Fuzzy Empiristic Implication, A New Approach

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Abstract The present paper is a brief introduction to logical fuzzy implication operators, the basic properties of a fuzzy implication function, and ways to construct new fuzzy implication functions. It is also argued that logical implication functions are defined in a rather rationalistic manner. Thus a new, empiristic approach is proposed, defining implication relations that are derived from data observation and with no regard to any preexisting constrains. A number of axioms are introduced to define a fuzzy empiristic implication relation, and a method of computing such a relation is proposed. It is argued that the proposed method is easy and with small time requirement even for very large data sets. Finally an application of the empiristic fuzzy implication relation is presented, the choice of a suitable logical fuzzy implication function to describe an "If…then…" fuzzy rule, when observed data exists. An empiristic fuzzy implication relation is computed according to the data, and through schemas of approximate reasoning, the difference of it to any logical fuzzy implication function is measured. The fuzzy implication function that is closer to the empiristic best resembles the observed "If…then…" fuzzy rule.

Keywords Fuzzy implication · Approximate reasoning

Introduction

The theory of fuzzy logic that has been presented by Zadeh [12] has developed rapidly both in theoretical and application basis. The main characteristic of this development is the abandonment of the binary classical logic of zero and one. In that sense a logical proposition can be true with any degree of truth from one, meaning it is absolutely true, to zero, meaning it is absolutely false. It also defines fuzzy sets, to which any element can belong with any value from zero to one. Special interest

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N. J. Daras, Th. M. Rassias (eds.), *Modern Discrete Mathematics and Analysis*,
Springer Optimization and Its Applications 131,
https://doi.org/10.1007/978-3-319-74325-7_16

is the generalization of operations between fuzzy propositions and fuzzy sets with the generalization of the classical implication to fuzzy logic which is significant.

The present paper describes the basic properties of fuzzy implication functions. Also a new type of fuzzy implication relationship constructed entirely from observation, which is called empiricist fuzzy implication relationship, is proposed. It is theoretically founded with proposed axioms, and a method of evaluating the relationship has been developed. Also a method of choosing a logical implication function is proposed in order to be suitable to observed data, using the empiristic fuzzy implication relationship.

Preliminaries

Classical Implication

In classical logic there are propositions, e.g., "The number three is odd" or "the number four is odd," which may be true or false, respectively. These propositions can be combined using logical operations as "and," "or," etc. A very important operation between the two proposals is the implication. For two logical propositions p and q, it is denoted that p implies q ($p \Rightarrow q$) if any time p is true, q must be also true. In the above case, proposition p is called cause or antecedent, and proposition p is called consequent. It represents an "If. . . Then. . . " rule [3].

Abiding classical logic, any proposition can be valid or not valid, so it is evaluated with one or zero, respectively. Thus the implication between two propositions, being a proposition itself, can be valid or not valid. Furthermore any implication ($p \Rightarrow q$) is equivalent to the proposition "negation of q or p." In the following Table 1, the Boolean table of the classical implication operation is presented.

Classical implication operation is directly applied on logical reasoning schemas as modus ponens, modus tollens, and hypothetical reasoning [1]. Modus ponens from the Latin modus ponendo ponens is proof inductive reasoning and has the following format: IF $p \Rightarrow q$ is true AND p is true, THEN q is true. A modus tollens schema (from Latin modus tollendo tollens) or rebuttal process follows these steps: IF $p \Rightarrow q$ is true and $n(q)$ is true, THEN $n(p)$ is true, where $n(p)$ is the negation of p, i.e., the proposal p is false. Finally, an example of hypothetical reasoning is this: IF $p \Rightarrow q$ is true and $q \Rightarrow r$ is true, THEN $p \Rightarrow r$ is true.

