Approximation by Lupaș–Kantorovich Operators

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Abstract The present article deals with the approximation properties of certain Lupaș-Kantorovich operators preserving e^{-x}. We obtain uniform convergence estimates which also include an asymptotic formula in quantitative sense. In the end, we provide the estimates for another modification of such operators, which preserve the function e^{-2x}.

Introduction

In the year 1995, Lupaș [9] proposed the Lupaș operators:

$$L_n(f,x)=\sum_{k=0}^{\infty}\frac{2^{-nx}(nx)_k}{k!2^k}f\left(\frac{k}{n}\right),$$

where $(nx)_k$ is the rising factorial given by

$$(nx)_k=nx(nx+1)(nx+2)\cdots(nx+k-1),\quad (nx)_0=1.$$

Four years later, Agratini [2] introduced the Kantorovich-type generalization of the operators L_n. After a decade Erençin and Taşdelen [4] considered a generalization of the operators discussed in [2] based on some parameters and established some approximation properties. We start here with the Kantorovich variant of Lupaș operators defined by

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$$K_n(f,x) = n\sum_{k=0}^{\infty} \frac{2^{-na_n(x)}(na_n(x))_k}{k!2^k} \int_{k/n}^{(k+1)/n} f(t)dt \qquad (1)$$

with the hypothesis that these operators preserve the function e^{-x}. Then using

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = (1-z)^{-a}, \, |z| < 1,$$

we write

$$\begin{aligned} e^{-x} &= n\sum_{k=0}^{\infty} \frac{2^{-na_n(x)}(na_n(x))_k}{k!2^k} \int_{k/n}^{(k+1)/n} e^{-t}dt \\ &= n\sum_{k=0}^{\infty} \frac{2^{-na_n(x)}(na_n(x))_k}{k!2^k} e^{-k/n}(1-e^{-1/n}) \\ &= n(1-e^{-1/n})(2-e^{-1/n})^{-na_n(x)}, \end{aligned}$$

which concludes

$$a_n(x) = \frac{x + \ln\left(n(1-e^{-1/n})\right)}{n \ln\left(2-e^{-1/n}\right)}. \qquad (2)$$

Therefore the operators defined by (1) take the following alternate form

$$K_n(f,x) = n\sum_{k=0}^{\infty} \frac{1}{k!2^k} 2^{-\frac{x+\ln\left(n(1-e^{-1/n})\right)}{\ln\left(2-e^{-1/n}\right)}} \left(\frac{x + \ln\left(n(1-e^{-1/n})\right)}{\ln\left(2-e^{-1/n}\right)}\right)_k \int_{k/n}^{(k+1)/n} f(t)dt.$$

These operators preserve constant and the function e^{-x}. The quantitative direct estimate for a sequence of linear positive operators was discussed and proved in [8] as the following result:

Theorem A ([8]) *If a sequence of linear positive operators* $L_n : C^*[0,\infty) \to C^*[0,\infty)$, *(where* $C^*[0,\infty)$ *be the subspace of all real-valued continuous functions, which has finite limit at infinity) satisfy the equalities*

$$\begin{aligned} ||L_n(e_0) - 1||_{[0,\infty)} &= \alpha_n \\ ||L_n(e^{-t}) - e^{-x}||_{[0,\infty)} &= \beta_n \\ ||L_n(e^{-2t}) - e^{-2x}||_{[0,\infty)} &= \gamma_n \end{aligned}$$

then

$$||L_n f - f||_{[0,\infty)} \leq 2\omega^* \left(f, \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right), \; f \in C^*[0,\infty),$$

where the norm is the uniform norm and the modulus of continuity is defined by

$$\omega^*(f,\delta) = \sup_{|e^{-x}-e^{-t}|\leq\delta, x,t>0} |f(t) - f(x)|.$$

Very recently Acar et al. [1] used the above theorem and established quantitative estimates for the modification of well-known Szász–Mirakyan operators, which preserve the function e^{2ax}, $a > 0$. Actually such a modification may be important to discuss approximation properties, but if the operators preserve e^{-x} or e^{-2x}, then such results may provide better approximation in the sense of reducing the error. In the present paper, we study Kantorovich variant of Lupaş operators defined by (1) with $a_n(x)$ as given by (2) preserving e^{-x}. We calculate a uniform estimate and establish a quantitative asymptotic result for the modified operators.

Auxiliary Results

In order to prove the main results, the following lemmas are required.

