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Homological and Combinatorial Methods in Algebra

SAA 4, Ardabil, Iran, August 2016

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Preface

The 4th SAA 2016, fourth in the series of international seminars on algebra and its applications, was held at the Department of Mathematics and Applications, University of Mohaghegh Ardabili, Iran, during 9–11 August, 2016. Following the tradition of its predecessors, this meeting gathered researchers around topics in present recent progress and new trends in algebra and its applications. A total number of 105 participants from several countries have attended the conference in the University of Mohaghegh Ardabili (UMA). The main goal of the seminar is to introduce Iranian faculty and graduate students to important ideas in the mainstream of algebra. A secondary goal is for Iranian mathematicians to open channels of communication with algebraists from around the globe and eventually begin collaborative research projects. The audience was multidisciplinary allowing the participants to exchange diversified ideas and to show the wide attraction of algebra and its applications. There were two kinds of lectures: invited talks of one hour presented by distinguished experts and half an hour contributions.

The Scientific Committee consisted of Kamran Divaani-Azar (Alzahra University), Hossien Abdolzadeh, Jafar Azami—Chair, Kamal Bahmanpour, Adel P. Kazemi, Ahmad Khojali, Majid Rahro-Zargar, Ahmad Yousefian Darani, and Naser Zamani all from UMA.

The Organizing Committee was constituted by Goudarz Sadeghi, Mohammad Narimani, Yousef Abbaspour, Daioush Latifi, Kazem Haghnejad, Hossein Abdolzadeh, Abbas Najati, and Ahmad Yousefian Darani (Chair) all from UMA.

We are particularly indebted to our plenary speakers: Moharam Aghapour (Arak University), Fariborz Azar Panah (Shahid Chamran University of Ahvaz), Ayman Badawi (American University of Sharjah), Reza Naghipour (University of Tabriz), Peyman Nasehpour (University of Tehran), Mohammad Reza Vedadi (Isfahan University of Technology), Roger Wiegand (University of Nebraska-Lincoln), Sylvia Wiegand (University of Nebraska Lincoln), Siamak Yassemi (University of Tehran), and Rahim Zaare-Nahandi (University of Tehran). Thanks are also due to the presenters of contributed papers, as well as everyone who attended for making the seminar a success. According to the evaluations of the scientific committee, there were several excellent talks presented by invited speakers.

The 4th SAA 2016 was sponsored by the UMA, and organized by the Faculty of Mathematics and Department of Mathematics and Applications, UMA. We would like to publicly acknowledge the financial support of the Vice-Chancellorship for Research and Technology of UMA, as well as the hospitality of the Faculty of Mathematics and Department of Mathematics and Applications of UMA. We are also very grateful for the secretarial help of Negin Karimi. Selected papers of 4th SAA 2016 are presented in the volume *Homological and Combinatorial Methods in Algebra* in the series Springer Proceedings of Mathematics & Statistics published by Springer. With the publication of this proceeding, we hope that a wider mathematical audience will benefit from the seminar research achievements and new contributions to the field of algebra and its applications. More details of the event can be found at http://uma.ac.ir//links/4saa.

Sharjah, United Arab Emirates — Ayman Badawi
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Contents

b-Symbol Distance Distribution of Repeated-Root Cyclic Codes

Hojjat Mostafanasab and Esra Sengelen Sevim

Abstract Symbol-pair codes, introduced by Cassuto and Blaum (Proc IEEE Int Symp Inf Theory, 988–992, 2010 [1]), have been raised for symbol-pair read channels. This new idea is motivated by the limitations of the reading process in high-density data storage technologies. Yaakobi et al. (IEEE Trans Inf Theory **62**(4):1541–1551, 2016 [8]) introduced codes for b-symbol read channels, where the read operation is performed as a consecutive sequence of $b > 2$ symbols. In this paper, we come up with a method to compute the b-symbol-pair distance of two n-tuples, where n is a positive integer. Also, we deal with the b-symbol-pair distances of some kind of cyclic codes of length p^e over F_{p^m}.

Keywords b-Symbol pair · Distance distribution · Cyclic codes

1 Introduction

Recently, it is possible to write information on storage devices with high resolution using advances in data storage systems. However, it causes a problem of the gap between write resolution and read resolution. Cassuto and Blaum [1, 2] laid out a framework for combating pair-errors, relating pair-error correction capability to a new metric called pair-distance. They proposed the model of symbol-pair read channels. Such channels are mainly motivated by magnetic-storage channels with high write resolution, due to physical limitations, each channel contains contributions from two adjacent symbols. Cassuto and Listsyn [3] studied algebraic construction of

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cyclic symbol-pair codes. Yaakobi et al. [9] proposed efficient decoding algorithms for the cyclic symbol-pair codes. Chee et al. [4, 5] established a Singleton-type bound for symbol-pair codes and constructed codes that meet the Singleton-type bound. Hirotomo et al. [7] proposed the decoding algorithm for symbol-pair codes based on the newly defined parity-check matrix and syndromes.

For this new channels, the codes defined as usual over some discrete symbol alphabet, but whose reading from the channel is performed as overlapping pairs of symbols. Let Ξ be the alphabet consisting of q elements. Each element in Ξ is called a symbol. We use Ξ^n to denote the set of all n-tuples, where n is a positive integer. In the symbol-pair read channel, there are in fact two channels. If the stored information is $x = (x_0, x_1, \ldots, x_{n-1}) \in \Xi^n$, then the symbol-pair read vector of x is

$$\pi(x) = [(x_0, x_1), (x_1, x_2), \ldots, (x_{n-2}, x_{n-1}), (x_{n-1}, x_0)],$$

and the goal is to correct a large number of the so called symbol-pair errors. The pair distance, $d_p(x, y)$, between two pair-read vectors x and y is the Hamming distance over the symbol-pair alphabet $(\Xi \times \Xi)$ between their respective pair-read vectors, that is, $d_p(x, y) = d_H(\pi(x), \pi(y))$. The minimum pair distance of a code $\mathscr{C}$ is defined as $d_p(\mathscr{C}) = \min\{d_p(x, y) | x, y \in \mathscr{C} \text{ and } x \neq y\}$. Accordingly, the pair weight of x is $\omega_p(x) = \omega_H(\pi(x))$. If $\mathscr{C}$ is a linear code, then the minimum pair-distance of $\mathscr{C}$ is the smallest pair-weight of nonzero codewords of $\mathscr{C}$. The minimum pair-distance is one of the important parameters of symbol-pair codes. This distance distribution is very difficult to compute in general, however, for the class of cyclic codes of length p^e over F_{p^m}, their Hamming distance has been completely determined in [6]. In [10], Zhu et al. investigated the symbol-pair distances of cyclic codes of length p^e over F_{p^m}.

For $b \geq 3$, the b-symbol read vector corresponding to the vector

$$x = (x_0, x_1, \ldots, x_{n-1}) \in \Xi^n$$

is defined as

$$\pi_b(x) = [(x_0, x_1, \ldots, x_{b-1}), (x_1, x_2, \ldots, x_b), \ldots, (x_{n-1}, x_0, \ldots, x_{b-2})] \in (\Xi^b)^n.$$

We refer to the elements of $\pi_b(x)$ as b-symbols. The b-symbol distance between x and y, denoted by $d_b(x, y)$, is defined as $d_b(x, y) = d_H(\pi_b(x), \pi_b(y))$. Similarly, we define the b-weight of the vector x as $\omega_H(\pi_b(x))$. In the analogy of the definition of symbol-pair codes, the minimum b-symbol distance of $\mathscr{C}$, $d_b(\mathscr{C})$, is given by $d_b(\mathscr{C}) = \min\{d_b(x, y) | x, y \in \mathscr{C} \text{ and } x \neq y\}$. For more information on these notions see [8].

We can rewrite [8, Proposition 9] for any arbitrary alphabet Ξ.

Proposition 1. *Let $x \in \Xi^n$ be such that $0 < \omega_H(x) \leq n - (b - 1)$. Then*

$$\omega_H(\mathscr{C}) + b - 1 \leq \omega_b(\mathscr{C}) \leq b \cdot \omega_H(\mathscr{C}).$$

Referring to Proposition 1, we see that:

Corollary 1. *Let $\mathscr{C}$ be a code. If $0 < d_H(\mathscr{C}) \le n-(b-1)$, then*

$$d_H(\mathscr{C}) + b - 1 \le d_b(\mathscr{C}) \le b \cdot d_H(\mathscr{C}).$$

In the next section we give a method to calculate the b-symbol distance of two n-tuples. We know that all cyclic codes of length p^e over a finite field of characteristic p are generated by a single "monomial" of the form $(x-1)^i$, where $0 \le i \le p^e$ (see [6]). Determining the b-symbol-pair distances of some kind of these cyclic codes is the main purpose of the next section.

2 Main Results

In the following theorem we give a formula to calculate the b-symbol distance of two n-tuples.

Theorem 1. *Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two vectors in Ξ^n with $0 < d_H(x, y) \le n-(b-1)$. Suppose that*

$$A = \{1, 2, \ldots, n\} \backslash \{r, r+1, r+2, \ldots, s \mid r, s \textit{ are such that } s - r \ge b-2 \textit{ and } x_i = y_i \\ \textit{for each } r \le i \le s \textit{ and indices may wrap around modulo} n\},$$

and $A = \cup_{l=1}^{L} B_l$ is a minimal partition of the set A to subsets of consecutive indices (every subset $B_l = [s_l, e_l]$ is the sequence of all indices between s_l and e_l, inclusive, and is the smallest integer that achieves such partition, also indices may wrap around modulo n). Then

$$d_b(x, y) = d_H(x, y) + e + L(b-1),$$

where $e = |\{i \mid i \in B_l \text{ for some } 1 \le l \le L \text{ such that } x_i = y_i\}|$.

Proof. Since the partition is minimal, there are no two indices $i, i+j$, where $j \in \{1, \ldots, b-1\}$, that belong to different subsets $B_l, B_{l'}$. The b-symbol distance between x and y is equal to the sum of the sizes of the b-tuple subsets

$$\{(s_l - b + 1, s_l - b + 2, \ldots, s_l), (s_l - b + 2, s_l - b + 3, \ldots, s_l, s_l + 1), \ldots, (s_l, s_l + 1, \ldots, s_l + b - 1), \\ (s_l + 1, s_l + 2, \ldots, s_l + b), \ldots, (e_l, e_l + 1, \ldots, e_l + b - 1)\}.$$

The number of b-tuples in each b-tuple subset equals $|B_l| + b - 1$, whence $d_b(x, y) = \sum_{l=1}^{L} B_l + L(b-1)$. Furthermore, it is easy to see that $\sum_{l=1}^{L} B_l = d_H(x, y) + e$ where $e = |\{i \mid i \in B_l \text{ for some } 1 \le l \le L \text{ such that } x_i = y_i\}|$.

Corollary 2. *Let $x = (x_1, x_2, \dots, x_n) \in \Xi^n$ with $0 < \omega_H(x) \le n - (b-1)$. Suppose that*

$$A = \{1, 2, \dots, n\} \backslash \{r, r+1, r+2, \dots, s \mid r, s \text{ are such that } s - r \ge b-2 \text{ and } x_i = 0 \text{ for each } r \le i \le s \text{ and indices may wrap around modulo } n\},$$

and $A = \cup_{l=1}^{L} B_l$ is a minimal partition of the set A to subsets of consecutive indices (every subset $B_l = [s_l, e_l]$ is the sequence of all indices between s_l and e_l, inclusive, and is the smallest integer that achieves such partition, also indices may wrap around modulo n). Then $\omega_b(x) = \omega_H(x) + e + L(b-1)$, where

$$e = |\{i \mid i \in B_l \text{ for some } 1 \le l \le L \text{ such that } x_i = 0\}|.$$

Example 1. Let $n = 15$, $b = 4$ and $x = (0, 0, 1, 3, 0, 5, 0, 0, 0, 2, 0, 7, 0, 0, 0) \in Z^{15}$. We list all of the 4-tuples as follows:

$$(0,0,1,3), (0,1,3,0), (1,3,0,5), (3,0,5,0), (0,5,0,0), (5,0,0,0), (0,0,0,2),$$
$$(0,0,2,0), (0,2,0,7), (2,0,7,0), (0,7,0,0), (7,0,0,0), (0,0,0,0), (0,0,0,1).$$

Hence $\omega_4(x) = 13$. On the other hand, $\omega_H(x) = 5$, $e = 2$ and $L = 2$. Therefore, the equation $\omega_b(x) = \omega_H(x) + e + L(b-1)$ holds.

Theorem 2. ([6, Theorem 6.4]) Let $\mathscr{C}$ be a cyclic code of length p^e over F_{p^m}. Then $\mathscr{C} = \langle (x-1)^i \rangle \subseteq \frac{F_{p^m}[x]}{\langle x^{p^e} - 1 \rangle}$, for $i \in \{0, 1, \dots, p^e\}$. Also,

(1) $d_H(\mathscr{C}) = 1$ if $i = 0$.
(2) $d_H(\mathscr{C}) = \beta + 2$ if $\beta p^{e-1} + 1 \le i \le (\beta+1)p^{e-1}$ where $0 \le \beta \le p - 2$.
(3) $d_H(\mathscr{C}) = (t+1)p^k$ if $p^e - p^{e-k} + (t-1)p^{e-k-1} + 1 \le i \le p^e - p^{e-k} + tp^{e-k-1}$, where $1 \le t \le p-1$, *and* $1 \le k \le e-1$.
(4) $d_H(\mathscr{C}) = 0$ if $i = p^e$.

From now on, in order to simplify the notation, for $i \in \{0, 1, \dots, p^e\}$, we denote each code $\langle (x-1)^i \rangle$ by $\mathscr{C}_i$.

Proposition 2. *If $b \le p^e$, then $d_b(\mathscr{C}_0) = b$.*

Proof. By Theorem 2, we have that $d_H(\mathscr{C}_0) = 1$. So, by Corollary 1, $b \ge d_b(\mathscr{C}_0) \ge d_H(\mathscr{C}_0) + b - 1 = b$. Hence $d_b(\mathscr{C}_0) = b$.

Proposition 3. *Let $b < p^e$. Then $b + 1 \le d_b(\mathscr{C}_i) \le 2b$ for every $1 \le i \le p^{e-1}$.*

Proof. By Theorem 2, $d_H(\mathscr{C}_i) = 2$ for every $1 \le i \le p^{e-1}$. Hence, $2b \ge d_b(\mathscr{C}_i) \ge 2 + (b-1) = b+1$, by Corollary 1.

Notice that, for two codes $\mathscr{C}, \mathscr{C}' \subseteq F_{p^m}^{p^e}$ with $\mathscr{C} \subseteq \mathscr{C}'$, we have $d_b(\mathscr{C}) \ge d_b(\mathscr{C}')$. We define $d_b(\mathscr{C}_{p^e}) = 0$.

Proposition 4. *Let $b \le p$ and $e = 1$. Then $d_b(\mathscr{C}_i) = i + b$ for each $0 \le i \le p - b$.*

Proof. By Theorem 2, $d_H(\mathscr{C}_i) = i + 1$ for $0 \leq i \leq p - 1$. Assume that $0 \leq i \leq p - b$. Hence, by Corollary 1, $d_b(\mathscr{C}_i) \geq i + 1 + b - 1 = i + b$. Moreover $\omega_b((x - 1)^i) = i + 1 + (b - 1) = i + b$. Then $d_b(\mathscr{C}_i) = i + b$.

Theorem 3. *Let $e \geq 2$ and $1 \leq i \leq p^{e-1}$ such that $i + b \leq p^e$ and $i \leq b$. Then $d_b(\mathscr{C}_i) = i + b$.*

Proof. Since $i + b \leq p^e$, then by Corollary 2, $\omega_b((x - 1)^i) = i + 1 + (b - 1) = i + b$. So, $d_b(\mathscr{C}_i) \leq i + b$. By Proposition 3, $d_b(\mathscr{C}_i) \geq b + 1$. Let $c(x)$ be a polynomial in $F_{p^m}[x]$. If $\omega_b(c(x)) = j + b$ for some $1 \leq j \leq i - 1$, then $i \leq b$ implies that $c(x) = x^t(a_0 + a_1x + \cdots + a_jx^j)$ where a_l's are in F_{p^m}, $a_0, a_j \neq 0$ and t is a non-negative integer. However $c(x) \notin \mathscr{C}_i$. So $d_b(\mathscr{C}_i) = i + b$.

Lemma 1. *Let e and k be two integers such that $e \geq 2$ and $1 \leq k \leq e - 1$. Suppose that $c(x) = (x - 1)^{p^e - p^{e-k}} g(x)$ where $g(x)$ is a nonzero polynomial in $F_{p^m}[x]$ with $d := deg(g(x)) < p^{e-k}$ and $b \leq p^e - d$. Then*

(1) If $d \leq p^{e-k} - b$ or $g_k = 0$ for every $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(c(x)) = p^k\omega_b(g(x))$.
(2) If $d > p^{e-k} - b$ and $g_k \neq 0$ for some $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(c(x)) = p^k\big(\omega_b(g(x)) - (b - 1) + \zeta\big)$ where $\zeta = p^{e-k} - d - 1$.

Proof. Assume that $g(x) = \sum_{j=0}^{d} g_jx^j$. Thus

$$c(x) = \sum_{i=0}^{p^k-1} x^{ip^{e-k}} g(x) = \sum_{i=0}^{p^k-1} \sum_{j=0}^{d} g_jx^{ip^{e-k}+j}.$$

As usual, we identify the polynomial $h(x) = h_0 + h_1x + \cdots + h_nx^n$ with the vector $h = (h_0, h_1, \ldots, h_n)$. Therefore, we have $c = (\overbrace{\widehat{g}, \ldots, \widehat{g}}^{p^k-\text{time}})$ where

$$\widehat{g} = (g_0, \ldots, g_d, \overbrace{0, \ldots, 0}^{(p^{e-k}-d-1)-\text{time}}).$$

We denote $\omega_b(\widehat{g}(x)) := \omega_b(\widehat{g})$. Since $\pi_b(c) = [\overbrace{\pi_b(\widehat{g}), \ldots, \pi_b(\widehat{g})}^{p^k-\text{time}}]$, then $\omega_b(c(x)) = p^k\omega_b(\widehat{g}(x))$. On the other hand, $\omega_b(g(x)) = \omega_b(g)$, where

$$g = (g_0, g_1, \ldots, g_d, \overbrace{0, \ldots, 0}^{(p^e-d-1)-\text{time}}).$$

We can check that:

(1) If $d \leq p^{e-k} - b$ or $g_k = 0$ for every $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(g) = \omega_b(\widehat{g})$, i.e., $\omega_b(g(x)) = \omega_b(\widehat{g}(x))$. Hence $\omega_b(c(x)) = p^k \omega_b(g(x))$.

(2) If $d > p^{e-k} - b$ and $g_k \neq 0$ for some $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(g) = \omega_b(\widehat{g}) + (b-1) - \zeta$ where $\zeta = p^{e-k} - d - 1$, i.e., $\omega_b(g(x)) = \omega_b(\widehat{g}(x)) + (b-1) - \zeta$. So, $\omega_b(c(x)) = p^k\big(\omega_b(g(x)) - (b-1) + \zeta\big)$.

Theorem 4. *Let e and k be two integers such that $e \geq 2$ and $1 \leq k \leq e-1$. If $0 \leq i \leq p^{e-k-1}$ such that $b + i \leq p^{e-k}$ and $i \leq b$, then $d_b(\mathscr{C}_{p^e - p^{e-k} + i}) = p^k(b+i)$.*

Proof. Fix $0 \leq i \leq p^{e-k-1}$ such that $b + i \leq p^{e-k}$ and $i \leq b$. Let $0 \neq c(x) \in \mathscr{C}_{p^e - p^{e-k} + i}$. Then, there exists $0 \neq f(x) \in F_{p^m}[x]$ such that $c(x) = (x-1)^{p^e - p^{e-k}} (x-1)^i f(x)$. Set $g(x) := (x-1)^i f(x)$ and $d := deg(g(x))$. Without loss of the generality we may assume that $d < p^{e-k}$. Notice that by Theorem 2, $\omega_H(g(x)) \geq 2$, and by Theorem 3, $\omega_b(g(x)) \geq b + i$. Regarding Lemma 1, we consider the following cases:

Case 1. If $d \leq p^{e-k} - b$ or $g_k = 0$ for every $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(c(x)) = p^k \omega_b(g(x)) \geq p^k(b+i)$.

Case 2. If $d > p^{e-k} - b$ and $g_k \neq 0$ for some $0 \leq k \leq b - p^{e-k} + d - 1$, then $\omega_b(c(x)) = p^k\big(\omega_b(g(x)) - (b-1) + \zeta\big)$ where $\zeta = p^{e-k} - d - 1$. If $\omega_H(g(x)) \geq b + i$, then Corollary 1 implies that $\omega_b(g(x)) \geq b + i + b - 1$. Hence $\omega_b(c(x)) \geq p^k\big(b + i + (b-1) - (b-1)\big) = p^k(b+i)$. Assume that $\omega_H(g(x)) = i + j$ for some $2 - i \leq j \leq b - 1$. It is easy to see that $\omega_H(g(x)) + z = d + 1$ where $z = |\{l \mid 0 \leq l \leq d \text{ and } g_l = 0\}|$. We claim that, $z \geq b - j - \zeta$. Otherwise $d + 1 < \omega_H(g(x)) + b - j - \zeta = i + j + b - j - (p^{e-k} - d - 1) = i + b - p^{e-k} + d + 1$. But $b + i \leq p^{e-k}$ leads us to a contradiction. Therefore the claim holds. So, $\omega_b(g(x)) \geq i + j + b - j - \zeta + (b-1)$. Thus $\omega_b(c(x)) \geq p^k\big(\omega_b(g(x)) - (b-1) + \zeta\big) = p^k(i+b)$. Hence $d_b(\mathscr{C}_{p^e - p^{e-k} + i}) \geq p^k(i+b)$. Moreover, by part (1) of Lemma 1, $\omega_b((x-1)^{p^e - p^{e-k} + i}) = p^k \omega_b((x-1)^i) = p^k(b+i)$. Consequently, $d_b(\mathscr{C}_{p^e - p^{e-k} + i}) = p^k(b+i)$.

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Bhargava Rings Over Subsets

I. Al-Rasasi and L. Izelgue

Abstract Let D be an integral domain with quotient field K and let E be any nonempty subset of K. The Bhargava ring over E at $x \in D$ is defined by $\mathbb{B}_x(E, D) := \{f \in K[X] \mid f(xX + e) \in D[X], \ \forall e \in E\}$. This ring is a subring of the ring of integer-valued polynomials over E. This paper studies $\mathbb{B}_x(E, D)$ for an arbitrary domain D. we provide information about its localizations and transfer properties, describe its prime ideal structure, and calculate its Krull and valuative dimensions.

Keywords Integer-valued polynomial · Bhargava ring · Prime ideal · Localization
Residue field · Krull dimension · Valuative dimension

1 Introduction

Throughout this paper we let D be an integral domain with quotient field K and E be a nonempty subset of K. The set of integer-valued polynomials on E is defined by

$$\text{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}.$$

Clearly, $\text{Int}(E, D)$ is a subring of $K[X]$ and if $E = D$, then $\text{Int}(E, D) = \text{Int}(D)$, the ring of integer-valued polynomials on D. These two rings, $\text{Int}(D)$ and $\text{Int}(E, D)$, were studied extensively for a long time and much is known about them. Reference [4] is a good reference on the algebraic properties of the rings of integer-valued polynomials.

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J. Yeramian,[10], studied a special class of subrings of $\text{Int}(D)$; namely the rings

$$\mathbb{B}_x(D) = \{f \in K[X];\ f(xX + d) \in D[X]\,\forall d \in D\},$$

where $0 \neq x \in D$. Continuing her work, we studied in [1], the prime ideal structure of $\mathbb{B}_x(D)$ in a general setting. Notice that for $x \neq 0$, $\mathbb{B}_x(D) = \bigcap_{a\in D} D[\frac{X-a}{x}]$ and $\text{Int}(D) = \bigcup_{0\neq x\in D} \mathbb{B}_x(D)$; i.e., the rings $\mathbb{B}_x(D)$ form a covering of $\text{Int}(D)$ (cf. [1], [10], [11]).

As $\text{Int}(E, D)$ is a generalization of $\text{Int}(D)$, it seems natural to introduce the ring

$$\mathbb{B}_x(E, D) := \{f \in K[X] \mid \forall e \in E,\ f(xX + e) \in D[X]\},$$

where $x \in D \setminus \{0\}$, as a generalization of $\mathbb{B}_x(D)$. We call such a ring, *the Bhargava ring over* E *at* x. In fact, if $E = D$, then $\mathbb{B}_x(E, D) = \mathbb{B}_x(D)$.

These rings were independently introduced in two different papers ([3, 5]). In [3], Bhargava, Cahen, and Yeramian defined such a ring as “the ring of integer-valued polynomials on E of modulus x”, denoted by $\text{Int}_x(E, D)$. Their study focused on the problem of finite generation for rings of integer-valued polynomials. Particularly, they stated that:

> For a Dedekind domain D and E any subset of K, $\mathbb{B}_x(E, D)$ is finitely generated over D and hence is Noetherian (cf. [3, Theorem 0.1]).

When investigating the problem of the flatness of $\text{Int}(E, D)$ as a D–module, Elliott [5] introduced such a ring as “the Bhargava ring over E at x”. Among other interesting results he stated that:

> For a subset E of D, if $\mathbb{B}_x(E, D)$ is D–flat for every nonzero $x \in D$ then Int(E,D) is also D–flat (cf. [5, Proposition 6.4]).

As a simple extension of what is known about $B_x(D)$, they all stated that

$$\mathbb{B}_x(E, D) = \bigcap_{a\in E} D[\frac{X-a}{x}]$$

which intersection may be restricted to a set of representatives of E modulo x. Also, as in the case of Bhargava rings (cf. [11, Theorem 1.4], the rings $B_x(E, D)$ form a covering of $\text{Int}(E, D)$; i.e.,

$$\text{Int}(E, D) = \bigcup_{0\neq x\in D} \mathbb{B}_x(E, D).$$

We will assume throughout, unless otherwise specifically stated, that x is a nonzero element of D.

Our goal in this paper is to study Bhargava rings over subsets. Thus, Sect. 2 is devoted to some basic properties about Bhargava rings over subsets. In particular, we characterize subsets E for which $\mathbb{B}_x(E, D)$ contains nonconstant polynomials (cf.

Proposition 3). In Sect. 3, we investigate the behavior of $\mathbb{B}_x(E, D)$ under localization. In Sect. 4, we study how the algebraic structure of D may transfer to $\mathbb{B}_x(E, D)$. In particular, we show that $\mathbb{B}_x(E, D)$ is (completely) integrally closed if and only if the same is true for D (cf. Propositions 13 and 15). In Sect. 5 we give a detailed study of the prime ideal structure of $\mathbb{B}_x(E, D)$. Thus, we introduce some particular forms of the prime ideals of $\mathbb{B}_x(E, D)$ and we study inclusion relationships among them (cf. Proposition 20). Also we study the lifting problem of the ideal $\mathfrak{p}[X] \in \text{Spec}(D[X])$ to $\mathbb{B}_x(E, D)$, when $E \subseteq D$ (cf. Proposition 22). In Sect. 6, we compute the valuative dimension of $\mathbb{B}_x(E, D)$ (that is the supermum of $\dim V$, where V runs over the valuation overrings of $\mathbb{B}_x(E, D)$). The main result of this section is: *For any ring B, such that $D \subset B \subseteq D + (X - a)K[X]$ then* $\dim_v B = 1 + \dim_v D$ (cf. Theorem 2).

As a corollary, we establish that $\dim_v \mathbb{B}_x(E, D) = 1 + \dim_v D$. This allows us to give conditions under which $\mathbb{B}_x(E, D)$ is a Jaffard domain; i.e., $\dim_v \mathbb{B}_x(E, D) = \dim \mathbb{B}_x(E, D)$.

2 Basic Properties

We start our study of Bhargava rings over subsets by listing some basic properties satisfied by these rings.

Proposition 1 *Let $D_1 \subseteq D_2$ be two domains with the same quotient field K, $E \subseteq F$ two nonempty subsets of K, and $0 \neq x \in D_1$. Then*

$$\mathbb{B}_x(F, D_1) \subseteq \mathbb{B}_x(E, D_2).$$

Proof straightforward.

Corollary 1 *Let D be a domain with quotient field K, $0 \neq x \in D$ and E be a subset of K. The following statements are equivalent:*

(i) *$E \subseteq D$;*
(ii) *$\mathbb{B}_x(D) \subseteq \mathbb{B}_x(E, D)$;*
(iii) *$D[X] \subseteq \mathbb{B}_x(E, D)$.*

Proof (i) ⇒ (ii). It follows from Proposition 1. Since $D[X] \subseteq \mathbb{B}_x(D)$, then (ii)⇒(iii). Now, statement (iii) implies that $f(X) = X \in D[X] \subseteq \mathbb{B}_x(E, D)$, and then $xX + e \in D[X]$ for every $e \in E$. Thus $e \in D$ for every $e \in E$ and hence (iii)⇒ (i).

Remark 1 (1) If x is a unit element of D and $E \cap D \neq \emptyset$, we have $\mathbb{B}_x(E, D) \subset D[X]$: indeed, let $a \in E \cap D$, then $\mathbb{B}_x(E, D) \subseteq D[\frac{X-a}{x}] \subseteq D[X]$. In contrast to $\mathbb{B}_x(D)$, which is then such that $\mathbb{B}_x(D) = D[X]$, It follows from Corollary 1 that we have $\mathbb{B}_x(E, D) \neq D[X]$, when E is not contained in D.

(2) If x is a non-unit element of D and $E \subseteq D$, then $D[X] \subseteq \mathbb{B}_x(D) \subseteq \mathbb{B}_x(E, D)$. By [1, Proposition 2.4], if the factor ring $D/(x)$ is finite then $D[X] \subset \mathbb{B}_x(D)$, and then $D[X] \subset \mathbb{B}_x(E, D)$ a proper inclusion. In general, if the set of representatives of E modulo (x) is finite, then arguing as in the proof of [1, Proposition 2.4], we see that $D[X]$ is a proper subset of $\mathbb{B}_x(E, D)$. If, moreover, D contains an infinite field then $D[X] = \mathbb{B}_x(D) \subset \mathbb{B}_x(E, D)$.

Now, it is easy to see that $\mathrm{Int}(\mathbb{Q}, \mathbb{Z})$ does not contain any nonconstant polynomial. A fortiori the same holds for $\mathbb{B}_x(\mathbb{Q}, \mathbb{Z})$, whatever $x \in \mathbb{Z}$. Hence the question arises to determine for which subsets of the domain D, $\mathbb{B}_x(E, D)$ does contain nonconstant polynomials. Partial answers are next given to this question. A sufficient condition is that E be fractional, that is if there exists a nonzero element d of D such that $dE \subseteq D$. In this case the isomorphism $K[X] \longrightarrow K[X]$ taking $f(X)$ to $f(X/d)$ yields an isomorphism $\mathbb{B}_x(E, D) \cong \mathbb{B}_{dx}(dE, D)$; that is $f(X) \in \mathbb{B}_x(E, D)$ if and only if $f(X/d) \in \mathbb{B}_{dx}(dE, D)$. By Corollary 1, $\mathbb{B}_x(E, D)$ contains an isomorphic copy of $D[X]$. So, if E is a fractional subset of D, we may as well assume that E is a subset of D.

Yet, the condition that E be a fractional subset of D is not necessary for $\mathbb{B}_x(E, D)$ to contain a nonconstant polynomial [4, Exercise 8, page 20]. Next thus comes a necessary condition, similar to [4, Proposition I.1.9]. Beforehand, we denote by D' the integral closure of D.

Proposition 2 *Let D be a domain, with quotient field K, and E be a nonempty subset of K. If $\mathbb{B}_x(E, D)$ contains a nonconstant polynomial, then E is a fractional subset of D'.*

Proof As recalled in the introduction, one has $\mathbb{B}_x(E, D) \subseteq \mathrm{Int}(E, D)$. Thus, if $\mathbb{B}_x(E, D)$ contains a nonconstant polynomial, then so does $\mathrm{Int}(E, D)$. By, [4, PropositionI.1.9] E must be a fractional subset of D'

For a fractional subset of D the situation is described in the following proposition:

Proposition 3 *Let D be a domain with quotient field K, and E be a nonempty subset of K. The following statements are equivalent:*

(i) *E is a fractional;*
(ii) $\mathbb{B}_x(E, D)$ *contains a polynomial of degree 1;*
(iii) $\mathrm{Int}(E, D)$ *contains a polynomial of degree 1.*

Proof (i) $\Longrightarrow$ (ii). If E is a fractional subset of D, then there exists $0 \neq d \in D$ such that $dE \subset D$. It follows that the polynomial $f(X) = dX$ belongs to $\mathbb{B}_x(E, D)$.
(ii) $\Longrightarrow$ (iii). Straightforward.
(iii) $\Longrightarrow$ (i). Let $f(X) = \frac{a}{s}X + \frac{b}{s} \in \mathrm{Int}(E, D)$, where $a, s \in D\backslash\{0\}$ and $b \in D$. Since $D \subset \mathrm{Int}(E, D)$, then $sf(X) = aX + b \in Int(E, D)$. It follows that for each $e \in E$, $ae + b \in D$ and thus $aE \subseteq D$. That is E is a fractional subset of D.

The condition given in Proposition 2 is not sufficient: there is an example where $\text{Int}(D', D)$ (a fortiori $\mathbb{B}_x(D', D)$ does not contain any nonconstant polynomial [4, Exercise 9, page 20]. However, putting together the two previous propositions, one derives immediately a characterization in case D is integrally closed:

Corollary 2 *Let D be an integrally closed domain. Let E be a nonempty subset of K and $0 \neq x \in D$. Then $\mathbb{B}_x(E, D)$ contains nonconstant polynomials if and only if E is a fractional subset of D.*

If $e \in E$, then $f(xX + e) \in D[X]$ if and only if $f(X) \in D[\frac{X-e}{x}]$. Thus we have

$$\mathbb{B}_x(E, D) = \bigcap_{e \in E} D[\frac{X-e}{x}].$$

We use this representation to prove the following proposition.

Proposition 4 *Let D be a domain, with quotient field K, and $\emptyset \neq E \subseteq K$. Let λ and x be nonzero elements of D. Then, $\mathbb{B}_x(E, D) \subseteq \mathbb{B}_{\lambda x}(E, D)$, with equality when λ is invertible in D.*

Proof For each $e \in E$, $\frac{X-e}{x} = \lambda \frac{X-e}{\lambda x}$ and hence $D[\frac{X-e}{x}] \subseteq D[\frac{X-e}{\lambda x}]$. This yields

$$\mathbb{B}_x(E, D) = \bigcap_{e \in E} D[\frac{X-e}{x}] \subseteq \bigcap_{e \in E} D[\frac{X-e}{\lambda x}] = \mathrm{B}_{\lambda x}(E, D).$$

If λ is invertible in D, then $\mathbb{B}_x(E, D) \subseteq \mathbb{B}_{\lambda x}(E, D) \subseteq \mathbb{B}_{\lambda^{-1}\lambda x}(E, D) = \mathbb{B}_x(E, D)$.

Definition 1 Let A be a nonempty subset of D. Define the set

$$\mathbb{B}_A(E, D) := \bigcap_{x \in A} \mathbb{B}_x(E, D) = \{f \in K[X] : \ \forall x \in A, \forall e \in E, f(xX + e) \in D[X]\}.$$

We note that

- $\mathbb{B}_A(E, D) = \bigcap_{0 \neq x \in A} \bigcap_{e \in E} D[\frac{X-e}{x}]$.
- $\mathbb{B}_A(E, D) \subseteq \text{Int}(E, D)$.

Proposition 5 *If A_1 and A_2 are two nonempty subsets of D, then:*

1. $\mathbb{B}_{A_1 \cup A_2}(E, D) = \mathbb{B}_{A_1}(E, D) \cap \mathbb{B}_{A_2}(E, D)$.
2. *If $A_1 \subseteq A_2$, then $\mathbb{B}_{A_2}(E, D) \subseteq \mathbb{B}_{A_1}(E, D)$.*

Proof Both assertions follow immediately from the definition.

Remark 2 If Dx is the principal ideal of D generated by x, then $\mathbb{B}_{Dx}(E, D) = \mathbb{B}_x(E, D)$. This follows from the definition and using Proposition 4.

Proposition 6 *Let D be a domain with quotient field K, E a subset of K, $0 \neq x \in D$, and $\mathfrak{p}$ be a prime ideal of D with infinite residue field. If E meets infinitely many cosets of $\mathfrak{p}$, then $\mathbb{B}_x(E, D) \subseteq D_{\mathfrak{p}}[X]$.*

Proof By [4, Proposition I.3.1], $\text{Int}(E, D) \subseteq D_{\mathfrak{p}}[X]$, a fortiori, $\mathbb{B}_x(E, D) \subseteq D_{\mathfrak{p}}[X]$.

Remark 3 It follows from Corollary 1 that if $E \subseteq D$, then $D[X] \subseteq \mathbb{B}_x(E, D)$. Thus, under the assumptions of Proposition 6, we obtain the equality $\mathbb{B}_x(E, D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$.

Proposition 7 *Let D be a domain with quotient field K, E be a subset of D, and $0 \neq x \in D$. If there exists a family $\mathscr{F}$ of prime ideals of D with infinite residue fields such that $D = \bigcap_{\mathfrak{p} \in \mathscr{F}} D_{\mathfrak{p}}$ and E meets infinitely many cosets of $\mathfrak{p}$ for each $\mathfrak{p} \in \mathscr{F}$, then*

$$\mathbb{B}_x(E, D) = D[X]$$

Proof Since E is a subset of D, by Corollary 1, $D[X] \subseteq \mathbb{B}_x(E, D)$. By Proposition 6, we obtain $\mathbb{B}_x(E, D) \subseteq \bigcap_{\mathfrak{p} \in \mathscr{F}} D_{\mathfrak{p}}[X] = D[X]$ and hence the equality follows.

3 Localization Properties

In this section, we study the behavior of $\mathbb{B}_x(E, D)$ under localization

Lemma 1 *Let S be a multiplicative subset of D, E be a subset of K, and $x \in D$. Then*

(1) $S^{-1}\mathbb{B}_x(E, D) \subseteq \mathbb{B}_x(E, S^{-1}D)$.
(2) *For each $s \in S$, $\mathbb{B}_x(E, D) \subseteq \mathbb{B}_{\frac{x}{s}}(E, S^{-1}D)$*

Proof (1). Let $\frac{1}{s} f \in S^{-1}\mathbb{B}_x(E, D)$, where $s \in S$ and $f \in \mathbb{B}_x(E, D)$. Then, $f(xX + e) \in D[X]$ for all $e \in E$. This implies $\frac{1}{s} f(xX + e) \in S^{-1}D[X]$ for all $e \in E$ and so $\frac{1}{s} f \in \mathbb{B}_x(E, S^{-1}D)$.

(2). By statement (1), $\mathbb{B}_x(E, D) \subseteq S^{-1}\mathbb{B}_x(E, D) \subseteq \mathbb{B}_x(E, S^{-1}D)$, and by Proposition 4, $\mathbb{B}_x(E, S^{-1}D) \subseteq \mathbb{B}_{\frac{x}{s}}(E, S^{-1}D)$ for all $s \in S$.

Proposition 8 *Let S be a multiplicative subset of D, E a subset of K, and I be an ideal of D. Then $S^{-1}\mathbb{B}_I(E, D) \subseteq \mathbb{B}_{S^{-1}I}(E, S^{-1}D)$.*

Proof We first show that $\mathbb{B}_I(E, D) \subseteq \mathbb{B}_{S^{-1}I}(E, S^{-1}D)$. For this, let $f \in \mathbb{B}_I(E, D) = \cap_{x \in I}\mathbb{B}_x(E, D)$. Then $f \in \mathbb{B}_x(E, D)$, for all $x \in I$. By statement (2) of Lemma 1, $f \in \mathbb{B}_{\frac{x}{s}}(E, S^{-1}D)$, for all $x \in I$ and $s \in S$, and hence $f \in \mathbb{B}_{S^{-1}I}(E, S^{-1}D)$.

Now let $\frac{1}{s} f \in S^{-1}\mathbb{B}_I(E, D)$, where $s \in S$ and $f \in \mathbb{B}_I(E, D)$. By the above observation, $f \in \mathbb{B}_{S^{-1}I}(E, S^{-1}D)$; that is, $f(yX + e) \in S^{-1}D[X]$ for all $y \in S^{-1}I$ and $e \in E$. This gives $\frac{1}{s} f(yX + e) \in S^{-1}D[X]$ for all $y \in S^{-1}I$ and $e \in E$. Thus $\frac{1}{s} f \in \mathbb{B}_{S^{-1}I}(E, S^{-1}D)$.

Proposition 9 *Let R be a subring of the domain D with quotient field K, S be a multiplicative subset of R, and I be an ideal of R. Then*

$$S^{-1}\mathbb{B}_I(R, D) \subseteq \mathbb{B}_{S^{-1}I}(R, S^{-1}D) = \mathbb{B}_{S^{-1}I}(S^{-1}R, S^{-1}D).$$

Proof By Proposition 8, $S^{-1}\mathbb{B}_I(R, D) \subseteq \mathbb{B}_{S^{-1}I}(R, S^{-1}D)$. Now, since $R \subseteq S^{-1}R$, then $\mathbb{B}_{S^{-1}I}(R, S^{-1}D) \supseteq \mathbb{B}_{S^{-1}I}(S^{-1}R, S^{-1}D)$. For the reverse inclusion, we first prove that, for any $x \in S^{-1}D \setminus \{0\}$, $\mathbb{B}_x(R, S^{-1}D) \subseteq \mathbb{B}_x(S^{-1}R, S^{-1}D)$. This can be shown by induction on the degree n of $f \in \mathbb{B}_x(R, S^{-1}D)$. Clearly the statement holds if $n = 0$. So, assume it holds for all polynomials of degree less than n. Let $f \in \mathbb{B}_x(R, S^{-1}D)$ be of degree n. We need to show that $f(xX + \frac{r}{s}) \in S^{-1}D[X]$, for all $r \in R$ and $s \in S$. Now, he polynomial $g(X) = s^n f(X) - f(sX)$ is of degree less than n and $g \in \mathbb{B}_x(R, S^{-1}D)$: in deed, for each $r \in R$, $g(xX + r) = s^n f(xX + r) - f(sxX + sr)$ and since $f \in \mathbb{B}_x(R, S^{-1}D)$, then $f(xX + r) \in S^{-1}D[X]$. Hence $s^n f(xX + r) \in S^{-1}D[X]$. Since $f \in \mathbb{B}_x(R, S^{-1}D) \subseteq \mathbb{B}_{sx}(R, S^{-1}D)$, then $f(sxX + sr) \in S^{-1}D[X]$. Thus, we have $g(xX + r) \in S^{-1}D[X]$ and hence $g \in \mathbb{B}_x(R, S^{-1}D)$. By the induction hypothesis, we conclude that $g \in \mathbb{B}_x(S^{-1}R, S^{-1}D)$. Now, we have that $s^n f(xX + \frac{r}{s}) = g(xX + \frac{r}{s}) + f(sxX + r)$ and $f \in \mathbb{B}_x(R, S^{-1}D) \subseteq \mathbb{B}_{sx}(R, S^{-1}D)$, then $f(sxX + r) \in S^{-1}D[X]$. Also, $g(xX + \frac{r}{s}) \in S^{-1}D[X]$ which implies that $s^n f(xX + \frac{r}{s}) \in S^{-1}D[X]$ and so $f(xX + \frac{r}{s}) \in S^{-1}D[X]$. Thus, $f \in \mathbb{B}_x(S^{-1}R, S^{-1}D)$. Now, let $f \in \mathbb{B}_{S^{-1}I}(R, S^{-1}D) = \bigcap_{0 \neq x \in S^{-1}I} \mathbb{B}_x(R, S^{-1}D)$. Then $f \in \mathbb{B}_x(R, S^{-1}D)$, for all $0 \neq x \in S^{-1}I$, and hence $f \in \mathbb{B}_x(S^{-1}R, S^{-1}D)$ for all $0 \neq x \in S^{-1}I$. Thus, $f \in \mathbb{B}_{S^{-1}I}(S^{-1}R, S^{-1}D)$.

Corollary 3 *Let S be a multiplicative subset of D and I be an ideal of D. Then*

$$S^{-1}\mathbb{B}_I(D) \subseteq \mathbb{B}_{S^{-1}I}(D, S^{-1}D) = \mathbb{B}_{S^{-1}I}(S^{-1}D).$$

Proof Take $R = D$ in Proposition 9.

Proposition 10 *(a) Let D be a Noetherian domain, E a fractional subset of D, and S a multiplicative subset of D. Then $S^{-1}\mathbb{B}_I(E, D) = \mathbb{B}_{S^{-1}I}(E, S^{-1}D)$.*

(b) Let D be a domain and R a Noetherian subring of D. Let S be a multiplicative subset of R and I an ideal of R. Then $S^{-1}\mathbb{B}_I(R, D) = \mathbb{B}_{S^{-1}I}(R, S^{-1}D) = \mathbb{B}_{S^{-1}I}(S^{-1}R, S^{-1}D)$.

Proof (a) By Proposition 8 and Proposition 9, we need only prove

$$\mathbb{B}_{S^{-1}I}(E, S^{-1}D) \subseteq S^{-1}\mathbb{B}_I(E, D).$$

Let $f \in \mathbb{B}_{S^{-1}I}(E, S^{-1}D)$. Since E is a fractional subset of D, we may assume E to be a subset of D. Let M be the D-module generated by the coefficients of the polynomials $f(xX + e)$, for all $x \in I$ ($\subseteq S^{-1}I$) and all $e \in E$ ($\subseteq D \subseteq S^{-1}D$). Then $M \subseteq S^{-1}D$ (as $f \in \mathbb{B}_{S^{-1}I}(E, S^{-1}D)$). Let $C(f)$ be the D-module generated by the coefficients of f. Then clearly $C(f)$ is a finitely generated

D-module. Since $M \subseteq S^{-1}D \cap C(f)$, $S^{-1}D \cap C(f)$ is a finitely generated D-module, and D is Noetherian, then $S^{-1}D \cap C(f)$ is a Noetherian D-module and hence M is a finitely generated D-module. So there exists $s \in S$ such that $sM \subseteq D$ and hence $sf(xX + e) \in D[X]$ for all $x \in I$ and $e \in E$. This yields $sf \in \mathbb{B}_I(E, D)$ and thus $f \in S^{-1}\mathbb{B}_I(E, D)$.

(b) By Proposition 8 and Proposition 9, we need only prove

$$\mathbb{B}_{S^{-1}I}(R, S^{-1}D) \subseteq S^{-1}\mathbb{B}_I(R, D).$$

This can be proved by proceeding as in part (a) by replacing E and D-module with R and R-module respectively.

Proposition 11 *Let D be a Krull domain, $\mathfrak{p}$ be a height-one prime ideal of D, E be a fractional subset of D, and I be an ideal of D. Then* $\mathbb{B}_I(E, D)_\mathfrak{p} = \mathbb{B}_{I_\mathfrak{p}}(E, D_\mathfrak{p})$.

Proof By Proposition 8, it is enough to prove the inclusion $\mathbb{B}_{I_\mathfrak{p}}(E, D_\mathfrak{p}) \subseteq \mathbb{B}_I(E, D)_\mathfrak{p}$. So, let $f \in \mathbb{B}_{I_\mathfrak{p}}(E, D_\mathfrak{p})$. Then there exists $0 \neq d \in D$ such that $df \in D[X]$. Let $\mathscr{T}$ be the set of height-one prime ideals that are different from $\mathfrak{p}$ and containing d. Since D is Krull, then $\mathscr{T}$ is finite. For each $\mathfrak{q} \in \mathscr{T}$, there exists an element $b_\mathfrak{q} \in D$ such that $b_\mathfrak{q} \in \mathfrak{q}$ and $b_\mathfrak{q} \notin \mathfrak{p}$. It follows that there exists an integer $n_\mathfrak{q}$ such that $v_\mathfrak{q}(b_\mathfrak{q}^{n_\mathfrak{q}}) > v_\mathfrak{q}(d)$, where $v_\mathfrak{q}$ is the valuation corresponding to the discrete valuation domain $D_\mathfrak{q}$. Set $b = \prod_{\mathfrak{q}\in\mathscr{T}} b_\mathfrak{q}^{n_\mathfrak{q}}$. Then $b \notin \mathfrak{p}$ and, for each $\mathfrak{q} \in \mathscr{T}$, d divides b in $D_\mathfrak{q}$. Thus, $bf \in D_\mathfrak{q}[X]$. Further, if $\mathfrak{q} \notin \mathscr{T}$, then d is invertible in $D_\mathfrak{q}$ and hence $f \in D_\mathfrak{q}[X]$. Since $b \in D$, then $bf \in D_\mathfrak{q}[X]$. Thus far we have shown that for each height-one prime ideal $\mathfrak{q} \neq \mathfrak{p}$, $bf \in D_\mathfrak{q}[X]$. As E is a fractional subset of D, we may assume that $E \subseteq D$. hence, we have for each $\mathfrak{q} \neq \mathfrak{p}$, $bf(xX + e) \in D_\mathfrak{q}[X]$ for each $x \in I$ and $e \in E$.
As $f \in \mathbb{B}_{I_\mathfrak{p}}(E, D_\mathfrak{p})$, then $f(xX + e) \in D_\mathfrak{p}[X]$, for each $x \in I$ and $e \in E$. Since $b \in D$, then $bf(xX + e) \in D_\mathfrak{p}[X]$ for each $x \in I$ and $e \in E$. We conclude that for each $x \in I$ and $e \in E$, $bf(xX + e) \in \bigcap_{\mathrm{ht}(\mathfrak{p})=1} D_\mathfrak{p}[X] = D[X]$. Thus $bf \in \mathbb{B}_I(E, D)$ and hence $f \in \mathbb{B}_I(E, D)_\mathfrak{p}$ as $b \notin \mathfrak{p}$.

Corollary 4 *Let D be a Krull domain, E be a fractional subset of D, and I be an ideal of D. Then* $\mathbb{B}_I(E, D) = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} \mathbb{B}_{I_\mathfrak{p}}(E, D_\mathfrak{p})$.

Proof By Proposition 11, $\mathbb{B}_I(E, D) \subseteq \bigcap_{\mathrm{ht}(\mathfrak{p})=1} \mathbb{B}_{I_\mathfrak{p}}(E, D_\mathfrak{p})$. For the reverse inclusion, let $f \in \bigcap_{\mathrm{ht}(\mathfrak{p})=1} \mathbb{B}_{I_\mathfrak{p}}(E, D_\mathfrak{p})$. Then for each height-one prime ideal $\mathfrak{p}$, $f(xX + e) \in D_\mathfrak{p}[X]$ for each $x \in I$ and $e \in E$. Thus $f(xX + e) \in \bigcap_{\mathrm{ht}(\mathfrak{p})=1} D_\mathfrak{p}[X] = D[X]$ and hence $f \in \mathbb{B}_I(E, D)$.

Proposition 12 *Let S be a multiplicative subset of D and E a subset of D. If the set of representatives of E modulo $(x) = xD$ is finite, then* $S^{-1}\mathbb{B}_x(E, D) = \mathbb{B}_x(E, S^{-1}D)$.

Proof If $\overline{E} = \{a_1, ..., a_n\}$ is a set of representatives of E modulo (x), then every element of E is a fortiori congruent to one a_i modulo $S^{-1}(x)$, and thus $\mathbb{B}_x(E, D) = \bigcap_{a\in E} D[\frac{X-a}{x}] = \bigcap_{i=1}^{i=n} D[\frac{X-a_i}{x}]$. Thus

$$S^{-1}\mathbb{B}_x(E,D) = S^{-1}\bigcap_{a_i\in \bar{E}} D[\frac{X-a_i}{x}] = \bigcap_{i=1}^{i=n} S^{-1}D[\frac{X-a_i}{x}].$$

On the other hand, if E_S denotes the set of representatives of E modulo $S^{-1}(x)$, then $\overline{E} \subseteq E_S$. Hence

$$\mathbb{B}_x(E,S^{-1}D) = \bigcap_{a\in E_S} S^{-1}D[\frac{X-a}{x}] \subseteq \bigcap_{a\in \bar{E}} S^{-1}D[\frac{X-a}{x}] = S^{-1}\mathbb{B}_x(E,D).$$

The equality follows from statement (1) of Lemma 1.

Corollary 5 *Let E be a subset of D such that the set of representatives of E modulo (x) is finite. Then, for each prime ideal $\mathfrak{p}$ of D, $\mathbb{B}_x(E,D)_{\mathfrak{p}} = \mathbb{B}_x(E,D_{\mathfrak{p}})$.*

4 Transfer Properties

In this section, we investigate the transfer of some algebraic properties between D and $\mathbb{B}_x(E,D)$.