Table 1 Boolean table of classical implication

p	q	$p \Rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Fuzzy Implication

Fuzzy logic is based on the idea that a proposition may be true to some degree of truth. It is immediately apparent that a generalization of the classical implication is required, such that firstly it is possible to evaluate the implication between fuzzy propositions, and secondly it is possible for an implication to be valid to a degree of truth. Thus, fuzzy implication can be defined as a function I (Eq. (1)):

$$I : [0, 1] \times [0, 1] \rightarrow [0, 1] \tag{1}$$

For the definition of a fuzzy logic implication operation, a set of axioms have been proposed in literature that any function has to fulfill in order to be considered as a fuzzy implication function. A fuzzy implication function is a function $I : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which satisfies the maximum of the following axioms [3]:

1. if $a \leq b$, then $I(a, x) \geq I(b, x)$ (left antitonicity)
2. if $a \leq b$ then $I(x, a) \leq I(x, b)$ (right isotonicity)
3. $I(0, a) = 1$
4. $I(1, a) = a$
5. $I(a, a) = 1$
6. $I(a, I(b, x)) = I(b, I(a, x))$
7. $I(a, b) \Leftrightarrow a \leq b$
8. $I(a, b) = I(n(b), n(a))$
9. A fuzzy implication function must be continuous on its domain

Construction of Fuzzy Implication Functions

In the literature there are many functions that have been proposed as fuzzy implication functions, satisfying some or all of the aforementioned axioms. In addition there are several ways to create implication functions based on logical rules of classical logic and using $t\text{-}norms$, $t\text{-}conorms$, or even other implications. Furthermore it is possible to create functions that satisfy the axiomatic restrictions on an algebraic basis. Some ways to construct a fuzzy implication function are as follows:

S-N Implications

As mentioned, the proposition p implies q in classical logic is identical with the proposition negation of p or q as in (Eq. (2)):

$$p \Rightarrow q \equiv n(p) \vee q \tag{2}$$

Thus, a fuzzy implication may be produced by a fuzzy negation N and a t-$conormS$ by the aforementioned tautology as follows [1] (Eq. (3)):

$$I(x, y) = S(N(x), y) \tag{3}$$

Reciprocal Implications

When for a given pair of fuzzy implication and negation operations, the rule of contraposition is not satisfied, it is possible for a new fuzzy implication function to be constructed with the following formula. The new fuzzy implication function paired with the negation utilized satisfies the rule of contraposition [1] (Eq. (4)).

$$I_N(x, y) = I(N(x), N(y)) \tag{4}$$

R Implications

The following formula of set theory, where A and B are subsets of a universal set X, obtained from the isomorphism of the classical set theory with the binary Boolean logic, can be used to construct new fuzzy implication functions (Eq. (5)).

$$A' \cup B = (A \setminus B)' = \cup\{C \subseteq X | A \cap C \subseteq B\} \tag{5}$$

Fuzzy implications created as a generalization of this rule in fuzzy logic are widely used in the intuitionistic fuzzy logic and are often found in the literature as R implications. For this purpose a t-$normT$ is needed, and the implications are calculated as follows (Eq. (6)):

$$I(x, y) = \sup\{t \in [0, 1] | T(x, t) < y\} \tag{6}$$

QL Implications

In quantum theory, a new tautology has prevailed to describe the implication operation. It is shown in the following equation (Eq. (7)), and it yields the same results with the rest known ways to describe the operation in classical logic.

$$p \Rightarrow q \equiv n(p) \vee (p \wedge q) \tag{7}$$

From the generalization (Eq. (7)) to fuzzy set theory arises a new class of implications that are found in literature as QL implications (quantum logic implications). These implication functions are created as in (Eq. (8)), when a fuzzy negation N, a fuzzy t-$conormS$, and a fuzzy t-$normT$ are applied.

$$I(x, y) = S(N(x)T(x, y)) \tag{8}$$

f and g Implications

For a function to be appropriate to be a fuzzy implication, it has to satisfy a set of axioms that have been mentioned above in the definition of fuzzy implication. So it is possible to create algebraic implication production techniques designed based on the restrictions, without any further association with logical rules as the other ways that have been reported [1]. Suppose a function $f : [0, 1] \to [0, \infty]$ that is strictly decreasing and continuous and that satisfies $f(1) = 0$. A fuzzy implication $I_f : [0, 1]^2 \to [0, 1]$ is defined as follows (Eq. (9)):

$$I_f(x, y) = f^{-1}(xf(y)) \tag{9}$$

The function f is called f-*generator* and the implication f-*generated*.

