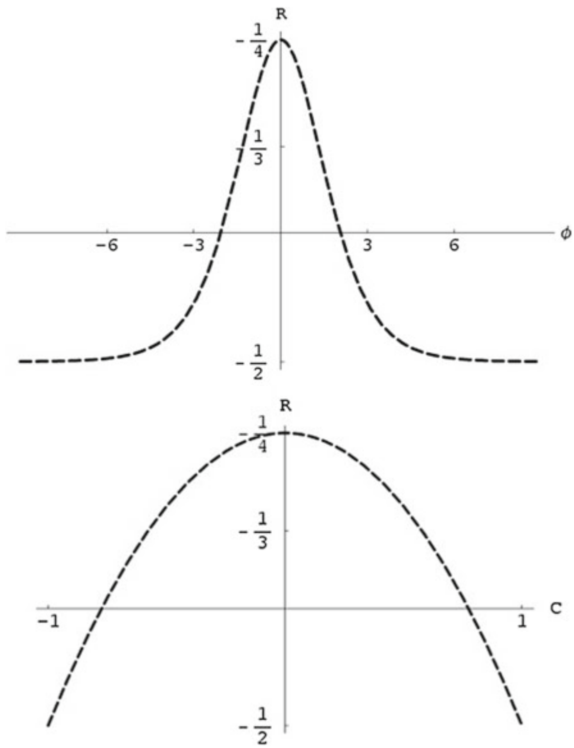


Fig. 3.7 In this figure we present the plot of the curvature of the surface Σ defined by Eq. (3.8.100) that has a hyperbolic isometry. The first picture displays the dependence of the curvature on the canonical coordinate ϕ, while the second picture displays its dependence on the coordinate C

$$R(\phi) = -\frac{\cosh(\phi)}{2(\cosh(\phi)+1)} \tag{3.8.101}$$

whose plot is presented in Fig. 3.7. In this case it is very simple to integrate the complex structure equation which defines the C-coordinate. We obtain:

$$C(\phi) = \tanh\left(\frac{\phi}{2}\right) \quad ; \quad \phi = 2\,\mathrm{ArcTanh}(C) \tag{3.8.102}$$

and we observe that in line with our general criteria for hyperbolic symmetry, the range of the C-coordinate is in this case finite:

$$C \in [-1\,,\ 1] \tag{3.8.103}$$

From the integration of Eq. (3.8.17) that defines the J-function and the Kähler potential we obtain:

$$J(\phi) = 2\,\phi\tanh\left(\frac{\phi}{2}\right) = J(C) = 4C\,\mathrm{ArcTanh}(C) \tag{3.8.104}$$

Calculating the metric coefficient from (3.8.104) we get:

$$\tfrac{1}{2}\,\frac{d^2 J}{dC^2} = \frac{4}{\left(C^2-1\right)^2} \quad ; \quad ds^2 = \frac{4}{\left(C^2-1\right)^2}\left(dC^2 + dB^2\right) \qquad (3.8.105)$$

displaying a polar singularity at both extrema of the C-range, namely at $C = \pm 1$.

In order to present a geometrical model of this Kähler manifold, we resort to the hyperbolic parametric surface encoded in formulae (3.8.57) and we calculate the relevant functions $f(\phi)$ and $g(\phi)$. In this case it is more convenient to express them in terms of the finite range coordinate C. We have:

$$f(\phi) = \cosh(\phi) + 1 = \frac{2}{1 - C^2} \qquad (3.8.106)$$

and inserting the result into Eq. (3.8.59) we get:

$$g(C) = \frac{1}{8}\left(\frac{2C\,\left(C^2-3\right)}{\left(C^2-1\right)^2} + \log(C-1) - \log(C+1)\right) \qquad (3.8.107)$$

The plots of these functions is presented in Fig. 3.8. In Fig. 3.9 we display the three dimensional shape of the parametric surface Σ realizing the desired Kähler manifold.

3.8.4 A Non Maximally Symmetric Kähler Manifold with Parabolic Isometry and Zero Curvature at One Boundary

As a final example we consider a parabolic model where the curvature at one of the two boundaries goes to zero so that the asymptotic behavior of the $J(C)$-function on that boundary becomes exceptional.