Lemma 1 *The following representation holds*

$$K_n(e^{At}, x) = \frac{n(e^{A/n} - 1)}{A}(2 - e^{A/n})^{-na_n(x)}.$$

Proof We have

$$\begin{aligned} K_n(e^{At}, x) &= n\sum_{k=0}^{\infty} \frac{2^{-na_n(x)}(na_n(x))_k}{k!2^k} \int_{k/n}^{(k+1)/n} e^{At}dt \\ &= n\sum_{k=0}^{\infty} \frac{2^{-na_n(x)}(na_n(x))_k}{k!2^k} \left[e^{At}(e^{A/n} - 1)\right] \\ &= \frac{n(e^{A/n} - 1)}{A}\left(2 - e^{A/n}\right)^{-na_n(x)}. \end{aligned}$$

Lemma 2 *If $e_r(t) = t^r$, $r \in N^0$, then the moments of the operators (1) are given as follows:*

$$K_n(e_0, x) = 1,$$

$$K_n(e_1, x) = a_n(x) + \frac{1}{2n},$$

$$K_n(e_2,x) = (a_n(x))^2 + \frac{3a_n(x)}{n} + \frac{1}{3n^2},$$
$$K_n(e_3,x) = (a_n(x))^3 + \frac{15(a_n(x))^2}{2n} + \frac{10a_n(x)}{n^2} + \frac{1}{4n^3},$$
$$K_n(e_4,x) = (a_n(x))^4 + \frac{14(a_n(x))^3}{n} + \frac{50(a_n(x))^2}{n^2} + \frac{53a_n(x)}{n^3} + \frac{1}{5n^4}.$$

Lemma 3 *If* $\mu_{n,m}(x) = K_n\left((t-x)^m, x\right)$, *then by using Lemma 2, we have*

$$\mu_{n,0}(x) = 1,$$
$$\mu_{n,1}(x) = a_n(x) + \frac{1}{2n} - x,$$
$$\mu_{n,2}(x) = (a_n(x)-x)^2 + \frac{3a_n(x)}{n} - \frac{x}{n} + \frac{1}{3n^2},$$
$$\mu_{n,4}(x) = (a_n(x)-x)^4 + \frac{14(a_n(x))^3 - 30x(a_n(x))^2 + 18x^2a_n(x) - 2x^3}{n}$$
$$+ \frac{50(a_n(x))^2 - 40xa_n(x) + 2x^2}{n^2} + \frac{53a_n(x) - x}{n^3} + \frac{1}{5n^4}.$$

Furthermore,

$$\lim_{n\to\infty} n\left[\frac{x + \ln\left(n(1-e^{-1/n})\right)}{n\ln\left(2-e^{-1/n}\right)} + \frac{1}{2n} - x\right] = x$$

and

$$\lim_{n\to\infty} n\left[\left(\frac{x + \ln\left(n(1-e^{-1/n})\right)}{n\ln\left(2-e^{-1/n}\right)} - x\right)^2 + \frac{3\left[x + \ln\left(n(1-e^{-1/n})\right)\right]}{n^2\ln\left(2-e^{-1/n}\right)}\right.$$
$$\left. - \frac{x}{n} + \frac{1}{3n^2}\right] = 2x.$$

Main Results

In this section, we present the quantitative estimates.

Theorem 1 *For* $f \in C^*[0,\infty)$, *we have*

$$||K_n f - f||_{[0,\infty)} \leq 2\omega^*\left(f, \sqrt{\gamma_n}\right),$$

where

$$\begin{aligned}\gamma_n &= ||K_n(e^{-2t}) - e^{-2x}||_{[0,\infty)}\\ &= \left|\left|\frac{2xe^{-2x}}{n} + \frac{(24x^2-48x-11)e^{-2x}}{12n^2} + O\left(\frac{1}{n^3}\right)\right|\right|_{[0,\infty)}.\end{aligned}$$

Proof The operators K_n preserve the constant and e^{-x}. Thus $\alpha_n = \beta_n = 0$. We only have to evaluate γ_n. In view of Lemma 1, we have

$$\begin{aligned}K_n(e^{-2t}, x) &= n\sum_{k=0}^{\infty}\frac{2^{-na_n(x)}(na_n(x))_k}{k!2^k}\int_{k/n}^{(k+1)/n} e^{-2t}dt\\ &= \frac{n(1-e^{-2/n})}{2}\left(2-e^{-2/n}\right)^{-na_n(x)},\end{aligned}$$

where $a_n(x)$ is given as

$$a_n(x) = \frac{x + \ln\left(n(1-e^{-1/n})\right)}{n\ln\left(2-e^{-1/n}\right)}.$$

Thus using the software Mathematica, we get at once

$$\begin{aligned}K_n(e^{-2t}, x) &= \frac{n(1-e^{-2/n})}{2}\left(2-e^{-2/n}\right)^{\left[-\frac{x+\ln\left(n(1-e^{-1/n})\right)}{\ln\left(2-e^{-1/n}\right)}\right]}\\ &= e^{-2x} + \frac{2xe^{-2x}}{n} + \frac{(24x^2-48x-11)e^{-2x}}{12n^2} + O\left(\frac{1}{n^3}\right).\end{aligned}$$

This completes the proof of the theorem.

