Mixed Unit Interval Bigraphs

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Abstract. The class of intersection bigraphs of unit intervals of the real line whose ends may be open or closed is called mixed unit interval bigraphs. This class of bigraphs is a strict superclass of the class of unit interval bigraphs. We provide several infinite families of forbidden induced subgraphs of mixed unit interval bigraphs. We also pose a conjecture concerning characterization of mixed unit interval bigraphs and verify parts of it.

Keywords: Interval bigraphs · Unit interval bigraphs
Mixed unit interval bigraphs

1 Introduction

Interval graphs are the intersection graphs of intervals of the real line. *Unit interval graphs* are interval graphs where all the intervals are of unit length. *Proper interval graphs* are interval graphs where no interval is properly contained in another. Interval graphs and their subclasses like unit/proper interval graphs have been extensively studied by several researchers from structural [7,10], algorithmic [2,3] and application [9] view point.

However, most of the researchers do not specify which type of interval is used, that is, whether the ends of the intervals are open, closed or semi-closed. This is acceptable because the class of graphs does not actually depend on this. Frankl and Meahara [8] observed that using only open intervals or only closed intervals leads to the same class of graphs. In [6] it was shown that this is even true when we allow all possible types of intervals in the intersection representation. This is no longer true for the class of unit interval graphs. Rautenbach and Szwarcfiter [15] showed that the class of intersection graphs of unit intervals of open and closed intervals is a strict superclass of the class of unit interval graphs. They also characterized this class of graphs, by a finite list of forbidden induced subgraphs. Dourado et al. [6] generalized the result of [15] to mixed unit interval graphs allowing all four distinct types of unit intervals. Felix Joos [12] gave a complete characterization of mixed unit interval graphs in terms of infinite families of forbidden induced subgraphs.

A bipartite graph (in short, bigraph) $B = (X, Y, E)$ is an *interval bigraph* if there exist a one-to-one correspondence between the vertex set $X \cup Y$ of B and a collection of intervals $\{I(v) : v \in X \cup Y\}$ on the real line such that two vertices

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are adjacent if and only if their corresponding intervals intersect and they belong to different partite sets. The collection of intervals $\{I(v) : v \in X \cup Y\}$ is called an interval representation of B. We simply denote the interval representation of B by I (which is a function from the vertex set $X \cup Y$ to a collection of intervals).

An interval bigraph is a *unit interval bigraph* if all the intervals in the interval representation are of unit length. An interval bigraph $B = (X, Y, E)$ is a *proper interval bigraph* if in the interval representation no interval is properly contained in another. Hell and Huang [11] proved that an interval bigraph is a unit interval bigraph if and only if it does not contain the bipartite claw (H_1), the bipartite net (H_2) or the bipartite tent (H_3) as an induced subgraph (see Fig. 1). In [4] we observe that the bigraphs H_1, H_2 and H_3 have intersection representation with unit open and closed intervals. In the same paper we give a characterization of the class of finite intersection bigraphs of unit open and closed intervals in terms of forbidden induced bigraphs.

In the present paper we generalize the results of [4] to the mixed unit interval bigraphs where we allow all four types of unit intervals namely closed, open, left closed-right open and right closed-left open unit interval in the interval representation. Here we show that the list of forbidden induced subgraphs for mixed unit interaval bigraphs is infinite.

In Sect. 2 we introduce basic definitions, terminology, and results related to our work. In Sect. 3 we give some forbidden induced subgraphs of mixed unit interval bigraphs. In Sect. 4 we pose a conjecture concerning characterization of mixed unit interval bigraphs and verify parts of it.

2 Preliminary Results

We consider only simple, finite and connected bigraphs. For a bigraph $B = (X, Y, E)$ the neighbourhood of a vertex $u \in X \cup Y$ is denoted by $N_B(u)$. Two distinct vertices u and v of B are copies if $N_B(u) = N_B(v)$. If no two vertices of B are copies then B is copy-free. If $\mathcal{F}$ is a set of graphs and a graph G does not contain a graph in $\mathcal{F}$ as an induced subgraph then G is $\mathcal{F}$-free.

Let $\mathcal{M}$ be a family of sets. An $\mathcal{M}$-intersection representation of a bigraph is a function $f : X \cup Y \to \mathcal{M}$ such that for any two distinct vertices u and v of a bigraph B, we have $uv \in E$ if and only if $f(u) \cap f(v) \neq \emptyset$. A bigraph is an $\mathcal{M}$-bigraph if it has an $\mathcal{M}$-intersection representation.