Let the moment map be the following one:

$$\mathscr{P}(\phi) = \exp[\nu\,\phi] + \mu\,\phi \qquad (3.8.108)$$

The corresponding $f(\phi)$-function is:

$$f(\phi) = \mathscr{P}'(\phi) = \nu\,\exp[\nu\,\phi] + \mu \qquad (3.8.109)$$

which has no zeros for finite ϕ if μ and ν have the same sign. If the two parameters have opposite signs there is such a zero and this creates a fixed point of the isometry $B \rightarrow B + c$ at finite ϕ which implies that the isometry is elliptic. Yet in case of opposite signs the curvature has a singularity so that any smooth Kähler manifold with a moment map of type (3.8.108) has a parabolic isometry group. Indeed using

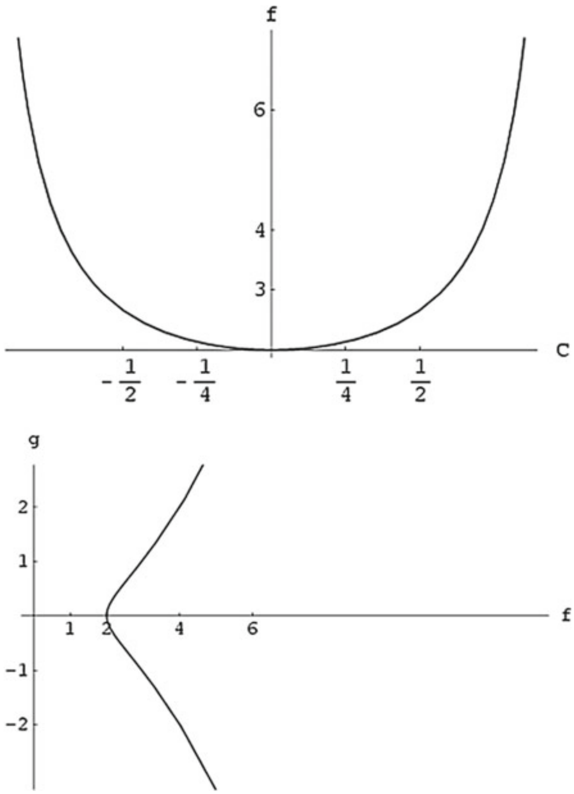


Fig. 3.8 In this picture we present the plots of the functions $f(C)$, $g(C)$ that define the realization of the Kähler manifold Σ associated with the potential (3.8.99) as a parametric surface in flat Minkowski three-dimensional space. The geometrical model is that appropriate to the hyperbolic character of the isometry $B \to B + c$. The first two pictures display the plot of g and f as functions of the VP coordinate C. The last plot is the parametric plot of the curve in the plane f, g. Geometrically this is the curve cut out by the surface Σ in any plane orthogonal to the axis X_2

Eq. (3.8.34) we can immediately calculate the curvature and we find:

$$R(\phi) = -\frac{e^{\nu\phi}\nu^3}{2\left(\mu + e^{\nu\phi}\nu\right)} \tag{3.8.110}$$

This shows what we just said. The manifold is smooth and singularity-free if and only if μ and ν have the same sign so that at no value of ϕ the denominator can develop a zero. Without loss of generality we can assume that $\nu > 0$ since the sign of ϕ can be flipped without changing its kinetic term. With this understanding it follows that also $\mu > 0$ for regularity.

Consider next the integral defining the VP coordinate C. We immediately obtain:

$$C(\phi) = \int \frac{1}{\mathscr{P}'(\phi)}\, d\phi = \frac{\phi}{\mu} - \frac{\log\left(\mu + e^{\nu\phi}\nu\right)}{\mu\nu} \tag{3.8.111}$$

The range of C is now easily determined considering the limits of the above function for $\phi = \pm\infty$. When $\mu > 0\,,\ \nu > 0$ we have:

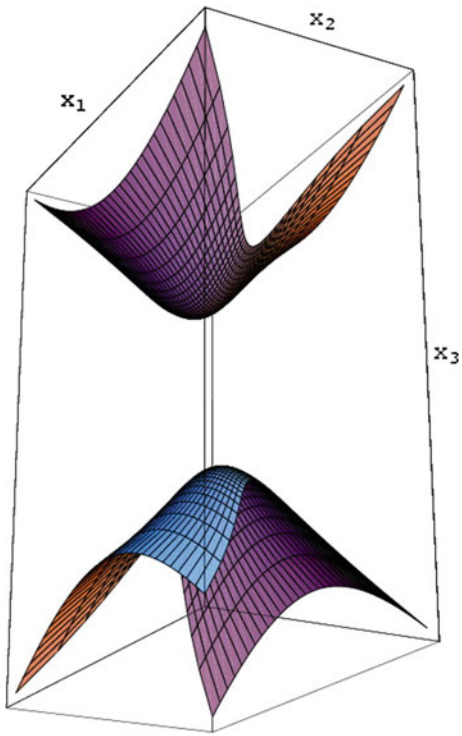


Fig. 3.9 In this figure we present the $3D$-plot of the surface Σ associated with the potential (3.8.99). The correct interpretation of the isometry in this case is that of a hyperbolic group. Indeed the hyperbolic embedding (3.8.57) in three-dimensional Minkowski space works beautifully and we have the smooth surface displayed here

$$C(-\infty) = -\infty \quad ; \quad C(\infty) = -\frac{\log[\nu]}{\mu\,\nu} \tag{3.8.112}$$

