Fuzzy Rings and Fuzzy Polynomial Rings

S. Melliani, I. Bakhadach and L. S. Chadli

Abstract In this paper, we introduce the notion of a ring of fuzzy points, and study some basic properties and the relationship between this set and the classical ring R. We also define the fuzzy polynomial rings and fuzzy algebraic elements.

Keywords Fuzzy points · Fuzzy subrings · Fuzzy polynomials

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [8] in 1965, which is a generalization of the crisp set. Since its conception, the theory of fuzzy set has developed in many directions and is finding applications in a wide variety of fields. Rosenfeld [7] used this concept to develop the theory of fuzzy subgroup. Liu [2] introduced the concept of fuzzy ring in 1982. Pu and Liu [6] introduced the notion of fuzzy points, Kyung ho kim [1] discussed the relation between the fuzzy interior ideals and the semigroup $\underline{R}$ the subset of all fuzy points of R. Based on these researches we have developed the notion of rings on the set of points defined by Pu and Liu [6]. We have also introduced and discussed the notion of polynomials on this ring.

Here is the summary of the paper. In Sect. 3, we define the subring consisting the set of all fuzzy points and discuss some basic properties of this ring. Based on the ring defined in Sect. 3, we introduce and investigate the fuzzy polynomial rings in Sect. 4.

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2 Preliminaries

In this section, we recall some definitions and results which will be used in the sequel.

Definition 1. [8] Let E be a non-empty set. A fuzzy subset of the E is a function $\mu : E \to [0, 1]$.

Definition 2. [5] Let μ be a fuzzy subset of E. For $\alpha \in [0, 1]$, define μ_α as follows:

$$\mu_\alpha = \{x | x \in R, \mu(x) \geq \alpha\}.$$

μ_α is called the α-**cut** (or α-**level set**) of μ.

Property 1. [5] Let $\mu, \nu \subset R$ be a fuzzy subsets. Then we have

1. $\mu \subseteq \nu, \alpha \in [0, 1] \Longrightarrow \mu_\alpha \subseteq \nu_\alpha$,
2. $\alpha \leq \beta \, \alpha, \beta \in [0, 1] \Longrightarrow \mu_\beta \subseteq \mu_\alpha$,
3. $\mu = \nu, \Leftrightarrow \mu_\alpha = \nu_\alpha$, for each $\alpha \in [0, 1]$.

Definition 3. [4] Let R be a ring with identity. Then $\mu \subset R$ is called a fuzzy subring if and only if

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$;
(ii) $\mu(xy) \geq \mu(x) \wedge \mu(y), \forall x, y \in R$ and
(iii) $\mu(1) = 1$.

Property 2. Let R be a ring and μ be a fuzzy subring of R. Then we have:

1. For each $x \in R$, $\mu(0) \geq \mu(x)$.
2. If $x, y \in R$ and $\mu(x - y) = 0$, then $\mu(x) = \mu(y)$.
3. For each $x \in R$, $\mu(x) = \mu(-x)$.

Definition 4. [6] Let A be a non-empty set and $x_\alpha : A \longrightarrow [0, 1]$ a fuzzy subset of A with $x \in A$ and $\alpha \in (0, 1]$ defined by

$$x_\alpha(y) = \begin{cases} \alpha & if \quad x = y \\ 0 & if \quad x \neq y \end{cases}$$

then x_α is called a fuzzy point (singleton).

Definition 5. Let μ be a fuzzy subring of R, and x_t be a fuzzy point of R. We write $x_t \in \mu$ to express that $\mu(x) \geq t$, by the principal extension of Zadeh we have

$$x_t + y_s = (x + y)_{t \wedge s}$$
$$x_s y_t = (xy)_{t \wedge s}.$$

Now we will first evolve some results on the fuzzy ring using the membership functions and we will also give a necessary and sufficient condition for $F_\mu(R)$, the set of fuzzy points of μ to be a ring.

3 Fuzzy Subrings

The following theorem gives us the relationship between a fuzzy subring and all of its α-cuts.

Theorem 1. *Let μ be a fuzzy subset of R, then μ is a fuzzy subring of R if and only if μ_t is a subring of R, for each $t \in [0, \mu(0)]$.*

Proof. It is clear that $\mu_t = \{x \in R, \quad \mu(x) \geq t\}$ is a non-empty subset of R.