Proposition 13 *Let E be a nonempty subset of K. Then $\mathbb{B}_x(E,D)$ is integrally closed if and only if D is integrally closed.*

Proof Clearly if $\mathbb{B}_x(E,D)$ is integrally closed, then so is D. Conversely, let $f \in K(X)$ be integral over $\mathbb{B}_x(E,D)$. Then $f^n + g_{n-1}f^{n-1} + \cdots + g_1 f + g_0 = 0$, where $g_i \in \mathbb{B}_x(E,D)$ for all i, $0 \le i \le n-1$. Since $g_i \in K[X]$, then f is integral over $K[X]$ and so $f \in K[X]$, as $K[X]$ is integrally closed. Let $e \in E$, then $f^n(xX+e) + g_{n-1}(xX+e)f^{n-1}(xX+e) + \cdots + g_1(xX+e)f(xX+e) + g_0$ $(xX+e) = 0$. Since $g_i(xX+e) \in D[X]$ for all i, $0 \le i \le n-1$, then $f(xX+e)$ is integral over $D[X]$ and hence, $f(xX+e) \in D[X]$ as $D[X]$ is integrally closed. Since e is arbitrary, then $f \in \mathbb{B}_x(E,D)$.

Remark 4 From the above proof, we deduce that $\mathbb{B}_x(E,D)' \subseteq \mathbb{B}_x(E,D')$: For if f is integral over $\mathbb{B}_x(E,D)$, then $f(xX+e)$ is integral over $D[X]$, for all $e \in E$. So $f(xX+e) \in (D[X])' = D'[X]$, for all $e \in E$. Thus, $f \in \mathbb{B}_x(E,D')$.

Proposition 14 *Let D be a Noetherian domain and E be a fractional subset of D. Then $\mathbb{B}_x(E,D')$ is almost integral over $\mathbb{B}_x(E,D)$.*

Proof Let $f \in \mathbb{B}_x(E,D')$, that is, for each $e \in E$, $f(xX+e) \in D'[X]$. Thus, for every positive integer n, $f^n(xX+e)$ is integral over $D[X]$ and hence $f^n(xX+e) \in R$, a finitely generated $D[X]$-module. Thus, the conductor $I = (D[X] : R) = \{g \in D[X] : gR \subseteq D[X]\} \neq (0)$. In fact, an element of R can be written as a linear combination of a finite number of polynomials $h_i \in D'[X]$, with coefficients in $D[X]$. As $D' \subseteq K$, then $h_i \in K[X]$. Let d be the common denominator of the coefficients of all h_i. Then $0 \neq d \in I$. It follows that $df^n(xX + e) \in D[X]$ and hence, $df^n \in \mathbb{B}_x(E, D)$.

Proposition 15 *Let D be a domain,* $0 \neq x \in D$ *and* $E \subseteq K$. *The domain* $\mathbb{B}_x(E, D)$ *is completely integrally closed if and only if D is completely integrally closed.*

Proof It is easy to see that if $\mathbb{B}_x(E, D)$ is completely integrally closed, then so is D. Conversely, let $f \in K(X)$ be almost integral over $\mathbb{B}_x(E, D)$. Then there is a polynomial $0 \neq h \in \mathbb{B}_x(E, D)$ such that $hf^n \in \mathbb{B}_x(E, D)$ for each positive integer n. Since f is almost integral over $K[X]$ and $K[X]$ is completely integrally closed, then $f \in K[X]$. As $hf^n \in \mathbb{B}_x(E, D)$ for each positive integer n, then, for each $e \in E$, $h(xX + e)f^n(xX + e) \in D[X]$ for each n. This implies that $f(xX + e)$ is almost integral over $D[X]$, for each $e \in E$. Since $D[X]$ is completely integrally closed (as D is, cf. [8, section 13]), then $f(xX + e) \in D[X]$ for each $e \in E$. Thus, $f \in \mathbb{B}_x(E, D)$ and so $\mathbb{B}_x(E, D)$ is completely integrally closed.

We point out that, in general, Proposition 15 does not hold for $\text{Int}(E, D)$ as indicated by [4, Exercise 10, page 153].

By [3, Proposition 3.13] the ACCP property transfers from D to any ring R such that $D \subseteq R \subseteq \text{Int}(E, D)$. As a consequence we can state:

Proposition 16 *Let D be a domain,* $0 \neq x \in D$ *and E an (infinite) subset of K. Then* $\mathbb{B}_x(E, D)$ *has ACCP if and only if D has ACCP.*

In Proposition 16 the word "infinite"may be omitted: indeed, $\mathbb{B}_x(E, D)$ may be described only by a set of representatives of E modulo x, which may be a finite set.

Proposition 17 *Let D be a domain,* $0 \neq x \in D$. *Unless D is a field,* $\mathbb{B}_x(D)$ *is never a Bezout domain.*

Proof Assume that D is not a field. If x is a unit, then $\mathbb{B}_x(D) = D[X]$ is not a Bezout domain since D is not a field. So assume x is not a unit. Assume $(x, X) = (f)$ for some $f \in \mathbb{B}_x(D)$. Then $f(X) = xg_1(X) + Xg_2(X)$ for some $g_1, g_2 \in \mathbb{B}_x(D)$. This implies that $f(0) = xg_1(0) := d \in D$. Further, we have $X = f(X)h_1(X)$ and $x = f(X)h_2(X)$ for some $h_1, h_2 \in \mathbb{B}_x(D)$. From $x = f(X)h_2(X)$, we deduce that the degree of f is zero and hence $f(X) = d$ (As $x = f(0)h_2(0) = dh_2(0)$, then $x \neq 0$ implies $d \neq 0$). From $X = f(X)h_1(X)$, it follows that the degree of h_1 is one. Write $h_1(X) = \alpha X + \beta$. Then, on one hand, we get $\alpha = h_1(1) - h_1(0) \in D$ and, on the other hand, we get $1 = d\alpha$ and hence $\alpha = \frac{1}{d} = \frac{1}{xg_1(0)} \notin D$ since x is not a unit. This contradiction proves that $(x, X) \neq (f)$ and hence $\mathbb{B}_x(D)$ is not a Bezout domain.

5 The Prime Ideal Structure

In this section, we determine and study the prime ideals of $\mathbb{B}_x(E, D)$. Then we generalize some results that hold in the case of $\mathbb{B}_x(D)$ (see [1] and [11]).

Since the prime ideals of $\mathbb{B}_x(E, D)$ are defined by means of those of $D[X]$, we better first recall the prime ideals of $D[X]$. In fact, any prime ideal P of $D[X]$ is either an expansion ideal of $\mathfrak{p} = P \cap D$, that is of the form $\mathfrak{p}[X] = \{f(X) = a_0 + ... + a_n X^n \mid a_i \in \mathfrak{p}\}$ or $P =< \mathfrak{p}, f >$ is an upper to $\mathfrak{p}$ in $D[X]$: that is $P = \varphi^{-1}(f\chi(p)[X])$, where $\chi(p) = qf(D/p)$, $\varphi : A[X] \longrightarrow \mathrm{qf}(D/\mathfrak{p})[X]$ the natural ring homomorphism and $f \in D[X]$ modulo $\mathfrak{p}D_{\mathfrak{p}}$ is irreducible in $\mathrm{qf}(D/\mathfrak{p})[X]$ (for convenience, we say that φ is irreducible modulo $\mathfrak{p}$) (cf. [9, Theorem 1 and 2]).

Proposition 18 *Let $\mathfrak{p}$ be a prime ideal of D, with infinite residue field, and E a fractional subset of D that meets infinitely many cosets of $\mathfrak{p}$. Then, the prime ideals of $\mathbb{B}_x(E, D)$ above $\mathfrak{p}$ are of the following types:*

1. *$\mathfrak{p}D_{\mathfrak{p}}[X] \bigcap \mathbb{B}_x(E, D)$, which is the set of polynomials of $\mathbb{B}_x(E, D)$ with coefficients in $\mathfrak{p}D_{\mathfrak{p}}$.*
2. *$(\mathfrak{p}, h)D_{\mathfrak{p}}[X] \bigcap \mathbb{B}_x(E, D)$, which is the set of polynomials of $\mathbb{B}_x(E, D)$ which are divisible by h modulo $\mathfrak{p}D_{\mathfrak{p}}$, where $h \in D_{\mathfrak{p}}[X]$ is irreducible modulo $\mathfrak{p}D_{\mathfrak{p}}$,*

Proposition 19 *Let E be a fractional subset of D and $K = \mathrm{qf}(D)$. The nonzero prime ideals of $\mathbb{B}_x(E, D)$ above (0) are in one-to-one correspondence with the monic irreducible polynomials of $K[X]$. To each monic irreducible polynomial φ of $K[X]$ corresponds the prime ideal*

$$\mathfrak{B}_{\varphi} = \varphi K[X] \bigcap \mathbb{B}_x(E, D).$$

Proof Since E is a fractional subset of D, then $\mathbb{B}_x(E, D)$ contains a nonconstant polynomial of degree 1 (cf. Proposition 3). It follows that $S^{-1}\mathbb{B}_x(E, D) = K[X]$, where $S = D \setminus (0)$. Hence $Spec(S^{-1}\mathbb{B}_x(E, D)) = Spec(K[X])$. In fact, if $\mathfrak{p}$ is a prime ideal of $\mathbb{B}_x(E, D)$ with $\mathfrak{p} \bigcap D = (0)$, then $S^{-1}\mathfrak{p}$ is a prime ideal of $K[X]$. So, there exists an irreducible polynomial $\varphi \in K[X]$, such that $S^{-1}\mathfrak{p} = \varphi K[X]$. Thus $\mathfrak{p} = \mathfrak{B}_{\varphi} = \varphi K[X] \bigcap \mathbb{B}_x(E, D)$.

Now, if E is a subset of D and $K = qf(D)$, then by Corollary 1

$$D[X] \subseteq \mathbb{B}_x(E, D).$$

So, if $S = D \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of D, then

$$D_{\mathfrak{p}}[X] = S^{-1}D[X] \subseteq S^{-1}\mathbb{B}_x(E, D) \subseteq \mathbb{B}_x(E, S^{-1}D).$$

Thus, if $x \notin \mathfrak{p}$, then x is a unit of $S^{-1}D = D_{\mathfrak{p}}$. Hence $\mathbb{B}_x(E, S^{-1}D) = D_{\mathfrak{p}[X]}$. It follows that $S^{-1}\mathbb{B}_x(E, D) = D_{\mathfrak{p}}[X]$. In this case, the prime ideals of $\mathbb{B}_x(E, D)$ above $\mathfrak{p} \in Spec(D)$ are in one–one correspondence with those of $D_{\mathfrak{p}}[X]$ above $\mathfrak{p}D_{\mathfrak{p}}$.

Notice, in this case, that the prime ideal $\mathfrak{p}[X] = \mathfrak{p}D[X]$ lifts to $\mathbb{B}_x(E, D)$.

On the other hand, $\{Q \in Spec(\mathbb{B}_x(E, D); x \notin Q\}$ is in one–one correspondence with $Spec(D[\frac{1}{x}, X])$, that is with $\{P \in Spec(D[X]); x \notin P\}$: in deed, $N = \{x^n; n \geq 0\}$ is a multiplicative subset of both $D[X]$ and $\mathbb{B}_x(E, D)$, with

$$N^{-1}D[X] \subseteq N^{-1}\mathbb{B}_x(E, D) = \mathbb{B}_x(E, N^{-1}D) = N^{-1}D[X] = D[\frac{1}{x}, X].$$

Now, as in the case of $\mathbb{B}_x(D)$ (cf. [1, Lemma 3.1]), with E any subset of $\mathrm{qf}(D) = K$, the following ring homomorphisms are useful in order to define some particular prime ideals of $\mathbb{B}_x(E, D)$.

Lemma 2 *Let D be a domain, $\mathfrak{p}$ a prime ideal of D, E be a subset of K and $a \in E$.*

(1) The map

$$\begin{array}{rcl} \Psi_a : \mathbb{B}_x(E, D) & \rightarrow & D[X] \\ f(X) & \longmapsto & f(xX + a) \end{array}$$

is an injective ring homomorphism.

(2) The map

$$\begin{array}{rcl} \Phi_a : \mathbb{B}_x(E, D) & \rightarrow & D/\mathfrak{p} \\ f(X) & \longmapsto & f(a) + \mathfrak{p} \end{array}$$

is a surjective ring homomorphism.

Next we define some particular types of prime ideals of $\mathbb{B}_x(E, D)$. We then investigate inclusion relations among them.

Let D be an integral domain and $\mathfrak{p}$ a prime ideal of D. Let E be a subset of K and $a \in E$.
To avoid the case $\mathbb{B}_x(E, D) = D$, we assume that, $B_x(E, D)$ contains nonconstant polynomials (for instance E is a fractional subset of D). Set

$$\mathfrak{B}_{\mathfrak{p},a} = \{f \in \mathbb{B}_x(E, D) \mid f(a) \in \mathfrak{p}\}$$

$$\mathfrak{p}_a[X] = \{f \in \mathbb{B}_x(E, D) \mid f(xX + a) \in \mathfrak{p}[X]\}$$

$$\mathfrak{M}_{<\mathfrak{p},\varphi>,a} = \{f \in \mathbb{B}_x(E, D) \mid f(xX + a) \in <\mathfrak{p}, \varphi>\}$$

Proposition 20 *Under the previous hypotheses and notations, we have:*

(1) *$\mathfrak{p}_a[X]$, $\mathfrak{B}_{\mathfrak{p},a}$ and $\mathfrak{M}_{<\mathfrak{p},\varphi>,a}$ are prime ideals of $\mathbb{B}_x(E, D)$ above $\mathfrak{p}$.*
(2) *$\mathfrak{p}_a[X] \subseteq \mathfrak{p}B_{\mathfrak{p},a}$ and $\mathfrak{p}_a[X] \subseteq \mathfrak{M}_{<\mathfrak{p},\varphi>,a}$.*
(3) *Let $\mathfrak{p} \subset \mathfrak{q}$ in $Spec(D)$. Then $\mathfrak{p}_a[X] \subset \mathfrak{q}_a[X]$, $\mathfrak{B}_{\mathfrak{p},a} \subset \mathfrak{B}_{\mathfrak{p},a}$ and for each $<\mathfrak{p}, \varphi> \subset <\mathfrak{q}, \psi>$, we have $\mathfrak{M}_{<\mathfrak{p},\varphi>,a} \subset \mathfrak{M}_{<\mathfrak{q},\psi>,a}$. Furthermore, if $<\mathfrak{p}, \varphi> \subset \mathfrak{q}[X]$, then $\mathfrak{M}_{<\mathfrak{p},\varphi>,a} \subset \mathfrak{q}_a[X]$.*

Proof Using the previous ring homomorphisms, the same proof as in [1, Proposition 3.3] holds, just replace $\mathbb{B}_x(D)$ with $\mathbb{B}_x(E, D)$.

Next, we will determine maximal ideals of $\mathbb{B}_x(E, D)$ among those of Proposition 20.

Proposition 21 *Let D be a domain, $x \in D \setminus \{0\}$, E a subset of K and $a \in E$.*

(1) *Let* $\mathfrak{p} \in \mathrm{Spec}(D)$. *Then* $\mathbb{B}_x(E, D)/\mathfrak{B}_{p,a} \backsimeq D/\mathfrak{p}$.
(2) *Let* $\mathfrak{m}$ *be a maximal ideal of* D *and* $< \mathfrak{m}, \varphi >$ *be an upper to* $\mathfrak{m}$ *in* $D[X]$. *Then* $\mathfrak{B}_{\mathfrak{m},a}$ *and* $\mathfrak{M}_{<\mathfrak{m},\varphi>,a}$ *are maximal ideals of* $\mathbb{B}_x(E, D)$.

Proof To avoid the trivial case of $\mathbb{B}_x(E, D) = D$, we assume that $\mathbb{B}_x(E, D)$ contains nonconstant polynomials. Then replace $B_x(D)$ with $B_x(E, D)$ in the proof of [1, Proposition 3.4].

Proposition 22 *Let* D *be a domain,* E *a subset of* D *and* $0 \neq x \in D$. *Let* $\mathfrak{p} \in \mathrm{Spec}(D)$ *and* $a \in E$.

(1) *If* $x \notin \mathfrak{p}$, *then:* $\mathfrak{p}_a[X] \cap D[X] = \mathfrak{p}[X]$. *Moreover,* $\mathfrak{p}_a[X]$ *is a proper subset of both* $\mathfrak{B}_{\mathfrak{p},a}$ *and* $\mathfrak{M}_{<\mathfrak{p},\eta>,a}$.
(2) *If* $x \in \mathfrak{p}$, *then* $\mathfrak{p}[X]$ *is a proper subset of* $\mathfrak{p}_a[X] \cap D[X]$. *Furthermore, if* $D/\mathfrak{p}$ *is infinite and* E *meets infinitely many cosets modulo* $\mathfrak{p}$, *then* $\mathfrak{p}[X]$ *lifts to a prime ideal* $\mathscr{P} \in \mathrm{Spec}(\mathbb{B}_x(E, D))$ *and* $\mathscr{P} \subset \mathfrak{p}_a[X] = \mathfrak{B}_{\mathfrak{p},a} = \mathfrak{M}_{<\mathfrak{p},\eta>,a}$, *for each* $\eta \in D[X]$ *with* η *irreducible modulo* $\mathfrak{p}$.

Proof We just reproduce the proof of [1, Proposition 3.4], with necessary changing.

First of all by Proposition 20, $\mathfrak{p}_a[X]$ is a subset of both $\mathfrak{B}_{\mathfrak{p},a}$ and $\mathfrak{M}_{<\mathfrak{p},\eta>,a}$, for each $\eta \in D[X]$ with η irreducible modulo $\mathfrak{p}$. On the other hand, Since E is a subset of D, by Corollary 1, $D[X] \subseteq \mathbb{B}_x(E, D)$.

(1). Assume that $x \notin \mathfrak{p}$. Localizing at $S_x = \{x^n | n \geq 0\}$, we get $S_x^{-1}\mathbb{B}_x(E, D) = D[\frac{1}{x}, X]$. Then $S_x^{-1}\mathfrak{p}_a[X] \subseteq S_x^{-1}\mathfrak{B}_{\mathfrak{p},a}$ are prime ideals of $D[\frac{1}{x}, X]$. Since $X - a \in D[\frac{1}{x}, X]$, then $S_x^{-1}\mathfrak{p}_a[X] = \mathfrak{p}D_\mathfrak{p}[\frac{1}{x}, X]$ and $S_x^{-1}\mathfrak{B}_{\mathfrak{p},a} =< \mathfrak{p}, X - a > D_\mathfrak{p}[\frac{1}{x}, X]$. Hence

$$\mathfrak{p}_a[X] \cap D[X] = S_x^{-1}\mathfrak{p}_a[X] \cap \mathbb{B}_x(E, D) \cap D[X] = \mathfrak{p}D_\mathfrak{p}[\frac{1}{x}, X] \cap D[X] = \mathfrak{p}[X].$$

In a similar way, we get $\mathfrak{B}_{\mathfrak{p},a} \cap D[X] =< \mathfrak{p}, X - a > D_\mathfrak{p}[\frac{1}{x}, X] \cap D[X] =< \mathfrak{p}, X - a >$. Since $x \notin \mathfrak{p}$, then $X - a \in \mathfrak{B}_{\mathfrak{p},a} \setminus \mathfrak{p}_a[X]$. Thus, $\mathfrak{p}_a[X] \subset \mathfrak{B}_{\mathfrak{p},a}$. On the other hand, since $x \notin \mathfrak{p}$, then x is a unit in $D_\mathfrak{p}$. Hence, the fact that $\mathfrak{p}_a[X] \subset \mathfrak{M}_{<\mathfrak{p},\eta>,a}$ is a consequence of Proposition 20 and Proposition 21. On the other hand, If $D/\mathfrak{p}$ is finite, then $\mathfrak{p}$ is a maximal ideal of D. By Proposition 21, $\mathfrak{p}_{<\mathfrak{p},\eta>,a}$ is a maximal ideal of $\mathbb{B}_x(E, D)$. Thus, $\mathfrak{p}_a[X] \subset \mathfrak{M}_{<\mathfrak{p},\eta>,a}$. If $D/\mathfrak{p}$ is infinite, by Remark 3, $\mathbb{B}_x(E, D_\mathfrak{p}) = \mathbb{B}_x(E, D)_\mathfrak{p} = D_\mathfrak{p}[X]$. With $S = D \setminus \mathfrak{p}$, it is easy to see that $S^{-1}\mathfrak{p}_a[X] = (S^{-1}\mathfrak{p})_a[X]$ and $S^{-1}\mathfrak{M}_{<\mathfrak{p},\eta>,a} = \mathfrak{M}_{<S^{-1}\mathfrak{p},\eta>,a}$. It follows that $\mathfrak{p}_a[X] \subset \mathfrak{M}_{<\mathfrak{p},\eta>,a}$.

(2). Since $x \in \mathfrak{p}$, then $X - a \in \mathfrak{p}_a[X]$. It follows that $X - a \in \mathfrak{p}_a[X] \cap D[X]$ and thus $\mathfrak{p}[X] \subset \mathfrak{p}_a[X] \cap D[X]$. Now, Assume that $D/\mathfrak{p}$ is infinite and E meets infinitely many cosets modulo $\mathfrak{p}$. By Remark 3, $\mathbb{B}_x(E, D)_\mathfrak{p} = \mathbb{B}_x(E, D_\mathfrak{p}) =$

$D_{\mathfrak{p}}[X]$. So, set $\mathscr{P} = \mathfrak{p}D_{\mathfrak{p}}[X] \cap \mathbb{B}_x(E, D)$. Then $\mathscr{P} \cap D[X] = \mathfrak{p}[X]$ and necessarily $\mathscr{P} \subset \mathfrak{p}_a[X]$.

On the other hand, $< \mathfrak{p}, X - a >= \mathfrak{p}[X] + (X - a)D[X] \subseteq \mathfrak{p}_a[X] \subseteq \mathfrak{B}_{\mathfrak{p},a}$. It follows that $S^{-1} < \mathfrak{p}, X - a > \subseteq S^{-1}\mathfrak{p}_a[X] \subseteq S^{-1}\mathfrak{B}_{\mathfrak{p},a}$ retracts to the same ideal $< \mathfrak{p}, X - a >$ of $D[X]$. It follows that $\mathfrak{p}_a[X] = \mathfrak{B}_{\mathfrak{p},a} = \mathfrak{M}_{<\mathfrak{p},\eta>,a}$, for each $\eta \in D[X]$ with η irreducible modulo $\mathfrak{p}$.

Next, if $E \subseteq D$, we show that for each $a \in E$ and for $x \in \mathfrak{p}$, the three prime ideals $\mathfrak{p}_a[X]$, $\mathfrak{B}_{\mathfrak{p},a}$, and $\mathfrak{M}_{<\mathfrak{p},\varphi>,a}$ of $\mathbb{B}_x(E, D)$ always have the same trace in $D[X]$.

Proposition 23 *Let D be a domain and $E \subseteq D$. Let $\mathfrak{p} \in Spec(D)$, $0 \neq x \in \mathfrak{p}$ and $a \in E$. Then*

$$\mathfrak{p}_a[X] \cap D[X] = \mathfrak{B}_{\mathfrak{p},a} \cap D[X] = \mathfrak{M}_{<\mathfrak{p},\varphi>,a} \cap D[X].$$

Proof Since $E \subseteq D$, then $D[X] \subseteq \mathbb{B}_x(E, D)$. The proof is then similar to that of [1, Proposition 3.6]

Remark 5 Let D be an integral domain, $0 \neq x \in D$ and $E \subseteq D$ such that $E \cap xD \neq \emptyset$. Then, for each $f \in \mathbb{B}_x(E, D)$, of degree n, $x^n f \in D[X]$: indeed, $E \cap xD \neq \emptyset$ implies that $f \in D[\frac{X}{x}]$ and thus, $x^n f \in D[X]$.

Proposition 24 *Let D be an integral domain, $0 \neq x \in D$ and $E \subseteq D$ such that $E \cap xD \neq \emptyset$. Let $\mathfrak{p} \in \mathrm{Spec}(D)$. if either $x \notin \mathfrak{p}$ or E meets infinitely $D/\mathfrak{p}$, then $\mathfrak{B}_{p,a} = \mathfrak{B}_{\mathfrak{p},b}$ if and only if $a \equiv b \mod \mathfrak{p}$.*

Proof The same as in the proof of [1, Proposition 3.9]: just replace Lemma 2.8 (resp., Proposition 2.2) with Remark 5 (resp., Proposition 6).

Proposition 25 *Let D be an integral domain, $0 \neq x \in D$ and $E \subseteq D$. Let $\mathfrak{p} \in \mathrm{Spec}(D)$ and $a, b \in E$ with $a - b \in xD$. Then $\mathfrak{p}_a[X] = \mathfrak{p}_b[X]$.*

Proof The fact that $a \equiv b \mod xD$ implies that $\frac{X-a}{x} = \frac{X-b}{x} + \alpha$, for some $\alpha \in D$. Thus $\mathfrak{p}[\frac{X-a}{x}] = \mathfrak{p}[\frac{X-b}{x}]$. The result follows by intersecting with $\mathbb{B}_x(E, D)$.

6 Krull and Valuative Dimension

In this section, we will compute the valuative dimension of $\mathbb{B}_x(E, D)$ and give conditions under which, $\mathbb{B}_x(E, D)$ is a Jaffard domain. But we first establish bounds for the Krull dimension of $\mathbb{B}_x(E, D)$.

Recall that the Krull dimension of a domain A, denoted by $\dim A$, is defined to be the largest length of all possible chains of prime ideals in A. The valuative dimension of A, denoted by $\dim_v A$, is defined to be the supremum of $\dim V$, where V runs over all valuation overrings of A. When $\dim A = \dim_v A$, then A is called a Jaffard domain

[2]. Noetherian, valuation, Dedekind and Prüfer domains are all examples of Jaffard domains.

Proposition 26 *Let D be a domain with quotient field K, E be a subset of K, and $0 \neq x \in D$. Then* $\dim \mathbb{B}_x(E, D) \geq \dim D$.

Furthermore, if E is a fractional subset of D, then $\dim \mathbb{B}_x(E, D) \geq \dim D + 1$.

Proof Let $\dim D = n$, $e \in E$ and let $(0) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of length n in D. Then $\mathfrak{B}_{\mathfrak{p}_0,e} \subset \mathfrak{B}_{\mathfrak{p}_1,e} \subset \cdots \subset \mathfrak{B}_{\mathfrak{p}_n,e}$ is a chain of prime ideals of length n in $\mathbb{B}_x(E, D)$ (for some $e \in D$). This yields the first inequality. If E is a fractional subset of D, then $dE \subseteq D$ for some $0 \neq d \in D$. Since $dX - de \in \mathfrak{B}_{\mathfrak{p}_0,e}$, then $(0) \subset \mathfrak{B}_{\mathfrak{p}_0,e}$ and hence the second inequality follows.

Theorem 1 *Let D be a domain with quotient field K, E be a subset of K, and $0 \neq x \in D$. Assume that $\mathbb{B}_x(E, D)$ contains nonconstant polynomials (for instance if E is a fractional subset of D). Then* $\dim \mathbb{B}_x(E, D) \geq \dim D[X] - 1$.

Proof The same proof as for [1, Theorem 4.2], just replace Proposition 3.4 (resp., 3.5) with Proposition 19 (resp., 2).

Remark 6 If $B_x(E, D)$ does not contain nonconstant polynomials, then the inequality in Theorem 1 fails to be true. For instance, [7, Example 4.4] give a non Jaffard two-dimensional domain D, with $\dim D[X] = 5$. So, Take $E = \mathrm{qf}(D)$, then for any $0 \neq x \in D$, one necessarily has $\mathbb{B}_x(E, D) = D$. It follows that, $\dim \mathbb{B}_x(E, D) = \dim D = 2 < 4 = \dim D[X] - 1$.

However, in case D is either a Jaffard of a Strong S–domain, then $\dim D[X] = \dim D + 1$. Thus even if $B_x(E, D)$ does not contain nonconstant polynomials, one has $\dim \mathbb{B}_x(E, D) \geq \dim D[X] - 1 = \dim D$ (see also Proposition 26).

Theorem 2 *Let D be a domain with quotient field K and $a \in K$. Let B be a domain such that $D \subset B \subseteq D + (X - a)K[X]$. If B contains a nonconstant polynomial, then:*

(1) $\dim(B) \geq 1 + \dim D$.
(2) $\dim_v(B) = 1 + \dim_v D$.
(3) *D is a Jaffard domain if, and only if B is a Jaffard domain and* $\dim B = 1 + \dim D$.

Proof Since B contains a nonconstant polynomial, then we can easily see that it may be of the form $f(X) = (X - a)h(X) \in D[X]\backslash\{0\}$.

(1.) Set $I = (X - a)K[X]$ and let $0 = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of D of length $n = \dim D$. By [6, Lemma 1.1], it follows that $0 \subset I \subset \mathfrak{p}_1 + I \subset \cdots \subset \mathfrak{p}_n + I$ is a chain of prime ideals of $D + (X - a)K[X]$ of length $n + 1 = \dim D + 1$. On the other hand, $f(X) \in B \cap I$ implies that $B \cap I \neq 0$ and hence $0 \subset I \cap B \subset (\mathfrak{p}_1 + I) \cap B \subset \cdots \subset (\mathfrak{p}_n + I) \cap B$ is a chain of prime ideals in B of length $n + 1 = \dim D + 1$. It follows that $\dim B \geq 1 + \dim D$.

(2). Note first that we have $D[f] \subseteq B \subseteq D + (X - a)K[X] \subseteq K(X)$. Clearly $K(X)$ is algebraic over $D[f]$ and thus, by [8, Theorem 30.8], we have $\dim_v D + (X - a)K[X] \le \dim_v B \le \dim_v D[f]$. By [6, Lemma 2.2], $\dim_v D + (X - a)K[X] = 1 + \dim_v D$. Since $D[f]$ is isomorphic to $D[X]$, then $\dim_v D[f] = 1 + \dim_v D$. Thus $\dim_v B = 1 + \dim_v D$.

(3). If D is a Jaffard domain, then $\dim_v D = \dim D$. It follows from statement (2) that $\dim_v B = 1 + \dim D$. From statement (1) we get

$$1 + \dim D \le \dim B \le \dim_v B = 1 + \dim D.$$

Thus $\dim B = 1 + \dim D = \dim_v B$ and B is a Jaffard domain. Conversely, if B is a Jaffard domain of dimension $1 + \dim D$, then, by statement (2), $\dim D = \dim_v D$ and D is Jaffard.

Corollary 6 *Let D be a domain with quotient field K and E be a subset of K. Let $0 \neq x \in D$ be such that $\mathbb{B}_x(E, D)$ contains a nonconstant polynomial (for instance, if E is a fractional subset of D). Then,*

(a) $\dim \mathrm{Int}(E, D) \ge 1 + \dim D$.
(b) $\dim \mathbb{B}_x(E, D) \ge 1 + \dim D$.
(c) $\dim_v \mathbb{B}_x(E, D) = \dim_v \mathrm{Int}(E, D) = 1 + \dim_v D$.
(d) *The following statements are equivalent:*

(i) *D is a Jaffard domain;*
(ii) $\mathrm{Int}(E, D)$ *is a Jaffard domain and* $\dim \mathrm{Int}(E, D) = 1 + \dim D$*;*
(iii) $\mathbb{B}_x(E, D)$ *is a Jaffard domain and* $\dim \mathbb{B}_x(E, D) = 1 + \dim D$.

Proof Let $a \in E$. Then

$$D \subseteq \mathbb{B}_x(E, D) \subseteq \mathrm{Int}(E, D) \subseteq \mathrm{Int}(\{a\}, D) = D + (X - a)K[X].$$

Thus, the result follows if we take either $B = \mathbb{B}_x(E, D)$ or $B = \mathrm{Int}(E, D)$ in Theorem 2.

Corollary 7 *Let D be a domain with quotient field K and E a subset of K. if $D[X]$ is a Jaffard domain then* $\dim \mathbb{B}_x(E, D) \le \dim D[X]$.

Lemma 3 *Let D be a domain, $0 \neq x \in D$ and $E \subseteq xD$. Then $\mathbb{B}_x(E, D) = D[\frac{X}{x}]$.*

Proof Since $E \subseteq xD$, so there exists $A \subseteq D$ such that $E = Ax$. Thus

$$\mathbb{B}_x(E, D) = \bigcap_{e \in E} D[\frac{X - e}{x}] = \bigcap_{a \in A} D[\frac{X - ax}{x}] = \bigcap_{a \in A} D[\frac{X}{x} - a] = D[\frac{X}{x}].$$

Remark 7 (1) By Lemma 3, For each domain D and each $0 \neq x \in D$, $\mathbb{B}_x(xD, D) = D[\frac{X}{x}] \simeq D[X]$. Thus, $\dim \mathbb{B}_x(xD, D) = \dim D[X]$.
(2) If further $D[X]$ is a Jaffard domain, then $\mathbb{B}_x(xD, D)$ is a Jaffard domain. So if D is not Jaffard, necessarily $\dim \mathbb{B}_x(xD, D) > 1 + \dim D$.

(3) The inequality $\dim B_x(E, D) \leq \dim D[X]$ (cf, Corollary 7) may hold even if $D[X]$ is not a Jaffard domain. for instance in [1, Example 5.2] we constructed a domain of the form: $D = k + M$ such that:

- $\dim D = 2$, $\dim_v D = 3$ and $\dim D[X] = 1 + \dim D = 3$,
- D and $D[X]$ are not Jaffard.

So taking $D/M = k$ finite and x a nonzero non-unit of D, then for each positive integer n, $\mathbb{B}_x(x^n D, D) \neq D[X]$ and $\dim \mathbb{B}_x(x^n D, D) = \dim D[X]$.

By Proposition 3, if E is a fractional subset of D, then $\mathbb{B}_x(E, D)$ contains nonconstant polynomials. However, if E is not a fractional subset of D, we may have $\mathbb{B}_x(E, D) = D$, for instance, take $D = \mathbb{Z}$, $E = S^{-1}\mathbb{Z}$ with S any nontrivial multiplicative subset of $\mathbb{Z}$. However, it may happen that $\mathbb{B}_x(E, D)$ contains a nonconstant polynomial, even if E is not a fractional subset of D, as the following example shows:

Example 1 (cf. [4, Exercise 8, p. 20]). We construct a domain D and a non-fractional subset E of D, such that:

(1) $\mathbb{B}_x(E, D)$ contains nonconstant polynomials (cf. Proposition 2 and 3).

(2) $\mathbb{B}_x(E, D)$ is a Jaffard domain, with $\dim \mathbb{B}_x(E, D) = \dim D[X] = 1 + \dim D$.

Let k be a field of characteristic $p \neq 0$ and $V = k[[t]]$ be the power series ring with coefficients in k: V is the ring of a discrete valuation v on the field $k((t))$. As stated in the reference above:

(i) Let $y \in V$ be such that y and t are algebraically independent over k. Set $K = k(t, y^p)$ and $L = k(t, y)$. Then $K \subset L \subset k((t))$. Further, L is a purely inseparable algebraic extension of degree p over K and $L^p \subseteq K$.
(ii) Let $W = V \cap K$ and $D = W[y]$. Then D is a one-dimensional Noetherian domain (W is DVR) with quotient field L.
(iii) The integral closure of D is $D' = V \cap L$.
(iv) Let w be the restriction of the valuation v to K. The ring of the valuation w is W and w extends uniquely to L. The ring of this extension is D'.
(v) D' is not a finitely generated D–module. However, $D = W[y]$ is a finitely generated W–module.

(1). By the statement (v), D' is not a fractional subset of D. Now let $x \in D$. Then for each $b \in D'$, we have $(xX + b)^p = x^p X^p + b^p$, (the remaining coefficients are multiples of p and hence are zero). On the other hand, $b^p \in D$ since $(D')^p \subseteq D$. It follows that the nonconstant polynomial X^p belongs to $\mathbb{B}_x(D', D)$.
(2). Now since D is Noetherian and one-dimensional, then D is one-dimensional Jaffard domain and by applying Corollary 6, we see that for each $x \in D$, the ring $\mathbb{B}_x(D', D)$ is a two-dimensional Jaffard domain, with $\dim \mathbb{B}_x(D', D) = \dim D[X] = 1 + \dim D$.

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On Commutativity of Banach *-Algebras with Derivation

Mohammad Ashraf and Bilal Ahmad Wani

Abstract The aim of this paper is to apply purely ring theoretic results to discuss the commutativity of a Banach algebra and Banach *-algebra via derivations. We prove that if $\mathfrak{A}$ is a semiprime Banach algebra and $\mathscr{G}$ a nonempty open subsets of $\mathfrak{A}$ which admits a nonzero continuous linear derivation $d : \mathfrak{A} \to \mathfrak{A}$ such that $d([x^m - x, y]) \in Z(\mathfrak{A})$ for each x in $\mathscr{G}$ and an integer $m = m(x) > 1$, then $\mathfrak{A}$ is commutative. Further, we discuss the commutativity of Banach *-algebra. In particular, it is shown that either a semiprime Banach *-algebra $\mathfrak{A}$ with continuous involution and derivation is commutative or the set of $x \in \mathfrak{A}$ for which $[d(x^k), d((x^k)^*)] \in Z(\mathfrak{A})$ for no positive integer $k \geq 1$, is dense in $\mathfrak{A}$. Finally, few more parallel results have been established about the commutativity of Banach and Banach *-algebras.

Keywords Commutativity · Derivations · Banach algebras · Banach *-algebras

1 Introduction

Let $\mathfrak{A}$ denote a Banach algebra over the complex field $\mathbb{C}$ with identity e, $Z(\mathfrak{A})$ denote the centre of $\mathfrak{A}$ and $\mathscr{M}$ be a closed linear subspace of $\mathfrak{A}$. Recall that an algebra $\mathfrak{A}$ is said to be prime if for any $a, b \in \mathfrak{A}$, $a\mathfrak{A}b = \{0\}$ implies $a = 0$ or $b = 0$, and $\mathfrak{A}$ is semiprime if for $a \in \mathfrak{A}$, $a\mathfrak{A}a = \{0\}$ implies $a = 0$. For any $x, y \in \mathfrak{A}$, the symbol $[x, y]$ will denote the commutator $xy - yx$. A linear mapping $x \mapsto x^*$ of $\mathfrak{A}$ into itself is called an involution on $\mathfrak{A}$ if it satisfies the conditions: (i) $(x^*)^* = x$, (ii) $(xy)^* = y^*x^*$ for all $x, y \in \mathfrak{A}$. A Banach algebra $\mathfrak{A}$ equipped with an involution * such that $\| x^* \| = \| x \|$ is called a Banach *-algebra. An element x of Banach *-algebra is said to be self-adjoint if $x^* = x$. We say that $x \in \mathfrak{A}$ is normal modulo

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$\mathscr{M}$ if $[x, x^*] \in \mathscr{M}$. A linear mapping $d : \mathfrak{A} \to \mathfrak{A}$ is said to be a derivation on $\mathfrak{A}$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathfrak{A}$.

In ring theory, much attention has been devoted to show that certain rings must be commutative as a consequence of conditions which are seemingly too weak to imply commutativity. This work was initiated largely by Jacobson, Kaplansky and especially Herstein and has continued up to the present time. Yood in [1, 2] and [3] pursued the same aim for Banach algebras.

One of the first mathematicians to follow-up Jacobson‘s result was Herstein [4] who proved that if there exists a positive integer n in a ring R such that $x^n - x$ is in the center of R, then R is commutative. In 1955 and 1957, Herstein [5, 6] proved that for a ring R, to be commutative the following conditions are necessary and sufficient:

(H_1) For all x and y in R there exists an integer $n = n(x, y) \geq 2$ such that $(x^n - x)y = y(x^n - x)$;

(H_2) For all x and y in R there exists an integer $n = n(x, y) \geq 2$ such that $xy - yx = (xy - yx)^n$;

If a ring is semisimple then the following are necessary and sufficient for commutativity:

(H_3) For all x and y in R there exists an integer $n = n(x, y) \geq 1$ such that $x^n y = yx^n$;

(H_4) For all x and y in R there exists an integer $n = n(x, y) \geq 2$ such that $(xy)^n = x^n y^n$.

Further several authors have done tremendous work in this area for reference see [7–15] etc. where more references can be found. Another technique for investigating commutativity of rings (algebras) is the use of additive mappings like derivations and automorphisms of the ring R. To indicate how strongly related a derivation is to commutativity, we say a derivation (or other function) $d : R \to R$ is commuting if $[d(x), x] = 0$ for all $x \in R$, and centralizing if $[d(x), x] \in Z(R)$ for all $x \in R$. The study of such mappings was initiated by Posner (Posner second theorem). In [16, Theorem 2], Posner proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. A number of authors have generalized Posner's result in the setting of rings and algebras (see [17–21]). Considerable attention has been paid to commutativity theorems for rings and algebras (see, for example, Chap. 2 of [22, 23] and Chap. 3 of [24], where further references can be found). Herstein [25] connected commutativity and derivations by proving that if a prime ring R admits a derivation $d \neq 0$ such that $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$ then, (i) if $char\, R = 2$, then R is a commutative integral domain, and (ii) if $char\, R = 2$, then R is commutative or an order in a simple algebra which is 4-dimensional over its center.

There has been a great deal of work concerning the relationship between the commutativity of rings and algebras and the existence of certain specified additive mappings like derivations (see [12, 13, 26–30] where further references can be found). The objective of this paper is to investigate the commutativity of Banach algebras and Banach *-algebras involving derivations.

2 Results on Banach Algebras

This research is motivated by the work of Yood [1]. Throughout this section $\mathfrak{A}$ will denote a Banach algebra over the complex field $\mathbb{C}$ with identity e and $Z(\mathfrak{A})$ denotes the centre of $\mathfrak{A}$. Here an important tool is the Baire category theorem. In this section we will use the ring theoretic results to study the commutativity of Banach algebra. We shall use several times the readily established fact that $p(t) = \sum_{r=0}^{n} b_r t^r$ is a polynomial in real variable t for infinite values of t and each $b_r \in \mathfrak{A}$. If $p(t) \in \mathscr{M}$, then each b_r lies in $\mathscr{M}$ (see [22]).

A motivating theorem for our study is the following result due to Yood [1, Theorem 2.5].

Theorem 1. *Let $u \in \mathfrak{A}$. Suppose that there is a nonempty open set G in $\mathfrak{A}$ such that, for each $x \in G$, we have a positive integer $n = n(x) > 1$ such that*

$$[x^n - x, u] \in \mathscr{M}.$$

For $M = (0)$ we have $u \in Z(\mathfrak{A})$. If $\mathfrak{A}$ has no nonzero nilpotent ideals and $M = Z(\mathfrak{A})$, then $u \in Z(\mathfrak{A})$.

We begin with the following results which are important for developing the proof of main result.

Lemma 1. *Let $\mathfrak{A}$ be a Banach algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a linear mapping. Suppose that $z \in \mathfrak{A}$ and $m \geq 1$ be a positive integer such that $d([z, x^m]) \in \mathscr{M}$ for all $x \in \mathfrak{A}$, then $d([z^m, x]) \in \mathscr{M}$ for all $x \in \mathfrak{A}$.*

Proof. Since $d([z, x^m]) \in \mathscr{M}$ for all $x \in \mathfrak{A}$. Therefore, for each real t, we have

$$d([z, (z + tx)^m]) \in \mathscr{M}.$$

The expression $d([z, (z + tx)^m])$ can be written as

$$d([z, (z + tx)^m]) = d([z, P_{m,0}(z, x)]) + d([z, P_{m-1,1}(z, x)])t + \cdots \\ + d([z, P_{1,m-1}(z, x)])t^{m-1} + d([z, P_{m,0}(z, x)])t^m.$$

Let i, j be nonnegative integers. If $i + j = m$, then $P_{i,j}(z, x)$ denotes the sum of all the terms in which z appears exactly i times and x appears exactly j times in the expansion of $(z + tx)^m$. The above expression is a polynomial in t and the coefficient of t in this polynomial is $d[z, \sum\limits_{k=0}^{m-1} z^k x z^{m-1-k}]$. Therefore, we have

$$d([z, \sum_{k=0}^{m-1} z^k x z^{m-1-k}]) = d([z^m, x]) \in \mathscr{M}.$$

Theorem 2. *Let $\mathfrak{A}$ be a Banach algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a continuous linear mapping. Suppose that there is a nonempty open set G in $\mathfrak{A}$ such that $d([x^m, y^n]) \in \mathscr{M}$ for each x, y in G and positive integers $m = m(x, y) \geq 1$ and $n = n(x, y) \geq 1$. Then there exists a fixed positive integer $k \geq 1$ such that $d([x^k, y]) \in \mathscr{M}$ for all $x, y \in \mathfrak{A}$.*

Proof. Fix $z \in G$. For each positive integers m, n; we define the set

$$V_{m,n} = \{y \in \mathfrak{A} | d([z^m, y^n]) \notin \mathscr{M}\}.$$

We claim that each $V_{m,n}$ is open in $\mathfrak{A}$. To show that $V_{m,n}$ is open we have to show that $V^c_{m,n}$ is closed. For this, we take a sequence $(w_k) \in V^c_{m,n}$ such that $w_k \to w$ as $k \to \infty$ and prove that $w \in V^c_{m,n}$. Since $w_k \in V^c_{m,n}$,

$$d([z^m, w_k^n]) \in \mathscr{M}.$$

Taking limit on k, we obtain

$$\lim_{k\to\infty} d([z^m, w_k^n]) \in \mathscr{M}.$$

Since d is continuous, we have

$$\lim_{k\to\infty} d([z^m, w_k^n]) = d([z^m, \lim_{k\to\infty} w_k^n]) = d([z^m, w^n])$$

is in $\mathscr{M}$. This implies that $w \in V^c_{m,n}$, so $V^c_{m,n}$ is closed and hence $V_{m,n}$ is open. If every $V_{m,n}$ is dense in $\mathfrak{A}$ then, by the Baire category theorem their intersection is also dense. But this would contradict the existence of G. Therefore there are positive integers r and s so that $V_{r,s}$ is not dense and a nonempty open set Q in the compliment of $V_{r,s}$ such that $d([z^r, u^s]) \in \mathscr{M}$ for all $u \in Q$. If $u_0 \in Q$ and $x \in \mathfrak{A}$, then $u_0 + tx \in Q$, for all sufficiently small real t. Therefore, we have

$$d([z^r, (u_0 + tx)^s]) \in \mathscr{M}.$$

It can be easily seen that

$$\begin{aligned} d([z^r, (u_0 + tx)^s]) &= d([z^r, A_{s,0}(u_0, x)]) + d([z^r, A_{s-1,1}(u_0, x)])t + \cdots \\ &\quad + d([z^r, A_{1,s-1}(u_0, x)])t^{s-1} + d([z^r, A_{0,s}(u_0, x)])t^s. \end{aligned}$$

Let i, j be nonnegative integers. If $i + j = s$, then $A_{i,j}(u_0, x)$ denotes the sum of all terms in which u_0 appears exactly i times and x appears exactly j times in the expansion of $(u_0 + tx)^s$. The above expression is a polynomial in t and the coefficient of t^s in this polynomial is $d([z^r, x^s])$. Therefore we obtain $d([z^r, x^s]) \in \mathscr{M}$ for all $x \in \mathfrak{A}$. By Lemma 1 we obtain $d([z^{rs}, x]) \in \mathscr{M}$ for all $x \in \mathfrak{A}$.

Next fix $y \in \mathfrak{A}$ and for each positive integer k, set $S_k = \{x \in \mathfrak{A} | d([x^k, y]) \notin \mathscr{M}\}$. Each S_k is open (as shown above). If every S_k is dense in $\mathfrak{A}$ then, by the Baire category theorem their intersection is also dense. But this would contradict the existence of G. Therefore there exists a positive integer m so that S_m is not dense and a nonempty open set P in the compliment of S_m such that $d[x^m, y] \in \mathscr{M}$ for all $x \in P$. If $z \in P$ and $w \in \mathfrak{A}$, then $z + tw \in P$ for all sufficiently small real t. Therefore, we have

$$d([(z + tw)^m, y]) \in \mathscr{M}.$$

Arguing in a similar manner, we see that $d([w^m, y]) \in \mathscr{M}$ for all $y \in \mathfrak{A}$ and $w \in \mathfrak{A}$.

Corollary 1. *Let $\mathfrak{A}$ be a semiprime Banach algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a nonzero continuous linear derivation. Suppose that there is a nonempty open set G in $\mathfrak{A}$ such that $d([x^m, y^n]) \in Z(\mathfrak{A})$ for each x, y in G and positive integers $m = m(x, y) \geq 1$ and $n = m(x, y) \geq 1$. Then $\mathfrak{A}$ is commutative.*

Proof. By using Theorem 2, there exists a positive integer k so that $d([x^k, y]) \in Z(\mathfrak{A})$ for all $x, y \in \mathfrak{A}$. Since $\mathfrak{A}$ is semiprime, by applying Theorem 2.5 of [31] we see that $\mathfrak{A}$ is commutative.

Theorem 3. *Let $\mathfrak{A}$ be a Banach algebra, $u \in \mathfrak{A}$ and $d : \mathfrak{A} \to \mathfrak{A}$ be a nonzero continuous linear mapping. Suppose that there is a nonempty open set G in $\mathfrak{A}$ such that $d([x^m - x, u]) \in \mathscr{M}$ for each x in G and an integer $m = m(x) > 1$. Then $d([x, u]) \in \mathscr{M}$ for all $x \in \mathfrak{A}$.*

Proof. For each positive integer $m > 1$, we define the set

$$Q_m = \{y \in \mathfrak{A} | d([y^m - y, u]) \notin \mathscr{M}\}.$$

We claim that each Q_m is open in $\mathfrak{A}$. To show that Q_m is open we have to show that Q_m^c is closed. For this, we take a sequence $(z_k) \in Q_m^c$ such that $z_k \to z$ as $k \to \infty$ and prove that $z \in Q_m^c$. Since $z_k \in Q_m^c$, then

$$d([z_k^m - z_k, u]) \in \mathscr{M}.$$

Taking limit on k, we obtain

$$\lim_{k\to\infty} d([z_k^m - z_k, u]) \in \mathscr{M}.$$

Since d is continuous, we have

$$\lim_{k\to\infty} d([z_k^m - z_k, u]) = d([\lim_{k\to\infty} z_k^m - \lim_{k\to\infty} z_k, u]) = d([z^m - z, u])$$

is in $\mathscr{M}$. This implies that $z \in Q_m^c$, so Q_m^c is closed and hence Q_m is open. If every Q_m is dense in A then, by the Baire category theorem their intersection is also dense.

But this would contradict the existence of G. Therefore there exists a positive integers r so that Q_r is not dense and a nonempty open set P in the compliment of Q_r such that

$$d([z^r - z, u]) \in \mathscr{M} \text{ for all } z \in P. \tag{1}$$

If $x \in \mathfrak{A}$, and $z \in P$, then $z + tx \in P$ for all sufficiently small real t. Therefore, we have

$$d([(z + tx)^r - (z + tx), u]) \in \mathscr{M}.$$

This can be written as

$$\begin{aligned} &d([A_{r,0}(z, x) + A_{r-1,1}(z, x)t + \cdots + A_{1,r-1}(z, x)t^{r-1} + A_{0,r}(z, x)t^r - (z + tx), u]) \\ &= d([A_{r,0}(z, x) - z, u]) + d([A_{r-1,1}(z, x) - x, u])t + \cdots \\ &+d([A_{1,r-1}(z, x), u])t^{r-1} + d([A_{0,r}(z, x)], u)t^r. \end{aligned}$$

Let i, j be nonnegative integers. If $i + j = r$, then $A_{i,j}(z, x)$ denotes the sum of all terms in which z appears exactly i times and x appears exactly j times in the expansion of $(z + tx)^r$. The above expression is a polynomial in t and the coefficient of t^r in this polynomial is $d([x^r, u])$. Therefore, we obtain

$$d([x^r, u]) \in \mathscr{M} \text{ for all } x \in \mathfrak{A}. \tag{2}$$

Combining (1) and (2), we have $d([z, u]) \in \mathscr{M}$ for all $z \in P$. If $z \in P$ and $v \in \mathfrak{A}$, then $z + tv \in P$, for all sufficiently small real t. Therefore, we have

$$d([z + tv, u]) \in \mathscr{M}.$$

Using similar arguments as above, we see that $d([v, u]) \in \mathscr{M}$ for all $v \in \mathfrak{A}$.

We close this section with the following corollary :

Corollary 2. *Let $\mathfrak{A}$ be a prime Banach algebra, $y \in \mathfrak{A}$ and $d : \mathfrak{A} \to \mathfrak{A}$ be a nonzero continuous derivation. Suppose that there exists a nonempty open set G in $\mathfrak{A}$, such that for each $x \in G$ there exists a positive integer $m = m(x) > 1$ such that $d([x^m - x, y]) = 0$. Then $\mathfrak{A}$ is commutative.*

Proof. By using Theorem 3 we get $d([x, y]) = 0$ for all $x, y \in \mathfrak{A}$. Now applying Theorem 2.5 of [32], we see that $\mathfrak{A}$ is commutative.