Hence $C \in \left[-\infty\,,\, -\frac{\log[\nu]}{\mu\,\nu}\right]$. The VP coordinate is always negative and it spans a semininfinite interval. Keeping this range in mind we can invert the relation (3.8.111) between ϕ and C obtaining:

$$\phi = -\frac{\log\left(\frac{e^{-C\mu\nu}}{\mu} - \frac{\nu}{\mu}\right)}{\nu} \tag{3.8.113}$$

The J-function is easily calculated from Eq. (3.8.17) and we find:

$$\mathscr{J}(\phi) = \frac{\nu^2\phi^2 + (2 - 2\nu\phi)\log\left(\frac{e^{\nu\phi}\nu}{\mu} + 1\right) - 2\mathrm{Li}_2\left(-\frac{e^{\nu\phi}\nu}{\mu}\right)}{\nu^2} \tag{3.8.114}$$

where $\mathrm{Li}_n(z)$ is the polylogarithmic function. Introducing in (3.8.114) the relation between ϕ and C, we get an explicit analytic expression for the $J(C)$ function, namely:

$$J(C) = \frac{1}{\nu^2}\left[\log^2\left(\frac{e^{-C\mu\nu}-\nu}{\mu}\right) + 2\left(\log\left(\frac{e^{-C\mu\nu}-\nu}{\mu}\right)+1\right)\log\left(\frac{1}{1-e^{C\mu\nu}\nu}\right)\right.$$
$$\left. -2\mathrm{Li}_2\left(1+\frac{1}{e^{C\mu\nu}\nu-1}\right)\right] \tag{3.8.115}$$

As for the metric, having the explicit expression (3.8.115), we easily calculate its second derivative and we find:

$$ds^2 = \frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}C^2}\left(\mathrm{d}C^2 + \mathrm{d}B^2\right) = \frac{\mu^2}{\left(e^{C\mu\nu}\nu-1\right)^2}\left(\mathrm{d}C^2 + \mathrm{d}B^2\right) \tag{3.8.116}$$

For $C \to -\infty$ the metric coefficient $\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}C^2}$ tends to a constant:

$$\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}C^2} \overset{C\to-\infty}{\approx} \mu^2 \quad \Rightarrow \quad J(C) \overset{C\to-\infty}{\approx} \frac{\mu^2}{2}C^2 \tag{3.8.117}$$

This asymptotic behavior differs from the usual logarithmic behavior of $J(C)$ at the boundary because at $C = -\infty$ and hence at $\phi = -\infty$ the curvature goes to zero.

In the other extremum of the C-range, namely for $C \to -\frac{\log[\nu]}{\mu\,\nu}$ the metric coefficient diverges and we have the standard logarithmic singularity. To see this, set $C = -\frac{\log[\nu]}{\mu\,\nu} - \xi$ and substitute it into the expression of the metric coefficient. We obtain:

$$\frac{1}{2}\frac{\mathrm{d}^2 J}{\mathrm{d}C^2} = \frac{\mu^2}{\left(e^{\mu\nu\left(-\xi-\frac{\log(\nu)}{\mu\nu}\right)}\nu-1\right)^2}$$
$$\overset{\xi\to 0}{\approx} \frac{1}{\nu^2\xi^2} + \frac{\mu}{\nu\xi} + \frac{5\mu^2}{12} + \frac{1}{12}\mu^3\nu\xi + \mathscr{O}\left(\xi^2\right) \tag{3.8.118}$$

and we conclude that, naming $C_0 = -\frac{\log[\nu]}{\mu\,\nu}$, we have:

$$J(C) \overset{C\to C_0}{\approx} \frac{2}{\nu^2}\log\left[C_0 - C\right] \tag{3.8.119}$$

This is the standard logarithmic singularity and the coefficient in front of the logarithm is indeed the inverse of the limiting curvature: $R_{C_0} = \frac{1}{2}\nu^2$.