Respectively, g-*generator* generator functions and g-*generated* generated implications can be defined where function $g : [0, 1] \to [0, \infty]$ is strictly increasing and continuous with $g(0) = 0$. The implication $I_g : [0, 1]^2 \to [0, 1]$ is computed as follows (Eq. (10)):

$$I_g(x, y) = g^{(-1)}\left(\frac{1}{x}g(y)\right) \tag{10}$$

where the function $g^{(-1)}$ pseudo-inverse of g, given by (Eq. (11)):

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \in [0, g(1)] \\ 1 & \text{if } x \in [g(1), \infty] \end{cases} \tag{11}$$

Convex Combinations

Another very usual way to produce a new fuzzy implication is from a convex combination of two old ones. Two fuzzy implication functions I and J and any real number λ from the closed interval [0, 1] can be combined in order to produce a new implication I^λ as follows (Eq. (12)):

$$I^\lambda(x, y) = \lambda I(x, y) + (1 - \lambda)J(x, y) \tag{12}$$

It turns out that since I and J implications satisfy the axioms of fuzzy implication definition, the same is true for the produced I^λ [7].

Symmetric Implication Functions

It is noteworthy that apart from the classical implication functions, symmetric implication functions are widely used and have been proposed in the literature [3]. Such examples are Mamdani and Larson implications that use *t-norms* to evaluate the degree of truth of the implication [8].

These functions are very useful for fuzzy control systems, particularly in some implemented to engineering problems. It is common for these problems to involve correlated variables, where it is not always clear to distinct the antecedent from the consequent. Furthermore in such a system, a decrease in the degree of truth of the antecedent should be followed by a decrease in the degree of truth of the consequent. Hence this kind of fuzzy implications, although cannot satisfy the established axioms, has significant advantages when modeling engineering problems. Hence they can be found in the literature as engineering implications (Eqs. (13)–(14)):

$$I_M(x, y) = \min(x, y) \tag{13}$$

$$I_L(x, y) = xy \tag{14}$$

Approximate Reasoning

In nature, as in engineering applications, a need emerges to make decisions when accurate data are not available. This can be due to failure of accurate forecasting (e.g., seismic actions on structures, road traffic volume), difficulty or cost of measurements (e.g., accurate traffic volume data), and even in some cases when there are no models to explain a phenomenon sufficiently (e.g., the seismic behavior of reinforced concrete frame structures with infill masonry). These inconsistencies make it necessary to develop approximate reasoning to judge the data and make decisions in a scientific and responsible manner.

Approximate reasoning is used to relate fuzzy propositions. The fuzzy inference schemas that have been used are an extension of the classical inference schemas. So generalized modus ponens (GMP), generalized modus tollens (GMT), and hypothetical reasoning can be used. For calculating these schemas, fuzzy implications are as significant as the classical implication is to classical inference.

The schemas are generalized as follows. Assume two variables x and y set in X and Y sets, respectively. Furthermore assume a relation R between x and y. Assume now that it is known that $x \in A$. Then we can reasonably assume that $y \in B$ where $B = \{y \in Y | \langle x, y\rangle \in R, x \in A\}$ This also applies when the sets A, B are fuzzy and R is a fuzzy relation with membership functions x_A, x_B, x_R, respectively, and the formula transformed as follows [6] (Eq. (15)):

$$x_B(y) = \sup_{x \in X} \min[x_A(x), x_R(x, y)] \tag{15}$$

This formula is known as compositional rule of inference [13] when $R(x, y)$ is a fuzzy implication relation $I(A(x), B(y))$.

Generalized modus ponens or any other approximate reason scheme can be calculated with the use of the compositional rule of inference. Assume for x and y that A implies B and that it is known that in a given instance $x \in A'$. Then $y \in B'$ and B' can be computed using the rule, as follows. It is apparent that if all involved sets are crisp, the formula is equivalent to classical modus ponens (Eq. (16)).

$$B'(y) = \sup_{x \in X} \min[A'(x), R(x, y)] \tag{16}$$

Choosing Fuzzy Implication Function

In the process of approximate reasoning, the selection of fuzzy operations can cause large variations in the results. Especially when constructing a fuzzy inference system for which an "IF...Then..." fuzzy rule base is the cornerstone, the rules are essentially fuzzy implication operators. The choice has to be from a large set of alternative functions. There are some well-known functions that can be found in most handbooks [6], and there are many methods to construct new functions. Some of the most significant have been noted.