Corollary 3. *Let $\mathfrak{A}$ be a semiprime Banach algebra, $y \in \mathfrak{A}$ and $d : \mathfrak{A} \to \mathfrak{A}$ be a nonzero continuous linear derivation. Suppose that there exists a nonempty open set G in $\mathfrak{A}$ such that for each $x \in G$ there exists a positive integer $m = m(x) > 1$ such that $d([x^m - x, y]) \in Z(\mathfrak{A})$. Then $\mathfrak{A}$ is commutative.*

Proof. By using Theorem 3 we get $d([x, y]) \in Z(\mathfrak{A})$ for all $x, y \in \mathfrak{A}$. Now by Corollary 2.6 of [31], we see that $\mathfrak{A}$ is commutative.

3 Results on Banach *-Algebras

Throughout this section $\mathfrak{A}$ will denote a Banach *-algebra over the complex field $\mathbb{C}$ with a continuous involution $x \mapsto x^*$ and $Z(\mathfrak{A})$ denotes the centre of $\mathfrak{A}$. Moreover $\mathscr{M}$ will denote a closed linear subspace of $\mathfrak{A}$. Recall that $x \in \mathfrak{A}$ is normal modulo $\mathscr{M}$ if $[x, x^*] \in \mathscr{M}$. In this section it is shown that either a semiprime Banach *-algebra $\mathfrak{A}$ with continuous involution and continuous derivation is commutative or the the set of $x \in \mathfrak{A}$ for which $[d(x^k), d((x^k)^*)] \in Z(\mathfrak{A})$ for no positive integer k, is dense in $\mathfrak{A}$. Further few more parallel results have been obtained to prove the commutativity of Banach *-algebra $\mathfrak{A}$.

We begin this section by obtaining some important results which will be used extensively to prove our main theorems.

Lemma 2. *Let $\mathfrak{A}$ be a Banach *-algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a linear mapping. Suppose $m \geq 1$ is a fixed positive integer such that $d(h^m) \in \mathscr{M}$ for all self-adjoint elements h. Then $d(x^m) \in \mathscr{M}$ for all x in $\mathfrak{A}$.*

Proof. Let h and k be self-adjoint in $\mathfrak{A}$. Since $d(h^m) \in \mathscr{M}$ for all self-adjoint elements h. Therefore, for each real t, we have

$$d((h + tk)^m) \in \mathscr{M}.$$

The expression $d((h + tk)^m)$ can be written as

$$\begin{aligned} d((h + tk)^m) = d(P_{m,0}(h, k)) + d(P_{m-1,1}(h, k))t + \cdots \\ + d(P_{1,m-1}(h, k))t^{m-1} + d(P_{m,0}(h, k))t^m. \end{aligned}$$

Let i, j be nonnegative integers. If $i + j = m$, then $P_{i,j}(h, k)$ denotes the sum of all the terms in which h appears exactly i times and k appears exactly j times in the expansion of $(h + tk)^m$. The above expression is a polynomial in t and therefore each coefficient in this polynomial lies in $\mathscr{M}$. Now consider $x = h + ik$. We have

$$d(x^m) = \sum_{r=0}^{m} i^r d(P_{m-r,r}) \in \mathscr{M}.$$

Lemma 3. *Let h and k be self-adjoint elements of Banach *-algebra $\mathfrak{A}$ and $d : \mathfrak{A} \to \mathfrak{A}$ be a linear mapping. Suppose that $d([x^m, (x^*)^m]) \in \mathscr{M}$ for all x in $\mathfrak{A}$ and fixed positive integer $m \geq 1$. Then $d[h^m, \sum_{j=0}^{m-1} h^j k h^{m-1-j}] \in \mathscr{M}$.*

Proof. Let h and k be self-adjoint and $t \neq 0$ be real. Since $d([x^m, (x^*)^m]) \in \mathcal{M}$ for all x in $\mathfrak{A}$. Substituting $x = h + itk$, we arrive at

$$d([(h+itk)^m, ((h+itk)^*)^m]) \in \mathcal{M}.$$

Let i, j be nonnegative integers. If $i + j = m$, then $P_{i,j}$ denotes the sum of all the terms in which h appears exactly i times and k appears exactly j times in the expansion of $(h + tk)^m$. Therefore the above expression can be written as

$$d([(h+itk)^m, ((h+itk)^*)^m]) = d([\sum_{r=0}^{m} i^r P_{m-r,r} t^r, \sum_{r=0}^{m} (-i)^r P_{m-r,r} t^r]) \in \mathcal{M}.$$

Let $\sigma(t)$ be the sum of the terms of this expression for even r and $\rho(t)$ be the sum for odd r. Then we have

$$\begin{aligned} d([(h+itk)^m, ((h+itk)^*)^m]) &= d([\sigma(t) + \rho(t), \sigma(t) - \rho(t)]) \\ &= d([\sigma(t), \rho(t)]) \in \mathcal{M}, \end{aligned}$$

for all real t. The above expression is a polynomial in t. Thus the coefficient of t must lie in $\mathcal{M}$. Therefore,

$$d[P_{m-0,0}, P_{m-1,1}] \in \mathcal{M}.$$

This implies that,

$$d[h^m, \sum_{j=0}^{m-1} h^j k h^{m-1-j}] \in \mathcal{M}.$$

Lemma 4. *Let $\mathfrak{A}$ be a Banach *-algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a continuous linear mapping. Suppose that the set of $x \in \mathfrak{A}$ for which there exists a positive integer $n = n(x) \geq 1$ so that $d([x^n, (x^*)^n]) \in \mathcal{M}$ has a nonempty interior. Then there exists a fixed positive integer $m \geq 1$ such that $d([x^m, (x^*)^m]) \in \mathcal{M}$ for all $x \in \mathfrak{A}$.*

Proof. For each positive integer n, we set

$$S_n = \{x \in \mathfrak{A} : d([x^n, (x^*)^n]) \notin \mathcal{M}\}.$$

We claim that s_n is open. To show that S_n is open we prove its complement, S_n^c is closed. For this, we take a sequence $(z_k) \in S_n^c$ such that $z_k \to z$ as $k \to \infty$ and prove that $z \in S_n^c$. Since $z_k \in S_n^c$, we have

$$d([z_k^n, (z_k^*)^n]) \in \mathcal{M}.$$

Taking limit on k, we obtain

$$\lim_{k\to\infty} d([z_k^n, (z_k^*)^n]) \in \mathscr{M}.$$

Since d and involution $*$ are continuous, we have

$$\lim_{k\to\infty} d([z_k^n, (z_k^*)^n]) = d([(\lim_{k\to\infty} z_k)^n, ((\lim_{k\to\infty} z_k)^*)^n]) = d([z^n, (z^*)^n]) \in \mathscr{M}.$$

This implies that $z \in S_n^c$, so S_n^c is closed and hence S_n is open. If every S_n is dense in $\mathfrak{A}$ then, by the Baire category theorem their intersection is also dense. But this would contradicts our hypothesis. Hence there exists a positive integer m and a nonempty open subset G of $\mathfrak{A}$ so that $d([w^m, (w^*)^m]) \in \mathscr{M}$ for all $w \in G$. If $u \in G$ and $x \in \mathfrak{A}$, then $u + tx \in G$, for all sufficiently small real t. Therefore, we have

$$d([(u+tx)^m, ((u+tx)^*)^m]) = d([(u+tx)^m, (u^* + tx^*)^m]) \in \mathscr{M}.$$

The above expression is a polynomial in t and the coefficient of t^{2m} is $d([x^m, (x^*)^m])$. Hence, we have

$$d([x^m, (x^*)^m]) \in \mathscr{M} \text{ for all } x \in \mathfrak{A}.$$

Theorem 4. *Let $\mathfrak{A}$ be unital Banach *-algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a continuous linear mapping. Then either $d([x, y]) \in \mathscr{M}$ for all x, y in $\mathfrak{A}$ or the set S of $x \in \mathfrak{A}$, for which $d([x^k, (x^k)^*]) \in \mathscr{M}$ for no positive integer $k \geq 1$, is dense in $\mathfrak{A}$.*

Proof. Let e be the identity of $\mathfrak{A}$. Suppose that the set S is not dense in $\mathfrak{A}$. Then, by Lemma 4, there exists a positive integer m such that $d([x^m, (x^*)^m]) \in \mathscr{M}$ for all $x \in \mathfrak{A}$. For $t \neq 0$, t real, set

$$u = t^{-1}((e+th)^m - e);$$

$$v = \sum_{j=1}^{m-1} (e+th)^j k (e+th)^{m-1-j}.$$

By using Lemma 3, we have $d([u, v]) \in \mathscr{M}$. Letting $t \to 0$, we see that $d([h, k]) \in \mathscr{M}$. Since h and k are self-adjoint elements in $\mathfrak{A}$, we have $d([x, y]) \in \mathscr{M}$ for all $x, y \in \mathfrak{A}$.

Corollary 4. *Let $\mathfrak{A}$ be a unital Banach *-algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a continuous linear derivation. If $\mathfrak{A}$ has no nonzero nilpotent ideal then either $\mathfrak{A}$ is commutative or the set S of $x \in \mathfrak{A}$, for which $d([x^k, (x^*)^k]) \in Z(\mathfrak{A})$ for no positive integer $k \geq 1$, is dense in $\mathfrak{A}$.*

Proof. By Theorem 4, we have $d([x, y]) \in Z(\mathfrak{A})$ for all x, y in $\mathfrak{A}$. Since $\mathfrak{A}$ has no nonzero nilpotent ideal. Then by Corollary 2.6 of [31], $\mathfrak{A}$ is commutative.

Lemma 5. *Let h and k be self-adjoint elements of Banach *-algebra $\mathfrak{A}$ and $d : \mathfrak{A} \to \mathfrak{A}$ be a linear mapping such that d commutes with *. Suppose that $d(x^m)$ is a normal modulo $\mathscr{M}$ for all x in $\mathfrak{A}$ and a fixed positive integer $m \geq 1$. Then $[d(h^m), d(\sum_{j=0}^{m-1} h^j k h^{m-1-j})] \in \mathscr{M}$.*

Proof. Let h and k be self-adjoint and $t \neq 0$ be real. Since $[d(x^m), d((x^*)^m)]) \in \mathscr{M}$ for all x in $\mathfrak{A}$. Substituting $x = h + itk$ we arrive at

$$[d((h + itk)^m), d(((h + itk)^*)^m)] \in \mathscr{M}.$$

The above expression can be written as

$$[d((h + itk)^m), d(((h + itk)^*)^m)] = [\sum_{r=0}^{m} i^r d(P_{m-r,r})t^r, \sum_{r=0}^{m} (-i)^r d(P_{m-r,r})t^r]) \in \mathscr{M}. \tag{3}$$

Let i, j be nonnegative integers. If $i + j = m$, then $P_{i,j}$ denotes the sum of all the terms in which h appears exactly i times and k appears exactly j times in the expansion of $(h + tk)^m$. Again consider,

$$d((h + itk)^m) = \sum_{r=0}^{m} i^r d(P_{m-r,r})t^r = \sigma(t) + \rho(t) \text{ for each real } t, \tag{4}$$

where $\sigma(t)$ be the sum of the terms of this expression for even r and $\rho(t)$ be the sum for odd r. Since * commutes with d, we get

$$(d((h + itk)^m))^* = d(((h + itk)^*)^m) = \sigma(t) - \rho(t). \tag{5}$$

Since $d((h + itk)^m)$ is normal modulo $\mathscr{M}$. Combining (3), (4) and (5) we get

$$\sigma(t)\rho(t) - \rho(t)\sigma(t) \in \mathscr{M}.$$

for all real t. The above expression is a polynomial in t. Thus the coefficient of t must lie in $\mathscr{M}$. Therefore,

$$[d(P_{m-0,0}), d(P_{m-1,1})] \in \mathscr{M}.$$

This implies that

$$[d(h^m), d(\sum_{j=0}^{m-1} h^j k h^{m-1-j})] \in \mathscr{M}.$$

Lemma 6. *Let $\mathfrak{A}$ be a Banach *-algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a continuous linear mapping such that d commutes with *. Suppose that the set of $x \in \mathfrak{A}$ for which there exists a positive integer $n = n(x) \geq 1$ so that $d(x^n)$ is normal modulo $\mathscr{M}$ has nonempty interior. Then there exists a fixed positive integer $m \geq 1$ such that $d(x^m)$ is normal modulo $\mathscr{M}$ for all $x \in \mathfrak{A}$.*

Proof. For each positive integer n, let

$$W_n = \{x \in \mathfrak{A} : d(x^n) \text{ is not normal modulo } \mathscr{M}\}.$$

Since d commutes with *, this can be written as

$$W_n = \{x \in \mathfrak{A} : [d(x^n), d((x^*)^n)] \notin \mathscr{M}\}.$$

We claim that W_n is open. To show that W_n is open we prove its complement, W_n^c is closed. For this, we take a sequence $(z_k) \in W_n^c$ such that $z_k \to z$ as $k \to \infty$ and prove that $z \in W_n^c$. Since $z_k \in W_n^c$, we have

$$[d(z_k^n), d((z_k^*)^n)] \in \mathscr{M}.$$

Taking limit on k, we obtain

$$\lim_{k\to\infty} [d(z_k^n), d((z_k^*)^n)] \in \mathscr{M}.$$

Since d and involution * are continuous, we have

$$\lim_{k\to\infty} [d(z_k^n), d((z_k^*)^n)] = [d((\lim_{k\to\infty} z_k)^n), d(((\lim_{k\to\infty} z_k)^*)^n)] = [d(z^n), d((z^*)^n)] \in \mathscr{M}.$$

This implies that $z \in W_n^c$, so W_n^c is closed and hence W_n is open. If every W_n is dense in $\mathfrak{A}$ then, by the Baire category theorem their intersection is also dense. But this would contradict our hypothesis. Hence there exists a positive integer m and a nonempty open subset G of $\mathfrak{A}$ so that $[d(u^m), (u^*)^m)] \in \mathscr{M}$ for all $u \in G$. If $w \in G$ and $x \in A$, then $w + tx \in G$, for all sufficiently small real t. Therefore, we have

$$[d((u + tx)^m), d(((u + tx)^*)^m)] = [d((u + tx)^m), d((u^* + tx^*)^m)] \in \mathscr{M}.$$

The above expression is a polynomial in t and the coefficient of t^{2m} is $[d(x^m), d((x^*)^m)]$. Thus, we have

$$[d(x^m), d((x^*)^m)] \in \mathscr{M} \text{ for all } x \in \mathfrak{A}.$$

Theorem 5. *Let $\mathfrak{A}$ be a unital Banach *-algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a continuous linear mapping such that d commutes with *. Then either $[d(x), d(y)] \in \mathscr{M}$ for all x, y in $\mathfrak{A}$ or the set S of $x \in \mathfrak{A}$ for which $[d(x^k), d((x^*)^k)]) \in \mathscr{M}$ for no positive integer $k \geq 1$, is dense in $\mathfrak{A}$.*

Proof. Let e be the identity of $\mathfrak{A}$. Suppose that the set S is not dense in $\mathfrak{A}$. Then, by Lemma 6, there is a positive integer m such that $[d(x^m), d((x^*)^m)] \in \mathscr{M}$ for all $x \in \mathfrak{A}$. For nonzero real t, set

$$u = t^{-1}((e + th)^m - e);$$

$$v = \sum_{j=1}^{m-1} (e + th)^j k(e + th)^{m-1-j}.$$

By using Lemma 5, we have $[d(u), d(v)] \in \mathscr{M}$. Letting $t \to 0$ and using the continuity of d, we see that $[d(h), d(k)] \in \mathscr{M}$. Since h and k are self-adjoint elements in $\mathfrak{A}$, we have $[d(x), d(y)] \in \mathscr{M}$ for all $x, y \in \mathfrak{A}$.

Corollary 5. *Let $\mathfrak{A}$ be a unital Banach *-algebra and $d : \mathfrak{A} \to \mathfrak{A}$ be a continuous linear derivation such that d commutes with *. If $\mathfrak{A}$ has no nonzero nilpotent ideals then either $\mathfrak{A}$ is commutative or the set S of $x \in \mathfrak{A}$ for which $[d(x^k), d((x^k)^*)] \in Z(\mathfrak{A})$ for no positive integer $k \geq 1$, is dense in $\mathfrak{A}$.*

Proof. By Theorem 4, we have $[d(x), d(y)] \in Z(\mathfrak{A})$ for all x, y in $\mathfrak{A}$. Since $\mathfrak{A}$ has no nonzero nilpotent ideals we have by Corollary 2.6 of [31] $\mathfrak{A}$ is commutative.

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An Application of Linear Algebra to Image Compression

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Abstract Nowadays the data are transmitted in the form of images, graphics, audio and video. These types of data require a lot of storage capacity and transmission bandwidth. Consequently, the theory of data compression becomes more significant for reducing the data redundancy in order to save more transfer and storage of data. In this context, this paper addresses the problem of the lossy compression of images. This proposed method is based on Block SVD Power Method that overcomes the disadvantages of Matlab's SVD function. The quantitative and visual results are showing the superiority of the proposed compression method over those of Matlab's SVD function and some different compression techniques in the state-of-the-art. In addition, the proposed approach is simple and can provide different degrees of error resilience, which gives, in a short execution time, a better image compression.

Keywords Image compression · Singular value decomposition · Block SVD Power Method · Lossy image compression · PSNR

1 Introduction

The Singular Value Decomposition (SVD) is a generalization of the eigen-decomposition used to analyze rectangular matrices. It plays an important role

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in many applications: physical and biological processes, mathematical models in economics, data mining applications, search engines to rank documents in very large databases, including the Web, image processing applications, etc. In this paper, we will study the SVD applied to the image compression.

Image compression is a type of data compression that involves encoding information in images using fewer bits than the original image representation. The main idea of image compression is reducing the redundancy of the image and the transferring data in an efficient form. The image compression takes an important place in several domains like web designing, in fact, maximally reduce an image allows us to create websites faster and saves bandwidth users, it also reduces the bandwidth of the servers and thus save time and money. When talking about compression, we generally take into account two aspects: image size in pixels and its degree of compression. The nature of the image is also playing a significant role. The main goal of such system is to reduce the storage quantity as much as possible while ensuring that the decoded image displayed in the monitor can be visually similar to the original image as much as it can be.

The rest of this paper is structured as follows. We briefly introduce a review of previous related work in Sect. 2. Section 3 describes the proposed system for image compression. The performance evaluation of the proposed algorithm is reported in Sect. 4. Finally, conclusions are drawn in Sect. 5.

2 Related Work

In recent years, numerous image compression schemes and their applications in image processing have been proposed. In this section, a brief review of some important contributions from the existing literature is presented.

In general, there are two approaches for image compression: lossy or lossless [1, 2].

A lossless compression is a kind of image compression method that allows no loss of data, and which retains the full information needed to reconstruct the original image. This type of compression is also known as entropy coding because of the fact that a compressed signal is generally more random than the original one and the patterns are removed when a signal is compressed. The lossless compression can be very useful for exact reconstruction of images. The compression ratio provided by this kind of methods is not sufficiently high to be truly used in image compression. Lossless image compression is particularly useful in image archiving as in the storage of legal or medical records. The lossless image compression methods include: Run-length coding, Bit-plane coding, Huffman coding [3], LZW (Lempel Ziv Welch) coding and Entropy coding.

Lossy compression is another type of image compression technique in which the original signal cannot be exactly reconstructed from the compressed data. The reason behind this is that much of the detail in an image can be discarded without greatly changing the appearance of the image. In lossy image compression, even a very fine detail of the images can be lost, but ultimately, the image size is drastically reduced.

Lossy image compressions are useful in many applications such as broadcast television, video conferencing, and facsimile transmission, in which a certain amount of error is an acceptable trade-off for increased compression performance. Among methods for lossy compression, we find: Fractal compression [4], Transform coding, Fourier-related transform, DCT (Discrete Cosine Transform) [5, 6] and Wavelet transform.

Generally, SVD is a lossy compression technique which achieves compression by using a smaller rank to approximate the original matrix representing an image. Furthermore, lossy compression yields good compression ratio comparing with lossless compression while the lossless compression gives good quality of compressed images.

When we give the definition of lossless or lossy methods, it is necessary to clarify that near lossless algorithms are theoretically lossless. However, they may suffer from numerical floating point accuracy reconstruction issues.

According to the state-of-the-art, there are several works suggested to use the SVD with other compression methods or with variation of SVD. Awwal et al. [7] presented new compression technique using SVD and the Wavelet Difference Reduction (WDR). The WDR used for further reduction. This technique has been tested with other techniques such as WDR and JPEG 2000 and gives a better result than these techniques. Furthermore, using WDR with SVD enhance the PSNR and compression ratio.

A technique based on Wavelet-SVD, which used a graph coloring technique in the quantization process, is presented in [8]. This technique worked well and enhanced the PSNR and compression ratio. The generated compression ratio by this work ranged between 50–60%, while the average PSNR ranged between 40–80db.

Ranade et al. [9] suggested a variation on SVD based image compression. This approach is a slight modification to the original SVD algorithm, which gives much better compression than the standard compression using SVD method. In addition, it performs substantially better than the SVD method. Typically, for any given compression quality, this approach needs about 30% fewer singular values and vectors to be retained.

Doaa et al. [10], proposed Block Truncation Coding (BTC), it is an image compression method proposed by Delp et al. [11, 12]. It is one of the simplest and easiest image compression algorithms, and also an efficient image coding method that has been adopted to obtain the statistical properties of a block in the compressed image.

The technique given by El Abbadi et al. [13], proposes to use SVD and MPQ-BTC, the input image is compressed by reducing the image matrix rank, by using the SVD process and then the result matrix compressed by using BTC.

Following the same objective of image compression using SVD, the most problem is which K rank to use for giving a better image compression. For this reason, the method presented in El Asnaoui et al. [14], introduces two new approaches: The first one is an improvement of the Block Truncation Coding method that overcomes the disadvantages of the classical Block Truncation Coding, while the second one describes how to obtain a new rank of SVD method, which gives a better image compression.

3 Image Compression Technique Using SVD

The main goal of studying the SVD of an image (matrix of $m * n$) is to create approximations of an image using the least amount of the terms of the diagonal matrix in the decomposition. This approximation of the matrix is the basis of image compression using SVD, since images can be viewed as matrices with each pixel being an element of a matrix.

The main idea of this section is to present two algorithms: The first one is the Matlab's SVD function, while the second one describes how to obtain a new SVD using Block SVD Power Method.

3.1 Algorithm of Matlab's SVD Function

Input: $A \in M_{m*n}(\mathbb{R})$
Output:$A = U_{m*m} * \sum_{m*n} * V^T_{n*n}$

$$\begin{cases} U \text{ and } V \text{ are unitary matrices} \\ \sum = diag[(\sigma_1, \sigma_2, ..., \sigma_p)] \\ where\ p = min(m, n) \\ r = rank(A) \\ and\ \sigma_1 \geq \sigma_1 \geq ... \geq \sigma_r \\ \sigma_r > \sigma_{r+1} = \sigma_{r+2} = ...\sigma_p = 0 \end{cases}$$

$[U, \sum, V] = svd(A)$; $(notation\ matlab\ is\ used)$

3.2 Algorithm of Block SVD Power Method [15]

Input: A matrix $A \in (\mathbb{R})^{n*m}$, $a\ block - vector$
$V = V^{(0)} \in \mathbb{R}^{m*s} and\ a\ tolerance\ tol$
Output: $An\ orthogonal\ matrices$
$U = [u_1, u_2, ..., u_s] \in \mathbb{R}^{n*s}$
$V = [v_1, v_2, ..., v_s] \in \mathbb{R}^{m*s}$
$and\ a\ positive\ diagonal\ matrix$
$\sum = diag(\sigma_1, \sigma_2, ..., \sigma_s)$
$such\ that : AV = U \sum$
While $(err > tol)$ **do**
$AV = QR(factorization\ QR)$,
$U \leftarrow Q(:, 1 : s)$
$(the\ s\ first\ vector\ colonne\ of\ Q)$
$A^T U = QR$,
$V \leftarrow Q(:, 1 : s)\ and\ \sum \leftarrow R(1 : s, 1 : s)$
$err = \|AV - U \sum\|$
End

3.3 Proposed Lossy Image Compression Technique

The contribution of this paper is the introduction of the concept of application of Block SVD Power Method to image compression, since the main idea of image compression is reducing the redundancy of the image and the transferring data in an efficient form.

In this section, we propose our contribution on which we integrate the Block SVD Power Method and adopt it to create an algorithm that compress an image. Figure 1 shows the main pipeline of the proposed method.

When the SVD is applied to an image, it is not compressed, but the data take a form in which the first singular value has a great amount of the image information. With this, we can use only a few singular values to represent the image with little differences from the original. The input image can be a color image with RGB color components or may be a grayscale image. Furthermore, for Creating new image with Matlab's SVD function as indicated in the Fig. 1, we use:

$$I_{comp} = U(:, 1 : K) * \sum(1 : K, 1 : K) * (V(:, 1 : K)^T) \tag{1}$$

Our contribution in this paper is to set up a new algorithm for image compression that overcomes some inconveniences encountered in existing methods that use Matlab's SVD function. Our modification consists of a computing the SVD for each component step, in which the entries in the image I are computed using Block SVD

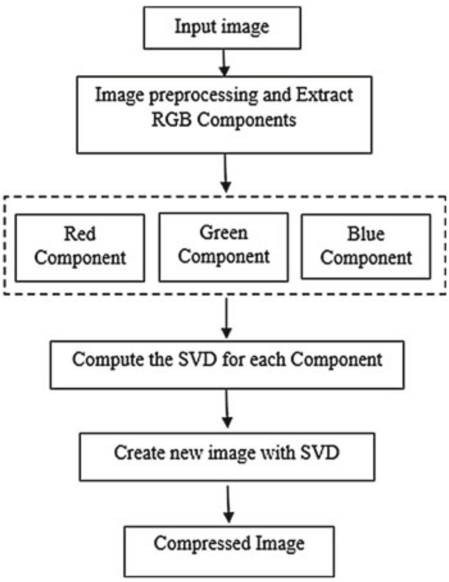

Fig. 1 Image pre-processing using SVD

Power Method obtained by [15] instead of Matlab's SVD function [14] and keeps the K rank determined by (see Eq. 5).

Thus for the same compression, we have better quality. We also provide a heuristic argument to justify our experimental finding.

To the best of our knowledge, this is the first work suggesting an image compression based on Block SVD Power Method. Most of methods focus on other methods and other variation of SVD. Moreover, our method is novel, efficient for solving our problem. It is general and many other computer visions can benefit from using it.

In the next section, the experimental results are reported. The results are clearly showing the superiority of the proposed lossy image compression technique over those of Matlab's SVD function and some different compression techniques in the state-of-the-art.

4 Experimental Results

Our work is aimed at image compression. For this purpose, our experiments were performed on several images available on Windows 7 Professional and numerical examples. Simulations were done in MATLAB 2009a using computer with Processor: Intel(R) Core (TM) 2 CPU T5200 @ 1.60 GHz, 1.60 GHz, 2Go RAM running on a Microsoft Windows 7 Professional (32-bit). In addition, the results and discussion of the proposed method are given in this section.

4.1 Parameters for comparison

To evaluate the performance of the proposed method, the quality of the image is estimated using several quality measurement variables like, Mean Square Error (MSE) and Peak Signal-to-Noise Ratio (PSNR). These variables are signal fidelity metrics and do not measure how viewers perceive visual quality of an image.

4.1.1 Compression Ratio

The degree of data reduction obtained by a compression method can be evaluated using the compression ratio (Q_{comp}) defined by the formula:

$$Q_{comp} = \frac{Size\ of\ original\ image}{Size\ of\ compressed\ image} \tag{2}$$

4.1.2 Mean Square Error (MSE)

MSE, which for two $M * N$ monochrome images X and Y where one of the images is considered noisy approximation of the other and is defined as follows:

$$e_{MSE} = \frac{1}{MN} \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} [X(i, j) - Y(i, j)]^2 \tag{3}$$

4.1.3 Peak Signal-to-Noise Ratio (PSNR)

PSNR is measured in decibels (dB), and is only meaningful for data encoded in terms of bits per sample bits per pixel.

For example, an image with 8 bits per pixel contains integers from 0–255. PSNR is given by the following equation:

$$PSNR = 10log_{10} \frac{(2^B - 1)^2}{e_{MSE}} \tag{4}$$

A high PSNR value indicates that there is less visual degradation in the compressed image.

4.2 Numerical Examples

The results of the image compression depend strongly on the goodness of the algorithm to compress an image. To illustrate it, first we checked if the chosen of Block SVD Power Method detects correctly the relative errors occurred when computing the singular values and CPU time (Figs. 2 and 3). For this purpose, we did several tests where we chose some numerical examples.

We have compared and tested in this section the numerical results obtained by algorithm [15] with Matlab's SVD function. Towards this end, Let $A \in (\mathbb{R})^{n*m}$ be a rectangular matrix defined as follows: $A \in Q \sum U^T$ where Q and U are random orthogonal matrices. We give below relative errors occurred when computing the singular values and the CPU time. The started block-vector in algorithm [15] is given by $V = V(0) = eye(m, s)$ (Matlab notation). The results are given from algorithm [15] after only at most $K = 1$ iteration. We have stopped the algorithm [15] whenever the error of the reduction: $err \| AV - U \sum \|$ is smaller than that achieved by Matlab's SVD function.

Example 1:

Let:

$\sum = diag([10^4, 10^4, 10^{-11}, 10^{-11}, 10^{-12}, 10^{-12}, 10^{-13}, 10^{-13}, 10^{-14}, 10^{-14}])$

$m = 1000, n = 1000, s = rank(A) = 10$. After only $K = 1$ iteration, we obtain:

	Matlab's SVD function	Algorithm [14]
Error $\|AV - U\sum\|$	2.3936e −011	9.1089e −012
CPU time (s)	19.3789	0.2830

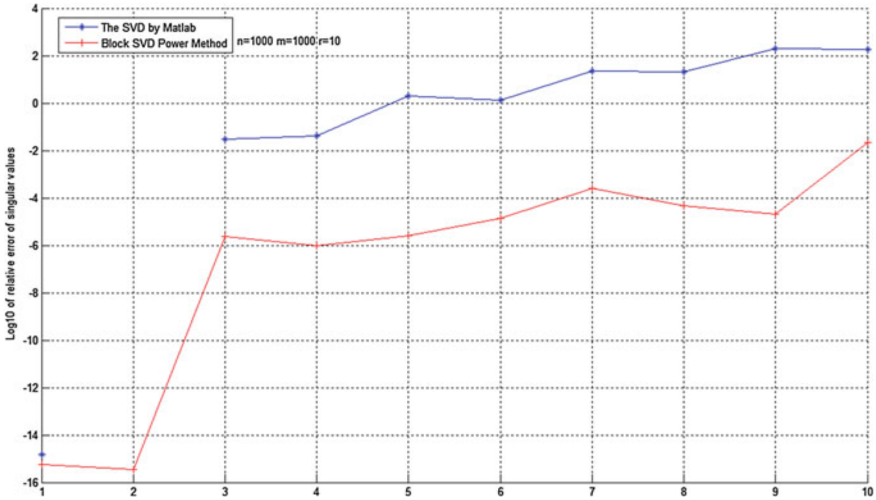

Fig. 2 Relative errors occurred when computing the singular values

Example 2:
Let:
$\sum = diag([10^3, 10^3, 10^3, 10^{-12}, 10^{-12}, 10^{-13}, 10^{-13}, 10^{-13}, 10^{-13}, 10^{-13}, 10^{-13}, 10^{-13}])$
$m = 1000, n = 1000, s = rank(A) = 12$. After only $K = 1$ iteration, we obtain:

	Matlab's SVD function	Algorithm [14]
Error $\|AV - U\sum\|$	2.6714e −012	1.2523e −012
CPU time (s)	21.0487	0.5318

4.3 Image Compression

To test our method, we develop a user interface. The method was applied to various and real images to demonstrate the performances of the proposed algorithm of image compression.

We use in this paper, 2 color images, Chrysantheme and Desert available in Windows 7 professional (32-bit), and 1 in grayscale. Figures 4, 5, 6 and 7 show the test images and the resulting compressed images using Matlab's SVD function [14] and the proposed compression method.

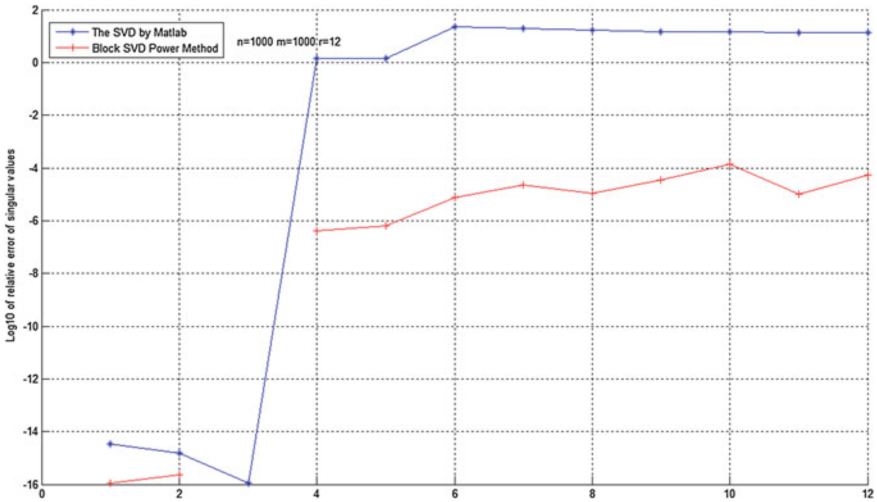

Fig. 3 Relative errors occurred when computing the singular values

We recall that our goal is to approximate an image (matrix of $m * n$) using the least amount of information. Thereby, to obtain a better quality of the compressed image using SVD, we use the K rank determined by El Asnaoui et al. [14]:

$$K = \frac{m * n}{m + n + 1} \tag{5}$$

Where m and n are the size of original image.

Fig. 4 Original images: **a**. Chrysantheme, **b**. Desert, **c**. grayscale

4.3.1 Test with Color Image

After rank $K = 438$, we obtain:

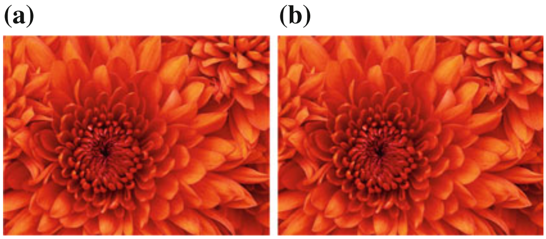

Fig. 5 Compressed results obtained by: **a**. Matlab's SVD function [14], **b**. Proposed method

Table 1 Compression results for Chrysantheme.jpg, 1024 × 768, 858Ko, by using:

	Matlab's SVD function [14]			Proposed method		
K	Q_{comp}	MSE	PSNR	Q_{comp}	MSE	PSNR
50	9.41	31.22	32.86	7.41	47.20	49.68
100	8.28	35.50	36.97	7.37	49.38	51.85
150	7.84	38.41	40.07	**7.35**	**51.07**	**53.48**
200	7.64	40.79	42.75	7.35	51.60	53.99
250	7.54	42.96	45.17	7.34	53.87	56.12
300	7.46	45.08	47.46	7.33	56.14	58.36
350	7.41	47.20	49.68	7.33	58.68	61.28
400	7.37	49.38	51.85	7.32	62.45	66.33
438	**7.35**	**51.07**	**53.48**	**7.32**	**70.90**	**76.92**

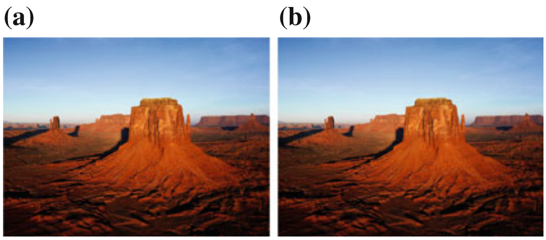

Fig. 6 Compressed results obtained by: **a**. Matlab's SVD function [14], **b**. Proposed method

4.3.2 Test with Grayscale Image

In order to compare this performance, we also applied the new method to the gray scale image. After rank $K = 548$, we obtain:

Table 2 Compression results for Desert.jpg, 1024 × 768, 826Ko, by using:

	Matlab's SVD function [14]			Proposed method		
K	Q_{comp}	MSE	PSNR	Q_{comp}	MSE	PSNR
50	9.24	28.40	35.10	6.98	44.51	48.44
100	7.98	31.29	37.49	6.94	47.82	51.20
150	7.49	33.83	39.60	**6.92**	**50.61**	**53.60**
200	7.25	36.32	41.65	6.92	51.56	54.43
250	7.10	38.86	43.75	6.89	55.93	58.88
300	7.02	41.56	45.99	6.88	60.11	63.26
350	6.98	44.51	48.44	6.87	64.43	67.19
400	6.94	47.82	51.20	6.87	70.78	74.64
438	**6.92**	**50.61**	**53.60**	**6.87**	**84.75**	**92.97**

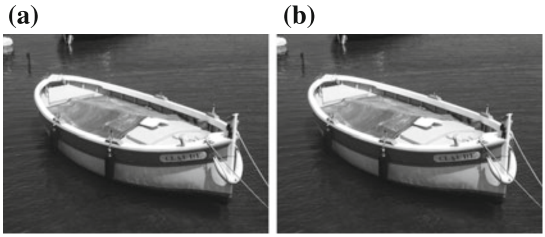

Fig. 7 Compressed results obtained on the: **a**. Matlab's SVD function, **b**. Proposed method

Table 3 Compression results for grayscale.jpg, 1280 × 960, 480Ko, by using:

	Matlab's SVD function [14]			Proposed method		
K	Q_{comp}	MSE	PSNR	Q_{comp}	MSE	PSNR
50	4.98	78.51	29.22	4.06	9.45	38.41
100	4.31	33.27	32.94	4.08	5.58	40.70
150	4.10	16.95	35.87	4.12	3.47	42.76
200	4.06	9.45	38.41	4.12	2.24	44.67
250	4.08	5.58	40.70	4.09	1.48	46.46
300	4.12	3.47	42.76	4.06	1.00	48.18
350	4.12	2.24	44.67	4.04	0.68	49.86
400	4.09	1.48	46.46	**4.02**	**0.47**	**51.46**
450	4.06	1.00	48.18	4.01	0.30	53.35
500	4.04	0.68	49.86	4.01	0.18	55.62
548	**4.02**	**0.47**	**51.46**	**4.01**	**0.08**	**58.94**

4.3.3 Test with Other Methods

To evaluate the robustness of our scheme, we test it with other methods like: [10, 13, 14]. Added experiment results for two images are listed in Table 4.

Table 4 Comparison against various algorithms

	Color image (Fig. 4a)			Grayscale image (Fig. 4c)		
	Q_{comp}	MSE	PSNR	Q_{comp}	MSE	PSNR
BTC method [13]	9.27	62.09	30.20	5.39	161.1	26.09
BTC method [10]	7.34	7.96	39.12	3.98	19.03	35.37
BTC method [14]	6.72	2.74	43.75	2.84	3.47	42.76
SVD method [14]	7.35	0.29	53.48	4.02	0.47	51.46
Proposed method	7.31	0.0013	**76.92**	4.01	0.08	**58.94**

4.4 Discussion

In this paper, the proposed algorithm is compared with the Matlab's SVD function [14] and the other state-of-the-art algorithms.

Numerical examples given above show the efficiency of the new SVD approach in computing the decomposition, the error and CPU time.

When applying the proposed method to image compression, Figs. 5, 6 and 7, it is clear that the compressed images by two approaches are perceptually similar to original images. However, the human visual response to image quality is insufficient.

In order to compare the performances of the proposed method, several values were used in this study to measure the quality of the compressed image. We will only discuss PSNR and MSE values, because, they are used to compare the squared error between the original image and the reconstructed image. There is an inverse relationship between PSNR and MSE. Therefore, a higher PSNR value indicates the higher quality of the image (better).

The above analysis shows the comparison when SVD and proposed method are applied on the real images. In these experiments, we used the K rank for different images. We see in this case that the compression ratio and PSNR, and other values of images varied when changing the rank of image during the SVD process as showed in Tables 1, 2 and 3, and it is evident that the proposed technique gives better performance compared to the SVD. In addition, for the Matlab's SVD function, the value of K which provides better PSNR value is the maximum value of $K = 438$, while for the proposed technique, a better, compression ratio, PSNR is provided from $K = 150$ for color images. We can say that in this case Matlab's SVD function [14] approximately present 1/3 of the proposed method in terms of K rank.

Concerning the grayscale image tested, it seems that the value of K which gives better PSNR value is the maximum value of $K = 548$, while for the proposed method, a better, compression ratio, PSNR is provided from $K = 400$.

We mainly compared the proposed algorithm with the other algorithms as illustrated in Table 4, because these algorithms are well-known and are mainly using Matlab's SVD function. Hence, we see that our proposed algorithm performs comparable to current state-of-the-art techniques, and is able to produce a compressed image with better visual quality, as indicated by its PSNR.

5 Conclusion

We suggest in this work a novel method for image compression. This approach is simple, and it can be used to overcome limitations of existing algorithms, that use the Matlab's SVD function. The results obtained indicate that the proposed approach might be considered as a solution for the development of image compression. Satisfactory compression of expected images is provided faster due to the lower number of iterations in the compression algorithm. Of course, Matlab's SVD method is accurate. However, in numerical analysis, we focus on always improving results.

Future Scope

This proposed method opens the door of lots of future work. For example, using the SVD for statistical applications to find relations between data, in the area of medical image denoising with different thresholding techniques associated with these multiwavelets, implements a compression technique using neural network. It is also useful with other techniques in image restoration.

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Intuitionistic Fuzzy Group With Extended Operations

S. Melliani, I. Bakhadach and L. S. Chadli

Abstract In this paper we give an extension in the intuitionistic fuzzy frame of a low of a crisp group $(G, *)$ by means of the extended form of the Zadeh's extension principle, And we build the conditions whose permit us to give an intuitionistic fuzzy group structure. Furthermore, we investigates some other properties for the intuitionistic fuzzy subgroups and homomorphisms for our set.

1 Introduction

L.A. Zadeh introduced the concept of fuzzy subsets of a well-defined set in his paper [16] for modeling the vague concepts in the real world. After him the concept of fuzzy group was introduced by Rosenfeld in 1971 [11], the theories and approaches on different fuzzy algebraic structures developed rapidly. Anthony and Sherwood [1] gave the definition of fuzzy subgroup based on t-norm. Yuan and Lee [15] defined the fuzzy subgroup and fuzzy subring based on the theory of falling shadows. Liu [7] gave the definition of fuzzy invariant subgroups. By far, two books on fuzzy algebra have been published [4, 9, 10].

K. Atanassov [2] introduced the concept of intuitionistic fuzzy sets in 1986. Since then, many researchers have investigated this topic such as intuitionistic fuzzy group [3]. It is well known that the intuitionistic fuzzy set and the interval-valued fuzzy set are equivalent [14], and consequently the results about interval-valued fuzzy sets can be generalized to the intuitionistic fuzzy sets. In his paper [6] Dubois has showed that the set of fuzzy numbers is just a semi-group. The purpose of this paper is to

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define the concept of intuitionistic fuzzy group based of the extended operation and develop the important results.

This paper is organized as follows. In Section 2 we recall some concept concerning the intuitionistic fuzzy set and give some basic results. The concept of extension operation of the binary operation of the group G takes place in Section 3. In Section 4 we show the associativity and the identity element of the extended operation. To complete the conditions, on the extended operation, which will allows us to define our intuitionistic fuzzy group we will demonstrate the symmetry of this operation in Section 5.

2 Preliminaries

First we give the concept of intuitionistic fuzzy set defined by Atanassov and we recall some elementary definitions that we use in the sequel. Assume that X is an arbitrary universe.

Definition 1. The intuitionistic fuzzy subsets (in shorts **IFSS**) are defined on a non-empty set X as objects having the form

$$A = \{< x, \mu(x), \nu(x) >: x \in X\}$$

where the functions $\mu : X \to [0, 1]$ and $\nu : X \to [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A respectively, and $0 \leq \mu(x) + \nu(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we shall use the symbol $< \mu, \nu >$ for the intuitionistic fuzzy subset $A = \{< x, \mu(x), \nu(x) >: x \in X\}$.

Definition 2. Let $A =< \mu_A, \nu_A >$ and $B =< \mu_B, \nu_B >$ be IFSS of X. Then
$A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$
$A = B$ iff $A \subset B$ and $B \subset A$
$A^c =< \nu_A, \mu_A >$
$A \cap B =< \mu_A \wedge \mu_B, \nu_A \vee \nu_B >$
$A \cup B =< \mu_A \vee \mu_B, \nu_A \wedge \nu_B >$
$[]A =< \mu_A, 1 - \mu_A >, <> A =< 1 - \nu_A, \nu_A >$.

Definition 3. [4] Let G and G' be groups and $f : G \to G'$ be a function. Then, f is called a homomorphism if

$$f(xy) = f(x)f(y) \tag{1}$$

for all $x, y \in G$.

Definition 4. [4] A non-empty subset A of a group G is said to be a subgroup of G if, under the operation in G, A itself forms a group.

Definition 5. [4] A semigroup is a non-empty set G together with a binary operation on G which is associative $a(bc) = (ab)c$, for all $a, b, c \in G$.

Definition 6. Let G be a classical group. Then $A =< x, \mu_{A(x)}, \nu_{A(x)} >\in$ **IFS**$[G]$ is called an intuitionistic fuzzy subgroup on G if the following conditions $(1) - (2)$ are satisfied for all $x, y \in G$,

1. $\mu_{A(xy)} \geq \mu_{A(x)}) \wedge \mu_{A(y)}, \qquad \nu_{A(xy)} \leq \nu_{A(x)} \vee \nu_{Ayx)};$
2. $\mu_{A(x^{-1})} \geq \mu_{A(x)}, \qquad \nu_{A(x^{-1})} \leq \nu_{A(x)}$

3 The Extended ∗ Operation

Let $*$ be an operation definied on a product set $X \times Y$ and taking values on a set Z. The $*$ operation can be extended to intuitionistic fuzzy sets by means of the following extension principle.

Definition 7. Let $A \in X$, $B \in Y$ be two intuitionistic fuzzy subsets, then the extension principle allows to define an intuitionistic fuzzy subset $C = A\tilde{*}B$ of Z as follows, in the case of noninteractive variables : $\forall z \in Z$

$$\mu_{A\tilde{*}B}(z) = \sup_{x*y=z} \mu_A(x) \wedge \mu_B(y) \qquad and \qquad \nu_{A\tilde{*}B}(z) = \inf_{x*y=z} \mu_A(x) \vee \mu_B(y) \quad (2)$$

where, as usual μ_A and ν_A are the membership and the nonmembership functions of **IFS** A, $\wedge$ and $\vee$ denotes the min and max operations.

A pratical application of our study is of course intuitionistic fuzzy arithmetic when A, B and C are intuitionistic fuzzy numbers, i.e. intuitionistic fuzzy subsets of $\mathbb{R}$, and when $*$ is one of the usual operations $+, -, \times, \div$. For example, $A + B = C$ does not imply $B = C - A$ as illustrated in Figure 1 using just the membership functions, where : $\forall z \in \mathbb{R}$

$$\mu_{A+B}(z) = \sup_{x+y=z} \mu_A(x) \wedge \mu_B(y) \qquad and \qquad \nu_{A+B}(z) = \inf_{x+y=z} \mu_A(x) \vee \mu_B(y) \quad (3)$$

$$\mu_{A-B}(z) = \sup_{x-y=z} \mu_A(x) \wedge \mu_B(y) \qquad and \qquad \nu_{A-B}(z) = \inf_{x-y=z} \mu_A(x) \vee \mu_B(y) \quad (4)$$

Example 1. 1. When A and B are crisp sets, A being a subset of X and B a subset of Y, $A\tilde{*}B$ reduces to a subset C of Z defined as

$$A * B = \{z \in Z, z = x * y, x \in Ay \in B\} \quad (5)$$

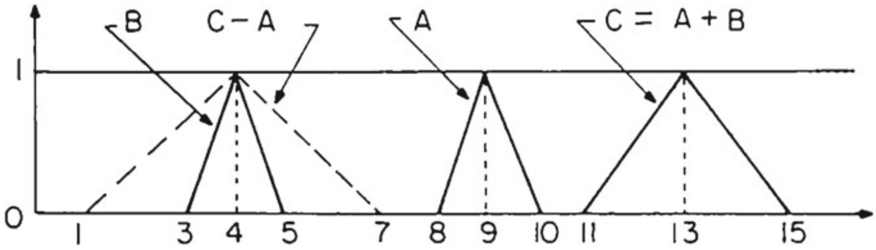

Fig. 1 Illustrative example for $A + B = C \nRightarrow A = C - B$.

2. When $X = Y = Z = \mathbb{R}$ and $*$ stands for addition on numbers, $A * B$ stands then for the addition $A + B$ of two sets as defined by H. Minkowski in 1911.
3. When $X = Y = Z = I(\mathbb{R})$, i.e. the set of interval numbers, or more simply intervals, in the real line $\mathbb{R}$, and when $*$ stands for example for addition, subtraction, multiplication or division, the study of $A * B$ corresponds to Interval Analysis [8].

For practical computations, let us show now one can transform the expression of $A * B$ given in 2.
Assume first that B is a (non-fuzzy) singleton identified with its unique element, say $b \in Y$, so that $\mu_B(y) = 1$ if $y = b$ and $\mu_B(y) = 0$ if $y \neq b$. Thus, equation 2 yields: $\forall z \in Z$

$$\mu_{A \tilde{*} b}(z) = \sup_{\substack{x \in X \\ x*b=z}} \mu_A(x) \qquad and \qquad \nu_{A \tilde{*} b}(z) = \inf_{\substack{x \in X \\ x*b=z}} \mu_A(x) \tag{6}$$

Equation 6 takes the following simpler form : $\forall z \in Z$

$$\forall z \in Z \qquad \mu_{A \tilde{*} B}(z) = \sup_{x*y=z} \mu_A(z-b) \qquad and \qquad \nu_{A \tilde{*} B}(z) = \inf_{x*y=z} \nu_A(z-b)$$

Let us return to 2

$$\begin{aligned} \forall z \in Z \qquad \mu_{A \tilde{*} B}(z) &= \sup_{\substack{x \in X, y \in Y \\ x*y=z}} \mu_A(x) \wedge \mu_B(y) \\ &= \sup_{y \in Y} \Big(\sup_{\substack{x \in X, \\ x*y=z}} \mu_A(x) \wedge \mu_B(y) \Big) \\ &= \sup_{y \in Y} \Big(\mu_B(y) \wedge \sup_{\substack{x \in X, \\ x*y=z}} \mu_A(x) \Big) \\ &= \sup_{y \in Y} \mu_B(y) \wedge \mu_{A \tilde{*} y}(z), \end{aligned}$$

and similary, we have

$$\begin{aligned}\forall z \in Z \qquad \nu_{A\tilde{*}B}(z) &= \inf_{\substack{x\in X, y\in Y\\ x*y=z}} \nu_A(x) \vee \nu_B(y)\\ &= \inf_{y\in Y}\Big(\inf_{\substack{x\in X,\\ x*y=z}} \nu_A(x) \vee \nu_B(y)\Big)\\ &= \inf_{y\in Y}\Big(\nu_B(y) \vee \inf_{\substack{x\in X,\\ x*y=z}} \nu_A(x)\Big)\\ &= \inf_{y\in Y} \nu_B(y) \vee \nu_{A\tilde{*}y}(z)\end{aligned}$$

from (6) with y playing the role of b. Hence, (2) is equivalent to

$$\mu_{A\tilde{*}B}(z) = \sup_{x\in X} \mu_A(x) \wedge \mu_{B\tilde{*}x}(z) \quad \textit{and} \quad \nu_{A\tilde{*}B}(z) = \inf_{x\in X} \nu_A(x) \vee \nu_{B\tilde{*}x}(z). \quad (7)$$

Analogously, exchanging the roles of A and B, we have

$$\mu_{A\tilde{*}B}(z) = \sup_{y\in Y} \mu_{A\tilde{*}y}(z) \wedge \mu_B(y) \quad \textit{and} \quad \nu_{A\tilde{*}B}(z) = \inf_{y\in Y} \nu_{A\tilde{*}y}(z) \vee \nu_B(y). \quad (8)$$

Note that with addition of intuitionistic fuzzy numbers, (7) takes the form

$$\mu_{A\tilde{*}B}(z) = \sup_{y\in Y} \mu_A(z-y) \wedge \mu_B(y) \quad \textit{and} \quad \nu_{A\tilde{*}B}(z) = \inf_{y\in Y} \nu_A(z-y) \vee \nu_B(y). \quad (9)$$

4 The Monoid Structure of (IFS(G); $\tilde{*}$)

In order to show that the intiuitionistic fuzzy set $IFS(G)$ is a group we will first demonstrate that is a monoid structure. Let A, B and C the intuitionistic fuzzy subset of $IFS(G)$, we have

$$\begin{aligned}\mu_{(A\tilde{*}B)\tilde{*}C}(z) &= \sup_{x*y=z} (\mu_{(A\tilde{*}B)}(x) \wedge \mu_C(y))\\ &= \sup_{x*y=z} (\sup_{a*b=x} (\mu_A(a) \wedge \mu_B(b)) \wedge \mu_C(y))\\ &= \sup_{(a*b)*y=z} (\mu_A(a) \wedge \mu_B(b)) \wedge \mu_C(y))\\ &= \sup_{a*(b*y)=z} \mu_A(a) \wedge (\mu_B(b) \wedge \mu_C(y)))\\ &= \sup_{a*\beta=z} \mu_A(a) \wedge \sup_{b*y=\beta} (\mu_B(b) \wedge \mu_C(y)))\\ &= \sup_{a*\beta=z} \mu_A(a) \wedge \mu_{B\tilde{*}C}(\beta)\\ &= \mu_{A\tilde{*}(B\tilde{*}C)}(z)\end{aligned}$$

$$\nu_{(A\tilde{*}B)\tilde{*}C}(z) = \inf_{x*y=z} \nu_{(A\tilde{*}B)}(x) \vee \nu_C(y)$$
$$\nu_{(A\tilde{*}B)\tilde{*}C}(z) = \inf_{x*y=z} (\inf_{a*b=x} \nu_A(a) \vee \nu_B(b)) \vee \mu_C(y))$$
$$\nu_{(A\tilde{*}B)\tilde{*}C}(z) = \inf_{(a*b)*y=z} (\nu_A(a) \vee \nu_B(b)) \vee \mu_C(y))$$
$$\nu_{(A\tilde{*}B)\tilde{*}C}(z) = \inf_{a*(b*y)=z} \nu_A(a) \vee (\nu_B(b) \vee \mu_C(y)))$$
$$\nu_{(A\tilde{*}B)\tilde{*}C}(z) = \inf_{a*\beta=z} \nu_A(a) \vee \inf_{b*y=\beta} (\nu_B(b) \vee \mu_C(y)))$$
$$\nu_{(A\tilde{*}B)\tilde{*}C}(z) = \inf_{a*\beta=z} \nu_A(a) \vee \mu_{(B\tilde{*}C)}(\beta)$$
$$\nu_{(A\tilde{*}B)\tilde{*}C}(z) = \nu_{A\tilde{*}(B\tilde{*}C)}(z).$$

and the identity element is given by:

$\forall y \in G\ \mu_e(y) = \begin{cases} 1 & if \quad e = y \\ 0 & if \quad e \neq y \end{cases}$ and $\nu_e(y) = \begin{cases} 0 & if \quad e = y \\ 1 & if \quad e \neq y \end{cases}$ Indeed

$$A\tilde{*}\tilde{e}(z) = \sup_{a*b=z} \mu_A(a) \wedge \mu_e(b)$$
$$= \begin{cases} \mu_A(z) & if \quad e = y \\ 0 & if \quad e \neq y \end{cases}$$

Now let we define a symetric element of an intuitionistic fuzzy element.