This result confirms once again the relation between the asymptotic behavior of the $J(C)$ function and the character of the isometry group. For a parabolic isometry the asymtotic behavior is just that anticipated in Eqs. (3.8.25), (3.8.26). For a vanishing limiting curvature the correct asymptotic is (3.8.26).

The present example is very paedagical in order to avoid possible misconceptions. If we looked at the expression (3.8.116) and we forgot the precisely defined range of the variable C which is determined by the integration of the complex structure equation, we might be tempted to consider the same metric also for positive values of C. We would conclude that when $C \rightarrow \infty$ the metric coefficient goes to zero as $\exp[-\nu C]$. Then we would dispute that the last mentioned behavior indicates an elliptic interpretation of the isometry and advocate that there is a clash with our a priori knowledge that the isometry is instead parabolic. In fact there is no clash since the positive range of C is excluded and it is not to be considered. At the extrema of the C-interval, the function $J(C)$ displays the expected asymptotic behavior foreseen for the parabolic case.

3.8.5 On the Topology of Isometries

In this last subsection we provide a mathematically more rigorous illustration of the criteria discriminating among elliptic, parabolic and hyperbolic isometries of a two dimensional manifold whose metric is written in the standard form utilized throughout this section, namely:

$$ds^2 = d\phi^2 + f(\phi)^2 \, dB^2, \tag{3.8.120}$$

In relation with the moment map issue, the function $f(\phi)$ is obviously the first derivative $\mathscr{P}'(\phi)$ with respect to the canonical coordinate ϕ of the moment map $\mathscr{P}(\phi)$. Considering the metric (3.8.120) as god-given, it obviously admits the one dimensional group of isometries $B \rightarrow B + c$ for any choice of the smooth function $f(\phi)$ parameterizing the metric coefficient and the question is what is the topology of such a group, is it compact or non-compact, and in the second case is it parabolic or hyperbolic. When we deal with a constant negative curvature manifold, namely with the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$ these questions have a precise answer within Lie algebra theory, since the considered one-dimensional group of isometries $\mathscr{G}_{iso}$ is necessarily a subgroup of $\mathrm{SL}(2,\mathbb{R})$ and as such its generator $\mathfrak{g} \in \mathfrak{sl}(2,\mathbb{R})$ can be of three types:

(a) $\mathfrak{g}$ is compact, which means that, as a matrix, in whatever representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ it is diagonalizable and its eigenvalues are purely imaginary. In this case the one-dimensional subrgroup is topologically a circle $\mathbb{S}^1$ and isomorphic to $\mathrm{U}(1)$. We name *elliptic* the isometry group $\mathscr{G}_{iso}$ generated by such a $\mathfrak{g}$.
(b) $\mathfrak{g}$ is non-compact and semisimple, which means that, as a matrix, in whatever representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, it is diagonalizable and its eigenvalues are real and non vanishing. In this case the one-dimensional subgroup is topologically a line $\mathbb{R}$ and it is isomorphic to $\mathrm{SO}(1,1)$. We name *hyperbolic* the isometry group $\mathscr{G}_{iso}$ generated by such a $\mathfrak{g}$.
(c) $\mathfrak{g}$ is non-compact and nilpotent, which means that, as a matrix, in whatever representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, it is nilpotent and its eigenvalues are zero. In this case the one-dimensional subrgroup is topologically a line $\mathbb{R}$. We name *parabolic* the isometry group $\mathscr{G}_{iso}$ generated by such a $\mathfrak{g}$.

The interesting question is whether the characterization of an isometry as *elliptic,parabolic* or *hyperbolic* can be reformulated in pure geometrical terms and applied to cases where there is no ambient Lie algebra for the unique one-dimensional continuous isometry $\mathscr{G}_{iso}$. In this respect it is useful to remark that a metric of type (3.8.120) implies a fibre-bundle structure of the underlying two-dimensional manifold Σ:

$$\Sigma = \mathrm{P}(\mathbb{R}, \mathscr{F}, \mathscr{G}_{iso}) \to \mathbb{R} \tag{3.8.121}$$

where the base manifold is the real line $\mathbb{R}$ spanned by the coordinate $\phi \in [-\infty\,,\,+\infty]$, the structural group is the one-dimensional isometry group $\mathscr{G}_{iso}$ and the standard fibre $\mathscr{F}$ is a one dimensional space on which $\mathscr{G}_{iso}$ has a transitive action. In other words the manifold Σ is fibered into orbits of the isometry group. An explicit geometrical realization of this fibration in the three cases was already provided in the previous subsections by means of the three types of parametric surfaces encoded in:

1. Equation (3.8.45) which realize a surface in three-dimensional Minkowski space which is fibered in circles S^1 representating the orbits of an elliptic isometry group $\mathscr{G}_{iso}$.
2. Equation (3.8.57) which realize a surface in three-dimensional Minkowski space which is fibered in hyperbolae representating the orbits of a hyperbolic isometry group $\mathscr{G}_{iso}$.
3. Equation (3.8.73) which realize a surface in three-dimensional Minkowski space which is fibered in parabolae representating the orbits of a parabolic isometry group $\mathscr{G}_{iso}$.