Although there is no unique and exact way of selecting the implication, several attempts have been made. It should be made clear that the fuzzy implication functions can be divided into groups. Thus, the literature is possible to give guidance regarding the selection of the initial implication group that matches according to theory in an optimal way. Thus the selection of the group may be based on the type of propositional reasoning schema that is utilized [6]. Additionally the implication function can be chosen from the $t\text{-}norm$, $t\text{-}conorm$, or negation utilized, or it could be chosen to fit optimal the observed data [9].

Special mention needs to be made to the fuzzy implication selection method based on data that was applied to describe the relationship of a country's economic status to the number of air transport movements in this country, after the creation of appropriate linguistic variables [4]. Recall that an implication function can be described as a relationship between two sets, the domain set of the antecedent and the consequent. Thus, each pair of observations can be considered as an element that certainly belongs to the relationship, as it is verified by the data.

Empiristic Fuzzy Implications

Empiricism and Fuzzy Implication

Fuzzy implication is perceived as a generalization of classical implication operator. Therefore any function utilized as a fuzzy implication relation between two linguistic variables should follow the axioms that have been determined as an extension of the rules of classical Boolean logic to fuzzy logic a priori. Those rules do not arise from observation, data collection, and professional experience but from philosophical reasoning. Thus it can be argued that those implication operators have been founded in a rationalistic manner. Rationalism in philosophy is the view that regards reason as the chief source and test of knowledge [2].

In the history of philosophy, starting from the controversy of Plato and Aristotle, there was a controversy between the two movements of rationalism and empiricism. Empiricism is the view that all concepts originate in experience [5]. Knowledge originates in what one can sense and there is no a priori knowledge. In this view human is described by John Locke as a blank table (*tabula rasa*), a phrase inspired by the works of Aristotle. From this scope, since a fuzzy implication function resembles knowledge about a correlation between two variables, it is possible the implication to be defined empiristicly.

Establishing a fuzzy implication relation between two fuzzy sets with the empiristic method requires data collection. The properties of this relationship must be constructed from the data, without a priori knowledge and constraints. Thus any implication function that abides to the aforementioned axioms cannot be considered as empiristic, but only rationalistic as the axioms effect the function significantly. Observed data can only come second, calibrating the function or distinguishing the most efficient from the available choices. Therefore any implication that is defined with the empiristic method will not satisfy the axioms, and so it will not be a logical implication function in the classical way of thinking.

Defining Empiristic Fuzzy Implication

Assume two variables $x \in X$ and $y \in Y$ and two linguistic variables $A : X \to [0, 1]$ and $B : Y \to [0, 1]$, with their corresponding membership functions. Suppose even that the variables x and y have some degree of correlation and there are n observed pairs of values, (x_i, y_i). It is possible to study the implication $x = A \Rightarrow y = B$ with the empiristic method, based on the data. If, for example, it is observed that for $a, b, c, d \in [0, 1]$ any time that the membership value of $A(x)$ resides in the closed interval $[a, b]$, $A(x) \in [a, b]$, then $B(y) \in [c, d]$, it is reasonable to assume that the fuzzy implication $A \Rightarrow B$ for $[a, b] \Rightarrow [c, d]$ is true with a degree of truth one, as it happens every time. Accordingly if there is no observation where $A(x) \in [a, b]$ and $B(y) \in [c, d]$, then the implication $[a, b] \Rightarrow [c, d]$ does not hold, i.e., it is true with degree zero. Furthermore if for $a, b, c, d, e, f \in [0, 1]$, when $A(x) \in [a, b]$, there are more observations for $B(y) \in [c, d]$ than for $B(y) \in [e, f]$, then the degree of truth for $[a, b] \Rightarrow [c, d]$ is greater than the degree of truth for $[a, b] \Rightarrow [e, f]$.