5 Symetric Element For $\tilde{*}$

We know that the symetric element of A is the solution of the equation $A\tilde{*}X = \tilde{e}$. to this end we will define the α operator. and solving the $\tilde{*}$-equation problem $i.e$ $A\tilde{*}X = C$ in general for any $* : X \times Y \longrightarrow Z$ operation.

5.1 The α operator

In order to solve the $\tilde{*}$-equation problem on intuitionistic fuzzy sets we need to recall the definition of the α operator which is characteristic of Brouwerian lattices. That α operator has proved to be useful in the resolution of composite fuzzy relation equations [[12],[13]] and we study here a particular composite, with a constraint expressed by the $*$ operation.

Definition 8. Given a and b in [0, 1], $a\alpha b$ is defined as the greatest element x in [0, 1] such that $a \wedge x \leq b$, i.e.

$$a\alpha b = \begin{cases} 1 & if \quad a \leq b \\ b & if \quad a > b \end{cases}$$

Here are some properties of the α operator that will be used in the sequel. We recall that, as usual, $\vee$ denotes the max operation. For all $a, b \in [0, 1]$ and for all family $(b_i)_i \in I$ of elements of [0, 1], we have

$$a \wedge (a\alpha b) \leq b \tag{10}$$

$$a\alpha(\sup_{i \in I} b_i) \geq a\alpha b \tag{11}$$

$$a\alpha(a \wedge b) \leq b \tag{12}$$

According to (8), properties (10) and (12) are directly verified. To check (11), it suffices to denote $c = \sup_{i \in I, i \neq j} b_i$, and to show that $a\alpha(c \vee b_i) \geq a\alpha b_j$.

Definition 9. Given $A \in \mathbf{IFS}(X)$ and $C \in \mathbf{IFS}(Z)$ and $\tilde{*} : \mathbf{IFS}(X) \times \mathbf{IFS}(Y) \rightarrow \mathbf{IFS}(Z)$, we define $\Diamond : \mathbf{IFS}(Z) \times \mathbf{IFS}(X) \rightarrow \mathbf{IFS}(Y)$ as follows $\forall y \in Y$
$\mu_{C \Diamond A}(z) = \inf_{x \tilde{*} y = z} \mu_A(x) \alpha \mu_C(z)$ and $\nu_{C \Diamond A}(z) = \sup_{x \tilde{*} y = z} \nu_A(x) \alpha \nu_C(z)$

As a property for this operation we have

$$C_1 \subseteq C_2 \Rightarrow C_1 \Diamond A \subseteq C_2 \Diamond A \tag{13}$$

this equation is simply verified after checking that in [0, 1], if $c_1 \leq c_2$ then $a\alpha c_1 \leq a\alpha c_2$

5.2 *Resolution Of* $\tilde{*}$ *Equation On* IFS

Theorem 1. *For every intuitionistic fuzzy set A of G, and for* $\tilde{*} : IFS(X) \times IFS(Y) \rightarrow IFS(Z)$, *we have*

$$A \tilde{*} (C \Diamond A) \subseteq C \tag{14}$$

In order terms, $C \Diamond A$ *is a particular solution to* $A \tilde{*} X = C$

Proof. Let $U = A \tilde{*} (C \Diamond A)$ and let $z \in G$. Then

$$\mu_U(z) = \sup_{x\tilde{*}y=z} \mu_A(x) \wedge \mu_{(C\Diamond A)}(y)$$
$$\mu_U(z) = \sup_{x\tilde{*}y=z} \mu_A(x) \wedge \inf_{x'\tilde{*}y=z'} \mu_A(x')\alpha\mu_{\tilde{e}}(z')$$
$$\mu_U(z) \leq \sup_{x\tilde{*}y=z} \mu_A(x) \wedge (\mu_A(x)\alpha\mu_C(z))$$
$$\mu_U(z) \leq \sup_{x\tilde{*}y=z} \mu_C(z)$$
$$\mu_U(z) \leq \mu_C(z)$$

and

$$\nu_U(z) = \inf_{x\tilde{*}y=z} \nu_A(x) \vee \nu_{(C\Diamond A)}(y)$$
$$\nu_U(z) = \inf_{x\tilde{*}y=z} \nu_A(x) \vee \sup_{x'\tilde{*}y=z'} \nu_A(x')\alpha\nu_{\tilde{e}}(z')$$
$$\nu_U(z) \geq \inf_{x\tilde{*}y=z} \nu_A(x) \vee (\nu_A(x)\alpha\nu_C(z))$$
$$\nu_U(z) \geq \inf_{x\tilde{*}y=z} \nu_C(z)$$
$$\nu_U(z) \geq \nu_C(z)$$

□

Theorem 2. *For every pair of intuitionistic fuzzy sets $A \in IFS(X)$ and $B \in IFS(Y)$ and for $\tilde{*} : IFS(X) \times IFS(Y) \to IFS(Z)$, we have*

$$B \subseteq (A * B)\Diamond A \tag{15}$$

Note that when, $A\tilde{*}B = C$, we have $B \subseteq C\Diamond A$

Proof. Let $V = (A * B)\Diamond A$ and let $y \in Y$.

$$\mu_V(y) = \inf_{x*y=z} \mu_A(x)\alpha\mu_{A\tilde{*}B}(z)$$
$$\mu_V(y) = \inf_{x*y=z} \mu_A(x)\alpha \sup_{x'*y'=z} \mu_A(x') \wedge \mu_B(y')$$
$$\mu_V(y) \geq \inf_{x*y=z} \mu_A(x)\alpha(\mu_A(x) \wedge \mu_B(y))$$
$$\mu_V(y) \geq \inf_{x*y=z} \mu_B(y)$$
$$\mu_V(y) \geq \mu_B(y)$$

and

$$\nu_V(y) = \sup_{x*y=z} \nu_A(x)\alpha\nu_{A\tilde{*}B}(z)$$

$$\nu_V(y) = \sup_{x*y=z} \nu_A(x)\alpha \inf_{x'*y'=z} \nu_A(x') \vee \nu_B(y')$$

$$\nu_V(y) \leq \sup_{x*y=z} \nu_A(x)\alpha(\nu_A(x) \vee \nu_B(y))$$

$$\nu_V(y) \leq \sup_{x*y=z} \nu_B(y)$$

$$\nu_V(y) \leq \nu_B(y)$$

□

Corollary 1. *Given $A \in IFS(X)$, $C \in IFS(Z)$ and $\tilde{*} : IFS(X) \times IFS(Y) \rightarrow IFS(Z)$, equation $A\tilde{*}X \subseteq C$ has always a greatest solution given by $C\Diamond A$. Moreover, the set of solutions of $A\tilde{*}X \subseteq C$ is a lattice.*

Proof. From (14), $C\Diamond A$ is a solution to $A\tilde{*}X \subseteq C$ let us show that is the greatest one. Let $B \in IFS(Y)$ such that $A\tilde{*}B \subseteq C$. From (13) we have $(A\tilde{*}B)\Diamond A \subseteq C\Diamond A$. Finally (15) yields $B \subseteq C\Diamond A$. The fact that the set of solutions of $A\tilde{*}X \subseteq C$ is a lattice was already pointed out as a result of

$$A\tilde{*}(B_1 \cup B_2) = (A\tilde{*}B_1) \cup (A\tilde{*}B_2) \tag{16}$$

and

$$A\tilde{*}(B_1 \cap B_2) \subseteq (A\tilde{*}B_1) \cup (A\tilde{*}B_2) \tag{17}$$

□

Corollary 2. *For $A \in IFS(X)$, $B \in IFS(Y)$ and $C \in IFS(Z)$*

$$A\tilde{*}B \subseteq C \quad iff \quad B \subseteq C\Diamond A \tag{18}$$

Proof. If "$A\tilde{*}B \subseteq C$ then $B \subseteq C\Diamond A$" was already shown in the proof of Corollary 1. Now assume that $B \subseteq C\Diamond A$. Hence from the fact that the extension $\tilde{*}$ of $*$ is inclusion monotonic $i.e.$ $B_1 \subseteq B_2 \Rightarrow A\tilde{*}B_1 \subseteq \tilde{*}B_2$ we have $A\tilde{*}B \subseteq A\tilde{*}(C\Diamond A)$. Finally from (14) yields $A\tilde{*}B \subseteq C$. □

Theorem 3. *Given an intuitionistic fuzzy set $A \in IFS(X)$, $C \in IFS(Z)$ and $\tilde{*} : IFS(X) \times IFS(Y) \rightarrow IFS(Z)$, the equation $A\tilde{*}X = C$ has a solution iff*

$$A\tilde{*}(C\Diamond A) = C$$

Moreover, when $C\Diamond A$ is a solution, then it is the greatest one and the set of solutions in an upper semi-lattice

An analogous theorem holds, of course, for equation $X\tilde{*}B = C$.

Proof. Let us assume that $B \in IFS(Y)$ is a solution to $A\tilde{*}X = C, i, eA\tilde{*}B = C$. Hence from (15) we have $B \subseteq C\Diamond A$. But (13) yields $A\tilde{*}B \subseteq A\tilde{*}(C\Diamond A)$, i.e $C \subseteq A\tilde{*}(C\Diamond A)$, so that $A\tilde{*}(C\Diamond A) = C$ from (14).
When $A\tilde{*}X = C$ has a solution, then $C\Diamond A$ is the greatest one as a direct application of (15).
The fact that the set of solutions $A\tilde{*}X = C$, when non void, is an upper semi-lattice was already pointed as a result of (16). □

Theorem 4. *Given two intuitionistic fuzzy sets $A, C \in IFS(G)$ and $\tilde{*} : IFS(G) \times IFS(G) \to IFS(G)$, th equation $A\tilde{*}X = \tilde{e}$ has an unique solution iff*

$$A\tilde{*}(\tilde{e}\Diamond A) = \tilde{e}$$

Proof. For the exictence of the solution is immidiatly by (3).
For the uniqueness.
Let B_1 and two intuitionistic fuzzy differents symetric element of A the we have $B_1 = \tilde{e}\Diamond A$ the same of B_2 we have $B_2 = \tilde{e}\Diamond A$ by definition of $\Diamond$ we have $B_1 = B_2$.
□

Example 2. In [6] we have $G = \mathbb{R}$ with the operation $+$ is just a semi group, but using the extended operation with the α oparator we have the same structur i.e. a group. Note that in this case we denote $A\Diamond B$ by $A \ominus B$.

Theorem 5. *Let $H \subseteq G$. We have H is a sub-group of G then $IFS(H)$ is an intuitionistic fuzzy sub-group of $IFS(G)$.*

Proof. Let H be a subgroup of G the by the previous theorem we have $IFS(H)$ is an intuitionistic fuzzy group with extended operation and we have $IFS(H) \subseteq IFS(G)$ that $IFS(H)$ is an intuitionistic fuzzy sub-group of $IFS(G)$. □

Theorem 6. *Let $f : (G, \star) \to (G', T)$ be a homomorphism then the extended operation $\tilde{f} : (IFS(G), \tilde{\star}) \to (IFS(G'), \tilde{T})$ is an ituitionistic fuzzy homomorphism.*

Proof. We have

$$\begin{aligned}
\tilde{f}(A\tilde{*}B)(z) &= \sup_{f(x*y)=z} A(x) \wedge B(y) \\
&= \sup_{f(x)Tf(y)=z} A(x) \wedge B(y) \\
&= \sup_{f(x)Tf(y)=z} A(x) \wedge B(y)
\end{aligned}$$

and we have

$$\begin{aligned}(\tilde{f}(A)\tilde{T}\tilde{f}(B))(z) &= \sup_{xTy=z} \tilde{f}A(x) \wedge \tilde{f}B(y) \\ &= \sup_{xTy=z} (\sup_{a=f(x)} A(x) \wedge \sup_{b=f(y)} A(y)) \\ &= \sup_{f(x)Tf(y)=z} A(x) \wedge B(y)\end{aligned}$$

Therefore $\tilde{f}$ is an intuitionistic fuzzy homomorphism. □

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Generalization of Quasi-modular Extensions

El Hassane Fliouet

Abstract Let K/k be a purely inseparable extension of characteristic $p > 0$. Let $lm(K/k)$ and $um(K/k)$ be the smallest extensions $k \longrightarrow lm(K/k) \longrightarrow K \longrightarrow um(K/k)$ such that $K/lm(K/k)$ and $um(K/k)/k$ are modular. In this note, we continue to study the locus problem of $lm(K/k)$ and $um(K/k)$ relative to K/k. Thus improving ([3], Theorem 1.4), we show that $lm(K/k)$ is nontrivial when K/k is of finite size, more precisely if K/k has a finite size and unbounded exponent, the same is true of $K/lm(K/k)$. However, if K/k is of unbounded size, it may well be that we lose this property by obtaining $lm(K/k) = K$. In the following, we will say that K/k is lq-modular (respectively, uq-modular) if $lm(K/k)/k$ (respectively, $um(K/k)/K$) has an exponent. The first study of these two concepts devoted to the extensions of finite size is in [4, 6, 7]. However, the object of the present work consists to generalize the results of finite size to any extension. In particular, we treat the stability questions of the lq-modularity and the uq-modularity relative to inclusion, intersection, and product. Furthermore, we are interested by the questions about existence of the smallest extensions which preserve these concepts in the ascendant or descendant sense, and also to the questions of existence of the maximal subextensions (closures).

Keywords Purely inseparable · q-finite modular extension · Lq-modular extension · Up-modular

1 Introduction

Let K/k be a purely inseparable extension of characteristic $p > 0$. We recall that K/k is modular if for each $n \in \mathbb{N}$, K^{p^n} and k are $k \cap K^{p^n}$-linearly disjoint. This notion was defined for the first time by Swedleer in [12]. In addition, the author characterizes the purely inseparable extensions which are tensor product over k of simple extensions of k. In the same order of ideas, Waterhouse in [13] shows that the modularity is

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stable by any intersection covering k or K, and that an increasing union of modular extensions is also modular. In particular, there exists smallest extensions denoted, respectively, by $lm(K/k)$ and $um(K/k)$ such that $k \longrightarrow lm(K/k) \longrightarrow K \longrightarrow um(K/k)$ with $K/lm(K/k)$ and $um(K/k)/k$ are modular. In this note, we continue to take an interest in the location problem of $lm(K/k)$ and $um(K/k)$ with respect to K/k. Let us thus improve ([3], Theorem 1.4), we show that $lm(K/k)$ is not trivial when K/k is of finite size. More precisely, if K/k is of finite size and of unbounded exponent, then the same holds also for $K/lm(K/k)$. However, if the size of K/k is infinite, it is highly probable that we lose this property by obtaining $lm(K/k) = K$. In the following, we will say that K/k is lq-modular (respectively, uq-modular) if $lm(K/k)/k$ (respectively, $um(K/k)/K$) has an exponent. Clearly in the case of finite size, the q-modularity is synonymous with the modularity up to finite extension. On the other hand, knowing that a first study of these two notions devoted to extensions of finite size is found in [4, 6, 7], the object of this work is to generalize the results of finite size to any extension. In particular, we deal with the stability issues of the lq-modularity and the uq-modularity with respect to inclusion, intersection, and product. We are also interested in the existence problems of the smallest extensions which retain these two notions either in the ascending or descending sense. We study as well the existence questions of the largest subextensions (the closure).

Finally, it should be noted that, during this note, we reiterate the notations and the elementary results from [8], since they are frequently used here.

2 Preliminary Notions and Terminologies

First, we will begin by giving a preliminary list of the most frequently-used symbols throughout this work:

- *k always designates a commutative field with characteristic $p > 0$, and Ω an algebraic closure of k.*
- *$k^{p^{-\infty}}$ indicates the purely inseparable closure of Ω/k.*
- *For any $a \in \Omega$, for every $n \in \mathbb{N}^*$, we symbolize the root of the polynomial $X^{p^n} - a$ in Ω* by *$a^{p^{-n}}$. In addition, we put $k(a^{p^{-\infty}}) = k(a^{p^{-1}}, \dots, a^{p^{-n}}, \dots) = \bigcup_{n \in \mathbb{N}^*} k(a^{p^{-n}})$ and $k^{p^{-n}} = \{a \in \Omega \mid a^{p^n} \in k\}$.*
- *For any family $B = (a_i)_{i \in I}$ of elements in Ω, we put $k(B^{p^{-\infty}}) = k(({a_i}^{p^{-\infty}})_{i \in I})$.*
- Finally, $|.|$ *designates the cardinal.*

It should also be pointed out that all extensions that intervene in this paper are purely inseparable subextensions of Ω, and it is convenient to denote $[k, K]$ the set of intermediate fields of an extension K/k.

2.1 Irrationality Degree

Definition 1 *Let K/k be a purely inseparable extension. A subset G of K is said to be r-generator of K/k if $K = k(G)$ and, if in addition, for each $x \in G$, $x \notin k(G \backslash x)$, then G will be called a minimal r-generator of K/k.*

Definition 2 *Given an extension K/k of characteristic $p > 0$ and a subset B of K. We say that B is a r-base (relative base) of K/k, if B is a minimal r-generator of $K/k(K^p)$. In the same order of ideas, B is said to be r-free (or r-independent) over k, if B is a r-base of $k(B)/k$, in the opposite case B is said to be r-dependent over k.*

By virtue of ([1], III, p. 49, Corollary 3) and by the exchange property of r-independence, we deduce that every extension has a r-base and that the cardinal of a r-base is invariant (cf. [8], Theorems 2.7 and 2.8). We recall that a purely inseparable extension K/k is said to have an exponent (or, to be of bounded exponent) if there exists nonnegative integer e such that $K^{p^n} \subseteq k$. Taking into account ([10], Corollary 1.6), if K/k has an exponent, then B is a r-base of K/k if and only if B is a minimal r-generator of K/k. However, a minimal r-generator may not exist in the general case (cf. [10], Lemma 1.16, Proposition 1.23).

We now have the required tool to define the irrationality degree of an extension K/k. Firstly, it is immediate that for any $n \in \mathbb{N}$, $k^{p^{-n}} \cap K/k$ has an exponent, let us therefore consider a r-base B_n of $k^{p^{-n}} \cap K/k$.

Definition 3 *The invariant $di(K/k) = \sup_{n \in \mathbb{N}}(|B_n|)$ will be called the irrationality degree of K/k.*

Here the sup is used in the sense from ([1], III, p. 25, Proposition 2). Moreover, for reasons of thematic specificity and coherence, $di(k/k^p)$ will be called the imperfection degree of k and will be denoted $di(k)$. Systematically these two invariants allow to control the size of K/k and the length of every field k. On the other hand, the size measurement of an extension is compatible with inclusion. In other words, we have:

Theorem 2.1 **([8], Theorem 3.8)** *For any family $k \subseteq L \subseteq L' \subseteq K$ of purely inseparable extensions, we have $di(L/L') \leq di(K/k)$.*

As an immediate consequence $di(K/k) = \sup(di(L/k))_{L \in [k,K]}$, that is to say the size measurement of K/k is seen as an inductive limit of the irrationality degree of these intermediate subextensions. We also deduce that any increasing family $(K_n)_{n \in \mathbb{N}}$ of purely inseparable extensions satisfies $di(\bigcup_{n \in \mathbb{N}}(K_n)/k) = \sup_{n \in \mathbb{N}}(di(K_n/k))$.

Corollary 2.1.1 **([8], Corollary 3.9)** *For any purely inseparable extension K/k, $di(K) \leq di(k)$.*

2.2 Relatively Perfect Extensions

In this section, we recall a few notions and results from [5]. A field k of characteristic p is said to be perfect if $k^p = k$. In the same order of ideas, we say that K/k is relatively perfect if $k(K^p) = K$. We can easily verify that:

- *The relatively perfect property is transitive, i.e., if K/L and L/k are relatively perfect, then K/k is also perfect.*
- If *K/k is relatively perfect, then the same is true of $L(K)/k(L)$.*
- *The relatively perfect property is stable by any product for k. In other words, for any family $(K_i/k)_{i\in I}$ of relatively perfect extensions, we then have $\prod_i K_i/k$ is also relatively perfect.*

So for every purely inseparable extension K/k, there exists a largest relatively perfect subextension contained in K. This is called the relatively perfect closure of K/k and it is denoted by $rp(K/k)$.

Proposition 2.2 **([5], Proposition 5.2)** *Let L be an intermediate field of a purely inseparable extension K/k, then $rp(rp(K/L)/k) = rp(K/k)$ and $rp(K/rp(L/k)) = rp(K/k)$.*

Corollary 2.2.1 *For every $L \in [k, K]$, we have K/L finie $\Longrightarrow rp(K/k) \subset L$.*

In particular, if K/k is relatively perfect, we have K/L *finite* $\Longrightarrow L = K$. Schematically we have a *hole*

$$\begin{array}{ccc} k \longrightarrow & & K; \\ & \uparrow & \\ & hole & \end{array}$$

and this *hole* characterizes the fact that K/k is relatively perfect. Indeed, suppose that K/k satisfies the *hole*, and let B be a r-base of K/k. Suppose $B \neq \emptyset$; let $x \in B$ and $L = k(K^p)(B \setminus \{x\})$; we have K/L is finite, so $K = L$, a contradiction.

Proposition 2.3 **([5], Lemma 2.1)** *Let K/k be a purely inseparable extension such that $[K : k(K^p)]$ is finite. Then we have:*

(i) *K is relatively perfect over a finite extension of k.*
(ii) *The decreasing sequence $(k(K^{p^n}))_{n\in\mathbb{N}}$ is stationary over $k(K^{p^{n_0}}) = rp(K/k)$.*

As a consequence of the preceding proposition, we have:

Proposition 2.4 **([5], Proposition 6.2)** *Let K/k be a purely inseparable extension such that $[K : k(K^p)]$ is finite. For every $L \in [k, K]$, we have $rp(K/L) = L(rp(K/k))$.*

2.3 Quasi-finite Extensions

Definition 4 *Any extension whose irrationality degree is finite is called q-finite (quasi-finite) extension.*

In other words, the q-finitude is synonymous of the horizontal finitude. However, the finitude is translated by the horizontal and vertical finitude, it is a finitude in terms of height and length, i.e., K/k is finite if and only if K/k is q-finite of bounded exponent. Furthermore, it is verified that *the irrationality degree of an extension K/k is 1 if only the set of intermediate fields of K/k is totally ordered.* Then, every extension that satisfies the previous statement will be called extension q-simple (quasi-simple).

Now let L/k be a subextension of a q-finite extension K/k, for each $n \in \mathbb{N}$, we always note $k_n = k^{p^{-n}} \cap K$. We verify immediately that:

(i) *The q-finitude is transitive, especially for each $n \in \mathbb{N}$, $K/k(K^{p^n})$ and k_n/k are finite.*
(ii) *There exists $n_0 \in \mathbb{N}$, for each $n \geq n_0$, $di(k_n/k) = di(K/k)$.*

In addition, here are some immediate applications of Proposition 2.3.

Proposition 2.5 *Let K/k be a q-finite extension. The sequence $(k(K^{p^n}))_{n\in\mathbb{N}}$ stops over $rp(K/k)$ from a n_0. In particular, $K/rp(K/k)$ is finite.*

Proposition 2.6 **([8], Proposition 4.3)** *For every q-finite extension K/k, there exists $n \in \mathbb{N}$ such that K/k_n is relatively perfect. Moreover, $k_n(rp(K/k)) = K$.*

The irrationality degree of a q-finite extension K/k checks the following equality.

Proposition 2.7 **([8], Proposition 4.8)** *For any sequence of relatively perfect subextensions $k = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n$ of a q-finite extension K/k, we have $di(K/k) = \sum_{i=0}^{n-1} di(K_{n+1}/K_n) + di(K/K_n)$.*

2.4 Exponents of a q-finite Extension

In this paragraph, we will use some basic definitions and notations as it is mentioned in [2]. Let K/k be a finite purely inseparable extension. For $x \in K$, put $o(x/k) = \inf\{m \in \mathbb{N} |\ x^{p^m} \in k\}$ and $o_1(K/k) = \inf\{m \in \mathbb{N} |\ K^{p^m} \subset k\}$. A r-base $B = \{a_1, a_2, \ldots, a_n\}$ of K/k is said to be canonically ordered if for $j = 1, 2, \ldots, n$, we have $o(a_j/k(a_1, a_2, \ldots, a_{j-1})) = o_1(K/k(a_1, a_2, \ldots, a_{j-1}))$. The integer $o(a_j/k(a_1, \ldots, a_{j-1}))$ thus defined satisfies $o(a_j/k(a_1, \ldots, a_{j-1})) = \inf\{m \in \mathbb{N} |\ di(k(K^{p^m})/k) \leq j-1\}$ (cf. [3], Lemma 1.3). We immediately deduce the result ([11], Satz 14) which confirm the independence of the integers $o(a_i/k(a_1, \ldots, a_{i-1}))$, $(1 \leq i \leq n)$, with respect to the choice of canonically ordered r-bases $\{a_1, \ldots, a_n\}$ of K/k.

Therefore, we put $o_i(K/k) = o(a_i/k(a_1, \ldots, a_{i-1}))$ if $1 \leq i \leq n$, and $o_i(K/k) = 0$ if $i > n$ where $\{a_1, \ldots, a_n\}$ is a canonically ordered r-base of K/k. The invariant $o_i(K/k)$ defined above is called the i-th exponent of K/k.

In the second step, we consider that K/k is q-finite. Recall that for each $n \in \mathbb{N}^*$, k_n always designates $k^{p^{-n}} \cap K$. By virtue of ([2], Proposition 6), for each $j \in \mathbb{N}^*$, the sequence of natural integers $(o_j(k_n/k))_{n\geq 1}$ is increasing, and thus $(o_j(k_n/k))_{n\geq 1}$ converges to $+\infty$, or $(o_j(k_n/k))_{n\geq 1}$ becomes constant after a certain rank. It is trivially obvious that, if $(o_j(k_n/k))_{n\geq 1}$ is bounded, then for each $t \geq j$, $(o_t(k_n/k))_{n\geq 1}$ is also bounded (and therefore stationary).

Definition 5 *Let K/k be a q-finite extension, and j a nonzero natural integer. We call the j-th exponent of K/k the invariant* $o_j(K/k) = \lim_{n\to+\infty}(o_j(k_n/k))$.

Lemma 2.1 **([8], Lemma 4.14)** *Let K/k be a q-finite extension, then $o_s(K/k)$ is finite if and only if there exists a natural integer n such that $di(k(K^{p^n})/k) < s$, and we have $o_s(K/k) = \inf\{m \in \mathbb{N} \mid di(k(K^{p^m})/k) < s\}$. In particular, $o_s(K/k)$ is infinite if and only if for each $m \in \mathbb{N}$, $di(k(K^{p^m})/k) \geq s$.*

The result below makes it possible to reduce the study of properties of exponents of a q-finite extension to a finite extension through the relatively perfect closure.

Theorem 2.8 **([8], Theorem 4.15)** *Let K_r/k be the relatively perfect closure of irrationality degree s of a q-finite extension K/k ($di(K_r/k) = s$), then we have:*

(i) *For each $t \leq s$, $o_t(K/k) = +\infty$.*
(ii) *For each $t > s$, $o_t(K/k) = o_{t-s}(K/K_r)$.*

In addition, $o_t(K/k)$ is finite if and only if $t > s$.

Here is a list of immediate consequences (cf. [8]).

Proposition 2.9 *Let K and L be two intermediate fields of a q-finite extension M/k. For every $j \in \mathbb{N}^*$, we have $o_j(L(K)/L) \leq o_j(K/k)$.*

Proposition 2.10 *Given q-finite extensions $k \subseteq L \subseteq K$. For each $j \in \mathbb{N}^*$, we have $o_j(L/k) \leq o_j(K/k)$.*

2.5 Modular Extensions

We recall that an extension K/k is said to be modular if and only if for each $n \in \mathbb{N}$, K^{p^n} and k are $K^{p^n} \cap k$-linearly disjoint. This notion has been for the first time by Swedleer in [12], she characterizes the purely inseparable extensions which are tensor product of simple extensions over k, it is the equivalent of the fundamental concept Galois theory. Furthermore, if there exists a subset B of a given field K such that $K \simeq \otimes_k(\otimes_k k(a))_{a\in B}$, necessarily B will be a r-base of K/k and it will be called subsequently modular r-base of K/k. In particular, according to Swedleer's theorem, if K/k has an exponent, it is equivalent to say that:

(i) *K/k has a modular r-base.*
(ii) *K/k is modular.*

Let m_j be the j-th exponent of a finite purely inseparable extension K/k and $\{\alpha_1, \dots, \alpha_n\}$ a canonically ordered r-base of K/k, therefore by ([3], Proposition 5.3), for each $j \in \{2, \dots, n\}$, there exists unique constants $C_\varepsilon \in k$ such that $\alpha_j{}^{p^{m_j}} = \sum_{\varepsilon \in \Lambda_j} C_\varepsilon (\alpha_1, \dots, \alpha_{j-1})^{p^{m_j}\varepsilon}$ where $\Lambda_j = \{(i_1, \dots, i_{j-1})$ such that $0 \leq i_1 < p^{m_1 - m_j}, \dots, 0 \leq i_{j-1} < p^{m_{j-1} - m_j}\}$. These relations that are due to G. Pickert (cf. [11]) will be called the definition equations of K/k.

The criterion below allows to test the modularity of an extension.

Theorem 2.11 (Modularity criterion, [3], Proposition 1.4) *Under the previous notions, the following properties are equivalent:*

(1) *K/k is modular.*
(2) *For every canonically ordered r-base $\{\alpha_1, \dots, \alpha_n\}$ of K/k, $C_\varepsilon \in k \cap K^{p^{m_j}}$ for each $j \in \{2, \dots, n\}$.*
(3) *There exists a canonically ordered r-base $\{\alpha_1, \dots, \alpha_n\}$ of K/k such that $C_\varepsilon \in k \cap K^{p^{m_j}}$ for each $j \in \{2, \dots, n\}$.*

The following result is an immediate consequence of the modularity.

Proposition 2.12 *Let $m, n \in \mathbb{Z}$ with $n \geq m$. If K/k is modular, then K^{p^m}/k^{p^n} is also modular.*

Proposition 2.13 ([3], Proposition 8.4) *Let K/k be a finite (respectively, and modular) purely inseparable extension, and let L/k be a subextension (respectively, and modular) of K/k with $di(L/k) = s$. If $K^p \subseteq L$, there exists a canonically ordered r-base (respectively, and modular) $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of K/k and $e_1, e_2, \dots, e_s \in \{1, p\}$ such that $(\alpha_1{}^{e_1}, \alpha_2{}^{e_2}, \dots, \alpha_s{}^{e_s})$ be a canonically ordered r-base (respectively, and modular) of L/k. In addition, for each $j \in \{1, \dots, s\}$, we have $o_j(K/k) = o_j(L/k)$ in which case $e_j = 1$, or $o_j(K/k) = o_j(L/k) + 1$ in which case $e_j = p$.*

The following theorem which is due to Waterhouse plays an important role in the study of modular extensions (cf. [13], Theorem 1.1).

Theorem 2.14 *Let $(K_j)_{j \in I}$ be a family of intermediate subfields of a commutative field Ω, and K an another subfield of Ω. If for each $j \in I$, K and K_j are $K \cap K_j$-linearly disjoint, then K and $\bigcap_j K_j$ are also $K \cap (\bigcap_j K_j)$-linearly disjoint.*

As a consequence the modularity is stable by any intersection covering a commutative field either above or below. More specifically, we have:

Corollary 2.14.1 *Under the same hypotheses of the above theorem, we have:*

(i) *If for each $j \in I$, K_j/k is modular, the same is also true for $\bigcap_j K_j/k$.*

(ii) *If for each $j \in I$, K/K_j is modular, the same is true for $K/\bigcap_j K_j$.*

According to Waterhouse's theorem, there exists a smallest subextension m/k of K/k (respectively, a smallest extension M/K) such that K/m (respectively, M/k) is modular. Henceforth, we denote $m = lm(K/k)$ and $M = um(K/k)$. However, the extension $um(K/k)$ will be called the modular closure of K/k.

The following result is well known (cf. [9]).

Proposition 2.15 *Let K/k be a purely inseparable modular extension, and let for each $n \in \mathbb{N}$, $K_n = k(K^{p^n})$. Then k_n/k, K/k_n, K_n/k and K/K_n are modular.*

3 Quasi-finitude and Modularity

Before stating the main results, we need to recall a few notions. Let K/k be a q-finite extension of unbounded exponent. For each $j \in \mathbb{N}^*$, we put $k_j = k^{p^{-j}} \cap K$, $U_s^j(K/k) = j - o_s(k_j/k)$, and $Ilqm(K/k)$ denotes the first nonzero natural integer i_0 for which the sequence $(U_{i_0}^j(K/k))_{j\in\mathbb{N}}$ is unbounded.

The above result is an immediate application of Proposition 2.10.

Proposition 3.1 *Given a q-finite extension K/k of unbounded exponent, then the sequence $(U_s^j(K/k))_{j\in\mathbb{N}}$ is increasing for each nonzero natural integer s.*

Proof. As ${k_{n+1}}^p \subseteq k_n$, it's clear that $o_s(k_n/k) \leq o_s(k_{n+1}/k) \leq o_s(k_n/k) + 1$, and so $n + 1 - o_s(k_{n+1}/k) \geq n - o_s(k_n/k)$, i.e., the sequence $(U_s^j(K/k))_{j\in\mathbb{N}}$ is increasing.

In addition, we verify immediately that:

(i) *For each $s \geq Ilqm(K/k)$, $\lim_{n\to+\infty}(U_s^n(K/k)) = +\infty$.*

(ii) *For each $s < Ilqm(K/k)$, the sequence $(U_s^j(K/k))_{j\in\mathbb{N}}$ is bounded*;

and, consequently, for all $n \geq \sup_{j\in\mathbb{N}}(\sup(U_s^j(K/k)))_{s<Ilqm(K/k)}$, we have $U_s^n(K/k) = U_s^{n+1}(K/k)$, in other words $o_s(k_{n+1}/k) = o_s(k_n/k) + 1$.

In the following, we set $e(K/k) = \sup_{j\in\mathbb{N}}(\sup(U_s^j(K/k)))_{s<Ilqm(K/k)}$, and for each $(s, j) \in \mathbb{N}^* \times \mathbb{N}^*$, $e_s^j = o_s(k_j/k)$.

Theorem 3.2 *Let K/k be a q-finite extension of unbounded exponent. Suppose that $di(rp(K/k)/k) = t$. The following statements are equivalent:*

(1) *K/k is modular over a finite extension of k.*
(2) *For each $s \in \{1, 2, \ldots, t\}$, the sequence $(U_s^j(K/k))_{j\in\mathbb{N}}$ is bounded.*
(3) *$Ilqm(K/k) = t + 1$.*

Proof. It's clear that (2) ⇔ (3). Furthermore, taking into account Proposition 2.6, there exists a natural integer j_0 such that K/k_{j_0} is relatively perfect, so $k_{j_0}(rp(K/k)) = K$, and we will have $di(K/k_{j_0}) = di(rp(K/k)/k) = t$. Assume next that (1) is satisfied. We distinguish two cases:

If K/k is modular, by virtue of ([8], Proposition 6.3), for each $j \geq j_0$, we have k_j/k_{j_0} is equiexponential of exponent $j - j_0$ and $di(k_j/k_{j_0}) = t$. Hence, for each $s \in \{1, \ldots, t\}$, we have $U_s^j(K/k) = U_s^{j+1}(K/k)$.

If K is modular over a finite extension L of k, having regard of finitude of L/k, there exists a natural integer n such that $L \subseteq k_n$. As a result, $L^{p^{-j}} \cap K \subseteq k_{n+j}$, and so $U_s^{n+j}(K/k) \leq n + U_s^j(K/L)$; from whence the sequence $(U_s^j(K/k))_j$ is stationary for each $s \in \{1, \ldots, t\}$ (namely $rp(K/L) = L(rp(K/k))$ and L/k is finite, therefore $di(rp(K/L)/L) = di(L(rp(K/k))/L) = di(rp(K/k)/k) = t$).

Conversely, if the condition (2) holds, there exists $m_0 \geq \sup(e(K/k), j_0)$, for each $j \geq m_0$, for each $s \in \{1, \ldots, t\}$, we have $o_s(k_{j+1}/k) = o_s(k_j/k) + 1$ (and $di(k_j/k_{m_0}) = t$). Consequently, k_j/k_{j_0} is equiexponential, and a fortiori modular. Hence, $K = \bigcup_{j>m_0} k_j$ is modular over k_{j_0}. □

Theorem 3.3 *The smallest subextension m/k of a q-finite extension K/k such that K/m is modular is not trivial ($m \neq K$). More precisely, if K/k is of unbounded exponent, the same is true of K/m.*

Proof. The case where K/k is not relatively perfect (in particular, the finite case) is trivially obvious, since $K/k(K^p)$ is modular. Thus, we are led to consider that K/k is relatively perfect of unbounded exponent. We next give a proof by recurrence on the integer $di(K/k) = t$. If $t = 1$, i.e., K/k is q-simple, it is immediate that K/k is modular. Suppose now that $t > 1$, if $Ilqm(K/k) = t + 1$, by virtue of Theorem 3.2, m/k is finite, and a fortiori K/m is of unbounded exponent. If $Ilqm(K/k) \leq t$, for each $j > e(K/k)$, for each $s \in [1; i - 1]$ where $i = Ilqm(K/k)$, we have $e_s^{j+1} = e_s^j + 1$. As $k_{j+1}^p \subseteq k_j$, by Proposition 2.13, there exists a canonically ordered r-base $(\alpha_1, \ldots, \alpha_n)$ of k_{j+1}/k, there exists $\varepsilon_i, \ldots, \varepsilon_t \in \{1, p\}$ such that $(\alpha_1^p, \ldots, \alpha_{i-1}^p, \alpha_i^{\varepsilon_i} \ldots, \alpha_t^{\varepsilon_t})$ is a canonically ordered r-base of k_j/k. In the sequel, for every $j > e(K/k)$, we denote $K_j = k(k_j^{p^{e_i^j}})$. Firstly, $K_j = k(\alpha_1^{p^{e_i^j+1}}, \ldots, \alpha_{i-1}^{p^{e_i^j+1}})$ and $K_{j+1} = k(\alpha_1^{p^{e_i^{j+1}}}, \ldots, \alpha_{i-1}^{p^{e_i^{j+1}}})$. Secondly, we have $e_i^{j+1} = e_i^j + \varepsilon$ with $\varepsilon = 0$ or 1, this leads to $K_j \subseteq K_{j+1}$. However, by definition of $Ilqm(K/k)$, we have $1 + e_i^j > e_i^{j+1}$ (i.e., $e_i^j = e_i^{j+1}$) for an infinity of values of j. For these values, we have $di(K_{j+1}/k) = i - 1$, otherwise by Lemma 2.1, $e_i^{j+1} = e_{i-1}^{j+1} = 1 + e_{i-1}^j = e_i^j$, and so $e_i^j > e_{i-1}^j$, which contradicts the definition of exponents. As $(di(K_j/k))_{j>e(K/k)}$ is an increasing sequence of integers bounded by $di(K/k)$, so it is eventually stationary in $Ilqm(K/k) - 1$. Furthermore, $K_j \neq K_{j+1}$, in effect if $K_j = K_{j+1} = k(K_{j+1}^p)$, as

K_{j+1}/k is of finite exponent, we will have $K_{j+1} = k$, a contradiction. We next put $H = \bigcup_{j > e(K/k)} K_j$. We check immediately that H/k is of unbounded exponent and $di(H/k) = i - 1$, in addition H/k is relatively perfect because $k(K_{j+1}^p) = K_j$ for an infinity of j. Moreover, according to Proposition 2.7, $di(K/H) < t$ and K/H is of unbounded exponent. From the recurrence hypothesis applied to K/H, we have K is modular over an intermediate subfield m' of K/H such that K/m' is of unbounded exponent; since $m \subseteq m'$, then K/m is also of unbounded exponent. □

An equivalent version of this in the particular case where $di(k)$ is finite is found in [3]. However, the above theorem is generally false if the q-finitude hypothesis is not satisfied (cf. [5], p. 149).

4 Generalization of Lower Quasi-modular Extensions

Definition 6 *A purely inseparable extension K/k is said to be lq-modular (lower quasi-modular) if K/k is modular up to extension of finite exponent, i.e., if there exists a subextension of a finite exponent L/k of K/k such that K/L is modular.*

As immediate consequences, we have:

- *K/k is lq-modular if and only if $lm(K/k)/k$ has an exponent.*
- *K/k is lq-modular if and only if the same is true for K/L for any subextension of finite exponent L/k* of K/k.

Let k be a commutative field of characteristic $p > 0$ and Ω an algebraic closure of k. In $[k : \Omega]$ we define the relation $\sim$ as follows: $k_1 \sim k_2$ if and only if $k_1 \subseteq k_2$ and k_2/k_1 has an exponent or $k_2 \subseteq k_1$ and k_1/k_2 has an exponent. We check immediately that $\sim$ is reflexive, symmetric, however $\sim$ is generally non-transitive. Moreover, the relation $\sim$ is compatible with the lower and upper modularity. More specifically, we have:

Proposition 4.1 *Let $k_1 \subseteq k_2 \subseteq K_1 \subseteq K_2$ be purely inseparable extensions. we have:*

(i) *If $k_1 \sim k_2$, then $lm(K_1/k_1) \sim lm(K_1/k_2)$.*
(ii) *If $K_1 \sim K_2$, then $um(K_1/k) \sim um(K_2/k)$.*

Proof. The proof results from the following considerations:

- *$lm(K_1/k_1) \subseteq lm(K_1/k_2)$, and if $o_1(k_2/k_1) = e_1$, then $lm(K_1/k_2) \subseteq (lm(K_1/k_1))^{p^{-e_1}} \cap K_1$ with $K_1/(lm(K_1/k_1))^{p^{-e_1}} \cap K_1$ is modular (cf. Proposition 2.15).*
- *Also, $um(K_1/k) \subseteq um(K_2/k)$, and if $o_1(K_2/K_1) = e_2$, then $um(K_2/k) \subseteq (um(K_1/k))^{p^{-e_2}}$ and $(um(K_1/k))^{p^{-e_2}}/k$ is modular (cf. Proposition 2.12).* □

Let P/k be the purely inseparable closure of an algebraic closure Ω of k, the above proposition can be interpreted in terms of correspondence as follows:

The mapping of lower modularity:

$$\begin{aligned} lm : [k : K_1] &\longmapsto [k : K_1] \\ L &\longrightarrow lm(K_1/L), \end{aligned}$$

and that of upper modularity:

$$\begin{aligned} um : [k : P] &\longmapsto [k : P] \\ L &\longrightarrow um(L/k), \end{aligned}$$

are compatible with the relation $\sim$.

As a consequence, the lq-modularity is compatible with the relation $\sim$. Furthermore, the lq-modularity is stable, up to extension of finite exponent. More specifically, we have:

Proposition 4.2 *Let k_1/k and k_2/k be two subextensions of a same purely inseparable extension K/k. If $k_1 \sim k_2$, then K/k_1 is lq-modular if and only if the same is true for K/k_2.*

Proof. Immediately follows from Proposition 4.1. □

In particular, we have:

Corollary 4.2.1 *Let n be a natural number. Then the following assertions hold true:*

(i) *K/k is lq-modular if and only if the same is true for $K/k^{p^{-n}} \cap K$.*
(ii) *K/k is lq-modular if and only if the same is true for K/k^{p^n}.*

Moreover, and unlike q-finite extensions, the lq-modularity is generally neither extended nor reduced. In other words, $K \sim K'$ does not necessarily imply $lm(K/k) \sim lm(K'/k)$, as the following example shows.

Example 1 *Let k_0 be a perfect field of characteristic $p > 0$ and $k = k_0(X, (Y_i, Z_i)_{i\geq 1})$ the field of rational fractions in indeterminates $(X, (Y_i, Z_i)_{i\geq 1})$. For each $i \in \mathbb{N}^*$, we denote by $\theta_i = Y_i^{p^{-1}} X^{p^{-i-1}} + Z_i^{p^{-1}}$ and $K = k(X^{p^{-\infty}}, (\theta_i)_{i\geq 1})$. Next, put $K' = k(X^{p^{-\infty}}, (Y_i^{p^{-1}}, Z_i^{p^{-1}})_{i\geq 1})$.*

It is immediately verified that:

- *$K' \simeq k(X^{p^{-\infty}}) \otimes_k (k(Y_i^{p^{-1}}) \otimes_k k(Z_i^{p^{-1}}))_{i\geq 1}$, and therefore K'/k is modular.*
- *Using the modularity criterion, we show that $lm(K/k) = k(X^{p^{-\infty}})$, and so K/k is not lq-modular.*

But for each $i \geq 1$, we have $\theta_i = {Y_i}^{p^{-1}} X^{p^{-i-1}} + {Z_i}^{p^{-1}}$. It follows that $\theta_i \in k(X^{p^{-i-1}}, {Y_i}^{p^{-1}}, {Z_i}^{p^{-1}}) \subseteq K'$. Hence $K \subseteq K'$ with $o_1(K'/K) = 1$, and thus $K \sim K'$; except that $\sim$ does not respect the lq-modularity. However, if we add the condition "to be relatively perfect" to one of this extensions, the lq-modularity will be respected as indicated by the following result.

Proposition 4.3 *Let K/k be a purely inseparable and relatively perfect extension. K/k is lq-modular if and only if the same holds true for $L(K)/k$ for every extension L/k such that $k \sim L$*

Proof. We put $m_0 = lm(K/k)$ and $m_1 = lm(L(K)/k)$. We now use a proof by induction. We start with the case $o_1(L/k) = 1$. Let G be a subset of L such that G is a r-base of $L(K)/K$, therefore $L(K) \simeq K \otimes_k (\otimes_k (k(a))_{a \in G}) \simeq K \otimes_{m_0} (\otimes_{m_0} (m_0(a))_{a \in G})$. Firstly, according to ([8], Proposition 5.11), we have $L(K)/m_0$ is modular, and so $m_1 \subseteq m_0$. On the other hand, taking into account Proposition 2.15, $m_1((L(K))^p) = m_1(K^p)$ is also modular over m_1. But K/k is relatively perfect and $m_1 \subseteq m_0 \subseteq K$, thus $K = m_1(K^p)$ is modular over m_1, and consequently $m_0 \subseteq m_1$. Hence $m_1 = m_0$, and it follows that K/k is lq-modular if and only if the same holds for $L(K)/k$.

If L/k has an exponent $e + 1$, then $k(L^p)/k$ is of exponent e. From the induction hypothesis, we have K/k is lq-modular if and only if the same is true for $k(L^p)(K)/k$ (in particular, the same hold for $k(L^p)(K)/k(L^p)$). As $o_1(L/k(L^p)) = 1$, according to the first case K/k is lq-modular if and only if the same is true for $L(K)/k(L^p)$, or again it is the same for $L(K)/k$. □

The lq-modularity is stable by a finite intersection covering k. More specifically, we have:

Proposition 4.4 *Let $(k_j)_{j \in I}$ be a finite family of purely inseparable subextensions of a same extension K/k. If K/k_j is lq-modular for any $j \in I$, then the same holds true for $K/\bigcap_{j \in I} k_j$.*

Proof. We reduce to the case where $I = \{1, 2\}$. For each $j = 1, 2$, we put $m_j = lm(K/k_j)$. In view of the lq-modularity, there exists $e \in \mathbb{N}$ such that $m_j \subseteq {k_j}^{p^{-e}} \cap K$ for $j = 1, 2$, and so $m_1 \cap m_2 \subseteq {k_1}^{p^{-e}} \cap {k_2}^{p^{-e}} \cap K = (k_1 \cap k_2)^{p^{-e}} \cap K$. It follows that $m_1 \cap m_2/k_1 \cap k_2$ is of finite exponent. Moreover, by virtue of Corollary 2.14.1, $K/m_1 \cap m_2$ is modular, therefore $K/k_1 \cap k_2$ is lq-modular. □

Remark 1 *The finiteness condition in the above proposition is essential, since any intersection covering k may not respect the lq-modularity, as shown in the following example.*

Example 2 *Let k_0 be a perfect field of characteristic $p > 0$ and $k = k_0((X_i, Y_i, Z_i)_{i \in \mathbb{N}^*})$ the field of rational fractions in indeterminates $(X_i, Y_i, Z_i)_{i \in \mathbb{N}^*}$. For every $i \geq 1$, we denote $\alpha_i = {X_i}^{p^{-i-1}}$ and $\theta_i = {Y_i}^{p^{-1}} \alpha_i + {Z_i}^{p^{-1}}$. Also, we put $K = k((\alpha_i, \theta_i)_{i \in \mathbb{N}^*})$, $m = lm(K/k)$, $S = k(({\alpha_i}^p)_{i \in \mathbb{N}^*})$, and for each $r \in \mathbb{N}^*$, $L_r = k({X_r}^{p^{-\infty}}, ({X_j}^{p^{-\infty}}, {Y_j}^{p^{-\infty}}, {Z_j}^{p^{-\infty}})_{j \in \mathbb{N}^* \setminus \{r\}})$ and $F_r = k({\alpha_1}^p, \ldots, {\alpha_r}^p)$.*

As $\theta_i = Y_i^{p^{-1}}\alpha_i + Z_i^{p^{-1}}$ for each $i \in \mathbb{N}^*$, we verify immediately that $\theta_i \in k(Y_i^{p^{-1}}, \alpha_i, Z_i^{p^{-1}}) \subseteq k(Y_i^{p^{-\infty}}, X_i^{p^{-\infty}}, Z_i^{p^{-\infty}})$, and so for every $r \in \mathbb{N}^*$, $K \subseteq k((X_j^{p^{-\infty}}, Y_j^{p^{-\infty}}, Z_j^{p^{-\infty}})_{j\in\mathbb{N}^*\setminus\{r\}}, \alpha_r, \theta_r) \subseteq L_r(\theta_r)$. We also verify using the modularity criterion that $m = S$. Indeed, if there exists $r \geq 1$ such that $\alpha_r{}^p \notin m$, or again the system $(1, \alpha_r{}^p)$ is linearly independent over m, whence it is remains in particular linearly independent over $m \cap K^p$. We complete this system to a linear basis B of K^p over $m \cap K^p$. As K^p and m are $m \cap K^p$-linearly disjoint (K/m is modular), we deduce that B is also a linear basis of $m(K^p)$ over m. But the definition equation of θ_r over m is written $\theta_r{}^p = Y_r\alpha_r{}^p + Z_r$, then by identification it results that $Y_r, Z_r \in m \cap K^p \subseteq K^p$. Moreover, $Y_r{}^{p^{-1}}, Z_r{}^{p^{-1}} \in K \subseteq L_r(\theta_r)$, and by Theorem 2.1, we will have $2 = di(L_r(Y_r{}^{p^{-1}}, Z_r{}^{p^{-1}})/L_r) \leq di(L_r(\theta_r)/L_r) = 1$. As a result, $2 \leq 1$, a contradiction. Hence, for every $r \in \mathbb{N}^*$, $\alpha_r{}^p \in m$, or again $S \subseteq m$. On the other hand, it is easy to check that K/S is modular, so $m = S$.