Let $x, y \in \mu_t$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. Since μ is a fuzzy subring of R, then we have $\mu(x - y) \geq \mu(x) \wedge \mu(y)$. This implies that $\mu(x - y) \geq t$, hence $x - y \in \mu_t$. Similarly, $\mu(xy) \geq \mu(x) \wedge \mu(y)$ then $\mu(xy) \geq t$. Hence $xy \in \mu_t$. Therefore, μ_t is a subring of R.

Conversely, let $x, y \in R$ and let $\mu(x) = t_1$ and $\mu(y) = t_2$. Then $x \in \mu_{t_1}$ and $y \in \mu_{t_2}$. Now suppose that $t_2 > t_1$, this implies that $\mu_{t_2} \subseteq \mu_{t_1}$. In this case, $y \in \mu_{t_2} \subseteq \mu_{t_1}$ since $x, y \in \mu_{t_1}$. So we have $x - y \in \mu_{t_1}$ and $xy \in \mu_{t_1}$; hence $\mu(x - y) \geq t_1 = \mu(x) \wedge \mu(y)$ and $\mu(xy) \geq t_1 = \mu(x) \wedge \mu(y)$.

Theorem 2. *Let μ be a fuzzy subset of R. Then μ is a fuzzy subring of R if and only if, for each point $x_t, y_s \in \mu$, we have $x_t - y_s \in \mu$ and $x_t.y_s \in \mu$.*

Proof. Suppose that μ is a fuzzy subring of R. Let $x, y \in R$ and $x_t, y_s \in \mu$. Then

$$\begin{aligned} \mu(x - y) &\geq \mu(x) \wedge \mu(y) \\ &\geq t \wedge s \end{aligned}$$

this implies that $x_t - y_s \in \mu$.
Similarly, since μ is a fuzzy subring of R,
we have

$$\begin{aligned} \mu(xy) &\geq \mu(x) \wedge \mu(y) \\ &\geq t \wedge s \end{aligned}$$

hence $x_t.y_s \in \mu$.

Conversely, let $x, y \in R$. We have

$$\mu(x) \geq \mu(x) \wedge \mu(y) \ \ and \ \ \mu(x) \geq \mu(x) \wedge \mu(y)$$

then

$$x_{\mu(x)\wedge\mu(y)} \in \mu \ \ and \ \ y_{\mu(x)\wedge\mu(y)} \in \mu.$$

Using the assumption we have

$$x_{\mu(x)\wedge\mu(y)} - y_{\mu(x)\wedge\mu(y)} \in \mu \ \ and \ \ x_{\mu(x)\wedge\mu(y)}.y_{\mu(x)\wedge\mu(y)} \in \mu$$

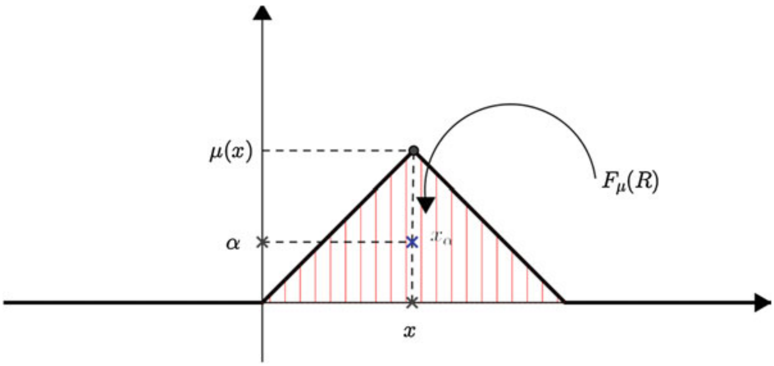

Fig. 1 Graphical representation of the set $F_\mu(R)$

This implies that

$$(x-y)_{\mu(x)\wedge\mu(y)} \in \mu \ \ and \ \ (xy)_{\mu(x)\wedge\mu(y)} \in \mu$$

Therefore, $\mu(x-y) \geq \mu(x) \wedge \mu(y) \ \ and \ \ \mu(xy) \geq \mu(x) \wedge \mu(y)$. Thus μ is a fuzzy subring of R.

Let $\underline{R}$ be the subset of all fuzzy points of R given by [6]. We set $F_\mu(R) = \{x_\alpha \in \underline{R} \mid \ \mu(x) \geq \alpha\}$ (Fig. 1).