As we argued in previous subsections, providing also some counterexamples, the subtle point is that the explicit geometric construction as a parametric surface fibered in circles, parabolae or hyperbolae, which a priori seems always possible, should lead to a smooth manifold having no singularity and being simply connected.

In more abstract terms the question was formulated by mathematicians for a single isometry Γ, even belonging to a discrete isometry group, not necessarily continuous and Lie, which can be characterized unambiguously as elliptic, parabolic, or hyperbolic, for Riemannian manifolds also of higher dimension than two, provided they are Hadamard manifolds.

Definition 3.8.1 A Hadamard manifold is a simply connected, geodesically complete Riemannian manifold $\mathscr{H} = (\mathscr{M}, g)$ whose scalar curvature $R(x)$ is **nonpositive definite and finite**, namely $-\infty < R(x) \le 0, \quad \forall x \in \mathscr{M}$.

The virtue of Hadamard manifolds is that they allow for what is usually not available in generic Riemannian manifolds, namely the definition of a bilocal distance function $d(x, y)$ providing the absolute distance between any two points $x,\ y \in \mathscr{H}$. As we teach our students when introducing (pseudo)-Riemanian geometry and General Relativity, the concept of absolute space-(time) distance is lost in Differential Geometry and we can only define the length of any curve $\beta^\mu(t)$ ($t \in [0\,,1]$), which starts at the point $x^\mu = \beta^\mu(0)$ and ends at the point $y^\mu = \beta^\mu(1)$. Given the metric $g_{\mu\nu}(x)$ we introduce the length functional which provides such a length:

$$\ell(\beta) = \int_0^1 \sqrt{g_{\mu\nu}\frac{d\beta^\mu}{dt}\frac{d\beta^\nu}{dt}}dt \tag{3.8.122}$$

The curves corresponding to extrema of the length functional are the geodesics, but in a generic Riemannian manifold there is no guarantee that for any two-points $x, y \in \mathscr{M}$ there is an arc of geodesic connecting them that is an absolute minimum of the length functional and that such minimum is unique and non-degenerate. Instead the hypotheses characterizing Hadamard manifolds guarantee precisely this (see, e.g. [16] and references therein) and one can define the distance function:

$$\forall x, y \in \mathscr{H} \quad : \quad d(x, y) = \text{infimum}\,[\ell(\beta)] \tag{3.8.123}$$

Hence restricting one's attention to Hadamard manifolds one can introduce a very useful geometrical concept that allows for a geometrical classification of isometries Γ:

$$\Gamma \quad : \quad \mathscr{M} \to \mathscr{M} \quad ; \quad \Gamma_\star\left[ds_g^2\right] = ds_g^2 \tag{3.8.124}$$

where $\Gamma_\star$ denotes the pull-back of Γ. The geometrical concept which provides the clue for such a classification is the displacement function defined below for any isometry Γ:

$$d_\Gamma(x) \equiv d(x, \Gamma x) \tag{3.8.125}$$

3.8.5.1 Classification of Isometries of Hadamard Manifolds $\mathscr{H} = (\mathscr{M}, g)$

The isometries of a Hadamerd manifold belong to the following types (see, e.g. [16] and references therein):

(a) **elliptic**, if $d_\Gamma(x)$ attains an absolute minimum of vanishing displacement $\min_{x\in\mathscr{M}} d_\Gamma(x) = 0$, or, to say it in other words, if and only if Γ has a fixed point $x_0 \in \mathscr{M}$ in the interior of the manifold for which $d\,(x_0,\ \Gamma x_0) = 0$.

(b) **hyperbolic**, if $d_\Gamma(x)$ attains an absolute minimum larger than zero $\min_{x\in\mathscr{M}} d_\Gamma(x) > 0$, or equivalently if Γ has two distinct fixed points on the boundary $\partial\mathscr{M}$ of $\mathscr{M}$

(c) **strictly parabolic**, if $d_\Gamma(x)$ never attains its infimum which is zero $\inf_{x\in\mathscr{M}} d_\Gamma(x) = 0$, or equivalently if Γ has just one fixed point on the boundary $\partial\mathscr{M}$ of $\mathscr{M}$;

(d) **mixed**, if $d_\Gamma(x)$ does not attain its the infimum which is larger than zero: $\inf_{x\in\mathscr{H}} d_\Gamma(x) > 0$.