Next, we denote by $k_i = k((\alpha_j{}^p)_{j\geq i})$. We show immediately that:

- *For each $i \in N^*$, K/k_i is lq-modular.*
- $\bigcap_{i\in\mathbb{N}^*} k_i = k$. *In fact, let $\theta \in \bigcap k_i$,* in particular $\theta \in k_1 = m$, *and therefore there exists* $j \geq 1$ such that $\theta \in k(\alpha_1{}^p, \ldots, \alpha_j{}^p) = F_j$. *Also $\theta \in k_{j+1}$, but as $m \simeq \otimes_k(k(\alpha_i{}^p))_{i\geq 1} \simeq F_j \otimes_k k_{j+1}$, then $F_j \cap k_{j+1} = k$; whence $\theta \in k$.*

Therefore $K/\bigcap k_i$ is not lq-modular (namely m/k has unbounded exponent), even if K/k_i is lq-modular for each $i \in N^*$.

Remark 2 *However if we add the q-finitude hypothesis, we will have any intersection covering k respect the lq-modularity. In addition, any q-finite extension K/k contains a smallest subextension m such that K/m is lq-modular. However, this property falls into default without the condition of q-finitude as shown in the example above.*

The result that follows gives a necessary and sufficient condition for the existence of the smallest subextension that respects the lq-modularity as the fixed ground field for a given extension. More specifically, we have

Proposition 4.5 *Let K/k be a purely inseparable extension and $lm(K/k) = m$. Then K/k has a smallest subextension m_1/k such that K/m_1 is lq-modular if and only if $m/rp(m/k)$ has an exponent. Moreover, if m_1 exists, we have $m_1 = rp(m/k)$.*

Proof. The necessary condition is obvious, just note that $K/k(m_1{}^p)$ is also lq-modular, and that $m_1 \leq m \leq lm(K/m_1)$ with $lm(K/m_1)/m_1$ is of finite exponent. Conversely, suppose that $m/rp(m/k)$ has an exponent, so $K/rp(m/k)$ is lq-modular. Let L/k be a subextension of K/k such that K/L is lq-modular, and let $L_1 = lm(K/L)$, therefore there exists $e \in \mathbb{N}$ such that $L_1 \subseteq L^{p^{-e}} \cap K$. Since $m = lm(K/k)$, then $m \subseteq L_1$; and thus $rp(m/k) \subseteq \bigcap_{i\in N} k(m^{p^i}) \subseteq \bigcap_{i\in N} k(L_1{}^{p^i}) \subseteq L$. □

Remark 3 *Unlike q-finite extensions, the product does not respect the lq-modularity as shown in the following example.*

Example 3 *We take the notations of Example 2 above, k_0 always designates a perfect field of characteristic $p > 0$ and $k = k_0((X_i, Y_i, Z_i)_{i\in\mathbb{N}^*})$ the field of rational fractions in indeterminates $(X_i, Y_i, Z_i)_{i\in\mathbb{N}*}$. We recall that for all $i \in \mathbb{N}^*$, $\alpha_i = X_i^{p^{-i-1}}$ and $\theta_i = Y_i^{p^{-1}}\alpha_i + Z_i^{p^{-1}}$. We also put that $K_1 = k((\alpha_i)_{i\in\mathbb{N}^*})$ and $K_2 = k((\theta_i)_{i\in\mathbb{N}^*})$.*

It is immediate that K_1/k and K_2/k are modular, therefore lq-modular. We also have $lm(K_1(K_2)/k) = k(K_1{}^p)$, and so $K_1(K_2)/k$ is not lq-modular.

5 Upper Quasi-modular Extensions

Definition 7 *A purely inseparable extension K/k is said to be uq-modular (upper quasi-modular) if there exists an extension K'/K of a finite exponent such that K'/k is modular.*

As immediate consequences, we have:

- *K/k is uq-modular if and only if $um(K/k)/K$ has an exponent.*
- *Taking into account Example 1, and unlike q-finite extensions, the uq-modularity generally does not implies the lq-modularity.*
- *The uq-modularity is compatible with the relation $\sim$. In addition, the uq-modularity is stable up to extension of finite exponent. More precisely, we have:*

Proposition 5.1 *Let K and K' be two purely inseparable subextensions of a same extension Ω/k. If $K \sim K'$, then K/k is uq-modular if and only if the same is true for K'/k.*

Proof. Immediately follows from Proposition 4.1. □

In particular,

Corollary 5.1.1 *Let e be a natural number, then the following statements are verified:*

(i) *K/k is uq-modular if and only if the same holds for $k(K^{p^e})/k$.*
(ii) *K/k is uq-modular if and only if the same holds for $K^{p^{-e}}/k$.*

Corollary 5.1.2 *If $k \sim L$, then K/k is uq-modular if and only if the same holds for $L(K)/k$.*

Let K/k be a purely inseparable extension. The smallest extension M/K such that M/k is uq-modular when it exists will be called the uq-modular closure of K/k and will be denoted by $uqm(K/k)$. Clearly $K((uqm(K/k))^p)/k$ is also uq-modular, and therefore $uqm(K/k)/K$ is relatively perfect.

Remark 4 *Contrary to the q-finite extensions, an extension may not have a uq-modular closure as shown in the following example.*

Example 4 *Let k_0 be a perfect field of characteristic $p > 0$ and $k = k_0((X_i, Y_i, Z_i)_{i \in \mathbb{N}^*})$ the field of rational fractions in indeterminates $(X_i, Y_i, Z_i)_{i \in \mathbb{N}^*}$. For each $i \in \mathbb{N}^*$, we put $\alpha_i = X_i^{p^{-i-1}}$, $\theta_i = Y_i^{p^{-i}}\alpha_i + Z_i^{p^{-i}}$, $K = k((\alpha_i, \theta_i)_{i \in \mathbb{N}*})$, and for every $j \in N^*$, $F_j = k(Y_1^{p^{-1}}, \dots, Y_j^{p^{-j}})$.*

It is immediate that:

- *By applying the criterion of modularity, we have $um(K/k) = K((Y_i^{p^{-i}}, Z_i^{p^{-i}})_{i \in \mathbb{N}^*})$. In addition, K/k is not uq-modular, (namely $um(K/k)/K$ has unbounded exponent).*
- *$um(K/k) = K((Y_i^{p^{-i}})_{i \in \mathbb{N}^*}) = K((Z_i^{p^{-i}})_{i \in \mathbb{N}^*}) \simeq K \otimes_k (\otimes_k k(Z_i^{p^{-i}})_{i \in \mathbb{N}^*}))$ $\simeq K \otimes_k (\otimes_k k((Y_i^{p^{-i}})_{i \in \mathbb{N}^*}))$.*
- *$um(K/k) = k((\alpha_i, Y_i^{p^{-i}}, Z_i^{p^{-i}})_{i \in \mathbb{N}^*}) \simeq (\otimes_k k((\alpha_i)_{i \in \mathbb{N}^*})) \otimes (\otimes_k k((Y_i^{p^{-i}})_{i \in \mathbb{N}^*}))$ $\otimes_k (\otimes_k k((Z_i^{p^{-i}})_{i \in \mathbb{N}^*}))$.*

Similarly, for every $i \in \mathbb{N}^*$, we put $M_i = K((Y_j^{p^{-j}})_{j \geq i})$. It is also verified that:

- *For each $i \in \mathbb{N}^*$, M_i/k is uq-modular.*
- *$\bigcap_{i \in \mathbb{N}^*} M_i/k$ is not uq-modular, it is sufficient to show that $\bigcap_{i \in \mathbb{N}^*} M_i = K$.*

Let $\beta \in \bigcap_{i \in \mathbb{N}^*} M_i$, in particular $\beta \in M_1$, so there exists $j \in \mathbb{N}^*$ tel que $\beta \in K(Y_1^{p^{-1}}, \dots, Y_j^{p^{-j}}) = K(F_j)$. We also have $\beta \in M_{j+1} = K((Y_i^{p^{-i}})_{i \geq j+1})$. But $M_1 = K((Y_i^{p^{-i}})_{i \in \mathbb{N}^*}) \simeq K \otimes_k (\otimes_k k(Y_i^{p^{-i}})_{i \in \mathbb{N}^*})) \simeq M_{j+1} \otimes_k F_j \sim M_{j+1} \otimes_K K(F_j)$, whence $\beta \in K(F_j) \cap M_{j+1} = K$.

The result which follows gives a necessary and sufficient condition for the existence of the uq-modular closure of a given extension. More specifically, we have:

Proposition 5.2 *Given a purely inseparable extension K/k and $M = um(K/k)$. Then K/k admits a uq-modular closure if and only if $M/rp(M/K)$ has an exponent. Moreover, when $ulqm(K/k)$ exists, then $ulqm(K/k) = rp(M/K)$.*

Proof. It is immediate that $rp(M/K)/k$ is uq-modular if $M/rp(M/K)$ has an exponent. Let S/K be a purely inseparable extension such that S/k is uq-modular, so the modular closure of S/k, denoted by S_1, has an exponent e over S. As a result, $M \subseteq S_1$. In particular, $K(M^{p^e}) \subseteq K(S_1^{p^e}) \subseteq S$, and consequently $rp(M/K) \subseteq S$. Hence $ulqm(K/k) = rp(M/K)$.

Conversely, we denote by $N = uqlm(K/k)$ and $M_1 = um(N/k)$. It is clear that $N \subseteq M \subseteq M_1$ and $M_1 \subseteq N^{p^{-s}}$ for some natural number s, and thus $K(M^{p^s}) \subseteq N$. Moreover N/K is relatively perfect since $K(N^p)/k$ is also uq-modular, but $K(M^{p^s})/k$ is uq-modular, we deduce that $K(M^{p^s}) = N$, and consequently $rp(M/K) = N$. □

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A Class of Finite 2-groups G with Every Automorphism Fixing $G/\Phi(G)$ Elementwise

Hossein Abdolzadeh and Reza Sabzchi

Abstract The family $G(m,n) = \langle x, y | x^2 = (xy^2)^2 = 1,\ \ y^{2^m} = (xy)^{2^n} \rangle$ of finite 2-groups will be introduced. The group $G(m,n)$ has order $2^{(m+n+1)}$, nilpotency class $1 + \max\{m,n\}$ and every automorphism of $G = G(m,n)$ fixes $G/\Phi(G)$ elementwise and therefore $Aut(G)$ is a 2-group. The parameterized presentation of $G = G(m,n)$ is efficient as the Schur multiplicator of G is non-trivial.

Keywords Finite 2-group · Automorphism group · Frattini subgroup

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1 Introduction

Presentation of a group is introducing the group by a set of generators and a sufficient set of relations between the generators. For a group G it is denoted by $G = \langle X|R \rangle$ in which X is the set of its generators and R is a set of relations. Such an expression of a group provides a short description of its associated group. Presentations arise from geometrical and topological vision of groups and is a central topic in group theory (see [6]). A group may has many presentations. A presentation $\langle X|R \rangle$ for the group G is said to be finite if X and R are both finite sets. A group is called finitely presented, if it has a finite presentation. A finite presentation $\langle X|R \rangle$ of a group G is called efficient if the set X is a minimal generating set of G and for any other presentation $\langle X|S \rangle$ for G, we have $|S| \geq |R|$.

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Let G be a group. By Z, G', Φ and $Aut(G)$, we denote the center, the commutator subgroup, the Frattini subgroup and the group of all automorphisms of G, respectively. Let N be a normal subgroup of G. By $Aut^N(G)$ we mean the subgroup of $Aut(G)$ consisting of all automorphisms of G, normalizing N and centralizing G/N, that is

$$Aut^N(G) = \{\alpha \in Aut(G) | \forall g \in G;\ g^{-1}\alpha(g) \in N\}.$$

Clearly for a characteristic subgroup N of a group G we have $Aut^N(G) \trianglelefteq Aut(G)$. The group $Aut^N(G)$ of automorphisms have been investigated by several authors where N is one of the characteristic subgroups $\Phi(G)$, $Z(G)$ or G' ([1–4]). By a well-known theorem of Burnside the group $Aut^{\Phi}(G)$ is a p-group whenever G is a finite p-group. Therefore for a finite p-group G, if $Aut^{\Phi}(G) = Aut(G)$ then the full automorphisms group, $Aut(G)$, is also a finite p-group. In [5] a class of 2-groups with this property is introduced.

Let $\alpha \in Aut(G)$ and let H be a normal subgroup of G. We say that α fixes G/H elementwise if and only if $\alpha \in Aut^H(G)$.

In this paper we construct an infinite family of finite 2-groups with high nilpotency class, defined by a single parameterized efficient presentation such that for every member, G of the family, all automorphisms of G fix $G/\Phi(G)$ elementwise and therefore $Aut(G)$ is a 2-group.

Let $m \geq 2$ and n be positive integers and let $G(m, n)$ be the group defined by the following presentation

$$G(m, n) = \langle x, y | x^2 = (xy^2)^2 = 1, y^{2^m} = (xy)^{2^n} \rangle.$$

Our main result is the following theorem.

Theorem 1. *Let $G := G(m, n)$, $m' = \min\{m, n\}$ and let $n' = \max\{m, n\}$. Then the following statements hold*

1. $|G| = 2^{m+n+1}$,
2. *Z and G' are of orders 2 and 2^{m+n-2}, respectively,*
3. $\Phi(G) \cong \mathbb{Z}_{2^{m'-1}} \times \mathbb{Z}_{2^{n'}}$,
4. *G is of nilpotency class $1 + n'$,*
5. *Every automorphism of G fixes $G/\Phi(G)$ elementwise,*
6. $|Aut(G)| = |Aut^{\Phi}(G)| = 2^{2(m+n-1)}$ *and* $|Aut^{G'}(G)| = 2^{2(m+n-2)}$,
7. $Aut^Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $Aut(G) \cong Aut^{G'}(G) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$,
8. *The Schur multiplicator, $M(G)$, of G is non-trivial and hence the presentation of G is efficient.*

2 The Center and The Frattini Subgroup of $G(m, n)$

The following lemma contains a list of necessary relations to exhibit the elements of $G(m, n)$ in a relevant form which helps to recognize, for example, the central elements and the non-generators of $G(m, n)$. By non-generators of a group we mean the elements of the Frattini subgroup.

Lemma 1. *Let $G := G(m, n)$ and let r, s be integers. Then the following relations hold in G*

1. $xy^{2r}x = y^{-2r}$,
2. $xy^{2r+1}x = (xy)^2 y^{-(2r+1)}$,
3. $(xy)^{2^{n+1}} = y^{2^{m+1}} = 1$,
4. $x(xy)^{2r}x = (xy)^{-2r}$ *and hence* $(yx)^{2r} = (xy)^{-2r}$,
5. $x(xy)^{2r+1}x = (xy)^{-(2r+1)}y^{-2}$ *and* $(yx)^{2r+1} = (xy)^{-(2r+1)}y^{-2}$,
6. $[(xy)^2, y^2] = 1$.

Proof. (1) By the second relation of G we have $xy^2x = y^{-2}$. Also $xy^{2r}x = (xy^2x)^r$, therefore $xy^{2r}x = y^{-2r}$.

(2) By the part (1) we see that $xy^{2r+1}x = xyy^{2r}x = xyxy^{-2r} = (xy)^2y^{-2r-1}$.

(3) By the third relation of G the element y^{2^m} is a power of xy, therefore y^{2^m} commutes with xy and hence also with x, that is $xy^{2^m}x = y^{2^m}$. On the other hand by (1) we have $xy^{2^m}x = y^{-2^m}$. Therefore $y^{2^m} = y^{-2^m}$. Hence $(xy)^{2^{n+1}} = y^{2^{m+1}} = 1$.

(4) It is obvious as $(xy)^2x(xy)^2x = 1$.

(5) Using part (4) we have

$$x(xy)^{2r+1}x = (xy)^{-2r}yx = (xy)^{-2r}(y^{-1}x^{-1}xy)yx = (xy)^{-(2r+1)}xy^2x = (xy)^{-(2r+1)}y^{-2}.$$

(6) By the part (1) we have $y^2x = xy^{-2}$. Hence y^2 commutes with xyx and therefore by $(xy)^2$

Corollary 1. *Every element of $G := G(m, n)$ could be uniquely written in the form $(xy)^iy^j$, where $0 \le i \le 2^n - 1,\ 0 \le j \le 2^{m+1} - 1$.*

Theorem 2. *Let $G := G(m, n)$, $m' = \min\{m, n\}$ and $n' = \max\{m, n\}$, then*

1. *G is of order 2^{m+n+1} and $\Phi(G) \cong \mathbb{Z}_{2^{m'-1}} \times \mathbb{Z}_{2^{n'}}$.*
2. *G' and Z is of order 2^{m+n-2} and 2, respectively.*

Proof. (1) We have $|G| = 2^{m+n+1}$ by Corollary 1. Consider the subgroup $H = \langle a = (xy)^2, b = y^2\rangle$ of G. By the parts (1), (4) and (5) of Lemma 1 the subgroup H is normal. Using Todd-Coxeter coset enumeration, H has the following presentation

$$H \cong \langle a, b|\ a^{2^n} = b^{2^m} = [a, b] = 1,\ a^{2^{n-1}} = b^{2^{m-1}}\ \rangle.$$

Obviously H is isomorphic to $\mathbb{Z}_{2^{m'-1}} \times \mathbb{Z}_{2^{n'}}$, where $m' = \min\{m, n\}$ and $n' = \max\{m, n\}$. Also $\Phi(G) = H$, since $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and G is two generated.
(2) It is easy to see that $G/G' \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Therefore $|G'| = 2^{m+n-2}$.
By Corollary 1 an element of G is in the form $(xy)^i y^j$. On the other hand by Lemma 1 the element $(xy)^i y^j$ of G is in $Z(G)$ if and only if $i = 0$ and $j = 2^m$. Therefore $Z = \langle y^{2^m} \rangle$ and hence $|Z| = 2$.

3 The Nilpotency Class of $G(m, n)$ and the Order of $Aut(G(m, n))$

In this section it's shown that an automorphism of $G(m, n)$ has a special form. We use this to compute the order of $Aut(G(m, n))$. Also the nilpotency class of $G(m, n)$ will be computed.

Theorem 3. *By the above notations, the nilpotency class of G is $1 + \max\{m, n\}$.*

Proof. Set $K = \langle [x, y], [y^{-1}, x] \rangle$ which is contained in G'. Note that $[x, y] = (xy)^2 y^{-2}$ and $[y^{-1}, x] = (xy)^{-2} y^{-2}$. The relations $x[x, y]x^{-1} = [x, y]^{-1}$, $y[x, y] y^{-1} = [y^{-1}, x]$, $x[y^{-1}, x]x^{-1} = [y^{-1}, x]^{-1}$ and $y[y^{-1}, x]y^{-1} = [x, y]$ hold. Hence $K \trianglelefteq G$ and $G/K \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. This means that $G' = K$. Also G' is abelian, since $[x, y][y^{-1}, x] = [y^{-1}, x][x, y]$. Consequently we have $G' = \gamma_2 = \langle (xy)^2 y^2, (xy)^2 y^{-2} \rangle$, where γ_i is the i-th member of the lower central series of G.

We show that $\gamma_i = \langle (xy)^{2^{i-1}}, y^{2^{i-1}} \rangle$ where $i \geq 3$. Clearly the relations $[x, [x, y]] = (xy)^{-4} y^4$, $[y, [x, y]] = (xy)^{-4}$, $[x, [y^{-1}, x]] = (xy)^4 y^4$ and $[y, [y^{-1}, x]] = (xy)^4$ hold in G. Hence we have

$$\gamma_3 = \langle (xy)^4, y^4 \rangle.$$

Now, the assertion follows easily by an induction method on i. If $n \geq m$ we have $\gamma_{n+1} = \langle (xy)^{2^n}, y^{2^n} \rangle \neq 1$ while $\gamma_{n+2} = \langle (xy)^{2^{n+1}}, y^{2^{n+1}} \rangle = 1$, hence in this case the nilpotency class of G is $n + 1$. If $m \geq n$, by a similar argument, we obtain that the nilpotency class of G is $m + 1$.

Lemma 2. *The orders of the elements of the cosets Φx, Φy and Φxy are $2, 2^{m+1}$ and 2^{n+1}, respectively.*

Proof. We do this for the coset Φxy. The others are similar to this case.

As noted in the proof of Theorem 2, we have $\Phi(G) = \langle a = (xy)^2, b = y^2 \rangle$. Let $a \in \Phi$. There are integers r, s such that $a = (xy)^{2s} y^{2r}$. We have

$$\begin{aligned} axy &= (xy)^{2s} y^{2r} xy \\ &= (xy)^{2s+1} y^{-2r} \qquad \text{(by Lemma 1 (1))} \end{aligned}$$

Now for $i \geq 0$ we have

$$\begin{aligned}(axy)^{2^{i+1}} &= \left((xy)^{2s+1}y^{-2r}(xy)^{2s+1}y^{-2r}\right)^{2^i}\\ &= \left((xy)^{4s+2}\right)^{2^i}\\ &= ((xy)^{2^{i+1}})^{(2s+1)}\end{aligned}$$

which shows that the orders of axy and xy are equal. Therefore the order of axy is equal to 2^{n+1}.

Lemma 3. *Let $G := G(m, n)$ and let $a, b \in \Phi$. Define the mapping $\alpha : \{x, y\} \longrightarrow G$ by $\alpha(x) = ax$ and $\alpha(y) = by$. Then, α extends to an automorphism of G.*

Proof. Using substitution test (Proposition 4.3 in [6]) we show that the result of substituting ax for x and by for y in all of the relations of G yields the identity of G and therefore α extends to a group homomorphism. By Lemma 1, there are positive integers r, s and r', s' such that $a = (xy)^{2s}y^{2r}$ and $b = (xy)^{2s'}y^{2r'}$.

1. Relation x^2: The element ax is in the coset Φx and by Lemma 2 every element of Φx has order 2 and hence $(ax)^2 = 1$.
2. Relation $(xy^2)^2$:
   We have

$$\begin{aligned}(by)^2 &= ((xy)^{2s'}y^{2r'}y)^2 = (xy)^{2s'}y^{2r'}y(xy)^{2s'}y^{2r'}y\\ &= (xy)^{2s'}(yx)^{2s'}y^{4r'+2} = y^{4r'+2}.\end{aligned}$$

   Now:

$$(ax(by)^2)^2 = axy^{4r'+2}axy^{4r'+2} = axay^{4r'+2}xy^{4r'+2} = (ax)^2 = 1.$$

3. Relation $y^{2^m}(xy)^{-2^n}$:
   By Lemma 1 we have:

$$((by)^{-1}(ax))^2 = ((xy)^{2(s'-s)-1}y^{2(r'-r)})^2 = (xy)^{4(s'-s)-2}.$$

   Hence

$$(by)^{2^m}((by)^{-1}(ax))^{2^n} = ((by)^{2^m}(xy)^{-2^n}) = 1.$$

It is enough to note that for every $a, b \in \Phi$ the set $\{ax, by\}$ is a generating set for the group G, and therefore the extension of α is surjective and hence is an automorphism of G.

Corollary 2. *Let $G := G(m, n)$. Then a map $\alpha : G \longrightarrow G$ is an automorphism if and only if there exist $a, b \in \Phi$ such that $\alpha(x) = ax$ and $\alpha(y) = by$.*

Corollary 3. *Let $G := G(m, n)$. Then, all the automorphisms of G fix G/Φ elementwise and the order of $Aut(G)$ is $2^{2(m+n-1)}$.*

A purely non-abelian group is a group which has no non-trivial abelian direct factor.

Theorem 4. *Let $G := G(m, n)$. Then $Aut^Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $Aut^{\Phi}(G) \cong Aut^{G'}(G) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.*

Proof. As $|Z| = 2$, the group G is purely non-abelian and $Aut^Z(G)$ is elementary abelian group (see [4]). Then $|Aut^Z(G)| = |Hom(G/G', Z)| = 4$ and $Aut^Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

By Corollary 2, $|Aut^{G'}(G)| = |G'|^2 = 2^{2(m+n-2)}$. Clearly the map θ defined by $\theta(x) = x$ and $\theta(y) = (xy)^{2^n+2}y$ is an automorphism of order 2, also there is an automorphism ρ of order 2 defined by $\rho(x) = x$ and $\rho(y) = y^{-1}$. Now it is easy to check that $Aut(G) \cong Aut^{G'}(G) \rtimes \langle \rho, \theta \rangle \cong Aut^{G'}(G) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Lemma 4. *Suppose that N is a normal subgroup of a finite group G. If $M(G) = 1$, then $M(G/N) \cong (G' \cap N)/[G, N]$.*

Proof. See Corollary 3.2.2 in [7].

Theorem 5. *Let $G := G(m, n)$, then $M(G) \neq 1$.*

Proof. Suppose that $M(G) = 1$. Set $N = \langle y^2 \rangle$, clearly N is a normal subgroup of G of order 2^m. It is easy to check that $[G, N] = \langle y^4 \rangle$. Obviously $G'N = \langle [x, y], [y^{-1}, x], y^2 \rangle$ and $[x, y]y^2 = (xy)^2$ thus $\Phi = G'N$ and so $|G'N| = 2^{m+n-1}$. Hence $|G' \cap N| = 2^{m-1}$ and therefore $G' \cap N = [G, N]$. On the other hand we have

$$\begin{aligned} G/N &\cong \langle x, y | x^2 = (xy^2)^2 = 1, y^{2^m} = (xy)^{2^n}, y^2 = 1 \rangle \\ &\cong \langle x, y | x^2 = y^2 = (xy)^{2^n} = 1 \rangle. \end{aligned}$$

Therefore $G/N \cong D_{2^{n+1}}$, the dihedral group of order 2^{n+1}, with $M(G/N) \cong \mathbb{Z}_2$, which is a contradiction by Lemma 4.

Now the proof of Theorem 1 follows by Theorems 2 and 3, Corollary 3, Theorems 4 and 5.

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Fuzzy Rings and Fuzzy Polynomial Rings

S. Melliani, I. Bakhadach and L. S. Chadli

Abstract In this paper, we introduce the notion of a ring of fuzzy points, and study some basic properties and the relationship between this set and the classical ring R. We also define the fuzzy polynomial rings and fuzzy algebraic elements.

Keywords Fuzzy points · Fuzzy subrings · Fuzzy polynomials

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [8] in 1965, which is a generalization of the crisp set. Since its conception, the theory of fuzzy set has developed in many directions and is finding applications in a wide variety of fields. Rosenfeld [7] used this concept to develop the theory of fuzzy subgroup. Liu [2] introduced the concept of fuzzy ring in 1982. Pu and Liu [6] introduced the notion of fuzzy points, Kyung ho kim [1] discussed the relation between the fuzzy interior ideals and the semigroup $\underline{R}$ the subset of all fuzy points of R. Based on these researches we have developed the notion of rings on the set of points defined by Pu and Liu [6]. We have also introduced and discussed the notion of polynomials on this ring.

Here is the summary of the paper. In Sect. 3, we define the subring consisting the set of all fuzzy points and discuss some basic properties of this ring. Based on the ring defined in Sect. 3, we introduce and investigate the fuzzy polynomial rings in Sect. 4.

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2 Preliminaries

In this section, we recall some definitions and results which will be used in the sequel.

Definition 1. [8] Let E be a non-empty set. A fuzzy subset of the E is a function $\mu : E \to [0, 1]$.

Definition 2. [5] Let μ be a fuzzy subset of E. For $\alpha \in [0, 1]$, define μ_α as follows:

$$\mu_\alpha = \{x | x \in R, \mu(x) \geq \alpha\}.$$

μ_α is called the α-**cut** (or α-**level set**) of μ.

Property 1. [5] Let $\mu, \nu \subset R$ be a fuzzy subsets. Then we have

1. $\mu \subseteq \nu, \alpha \in [0, 1] \Longrightarrow \mu_\alpha \subseteq \nu_\alpha,$
2. $\alpha \leq \beta \, \alpha, \beta \in [0, 1] \Longrightarrow \mu_\beta \subseteq \mu_\alpha,$
3. $\mu = \nu, \Leftrightarrow \mu_\alpha = \nu_\alpha$, for each $\alpha \in [0, 1]$.

Definition 3. [4] Let R be a ring with identity. Then $\mu \subset R$ is called a fuzzy subring if and only if

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y);$
(ii) $\mu(xy) \geq \mu(x) \wedge \mu(y), \forall x, y \in R$ and
(iii) $\mu(1) = 1.$

Property 2. Let R be a ring and μ be a fuzzy subring of R. Then we have:

1. For each $x \in R$, $\mu(0) \geq \mu(x)$.
2. If $x, y \in R$ and $\mu(x - y) = 0$, then $\mu(x) = \mu(y)$.
3. For each $x \in R$, $\mu(x) = \mu(-x)$.

Definition 4. [6] Let A be a non-empty set and $x_\alpha : A \longrightarrow [0, 1]$ a fuzzy subset of A with $x \in A$ and $\alpha \in (0, 1]$ defined by

$$x_\alpha(y) = \begin{cases} \alpha & if \quad x = y \\ 0 & if \quad x \neq y \end{cases}$$

then x_α is called a fuzzy point (singleton).

Definition 5. Let μ be a fuzzy subring of R, and x_t be a fuzzy point of R. We write $x_t \in \mu$ to express that $\mu(x) \geq t$, by the principal extension of Zadeh we have

$$x_t + y_s = (x + y)_{t \wedge s}$$
$$x_s y_t = (xy)_{t \wedge s}.$$

Now we will first evolve some results on the fuzzy ring using the membership functions and we will also give a necessary and sufficient condition for $F_\mu(R)$, the set of fuzzy points of μ to be a ring.

3 Fuzzy Subrings

The following theorem gives us the relationship between a fuzzy subring and all of its α-cuts.

Theorem 1. *Let μ be a fuzzy subset of R, then μ is a fuzzy subring of R if and only if μ_t is a subring of R, for each $t \in [0, \mu(0)]$.*

Proof. It is clear that $\mu_t = \{x \in R, \quad \mu(x) \geq t\}$ is a non-empty subset of R.

Let $x, y \in \mu_t$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. Since μ is a fuzzy subring of R, then we have $\mu(x - y) \geq \mu(x) \wedge \mu(y)$. This implies that $\mu(x - y) \geq t$, hence $x - y \in \mu_t$. Similarly, $\mu(xy) \geq \mu(x) \wedge \mu(y)$ then $\mu(xy) \geq t$. Hence $xy \in \mu_t$. Therefore, μ_t is a subring of R.

Conversely, let $x, y \in R$ and let $\mu(x) = t_1$ and $\mu(y) = t_2$. Then $x \in \mu_{t_1}$ and $y \in \mu_{t_2}$. Now suppose that $t_2 > t_1$, this implies that $\mu_{t_2} \subseteq \mu_{t_1}$. In this case, $y \in \mu_{t_2} \subseteq \mu_{t_1}$ since $x, y \in \mu_{t_1}$. So we have $x - y \in \mu_{t_1}$ and $xy \in \mu_{t_1}$; hence $\mu(x - y) \geq t_1 = \mu(x) \wedge \mu(y)$ and $\mu(xy) \geq t_1 = \mu(x) \wedge \mu(y)$.

Theorem 2. *Let μ be a fuzzy subset of R. Then μ is a fuzzy subring of R if and only if, for each point $x_t, y_s \in \mu$, we have $x_t - y_s \in \mu$ and $x_t.y_s \in \mu$.*

Proof. Suppose that μ is a fuzzy subring of R. Let $x, y \in R$ and $x_t, y_s \in \mu$. Then

$$\begin{aligned} \mu(x - y) &\geq \mu(x) \wedge \mu(y) \\ &\geq t \wedge s \end{aligned}$$

this implies that $x_t - y_s \in \mu$.
Similarly, since μ is a fuzzy subring of R,
we have

$$\begin{aligned} \mu(xy) &\geq \mu(x) \wedge \mu(y) \\ &\geq t \wedge s \end{aligned}$$

hence $x_t.y_s \in \mu$.

Conversely, let $x, y \in R$. We have

$$\mu(x) \geq \mu(x) \wedge \mu(y) \ \ and \ \ \mu(x) \geq \mu(x) \wedge \mu(y)$$

then

$$x_{\mu(x)\wedge\mu(y)} \in \mu \ \ and \ \ y_{\mu(x)\wedge\mu(y)} \in \mu.$$

Using the assumption we have

$$x_{\mu(x)\wedge\mu(y)} - y_{\mu(x)\wedge\mu(y)} \in \mu \ \ and \ \ x_{\mu(x)\wedge\mu(y)}.y_{\mu(x)\wedge\mu(y)} \in \mu$$

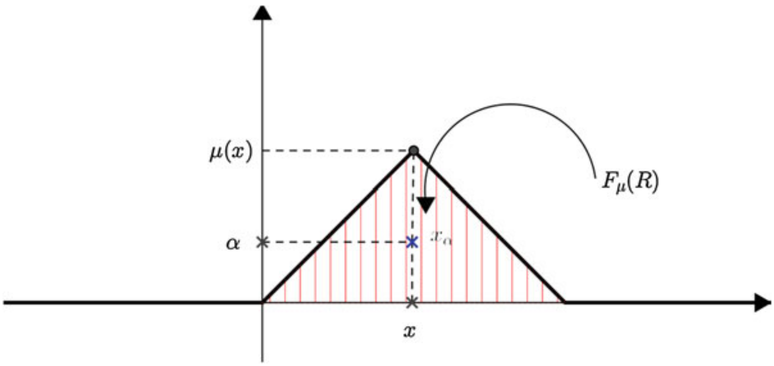

Fig. 1 Graphical representation of the set $F_\mu(R)$

This implies that

$$(x - y)_{\mu(x)\wedge\mu(y)} \in \mu \;\; and \;\; (xy)_{\mu(x)\wedge\mu(y)} \in \mu$$

Therefore, $\mu(x - y) \geq \mu(x) \wedge \mu(y)$ *and* $\mu(xy) \geq \mu(x) \wedge \mu(y)$. Thus μ is a fuzzy subring of R.

Let $\underline{R}$ be the subset of all fuzzy points of R given by [6]. We set $F_\mu(R) = \{x_\alpha \in \underline{R} \mid \mu(x) \geq \alpha\}$ (Fig. 1).

Theorem 3. *Let R be a ring with identity, and let μ be a fuzzy subset of R. If μ is a fuzzy subring of R, then $(F_\mu(R), +, \times)$ is a ring.*

Proof. Let $x_t, y_s, z_u \in F_\mu(R)$. We have $x_t + y_s = (x + y)_{t\wedge s} \in F_\mu(R)$. Hence $F_\mu(R)$ is closed under the operation $+$. For associativity of $+$ we have

$$\begin{aligned} x_t + (y_s + z_u) &= x_t + (y + z)_{s\wedge u} \\ &= (x + (y + z))_{t\wedge(s\wedge u)} \\ &= ((x + y) + z)_{(t\wedge s)\wedge u} \\ &= (x_t + y_s) + z_u. \end{aligned}$$

Then $+$ is associative.

We have also $\mu(0) \geq \mu(1) = 1$. Therefore, $0_s \in F_\mu(R)$ for all $s \in (0, 1]$, for the symmetric element, we have $\mu(-x) \geq \mu(x) \geq t$, then $-x_t \in F_\mu(R)$ and $x_t - x_t = (x - x)_t = 0_t$ for all $t \in (0, 1]$.

Furthermore,

$$x_t + y_s = (x + y)_{t\wedge s} = (y + x)_{s\wedge t} = y_s + x_t.$$

Thus $(F_\mu(R), +)$ is an abelian group.

Since $x_t \times y_s = (xy)_{t\wedge s} \in F_\mu(R)$, so $F_\mu(R)$ is closed under "×". Finally, as we have

$$\begin{aligned} x_t \times (y_s \times z_u) &= x_t \times (y \times z)_{s\wedge u} \\ &= (x \times (y \times z))_{t\wedge(s\wedge u)} \\ &= ((x \times y) \times z)_{(t\wedge s)\wedge u} \\ &= (x_t \times y_s) \times z_u \end{aligned}$$

and

$$\begin{aligned} x_t \times (y_s + z_u) &= (x \times (y + z))_{t\wedge(s\wedge u)} \\ &= (xy + xz)_{t\wedge s\wedge u} \\ &= (xy)_{t\wedge s} + (xz)_{t\wedge u}. \end{aligned}$$

it follows that $(F_\mu(R), +, \times)$ is a ring.

Proposition 1. *Let R be a commutative ring. Let μ and ν be two fuzzy subrings of R such that $\mu \subset \nu$. Then $F_\mu(R)$ is a subring of $F_\nu(R)$.*

Proof. Since μ, ν are fuzzy subrings of R, so $F_\mu(R)$ and $F_\nu(R)$ are rings by Theorem 3. Let $x_t \in F_\mu(R)$. Then $\mu(x) \geq t$, since $\mu \subset \nu$, then $\nu(x) \geq t$. This implies that $F_\mu(R) \subset F_\nu(R)$. In addition, we have $1_1 \in F_\mu(R)$.

Definition 6. Let μ be a fuzzy subring of R. Then the singleton $a_t \neq 0_t \in F_\mu(R)$ with $t \in (0, 1]$, is called a fuzzy zero-divisor if there exists a nonzero fuzzy singleton $b_s \in F_\mu(R)$ such that $a_t.b_s = 0_\lambda$ where $\lambda = \min(s, t)$.

Definition 7. Let $F_\mu(R)$ be a ring. We say that $F_\mu(R)$ is an integral ring if it has no zero-divisor fuzzy singletons, that is if $(x_t.y_s = 0_{t\wedge s}$, then $x_t = 0_t$ or $y_s = 0_s)$.

Theorem 4. *$F_\mu(R)$ is an integral ring if and only if R is an integral domain.*

Proof. Let $x_t, y_t \in F_\mu(R)$ with $x_t.y_s = 0_{t\wedge s}$. We must show that $x_t = 0_t$ or $y_s = 0_s$. Note that $x_t.y_s = 0_{t\wedge s}$ implies that, for all $z \in R$, $(xy)_{t\wedge s}(z) = 0_{t\wedge s}(z)$. Hence

$$(t \wedge s)\chi_{\{xy\}}(z) = (t \wedge s)\chi_{\{0\}}(z)$$

Since, for each $z \in R$,

$$\chi_{\{xy\}}(z) = \chi_{\{0\}}(z)$$

so $xy = 0$ and since R is an integral domain we have $x = 0$ or $y = 0$. Hence $x_t = 0_t$ or $y_s = 0_s$ for all $t, s \in (0, 1]$.

Conversely, suppose that $F_\mu(R)$ is an integral ring. Let $xy = 0$ for some $x, y \in R$. Since $xy = 0$ we have $(xy)_t = 0_t$ for every $t \in (0, 1]$. This implies that $x_t = 0_t$ or $y_t = 0_t$. So, for each $u, v \in R$, $x_t(u) = 0_t(u)$ or $y_t(v) = 0_t(v)$. Consequently, we have

$$x_t(u) = \begin{cases} t & if \quad x = u \\ 0 & if \quad x \neq u \end{cases} = \begin{cases} t & if \quad 0 = u \\ 0 & if \quad 0 \neq u \end{cases} = 0_t(u)$$

or

$$y_t(v) = \begin{cases} t & if \quad y = v \\ 0 & if \quad y \neq v \end{cases} = \begin{cases} t & if \quad 0 = v \\ 0 & if \quad 0 \neq v \end{cases} = 0_t(v)$$

Hence

$$x_t(u) = 0_t(u) \begin{cases} t & if \quad x = u = 0 \\ 0 & if \quad x \neq u \neq 0 \end{cases}$$

or

$$y_t(v) = 0_t(v) \begin{cases} t & if \quad y = v = 0 \\ 0 & if \quad y \neq v \neq 0 \end{cases}.$$

Therefore, $x = 0$ or $y = 0$.

4 Fuzzy Polynomials Ring

In this section, we will give a new definition of a fuzzy polynomials based on the ring of fuzzy points defined in Sect. 3. Then we will discuss some basic properties of this new concept.

Definition 8. A fuzzy polynomial with one indeterminate on $F_\mu(R)$ is a set of sequences $(a_{t_0}, a_{t_1}, a_{t_2} ....) = (a_{t_k})_{k\in\mathbb{N}}$ with $a_{t_k} \in F_\mu(R)$ such taht there exists $n \in \mathbb{N}$ for all $p \geq n,\ a_{t_p} = 0_{t_p}$. So the fuzzy polynomial is defined as $(a_{t_0}, a_{t_1}, a_{t_2}, ...a_{t_i},$ $0_s, ..., 0_s)$ with $t_i, s \in (0, 1]$. The set of all fuzzy polynomials with one indeterminate on $F_\mu(R)$ is denoted by $F_\mu(R)[X]$.

Let us now define some operations on the fuzzy polynomials.

Let $P, Q \in F_\mu(R)[X]$. Then, $P = (a_{t_k})_{k\in\mathbb{N}}$ such that there exists $n \in \mathbb{N}$ with $a_{t_p} = 0_{t_p}$ for each $p > n$, and $Q = (b_{s_k})_{k\in\mathbb{N}}$ such that there exists $m \in \mathbb{N}$ with $b_{s_p} = 0_{s_p}$ for all $p > m$.

Addition "(+)"
Define $P + Q = (c_{\alpha_k})_{k\in\mathbb{N}}$ such that $c_{\alpha_k} = a_{t_k} + b_{s_k} = (a + b)_{t_k \wedge s_k}$ and $c_{\alpha_k} = 0_{\alpha_k}$ for all $p > \max(n, m)$. It is obvious that $P + Q \in F_\mu(R)[X]$.

Multiplication "$(\times)$"
The multiplication $P \times Q$ is defined by $P \times Q = (d_{\beta_k})_{k \in \mathbb{N}}$ such that $d_{\beta_k} = \sum_{i+j=p} a_{t_i} b_{s_j}$ with $\beta_k = \min_{0 \geq i, j \geq k} \{t_i, s_j\}$ with $d_{\beta_p} = 0_{\beta_p}$ for each $p > n + m$ because $p = i + j > n + m$ implies $i > n$ or $j > m$. This implies that $a_{t_i} = 0_{t_i}$ or $b_{s_j} = 0_{s_j}$.

Remark 1. Let $P, Q \in F_\mu(R)[X]$ be two fuzzy polynomials. Then $P = Q$ if and only if $a_{t_i} = b_{s_i}$, for each $i \in \mathbb{N}$. The zero fuzzy polynomial is defined as $(a_{t_i})_{i \in \mathbb{N}}$ such that $a_{t_i} = 0_{t_i}$, for each $i \in \mathbb{N}$.

Proposition 2. $(F_\mu(R)[X], +, \times)$ *is a comutative ring.*

Proof. The zero element is $(0_s, 0_s, 0_s, ..., 0_s)$ with $s \in (0, 1]$. For all $P, Q, R \in F_\mu(R)[X]$,

$$\begin{aligned}(P + Q) + R &= (a_{t_i} + b_{s_i}) + c_{k_i} \\ &= (a + b)_{t_i \wedge s_i} + c_{k_i} \\ &= \big((a + b) + c\big)_{(t_i \wedge s_i) \wedge k_i} \\ &= \big(a + (b + c)\big)_{t_i \wedge (s_i \wedge k_i)} \\ &= a_{t_i} + (b + c)_{s_i \wedge k_i} \\ &= a_{t_i} + (b_{s_i} + c_{k_i}) \\ &= P + (Q + R).\end{aligned}$$

Hence $+$ is associative. In a similar way, we can prove that $P + Q = Q + P$. The symmetrical element is given by

$$-P = (-a_{t_k})_{k \in \mathbb{N}} \in F_\mu(R)[X]$$

Indeed

$$P + (-P) = (0_s, 0_s, 0_s, ..., 0_s)$$

with $s \in (0, 1]$. In addition, $(F_\mu(R)[X], \times)$ is a semigroup. Using the fact that $(F_\mu(R), \times)$ is a semigroup and the definition of "$\times$" in $F_\mu(R)[X]$ we can easily show that

$$P \times (Q \times R) = (P \times Q) \times R$$

and $P \times Q = Q \times P$ and $P \times (Q + R) = P \times Q + P \times R$. Consequently $(F_\mu(R)[X], +, \times)$ is a commutative ring with identity. The identity is given by $(1_1, 0_s, 0_s, ..., 0_s)$ since

$$P \times (1_1, 0_s, 0_s, ..., 0_s) = (a_{t_0}, a_{t_1}, ..., a_{t_n}, 0_s, ..., 0_s) = P.$$

Denote by $X = (0_s, 1_1, 0_s, ..., 0_s)$, with $s \in (0, 1]$ and call it one indeterminate. By convention,

$$X^0 = 1_1 = (1_1, 0_s, 0_s, ..., 0_s);\ X^2 = XX = (0_s, 1_1, 0_s, ..., 0_s)(0_s, 1_1, 0_s, ..., 0_s) = (0_s, 0_s, 1_1, 0_s, ..., 0_s)$$

and

$$X^n = (\overbrace{0_s, , 0_s, ...0_s}^{n\, times\, 0_s}, 1_1, ..., 0_s).$$

Let $P = (a_{t_k})_{k\in\mathbb{N}} \in F_\mu(R)[X]$. Then

$$P = (a_{t_0}, a_{t_1}, a_{t_2}, 0_s, ..., 0_s) = a_{t_0}(1_1, 0_s, 0_s, ..., 0_s) + a_{t_1}(0_s, 1_1, 0_s, ..., 0_s) + ... + a_{t_n}(\overbrace{0_s, ...0_s}^{n\, times\, 0_s}, 1_1, ..., 0_s).$$

Hence, the fuzzy polynomial P is written in the form $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + ... + a_{t_n}X^n$.

Definition 9. We say that $P \in F_\mu(R)[X]$ is a fuzzy polynomial on $F_\mu(R)$ if there exists $a_{t_i} \in F_\mu(R)$ such that $P = \sum_{i=0}^{i=n} a_{t_i}X^i$.

Definition 10. Let $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + ... + a_{t_n}X^n$ be a nonzero fuzzy polynomial. Then there exists a nonzero coefficient of $a_{t_0}, a_{t_1}, ..., a_{t_n}$.

Definition 11. (fuzzy degree) Let $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + ... + a_{t_n}X^n \in F_\mu(R)[X]$. The fuzzy degree of P denoted by $deg(P)$ or d^o is defined as the maximal number n such that $a_{t_n} \neq 0_{t_n}$. In this case a_{t_n} is called the leading coefficient of P.

Proposition 3. *Let $F_\mu(R)$ be an integral ring and P and Q be two polynomials of $F_\mu(R)[X]$. Then, we have*

(a) $d^o(P + Q) \leq \max(d^o(P), d^o(Q))$.
(b) $d^o(P.Q) = (d^o(P) + d^o(Q))$.

Proof. (a) Suppose that $d^o(P) = n$ and $d^o(Q) = p$. Then $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + ... + a_{t_n}X^n$ and $Q = b_{s_0} + b_{s_1}X + b_{s_2}X^2 + ... + b_{s_p}X^p$. Suppose that $n > p$. Then

$$P + Q = (a + b)_{t_0\wedge s_0} + (a + b)_{t_1\wedge s_1}X + ... + a_{t_n}X^n$$

Hence $d^o(P + Q) = n = \max d^o(P), d^o(Q)$. If $n < p$ we have $d^o(P + Q) = p = \max d^o(P), d^o(Q)$. If $n = p$, then

$$P + Q = (a + b)_{t_0\wedge s_0} + (a + b)_{t_1\wedge s_1}X + ... + (a + b)_{t_n\wedge s_n}X^n.$$

Consider the following cases:

(i) if $(a+b)_{t_n \wedge s_n} \neq 0_{t_n \wedge s_n}$, then $d^o(P+Q) = n = \max d^o(P), d^o(Q)$.
(ii) if $(a+b)_{t_n \wedge s_n} = 0_{t_n \wedge s_n}$, then $P+Q = (a+b)_{t_0 \wedge s_0} + (a+b)_{t_1 \wedge s_1} X + \dots + (a+b)_{t_{n-1} \wedge s_{n-1}} X^{n-1}$, hence $d^o(P+Q) \leq n-1 \leq \max(d^o(P), d^o(Q))$.

(b) $P \times Q = (ab)_{t_0 \wedge s_0} + ((ab)_{t_0 \wedge s_1} + (ab)_{t_1 \wedge s_0})X + \dots + (ab)_{t_n \wedge s_p} X^{n+p}$, since $a_{t_n} \neq 0_{t_n}$ and $b_{s_p} \neq 0_{s_p}$ then $a_{t_n} b_{s_p} \neq 0_{t_n \wedge s_p}$. Therefore, $d^o(P \times Q) = d^o(P) + d^o(Q)$.

Remark 2. If P is a zero polynomial we denote by convention $d^o(P) = -\infty$. If $F_\mu(R)$ is a non integral ring, then $d^o(PQ) \leq d^o(P) + d^o(Q)$.

Definition 12. Let μ be a fuzzy subring of R. An extension of μ is a fuzzy subring ν of R, such that $\mu \subset \nu$.

Example 1. Define μ and ν as follows:

$$\nu : \begin{cases} \mathcal{M}_2(\mathbb{R}) \longrightarrow [0,1] \\ x \longmapsto \begin{cases} 1 & if \quad x = 0 \\ \frac{1}{2} & if \quad x \neq 0 \end{cases} \end{cases}$$

$$\mu : \begin{cases} \mathcal{M}_2(\mathbb{R}) \longrightarrow [0,1] \\ x \longmapsto \begin{cases} 1 & if \quad x = 0 \\ \frac{1}{4} & if \quad x \neq 0 \end{cases} \end{cases}$$

It is easy to show that $\mu \subset \nu$. Hence ν is an extension of μ.

Definition 13. We say that $\alpha_s \in F_\mu(R)$ is a zero of $P \in F_\mu(R)[X]$ if $P(\alpha_s) = \sum_{i=0}^{i=n} a_{t_i} \alpha_s^i = 0_\beta$ such that $\beta \leq s$.

Let $I(b_t) = \{P \in F_\mu(R)[X], P(b_t) = 0_s\}$. It is clear that $I(b_t)$ is an ideal of $F_\mu(R)[X]$.

Definition 14. $b_t \in \nu$ is called an algebraic element if $I(b_t) \neq \{0\}$. Otherwise b_t is called a transcendent element.

Note that if $a_{t_i} = 1_1$ then $b_t \in \nu$ is called an integral element.

Theorem 5. *Let R be a ring. Then R is an integral domain if and only if $F_\mu(R)[X]$ is an integral ring.*

Proof. Suppose that R is an integral domain. According to the Theorem 4, $F_\mu(R)$ is an integral domain. Let $P, Q \in F_\mu(R)[X]$ be such that $P \neq 0$ and $Q \neq 0$. let $a_t X^p$ and $b_s X^q$ be the monomials of more high degrees of P and Q, respectively. The term of degree $p+q$ of QP is $a_t b_s X^{p+q}$. Conversely, let $a_t, b_s \in F_\mu(R)$ be such that $a_t b_s = 0_{t \wedge s}$. We have $a_t, b_s \in F_\mu(R)[X]$ hence $a_t = 0_t$ or $b_s = 0_s$ because $F_\mu(R)[X]$ is an integral ring. So we have the result.

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On (Completely) Weak* Rad-⊕-Supplemented Modules

Manoj Kumar Patel

Abstract In this paper, we establish various properties of weak* Rad-⊕-supplemented modules and completely weak* Rad-⊕-supplemented modules, which are the generalizations of ⊕–supplemented and Rad-⊕-supplemented modules. Our main focus is to characterize the weak* Rad-⊕-supplemented modules in terms of radical modules, modules having property (p^*) and w–local modules.