Theorem 2 *Let* $f, f'' \in C^*[0,\infty)$. *Then the inequality*

$$\begin{aligned}&\left|n\left[K_n(f,x) - f(x)\right] - x[f'(x) + f''(x)]\right|\\ &\quad\le |p_n(x)||f'| + |q_n(x)||f''| + 2\left(2q_n(x) + 2x + r_n(x)\right)\omega^*\left(f'', n^{-1/2}\right)\end{aligned}$$

holds for any $x \in [0,\infty)$, *where*

$$\begin{aligned}p_n(x) &= n\mu_{n,1}(x) - x,\\ q_n(x) &= \frac{1}{2}\left(n\mu_{n,2}(x) - 2x\right),\\ r_n(x) &= n^2\sqrt{K_n\left((e^{-x}-e^{-t})^4, x\right)}\sqrt{\mu_{n,4}(x)},\end{aligned}$$

and $\mu_{n,1}(x)$, $\mu_{n,2}(x)$, *and* $\mu_{n,4}(x)$ *are given in Lemma 3.*

Proof By Taylor's expansion, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \varepsilon(t,x)(t-x)^2, \tag{3}$$

where

$$\varepsilon(t,x) = \frac{f''(\eta) - f''(x)}{2}$$

and η is a number lying between x and t. If we apply the operator K_n to both sides of (3), we have

$$\left| K_n(f,x) - f(x) - \mu_{n,1}(x)f'(x) - \frac{1}{2}\mu_{n,2}(x)f''(x) \right| \\ \leq |K_n(\varepsilon(t,x)(t-x)^2, x)|,$$

Applying Lemma 2, we get

$$\left| n[K_n(f,x) - f(x)] - x[f'(x) + f''(x)] \right| \\ \leq \left| n\mu_{n,1}(x) - x \right| |f'(x)| + \frac{1}{2} \left| n\mu_{n,2}(x) - 2x \right| |f''(x)| \\ + |nK_n(\varepsilon(t,x)(t-x)^2, x)|.$$

Put $p_n(x) := n\mu_{n,1}(x) - x$ and $q_n(x) := \frac{1}{2}[n\mu_{n,2}(x) - 2x]$. Thus

$$\left| n[K_n(f,x) - f(x)] - x[f'(x) + f''(x)] \right| \\ \leq |p_n(x)|.|f'(x)| + |q_n(x)|.|f''(x)| + |nK_n(\varepsilon(t,x)(t-x)^2, x)|.$$

In order to complete the proof of the theorem, we must estimate the term $|nK_n(\varepsilon(t,x)(t-x)^2, x)|$. Using the property

$$|f(t) - f(x)| \leq \left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right) \omega^*(f,\delta),\ \delta > 0,$$

we get

$$|\varepsilon(t,x)| \leq \left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right) \omega^*(f'',\delta).$$

For $|e^{-x} - e^{-t}| \leq \delta$, one has $|\varepsilon(t,x)| \leq 2\omega^*(f'', \delta)$. In case $|e^{-x} - e^{-t}| > \delta$, then $|\varepsilon(t,x)| < 2\frac{(e^{-x}-e^{-t})^2}{\delta^2}\omega^*(f'', \delta)$. Thus

$$|\varepsilon(t,x)| \leq 2\left(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\omega^*(f'', \delta)\right).$$

Obviously using this and Cauchy–Schwarz inequality after choosing $\delta = n^{-1/2}$, we get

$$\begin{aligned} nK_n(|\varepsilon(t,x)|(t-x)^2, x) &\leq 2\omega^*(f''(x), n^{-1/2})\left[n\mu_{n,2}(x) + r_n(x)\right] \\ &= 2\omega^*(f''(x), n^{-1/2})\left[2q_n(x) + 2x + r_n(x)\right], \end{aligned}$$

where $r_n(x) = n^2[K_n((e^{-x} - e^{-t})^4, x).\mu_{n,4}(x)]^{1/2}$ and

$$\begin{aligned} K_n((e^{-x} - e^{-t})^4, x) = &-\frac{n}{4}(e^{-4/n} - 1)(2 - e^{-4/n})^{-na_n(x)} \\ &+\frac{4n}{3}e^{-x}(e^{-3/n} - 1)(2 - e^{-3/n})^{-na_n(x)} \\ &-3ne^{-2x}(e^{-2/n} - 1)(2 - e^{-2/n})^{-na_n(x)} \\ &+4ne^{-3x}(e^{-1/n} - 1)(2 - e^{-1/n})^{-na_n(x)} + e^{-4x}. \end{aligned}$$

This completes the proof of the result.