Definition 1 Let two variables $x \in X$ and $y \in Y$, two linguistic variables $A : X \to [0, 1]$ and $B : Y \to [0, 1]$, and there is n number of data pairs (x_i, y_i). Define E_{AB} the empiricist implication of linguistic variables based on the sample, a fuzzy relation between two sets of intervals that are inside the interval [0, 1] such that:

α) If for A' and B' intervals inside [0, 1]

$$E_{AB}(A', B') = 1 \Leftrightarrow \forall x_i : A(x_i) \in A' \Rightarrow B(y_i) \in B'$$

β) If for A' and B' intervals inside [0, 1]

$$E_{AB}(A', B') = 0 \Leftrightarrow \forall x_i : A(x_i) \in A' \Rightarrow B(y_i) \notin B'$$

γ) If for A', B', and C' intervals inside [0, 1]

$$E_{AB}(A', B') < E_{AB}(A', C')$$
$$\Leftrightarrow |\{x_i : A(x_i) \in A' \text{ and } B(y_i) \in B'\}| < |\{x_i : A(x_i) \in A' \text{ and } B(y_i) \in C'\}|$$

The first two of the axioms are based on classical logic, which must be verified by the fuzzy empiricist implication relation. Specifically, the first axiom means that two propositions $p(A(x_i) \in A')$ and $q(B(y_i) \in B')$ if it is true that whenever p is true, q is also true, then based on the data, the implication $p \Rightarrow q$ is true. In contrast if whenever p is true, q is not true, the implication $p \Rightarrow q$ is not true, and it stands for the second of the axioms. The third axiom includes three logical propositions $p(A(x_i) \in A')$, $q(B(y_i) \in B')$, and $r(B(y_i) \in C')$. If based on observations, when p is true, it is more likely r to be true than q to be true, then the implication $p \Rightarrow r$ is valid with a greater degree of truth than the implication $p \Rightarrow q$.

Computation of Fuzzy Empiristic Implication Relation

A method to calculate the membership function of a fuzzy empiristic implication relation derived from data observation is proposed. The computation is easy to comprehend and use and can be evaluated in short time when using a computer, even for large amounts of data.

Assume two variables $x \in X$ and $y \in Y$ and two linguistic variables $A : X \rightarrow [0, 1]$ and $B : Y \rightarrow [0, 1]$. Also there are pairs of observations (x_i, y_i) for $i = 1, 2 \dots n$. In order to evaluate the implication relation $A \Rightarrow B$, the observation pairs should be transformed to pairs of membership values $(A(x_i), B(y_i))$. The characteristic functions $A(x)$, $B(y)$ are considered known.

The process can be done even if the original data are not in the form (x_i, y_i), but they are already in the form $(A(x_i), B(y_i))$. The reason is that the implication does not examine directly the correlation between the variables x and y, but the correlation between the linguistic variables that arises. Furthermore, an empiristic implication relation can be calculated for linguistic variables that are based on qualitative rather than quantitative data, e.g., bad weather implies a desire for hot beverage. In such cases it is possible to find a connection directly to the fuzzy data that can be collected, for example, from questionnaires, without the study of the relationship of the linguistic variable "bad weather" to quantitative data such as temperature, sunshine, humidity, etc. So from qualitative data, it is possible to obtain useful conclusions.

Initially the pairs of membership values $(A(x_i), B(y_i))$ are calculated if they are not available, from the observation pairs (x_i, y_i). Afterward the pairs of membership values $(A(x_i), B(y_i))$ are divided into k and l in incompatible intervals shaping rectangular areas of the $[0, 1] \times [0, 1]$ The numbers of intervals k and l can be equal or not, and the determination of k and l can be made with binning techniques. The division is shown in Fig. 1, where the dots resemble pairs of membership values of observations and the lines form a possible division. So for $A(x)$ there has been a fragmentation in $A_1, A_2 \dots A_k$ intervals inside $[0, 1]$, and for $B(y)$ there has been a fragmentation in $B_1, B_2 \dots B_l$ intervals inside $[0, 1]$. The intervals are proposed to be of the same diameter, but it is not necessary, so if there is a