Keywords Rad-⊕-supplemented module · Weak* Rad-⊕-supplemented module
Completely weak* Rad-⊕-supplemented module

1 Introduction

Throughout this paper, R will be an associative ring with identity and all modules are unitary left R–modules unless otherwise specified. Let M be an R–module. The notation $N \subseteq M$ means that N is a submodule of M and $Rad(M)$ will indicate the Jacobson radical of M. A submodule N of a module M is called small in M (denoted by $N \ll M$), if $M \neq N + K$ for every proper submodule K of M. A nonzero module M is said to be hollow if every proper submodule of M is small in M, and it is said to be local if the sum of all proper submodules of M is also a proper submodule of M, equivalently $RadM$ is the unique maximal submodule of M and $RadM \ll M$, equivalently M is hollow and finitely generated. A nonzero module M is said to be w–local, if it has a unique maximal submodule.

A module M is said to have property (p^*), if for every submodule N of M, there exists a direct summand K of M such that $K \subseteq N$ and $N/K \subseteq Rad(M/K)$, or equivalently, for every submodule $N \subseteq M$ there exists a decomposition $M = K \oplus L$ with $K \subseteq N$ such that $N \cap L \subseteq RadL$ [1]. Recall that a module M is called radical if M has no maximal submodule i.e. $RadM = M$ [5]. Every divisible Z–

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module is a radical module. For a module M, $P(M)$ will indicate the sum of all radical submodules of M. Note that $P(M)$ is the largest radical submodule of M. If $P(M) = 0$, then M is called reduced. Also $M/P(M)$ is reduced for every module M.

If N and L are submodules of M, then N is called a supplement of L, if $N + L = M$ and $N \cap L \ll N$. A module M is called supplemented if each of its submodules has a supplement in M. A module M is called $\oplus$–supplemented (completely $\oplus$–supplemented) if every submodule (direct summand) of M has a supplement that is a direct summand of M [3–5, 7, 8]. A module M is called amply supplemented if for every submodules N and L of M with $M = N + L$, L contains a supplement of N in M.

A submodule N of a module M has a Rad-supplement K in M if $N + K = M$ and $N \cap K \subseteq Rad K$. A module M is called Rad-supplemented if every submodule of M has a Rad-supplement [4, 9]. M is called Rad-$\oplus$-supplemented if every submodule of M has a Rad-supplement that is a direct summand of M. The Z–module Q is Rad-$\oplus$-supplemented but not $\oplus$–supplemented where Z and Q denote the ring of integers and rational numbers respectively. Every module with (p^*) is Rad-$\oplus$-supplemented. A module M is called completely Rad-$\oplus$-supplemented if every direct summand of M is Rad-$\oplus$-supplemented [4].

2 Weak* Rad-⊕-Supplemented Modules

Definition 1. An R–module M is called a weak* Rad-$\oplus$-supplemented module if every semi-simple submodule of M has a Rad-supplement that is a direct summand of M. An R–module M is called completely weak* Rad-$\oplus$-supplemented module if every direct summand of M is a weak* Rad-$\oplus$-supplemented module i.e. every direct summand of M has a Rad-supplement which is a direct summand of M [4].

For example, hollow modules and modules with (p^*) are weak* Rad–$\oplus$–supplemented modules. Also, hollow modules are completely weak* Rad–$\oplus$–supplemented modules. Clearly, every Rad-$\oplus$-supplemented module is a weak* Rad-$\oplus$-supplemented module but the converse is not true in general; for counter examples see [4]. Thus we have the following implications, but in general the reverse implications do not hold.

Lifting $\Rightarrow$ $\oplus$–supplemented $\Rightarrow$ Rad-$\oplus$-supplemented $\Rightarrow$ weak* Rad-$\oplus$-supplemented $\Rightarrow$ Rad-supplemented.

Let M be an R–module. We consider the following conditions.

(D1) For every submodule N of M, there exists a decomposition of $M = M_1 + M_2$, such that $M_1 \subseteq N$ and $M_2 \cap N$ is small in M_2.

(D2) If N is a submodule of M such that M/N is isomorphic to a direct summand of M, then N is a direct summand of M.

(D3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

An R–module M is called a lifting module if M satisfies $(D1)$; M is called a discrete module if it satisfies $(D1)$ and $(D2)$; and quasi-discrete if it satisfies $(D1)$ and $(D3)$.

For a submodule N of M, $N \cap Rad M \neq Rad N$ in general [8]. But the equality holds if N is a supplement or a Rad-supplement submodule of M.

Lemma 1. [8] *Let M be an R–module and N be a supplement (or Rad-supplement) submodule of M. Then $N \cap Rad M = Rad N$.*

Lemma 2. *Let M be a module and N be a semi-simple submodule of M. If K is a weak* Rad−⊕ –supplement of N in M, then $(K + L)/L$ is a weak* Rad−⊕ –supplement of N/L in M/L for every submodule L of N.*

Proof. Assume K is a weak* Rad–⊕–supplement of N in M. Then, by definition we have $M = N + K$, $N \cap K \subseteq Rad(K)$ and K is direct summand of M i.e. there exists $X \subseteq M$ such that $M = K \oplus X$. Hence $M/L = N/L + (K + L)/L$ for every submodule L of N. Consider the natural epimorphism $f : K \to (K + L)/L$. Then by properties of radical ([5], 2.8(1)), $f(Rad(K)) \subseteq Rad((K + L)/L)$. Since $N \cap K \subseteq Rad(K)$, we have $(N/L) \cap (K + L)/L = (L + (N \cap K))/L = f(N \cap K) \subseteq f(Rad(K)) \subseteq Rad((K + L)/L)$. Clearly $(K + L)/L$ is a direct summand of M/L. Hence $(K + L)/L$ is a weak* Rad−⊕–supplement of N/L in M/L.

Proposition 1. *Let M be a module. If M is weak* Rad-⊕-supplemented, then the factor module $M/P(M)$ of M is weak* Rad-⊕-supplemented.*

Proof. By ([5], 2.8(1)) we have $f(Rad(P(M))) \subseteq Rad(P(M)$ for any endomorphism f of M. Also we have $Rad(P(M)) = P(M)$. Therefore $f(P(M)) \subseteq P(M)$ for any endomorphism f of M. Since M is weak* Rad-⊕-supplemented, by definition, for semi-simple submodule N of M, there exists submodules L and L' of M such that $M = N + L$, $N \cap L \subseteq Rad(L)$ and $M = L \oplus L'$. For submodule $N/P(M)$ of $M/P(M)$, by Lemma 2, $(L + P(M))/P(M)$ is a weak* Rad–⊕–supplement of $N/P(M)$ in $M/P(M)$. Since $f(P(M)) \subseteq P(M)$ for any endomorphism f of M, we have $P(M) = (L \cap P(M)) \oplus (L' \cap P(M))$. Hence $(L + P(M)) \cap (L' + P(M)) \subseteq P(M)$ and so $(L + P(M))/P(M) \cap (L' + P(M))/P(M) = 0$ i.e. $(L + P(M))/P(M)$ is a direct summand of $M/P(M)$. Hence $M/P(M)$ is a weak* Rad-⊕-supplemented module.

Corollary 1. *The largest radical submodule $P(M)$ of M is completely weak* Rad-⊕-supplemented, for every module M.*

Proof. We know that every radical module is weak* Rad-⊕-supplemented by ([4], Lemma 8). Hence it remains to show that any direct summand K of $P(M)$ is radical. Let $P(M) = K \oplus L$ for some summand L of $P(M)$. By ([5], 2.8(5)), we have $P(M) = Rad(P(M)) = Rad(K \oplus L) = Rad(K) \oplus Rad(L)$. Applying the modular law, $K = K \cap P(M) = K \cap (Rad(K) \oplus Rad(L)) = Rad(K) \oplus Rad(L) \cap K = Rad(K)$. Thus $K = Rad(K)$ i.e. K is radical. Therefore $P(M)$ is completely weak*Rad-⊕-supplemented.

Proposition 2. *If the module M has property (p^*), then M is completely weak* Rad-$\oplus$-supplemented.*

Proof. For completeness, we prove that every direct summand N of M is weak* Rad-$\oplus$-supplemented. Let K be a semi-simple submodule of N. Since the module M has property (p^*), there exists a decomposition of $M = L \oplus L'$ such that $L \subseteq K$ and $K \cap L' \subseteq Rad(L')$. Applying the modular law, we have $N = (N \cap L) \oplus (N \cap L') = L \oplus (N \cap L')$ which shows that $N \cap L'$ is a direct summand of N. Hence $N = K + (N \cap L')$. Now it remains to show that $K \cap (N \cap L') = K \cap L' \subseteq Rad(N \cap L')$. Let x be any element of $K \cap L'$. Since $K \cap L' \subseteq Rad(L')$ by ([6], 9.13(a)), we get $Rx \ll L'$ so that $Rx \ll M$. Using ([5], 2.2(6)), we get $Rx \ll N$, again using ([5], 2.2(6)), we obtain $Rx \ll N \cap L'$, by ([6], 9.13(a)), we have $K \cap L' \subseteq Rad(N \cap L')$.

Recall that ([8], 41.15), a π–projective module is $\oplus$–supplemented if and only if the module is lifting. A similar characterization for weak* Rad-$\oplus$-supplemented module is not true in general but it holds partially, which is given in the following proposition.

Proposition 3. *A π–projective module M is weak* Rad-$\oplus$-supplemented if and only if it has the property (p^*).*

Proof. Assume that the π–projective module M is weak* Rad-$\oplus$-supplemented. Let K be a semi-simple submodule of M. Then by definition, there exists a direct summand N of M such that $M = K + N$, $K \cap N \subseteq Rad(N)$ and $M = N \oplus N'$ for some summand N' of M. Since M is π–projective, by ([5], 4.14(1)), we have $M = N \oplus K'$ for some submodule K' of K. It follows that, for the semi-simple submodule K of M, there exists a decomposition of $M = N \oplus K'$ such that $K' \subseteq K$ and $K \cap N \subseteq Rad(N)$, which shows that M has the property (p^*) only for the semi-simple submodule K of M. The converse is clear by Proposition 2.

Note 1. The necessary part of Proposition 3 is true for semi-simple submodules. But it holds for all submodules if the base ring is semi-simple.

Every lifting module has the property (p^*) but the converse need not be true. However it holds for projective modules, which is shown in the following proposition.

Proposition 4. *If a projective module M has the property (p^*), then M is lifting.*

Proof. Using Proposition 2, M is weak* Rad-$\oplus$-supplemented. Now by ([4], Proposition 5), it is $\oplus$–supplemented. Since M is projective, it is π–projective. Thus M is lifting by ([8], 41.15).

Proposition 5. *For a projective module M, the following statements are equivalent:*

(i) *M is supplemented;*
(ii) *M is $\oplus$–supplemented;*
(iii) *M is Rad-$\oplus$-supplemented;*
(iv) *M is weak* Rad-$\oplus$-supplemented;*

(v) *M has the property* (p^*)*;*
(vi) *M is lifting;*
(vii) *M is discrete;*
(viii) *M is quasi-discrete.*

Proof. $(i) \Rightarrow (ii)$ is an immediate consequence of ([8], 41.15). $(ii) \Rightarrow (iii)$ follows from ([4], Proposition 5). $(iii) \Rightarrow (iv)$ is obvious. $(iv) \Rightarrow (v)$ partially holds by Proposition 3 and is clear for semi-simple modules. $(v) \Rightarrow (vi)$ is shown in Proposition 4. $(vi) \Rightarrow (vii)$ is clear, since projective module satisfies $(D2)$ property. $(vii) \Rightarrow (viii)$ is obvious as property $(D2) \Rightarrow (D3)$. $(viii) \Rightarrow (i)$ is obvious by ([5], 26.6).

Proposition 6. *For a ring R, the following statements are equivalent:*

(i) *Every R–module has property* (p^*)*;*
(ii) *Every R–module is lifting;*
(iii) *R is an artinian serial ring and* $J^2 = 0$*, where J is the Jacobson radical of R.*

Proof. $(i) \Rightarrow (ii)$ Let M be any projective module. Since M is projective, it is π–projective and so weak* Rad-⊕-supplemented by Proposition 3. It follows from ([4], Corollary 7), that R is a left perfect ring. By assumption M has property (p^*), so for every submodule K of M, there exists a decomposition of $M = L \oplus L'$ such that $L \subseteq K$ and $K \cap L' \subseteq Rad(L')$. Since R is left perfect, we have $K \cap L' \ll L'$, which shows that M is lifting. $(ii) \Rightarrow (iii)$ See ([5], 29.10). $(iii) \Rightarrow (i)$ is obvious.

Proposition 7. *For a reduced w–local module M, the following statements are equivalent:*

(i) *M is Rad-⊕-supplemented;*
(ii) *M is weak* Rad-⊕-supplemented;*
(iii) *M is Rad-supplemented;*
(iv) *M is a local module;*
(v) *M is supplemented;*
(vi) *M is amply supplemented.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ Straight forward. $(iii) \Rightarrow (iv)$ Let $x \in M/Rad(M)$ and K be a Rad-supplement of Rx in M. Since $Rx + K = M$, we have $M/K = (Rx + K)/K \cong Rx/(Rx \cap K)$. Thus M/K has a maximal submodule. But M is w–local, so $Rad(M)$ is the only maximal submodule of M. So $K \subseteq Rad(M)$. Since K is a Rad-supplement submodule of M, by Lemma 1, it follows that $Rad(K) = K \cap Rad(M)$. Therefore $Rad(K) = K$. But M is reduced, so $K = 0$, which gives $M = Rx$ is a local module. $(iv) \Rightarrow (v)$ is clear. $(vi) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (vi)$ is straight forward.

Proposition 8. *For a w–local module M, the following statements are equivalent:*

(i) *M is Rad-⊕-supplemented;*
(ii) *M is weak* Rad-⊕-supplemented;*
(iii) $M/P(M)$ *is weak* Rad-⊕-supplemented;*

(iv) $M/P(M)$ *is Rad-supplemented;*
(v) $M/P(M)$ *is supplemented;*
(vi) $M/P(M)$ *is local module;*
(vii) *For every* $x \in M/Rad(M)$, $M = P(M) + Rx$ *and the ring* R/I_x *is a local ring, where* $I_x = \{r \in R | rx \in P(M)\}$*;*
(viii) *There exists* $x \in M$ *such that* $M = P(M) + Rx$ *and the ring* R/I_x *is a local ring, where* $I_x = \{r \in R | rx \in P(M)\}$.

Proof. $(i) \Rightarrow (ii)$ is obvious. $(ii) \Rightarrow (iii)$ is done in Proposition 1. $(iii) \Rightarrow (iv)$ is clear. $(ii) \Rightarrow (vi)$ $M/P(M)$ is weak* Rad-$\oplus$-supplemented by Proposition 1. It is clearly seen that $M/P(M)$ is reduced w–local. Now applying Proposition 7, we get $M/P(M)$ is a local module. $(vi) \Rightarrow (vii)$ Let $x \in M/Rad(M)$. Since M is w–local, $Rad(M)$ is a maximal submodule of M; so $M = Rad(M) + Rx$. As $P(M) \subseteq Rad(M)$, $M/P(M) = Rad(M)/P(M) + (Rx + P(M))/P(M)$. By assumption $M/P(M)$ is local and $M \neq Rad(M)$, so we get $M = P(M) + Rx$. Moreover $M/P(M) \cong Rx/(Rx \cap P(M))$ and $Ann_R(M/P(M)) = I_x$. Thus R/I_x is a local ring. $(vii) \Rightarrow (viii)$ is obvious. $(viii) \Rightarrow (v)$ is clear from the fact that $M/P(M) \cong Rx/(Rx \cap P(M)) \cong R/I_x$. $(v) \Rightarrow (iv)$ is obvious. $(iv) \Rightarrow (i)$ is clear.

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When Is Int(E, D) a Locally Free D–Module

Lahoucine Izelgue and Ali Tamoussit

Abstract Let D be an integral domain with quotient field K, E a subset of K and X an indeterminate over K. The set of *integer–valued polynomials on* E is defined by $\text{Int}(E, D) = \{f \in K[X] : f(E) \subseteq D\}$. Clearly, $\text{Int}(E, D)$ is a subring of $K[X]$ and $\text{Int}(D, D) = \text{Int}(D)$, the *ring of integer–valued polynomials over* D. In this paper, we investigate some conditions under which $\text{Int}(E, D)$ is locally free, or at least flat, as a D–module. Particularly, we are interested in domains that are locally essential with subsets E residually cofinite.

Keywords Integer–valued polynomials · Flat modules · Locally free modules
Residue field · Locally essential domains

1 Introduction

Let D be an integral domain with quotient field K. The ring of *integer–valued polynomials over* D is defined by $\text{Int}(D) = \{f \in K[X] : f(D) \subseteq D\}$, a subring of $K[X]$ with $D[X] \subseteq \text{Int}(D) \subseteq K[X]$. The fact that there may exist subsets $E \subsetneq D$ such that $\text{Int}(D) = \{f \in K[X] : f(E) \subseteq D\}$ led Gilmer to introduce and study these sets in [16]. That give rise to a large class of integral domains, known as rings of integer–valued polynomials on a subset $E \subseteq K$ and defined by $\text{Int}(E, D) = \{f \in K[X] : f(E) \subseteq D\}$, a natural generalization of $\text{Int}(D)$ as $\text{Int}(D, D) = \text{Int}(D)$ [4, 5]. That also generate new concepts, such as polynomially equivalent (resp., polynomially dense, polynomial closure of) subsets. Remarkably, a circle of investigations then began about these concepts and also about ring-theoretic properties, the module

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structure, the (prime) ideal structures and the calculation of the Krull dimensions of $\mathrm{Int}(E, D)$.

Polya [23] established that for any ring D, of integers of a number field, $\mathrm{Int}(D)$ is a free D–module. Later, Cahen and Chabert [6] established that $\mathrm{Int}(D)$ is a faithfully flat, D–module, for any Dedekind domain D. Then Cahen [3] showed that it is, in fact, free with a regular basis, (see also [7, Remark II.3.7(iii) and Proposition II.3.14]).

Recently, Elliott investigated these rings in a category-theoretic viewpoint [11–14]. Many of his works are based or dealt with the questions of when $\mathrm{Int}(D)$ is either locally free or flat as a D–module.

Particularly, he established that under certain general conditions, such as [13, Theorem 1.2 and Lemma 2.9], the domain $\mathrm{Int}(D)$ is locally free, hence flat, as a D–module. This includes the case of when D is a Krull domain or more generally a TV PvMD.

Notice also that Chabert et al. [10] showed that $\mathrm{Int}(\mathbb{P}, \mathbb{Z})$ is a free $\mathbb{Z}$–module (with a regular basis), where $\mathbb{P}$ denotes the set of all prime integers. Then, Bourlanger et al. [2] proved that, for E an infinite subset of a discrete valuation domain V, $\mathrm{Int}(E, V)$ is a free V–module.

However there is no example, in the literature, of an integral domain D such that $\mathrm{Int}(E, D)$ is not flat or free as a D–module, even when $E = D$.

Thus, in parallel to [8, Problem 19], we pose to study the problem of when $\mathrm{Int}(E, D)$ is locally free, or at least flat, as a D–module.

On the contrary to the case of $\mathrm{Int}(D)$, while D is always a subring of $\mathrm{Int}(E, D)$, $D[X]$ need not be so. However, if E is a subset of D, then $D[X] \subseteq \mathrm{Int}(D) \subseteq \mathrm{Int}(E, D)$ ([5, Proposition 1.2]). On the other hand, if $(D, \mathfrak{m})$ is a local domain and E meets infinitely many cosets of $\mathfrak{m}$, then $\mathrm{Int}(E, D) = \mathrm{Int}(D) = D[X]$ [7, Proposition IV.1.20]. The D–module $\mathrm{Int}(E, D)$ is then free, and thus flat, over D. Also, if $R \subseteq D$ is an extension of integral domains such that R has infinite residue fields, then $\mathrm{Int}(R, D) = \mathrm{Int}(D) = D[X]$ (cf. [7, Corollary IV.1.21]). Hence, $\mathrm{Int}(R, D)$ is a free D–module.

On the other hand, if a fractional subset E is polynomially dense in D, that is $\mathrm{Int}(E, D) = \mathrm{Int}(D)$, then we are led to the case of $\mathrm{Int}(D)$ already treated in [19].

In this paper, we pose to classify locally essential domains D such that $\mathrm{Int}(E, D)$ is a locally free, or at least flat, D–module.

Thus, we first establish that flatness always hold over any Prüfer domain D and any subset E of K. Particularly, when D is (Almost) Dedekind. A similar result holds over locally essential domains with $E \subseteq D$ residually cofinite with D. Our main result shows that with this last hypothesis if E is infinite and D Almost Krull, then $\mathrm{Int}(E, D)$ is a locally free over D. The same result is then established over a class of Krull–type domains.

2 When Is Int(E,D) Locally Free, or at Least Flat, Over D?

Most of the results established in the case of Int(D) are due to the fact that, for each $\mathfrak{p} \in \text{Spec}(D)$ with infinite residue field, $\text{Int}(D_\mathfrak{p}) \subseteq D_\mathfrak{p}[X]$. However, the inclusion $\text{Int}(E, D_\mathfrak{p}) \subseteq D_\mathfrak{p}[X]$, may not hold. That makes it harder to characterize flatness of Int(E, D) as a D–module.

Recall that Int(E, D) is said to be a *locally free* D–module if $\text{Int}(E, D)_\mathfrak{m}$ is a free $D_\mathfrak{m}$–module, for each maximal ideal $\mathfrak{m}$ of D.

Now, [2, Proposition 2.2] establishes that for any discrete valuation domain V and any infinite subset E of V, the V–module Int(E, V) is free, and thus flat. Next, we see that Int(E, V) is V–flat for any subset E of qf.(V). In fact, any torsion–free module over a Prüfer domain D is flat (cf. [15, Theorem 1.4, page 71]), thus we have:

Proposition 1. *Let D be a Prüfer domain and E be a subset of $K = \text{qf}.(D)$. Then* Int(E, D) *is a flat D–module.*

Since (Almost) Dedekind domains are Prüfer, so we have:

Corollary 1. *Let D be an (Almost) Dedekind domain with quotient field K. Then* Int(E, D) *is a flat D–module for any subset E of K. If moreover E is assumed to be an infinite subset of D, then* Int(E, D) *is a locally free D–module.*

Proof. The first affirmation follows from Proposition 1.

Now, since D is (Almost) Dedekind, then for each maximal ideal $\mathfrak{m}$ of D, $D_\mathfrak{m}$ is a DVR and thus a PID. So, if E is infinite, by [2, Proposition 2.2], Int$(E, D_\mathfrak{m})$ is a free $D_\mathfrak{m}$–module and by [24, Theorem 9.8, page 650] $\text{Int}(E, D)_\mathfrak{m}$ is also free. It follows that Int(E, D) is a locally free, and thus faithfully flat, D–module.

Next we recover an example due to Chabert et al. [10].

Example 1. For any subset E of $\mathbb{Q}$, Int$(E, \mathbb{Z})$ is a flat $\mathbb{Z}$–module. In particular, since $\mathbb{P}$, the set of all prime integers, is an infinite subset of $\mathbb{Z}$, by Corollary 1, Int$(\mathbb{P}, \mathbb{Z})$ is a locally free $\mathbb{Z}$–module.

If D is infinite, the previous two results generalize [19, Theorem 2.2 and Corollary 2.3].

Corollary 2. *Let D be an (Almost) Dedekind domain. Then* Int(D) *is a locally free, and thus faithfully flat, D–module.*

Recall that a nonempty subset E of a domain D is *residually cofinite* with D if it possesses the property that for each prime ideal P of D, $|E/P| < \infty$ implies that $|D/P| < \infty$ [21]. For instance, D is cofinite with itself. On the other hand, if a subset E of D is residually cofinite with D, then E remains so with $S^{-1}D$, for each multiplicatively closed subset of D (cf. [21, Lemma 3 (i)]). Now, if all residue fields of D are infinite and E is residually cofinite with D, by [21, Lemma 4 (ii)], $\text{Int}(E, D) = \text{Int}(D) = D[X]$.

On the other hand, an integral domain D is said to be *essential* if $(\star)$ $D = \bigcap_{\mathfrak{p}\in\mathscr{P}} D_{\mathfrak{p}}$, for some $\mathscr{P} \subseteq \mathrm{Spec}(D)$ with $D_{\mathfrak{p}}$ a valuation domain for each $\mathfrak{p} \in \mathscr{P}$. This notion does not carry up to localizations. Thus, D is said to be a *locally essential domain* if $D_{\mathfrak{q}}$ is an essential domain for each $\mathfrak{q} \in \mathrm{Spec}(D)$.

A *Krull–type* domain is an essential domain of finite character, that is each nonzero element of D belongs to only finitely many prime ideals $\mathfrak{p} \in \mathscr{P}$ (cf. the intersection $(\star)$). An integral domain D is said to be *Almost Krull*, if $D_{\mathfrak{m}}$ is a Krull domain for each $\mathfrak{m} \in \mathrm{Max}(D)$. Notice that domains that are Krull, almost Krull or of Krull–type are also locally essential.

Theorem 1. *Let $D = \bigcap_{\mathfrak{p}\in\mathscr{P}} D_{\mathfrak{p}}$, where $\mathscr{P} \subseteq \mathrm{Spec}(D)$, be a locally essential domain and $E \subseteq D$ be residually cofinite with D. Then* $\mathrm{Int}(E, D)$ *is a flat D–module.*

Proof. If $\mathfrak{m} \in \mathrm{Max}(D) \cap \mathscr{P}$, then $D_{\mathfrak{m}}$ is a valuation domain. Also $\mathrm{Int}(E, D_{\mathfrak{m}})$ and thus $\mathrm{Int}(E, D)_{\mathfrak{m}}$ is a torsion-free $D_{\mathfrak{m}}$–module. By [15, Theorem 1.4, page 71], $\mathrm{Int}(E, D)_{\mathfrak{m}}$ is a flat $D_{\mathfrak{m}}$–module. If $\mathfrak{m} \in \mathrm{Max}(D) \setminus \mathscr{P}$, since D is a locally essential domain, we can write $D_{\mathfrak{m}} = \bigcap_{\mathfrak{p}\in\mathscr{P},\mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}$. Then, for each $\mathfrak{p} \in \mathscr{P}$ with $\mathfrak{p} \subsetneq \mathfrak{m}$, $D_{\mathfrak{p}}$ has an infinite residue field and hence $\mathrm{Int}(E, D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ [21, Lemmas 3 (i) and 4 (ii)]. Thus

$$\mathrm{Int}(E, D_{\mathfrak{m}}) = \bigcap_{\mathfrak{p}\in\mathscr{P},\mathfrak{p}\subsetneq\mathfrak{m}} \mathrm{Int}(E, D_{\mathfrak{p}}) = \bigcap_{\mathfrak{p}\in\mathscr{P},\mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}[X] = D_{\mathfrak{m}}[X].$$

Since, $D_{\mathfrak{m}}[X] \subseteq \mathrm{Int}(E, D)_{\mathfrak{m}} \subseteq \mathrm{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$, then $\mathrm{Int}(E, D)_{\mathfrak{m}} = \mathrm{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$. Thus, $\mathrm{Int}(E, D)_{\mathfrak{m}}$ is a flat $D_{\mathfrak{m}}$–module. By [1, Proposition 3.10, page 41], $\mathrm{Int}(E, D)$ is flat as a D–module.

Since D is residually cofinite with itself, we recover [19, Theorem 2.5] in the following:

Corollary 3. *For any locally essential domain D,* $\mathrm{Int}(D)$ *is a flat D–module.*

Theorem 2. *Let D be an Almost Krull domain and $E \subseteq D$ be infinite and residually cofinite with D. Then* $\mathrm{Int}(E, D)$ *is a locally free D–module.*

Proof. By [22, Proposition 2.6], $D = \bigcap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}$. Thus, if $\mathfrak{m} \in \mathrm{Max}(D)$, is of height one, then $D_{\mathfrak{m}}$ is a one dimensional local Krull domain, and hence a DVR. By [2, Proposition 2.2], $\mathrm{Int}(E, D_{\mathfrak{m}})$ is a free $D_{\mathfrak{m}}$–module. Since $\mathrm{Int}(E, D)_{\mathfrak{m}}$ is a submodule of $\mathrm{Int}(E, D_{\mathfrak{m}})$ and $D_{\mathfrak{m}}$ is a PID, $\mathrm{Int}(E, D)_{\mathfrak{m}}$ is also free as a $D_{\mathfrak{m}}$–module (cf. [24, Theorem 9.8, page 650]). If $\mathfrak{m} \in \mathrm{Max}(D) \setminus X^1(D)$, we can write $D_{\mathfrak{m}} = \bigcap_{\mathfrak{p}\in X^1(D),\mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}$, since D is locally essential. Now, for each $\mathfrak{p} \subsetneq \mathfrak{m}$, $D_{\mathfrak{p}}$ has an infinite residue field and hence $\mathrm{Int}(E, D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ [21, Lemmas 3 (i) and 4 (ii)]. It follows that

$$\mathrm{Int}(E, D_{\mathfrak{m}}) = \bigcap_{\mathfrak{p}\in X^1(D),\mathfrak{p}\subsetneq\mathfrak{m}} \mathrm{Int}(E, D_{\mathfrak{p}}) = \bigcap_{\mathfrak{p}\in X^1(D),\mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}[X] = D_{\mathfrak{m}}[X].$$

Since, $D_{\mathfrak{m}}[X] \subseteq \text{Int}(E, D)_{\mathfrak{m}} \subseteq \text{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$, then $\text{Int}(E, D)_{\mathfrak{m}} = \text{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$.

Therefore, $\text{Int}(E, D)_{\mathfrak{m}}$ is a free $D_{\mathfrak{m}}$–module for each $\mathfrak{m} \in \text{Max}(D)$. That ends the proof.

If $D = \bigcap_{\mathfrak{p}\in\mathscr{P}} D_{\mathfrak{p}}$ a locally finite intersection of localizations, then for each $\mathfrak{m} \in \text{Max}(D)$, $D_{\mathfrak{m}} = \bigcap_{\mathfrak{p}\in\mathscr{P}} (D_{\mathfrak{p}})_{\mathfrak{m}}$ [17, Proposition 43.5]. As a consequence, we have:

Theorem 3. *Let $D = \bigcap_{\mathfrak{p}\in\mathscr{P}} D_{\mathfrak{p}}$, where $\mathscr{P} \subseteq \text{Spec}(D)$, be a Krull–type domain and $E \subseteq D$ be infinite and residually cofinite with D. If for each $\mathfrak{p} \in \mathscr{P}$ with $D/\mathfrak{p}$ finite, $D_{\mathfrak{p}}$ is a DVR, then* $\text{Int}(E, D)$ *is a locally free D–module.*

Proof. Let $\mathfrak{m} \in \text{Max}(D)$, then $\text{Int}(E, D)_{\mathfrak{m}} = \text{Int}(E, D_{\mathfrak{m}})$ (cf. [18, Theorem 3.11]). Now, if $\mathfrak{m} \in \text{Max}(D) \setminus \mathscr{P}$, then $D_{\mathfrak{m}} = \bigcap_{\mathfrak{p}\in\mathscr{P},\mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}$, as D is a locally finite intersection. Since, for each $\mathfrak{p} \subsetneq \mathfrak{m}$, $D_{\mathfrak{p}}$ has an infinite residue field and hence $\text{Int}(E, D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ [21, Lemma 3(i) and 4(ii)], then

$$\text{Int}(E, D)_{\mathfrak{m}} = \text{Int}(E, D_{\mathfrak{m}}) = \bigcap_{\mathfrak{p}\in\mathscr{P},\mathfrak{p}\subsetneq\mathfrak{m}} \text{Int}(E, D_{\mathfrak{p}}) = \bigcap_{\mathfrak{p}\in\mathscr{P},\mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}[X] = D_{\mathfrak{m}}[X].$$

If $\mathfrak{m} \in \text{Max}(D) \cap \mathscr{P}$, then $D_{\mathfrak{m}}$ is a valuation domain. Thus, either $D/\mathfrak{m}$ infinite and so, by [7, Remark I.3.5 (ii)], $\text{Int}(E, D)_{\mathfrak{m}} = \text{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$ a free $D_{\mathfrak{m}}$–module, or $D/\mathfrak{m}$ is finite and thus $D_{\mathfrak{m}}$ is a DVR. By [2, Proposition 2.2], $\text{Int}(E, D)_{\mathfrak{m}} = \text{Int}(E, D_{\mathfrak{m}})$ is a free $D_{\mathfrak{m}}$–module. It follows that $\text{Int}(E, D)$ is locally free as a D–module.

Corollary 4. *Let D be a Krull–type domain such that* $\text{Int}(D)$ *is a PvMD, then* $\text{Int}(D)$ *is a locally free D–module.*

Proof. It follows from Theorem 3 and [25, Theorem 3.2].

A locally finite intersection $D = \bigcap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}$, is said to be *infra–Krull*, if $D_{\mathfrak{p}}$ is Noetherian for each $\mathfrak{p} \in X^1(D)$. For instance any Krull domain is infra–Krull.

Proposition 2. *Let D be an infra–Krull domain and $E \subseteq D$ be infinite and residually cofinite with D. Assume that for each $\mathfrak{p} \in \text{Max}(D) \cap X^1(D)$ with finite residue field $\mathfrak{p}D_{\mathfrak{p}}$ is principal. Then* $\text{Int}(E, D)$ *is a locally free D–module.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of D. By [21, Lemma 3 (i)], E remains residually cofinite with $D_{\mathfrak{m}}$ and by [9, Proposition 2.2 (1)] $\text{Int}(E, D)_{\mathfrak{m}} = \text{Int}(E, D_{\mathfrak{m}})$. Now, if $\mathfrak{m} \in X^1(D)$ then either:

$\mathfrak{m}$ is of infinite residue field, so by [21, Lemma 4 (ii)], $\text{Int}(E, D)_{\mathfrak{m}} = D_{\mathfrak{m}}[X]$, a free $D_{\mathfrak{m}}$–module. Or $\mathfrak{m}D_{\mathfrak{m}}$ is principal, in which case $D_{\mathfrak{m}}$ is a DVR (cf. [20, Theorem 11.2]). Hence, $\text{Int}(E, D_{\mathfrak{m}})$ is a free $D_{\mathfrak{m}}$–module (cf. [2, Proposition 2.2]). Other ways $\mathfrak{m} \notin X^1(D)$ and then, $D_{\mathfrak{m}} = \bigcap_{\mathfrak{p}\in X^1(D),\mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}$. It follows that $\text{Int}(E, D_{\mathfrak{m}}) = \bigcap_{\mathfrak{p}\in X^1(D),\mathfrak{p}\subsetneq\mathfrak{m}} \text{Int}(E, D)_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in X^1(D),\mathfrak{p}\subsetneq\mathfrak{m}} \text{Int}(E, D_{\mathfrak{p}}) = \bigcap_{\mathfrak{p}\in X^1(D),\mathfrak{p}\subsetneq\mathfrak{m}} D_{\mathfrak{p}}[X] = D_{\mathfrak{m}}[X]$, a free $D_{\mathfrak{m}}$–module.

Corollary 5. *Let D be a Krull domain and $E \subseteq D$ be infinite and residually cofinite with D. Then* $\text{Int}(E, D)$ *is a locally free D–module.*

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Pairs of Rings Whose All Intermediate Rings Are G–Rings

Lahoucine Izelgue and Omar Ouzzaouit

Abstract A *G*–ring is any commutative ring R with a nonzero identity such that the total quotient ring $\mathbf{T}(R)$ is finitely generated as a ring over R. A G–ring pair is an extension of commutative rings $A \hookrightarrow B$, such that any intermediate ring $A \subseteq R \subseteq B$ is a G–ring. In this paper we investigate the transfer of the G–ring property among pairs of rings sharing an ideal. Our main result is a generalization of a theorem of David Dobbs about G–pairs to rings with zero divisors.

Keywords G–domain · G–ring · G–ring pair · Amalgamated duplication

1 Introduction

All rings considered in this paper are commutative with unit. An integral domain R is said to be a G–domain if the quotient field K of R is a finitely generated ring over R. This is equivalent to saying that the quotient field K is of the form $R\left[\frac{1}{t}\right]$ for some nonzero element $t \in R$ (cf. [6, Theorem 18]). An integral domain with only finitely many prime ideals is a G–domain. However, the polynomial ring with coefficients in R is never a G–domain [6]. Notice also that an infinite G–domain R has the same cardinality as its set of units $U(R)$ [2]. Thus, any infinite ring with a finite group of units, such as $\mathbb{Z}$, is not a G–domain.

On the other hand, Adams [1], introduced the concept of a G–ring as a generalization of Kaplansky's definition of a G–domain to rings with zero divisors. He defined a G–ring to be any commutative ring R, with a nonzero identity, such that the total quotient ring $\mathbf{T}(R)$ is finitely generated as a ring over R. He then pointed out that $\mathbf{T}(R)$ is finite over R if, and only if, $\mathbf{T}(R) = R[u^{-1}]$, for some regular element u in

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R [1, page 1]. Also, he studied the transfer of the G–ring property between a ring and some of its extensions by way of direct products, polynomials, power series, and completions. Particularly, as for G–domains, The polynomial ring over a not necessarily integral domain is never a G-ring [1, Proposition 3.2].

In [6, Theorem 20] it was established that any overring of a G–domain R is of the form $R\left[\frac{1}{t}\right]$ for some element $0 \neq t \in R$. Moreover, every overring of a G–domain is a G–domain: that is every domain between R and its quotient field is a G–domain. This fact motivated Dobbs [4] to introduce the notion of "*G–domain pair*" as any pair of integral domains (A, B) such that for each intermediate ring $A \subseteq R \subseteq B$, R is a G–domain. Thus it seems natural to extend this definition to the case of rings with zero divisors and define a *G–ring pair* (in the sense of Adams). For instance, if A is a G–ring, with total quotient ring $\mathbf{T}(A)$, then $(A, \mathbf{T}(A))$ is a G–ring pair and, obviously, any G–domain pair is a G–ring pair. In this paper, we study the transfer of the G–ring property among rings that share a common regular ideal. In fact, we generalize [5, Theorem 3.1] to rings with zero divisors. Our main result Theorem 2 generalizes [4, Theorem 2.1], in which Dobbs gave a characterization of G–domain pairs, to the case of rings that are not necessarily integral.

2 G–Ring Pairs in the Sense of Adams

To establish our main results, we first need to extend some results from Kaplansky's book [6] to the case of rings with zero divisors. Thus the following result generalizes [6, Theorem 19]:

Proposition 1. *Let R be a ring with total quotient ring $\mathbf{T}(R)$. For any regular element $u \in R$, the following statements are equivalent:*

(i) *Any regular prime ideal of R contains u;*
(ii) *Any regular ideal of R contains a power of u;*
(iii) $\mathbf{T}(R) = R[u^{-1}]$.

Proof. The proof is parallel to that of [6, Theorem 19], just replace $u \neq 0$ by u is regular.

Next we state two useful lemmas.

Lemma 1. *Let $A \subset B$ be a ring extension. If a regular element $y \in B$ satisfies an equation of integral dependence, with minimal degree, say $r_0 + r_1 y + \ldots + r_{n-1} y^{n-1} + y^n = 0$, with $r_i \in A$ for all i, then the constant coefficient r_0 is regular.*

Proof. Straightforward, since the integral dependence equation implies that $r_0 = y(-r_1 - \ldots - r_{n-1} y^{n-2} - y^{n-1})$. Thus, r_0 and y are simultaneously regular.

We say that a ring $\mathbf{T}$ is a total quotient ring, if any regular element of $\mathbf{T}$ is invertible. The following lemma is an extension of [6, Theorem 16] to rings with zero divisors.

Lemma 2. *Let $A \subset B$ be a ring extension with B integral over A. Then A is a total quotient ring if and only if so is B.*

Proof. Assume that each regular element of A is invertible and let $x \in B$ be regular. By integral dependence, x satisfies an equation of minimal degree, $r_0 + r_1 x + ... + r_{n-1}x^{n-1} + x^n = 0$, with $r_i \in A$ for all i. By Lemma 1, r_0 is regular in A and thus r_0 is invertible. So multiplying $r_0 = x(-r_1 - ... - r_{n-1}x^{n-2} - x^{n-1})$ by r_0^{-1}, we get $1 = x(-r_1 - ... - r_{n-1}x^{n-2} - x^{n-1})r_0^{-1}$. It follows that x is invertible in B.

The converse is a consequence of [6, Theorem 15].

Next we generalize [5, Theorem 3.1] to rings with zero divisors.

Theorem 1. *Let $R \subseteq B$ be an extension of rings, sharing a regular common ideal I. Then R is a G-ring if and only if so is B.*

Proof. Since R and B share the ideal I, they have the same total quotient ring $\mathbf{T}$ and thus B is an overring of R. So if R is a G-ring then naturally so is B. Conversely, assume that B is a G–ring, so $\mathbf{T} = B[u^{-1}]$ for some regular element $u \in B$. By Proposition 1, some power of u is in I, say u^r, $r > 0$. We claim that $B[u^{-1}] = I[u^{-1}]$: indeed, $I[u^{-1}] \subseteq B[u^{-1}]$, so let $\theta = a_0 + a_1u^{-1} + ... + a_nu^{-n} \in B[u^{-1}]$. Thus, $u^n\theta \in B$. Since I is an ideal of B and $u^r \in I$, then $u^n\theta u^r \in I$. Hence, $\theta = (u^n\theta u^r)u^{-(n+r)} \in I[u^{-1}]$, i.e., $B[u^{-1}] = I[u^{-1}]$. It follows that $R[u^{-1}] = B[u^{-1}] = \mathbf{T}$, the common total quotient ring of both R and B.

As an immediate consequence we have:

Corollary 1. *Let B be an integral domain, I an ideal of B and D a subring of B/I. The ring $D + I$ is a G-ring if and only if so is B.*

Remark 1. If a commutative ring A is a G–ring then so is each of its overrings. But as stated in [4] it is not characterized when does A admit a subring that is a G-ring. In that context, the previous result allows us to construct subrings of a G-ring that are G-rings.

Now, let $A \subseteq B$ be an extension of rings. Following Snapper [8] we say that a regular element $x \in B$ is algebraic over A, if there exists a regular polynomial $f(X) \in A[X]$, such that $f(x) = 0$. In that case, we say that "$f(x) = 0$"is an algebraicity equation and that "x satisfies the polynomial $f(X)$".

Recall that a polynomial of $A[X]$ is said to be regular if it is a regular element of $A[X]$. On the other hand, it is well known that if $h(X) \in A[X]$ is a zero divisor then each coefficient of $h(X)$ is a zero divisor, and hence there exists $\alpha \in A\backslash\{0\}$ such that $\alpha h(X) = 0$.

Lemma 3. *Let $A \subseteq B$ be an extension of rings and let $x \in B$ be regular. If x is algebraic over A, then x satisfies an algebraicity equation $a_0 + a_1x + ... + a_{n-1}x^{n-1} + a_nx^n = 0$, such that a_i is a regular element of A, for each $i \in \{1, ..., n\}$.*

Proof. Since $x \in B$ is algebraic over A, then $f(x) = 0$ for some regular element $f(X) \in A[X]$ say, $(\star)$: $b_0 + b_1 x + ... + b_{n-1}x^{n-1} + b_n x^n = 0$, where the b_i's are elements of A, some of which are regular.

Let us write $\{0, ..., n\} = \Delta_1 \cup \Delta_2$ such that, for each $i \in \Delta_1$, b_i is regular and for each $i \in \Delta_2$, b_i is a zero divisor. Thus from $(\star)$ we deduce $\Sigma_{i\in\Delta_1} b_i x^i$= -$\Sigma_{i\in\Delta_2} b_i x^i$. Multiplying by an appropriate $\beta \in A$ we get $\beta\Sigma_{i\in\Delta_1} b_i x^i = 0$. Since each $b_i \in \Delta_1$ is regular, necessarily one has $\Sigma_{i\in\Delta_1} b_i x^i = 0$. A simplification by an appropriate power of x ends the proof.

In what follows we denote the set of units of a ring R by $\mathrm{U}(R)$.

Lemma 4. *Let $A \hookrightarrow B$ be an algebraic extension of rings and $\overline{A}$ be the integral closure of A in B. Then $\mathbf{T}(\overline{A})$and$\mathbf{T}(B)$ have the same set of units.*

Proof. Clearly $\mathrm{U}(\mathbf{T}(\overline{A})) \subseteq \mathrm{U}(\mathbf{T}(B)))$. So, let $x \in B$ be regular, it satisfies an algebraicity equation (#):$a_0 + a_1 x + ... + a_{n-1}x^{n-1} + a_n x^n = 0$, of minimal degree, with $a_i \in A$ for each $i \in \{1, ..., n\}$. By Lemma 3, for each $i = 1, ..., n$, the coefficient a_i is a regular element of A. Thus multiplying in (#) by a_n^{n-1} we get $a_0 a_n^{n-1} + a_1 a_n^{n-2}(a_n x) + ... + a_{n-1}(a_n x)^{n-1} + (a_n x)^n = 0$, that is $a_n x$ is integral over A. It follows that $a_n x \in \overline{A}$ and hence $x = a_n^{-1} a_n x \in \mathbf{T}(\overline{A})$, that is, $\mathbf{T}(B)$ and $\mathbf{T}(\overline{A})$ have the same invertible elements.

We are now ready to state our generalization of [4, Theorem 2.1] to rings with zero divisors.

Theorem 2. *Let $A \subset B$ be an extension of rings. The following assertions are equivalent:*

(i) (A, B) *is a G–ring pair;*
(ii) *A is a G–ring and each regular element of B is algebraic over A.*

Proof. (i)⇒ (ii). Let $\mathbf{T}(A)$ denotes the total ring of quotients of A and assume that (A, B) is a G–ring pair. If some regular element $\alpha \in B$ is not algebraic over A. Then, α is transcendental over A and hence $A[\alpha]$ is isomorphic to $A[X]$ and hence, $A[\alpha]$ is not a G–ring. A contradiction, Since $A \subset A[\alpha] \subset B$.

(ii)⇒ (i). Assume A is a G–ring and that any regular element of B is algebraic over A. We first show that B is a G–ring. For, let $\overline{A}$ denotes the integral closure of A in B. Since any total ring of quotients is a G-ring, we can assume that A in not a total quotient ring (cf. Lemma 2). Hence some regular element $\omega \in A$ lies in each regular prime ideal of A (cf. Proposition 1).

Now, let $Q \in \mathrm{Spec}(\overline{A})$ be regular, then it contains a regular element θ. Moreover, θ satisfies an integral dependence equation of minimal degree $b_0 + b_1\theta + ... + b_{n-1}\theta^{n-1} + \theta^n = 0$, with $b_i \in A$ for all i. By minimality of the degree, b_0 is regular (cf. Lemma 1), and since $b_0 = \theta(-b_1 - ... - b_{n-1}\theta^{n-2} - \theta^{n-1}) \in \theta\overline{A} \subseteq Q$, then $b_0 \in Q \cap A$. That is $Q \cap A$ is a regular prime ideal of A. Hence, it contains ω.

We showed that each regular prime ideal of $\overline{A}$ contains ω, thus $\mathbf{T}(\overline{A}) = \overline{A}[\omega^{-1}]$. That is $\overline{A}$ is a G–ring. By Lemma 4, we have $\mathrm{U}(\mathbf{T}(\overline{A})) = \mathrm{U}(\mathbf{T}(B))$ that is,

$$\mathrm{U}(B[\omega^{-1}]) \subseteq \mathrm{U}(\mathbf{T}(B)) = \mathrm{U}(\overline{A}[\omega^{-1}]) \subseteq \mathrm{U}(B[\omega^{-1}]).$$

It follows that $\mathbf{T}(B) = B[\omega^{-1}]$ and hence B is a G–ring.

Let A be a ring and I an ideal of A. The ring $A \bowtie I := \{(a, a + j) \mid a \in A, \text{ and } j \in I\}$ was introduced and studied in [3] as *the amalgamated duplication of A along I*. Next we characterize G–ring pairs issued from these constructions.

Proposition 2. *Let A be a ring and $I \subseteq J$ two regular ideals of A. Then $(A \bowtie I, A \bowtie J)$ is a G–ring pair if, and only if, A is a G–ring.*

Proof. It is a consequence of [7, Lemma 3.2.1] and [5, Corollary 4.1].

Proposition 3. *Let $A \subseteq B$ be a ring extension, with a common regular ideal J. Then, $(A \bowtie J, B \bowtie J)$ is a G–ring pair if, and only if, (A, B) is a G–ring pair.*

Proof. Assume (A, B) is a G–ring pair. So, by [5, Corollary 4.1], $A \bowtie J$ and $B \bowtie J$ are G–rings. Let R be a ring such that $A \bowtie J \subseteq R \subseteq B \bowtie J$. By [7, Lemma 3.2.13], $R = S \bowtie J$ for some intermediate ring $A \subseteq S \subseteq B$. As (A, B) is G–ring pair, so S is a G–ring and again by [5, Corollary 4.1], R is a G–ring. Conversely, If $(A \bowtie J, B \bowtie J)$ is a G–ring pair, then, by [5, Corollary 4.1], A and B are G–rings. Now, Let S be a ring such that $A \subseteq S \subseteq B$, then $A \bowtie J \subseteq S \bowtie J \subseteq B \bowtie J$. Hence, $S \bowtie J$ is a G–ring and then S is a G–ring (cf. [5, Corollary 4.1]).

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Weakly Finite Conductor Property in Amalgamated Algebra

Haitham El Alaoui

Abstract Let $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal of B. In this paper, we investigate the transfer of weakly finite conductor property in amalgamation of A with B along J with respect to f (denoted by $A\bowtie^f J$), introduced and studied by D'Anna, Finocchiaro and Fontana in 2009 (see D'Anna et al. (Commutative Algebra and Applications. Walter De Gruyter Publisher, Berlin, pp. 55–172, 2009), D'Anna et al. (J Pure Appl Algebra 214:1633–1641, 2010)). Our results generate original examples which enrich the current literature with new families of examples of nonfinite conductor weakly finite conductor rings.

Keywords Weakly finite conductor · Finite conductor ring · Coherent ring
Amalgamated duplication · Amalgamated algebra

1 Introduction

All rings considered in this paper are assumed to be commutative, and have identity element and all modules are unitary.

Let A and B be two rings and J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of A×B:

$$A \bowtie^f J = \{(a, f(a) + j) /a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [4, 8]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [5–7] and denoted by $A \bowtie I$). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions)

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can be studied as particular cases of the amalgamation ([4, Examples 2.5 and 2.6]) and other classical constructions, such as Nagata's idealizations (cf. [15, page 2]), and the CPI extensions (in the sense of Boisen and Sheldon [3]) are strictly related to it ([4, Example 2.7 and Remark 2.8]). On the other hand, the amalgamation is related to a construction proposed by Anderson in [1] and motivated by a classical construction due to Dorroh [10], concerning the embedding of a ring without identity in a ring with identity. In [4], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation.

Let R be a commutative ring. For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules:

$$F_n \longrightarrow F_{n-1} \longrightarrow \dots F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-module are respectively, finitely generated and finitely presented R-module.

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent ring are Noetherian ring, Boolean algebras, von Neumann regular rings, and prüfer/semi-hereditary rings. For instance see [12].

An ideal I of R is called n-generated ideal if I can be generated by n-elements. Glaz (2000) extended the definition of a finite conductor domains to rings with zero divisors. A ring R is called finite conductor if $Ra \cap Rb$ and $(0 : c)$ are finitely generated ideals of R for all elements a, b and c of R (see [2, 13, 17]). Also, Glaz shows that R is a finite conductor ring if and only if each 2-generated ideal of R is finitely presented ([13, Proposition 2.1]). We say that R is a weakly finite conductor ring if $Ra \cap Rb$ is a finitely generated ideal of R for each pair $a, b \in R$. Hence, if R is a domain, then R is finite conductor if and only if R is a weakly finite conductor. For instance, any coherent ring is a finite conductor ring and so it is weakly finite conductor.

2 Main Result

In this paper, we characterize $A \bowtie^f J$ to be weakly finite conductor ring for some classes of ideals J and homomorphisms f. Thereby, new examples are provided which particularly, enriches the current literature with new classes of nonfinite conductor weakly finite conductor rings.

Then, before announcing the main result, we recall by the following remark.

Remark 1. Let $f : A \longrightarrow B$ be a ring homomorphism and M be an B-module. Then M is a module over A, via f. Precisely, $a.m = f(a)m$ for each $a \in A$ and $m \in M$.

Theorem 1. *Let $f : A \longrightarrow B$ be a ring homomorphism and J be a proper ideal of B.*

1. *If $A \bowtie^f J$ is a weakly finite conductor ring, then so is A.*
2. *Assume that A is a local ring with maximal ideal M such that $MJ = 0$.*
    a. *If $Ma \cap Mb$ and $Jk \cap Jl$ are finitely generated A-modules, for all $a, b \in M$ and $k, l \in J$ and $J \subseteq Rad(B)$, then $A \bowtie^f J$ is a weakly finite conductor ring.*
    b. *If A is a domain and $J^2 = 0$, then $A \bowtie^f J$ is a weakly finite conductor ring if and only if so is A, and M and J are finitely generated A-modules.*

Before proving main result, we establish the following Lemmas.