Theorem 3. *Let R be a ring with identity, and let μ be a fuzzy subset of R. If μ is a fuzzy subring of R, then $(F_\mu(R), +, \times)$ is a ring.*

Proof. Let $x_t, y_s, z_u \in F_\mu(R)$. We have $x_t + y_s = (x+y)_{t\wedge s} \in F_\mu(R)$. Hence $F_\mu(R)$ is closed under the operation $+$. For associativity of $+$ we have

$$\begin{aligned} x_t + (y_s + z_u) &= x_t + (y+z)_{s\wedge u} \\ &= (x + (y+z))_{t\wedge(s\wedge u)} \\ &= ((x+y)+z)_{(t\wedge s)\wedge u} \\ &= (x_t + y_s) + z_u. \end{aligned}$$

Then $+$ is associative.

We have also $\mu(0) \geq \mu(1) = 1$. Therefore, $0_s \in F_\mu(R)$ for all $s \in (0, 1]$, for the symmetric element, we have $\mu(-x) \geq \mu(x) \geq t$, then $-x_t \in F_\mu(R)$ and $x_t - x_t = (x-x)_t = 0_t$ for all $t \in (0, 1]$.

Furthermore,

$$x_t + y_s = (x+y)_{t\wedge s} = (y+x)_{s\wedge t} = y_s + x_t.$$

Thus $(F_\mu(R), +)$ is an abelian group.

Since $x_t \times y_s = (xy)_{t\wedge s} \in F_\mu(R)$, so $F_\mu(R)$ is closed under "$\times$". Finally, as we have

$$\begin{aligned} x_t \times (y_s \times z_u) &= x_t \times (y \times z)_{s\wedge u} \\ &= (x \times (y \times z))_{t\wedge(s\wedge u)} \\ &= ((x \times y) \times z)_{(t\wedge s)\wedge u} \\ &= (x_t \times y_s) \times z_u \end{aligned}$$

and

$$\begin{aligned} x_t \times (y_s + z_u) &= (x \times (y + z))_{t\wedge(s\wedge u)} \\ &= (xy + xz)_{t\wedge s\wedge u} \\ &= (xy)_{t\wedge s} + (xz)_{t\wedge u}. \end{aligned}$$

it follows that $(F_\mu(R), +, \times)$ is a ring.

Proposition 1. *Let R be a commutative ring. Let μ and ν be two fuzzy subrings of R such that $\mu \subset \nu$. Then $F_\mu(R)$ is a subring of $F_\nu(R)$.*

Proof. Since μ, ν are fuzzy subrings of R, so $F_\mu(R)$ and $F_\nu(R)$ are rings by Theorem 3. Let $x_t \in F_\mu(R)$. Then $\mu(x) \geq t$, since $\mu \subset \nu$, then $\nu(x) \geq t$. This implies that $F_\mu(R) \subset F_\nu(R)$. In addition, we have $1_1 \in F_\mu(R)$.

Definition 6. Let μ be a fuzzy subring of R. Then the singleton $a_t \neq 0_t \in F_\mu(R)$ with $t \in (0, 1]$, is called a fuzzy zero-divisor if there exists a nonzero fuzzy singleton $b_s \in F_\mu(R)$ such that $a_t.b_s = 0_\lambda$ where $\lambda = \min(s, t)$.

Definition 7. Let $F_\mu(R)$ be a ring. We say that $F_\mu(R)$ is an integral ring if it has no zero-divisor fuzzy singletons, that is if ($x_t.y_s = 0_{t\wedge s}$, then $x_t = 0_t$ or $y_s = 0_s$).

Theorem 4. *$F_\mu(R)$ is an integral ring if and only if R is an integral domain.*

Proof. Let $x_t, y_t \in F_\mu(R)$ with $x_t.y_s = 0_{t\wedge s}$. We must show that $x_t = 0_t$ or $y_s = 0_s$. Note that $x_t.y_s = 0_{t\wedge s}$ implies that, for all $z \in R$, $(xy)_{t\wedge s}(z) = 0_{t\wedge s}(z)$. Hence

$$(t \wedge s)\chi_{\{xy\}}(z) = (t \wedge s)\chi_{\{0\}}(z)$$

Since, for each $z \in R$,

$$\chi_{\{xy\}}(z) = \chi_{\{0\}}(z)$$

so $xy = 0$ and since R is an integral domain we have $x = 0$ or $y = 0$. Hence $x_t = 0_t$ or $y_s = 0_s$ for all $t, s \in (0, 1]$.