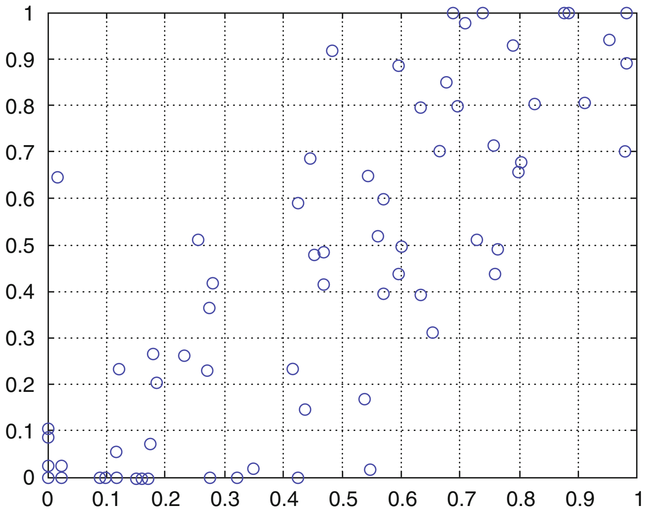

Fig. 1 Membership values of observation pairs and division

Table 2 Fuzzy empiristic implication relation matrix

	B_1	B_2	...	B_l
A_1	$I(A_1, B_1)$	$I(A_1, B_2)$		$I(A_1, B_l)$
A_2	$I(A_2, B_1)$	$I(A_2, B_2)$		$I(A_2, B_l)$
...				
A_k	$I(A_k, B_1)$	$I(A_k, B_2)$		$I(A_k, B_l)$

practical purpose, it can be uneven. Afterward for every possible pair of $A_i, B_j, i = 1, 2 \ldots k, j = 1, 2 \ldots l$, the degree of truth of the implication $E_{AB}(A_i, B_j)$ is computed as shown in (Eq. (17)). The formula chosen is quite similar to the law of conditional probability (Eq. (18)). The implications that are created this way satisfy the proposed axioms of the definition of empiristic implication relation.

$N((x, y) : A(x) \in A_i) \cap B(y) \in B_j$ is the number of pairs of observations for which $A(x) \in A_i$ and $B(y) \in B_j$, while $N(x : A(x) \in A_i)$ is the number of pairs of observations for which $A(x) \in A_i$. Thus the empiristic implication can be evaluated, and the matrix of the fuzzy relation is shown in Table 2.

$$E_{AB}(A_i, B_j) = \frac{N((x, y) : (A(x) \in A_i) \cap (B(y) \in B_j))}{N(x : A(x) \in A_i)} \tag{17}$$

$$P(A/B) = \frac{P(A \cap B)}{P(A)} \tag{18}$$

To carry out the required calculations, first it is necessary to create another matrix, counting the number of the instances for any possible pair of A_i, B_j. The divided data matrix is shown in Table 3 where $N(A_i, B_j)$ is the number of instances

Table 3 Divided data matrix

	B_1	B_2	...	B_l
A_1	$N(A_1, B_1)$	$N(A_1, B_2)$		$N(A_1, B_l)$
A_2	$N(A_2, B_1)$	$N(A_2, B_2)$		$N(A_2, B_l)$
...				
A_k	$N(A_k, B_1)$	$N(A_k, B_2)$		$N(A_k, B_l)$

where $A(x) \in A_i$ and $B(y) \in B_j$. Subsequently every element of the matrix is divided by the sum of the column, and thus the empiristic implication operator is calculated.

The separation in intervals as mentioned is another important step for calculating an implication that adequately describes the relationship between cause and effect. It is natural that in every case, the ideal number of intervals, and perhaps their diameters, when we have unequal intervals, cannot be determined a priori. This is because the ideal division depends on the number of the observations and on the relation between antecedent and consequent. If the division is done to a small number of intervals, the implication constructed becomes too biased and cannot provide much information. On the other hand, if the division is done to a larger number of intervals, the constructed implication may have significant variance, making any resulting model over-fitted, and thus unable to generalize and interpolate.