The above classification of isometries is a generalisation to a nonconstant curvature case of the classification of isometries of the very particular constant curvature case, namely the Poincaré-Lobachevsky plane $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{O}(2)}$, where only the isometries (a), (b) and (c) are realized.

3.8.5.2 Application to the Kähler Surfaces considered in this Section

Not all Kähler surfaces Σ defined by Eq. (3.8.12) are Hadarmard since the curvature sometimes becomes positive in the interior of the manifold but most of them are such and moreover the limiting curvature of the boundary is non positive for all models. Therefore it makes sense to utilize the above geometric classification of isometries and verify that it just agrees with the criteria based on asymptotic expansions of the function $J(C)$ utilized in the previous subsections in order to discriminate among elliptic, parabolic and hyperbolic groups. Negative curvature guarantees the existence of a distance function, but probably in all considered examples such a distance function is well defined in spite of the existence of positive curvature domains in the deep interior of the manifold.

Hence with reference to the metric (3.8.120) let us consider the isometry Γ corresponding to B-shifts:

$$B \rightarrow \Gamma B = B + \delta\,, \tag{3.8.126}$$

where δ is a constant parameter, let us assume that the curvature

$$R = -\frac{\frac{d^2}{d\phi^2} f(\phi)}{f(\phi)}, \tag{3.8.127}$$

fulfills the Hadamard condition: $-\infty < R \leq 0$ and let us apply the classification scheme introduced above.

The first observation is the following. If the function $f(\phi)$ has neither a singularity nor a zero (i.e., if $f(\phi) \neq \pm\infty$ *and* $f(\phi) \neq 0$) both in the range of the coordinates $\{\phi,\ B\}$ corresponding to the interior of the manifold $\mathscr{M}$ and for those limiting values corresponding to the boundary $\{\phi,\ B\} \in \partial\mathscr{M}$ then the metric (3.8.120) has no coordinate singularity and the isometry (3.8.126) admits only one fixed point $B = \infty \in \partial\mathscr{M}$ on the boundary of the manifold. In this case the isometry Γ is strictly parabolic, according to item (c) of the above classification.

On the other hand, if the function $f(\phi)$ possesses a coordinate singularity at some value of $\phi = \phi_0 \in \mathscr{M}$ in the interior of $\mathscr{M}$, then in order to establish which is the type of the isometry Γ one has to introduce a new coordinate system $\{\phi,\ B\} \rightarrow \{\widetilde{\phi},\ \widetilde{B}\}$ such that the metric expressed in terms of the new coordinates is non-singular in the vicinity of the former coordinate singularity. The existence of such a coordinate system is guaranteed by the non-singularity of the curvature and by the smoothness of the manifold. If in the newly constructed coordinate system the isometry has a fixed point corresponding to the former coordinate singularity then, according to item a) of the above classification, it is elliptic. Since this happens for all elements of the isometry group $\mathcal{G}_{iso}$, this latter is a compact U(1) and the appropriate complex structure is $\mathfrak{z} = \zeta = \exp[\delta(c - \mathrm{i}B)]$. Otherwise the isometry is certainly not elliptic and non-compact.

Summarizing, the necessary condition for the isometry Γ to be elliptic is that the function $f(\phi)$ has a zero or a pole in the interior of $\mathcal{M}$ at some $\phi = \phi_0 \equiv -\frac{a_1}{a_2}$, where a_1 and $a_2 > 0$ are arbitrary constant parameters. In case such a singularity is power-like, we conclude that in a neighborhood $\mathbb{U}_{\phi_0}$ of ϕ_0 we have:

$$f(\phi)|_{\phi\in\mathbb{U}_{\phi_0}} = (a_2\phi + a_1)^n \tag{3.8.128}$$

where n is a positive or negative integer. Comparing Eq. (3.8.127) we see that the condition of a regular and finite curvature is fulfilled if and only if $n = 1$. In other words the function $f(\phi)$ has the following behavior at $\phi = \phi_0$:

$$f(\phi)|_{\phi\in\mathbb{U}_{\phi_0}} = a_2\phi + a_1 + \mathcal{O}\left[(\phi - \phi_0)^3\right] \tag{3.8.129}$$