Conversely, suppose that $F_\mu(R)$ is an integral ring. Let $xy = 0$ for some $x, y \in R$. Since $xy = 0$ we have $(xy)_t = 0_t$ for every $t \in (0, 1]$. This implies that $x_t = 0_t$ or $y_t = 0_t$. So, for each $u, v \in R$, $x_t(u) = 0_t(u)$ or $y_t(v) = 0_t(v)$. Consequently, we have

$$x_t(u) = \begin{cases} t & if \quad x = u \\ 0 & if \quad x \neq u \end{cases} = \begin{cases} t & if \quad 0 = u \\ 0 & if \quad 0 \neq u \end{cases} = 0_t(u)$$

or

$$y_t(v) = \begin{cases} t & if \quad y = v \\ 0 & if \quad y \neq v \end{cases} = \begin{cases} t & if \quad 0 = v \\ 0 & if \quad 0 \neq v \end{cases} = 0_t(v)$$

Hence

$$x_t(u) = 0_t(u) \begin{cases} t & if \quad x = u = 0 \\ 0 & if \quad x \neq u \neq 0 \end{cases}$$

or

$$y_t(v) = 0_t(v) \begin{cases} t & if \quad y = v = 0 \\ 0 & if \quad y \neq v \neq 0 \end{cases}.$$

Therefore, $x = 0$ or $y = 0$.

4 Fuzzy Polynomials Ring

In this section, we will give a new definition of a fuzzy polynomials based on the ring of fuzzy points defined in Sect. 3. Then we will discuss some basic properties of this new concept.

Definition 8. A fuzzy polynomial with one indeterminate on $F_\mu(R)$ is a set of sequences $(a_{t_0}, a_{t_1}, a_{t_2}....) = (a_{t_k})_{k\in\mathbb{N}}$ with $a_{t_k} \in F_\mu(R)$ such taht there exists $n \in \mathbb{N}$ for all $p \geq n,\ a_{t_p} = 0_{t_p}$. So the fuzzy polynomial is defined as $(a_{t_0}, a_{t_1}, a_{t_2}, ...a_{t_i},$ $0_s, ..., 0_s)$ with $t_i, s \in (0, 1]$. The set of all fuzzy polynomials with one indeterminate on $F_\mu(R)$ is denoted by $F_\mu(R)[X]$.

Let us now define some operations on the fuzzy polynomials.

Let $P, Q \in F_\mu(R)[X]$. Then, $P = (a_{t_k})_{k\in\mathbb{N}}$ such that there exists $n \in \mathbb{N}$ with $a_{t_p} = 0_{t_p}$ for each $p > n$, and $Q = (b_{s_k})_{k\in\mathbb{N}}$ such that there exists $m \in \mathbb{N}$ with $b_{s_p} = 0_{s_p}$ for all $p > m$.

Addition "(+)"
Define $P + Q = (c_{\alpha_k})_{k\in\mathbb{N}}$ such that $c_{\alpha_k} = a_{t_k} + b_{s_k} = (a + b)_{t_k \wedge s_k}$ and $c_{\alpha_k} = 0_{\alpha_k}$ for all $p > \max(n, m)$. It is obvious that $P + Q \in F_\mu(R)[X]$.

Multiplication "($\times$)"

The multiplication $P \times Q$ is defined by $P \times Q = (d_{\beta_k})_{k\in\mathbb{N}}$ such that $d_{\beta_k} = \sum_{i+j=p} a_{t_i} b_{s_j}$ with $\beta_k = \min_{0\geq i,j\geq k}\{t_i, s_j\}$ with $d_{\beta_p} = 0_{\beta_p}$ for each $p > n+m$ because $p = i + j > n+m$ implies $i > n$ or $j > m$. This implies that $a_{t_i} = 0_{t_i}$ or $b_{s_j} = 0_{s_j}$.

Remark 1. Let $P, Q \in F_\mu(R)[X]$ be two fuzzy polynomials. Then $P = Q$ if and only if $a_{t_i} = b_{s_i}$, for each $i \in \mathbb{N}$. The zero fuzzy polynomial is defined as $(a_{t_i})_{i\in\mathbb{N}}$ such that $a_{t_i} = 0_{t_i}$, for each $i \in \mathbb{N}$.