An efficient way to make the choice is with the sample split into two groups randomly and calculating the relation for both groups separately. Starting from small k and l numbers, the two matrices that describe the same phenomenon should have few differences. In large numbers of k and l, the two matrices should be very different from one another indicating that the created relation is over-fitted. By successively increasing the numbers k and l, calculating the relations and their differences, it is possible to find the ideal division for the data.

If the sample is limited, the use of rule of Sturges is proposed to calculate the appropriate number of intervals based on (Eqs. (19)–(20)), where n is the sample size [10, 11]. In such problems it is also suggested for k and l to be equal and for the intervals to be of equal diameter. In this case, when separating the [0, 1] interval into same length intervals, the diameter d of each interval is given by (Eq. (21)):

$$k = 1 + \log_2 n \tag{19}$$

or

$$k = 1 + 3.32 \log n \tag{20}$$

$$d = \frac{1}{k} \tag{21}$$

Finally, it should be clear that the results of such a process cannot be considered absolutely safe in every case because the sample used has an important role in the outcome. So the quality of the sample firstly and the quality of the fuzzification secondly impact the quality of the resulting implication relation. Of course, most methods that have to do with data are vulnerable to biased or incorrect data.

Application

Selection of Logical Fuzzy Implication Through the Empiricist

As it has been noted, the empiristic fuzzy implication relation that has been defined does not satisfy the established logical fuzzy implication axioms, and therefore it cannot resemble a logical fuzzy implication operator. Furthermore the empiristic implication is a relation between intervals and not a relation between numbers as the logical fuzzy implication functions are. So there can be no direct comparison between the two. However the two can be compared in the procedures of approximate reasoning.

Assume that there are two variables $x \in X$, $y \in Y$ and two linguistic variables $A(x) : X \to [0, 1]$, $B(y) : Y \to [0, 1]$. Also there is a relation $R(x, y) = I(A(x), B(y))$, where $I(A(x), B(y))$ is a logical fuzzy implication function. Then using the generalized modus ponens scheme, approximate reasoning has provided the compositional rule of inference, so if $x \in A'$, where A' is a fuzzy set, y belongs to a fuzzy set B' computed by (Eq. (22)).

$$B'(y) = \sup_{x \in X} \min[A'(x), R(x, y)] \tag{22}$$

Assume that for the aforementioned variables x, y and for the linguistic variables $A(x), B(y)$ there is a data set of pairs of observations. In this case it is possible to calculate an empiristic fuzzy implication relation. Utilizing the empiristic implication, it is possible to argue that when $\forall x : A(x) \in A'$, y is such that $B(y) \in B_1$ with a degree of truth $E_{AB}(A', B_1)$, $B(y) \in B_2$ with a degree of truth $E_{AB}(A', B_2)$, etc. While the values calculated by the compositional rule of inference are results of a rationalistic manner of reasoning, the corresponding values of the empiristic implication, when the data set is proper, are results of observation. Thus the empiristic implication better resembles reality. So it is possible to measure the difference between the logical and the empiristic implication, for any A_i and B_j pair of the empiristic relation.

It is plausible that this process can be a criterion for choice of logical fuzzy implication functions. So to make such a choice between some logical fuzzy implication functions in a particular problem, for which observation data is available, the difference of each of the logical implication functions to the empiristic implication can be measured, and the one that is closer to the empiristic implication should be the appropriate logical implication function. Such a calculation can be made with results of the compositional rule of inference and any approximate reasoning scheme as modus ponens, modus tollens, hypothetical reasoning, etc. Assuming that the implication to be chosen will be used on a real-life problem, it is preferable to use the reasoning scheme that is useful for the specific problem.

This implication selection process is of a great theoretical and practical interest. The logical implication decided by this process results is something of a compromise between rationalistic and empiricist logic, since it is a relationship structured with all the constraints dictated by the logical reasoning, but selected for good behavior with respect to the data observed in the physical world.

Algorithm

First step of the process is to calculate the empiristic fuzzy implication relationship based on the data as discussed in the previous subchapter. Result of the process is the relationship matrix of the implication.