Lemma 1. *Let $f : A \longrightarrow B$ be a ring homomorphism and J be a proper ideal of B. If $A \bowtie^f J(a, f(a)) \cap A \bowtie^f J(b, f(b))$ is a finitely generated ideal of $A \bowtie^f J$ for all $a, b \in A$, then so is $Aa \cap Ab$.*

Proof. Let $a, b \in A$. Our aim is to show that $Aa \cap Ab$ is a finitely generated ideal of A. If $A \bowtie^f J(a, f(a)) \cap A \bowtie^f J(b, f(b)) = \sum_{i=1}^{i=n} A \bowtie^f J(a_i, f(a_i) + e_i)$, where $a_i \in A$, $e_i \in J$ and n is a positive integer, then $Aa \cap Ab = \sum_{i=1}^{i=n} Aa_i$. Indeed, let $x \in Aa \cap Ab$ so there exists $\alpha, \beta \in A$ such that $x = \alpha a = \beta b$. Then:

$$\begin{aligned}(x, f(x)) &= (\alpha a, f(\alpha)f(a)) = (\beta b, f(\beta)f(b)) \\ &= (\alpha, f(\alpha))(a, f(a)) = (\beta, f(\beta))(b, f(b)).\end{aligned}$$

Hence, $(x, f(x)) = \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i) + j_i)(a_i, f(a_i) + e_i) = (\sum_{i=1}^{i=n} \alpha_i a_i, \sum_{i=1}^{i=n}(f(\alpha_i) + j_i)(f(a_i) + e_i)$, where $(\alpha_i, f(\alpha_i) + j_i)_{i=1}^{i=n} \in (A \bowtie^f J)^n$. Therefore, $x = \sum_{i=1}^{i=n} \alpha_i a_i \in \sum_{i=1}^{i=n} Aa_i$. On the other hand, the $(a_i, f(a_i) + e_i) \in A \bowtie^f J(a, f(a)) \cap A \bowtie^f J(b, f(b))$, so there exists $(\beta_j, f(\beta_j) + k_j), (\gamma_j, f(\gamma_j) + l_j) \in A \bowtie^f J$ such that $(a_i, f(a_i) + e_i) = (\beta_j, f(\beta_j) + k_j)(a, f(a)) = (\gamma_j, f(\gamma_j) + l_j)(b, f(b))$ with $j \in \{1, \dots, n\}$, then $a_i = \beta_j a = \gamma_j b$. Therefore, $a_i \in Aa \cap Ab$ for all $i \in \{1, \dots, n\}$. Hence, $Aa \cap Ab = \sum_{i=1}^{i=n} Aa_i$. □

Lemma 2. *Let $f : A \longrightarrow B$ be a ring homomorphism, and J be an ideal proper of B. Let I and K be two ideals of A and B respectively such that $K \subseteq J$.*

1. *Assume that $IJ \subseteq K$. Then:*
    a. *$I \bowtie^f K = \{(i, f(i) + k)/i \in I, k \in K\}$ is an ideal of $A \bowtie^f J$.*
    b. *If I and K are finitely generated A-modules. Then, $I \bowtie^f K$ is a finitely generated ideal of $A \bowtie^f J$.*
2. *Assume that $IJ = 0$ and $J^2 = 0$. Then, $I \bowtie^f K$ is a finitely generated ideal of $A \bowtie^f J$ if and only if I and K are finitely generated A-modules.*

Proof. 1. Assume that $IJ \subseteq K$.

a. It is clear that $I \bowtie^f K$ is an ideal of $A \bowtie^f J$. Indeed:
- $(i, f(i)+k)+(i', f(i')+k') = (i+i', f(i+i')+k+k') \in I \bowtie^f K$ for all $(i, f(i)+k), (i', f(i')+k') \in I \bowtie^f K$.
- $(a, f(a)+j)(i, f(i)+k) = (ai, f(ai)+jf(i)+kf(a)+kj) = (ai, f(ai)+ij+ak+kj)$ by Remark 1, so $(a, f(a)+j)(i, f(i)+k) \in I \bowtie^f K$ for all $(a, f(a)+j) \in A \bowtie^f J$ and $(i, f(i)+k) \in I \bowtie^f K$, since $IJ \subseteq K$.

b. Assume that $I := \sum_{i=1}^{i=n} Au_i$ is a finitely generated ideal of A, where $u_i \in I$ for all $i \in \{1, \ldots, n\}$ and $K = \sum_{i=1}^{i=m} Ae_i$, where $e_i \in K$ for all $i \in \{1, \ldots, m\}$. Let $(x, f(x)+k) \in I \bowtie^f K$, where $x \in I$ and $k \in K$, so there exists $(\alpha_i)_{i=1}^{i=n} \in A^n$ and $(\beta_i)_{i=1}^{i=m} \in A^m$ such that $x = \sum_{i=1}^{i=n} \alpha_i u_i$ and $k = \sum_{i=1}^{i=m} \beta_i e_i = \sum_{i=1}^{i=m} f(\beta_i)e_i$ by Remark 1. So, we obtain:

$$\begin{aligned}(x, f(x)+k) &= (\sum_{i=1}^{i=n} \alpha_i u_i, \sum_{i=1}^{i=n} f(\alpha_i)f(u_i) + \sum_{i=1}^{i=m} f(\beta_i)e_i) \\ &= (\sum_{i=1}^{i=n} \alpha_i u_i, \sum_{i=1}^{i=n} f(\alpha_i)f(u_i)) + (0, \sum_{i=1}^{i=m} f(\beta_i)e_i) \\ &= \sum_{i=1}^{i=n} (\alpha_i, f(\alpha_i))(u_i, f(u_i)) + \sum_{i=1}^{i=m} (\beta_i, f(\beta_i))(0, e_i).\end{aligned}$$

Consequently, $(x, f(x)+k) \in \sum_{i=1}^{i=n} A \bowtie^f J(u_i, f(u_i)) + \sum_{i=1}^{i=m} A \bowtie^f J(0, e_i)$ since $(\alpha_i, f(\alpha_i)) \in A \bowtie^f J$ for all $i \in \{1, \ldots, n\}$ and $(\beta_i, f(\beta_i)) \in A \bowtie^f J$ for all $i \in \{1, \ldots, m\}$. Therefore, $I \bowtie^f K \subseteq \sum_{i=1}^{i=n} A \bowtie^f J(u_i, f(u_i)) + \sum_{i=1}^{i=m} A \bowtie^f J(0, e_i)$. Conversely, $\sum_{i=1}^{i=n} A \bowtie^f J(u_i, f(u_i)) + \sum_{i=1}^{i=m} A \bowtie^f J(0, e_i) \subseteq I \bowtie^f K$ since $(u_i, f(u_i)) \in A \bowtie^f J$ for all $i \in \{1, \ldots, n\}$, $(0, e_i) \in A \bowtie^f J$ for all $i \in \{1, \ldots, m\}$ and $I \bowtie^f K$ is a ideal of $A \bowtie^f J$. Hence, $I \bowtie^f K = \sum_{i=1}^{i=n} A \bowtie^f J(u_i, f(u_i)) + \sum_{i=1}^{i=m} A \bowtie^f J(0, e_i)$ is a finitely generated ideal of $A \bowtie^f J$.

2. Let $I \bowtie^f K$ is a finitely generated ideal of $A \bowtie^f J$, i.e. $I \bowtie^f K = \sum_{i=1}^{i=r} A \bowtie^f J(u_i, f(u_i)+e_i)$ where $u_i \in I$ and $e_i \in K$ for all $i \in \{1, \ldots, r\}$. Let $x \in I$ and $k \in K$, so $(x, f(x)+k) \in I \bowtie^f K$. Then, there exists $(\alpha_i, f(\alpha_i)+j_i)_{i=1}^{i=r} \in (A \bowtie^f J)^r$ such that $(x, f(x)+k) = \sum_{i=1}^{i=r}(\alpha_i, f(\alpha_i)+j_i)(u_i, f(u_i)+e_i)$= $(\sum_{i=1}^{i=r} \alpha_i u_i, \sum_{i=1}^{i=r}(f(\alpha_i)+j_i)(f(u_i)+e_i)$. Therefore, $x = \sum_{i=1}^{i=r} \alpha_i u_i$, hence I is a finitely generated ideal of A. On the other hand, let $k \in K$. So, $(0, k) \in I \bowtie^f K$ i.e., there exists $(\beta_i, f(\beta_i)+k_i)_{i=1}^{i=r} \in (A \bowtie^f J)^r$ such that $(0, k) = \sum_{i=1}^{i=r}(\beta_i, f(\beta_i)+k_i)(u_i, f(u_i)+e_i) = (\sum_{i=1}^{i=r} \beta_i u_i, \sum_{i=1}^{i=r} f(\beta_i)f(u_i) + \sum_{i=1}^{i=r} f(u_i)k_i + \sum_{i=1}^{i=r}(f(\beta_i)+k_i)e_i$. Then, $\sum_{i=1}^{i=r} \beta_i u_i = 0$ and $k = \sum_{i=1}^{i=r} f(u_i)k_i + \sum_{i=1}^{i=r}(f(\beta_i)+k_i)e_i$. Moreover, we have $u_i \in I$ for all $i \in \{1, \ldots, r\}$, so $f(u_i)k_i = 0$ for all $i \in \{1, \ldots, r\}$, since $(IJ = 0)$. Hence, $K = \sum_{i=1}^{i=r} f(\beta_i)e_i$

because $J^2 = 0$. Therefore, K is a finitely generated A-module, by Remark 1. Conversely, let I and K are finitley generated A-modules. So, $I \bowtie^f K$ is a finitely generated ideal of $A \bowtie^f J$, by (b) as desired.

□

Lemma 3. *Let A be a local ring with maximal ideal M, $f : A \longrightarrow B$ be a ring homomorphism and J be a proper ideal of B such that $J \subseteq Rad(B)$. Then, $U(A \bowtie^f J) = U(A) \bowtie^f J$.*

Proof. By [8, Proposition 2.6 (5)], $Max(A \bowtie^f J) = \{m \bowtie^f J / m \in Max(A)\} \cup \{\overline{Q}^f\}$ with $Q \in Max(B)$ not containing $V(J)$ and $\overline{Q}^f = \{(a, f(a) + j) \in A / j \in J, f(a) + j \in Q\}$, and since $J \subseteq Rad(B)$ then $J \subseteq Q$ for all $Q \in Max(B)$. Hence, $Max(A \bowtie^f J) = \{m \bowtie^f J / m \in Max(A)\} = M \bowtie^f J$ since A is a local ring. Therefore, $A \bowtie^f J$ is a local ring with maximal ideal $M \bowtie^f J$. Thus, $U(A \bowtie^f J) = (A \bowtie^f J) - (M \bowtie^f J) = (A - M) \bowtie^f J = U(A) \bowtie^f J$ as desired. □

Lemma 4. *Let A be a local ring with maximal ideal M, $f : A \longrightarrow B$ be a ring homomorphism, and J be an ideal proper of B such that $MJ = 0$ and $J \subseteq Rad(B)$. And let $aM \cap bM$ and $lJ \cap kJ$ are finitely generated A-modules, for all $a, b \in M$ and $k, l \in J$. Then, $A \bowtie^f J(a, f(a) + k) \cap A \bowtie^f J(b, f(b) + l)$ is a finitely generated ideal of $A \bowtie^f J$.*

Proof. If $A \bowtie^f J(a, f(a) + k) \subseteq A \bowtie^f J(b, f(b) + l)$ or $A \bowtie^f J(b, f(b) + l) \subseteq A \bowtie^f J(a, f(a) + k)$, nothing to demonstrate. Otherwise, $A \bowtie^f J(a, f(a) + k) \cap A \bowtie^f J(b, f(b) + l) \neq A \bowtie^f J(a, f(a) + k)$ (for example). Let $(x, f(x) + e) \in A \bowtie^f J(a, f(a) + k) \cap A \bowtie^f J(b, f(b) + l)$ i.e. there exists $\alpha, \beta \in A$ and $j_1, j_2 \in J$ such that,

$$\begin{aligned}(x, f(x) + e) &= (\alpha, f(\alpha) + j_1)(a, f(a) + k) = (\beta, f(\beta) + j_2)(b, f(b) + l) \\ &= (\alpha a, f(\alpha a) + f(\alpha)k + aj_1 + kj_1) = (\beta b, f(\beta b) + f(\beta)k + bj_2 + lj_2).\end{aligned}$$

It is clear that $\alpha \in M$. Otherwise, α invertible in A, then $(\alpha, f(\alpha) + j_1)$ is invertible in $A \bowtie^f J$ by Lemma 3. Therefore, $(a, f(a) + k) = (\alpha, f(\alpha) + j_1)^{-1}(x, f(x) + e)$ so $(a, f(a) + k) \in K$ then $A \bowtie^f J(a, f(a) + k) \subseteq K$. A contradiction, by symmetry $\beta \in M$ hence $(x, f(x) + e) = (\alpha a, f(\alpha a) + kj_1) = (\beta b, f(\beta b) + lj_2)$ (because $MJ = 0$). Therefore, $A \bowtie^f J(a, f(a) + k) \cap A \bowtie^f J(b, f(b) + l) \subseteq (Ma \cap Mb) \bowtie^f (Jk \cap Jl)$. On the other hand, let $y = (m_1 a, f(m_1 a) + j_1 k) = (m_2 b, f(m_2 b) + j_2 l) \in (Ma \cap Mb) \bowtie^f (Jk \cap Jl)$, where $m_1, m_2 \in M$ and $j_1, j_2 \in J$. Then:

$$\begin{aligned}y &= (m_1, f(m_1) + j_1)(a, f(a) + k) \in A \bowtie^f J(a, f(a) + k) \\ &= (m_2, f(m_2) + j_2)(b, f(b) + l) \in A \bowtie^f J(b, f(b) + l).\end{aligned}$$

So, $A \bowtie^f J(a, f(a) + k) \cap A \bowtie^f J(b, f(b) + l) = (Ma \cap Mb) \bowtie^f (Jk \cap Jl)$. Therefore, $A \bowtie^f J(a, f(a) + k) \cap A \bowtie^f J(b, f(b) + l)$ is a finitely generated ideal of $A \bowtie^f J$ by Lemma 2 (1) (b). □

Lemma 5. *Let A be a local ring with maximal ideal M, contains a regular elements a, $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal proper of B such that $MJ = 0$ and $J^2 = 0$. Then, $A \bowtie^f J(a, f(a) + k) \cap A \bowtie^f J(a, f(a) + l) = M \bowtie^f J(a, f(a))$ for all $k \neq l \in J$.*

Proof. Let $l \neq k \in J$ and $(\alpha, f(\alpha) + j)(a, f(a) + k) = (\beta, f(\beta) + e)(a, f(a) + l) \in A \bowtie^f J(a, f(a) + k) \cap A \bowtie^f J(a, f(a) + l)$, where $(\alpha, f(\alpha) + j)$, $(\beta, f(\beta) + e) \in A \bowtie^f J$. Then, $\alpha a = \beta b$ and $\alpha k = f(\alpha)k = f(\beta)l = \beta l$, because $a \in M$ and $MJ = 0$, so $\bar{\alpha}k = \alpha k = \beta l = \bar{\beta}l$ since J is a A/M-vector space. Therefore, $\alpha = \beta$ since a is a regular element, so $\bar{\alpha}(k - l) = 0$, hence $\alpha \in M$ since $k - l \neq 0$ and J is an A/M-vector space. Therefore, $(\alpha, f(\alpha) + j)(a, f(a) + k) = (\alpha, f(\alpha) + j)(a, f(a)) \in M \bowtie^f J(a, f(a))$. Conversely, let $(m, f(m) + e)(a, f(a)) \in M \bowtie^f J(a, f(a))$ where $m \in M$ and $e \in J$. Clearly, $(m, f(m) + e)(a, f(a)) = (m, f(m) + e)(a, f(a) + k) = (m, f(m) + e)(a, f(a) + l)$ since $m \in M$, so $(ma, f(ma)) \in A \bowtie^f J(a, f(a) + k) \cap A \bowtie^f J(a, f(a) + l)$. □

Proof of Theorem 1.

1. By Lemma 1.
2. Assume that A is a local ring with maximal ideal M such that $MJ = 0$.

   a. By Lemma 4.
   b. If $A \bowtie^f J$ is a weakly finite conductor ring, then $M \bowtie^f J$ is a finitely generated ideal of $A \bowtie^f J$ by Lemma 5. Since A is a domain. Otherwise, $M \bowtie^f J(a, f(a))$ is not finitely generated ideal of $A \bowtie^f J$, a contradiction. Hence, M and J are finitely generated A-modules by Lemma 2 (2). Conversely, Let $I = A \bowtie^f J(a, f(a) + e)$ and $K = A \bowtie^f J(b, f(b) + j)$ be two proper ideals of $A \bowtie^f J$ i.e. $a, b \in M$ by Lemma 3. Our aim is to show that $I \cap K$ is a finitely generated ideal of $A \bowtie^f J$. Let $x \in I \cap K$. Three cases are then possible.
   Case 1: If $a = b = 0$ in that case $x = (\alpha, f(\alpha) + u)(0, e) = (\beta, f(\beta) + v)(0, j)$ where $(\alpha, f(\alpha) + u)$ and $(\beta, f(\beta) + v) \in A \bowtie^f J$, so $\alpha e = f(\alpha) e = f(\beta)j = \beta j$ and since J is an (A/M)-vector space then $\bar{\alpha}e = \alpha e = \beta j = \bar{\beta}j$. So, two cases are then possible:
   If $\{e, j\}$ are linearly independent, then $\bar{\alpha} = \bar{\beta} = 0$ i.e. $\alpha, \beta \in M$ hence $x = (0, 0)$. Therefore, $I \cap K = 0$, thus a finitely generated ideal of $A \bowtie^f J$.
   If $\{e, j\}$ are linearly dependent, then there exist $\omega \in A$ such that $e = \bar{\omega}j$ so $(0, e) = (0, \omega j) = (\omega, f(\omega))(0, j) \in A \bowtie^f J(0, j)$, then $I \subseteq K$ hence $I \cap K = I$. Therefore, $I \cap K$ is a finitely generated ideal of $A \bowtie^f J$ in this cases.
   Case 2: a and b are comparable. Assume for example that $a = cb$, where $c \in A$. Two cases are then possible:
   If $c \in M$, we claim that $I \cap K$ is a finitely generated ideal of $A \bowtie^f J$. Indeed, let $(\alpha, f(\alpha) + u)(a, f(a) + e) = (\beta, f(\beta) + v)(b, f(b) + j) \in I \cap K$, where $(\alpha, f(\alpha) + u)$, $(\beta, f(\beta) + v) \in A \bowtie^f J$. Then, $\alpha a = \beta b = \alpha cb$ and $\bar{\alpha}e = \bar{\beta}j$ since $a, b \in M$. But, $\beta b = \alpha cb$ implies $\beta = \alpha c \in M$

since A is a domain; so $\bar{\alpha}e = \bar{\beta}j = 0$. Two cases are then possible: $e = 0$ or $e \neq 0$.
Assume that $e = 0$, then $I \cap K = A \bowtie^f J(a, f(a)) \cap A \bowtie^f J(b, f(b) + j) = A \bowtie^f J(a, f(a))$. Indeed, let $y \in A \bowtie^f J(a, f(a))$ so there exist $(\lambda, f(\lambda) + g) \in A \bowtie^f J$ such that, $y = (\lambda, f(\lambda) + g)(a, f(a)) = (\lambda, f(\lambda) + g)(bc, f(bc)) = (\lambda, f(\lambda) + g)(c, f(c))(b, f(b)) = (bc, f(bc))$ $(\lambda c, f(\lambda c))(b, f(b) + j)$, so $A \bowtie^f J(a, f(a)) \subseteq A \bowtie^f J(b, f(b) + j)$, hence $I \cap K = A \bowtie^f J(a, f(a))$.
Assume that $e \neq 0$. Hence, $\alpha \in M$ since $\bar{\alpha}e = 0$ and so $I \cap K \subseteq M \bowtie^f J(a, f(a))$. Conversely, let $y = (m, f(m) + g)(a, f(a)) \in M \bowtie^f J(a, f(a))$, with $m \in M$ and $g \in J$, so $y = (m, f(m) + g)(a, f(a)) = (m, f(m) + g)(a, f(a) + e)$ because $MJ = 0$ and $J^2 = 0$, then $y \in A \bowtie^f J(a, f(a) + e)$. On the other hand, $a = bc$ so $y = (m, f(m) + g)(bc, f(bc)) = (m, f(m) + g)(c, f(c))(b, f(b)) = (mc, f(mc))(b, f(b))$ since $(c \in M)$, hence $y = (mc, f(mc))(b, f(b) + j)$, because $MJ = 0$, therefore $I \cap K = M \bowtie^f J(a, f(a))$, and since M and J are finitely generated A-modules then $I \cap K$ is a finitely generated ideal of $A \bowtie^f J$.
If $c \notin M$, then c is invertible in A, so $A \bowtie^f J(b, f(b) + j) = A \bowtie^f J(ac^{-1}, f(ac^{-1}) + j) = A \bowtie^f J(ac^{-1}, (f(a) + jf(c))f(c^{-1})) = A \bowtie^f J(c^{-1}, f(c^{-1}))A \bowtie^f J(a, f(a) + f(c)j) = A \bowtie^f J(a, f(a) + cj)$, because $(c^{-1}, f(c^{-1}))$ is invertible in $A \bowtie^f J$ by Lemma 3. Therefore, $I \cap K = A \bowtie^f J(a, f(a) + e) \cap A \bowtie^f J(a, f(a) + cj)$. If $e \neq cj$, then $I \cap K = M \bowtie^f J(a, f(a))$ by Lemma 5, and since M and J are finitely generated A-modules, so $M \bowtie^f J$ is a finitely generated ideal in $A \bowtie^f J$ by Lemma 2 (1)(b); hence $I \cap K$ is a finitely generated ideal of $A \bowtie^f J$. Otherwise $I \cap K = K$ which is finitely generated ideal of $A \bowtie^f J$.
Case 3: a and b are not comparable. We claim that $I \cap K$ is a finitely generated ideal of $A \bowtie^f J$. Indeed, let $x \in I \cap K = A \bowtie^f J(a, f(a) + e) \cap A \bowtie^f J(b, f(b) + j)$, then there exist $(\alpha, f(\alpha) + k)$ and $(\beta, f(\beta) + l) \in A \bowtie^f J$ such that, $x = (\alpha, f(\alpha) + k)(a, f(a) + e) = (\beta, f(\beta) + l)(b, f(b) + j)$, so $\alpha a = \beta b$, hence $\alpha a \in Aa \cap Ab$ and since A is a weakly finite conductor ring, then $Aa \cap Ab = \sum_{i=1}^{i=n} Aa_i$ where $a_i \in Aa \cap Ab$ for all $i \in \{1, \ldots, n\}$. So, there exist c_i and $d_i \in A$ such that $a_i = c_i a = d_i d$, moreover $c_i \in M$ for all $i \in \{1, \ldots, n\}$. Otherwise, there exist $j \in \{1, \ldots, n\}$ such that $c_j \notin M$, then c_j invertible in A, so $Aa_j = Ac_j = Aa$, and since $a_j \in Aa \cap Ab$, then $Aa_j \subseteq Aa \cap Ab$, so $Aa = Aa_j \subseteq Aa \cap Ab \subseteq Ab$, contradiction (because a and b are not comparable). Therefore, $c_i \in M$ for all $i \in \{1, \ldots, n\}$, and by symmetry $d_i \in M$ for all $i \in \{1, \ldots, n\}$. On the other hand, $\alpha a \in Aa \cap Ab = \sum_{i=1}^{i=n} Aa_i a = a \sum_{i=1}^{i=n} Ac_i$, so $\alpha \in \sum_{i=1}^{i=n} Ac_i$ i.e. there exists $(\alpha_i)_{i=1}^{i=n} \in A^n$ such that $\alpha = \sum_{i=1}^{i=n} \alpha_i c_i$, and since the $c_i \in M$, so:

$$x = (\alpha, f(\alpha)+k)(a, f(a)+e) = (\alpha a, f(\alpha)f(a)) = (a\sum_{i=1}^{i=n}\alpha_i c_i, f(a)\sum_{i=1}^{i=n} f(\alpha_i)f(c_i))$$
$$= \sum_{i=1}^{i=n}(a\alpha_i c_i, f(a)f(\alpha_i f(c_i)) = \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i))(ac_i, f(a)f(c_i))$$
$$= \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i))(a_i, f(a_i)) \in \sum_{i=1}^{i=n} A \bowtie^f J(a_i, f(a_i)).$$

Therefore, $I \cap K \subseteq \sum_{i=1}^{i=n} A \bowtie^f J(a_i, f(a_i))$. Conversely, let $y = \sum_{i=1}^{i=n}$ $(\alpha_i, f(\alpha_i) + g_i)(a_i, f(a_i)) \in \sum_{i=1}^{i=n} A \bowtie^f J(a_i, f(a_i))$, where $(\alpha_i, f(\alpha_i)$ $+ g_i) \in A \bowtie^f J$ for all $i \in \{1, \dots, n\}$, then $y = \sum_{i=1}^{i=n}(\alpha_i a_i, f(\alpha_i)f(a_i))$ (because $a_i \in Aa \cap Ab \subseteq M$), so $\sum_{i=1}^{i=n}(\alpha_i c_i a, f(\alpha_i)f(c_i)f(a)) = \sum_{i=1}^{i=n}$ $(\alpha_i c_i, f(\alpha_i)f(c_i))(a, f(a)+e)$ (since the $c_i \in M$), therefore $\sum_{i=1}^{i=n}(\alpha_i d_i b,$ $f(\alpha_i)f(d_i)f(b)) = \sum_{i=1}^{i=n}(\alpha_i d_i, f(\alpha_i)f(d_i))(b, f(b)+j)$. Hence, $I \cap K$ $= \sum_{i=1}^{i=n} A \bowtie^f J(a_i, f(a_i))$ which is a finitely generated ideal of $A \bowtie^f J$. Therefore, $I \cap K$ is a finitely generated ideal of $A \bowtie^f J$ in all cases. So, $A \bowtie^f J$ is a weakly finite conductor ring.

□

The following Corollaries are an immediate consequence of Theorem 1.

Corollary 1. *Let A be a local domain with maximal ideal M and let $B = A/M^2$ and $J = M/M^2$. Consider the canonical homomorphism $f : A \longrightarrow B$ ($f(x) = \bar{x}$). Then, $A \bowtie^f J$ is a weakly finite conductor ring if and only if so is A, and M is a finitely generated ideal of A.*

Corollary 2. *Let $f : A \longrightarrow B$ be a ring homomorphism and J be a proper ideal of B.*

1. *If $A \bowtie^f J$ is weakly finite conductor, then so is A.*
2. *Assume that A is a local ring with maximal ideal M such that $MJ = 0$ and $J^2 = 0$.*
   a. *If M is principal and A weakly finite conductor, then $A \bowtie^f J$ is weakly finite conductor.*
   b. *If A is a domain and $M^2 = 0$, then $A \bowtie^f J$ is a weakly finite conductor ring if and only if M and J are finitely generated A-modules.*

Proof. 1. By Lemma 1.
2. Assume that A is a local ring with maximal ideal M such that $MJ = 0$ and $J^2 = 0$.
   a. Clear.
   b. By [14, Example 2.4] and Theorem 1 (b).

□

The next Corollary examines the case of the amalgamated duplication.

Corollary 3. *Let A be a ring and I be a proper ideal of A.*

1. *If $A \bowtie I$ is a weakly finite conductor ring, then so is A.*
2. *Assume that A is a local ring with maximal ideal M such that $MI = 0$.*
   a. *If $Ma \cap Mb$ is a finitely generated A-module, for all $a, b \in M$, then $A \bowtie I$ is a weakly finite conductor ring.*
   b. *If A is a domain, then $A \bowtie I$ is a weakly finite conductor ring if and only if so is A, and M and I are finitely generated A-modules.*

Now, we give examples of a weakly finite conductor ring which is not a finite conductor ring and so not a coherent ring.

Example 1. Let A be a local ring with principal maximal ideal M (for instance $A = \mathbb{Z}/8\mathbb{Z}$ and $M = 2\mathbb{Z}/8\mathbb{Z}$) and let E be a A/M-vector space with infinite rank, $B = A \propto E$ and $J = 0 \propto E$. Consider the ring homomorphism $f : A \longrightarrow B$ $(f(a) = (a, 0))$. Then:

1. $A \bowtie^f J$ is a weakly finite conductor ring.
2. $A \bowtie^f J$ is not a finite conductor ring.

Proof. 1. A is a weakly finite conductor ring by [12, Theorem 2.4.1(1)]. So, $A \bowtie^f J$ is weakly finite conductor by Corollary 2 (2) (a).

2. Let $c = (a, f(a)) \in A \bowtie^f J$, where $a \neq 0 \in M$ (if $a \notin M$ then c is invertible, and so $(0 : c) = 0$). Then, $(0 : c) = (0 : a) \bowtie^f J$ is not a finitely generated ideal of $A \bowtie^f J$ by Lemma 2 (2) since E is a A/M-vector space with infinite rank. Therefore, $A \bowtie^f J$ is not a finite conductor ring.

□

Example 2. Let $A = K$ where K is a field and E be a K-vector space with infinite rank and let $B = K \propto E$ and $J = 0 \propto E$. Consider the ring homomorphism $f : A \longrightarrow B$ $(f(a) = (a, 0))$. Then:

1. $K \bowtie^f J$ is a weakly finite conductor ring.
2. $K \bowtie^f J$ is not a finite conductor ring.

Proof. 1. We claim that $K \bowtie^f J$ is a weakly finite conductor ring. Indeed, let $I = K \bowtie^f J(0, j) = 0 \bowtie^f Kj$ and $L = K \bowtie^f J(0, g) = 0 \bowtie^f Kg$ be two principal proper ideals of $K \bowtie^f J$, where $j, g \in J - \{0\}$. Hence, $I \cap L = 0 \bowtie^f (Kj \cap Kg)$. But, $Kj \cap Kg$ is a K-vector space of rang at most 1, so $Kj \cap Kg = Kh$, where $h \in J$. Therefore, $I \cap L = 0 \bowtie^f (Kj \cap Kg) = 0 \bowtie^f Kh = K \bowtie^f J(0, h)$ is a finitely generated ideal of $K \bowtie^f J$ and so $K \bowtie^f J$ is a weakly finite conductor ring.

2. Our aim is to show that $K \bowtie^f J$ is not a finite conductor ring. Let $e \in J - \{0\}$ and let $I = K \bowtie^f J(0, e)$ be an ideal of $K \bowtie^f J$. It suffices to show that I is not finitely presented. Consider the exact sequence of $K \bowtie^f J$-modules:

$$0 \longrightarrow Ker(U) \longrightarrow K \bowtie^f J \xrightarrow{U} I \longrightarrow 0$$

where $U(b, f(b) + j) = (b, f(b) + j)(0, e) = (0, f(b)e) = (0, be)$. Clearly, $Ker(U) = 0 \bowtie^f J$ which is not finitely generated by Lemma 2 (2) since E is a K-vector space with infinite rank. Therefore, I is not finitely presented and so $K \bowtie^f J$ is not a finite conductor ring.

□

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Coherence in Bi-amalgamated Algebras Along Ideals

Mounir El Ouarrachi and Najib Mahdou

Abstract Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. In this paper, we investigate the transfer of the property of coherence in the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) (denoted by $A \bowtie^{f,g} (J, J')$), introduced and studied by Kabbaj, Louartiti, and Tamekkante in 2013. We provide necessary and sufficient conditions for $A \bowtie^{f,g} (J, J')$ to be a coherent ring.

Keywords Bi-amalgamated algebra · Amalgamated algebra · Coherence

1 Introduction

Throughout this paper, all rings are commutative with identity elements, and all modules are unitary.

Let R be a commutative ring. We say that an ideal is regular if it contains a regular element, i.e.; a nonzero divisor element.

For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules

$$F_n \longrightarrow F_{n-1} \longrightarrow ...F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules.

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A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, and Prüfer/semihereditary rings. For instance see [1, 15, 20].

Recall that an R-module M is called a coherent R-module if it is finitely generated and every finitely generated submodule of M is finitely presented.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. In this setting, we can consider the following subring of $B \times C$:

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j')|a \in A, j \in J, j' \in J'\}$$

called the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) (introduced and studied by Kabbaj et al. [19]). This construction is a generalization of the amalgamated algebra along an ideal (introduced and studied by D'Anna and Fontana in [10, 11].) Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [10, Examples 2.5 and 2.6]and other classical constructions, such as the Nagata's idealization ([21, page2]), and the CPI extensions are strictly related to it ([10, Example 2.7 and Remark 2.8]). In [19], the authors studied the basic properties of this construction (e.g., characterized for $A \bowtie^{f,g} (J, J')$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as a bi-amalgamation. Moreover, they pursued the investigation on the structure of the rings of the form $A \bowtie^{f,g} (J, J')$, with particular attention to the prime spectrum.

This paper investigates the property of coherence in bi-amalgamated algebra along ideals. Our results generate original examples which enrich the current literature with new families of non-Noetherian coherent rings.

2 Main Results

This section characterizes the bi-amalgamated algebra along ideals $A \bowtie^{f,g} (J, J')$ to be a coherent ring. The main result (Theorem 1) examines the property of coherence that the amalgamation $A \bowtie^{f,g} (J, J')$ might inherit from the rings $f(A) + J, g(A) + J$ for some classes of ideals J, J', and homomorphisms f, g and hence generates new examples of non-Noetherian coherent rings.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$ and let n be a positive integer. Consider the functions $f^n : A^n \longrightarrow B^n$ defined by $f^n((\alpha_i)_{i=1}^{i=n}) = (f(\alpha_i))_{i=1}^{i=n}$ and $g^n : A^n \longrightarrow C^n$ defined by $g^n((\alpha_i)_{i=1}^{i=n}) = (g(\alpha_i))_{i=1}^{i=n}$. Obviously, f^n and g^n are ring homomorphisms and J^n, J'^n are ideals of B^n and C^n, respectively . This allows us to defined $A^n \bowtie^{f^n,g^n} (J^n, J'^n)$.

Moreover, let $\phi : (A \bowtie^{f,g} (J, J'))^n \longrightarrow A^n \bowtie^{f^n,g^n} (J^n, J'^n)$ defined by

$$\phi((f(\alpha_i) + j_i, g(\alpha_i) + j'_i)_{i=1}^{i=n}) = (f^n((\alpha_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n}, g^n((\alpha_i)_{i=1}^{i=n}) + (j'_i)_{i=1}^{i=n}).$$

It is easily checked that ϕ is a ring isomorphism. So $A^n \bowtie^{f^n,g^n} (J^n, J'^n)$ and $(A \bowtie^{f,g} (J, J'))^n$ are isomorphic as rings.

Let U be a submodule of A^n. Then $U \bowtie^{f^n,g^n} (J^n, J'^n) := \{(f^n(u) + j, g^n(u) + j') \in A^n \bowtie^{f^n,g^n} (J^n, J'^n)/u \in U, j \in J^n, j' \in J'^n\}$ is a submodule of $A^n \bowtie^{f^n,g^n} (J^n, J'^n)$.

Remark 1. 1. Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. Then $f^n(\alpha a) = f(\alpha)f^n(a)$ for all $\alpha \in A$ and $a \in A^n$, where f^n is the homomorphism defined as follows $f^n((a_i)_{i=1}^{i=n}) = (f(a_i))_{i=1}^{i=n}$
2. If $f^{-1}(J) = g^{-1}(J') = 0$, then A is a module retract of $A \bowtie^{f,g} (J, J')$.
3. If g is injective and $J' \subseteq g(A)$, then A is a module retract of $A \bowtie^{f,g} (J, J')$.

Proof. 1. Straightforward
2. Let $\varphi : A \longrightarrow A \bowtie^{f,g} (J, J')$ defined by $\varphi(a) = (f(a), g(a))$ and $\psi : A \bowtie^{f,g} (J, J') \longrightarrow A$ defined by $\psi(f(a) + j, g(a) + j') = a$. ψ is well defined since $f^{-1}(J) = g^{-1}(J') = 0$ and the conclusion now is straightforward.
3. let $\varphi : A \longrightarrow A \bowtie^{f,g} (J, J')$ defined by $\varphi(a) = (f(a), g(a))$ and $\psi : A \bowtie^{f,g} (J, J') \longrightarrow A$ defined by $\psi(f(a) + j, g(a) + j') = a + t$, where t is the unique element such that $g(t) = j'$. ψ is well defined and the conclusion now is straightforward.

Next, before we announce the main result of this section (Theorem 1), we establish the following lemmas.

Lemma 1. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be a proper ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. Then:*

1. *$\{0\} \times J'$ (resp., $J \times \{0\}$) is a finitely generated ideal of $A \bowtie^{f,g} (J, J')$ if and only if J' (resp., J) is a finitely generated ideal of $g(A) + J'$ (resp., $f(A) + J$).*
2. *If $A \bowtie^{f,g} (J, J')$ is a coherent ring and J, J' are finitely generated ideals of $f(A) + J$ and $g(A) + J'$, respectively, then $f(A) + J$ and $g(A) + J'$ are coherent rings.*

Proof. 1. Assume that $J' := \sum_{i=1}^{i=n}(g(A) + J')k_i$ is a finitely generated ideal of $g(A) + J'$, where $k_i \in J'$. It is clear that $\sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(0, k_i) \subseteq \{0\} \times J'$. Let $x := (0, \sum_{i=1}^{i=n}(g(\alpha_i) + j'_i)k_i) \in \{0\} \times J'$, where $\alpha_i \in A$ and $j'_i \in J'$. Hence, $x = (0, \sum_{i=1}^{i=n}(g(\alpha_i) + j'_i)k_i) = \sum_{i=1}^{i=n}(0, (g(\alpha_i) + j'_i)k_i) = \sum_{i=1}^{i=n}(f(\alpha_i), g(\alpha_i) + j'_i)(0, k_i) \in \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(0, k_i)$. Therefore, $\{0\} \times J' \subseteq \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(0, k_i)$ and so $\{0\} \times J' = \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(0, k_i)$. Conversely, assume that $\{0\} \times J' = \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(0, k_i)$ is a finitely generated ideal of $A \bowtie^{f,g} (J, J')$, where $k_i \in J'$. It is readily seen that $J' = \sum_{i=1}^{i=n}(g(A) + J')k_i$, as desired.

2. Assume that $A \bowtie^{f,g} (J, J')$ is a coherent ring and $J \times \{0\}$ (resp., $\{0\} \times J'$) is a finitely generated ideal of $A \bowtie^{f,g} (J, J')$. Then $f(A) + J (\cong \frac{A \bowtie^{f,g}(J,J')}{\{0\}\times J'})$ and $g(A) + J' (\cong \frac{A \bowtie^{f,g}(J,J')}{J\times\{0\}})$ by [19, Proposition 4.1(b)] are coherent rings by [15, Theorem 2.4.1], as desired.

Lemma 2. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$ and let U be a submodule of A^n.*

Assume that U is a finitely generated $A-$module and J, J' are finitely generated ideals of $f(A) + J$ and $g(A) + J'$, respectively. Then $U \bowtie^{f^n,g^n} (J^n, J'^n)$ is a finitely generated $A \bowtie^{f,g} (J, J')-$ module.

Proof. Assume that $U := \sum_{i=1}^{i=n} Au_i$ is a finitely generated $A-$module, where $u_i \in U$ for all $i \in \{1, ..., n\}$, $J^n := \sum_{i=1}^{i=n}(f(A) + J)e_i$ and $J'^n := \sum_{i=1}^{i=n}(g(A) + J)d_i$ are finitely generated $(f(A) + J)-$module and $(g(A) + J')-$module, respectively, where $e_i \in J^n$ and $d_i \in J'^n$ for all $i \in \{1, ..., n\}$. We claim that

$U \bowtie^{f^n,g^n} (J^n, J'^n) = \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(f^n(u_i), g^n(u_i)) + \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(e_i, 0) + \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(0, d_i)$.

Indeed, $\sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(f^n(u_i), g^n(u_i)) + \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(e_i, 0) + \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(0, d_i) \subseteq U \bowtie^{f^n,g^n} (J^n, J'^n)$ since $(f^n(u_i), g^n(u_i)) \in U \bowtie^{f^n,g^n} (J^n, J'^n)$, $(e_i, 0) \in U \bowtie^{f^n,g^n} (J^n, J'^n)$ and $(0, d_i) \in U \bowtie^{f^n,g^n} (J^n, J'^n)$ for all $i \in \{1, ..., n\}$.

Conversely, let $(f^n(x) + j, g^n(x) + j') \in U \bowtie^{f^n,g^n} (J^n, J'^n)$, where $x \in U$, $j \in J^n$ and $j' \in J'^n$. Hence, $x = \sum_{i=1}^{i=n} \alpha_i u_i$, for some $\alpha_i \in A$ $(i \in \{1, ..., n\})$, $j = \sum_{i=1}^{i=n}(f(\beta_i) + j_i)e_i \in J^n$ and $j' = \sum_{i=1}^{i=n}(g(\lambda_i) + j'_i)d_i \in J'^n$ for $\beta_i, \lambda_i \in A$, $j_i \in J$ and $j'_i \in J'$ $(i \in \{1, ..., n\}$.

We obtain

$$\begin{aligned}
(f^n(x) + j, g^n(x) + j') &= (f^n(\sum_{i=1}^{i=n} \alpha_i u_i) + j, g^n(\sum_{i=1}^{i=n} \alpha_i u_i) + j') \\
&= (\sum_{i=1}^{i=n} f(\alpha_i) f^n(u_i), \sum_{i=1}^{i=n} g(\alpha_i) g^n(u_i)) + (j, 0) + (0, j') \\
&= \sum_{i=1}^{i=n}(f(\alpha_i), g(\alpha_i))(f^n(u_i), g^n(u_i)) + (\sum_{i=1}^{i=n}(f(\beta_i) + j_i)e_i, 0) \\
&+ (0, \sum_{i=1}^{i=n}(g(\lambda_i) + j'_i)d_i) \\
&= \sum_{i=1}^{i=n}(f(\alpha_i), g(\alpha_i))(f^n(u_i), g^n(u_i)) + \sum_{i=1}^{i=n}(f(\beta_i) + j_i, 0)(e_i, 0)
\end{aligned}$$

$$+\sum_{i=1}^{i=n}(0, g(\lambda_i)+j'_i)(0, d_i)$$

$$=\sum_{i=1}^{i=n}(f(\alpha_i), g(\alpha_i))(f^n(u_i), g^n(u_i))+\sum_{i=1}^{i=n}(f(\beta_i)+j_i, g(\beta_i))(e_i, 0)$$

$$+\sum_{i=1}^{i=n}(f(\lambda_i), g(\lambda_i)+j'_i)(0, d_i).$$

Consequently, $(f^n(x)+j, g^n(x)+j') \in \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(f^n(u_i), g^n(u_i)) + \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(e_i, 0) + \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(0, d_i)$
since $(f(\alpha_i), g(\alpha_i)), (f(\beta_i)+j_i, g(\beta_i)), (f(\lambda_i), g(\lambda_i)+j'_i) \in A \bowtie^{f,g} (J, J')$ for all $i \in \{1, ..., n\}$. Hence $U \bowtie^{f^n,g^n} (J^n, J'^n) = \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(f^n(u_i), g^n(u_i)) + \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(e_i, 0) + \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(0, d_i)$ is a finitely generated $(A \bowtie^{f,g} (J, J'))$-module, as desired.

Lemma 3. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. Assume that J (resp., J') is a finitely generated ideal of $f(A)+J$ (resp., $g(A)+J'$) and $J \subseteq f(A)$. Then $J \times \{0\}$ is a coherent $(A \bowtie^{f,g} (J, J'))$-module provided $f(A)+J$ is a coherent ring.*

Proof. Since $J \times \{0\}$ is a finitely generated $(A \bowtie^{f,g} (J, J'))$-module, it remains to show that every finitely generated submodule of $J \times \{0\}$ is finitely presented. Assume that $f(A)+J$ is a coherent ring and let N be a finitely generated submodule of $J \times \{0\}$. It is clear that $N = I \times \{0\}$, where $I = \sum_{i=1}^{i=n}(f(A)+J)b_i$ for some integer n and $b_i \in I$. Consider the exact sequence of $(f(A)+J)$-modules

$$0 \longrightarrow \ker v \longrightarrow (f(A)+J)^n \longrightarrow I \longrightarrow 0 \tag{1}$$

where $v((f(\alpha_i)+j_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i$. Then,

$$\ker v = \{(f(\alpha_i)+j_i)_{i=1}^{i=n}) \in (f(A)+J)^n / \sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i = 0\}$$

$$= \{(f(c_i))_{i=1}^{i=n}) \in (f(A))^n / \sum_{i=1}^{i=n}(f(c_i))b_i = 0\}$$

where $c_i = \alpha_i + k_i$ and $f(k_i) = j_i$ for some $k_i \in A$ (since $J \subseteq f(A)$).

The $(f(A)+J)$-module $\ker v$ is finitely generated since $f(A)+J$ is a coherent ring. Let $\{f^n((c_i^1))_{i=1}^{i=n}), f^n((c_i^2))_{i=1}^{i=n}), ..., f^n((c_i^m))_{i=1}^{i=n})\}$ be a generating set of $\ker v$. On the other hand, it is easily verified that $N = \sum_{i=1}^{i=n}(A \bowtie^{f,g} (J, J'))(b_i, 0)$. Consider the exact sequence of $(A \bowtie^{f,g} (J, J'))$-modules

$$0 \longrightarrow \ker u \longrightarrow (A \bowtie^{f,g} (J, J'))^n \longrightarrow N \longrightarrow 0 \tag{2}$$

where $u((f(\alpha_i)+j_i, g(\alpha_i)+j'_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n}(f(\alpha_i)+j_i, g(\alpha_i)+j'_i))(b_i, 0)$. Then,

$$\begin{aligned}\ker u &= \{((f(\alpha_i)+j_i, g(\alpha_i)+j'_i)_{i=1}^{i=n}) \in (A \bowtie^{f,g} (J, J'))^n / \sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i = 0\} \\ &= \{((f(d_i), g(d_i)+k_i)_{i=1}^{i=n}) \in (A \bowtie^{f,g} (J, J'))^n / \sum_{i=1}^{i=n}(f(d_i))b_i = 0\}\end{aligned}$$

where $d_i = \alpha_i + t_i$ and $f(t_i) = j_i$ for some $t_i \in A$ (since $J \subseteq f(A)$).

Let U be the the submodule of A^n generated by $\{((c_i^1)_{i=1}^{i=n}), ((c_i^2)_{i=1}^{i=n}), \ldots, ((c_i^m)_{i=1}^{i=n})\}$, we claim that $\ker u = U \bowtie^{f^n,g^n} (0, J'^n)$.

Indeed, let $x = (f^n((d_i)_{i=1}^{i=n}), g^n((d_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n}) \in U \bowtie^{f^n,g^n} (0, J'^n)$, so $((d_i)_{i=1}^{i=n}) = \sum_{j=1}^{j=m} a_j((c_i^j)_{i=1}^{i=n}) = (\sum_{j=1}^{j=m} a_j c_i^j)_{i=1}^{i=n}$. We have

$$\sum_{i=1}^{i=n} f(d_i)b_i = \sum_{i=1}^{i=n} f(\sum_{j=1}^{j=m} a_j c_i^j)b_i = \sum_{i=1}^{i=n}(\sum_{j=1}^{j=m} f(a_j c_i^j))b_i = \sum_{j=1}^{j=m} f(a_j)(\sum_{i=1}^{i=n} f(c_i^j)b_i) = 0.$$

Consequently, $x \in \ker u$.

Conversely, let $x \in \ker u$, so $x = (f^n((d_i)_{i=1}^{i=n}), g^n((d_i)_{i=1}^{i=n}) + (k_i)_{i=1}^{i=n})$ such that $\sum_{i=1}^{i=n} f(d_i)b_i = 0$. Then, $f^n((d_i)_{i=1}^{i=n}) \in \ker v$ hence

$$\begin{aligned}f^n((d_i)_{i=1}^{i=n}) &= \sum_{j=1}^{j=m} f(a_j) f^n((c_i^j)_{i=1}^{i=n}) \\ &= \sum_{j=1}^{j=m} f^n((a_j c_i^j)_{i=1}^{i=n}) \\ &= f^n((\sum_{j=1}^{j=m} a_j c_i^j)_{i=1}^{i=n})\end{aligned}$$

consequently, $x = (f^n((\sum_{j=1}^{j=m} a_j c_i)_{i=1}^{i=n}), g^n((\sum_{j=1}^{j=m} a_j c_i^j)_{i=1}^{i=n}) + (k_i)_{i=1}^{i=n})$ where $(\sum_{j=1}^{j=m} a_j c_i^j)_{i=1}^{i=n} \in U$ which implies that $x \in U \bowtie^{f^n,g^n} (0, J'^n)$. Since U is a finitely generated A-module and J is a finitely generated ideal of B, then $U \bowtie^{f^n,g^n} (0, J'^n)$ is a finitely generated $(A \bowtie^{f,g} (J, J'))$-module (by Lemma 2). Therefore, N is a finitely presented $(A \bowtie^{f,g} (J, J'))$-module by the sequence (2) and hence $J \times \{0\}$ is a coherent $(A \bowtie^{f,g} (J, J'))$-module, to complete the proof of lemma 3.

Lemma 4. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$. Assume that J (resp., J') is a finitely generated ideal of $f(A) + J$ (resp., $g(A) + J'$) and $J^2 = 0$. Then $J \times \{0\}$ is a coherent $(A \bowtie^{f,g} (J, J'))$-module provided $f(A) + J$ is a coherent ring.*

Proof. Since $J \times \{0\}$ is a finitely generated $(A \bowtie^{f,g} (J, J'))$-module, it remains to show that every finitely generated submodule of $J \times \{0\}$ is finitely presented.

Assume that $f(A)+J$ is a coherent ring and let N be a finitely generated submodule of $J\times\{0\}$. It is clear that $N=I\times\{0\}$, where $I=\sum_{i=1}^{i=n}(f(A)+J)b_i$ for some integer n and $b_i\in I$. Consider the exact sequence of $(f(A)+J)$-modules

$$0\longrightarrow \ker v\longrightarrow (f(A)+J)^n\longrightarrow I\longrightarrow 0 \qquad (1)$$

where $v((f(\alpha_i)+j_i)_{i=1}^{i=n})=\sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i$. Then,

$$\begin{aligned}\ker v&=\{(f(\alpha_i)+j_i)_{i=1}^{i=n})\in(f(A)+J)^n/\sum_{i=1}^{i=n}(f(\alpha_i)+j_i))b_i=0\}\\ &=\{(f(\alpha_i)+j_i)_{i=1}^{i=n})\in(f(A)+J)^n/\sum_{i=1}^{i=n}(f(\alpha_i))b_i=0\}\end{aligned}$$

since $I\subseteq J$ and $J^2=0$.

The $(f(A)+J)$-module $\ker v$ is finitely generated since $f(A)+J$ is a coherent ring. Let $\{f^n((\alpha_i^1))_{i=1}^{i=n})+(j_i^1)_{i=1}^{i=n}, f^n((\alpha_i^2))_{i=1}^{i=n})+(j_i^2)_{i=1}^{i=n}, ..., f^n((\alpha_i^m))_{i=1}^{i=n})+(j_i^m)_{i=1}^{i=n}\}$ be a generating set of $\ker v$. On the other hand, it is easily verified that $N=\sum_{i=1}^{i=n}(A\bowtie^{f,g}(J,J'))(b_i,0)$. Consider the exact sequence of $(A\bowtie^{f,g}(J,J'))$-modules

$$0\longrightarrow \ker u\longrightarrow (A\bowtie^{f,g}(J,J'))^n\longrightarrow N\longrightarrow 0 \qquad (2)$$

where $u((f(d_i)+j_i, g(d_i)+j_i')_{i=1}^{i=n})=\sum_{i=1}^{i=n}(f(d_i)+j_i, g(d_i)+j_i'))(b_i,0)$. Then,

$$\begin{aligned}\ker u&=\{(f(d_i)+j_i, g(d_i)+j_i')_{i=1}^{i=n})\in(A\bowtie^{f,g}(J,J'))^n/\sum_{i=1}^{i=n}(f(d_i)+j_i))b_i=0\}\\ &=\{(f(d_i)+j_i, g(d_i)+k_i)_{i=1}^{i=n})\in(A\bowtie^{f,g}(J,J'))^n/\sum_{i=1}^{i=n}(f(d_i))b_i=0\}\end{aligned}$$

since $I\subseteq J$ and $J^2=0$.

Let U be the submodule of A^n generated by $\{((\alpha_i^1)_{i=1}^{i=n}), ((\alpha_i^2)_{i=1}^{i=n}), ..., ((\alpha_i^m)_{i=1}^{i=n})\}$, we claim that $\ker u=U\bowtie^{f^n,g^n}(J^n,J'^n)$.