Correspondingly the curvature is zero at leading order:

$$R|_{\phi\in\mathbb{U}_{\phi_0}} = 0 + \mathcal{O}\left[(\phi - \phi_0)^3\right] \tag{3.8.130}$$

In the new coordinate system $\{x, y\}$, $\{\phi, B\} \to \{x, y\}$, defined by

$$x = \left(\phi + \frac{a_1}{a_2}\right)\cos(a_2 B), \quad y = \left(\phi + \frac{a_1}{a_2}\right)\sin(a_2 B), \tag{3.8.131}$$

the metric (3.8.120) becomes

$$\begin{aligned} ds^2|_{\phi\in\mathbb{U}_{\phi_0}} &\simeq d\phi^2 + (a_2\phi + a_1)^2 dB^2 \\ &= dx^2 + dy^2, \end{aligned} \tag{3.8.132}$$

and the isometry transformations (3.8.126) takes the following form:

$$\{x, y\} \to \{x\cos\delta + y\sin\delta, -x\sin\delta + y\cos\delta\}, \tag{3.8.133}$$

The original coordinate singularity has disappeared, but in the new coordinates (3.8.131) the isometry (3.8.133) acquires the fixed point $\{x_0 = 0, y_0 = 0\}$, $\{0, 0\} \to \{0, 0\}$, in the interior of $\mathcal{M}$. Hence if the above situation is verified according to item a) of the above classification the isometry group is elliptic.

Consider next the behavior of the C-coordinate, defined by Eq. (3.8.15), in the neighborhood of ϕ_0. To leading order we have

$$\phi \to C \simeq \frac{1}{a_2} \ln(a_2 \phi + a_1) + \mathscr{O}\left[(\phi - \phi_0)^{-1}\right] \quad \Rightarrow \quad \phi_0 \Leftrightarrow C_0 = -\infty \tag{3.8.134}$$

so that the metric (3.8.120) becomes

$$ds^2|_{C \in \mathbb{U}_{C_0}} \simeq e^{2 a_2 C} \left(\mathrm{d}B^2 + \mathrm{d}C^2\right) \tag{3.8.135}$$

in the C_0-neighborhood $C \in \mathbb{U}_{C_0}$. Inspection of the latter formula shows that it reproduces the criterion to decide that the isometry is elliptic advocated in Eq. (3.8.22).

$$\tfrac{1}{2} \frac{d^2}{dC^2} J(C)|_{C \in \mathbb{U}_{C_0}} = e^{2 a_2 C}|_{C \in \mathbb{U}_{C_0}} \to 0 \tag{3.8.136}$$

Let us stress that the fixed point in the interior of the manifold required for an elliptic interpretation of the isometry group is just the origin of the manifold where the Kähler metric becomes approximately the flat one.

Let us now turn to the case where the singularity of the metric coefficient is of the exponential type, namely for $\phi_0 = \infty$ and for $\phi \in \mathbb{U}_{\phi_0}$, we have

$$f(\phi)|_{\phi \in \mathbb{U}_{\phi_0}} = a_1 e^{a_2 \phi}, \quad a_2 > 0 \tag{3.8.137}$$

this behavior is also consistent with the regularity of the curvature R (see Eq. (3.8.127)), which, in this case takes a finite negative value in the leading order approximation:

$$R|_{\phi \in \mathbb{U}_{\phi_0}} \simeq -a_2^2 + \text{subleading terms} \tag{3.8.138}$$

The metric (3.8.120) reproduces locally the metric of the hyperbolic (Poincaré - Lobachevsky) plane

$$ds^2|_{\phi \in \mathbb{U}_{\phi_0}} \approx d\phi^2 + a_1^2 e^{2 a_2 \phi} dB^2 \tag{3.8.139}$$

for which it is well known that the value of $\phi_0 = \infty$ corresponds to the boundary $\partial\mathscr{M}$. If the function $f(\phi)$ does not have other singularities of the exponential type, but (3.8.137), then one can immediately conclude that the isometry (3.8.126) is strictly parabolic according to item c) of the above classification, since it possesses just a single fixed point $B = \infty$ on the boundary $\partial\mathscr{M}$.

If besides the singularity (3.8.137) the function $f(\phi)$ possesses a second exponential singularity at $\widetilde{\phi}_0 = -\infty$ for $\phi \in \mathbb{U}_{\widetilde{\phi}_0}$, namely

$$f(\phi)|_{\phi \in \mathbb{U}_{\widetilde{\phi}_0}} = \widetilde{a}_1 e^{-\widetilde{a}_2 \phi}, \quad \widetilde{a}_2 > 0, \tag{3.8.140}$$

then by the same token as above we come to the conclusion that the point $\widetilde{\phi}_0$ belongs to the boundary of another hyperbolic plane locally isomorphic to the neighborhood $U_{\widetilde{\phi}_0} \subset \mathscr{H}$ and that isometry (3.8.126) possesses a second fixed point on such a boundary. Hence the isometry is hyperbolic according to item (b) of the above classification and since this applies to all elements of the isometry group $\mathscr{G}_{iso}$ this latter is hyperbolic and isomorphic to SO(1, 1).