Proposition 2. $(F_\mu(R)[X], +, \times)$ *is a comutative ring.*

Proof. The zero element is $(0_s, 0_s, 0_s, ..., 0_s)$ with $s \in (0,1]$. For all $P, Q, R \in F_\mu(R)[X]$,

$$\begin{aligned}(P+Q)+R &= (a_{t_i} + b_{s_i}) + c_{k_i}\\ &= (a+b)_{t_i\wedge s_i} + c_{k_i}\\ &= \big((a+b)+c\big)_{(t_i\wedge s_i)\wedge k_i}\\ &= \big(a+(b+c)\big)_{t_i\wedge(s_i\wedge k_i)}\\ &= a_{t_i} + (b+c)_{s_i\wedge k_i}\\ &= a_{t_i} + (b_{s_i} + c_{k_i})\\ &= P + (Q+R).\end{aligned}$$

Hence $+$ is associative. In a similar way, we can prove that $P+Q = Q+P$. The symmetrical element is given by

$$-P = (-a_{t_k})_{k\in\mathbb{N}} \in F_\mu(R)[X]$$

Indeed

$$P + (-P) = (0_s, 0_s, 0_s, ..., 0_s)$$

with $s \in (0,1]$. In addition, $(F_\mu(R)[X], \times)$ is a semigroup. Using the fact that $(F_\mu(R), \times)$ is a semigroup and the definition of "$\times$" in $F_\mu(R)[X]$ we can easily show that

$$P \times (Q \times R) = (P \times Q) \times R$$

and $P \times Q = Q \times P$ and $P \times (Q+R) = P \times Q + P \times R$. Consequently $(F_\mu(R)[X], +, \times)$ is a commutative ring with identity. The identity is given by $(1_1, 0_s, 0_s, ..., 0_s)$ since

$$P \times (1_1, 0_s, 0_s, ..., 0_s) = (a_{t_0}, a_{t_1}, ..., a_{t_n}, 0_s, ..., 0_s) = P.$$

Denote by $X = (0_s, 1_1, 0_s, ..., 0_s)$, with $s \in (0, 1]$ and call it one indeterminate. By convention,

$$X^0 = 1_1 = (1_1, 0_s, 0_s, ..., 0_s);\ X^2 = XX = (0_s, 1_1, 0_s, ..., 0_s)(0_s, 1_1, 0_s, ..., 0_s) = (0_s, 0_s, 1_1, 0_s, ..., 0_s)$$

and

$$X^n = (\overbrace{0_s, , 0_s, ...0_s}^{n\, times\, 0_s}, 1_1, ..., 0_s).$$

Let $P = (a_{t_k})_{k\in\mathbb{N}} \in F_\mu(R)[X]$. Then

$$P = (a_{t_0}, a_{t_1}, a_{t_2}, 0_s, ..., 0_s) = a_{t_0}(1_1, 0_s, 0_s, ..., 0_s) + a_{t_1}(0_s, 1_1, 0_s, ..., 0_s) + ... + a_{t_n}(\overbrace{0_s, ...0_s}^{n\, times\, 0_s}, 1_1, ..., 0_s).$$

Hence, the fuzzy polynomial P is written in the form $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + ... + a_{t_n}X^n$.

Definition 9. We say that $P \in F_\mu(R)[X]$ is a fuzzy polynomial on $F_\mu(R)$ if there exists $a_{t_i} \in F_\mu(R)$ such that $P = \sum_{i=0}^{i=n} a_{t_i}X^i$.

Definition 10. Let $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + ... + a_{t_n}X^n$ be a nonzero fuzzy polynomial. Then there exists a nonzero coefficient of $a_{t_0}, a_{t_1}, ..., a_{t_n}$.

Definition 11. (fuzzy degree) Let $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + ... + a_{t_n}X^n \in F_\mu(R)[X]$. The fuzzy degree of P denoted by $deg(P)$ or d^o is defined as the maximal number n such that $a_{t_n} \neq 0_{t_n}$. In this case a_{t_n} is called the leading coefficient of P.

Proposition 3. *Let $F_\mu(R)$ be an integral ring and P and Q be two polynomials of $F_\mu(R)[X]$. Then, we have*

(a) $d^o(P + Q) \leq \max(d^o(P), d^o(Q))$.
(b) $d^o(P.Q) = (d^o(P) + d^o(Q))$.