Remark 1 From the Lemma 3, $p_n(x) \to 0, q_n(x) \to 0$ as $n \to \infty$ and using Mathematica, we get

$$\lim_{n\to\infty} n^2\mu_{n,4}(x) = 12x^2.$$

Furthermore

$$\lim_{n\to\infty} n^2 K_n\left((e^{-t} - e^{-x})^4, x\right) = 12e^{-4x}x^2.$$

Thus in the above Theorem 2, convergence occurs for sufficiently large n.

Corollary 1 *Let $f, f'' \in C^*[0,\infty)$. Then, the inequality*

$$\lim_{n\to\infty} n\left[K_n(f,x) - f(x)\right] = x[f'(x) + f''(x)]$$

holds for any $x \in [0,\infty)$.

Remark 2 In case the operators (1) preserve the function e^{-2x}, then in that case using Lemma 1, we have

$$e^{-2x} = \frac{n(1-e^{-2/n})}{2}\left(2-e^{-2/n}\right)^{-na_n(x)},$$

which implies

$$a_n(x) = \frac{2x + \ln\left(\frac{n(1-e^{-2/n})}{2}\right)}{n\ln(2-e^{-2/n})} \tag{4}$$

Also, for this preservation corresponding limits of Lemma 3 takes the following forms:

$$\lim_{n\to\infty} n\left[\frac{2x + \ln\left(\frac{n(1-e^{-2/n})}{2}\right)}{n\ln(2-e^{-2/n})} + \frac{1}{2n} - x\right] = 2x$$

and

$$\lim_{n\to\infty} n\left[\left(\frac{2x + \ln\left(\frac{n(1-e^{-2/n})}{2}\right)}{n\ln(2-e^{-2/n})} - x\right)^2 + \frac{3(2x + \ln\left(\frac{n(1-e^{-2/n}))}{2}\right)}{n^2\ln(2-e^{-2/n})} - \frac{x}{n} + \frac{1}{3n^2}\right] = 2x$$

and we have the following Theorems 1and 2 and Corollary 1 taking the following forms:

Theorem 3 *For $f \in C^*[0,\infty)$, we have*

$$||K_n f - f||_{[0,\infty)} \le 2\omega^*\left(f, \sqrt{2\beta_n}\right),$$

where

$$\begin{aligned}\beta_n &= ||K_n(e^{-t}) - e^{-x}||_{[0,\infty)}\\ &= \left|\left|\frac{-xe^{-x}}{n} + \frac{(12x^2+24x+11)e^{-x}}{24n^2} + O\left(\frac{1}{n^3}\right)\right|\right|_{[0,\infty)}.\end{aligned}$$

Theorem 4 *Let $f, f'' \in C^*[0,\infty)$. Then the inequality*

$$\begin{aligned}&\left|n\left[K_n(f,x) - f(x)\right] - x[2f'(x) + f''(x)]\right|\\ &\le |\hat{p}_n(x)||f'| + |\hat{q}_n(x)||f''| + 2\left(2\hat{q}_n(x) + 2x + \hat{r}_n(x)\right)\omega^*\left(f'', n^{-1/2}\right)\end{aligned}$$

holds for any $x \in [0, \infty)$, *where*

$$\hat{p}_n(x) = n\mu_{n,1}(x) - x,$$

$$\hat{q}_n(x) = \frac{1}{2}\left(n\mu_{n,2}(x) - 4x\right),$$

$$\hat{r}_n(x) = n^2\sqrt{K_n\left((e^{-x} - e^{-t})^4, x\right)}\sqrt{\mu_{n,4}(x)}.$$

and $\mu_{n,1}(x)$, $\mu_{n,2}(x)$ *and* $\mu_{n,4}(x)$ *are given in Lemma 3, with values of* $a_n(x)$, *given by (4).*

Corollary 2 *Let* $f, f'' \in C^*[0, \infty)$. *Then, the inequality*

$$\lim_{n\to\infty} n\left[K_n(f, x) - f(x)\right] = x[2f'(x) + f''(x)]$$

holds for any $x \in [0, \infty)$.

Remark 3 Several other operators, which are linear and positive, can be applied to establish analogous results. Also, some other approximation properties for the operators studied in [3, 5–7, 10] and references therein may be considered for these operators.

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