Second step is to calculate a corresponding table for each of the logical implication functions in order to be compared. The calculation will be done using the compositional rule of inference, for a reasoning scheme as generalized modus ponens. The table has to have the same dimensions as the empiricist implication relation.

Third step is to count the difference between the matrices using some metric and chose the optimal implication function.

In the relation matrix of the empiristic fuzzy implication, the element that resides in the ith row and the jth column is the degree of truth in implying that $E(A_i, B_j)$. So when x is such that $A(x) \in A_i$, it implies that y is such that $B(y) \in B_j$, with a degree of truth $E(A_i, B_j)$. It is noted that any A_i, B_j set for $i = 1, 2 \ldots k$, $j = 1, 2 \ldots l$ is a classical, crisp set. When $A(x)]inA_i$, from the CRI (Eq. (23)) is:

$$B'(y) = \sup_{x \in X} \min[A'_i, I(A(x) \in A_i, B(y))] \tag{23}$$

This formula, for crisp sets A_i, becomes (Eq. (24)):

$$B'(y) = \sup_{x \in X}[I(A(x), B(y))] \tag{24}$$

Note that for the compositional rule of inference formula, any t-$norm$ is applicable. Since sets A_i are crisp and for any t-$norm$ $T(0, a) = a$, the selection of t-$norm$ is of no consequence.

Most logical implication functions meet the first two axioms (1 and 2), so they are decreasing for the first variable and increasing for the second. The algorithm is developed for implication functions for who this is valid. Otherwise the complexity of operations is increased, although not very much. Thus, because any A_i interval is bounded, with an infimum (assume a_i) on the left boarder, (Eq. (24)) becomes (Eq. (25)):

$$B'(y) = I(a_i, B(y)) \tag{25}$$

So when $x \in A'_i$, $B'(y)$ is calculated according to the formula above. But to describe the implication relation matrix corresponding to each element of the empiricist implication relation matrix the question: "When $x \in A'_i$, with what degree of truth is it implied that $y \in B'_j$?". So the degree of truth $I(A_i, B_j)$ is calculated as follows (Eq. (26)):

$$I(A_i, B_j) = \cup_{B(y) \in B_j} I(a_i, B(y)) \tag{26}$$

As *t-conorm* can be used the maximum. This maximum is very easy to find for logical implication functions for which the second axiom is satisfied, i.e., increasing the second variable. Then, since $B(y)$ belongs in a bounded interval with supremum being the right boarder (assume b_j), (Eq. (26)) becomes:

$$I(A_i, B_j) = I(a_i, b_j) \tag{27}$$

Thus, the appropriate matrix is calculated for each logical implication function from (Eq. (27)), so that the element located at the ith row and jth column has the value $I(a_i, b_j)$.

Finally the deviation of any logical implication function from the empiristic has to be calculated. It is proposed that the norm of the difference of the two matrices is calculated. The implication function with the smallest deviation from the empiristic is selected to be the one that represents more accurately the relationship between linguistic variables according to the given data set.

Conclusions and Further Research

The fuzzy implication functions that can be used are numerous, and their properties have important influence on the quality of the models and the understanding of the phenomena described. A new class of fuzzy implication relations is founded, named empiristic fuzzy implication relations. These relations are inspired by the philosophical movement of empiricism, constructed only from observed data, with no a priori restrictions and conditions. An easy and clear way of calculating an empiristic fuzzy implication relation is described that can be functional even for huge volumes of data.

Finally fuzzy implication function selection problem was analyzed. The problem has been studied by many researchers in the world, and there is no unanimous solution to address it. A fuzzy implication function selection method is recommended, based on empiristic implication relations. For comparison, observational data is needed based on which the empiricist implication relations are calculated, and the logical implication relationship with the less deviation from the empiristic is chosen to be the one that accurately represents the data.

In the future the construction of fuzzy inference systems will be studied, using only the fuzzy empiristic implication relation. These systems will consist of a base of fuzzy rules "If ... then ...," each one of which will be an empiristic implication, computed from data observations.

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