Proof. (a) Suppose that $d^o(P) = n$ and $d^o(Q) = p$. Then $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + ... + a_{t_n}X^n$ and $Q = b_{s_0} + b_{s_1}X + b_{s_2}X^2 + ... + b_{s_p}X^p$. Suppose that $n > p$. Then

$$P + Q = (a + b)_{t_0\wedge s_0} + (a + b)_{t_1\wedge s_1}X + ... + a_{t_n}X^n$$

Hence $d^o(P + Q) = n = \max d^o(P), d^o(Q)$. If $n < p$ we have $d^o(P + Q) = p = \max d^o(P), d^o(Q)$. If $n = p$, then

$$P + Q = (a + b)_{t_0\wedge s_0} + (a + b)_{t_1\wedge s_1}X + ... + (a + b)_{t_n\wedge s_n}X^n.$$

Consider the following cases:

(i) if $(a+b)_{t_n \wedge s_n} \neq 0_{t_n \wedge s_n}$, then $d^o(P+Q) = n = \max d^o(P), d^o(Q)$.
(ii) if $(a+b)_{t_n \wedge s_n} = 0_{t_n \wedge s_n}$, then $P+Q = (a+b)_{t_0 \wedge s_0} + (a+b)_{t_1 \wedge s_1} X + ... + (a+b)_{t_{n-1} \wedge s_{n-1}} X^{n-1}$, hence $d^o(P+Q) \leq n-1 \leq \max(d^o(P), d^o(Q))$.

(b) $P \times Q = (ab)_{t_0 \wedge s_0} + ((ab)_{t_0 \wedge s_1} + (ab)_{t_1 \wedge s_0})X + ... + (ab)_{t_n \wedge s_p} X^{n+p}$, since $a_{t_n} \neq 0_{t_n}$ and $b_{s_p} \neq 0_{s_p}$ then $a_{t_n} b_{s_p} \neq 0_{t_n \wedge s_p}$. Therefore, $d^o(P \times Q) = d^o(P) + d^o(Q)$.

Remark 2. If P is a zero polynomial we denote by convention $d^o(P) = -\infty$. If $F_\mu(R)$ is a non integral ring, then $d^o(PQ) \leq d^o(P) + d^o(Q)$.

Definition 12. Let μ be a fuzzy subring of R. An extension of μ is a fuzzy subring ν of R, such that $\mu \subset \nu$.

Example 1. Define μ and ν as follows:

$$\nu : \begin{cases} \mathcal{M}_2(\mathbb{R}) \longrightarrow [0,1] \\ x \longmapsto \begin{cases} 1 & if \quad x = 0 \\ \frac{1}{2} & if \quad x \neq 0 \end{cases} \end{cases}$$

$$\mu : \begin{cases} \mathcal{M}_2(\mathbb{R}) \longrightarrow [0,1] \\ x \longmapsto \begin{cases} 1 & if \quad x = 0 \\ \frac{1}{4} & if \quad x \neq 0 \end{cases} \end{cases}$$

It is easy to show that $\mu \subset \nu$. Hence ν is an extension of μ.

Definition 13. We say that $\alpha_s \in F_\mu(R)$ is a zero of $P \in F_\mu(R)[X]$ if $P(\alpha_s) = \sum_{i=0}^{i=n} a_{t_i} \alpha_s^i = 0_\beta$ such that $\beta \leq s$.

Let $I(b_t) = \{P \in F_\mu(R)[X], P(b_t) = 0_s\}$. It is clear that $I(b_t)$ is an ideal of $F_\mu(R)[X]$.

Definition 14. $b_t \in \nu$ is called an algebraic element if $I(b_t) \neq \{0\}$. Otherwise b_t is called a transcendent element.

Note that if $a_{t_i} = 1_1$ then $b_t \in \nu$ is called an integral element.

Theorem 5. *Let R be a ring. Then R is an integral domain if and only if $F_\mu(R)[X]$ is an integral ring.*

Proof. Suppose that R is an integral domain. According to the Theorem 4, $F_\mu(R)$ is an integral domain. Let $P, Q \in F_\mu(R)[X]$ be such that $P \neq 0$ and $Q \neq 0$. let $a_t X^p$ and $b_s X^q$ be the monomials of more high degrees of P and Q, respectively. The term of degree $p+q$ of QP is $a_t b_s X^{p+q}$. Conversely, let $a_t, b_s \in F_\mu(R)$ be such that $a_t b_s = 0_{t \wedge s}$. We have $a_t, b_s \in F_\mu(R)[X]$ hence $a_t = 0_t$ or $b_s = 0_s$ because $F_\mu(R)[X]$ is an integral ring. So we have the result.

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