One can not exclude the existence of more sophisticated types of $f(\phi)$ singularities, besides the above described power-like and exponential one, that might be consistent with the regularity of the curvature R (3.8.127), yet in all examples considered in previous subsections no other singularities than these two are met.

Relying on these results we can summarize the geometric criteria for the classification of isometries in two-manifolds with a metric of type (3.8.120) which are of the Hadamard type

(a) **elliptic**, if the function $f(\phi)$ possesses a first order zero, i.e. $f(\phi)|_{\phi\in\mathbb{U}_{\phi_0}} = a_2(\phi - \phi_0)$;
(b) **hyperbolic**, if the function $f(\phi)$ possesses two different leading exponential singularities at $\phi_0^{(\pm)} = \pm\infty$, i.e. $f(\phi)|_{\phi\in\mathbb{U}_{\phi_0^{(\pm)}}} = a_1^{(\pm)}\, e^{\pm a_2^{(\pm)}\phi}$ and $a_2^{(\pm)} > 0$;
(c) **strictly parabolic**, if the function $f(\phi)$ possesses a single leading exponential singularity at either $\phi_0^{(+)} = +\infty$ or $\phi_0^{(-)} = -\infty$, i.e. $f(\phi)|_{\phi\in\mathbb{U}_{\phi_0^{(+)}}} = a_1^{(+)}\, e^{+a_2^{(+)}\phi}$ or $f(\phi)|_{\phi\in\mathbb{U}_{\phi_0^{(-)}}} = a_1^{(-)}\, e^{-a_2^{(-)}\phi}$ and $a_2^{(\pm)} > 0$.

The above characterization yields exactly the same result as the criteria based on the asymptotic behavior of $J(C)$ that have been utilized in the previous subsections and this happens also for such models that do not lead to exactly Hadamard manifolds, the curvature attaining somewhere also positive values. As an exemplification of the use of the above concepts we briefly reconsider from this point of view the flat models and the constant curvature models.

3.8.5.3 Flat Models

The flat metric

$$ds^2 = d\phi^2 + (a_2\,\phi + a_1)^2\, dB^2 \tag{3.8.141}$$

in case $a_2 \neq 0$ possesses a coordinate singularity at

$$\phi = -\frac{a_1}{a_2} \tag{3.8.142}$$

corresponding to a first order zero $f(\phi)$ at finite ϕ. According to the above classification this implies that the isometry $B \to B + \delta$ is elliptic.

In the case $a_2 = 0$ the metric (3.8.141) becomes

$$ds^2 = d\phi^2 + a_1^2\, dB^2 \tag{3.8.143}$$

and does not possess a coordinate singularity at all. This implies that the isometry $B \to B + \delta$ is strictly parabolic.

3.8.5.4 Constant Negative Curvature Models

Case (A)

$$ds^2 = d\phi^2 + \sinh^2(\nu\,\phi)\; dB^2 \tag{3.8.144}$$

This metric possesses a coordinate singularity at $\phi = 0$. In the neighborhood of $\phi = 0$ at leading order it behaves as follows

$$ds^2 \approx d\phi^2 + \nu^2\,\phi^2\,dB^2 \tag{3.8.145}$$

which modulo an inessential rescaling of the coordinate B and a shifting the coordinate ϕ reproduces the metric (3.8.141). Hence its isometry (3.8.126) is elliptic in this case.

Case (B)

$$ds^2 = d\phi^2 + \cosh^2(\nu\,\phi)\; dB^2 \tag{3.8.146}$$

This metric does not possess a coordinate singularity in the finite range of ϕ, but it has two exponential singularities of the type (3.8.137) and (3.8.140). Hence the isometry (3.8.126) is hyperbolic in this case.

Case (C)

$$ds^2 = d\phi^2 + e^{2\,\nu\,\phi}\,dB^2 \tag{3.8.147}$$

This metric does not possess a coordinate singularity in the finite range of ϕ, but it possesses a single exponential singularity either of the type (3.8.137) or of the type (3.8.140). Hence the isometry (3.8.126) is strictly parabolic in this case.

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