Indeed, let $x=(f^n((d_i)_{i=1}^{i=n})+(j_i)_{i=1}^{i=n}, g^n((d_i)_{i=1}^{i=n})+(j_i')_{i=1}^{i=n})\in U\bowtie^{f^n,g^n}(J^n,J'^n)$, so $((d_i)_{i=1}^{i=n})=\sum_{j=1}^{j=m}a_j((\alpha_i^j)_{i=1}^{i=n})=(\sum_{j=1}^{j=m}a_j\alpha_i^j)_{i=1}^{i=n}$. We have

$$\sum_{i=1}^{i=n}f(d_i)b_i=\sum_{i=1}^{i=n}f(\sum_{j=1}^{j=m}a_j\alpha_i^j)b_i=\sum_{i=1}^{i=n}(\sum_{j=1}^{j=m}f(a_j\alpha_i^j))b_i=\sum_{j=1}^{j=m}f(a_j)(\sum_{i=1}^{i=n}f(\alpha_i^j)b_i)=0.$$

Consequently, $x\in\ker u$.

Conversely, let $x\in\ker u$, so $x=(f^n((d_i)_{i=1}^{i=n})+(k_i)_{i=1}^{i=n}, g^n((d_i)_{i=1}^{i=n})+(k_i')_{i=1}^{i=n})$ such that $\sum_{i=1}^{i=n}f(d_i)b_i=0$, then $f^n((d_i)_{i=1}^{i=n})+(k_i)_{i=1}^{i=n}\in\ker v$ hence

$$\begin{aligned} f^n((d_i)_{i=1}^{i=n}) + (k_i)_{i=1}^{i=n} &= \sum_{s=1}^{s=m}(f(a_s)+t_s)(f^n((\alpha_i^s)_{i=1}^{i=n}) + (j_i^s)_{i=1}^{i=n}) \\ &= \sum_{s=1}^{s=m} f^n((a_s\alpha_i^s)_{i=1}^{i=n}) + (l_i)_{i=1}^{i=n} \\ &= f^n((\sum_{s=1}^{s=m} a_s\alpha_i^s)_{i=1}^{i=n}) + (l_i)_{i=1}^{i=n} \end{aligned}$$

consequently, $x = (f^n((\sum_{s=1}^{s=m} a_s\alpha_i^s)_{i=1}^{i=n}) + (l_i)_{i=1}^{i=n}, g^n((\sum_{s=1}^{s=m} a_s\alpha_i^s)_{i=1}^{i=n}) + (l'_i)_{i=1}^{i=n})$ where $(\sum_{s=1}^{s=m} a_s\alpha_i^s)_{i=1}^{i=n} \in U$ which implies that $x \in U \bowtie^{f^n,g^n} (J^n, J'^n)$. Since U is a finitely generated A-module and J (resp., J') is a finitely generated ideal of B (resp., C), then $U \bowtie^{f^n,g^n} (0, J'^n)$ is a finitely generated $(A \bowtie^{f,g} (J, J'))$-module (by Lemma 2). Therefore, N is a finitely presented $(A \bowtie^{f,g} (J, J'))$-module by the sequence (2) and hence $J \times \{0\}$ is a coherent $(A \bowtie^{f,g} (J, J'))$-module, to complete the proof of lemma 4.

Lemma 5. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$.*

1. *If $(A \bowtie^{f,g} (J, J'))$ is a coherent ring and J is regular, then J' is a finitely generated ideal of $g(A) + J'$.*
2. *Assume that J, J' are regular ideals of $f(A) + J$ and $g(A) + J'$, respectively. If $A \bowtie^{f,g} (J, J')$ is a coherent ring then so is $f(A) + J$ and $g(A) + J'$.*

Proof. 1. Assume that $(A \bowtie^{f,g} (J, J'))$ is a coherent ring and J contains a regular element k. Set $c = (k, 0) \in A \bowtie^{f,g} (J, J')$. One can easily check that

$$\begin{aligned} (0:c) &= \{(f(a)+j, g(a)+j') \in A \bowtie^{f,g} (J, J')/(f(a)+j, g(a)+j')(k,0) = 0\} \\ &= \{(f(a)+j, g(a)+j') \in A \bowtie^{f,g} (J, J')/(f(a)+j)k = 0\} \\ &= \{(f(a)+j, g(a)+j') \in A \bowtie^{f,g} (J, J')/(f(a)+j) = 0\} \\ &= \{(0, g(a)+j') \in A \bowtie^{f,g} (J, J')/a \in f^{-1}(J) = g^{-1}(J')\} \\ &= \{0\} \times J' \end{aligned}$$

Since $(A \bowtie^{f,g} (J, J'))$ is a coherent ring, then $(0 : c) = \{0\} \times J'$ is a finitely generated ideal of $A \bowtie^{f,g} (J, J')$. Therefore, J' is a finitely generated ideal of $g(A) + J'$, as desired.

2. Follows immediately from (1) and Lemma 1.

Now, to the main result:

Theorem 1. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be a proper ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$.*

1. *Assume that J, J' are finitely generated ideals of $f(A) + J$ and $g(A) + J'$, respectively, and $J \subseteq f(A)$. Then*
   $A \bowtie^{f,g} (J, J')$ is a coherent ring if and only if $f(A) + J$ and $g(A) + J'$ are coherent rings.

2. *Assume that* J, J' *are finitely generated ideals of* $f(A)+J$ *and* $g(A)+J'$, *respectively, and* $J^2=0$. *Then*
   $A \bowtie^{f,g} (J, J')$ *is a coherent ring if and only if* $f(A)+J$ *and* $g(A)+J'$ *are coherent rings.*
3. *Assume that* J, J' *are regular ideals of* $f(A)+J$ *and* $g(A)+J'$, *respectively, and* $J \subseteq f(A)$. *Then*
   $A \bowtie^{f,g} (J, J')$ *is a coherent ring if and only if* $f(A)+J$ *and* $g(A)+J'$ *are coherent rings and* J, J' *are finitely generated ideals of* $f(A)+J$ *and* $g(A)+J'$, *respectively.*
4. *Assume that* J *is a regular finitely generated ideal of* $f(A)+J$ *and* $J' \subseteq g(A)$. *Then*
   $A \bowtie^{f,g} (J, J')$ *is a coherent ring if and only if* $f(A)+J$ *and* $g(A)+J'$ *are coherent rings and* J' *is a finitely generated ideal of* $g(A)+J'$.
5. *Assume that* J *is a regular finitely generated ideal of* $f(A)+J$ *and* $J'^2=0$. *Then*
   $A \bowtie^{f,g} (J, J')$ *is a coherent ring if and only if* $f(A)+J$ *and* $g(A)+J'$ *are coherent rings and* J' *is a finitely generated ideal of* $g(A)+J'$.

Proof. 1. If $A \bowtie^{f,g} (J, J')$ is a coherent ring, then so are $f(A)+J$ and $g(A)+J'$ by (lemma 1(2)) since J, J' are finitely generated ideals of $f(A)+J$ and $g(A)+J'$, respectively. Conversely, assume that $f(A)+J$ and $g(A)+J'$ are coherent rings. Since $f(A)+J(\cong \frac{A\bowtie^{f,g}(J,J')}{\{0\}\times J'})$ and $g(A)+J'(\cong \frac{A\bowtie^{f,g}(J,J')}{J\times\{0\}})$ by [19, Proposition 4.1(b)] and $J\times\{0\}$ is a coherent $(A \bowtie^{f,g} (J, J'))$-module (by Lemma 3), then $A \bowtie^{f,g} (J, J')$ is a coherent ring by [15, Theorem 2.4.1] .

2. If $A \bowtie^{f,g} (J, J')$ is a coherent ring, then so are $f(A)+J$ and $g(A)+J'$ by (lemma 1(2)) since J, J' are finitely generated ideals of $f(A)+J$ and $g(A)+J'$, respectively. Conversely, assume that $f(A)+J$ and $g(A)+J'$ are coherent rings. Since $f(A)+J(\cong \frac{A\bowtie^{f,g}(J,J')}{\{0\}\times J'})$ and $g(A)+J'(\cong \frac{A\bowtie^{f,g}(J,J')}{J\times\{0\}})$ by [19, Proposition 4.1 (b)] and $J\times\{0\}$ is a coherent $(A \bowtie^{f,g} (J, J'))$-module (by Lemma 1), then $A \bowtie^{f,g} (J, J')$ is a coherent ring by [15, Theorem 2.4.1] .
3. Follows immediately from theorem 1(1) and Lemma 5.
4. Follows immediately from theorem 1(1) and Lemma 5.
5. Follows immediately from theorem 1(2) and Lemma 5.

Recall that the amalgamation of A with B along J with respect to f is given by

$$A \bowtie^{f} J := \{(a, f(a)+j) \mid a \in A, j \in J\}$$

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^{f} J = A \bowtie^{Id,f} (f^{-1}(J), J)$. Accordingly, Theorem 1 covers the special case of amalgamation [1], as recorded below.

Corollary 1. *Let* $f: A \longrightarrow B$ *be a ring homomorphism and let* J *be a proper ideal of* B.

1. *If* $A \bowtie^{f} J$ *is a coherent ring, then so is* A.

2. *Assume that J and $f^{-1}(J)$ are finitely generated ideals of $f(A)+J$ and A, respectively. Then $A \bowtie^f J$ is a coherent ring if and only if $f(A)+J$ and A are coherent rings.*
3. *Assume that J is a regular finitely generated ideal of $f(A)+J$. Then $A \bowtie^f J$ is a coherent ring if and only if $f(A)+J$ and A are coherent rings and $f^{-1}(J)$ is a finitely generated ideal of A.*

The following Corollary is an immediate consequence of Theorem 1(3)(4).

Corollary 2. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J (resp., J') be an ideal of B (resp., C) such that $f^{-1}(J) = g^{-1}(J')$.*

1. *If B is an integral domain, J is a finitely generated ideal of $f(A)+J$ and $J'^2 = 0$, then:*
   $A \bowtie^{f,g} (J, J')$ is a coherent ring if and only if $f(A)+J$ and $g(A)+J'$ are coherent rings and J' is a finitely generated ideal of $g(A)+J'$.
2. *If B and C are integral domains and $J \subseteq f(A)$, then::*
   $A \bowtie^{f,g} (J, J')$ is a coherent ring if and only if $f(A)+J$ and $g(A)+J'$ are coherent rings and J, J' are finitely generated ideals of $f(A)+J$ and $g(A)+J'$, respectively.

Example 1. Let A be a non-Noetherian coherent ring, I and K are finitely generated ideals of A such that $I \subseteq K$. Let $f : A \to A/I$ be the canonical homomorphism and $g : A \to A \times A$ be the injective homomorphism defined by $g(a) = (a, 0)$, $J = K/I$ and $J' = K \times 0$. Then $A \bowtie^{f,g} (J, J')$ is a non-Noetherian coherent ring.

Proof. By Theorem 1, $A \bowtie^{f,g} (J, J')$ is a coherent ring since $f(A)+J = A/I$ and $g(A)+J' = A \times A$ are both coherent rings and J (resp., J') is a finitely generated ideal of $f(A)+J$ (resp., $g(A)+J'$) and $J = K/I \subseteq f(A) = A/I$. On the other hand, $A \bowtie^{f,g} (J, J')$ is a non-Noetherian ring by [19, Proposition 4.2] since $g(A)+J' = A \times A$ is non-Noetherian ring.

Example 2. Let (A, M) be a non-Noetherian local coherent ring such that M is a finitely generated ideal and E be an A/M-vector space of finite rank. Let $f : A \to A \ltimes E$ be the injective homomorphism defined by $f(a) = (a, 0)$ and $g : A \to A/M[X_1, X_2, ..., X_n]$ defined by $g(a) = \overline{a}$, $J = 0 \ltimes E$ and $J' = (X_1, X_2, ..., X_n)$. Then $A \bowtie^{f,g} (J, J')$ is a non-Noetherian coherent ring.

Proof. By Theorem 1, $A \bowtie^{f,g} (J, J')$ is a coherent ring since $f(A)+J = A \ltimes E$ is a coherent ring by [20, Theorem 2.6] and $g(A)+J' = A/M[X_1, X_2, ..., X_n]$ is a coherent ring (Noetherian) and J (resp., J') is a finitely generated ideal of $f(A)+J$ (resp., $g(A)+J'$) and $J^2 = 0$. On the other hand, $A \bowtie^{f,g} (J, J')$ is a non-Noetherian ring by [19, Proposition 4.2] since $f(A)+J = A \times E$ is non-Noetherian ring.

Example 3. Let $A = \mathbb{Z}[X]$, $B = \mathbb{Z} + X\mathbb{Q}[X]$, $C = \mathbb{Z}$ and let $J = n_0\mathbb{Z} + X\mathbb{Q}[X]$, $J' = n_0\mathbb{Z}$ ideals of B and C, respectively. Let $f : A \longrightarrow B$ be the homomorphism defined by $f(P(X)) = P(X)$ and $g : A \longrightarrow C$ be the homomorphism defined by $f(P(x)) = P(0)$. Then $A \bowtie^{f,g} (J, J')$ is a non-Noetherian coherent ring.

Proof. By Theorem 1, $A \bowtie^{f,g} (J, J')$ is a non-Noetherian coherent ring since $f(\mathbb{Z}[X]) + J = \mathbb{Z} + X\mathbb{Q}[X]$ is a non-Noetherian coherent ring and $g(\mathbb{Z}[X]) + J' = \mathbb{Z}$ is a coherent ring (Noetherian) and $J' \subseteq g(A)$.

Example 4. Let $A = \mathbb{Z}[X]$, $B = \mathbb{Z} + X\mathbb{Q}[X]$, $C = \mathbb{Z} + i\mathbb{Z}[i] = \mathbb{Z}[i]$, and let $J = n_0\mathbb{Z} + X\mathbb{Q}[X]$, $J' = n_0\mathbb{Z} + i\mathbb{Z}[i]$ ideals of B and C, respectively. Let $f : A \longrightarrow B$ be the homomorphism defined by $f(P(X)) = P(0)$ and $g : A \longrightarrow C$ be the homomorphism defined by $f(P(x)) = P(i)$. Then $A \bowtie^{f,g} (J, J')$ is a non-Noetherian coherent ring.

Proof. By Theorem 1, $A \bowtie^{f,g} (J, J')$ is a non-Noetherian coherent ring since $f(\mathbb{Z}[X]) + J = \mathbb{Z} + X\mathbb{Q}[X]$ is a non-Noetherian coherent ring and $g(\mathbb{Z}[X]) + J' = \mathbb{Z}[i]$ is a coherent ring (Noetherian) and $J' \subseteq g(A)$.

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On the Set of Intermediate Artinian Subrings

Driss Karim

Abstract The paper contributes to the investigation of intermediate Artinian subrings between R and T, where $R \hookrightarrow T$ is an extension of rings.

Keywords Artinian ring · Intermediate Artinian subring · Directed union of Artinian subrings · Infinite product · Reduced ring · Residue fields Semi-quasilocal ring · Von Neumann regular ring · Zero-dimensional ring

1 Introduction

The study of zero-dimensionality in commutative rings has been widely treated in the literature (see [2, 3, 6, 9, 10]). In particular, many recent papers investigate zero-dimensional overrings, zero-dimensional subrings, and Artinian subrings of a commutative ring. The aim of this paper is to contribute to the investigation of rings which can be embeddable in an Artinian ring. Recall that Artinian rings form an important class of zero-dimensional rings. Moreover, an Artinian ring has only finitely many idempotent elements. Essentially, the characterization of the set of Artinian subrings of a commutative ring is known (see [9]). Now, we are interested in the Artinian overring of pair of rings, that means, we are looking for intermediate Artinian rings between R and T, where R is a subring of a ring T. Of particular interest is [9, Theorem 2.1], which shows that any zero-dimensional ring with only finitely many distinct characteristics of its residue fields contains an Artinian subring.

We give a short overview of the paper. In Sect. 1, the basic notions and technical tools, which touch upon various aspects of zero-dimensionality, are introduced and developed. In particular, we provide some results concerning the minimal zero-dimensional subring of T containing R, where R is a subring of T. Section 2 deals

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with the problem of determining conditions under which the pair of rings (R, T) contains an intermediate Artinian ring. We show in Theorem 1 that if R is reduced then there exists an indeterminate Artinian ring between R and T if and only if $Idem(R^0)$ is finite. Finally, we offer a partial answer to the problem (**P**).

Find necessary and sufficient conditions on a pair (R, S) of rings such that there exists an intermediate Artinian subring.

All rings considered in this paper are commutative with identity element and, unless otherwise specified, are assumed to be nonzero. All ring-homomorphisms are unital. If R is a subring of a ring S, we assume that the unity of S belongs to R. Throughout, we use $Idem(R)$, $Spec(R)$, and $\mathscr{C}(R)$, respectively, to denote the set of idempotent elements of R, the set of prime ideals of R, and the set $\{char(R/M)$: M ranges over the maximal ideals of $R\}$.

2 Preliminaries and General Results

Let R be a ring, we recall that R is reduced if $\bigcap_{P\in Min(R)} P = (0)$, where $Min(R)$ is the set of minimal prime ideals of R, and zero-dimensional if all prime ideals are maximal. It is worthwhile recalling that any Artinian ring T is zero-dimensional but the converse is not true, in general. For example, the ring $T = \mathbb{Q}^{\mathbb{N}}$ the countable direct product of copies of $\mathbb{Q}$, is zero-dimensional but not Artinian since $Spec(T)$ is infinite.

Let S be a ring, a subring of R is assumed to contain the prime subring of S. We denote by $\mathscr{Z}(S)$ and $\mathscr{A}(S)$, respectively, the set of zero-dimensional and Artinian subrings of S. Let $\mathscr{Z}(R, S)$ denote the set of zero-dimensional overrings of R contained in S and finally, $\mathscr{A}(R, S)$ denotes the set of all intermediate Artinian subrings of S containing R.

Under the assumption that $\mathscr{Z}(R, T)$ is nonempty, Arapovic constructed the unique minimal zero-dimensional subring R^0 of T containing R (see [3]). If $\{R_\alpha\}_{\alpha\in A}$ is a the family of zero-dimensional subrings of T containing R, then $\bigcap_{\alpha\in A} R_\alpha$ is a zero-dimensional ring. Hence if $\mathscr{Z}(R, T)$ is nonempty, then there exists a unique minimal zero-dimensional subring of T containing R. We will denote the minimal zero-dimensional subring of T containing R, where $\mathscr{Z}(R, T) \neq \emptyset$, by R^0, but R^0 is not determined up to isomorphism by R alone; R^0 also depends upon T, so notation such as $R^0(T)$ would more accurately reflect the situation. Next, we give preliminaries and some properties of minimal zero-dimensional subrings.

Lemma 1. *Let $\{(R_i, T_i)\}_{i=1}^n$ be a finite family of pairs of rings, and R_i^o be the minimal zero-dimensional subring of T_i containing R_i, for each $i = 1, \dots, n$. Then $\prod_{i=1}^n R_i^o$ is the minimal zero-dimensional subring of $\prod_{i=1}^n T_i$ containing $\prod_{i=1}^n R_i$.*

Proof. We denote by $R = \prod_{i=1}^n R_i$ and $T = \prod_{i=1}^n T_i$. We observe that $R = \prod_{i=1}^n R_i^0$ is a zero-dimensional ring since $dim(\prod_{i=1}^n R_i^0) = Sup\{dim(R_i^0)\ : 1 \le i \le n\} = 0$. Hence $\mathscr{Z}(R, T) \neq \emptyset$. Let R^0 be the minimal zero-dimensional subring of T containing R. We denote by e_i the primitive idempotent with support $\{i\}$. Then

$\{e_i\}_{i=1}^n$ is an orthogonal family of idempotent elements. Consider $\phi_i : T \to T_i$ the canonical projection homomorphism with $\phi_i(T) = T_i$ and $\phi_i(R) = R_i$ for each $i = 1, \ldots, n$. Let $\phi_i(R^0) = W_i$ be a zero-dimensional subring of T_i containing R_i for each $i = 1, \ldots, n$. Then $R_i^0 \subseteq W_i$ for each $i = 1, \ldots, n$. Suppose that there exists $i_0 \in \{1, \ldots, n\}$ such that $R_i^0 \subset W_i$ which means that $R \subseteq \prod_{i=1}^n R_i^0 \subset \prod_{i=1}^n W_i = R_i^0 \subseteq T$, that is a contradiction with R^0 is the minimal zero-dimensional subring of T containing R.

Besides Artinian rings another distinguished class of zero-dimensional rings is that von Neumann regular rings. We recall that a ring R is said to be von Neumann regular if for each element t of R there is an element x in R such that $t = t^2x$. We will use the knowing lemmas throughout this paper.

Lemma 2. *[10, Proposition 2.3] A ring R is a von Neumann regular ring if and only if R is zero-dimensional and reduced.*

Lemma 3. *Any zero-dimensional ring R with only finitely many idempotents is semi-quasilocal.*

Proof. Let $M_1, \ldots, M_{r+1}$ be distinct maximal ideals of R. Let $x \in M_{r+1} \setminus (\cup_{i=1}^r M_i)$, since $\dim R = 0$ by [6, Theorem 3.1], there exists $t \in \mathbb{Z}^+$ and e an idempotent element of R such that $x^t R = eR$. Hence $e \in M_{r+1} \setminus (\cup_{i=1}^r M_i)$. It follows that if R has n maximal ideals, it has at least $n - 1$ idempotents. Therefore, R is necessarily semi-quasilocal.

Lemma 4. *Any von Neumann regular ring R with only finitely many maximal ideals is finite product of fields.*

Lemma 5. *Let $\{R_i\}_{i=1}^n$ be a finite family of rings. Then $\prod_{i=1}^n R_i$ is Artinian if and only if R_i is Artinian for each $i = 1, \ldots, n$.*

Proof. Suppose that $R = \prod_{i=1}^n R_i$ is Artinian, by Lemma 3, $|Idem(R)| < k$, for some $k \in \mathbb{Z}^+$. Let $\pi_i : R \to R_i$ be the natural surjective homomorphism for each $i = 1, \ldots, n$. If we denote by $I_i = Ker(\pi_i)$, then $\frac{R}{I_i} \simeq R_i$ and hence R_i is an Artinian ring, for each $i = 1, \ldots, n$. Conversely, let $\prod_{i=1}^n R_i$ be a finite direct product of Artinian rings. It is clear that R is Noetherian as each R_i is Noetherian. Since $dim(R) = Sup\{dim(R_i) : i = 1, \ldots, n\} = 0$. By [4, Theorem 8.5], the ring R is Artinian.

Remark 1. If $R = \prod_{\alpha \in A} R_\alpha$ is an infinite direct product of Artinian rings, then R is not Artinian since $|Idem(R)|$ is infinite.

Let R be a commutative ring and X an indeterminate over R. For a polynomial $f \in R[X]$, denote by $\sigma(f)$ the so-called the content ideal of f, that is, the ideal of R generated by the coefficients of f. The set $S = \{f \in R[X] : \sigma(f) = R\} = R[X] \setminus \bigcup\{M[X] : \text{M is maximal of R}\}$ is a multiplicatively closed subset of $R[X]$. The localization $R(X) = S^{-1}R[X]$ is called the Nagata ring in one variable over R. (See also [8, Chapter IV]).

Lemma 6. *If R^o is the minimal zero-dimensional ring of $\mathscr{Z}(R,T)$, then $R^o(X)$ is the minimal zero-dimensional of $T(X)$ containing $R(X)$.*

Proof. If we denote $R^{'} = R[\{e : \text{e an idempotent of R}\}]$, then $R^0 = T(R^{'})$ (cf. [6, Remark 3.5]). Since R^0 is zero-dimensional, by [8, Corollary 2.12] R^0 satisfies Property A[1] and by [8, Corollary 2.6] $R^{'}$ satisfies Property A. It follows that $R^0(X) = T(R^{'}(X)) = T(R^{'}(X))$ (cf. [8, Theorem 16.4]). According to [8, Theorem 14.7], T and $T(X)$ have the same idempotent elements, hence

$$(R(X))^{'} = R(X)[\{e;\ \text{e an idempotent of T}\}] = R[\{e;\ \text{e an idempotent of T}\}](X) = R^{'}(X).$$

It follows that

$$R^0(X) = T(R(X)^{'}) = (R(X))^0.$$

By using the correspondence of ideals one can also show that if $\mathscr{A}(R,T) \neq \emptyset$ and then so is $\mathscr{A}(S^{-1}R, S^{-1}T) \neq \emptyset$, where S is a multiplicatively closed subset of R. Since any intermediate Artinian subring A in $\mathscr{A}(R,T)$, we have $S^{-1}A$ is also Artinian ring such that $S^{-1}R \subseteq S^{-1}A \subseteq S^{-1}T$ (see [4]). It follows that, $\mathscr{A}(S^{-1}R, S^{-1}T) \neq \emptyset$.

Lemma 7. *If R is an Artinian ring, then $R(X)$ is also Artinian ring.*

Proof. Assume that R is Artinian, by [11, (6.17)] $R(X)$ is Noetherian since R is Noetherian. Each $R(X)$ is zero-dimensional as it is R (cf. [1, Proposition 1.21]). Therefore, $R(X)$ is both zero-dimensional and Noetherian, by [4, Theorem 8.5], $R(X)$ is Artinian.

Lemma 8. *Let R be a zero-dimensional ring with finite spectrum, then R is expressible as a finite product of zero-dimensional quasilocal subrings.*

Proof. Let $Spec(R) = \{M_i\}_{i=1}^n$ be the spectrum of R. Let $S_{M_i}(0)$ to denote $Ker\varphi_i$ for each $i = 1, ..., n$, where $\varphi_i : R \to R_{M_i}$ and $\varphi_i(r) = \frac{r}{1}$, is the canonical homomorphism. Since $Rad(S_{M_i}(0)) = M_i$, $S_{M_i}(0)$ is a primary ideal. Note that $\cap_{i=1}^n S_{M_i}(0) = (0)$ and $S_{M_i}(0) + S_{M_j}(0) = R$ for each $i \neq j$ in { 1,..., n}. Therefore, $R \simeq \frac{R}{\cap_{i=1}^n S_{M_i}(0)}$. By the Chinese remainder theorem, $R \simeq \prod_{i=1}^n \frac{R}{S_{M_i}(0)}$, where $\frac{R}{S_{M_i}(0)}$ is quasilocal and zero-dimensional, for $i = 1, ..., n$.

If $Spec(R)$ is finite, then so is $Spec(R^0)$, and hence $Idem(R^0)$ is a finite set. By Lemma 8, we can write $R^0 = W_1 \oplus \cdots \oplus W_n$, where each W_i is a zero-dimensional quasilocal ring. Now, assume R^0 is quasilocal. In this case let M be the maximal ideal of R^0 and let $P_0 = M \cap R$. Then P_0 is the unique minimal prime of R. We conclude this section with a characterization of rings which are embeddable in a zero-dimensional ring by using the annihilator of ideals.

Proposition 1. *Let R be a subring of a ring T. Then $\mathscr{Z}(R,T) \neq \emptyset$ if and only if for each finitely generated ideal I, the set $\{Ann_R(I^j)\}_{j=1}^{\infty}$ stabilizes for some $m \in \mathbb{Z}^+$.*

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[1] A ring R satisfies Property A if each finitely generated ideal $I \subseteq Z(R)$ has a nonzero annihilator.

To prove this result, we need the following Lemma.

Lemma 9. *[5, Theorem 4.1] If R admits a zero-dimensional extension ring, then for each x in R, there exists a positive integer m_x such that x^{m_x} and x^{m_x+1} have the same annihilator in R.*

Proof of Proposition 1. Let $I = (r_1, \dots, r_k)$ be a finitely generated ideal of R. Let $X_1, \dots, X_k$ be indeterminates over R and let S_n be the set of all homogeneous polynomials $f \in R[X_1, \dots, X_k]$ of degree n such that $f(r_1, \dots, r_k) \in I^n$ (All terms of f have degree n). Let $S = \bigcup_{i=1}^{\infty} S_i$. Let $z \in I^n$, then $z = f(r_1, \dots, r_k)$, i.e., $z = \sum_{i=1}^{l} f_i(r_1, \dots, r_k)$, where $f_i \in S_n$ is a monomial for each $i = 1, \dots, l$. That means that $f_i(r_1, \dots, r_k) = r_1^{l_1} \dots r_k^{l_k}$ such that $l_1 + \dots + l_k = n$. Since $\mathscr{Z}(R, T) \neq \emptyset$, for each $i = 1, \dots, k$, there exists $m_i \in \mathbb{Z}^+$ such that $Ann_R(r_i^{m_i}) = Ann_R(r_i^{m_i+1})$. Let $s = m_1 + \dots + m_k = l_1 + \dots + l_k$, we have

$$f_i(r_1, \dots, r_k) = r_1^{l_1} \dots r_i^{m_i} \dots r_k^{l_k} \in I^{l_1+\dots+m_i+\dots+l_k}$$

for each $f_i \in S_{l_1+\dots+m_i+\dots+l_k}$. We observe that

$$Ann_R(r_1^{l_1} \dots r_i^{m_i} \dots r_k^{l_k}) = Ann_R(r_1^{l_1} \dots r_i^{m_i+1} \dots r_k^{l_k})$$

for each $i = 1, \dots, k$ and $r_1^{l_1} \dots r_i^{m_i} \dots r_k^{l_k} \in I^{l_1+\dots+m_i+\dots+l_k}$. In other words, by induction, we state that

$$Ann_R(f_i(r_1 \dots r_k)) = Ann_R(f_{i+n}(r_1 \dots r_k))$$

for each $i \in \mathbb{Z}^+$, and for each $f_i \in S_{m_1+\dots+m_k}$, i.e., $Ann_R(I^{m_1+\dots+m_k}) = Ann_R(I^{m_1+\dots+m_k+n})$ for each $n \in \mathbb{Z}^+$. We denote by $J = m_1 + \dots + m_k$, we have $Ann_R(I^j) = Ann_R(I^{j+n})$, for each $n \in \mathbb{Z}^+$. It follows that the sequence $\{Ann_R(I^j)\}_{j=1}^{\infty}$ stabilizes, which is what we wished to show. □

3 The Set of Intermediate Reduced Artinian Subrings

Artinian rings are the simplest kind of ring after a field, and we study them not because of their generality but because of their simplicity. Since any Artinian ring is zero-dimensional, before working over $\mathscr{A}(R, T)$ we have to recall some basic properties concerning the set $\mathscr{Z}(R, T)$. We denote that almost all rings are subrings of zero-dimensional rings. Among these rings which are included

(i) Integral domains, since each integral domain is a subring of its quotient field.
(ii) Reduced rings, these are precisely subrings of product of fields.

(iii) Any ring in which (0) is a finite intersection of primary ideals (this includes Noetherian rings and Laskerain rings).[2]

If R is an Artinian quasilocal ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is the only prime ideal of R and therefore $\mathfrak{m}$ is the nilradical of R. Hence every element of $\mathfrak{m}$ is nilpotent, and $\mathfrak{m}$ itself is nilpotent. Every element of R is either a unit or is nilpotent. In this section, we are interested in the problem (P). Next, we try to obtain results concerning the emptiness of $\mathscr{A}(R, T)$ for special pairs of rings.

Proposition 2. *Let R be a subring of T. Assume that $\mathscr{Z}(R, T) \neq \emptyset$. Then $\mathscr{A}(R, T) \neq \emptyset$ if and only if $\mathscr{A}(R^o, T) \neq \emptyset$.*

Proof. Since each Artinian ring is zero-dimensional, we have $\mathscr{A}(R, T) \subseteq \mathscr{Z}(R, T)$. Assume that $\mathscr{A}(R, T) \neq \emptyset$, that means that there exists an intermediate Artinian subring A between R and T, then $A \in \mathscr{Z}(R, T)$ and hence $R^o \subseteq A$, because R^o is the smallest zero-dimensional subring of T containing R. Therefore, $A \in \mathscr{A}(R^o, T)$. The converse is immediate from the fact that any Artinian subring of T containing R contains R^0.

Proposition 3. *Let $\{(R_\alpha, T_\alpha)\}_{\alpha \in A}$ be a family of pairs of rings. Then $\mathscr{A}(\prod_{\alpha \in A} R_\alpha, \prod_{\alpha \in A} T_\alpha) \neq \emptyset$ if and only if A is a finite set and $\mathscr{A}(R_\alpha, T_\alpha) \neq \emptyset$ for each $\alpha \in A$.*

Proof. Suppose that $\mathscr{A}(\prod_{\alpha \in A} R_\alpha, \prod_{\alpha \in A} T_\alpha) \neq \emptyset$, let S be an Artinian subring of $\prod_{\alpha \in A} T_\alpha$ containing $\prod_{\alpha \in A} R_\alpha$. Let $\pi_\beta : \prod_{\alpha \in A} T_\alpha \to T_\beta$ be the canonical projection homomorphism for each $\beta \in A$. Let $I_\beta = Ker(\pi_\beta)$, then $\pi_\beta(S) = S_\beta \simeq \frac{S}{I_\beta \cap S}$ and hence S_β is Artinian which satisfies $\pi_\beta(\prod_{\alpha \in A} R_\alpha) \subseteq S_\beta \subseteq \pi_\beta(\prod_{\alpha \in A} T_\alpha)$, i.e., $R_\beta \subseteq S_\beta \subseteq T_\beta$. It follows that, $S_\beta \in \mathscr{A}(R_\beta, T_\beta)$ for each $\beta \in A$. If A is infinite, then $\prod_{\alpha \in A} R_\alpha$ has an infinite idempotent elements. Indeed, let e_α be an element of $\prod_{\alpha \in A} R_\alpha$ defined by $e_\alpha(\beta) = 0$ if $\alpha \neq \beta$ and $e_\alpha(\alpha) = 1$. Then $e_\alpha^2 = e_\alpha$, i.e., $e_\alpha \in Idem(\prod_{\alpha \in A} R_\alpha)$ and $|Idem(\prod_{\alpha \in A} R_\alpha)|$ is infinite. We observe that for each $\alpha \in A$, $e_\alpha \in S$. That means that $|Idem(S)| > |Idem(\prod_{\alpha \in A} R_\alpha)|$. In other words, $Idem(S)$ is infinite. This is a contradiction with S is Artinian (see Lemma 3). Thus, A is finite. Conversely, since each $\mathscr{A}(R_\alpha, T_\alpha)$ is a non-empty set, let S_α be an Artinian subring of T_α containing R_α. By Lemma 5, $\prod_{\alpha \in A} S_\alpha$ is an Artinian subring of $\prod_{\alpha \in A} T_\alpha$ containing $\prod_{\alpha \in A} R_\alpha$, this is because of the finiteness of the set A.

As we saw in the previous section that there is a nice relationship between the finiteness of the set of idempotent elements and the cardinality of the spectrum of a commutative ring. More precisely, if a ring R has only finitely many prime ideals, then the set of idempotents is also finite.

When we turn to Artinian rings, it worth noticing that if R has infinite many distinct idempotent elements, then R is not embeddable in an in an Artinian ring, because each idempotent element of R is also an idempotent element of any extension of rings. In this section, for convenience, we assume that the set of idempotent elements is finite.

[2] A ring R is said to be Laskerian if each ideal of R admits a shortest primary representation.

Theorem 1. *Let R be a reduced subring of T. Then $\mathscr{A}(R,T) \neq \emptyset$ if and only if $Idem(R^o)$ is finite.*

To prove this result we need the following lemma.

Lemma 10. *Let R be a subring of a ring T and assume that $\mathscr{Z}(R,T) \neq \emptyset$. Then R is reduced implies that R^o is also reduced.*

Proof. If $\mathscr{Z}(R,T)$ is nonempty, then there exists a unique smallest zero-dimensional subring of T containing R, say R^o. The ring R^o is generated by R and idempotents $e_x \in T$ for each $x \in R$, i.e., $R^o = R[e_x/x \in R]$. Let E be the Boolean algebra of idempotents generated by the idempotents e_x for each $x \in R$. Then for each $x \in R$, x takes the following form:

$$x = r_1 f_1 + \cdots + r_n f_n,$$

where $f_i \in E$ are orthogonal and $r_i \in R$, for each $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, setting $e_{r_i} \in Idem(T)$. Let

$$e = e_{r_1} f_1 + \cdots + e_{r_n} f_n.$$

For each $s \in \mathbb{Z}^+$,

$$x^s = r_1^s f_1 + \cdots + r_n^s f_n.$$

As $f_i^2 = f_i$ and $f_i f_j = 0$ for each $i \neq j$. Then $x^s \neq 0$ for each $s \in \mathbb{Z}^+$, since R is a reduced ring. Hence R^o is reduced.

Proof of Theorem 1. Assume that $\mathscr{A}(R,T) \neq \emptyset$, i.e., let $A \in \mathscr{A}(R,T)$. Since R^o is the smallest zero-dimensional subring of T containing R, we may have $R^0 \subseteq A$. Then $|Idem(R^0)| < |Idem(A)|$. As A is Artinian, there exists $k \in \mathbb{Z}^+$ such that $|Idem(A)| < k$, that means that $|Idem(R^0)| < k$. It follows that $Idem(R^0)$ is a finite set. Conversely, suppose that $Idem(R^0)$ is finite. Since R is reduced, by Lemma 10, R^0 is reduced. By Lemma 2, R^0 is a von Neumann regular ring. Our hypothesis implies that R^0 is a von Neumann regular ring with finite spectrum. By Lemma 4, R^0 is a finite product of fields. Therefore, R^0 is Artinian. Thus $\mathscr{A}(R,T) \neq \emptyset$. □

An important remark: given $R \hookrightarrow T$ an extension of rings, if $\mathscr{C}(T) = \{char(\frac{T}{M}) : M \in Max(T)\}$ is infinite, by [9, Theorem 2.1], the set $\mathscr{A}(T)$ is empty. This shows the relationship between the set of Artinian intermediate subrings and the set of characteristics of residue fields of T.

Lemma 11. *[9, Theorem 2.1] Let R be a ring, then $\mathscr{A}(R) \neq \emptyset$ if and only if $\mathscr{Z}(R) \neq \emptyset$ and $\mathscr{C}(R)$ is a finite set.*

Using this result we state the relationship between Artinian subrings and zero-dimensional subrings since any Artinian ring is zero-dimensional, i.e., $\mathscr{A}(R) \subseteq \mathscr{Z}(R)$. Further, we note that if a ring R has $\mathscr{C}(R)$ is infinite, then R has no Artinian subrings. Assume that $\mathscr{C}(T)$ is a finite set. Since R^0 is zero-dimensional, we have $\mathscr{C}(T) = \mathscr{C}(R^0)$. In other words, $\mathscr{C}(R^0)$ is a finite set. Observe that if R is a zero-dimensional ring of characteristic zero, then $0 \in \mathscr{C}(R)$. This is clear from the fact that $\mathbb{Z} \subseteq R$ impies that there exists a prime of R lying over (0) in $\mathbb{Z}$.

Now, let R be a ring of characteristic 0. That means that R contains π, the prime subring, which is isomorphic to $\mathbb{Z}$. Then R is an extension of $\mathbb{Z}$. Therefore, it is desirable to find the existence of intermediate Artinian subring in $\mathscr{A}(\mathbb{Z}, R)$.

Proposition 4. *For each Artinian subring in $A \in \mathscr{A}(\mathbb{Z}, R)$, we have A contains $\mathbb{Q}$ the field of rational numbers.*

Proof. Assume that $\mathscr{A}(\mathbb{Z}, R) \neq \emptyset$ and let A be an Artinian subring of R containing $\mathbb{Z}$. Since each Artinian is a finite direct product of quasilocal Artinian rings, we can assume without loss of generality that A is a quasilocal ring. It is well known that a quasilocal ring of characteristic zero that contains a zero-dimension subring contains $\mathbb{Q}$. That implies that A contains $\mathbb{Q}$.

From this result we state the following.

Corollary 1. *Let R be a ring of characteristic* 0. *Then $\mathscr{A}(\mathbb{Z}, R) \neq \emptyset$ if and only if $\mathbb{Q} \subseteq R$.*

Remark 2. If $\mathscr{A}(\mathbb{Z}, R)$ is a non-empty set, then R needs not be semi-quasilocal. Indeed, let $R = \mathbb{Q}^{w_0}$ be a countable direct product of $\mathbb{Q}$. We observe that $\mathscr{A}(\mathbb{Z}, R) \neq \emptyset$ since $\mathbb{Q}^{(i)} = \{\{x_i\}_{i=1}^{\infty} : x_i = x_{i+1} = \dots\}$. It is not difficult to see that $\mathbb{Q}^{(i)} = \mathbb{Q}^i$ is the finite direct product of the field $\mathbb{Q}$, and hence $\mathbb{Q}^{(i)}$ is an Artinian subring of R. Thus $\mathscr{A}(\mathbb{Z}, R) \neq \emptyset$.

Finally, if R is a finitely generated extension of $\mathbb{Z}$, then R does not contain a zero-dimensional subring (cf. [6, Proposition 2.2]). This is an example of a ring which cannot be embedded in zero-dimensional extension. Let R be a subring of $\mathbb{Z}$, if $\mathscr{C}(R)$, consists of all characteristics of residue fields of R, is infinite, then $\mathbb{Z}_V$ is a Hilbert ring [3] of positive dimension, and by [6, Proposition 2.2], R has not zero-dimensional subring if R is finitely generated over $\mathbb{Z}_V$. Thus we can state the following result.

Corollary 2. *Let R be an overring of $\mathbb{Z}$. Assume that $\mathscr{C}(R)$ is finite. Then $\mathscr{Z}(R) \neq \emptyset$ if and only if $\mathscr{A}(R) \neq \emptyset$. That means that if R contains a zero-dimensional subring, then R also contains an Artinian subring. When $\mathscr{C}(R)$ is infinite, we have $\mathscr{Z}(R) = \emptyset$ and hence $\mathscr{A}(R) = \emptyset$.*

Remark 3. A subring of a Noetherian ring is not necessarily Noetherian. Indeed, consider $R = k[X_1, X_2, \dots]$ the polynomial ring of infinitely many indeterminates, with k is a field. There is an ascending chain of ideals

$$(X_1) \subset (X_1, X_2) \subset \cdots \subset (X_1, X_2, \dots, X_i) \subset \dots .$$

So R is not Noetherian. But clearly K, its quotient fields, is Noetherian.

[3] A ring R is said to be Hilbert if for each proper prime ideal of R is an intersection of maximal ideals of R.

Intermediate Artinian subring is preserved by taking quotient and ring-homomorphism.

Lemma 12. *Let R be a subring of a ring T and $\varphi : T \to T'$ be a ring-homomorphism. If $\mathscr{A}(R, T) \neq \emptyset$, then $\mathscr{A}(\varphi(R), \varphi(T)) \neq \emptyset$.*

Proof. Let S be an Artinian subring of $\varphi(T)$ containing $\varphi(R)$, i.e., $\varphi(R) \subseteq S \subseteq \varphi(T)$. Let A be the inverse image of S by φ, so $R \subseteq A \subseteq T$. Since $S = \varphi(A) = \frac{A}{Ker(\varphi)}$ is an Artinian ring, so we have $\mathscr{A}(\varphi(R), \varphi(T)) \neq \emptyset$.

Example 1. Let p be a positive prime number. Let $R = \prod_{n=1}^{\infty}(\frac{\mathbb{Z}}{p\mathbb{Z}})^n$ be an infinite direct product of $\frac{\mathbb{Z}}{p\mathbb{Z}}$. Since $char(R) = p$, we have $\mathbb{Z}_p^*$ is isomorphic to $\mathbb{Z}_p$ which means that $\mathbb{Z}_p^*$ is Artinian ring. Therefore, $\mathscr{A}(\mathbb{Z}_p^*, R) \neq \emptyset$. However, $Idem(R)$ is an infinite set.

Proposition 5. *Let R be a subring of a ring T. If R^0 is semi-quasilocal, then R has exactly many minimal primes. In particular, R^0 is the total quotient ring of R.*

Proof. Let $\{M_i\}_{i=1}^n$ be the set of maximal primes of R^0. Then $P_i = M_i \cap R$ is the set of minimal primes of R. In other words, $Z(R) = \bigcup_{i=1}^n P_i$ is the set of zero-divisors of R. Thus the total quotient ring of R is $Q(R) = S^{-1}R$, where $S = R \setminus (\bigcup_{i=1}^n P_i)$, which is zero-dimensional and is contained in R^0. Since R^0 is the smallest zero-dimensional subring of T containing R. Then $Q(R) = R^0$.

Proposition 6. *Let R be a reduced ring in which* (0) *has a finite primary decomposition. If* $(0) = Q_1 \cap \cdots \cap Q_n$ *be a reduced primary decomposition of* (0)*, where Q_i is a P_i- primary ideal in R. Then R is embeddable in an Artinian overring.*

Proof. Assume that $(0) = Q_1 \cap \cdots \cap Q_n$ is a finite primary decomposition. As $(0) = Q_1 \cap \cdots \cap Q_n$, the ring-homomorphism $\varphi : R \to \frac{R_{P_1}}{Q_1 R_{P_1}} \bigoplus \cdots \bigoplus \frac{R_{P_n}}{Q_n R_{P_n}} = T$ defined by $r \mapsto (\frac{r}{1} + Q_1 R_{P_1}, \ldots, \frac{r}{1} + Q_n R_{P_n})$ is injective. Then we can regard R as a subring of T. We have each $\frac{R_{P_i}}{Q_i R_{P_i}}$ is also reduced for each $i = 1, \ldots, n$. It follows that $\frac{R_{P_i}}{Q_i R_{P_i}}$ is a von Neumann regular quasilocal ring for each $i = 1, \ldots, n$. Therefore, $\frac{R_{P_i}}{Q_i R_{P_i}}$ is a field for each $i = 1, \ldots, n$, since each any von Neumann regular with only many maximal ideals is a field. Hence $\frac{R_{P_1}}{Q_1 R_{P_1}} \bigoplus \cdots \bigoplus \frac{R_{P_n}}{Q_n R_{P_n}}$ is a finite direct product of fields. Then R is embeddable in a Artinian overring. In other words, R is a subring of an Artinian ring.

Remark 4. If $(0) = \bigcap_{\alpha \in A} Q_\alpha$, where $\{Q_\alpha\}_{\alpha \in A}$ is an infinite family of primary ideals, then R needs not be a subring of an Artinian as shows the following example: Let p be a prime number. We have $\bigcap_{i=1}^{\infty} p^i\mathbb{Z} = (0)$ and each $p^i\mathbb{Z}$ is $p\mathbb{Z}$-primary ideal of $\mathbb{Z}$. We observe that the ring-homomorphism $\varphi : \mathbb{Z} \to \prod_{i=1}^{\infty} \frac{\mathbb{Z}}{p^i\mathbb{Z}}$ defined by $\varphi(n) = n + p^i\mathbb{Z}$ is injective. We state that $\mathscr{A}(\mathbb{Z}, \prod_{i=1}^{\infty} \frac{\mathbb{Z}}{p^i\mathbb{Z}}) = \emptyset$, otherwise let $A \in \mathscr{A}(\mathbb{Z}, \prod_{i=1}^{\infty} \frac{\mathbb{Z}}{p^i\mathbb{Z}})$, which means that $\mathscr{Z}(\mathbb{Z}, \prod_{i=1}^{\infty} \frac{\mathbb{Z}}{p^i\mathbb{Z}}) \neq \emptyset$, by [9, Theorem 2.1], this a contradiction with $\prod_{i=1}^{\infty} \frac{\mathbb{Z}}{p^i\mathbb{Z}}$ has no zero-dimensional subring. Thus, $\mathscr{A}(\mathbb{Z}, \prod_{i=1}^{\infty} \frac{\mathbb{Z}}{p^i\mathbb{Z}}) = \emptyset$.

Remark 5. If $Spec(R^0)$ is finite, then the set of idempotents of R^0 is finite (cf. Lemma 3).We can write R^0 as direct sum $R^0 = R^0e_1 \oplus \cdots \oplus R^0e_n$, where each e_i is a nonzero idempotent, each R^0e_i is a quasilocal ring. We can replace R by $R[e_1, \ldots, e_n] = Re_1 \oplus \cdots \oplus Re_n$.

Proposition 7. *Let R be a subring of a ring T. Assume R^0 is a quasilocal ring. If R is Noetherian then R is embeddable in Artinian overring.*

Proof. Assume that R is a Noetherian and $\mathscr{Z}(R, T) \neq \emptyset$. Let R^0 be the minimal zero-dimensional subring of T containing R. Since R^0 is quasilocal with maximal ideal $\mathfrak{m}$, we may have $P = \mathfrak{m} \cap R$ is the unique minimal ideal of R. In other words, $dim(R_P) = 0$ and $R_P \subseteq R^0$, that means, $R^0 = R_P$. As R is Noetherian, we have R^0 is also Noetherian. By [4, Theorem 8.5], R^0 is Artinian.

Theorem 2. *Let R be a subring of a ring T. Assume that the nilradical $N(T)$ of T is a finitely generated $R-$ module. If $\mathscr{Z}(R, T) \neq \emptyset$ then $\mathscr{A}(R, T) \neq \emptyset$.*

Proof. If $\mathscr{Z}(R, T)) \neq \emptyset$, we consider R^0 the minimal zero-dimensional subring of T containing R. To show that $\mathscr{A}(R, T))$ is nonempty, it suffices to prove that R^0 is an Artinian subring of T containing R. Also to show R^0 is Artinian it suffices, by Cohen's Theorem, to show that $N(R^0)$ is finitely generated as an ideal of R^0. This is because $\frac{R^0}{N(R^0)}$ is a reduced zero-dimensional with only finitely many idempotent elements. In other words, is von Neumann regular with only finitely many idempotents which means that $\frac{R^0}{N(R^0)}$ is finite direct product of fields and hence is Artinian. Furthermore, observe that $N(R^0) = N(T) \cap R^0$ is a finitely generated as R^0- module. By Cohen's Theorem, R^0 is an Artinian subring of T containing R. Thus $\mathscr{A}(R, T) \neq \emptyset$.

Remark 6. If R is not finitely generated as R-module, then $N(R)$ needs not be finitely generated. Indeed, let K be a field and $\{X_i\}_{i=1}^{\infty}$ be an infinite family of indeterminates over K. We consider I as an ideal of $K[[X_1, X_2, \ldots]]$, the ring of formal power series in variables $\{X_i\}_{i=1}^{\infty}$, generated by $(X_iX_jX_k, X_i^2,$ all $i, j, k \in \mathbb{Z}^+)$. Observe that $S = \frac{K[[X_1, X_n, \ldots]]}{I}$ is a quasilocal zero-dimensional ring with maximal ideal $\overline{M} = \frac{(X_1, X_2, \ldots)}{I}$. Then $N(S) = (x_1, x_2, \ldots)$ is not finitely generated ideal of S, where $x_i = X_i + I$ for each $i \in \mathbb{Z}^+$. Let's construct T that satisfies the condition that $N(T)$ is finitely generated. Let $T = \frac{S[[t_1, t_2, Y_{n_i} : i=1, 2\ldots, n=1, 2, \ldots]]}{(t_1^2, t_2^2, X_i - Y_{n_1}t_1 - Y_{n_2}t_2)}$, all t_i and Y_{n_i} are indeterminates over S. Let $f : S \to T$ be the natural ring-homomorphism, it is an inclusion. Further, the ring T obviously has $N(T) = (\overline{t_1}, \overline{t_2})$.

Proposition 8. *Let R be a subring of a ring S. Assume that $\mathscr{Z}(R, S) \neq \emptyset$ and $Idem(S)$ is finite. If $N(R)$ is a finitely generated ideal then $\mathscr{A}(R, S) \neq \emptyset$.*

Proof. Let $Idem(S) = \{e_i\}_{i=1}^{n}$, then we can express S as a finite direct sum of indecomposable rings,[4] i.e., $S = \bigoplus_{i=1}^{n} Se_i$, where e_i is an idempotent element of S, for each $i = 1, \ldots, n$. Since $\mathscr{Z}(R, S) \neq \emptyset$, let R^0 be the minimal zero-dimensional subring of S containing R. As Se_i is a quasilocal ring for each

[4] A ring R is said to be indecomposable if it cannot be written as a direct sum of nontrivial rings.

$i = 1, \ldots, n$. Then $R^0e_1 \oplus \cdots \oplus R^0e_n = R^0[e_1, \ldots, e_n]$ is a integral extension of R^0, then $dim(R^0[e_1, \ldots, e_n]) = 0$. Observe that R^0e_i is a zero-dimensional subring of Se_i containing Re_i, by Lemma 2, R^0e_i is the minimal zero-dimensional subring of Se_i containing Re_i, for each e_i. We can assume without loss of generality that R^0 is quasilocal with maximal ideal $\mathfrak{m}$. We have $N(R^0) = \mathfrak{m}$ and $N(R) = \mathfrak{m} \cap R = P$ is the unique minimal prime ideal of R. Then R_P is a zero-dimensional subring of S containing R. By the minimality of R^0, we have $R^0 = R_P$ and $PR_P = \mathfrak{m}$, and hence $\mathfrak{m}$ is a finitely generated ideal as P is a finitely generated ideal of R. According to Cohen's Theorem, R^0 is a Noetherian ring. Then $\mathscr{A}(R, S) \neq \emptyset$.

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