For two real numbers a and b, we denote the open interval $\{x \in \mathbb{R} | a < x < b\}$ by (a, b), the closed interval $\{x \in \mathbb{R} | a \leq x \leq b\}$ by $[a, b]$, the open-closed interval $\{x \in \mathbb{R} | a < x \leq b\}$ by $(a, b]$ and the closed-open interval $\{x \in \mathbb{R} | a \leq x < b\}$ by $[a, b)$. For an interval I, let $l(I) = \inf(I)$ and $r(I) = \sup(I)$. We suppose $\mathcal{I}^{++}$ is the set of closed intervals, $\mathcal{I}^{--}$ is the set of open intervals, $\mathcal{I}^{+-}$ is the set of closed-open intervals and $\mathcal{I}^{-+}$ is the set of open-closed intervals. Also suppose $\mathcal{U}^{++}$ is the set of unit closed intervals, $\mathcal{U}^{--}$ is the set of unit open intervals, $\mathcal{U}^{+-}$ is the set of unit closed-open intervals and $\mathcal{U}^{-+}$ is the set of unit open-closed intervals. In addition, let $\mathcal{I}^{\pm} = \mathcal{I}^{++} \cup \mathcal{I}^{--}$, $\mathcal{U}^{\pm} = \mathcal{U}^{++} \cup \mathcal{U}^{--}$, $\mathcal{I} = \mathcal{I}^{++} \cup \mathcal{I}^{--} \cup \mathcal{I}^{+-} \cup \mathcal{I}^{-+}$, and $\mathcal{U} = \mathcal{U}^{++} \cup \mathcal{U}^{--} \cup \mathcal{U}^{+-} \cup \mathcal{U}^{-+}$.

Our first result shows that as in the case of interval graphs, the class of interval bigraphs does not depend on the type interval used in the intersection representation.

Proposition 1. *The classes of $\mathcal{I}^{++}$-bigraphs, $\mathcal{I}^{--}$-bigraphs, $\mathcal{I}^{\pm}$-bigraphs, $\mathcal{I}^{+-}$-bigraphs, $\mathcal{I}^{-+}$-bigraphs and $\mathcal{I}$-bigraphs are the same.*

The following proposition extends the result of Proposition 2 of [6] which showed that a bigraph is a $\mathcal{U}^{++}$-bigraph if and only if it is a $\mathcal{U}^{--}$-bigraph.

Proposition 2. *The classes of $\mathcal{U}^{++}$-bigraphs, $\mathcal{U}^{--}$-bigraphs, $\mathcal{U}^{+-}$-bigraphs, $\mathcal{U}^{-+}$-bigraphs and $\mathcal{U}^{+-} \cup \mathcal{U}^{-+}$-bigraphs are the same.*

The proofs of the above propositions are similar to the proof of Dourado et al. [6] and so omitted.

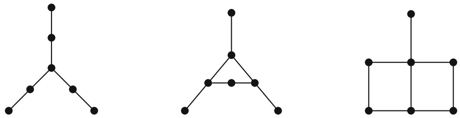

Fig. 1. The bipartite claw (H_1), net (H_2) and tent (H_3)

Interval bigraphs coincide with $\mathcal{I}^{++}$-bigraphs and unit interval bigraphs coincide with $\mathcal{U}^{++}$-bigraphs. Following theorem relates the class of interval bigraphs, unit interval bigraphs and proper interval bigraphs.

Theorem 3 **([11,13,16]).** *An interval bigraph is a unit interval bigraph if and only if it is a proper interval bigraph if and only if it does not contain H_1, H_2 or H_3 as an induced subgraph.*

As mentioned in the introduction, the bigraphs H_1, H_2 and H_3 have $\mathcal{U}$-intersection representation; see Figs. 2, 3 and 4.

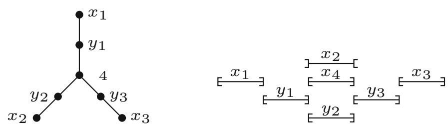

Fig. 2. The bipartite claw H_1 and its $\mathcal{U}$-intersection representation.

As observed in [4] each intersection representation of H_1, H_2 and H_3 is unique upto trivial modifications (these trivial modifications include suitable interval shifts that preserve intersections and relative positions between intervals, changes in the types (open, closed or half closed) of some intervals, reflection of the

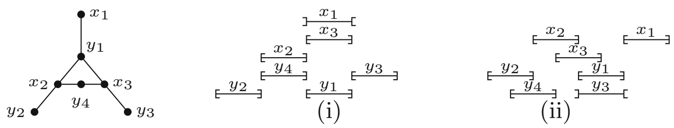

Fig. 3. The bipartite Net H_2 and its two $\mathcal{U}$-intersection representation.

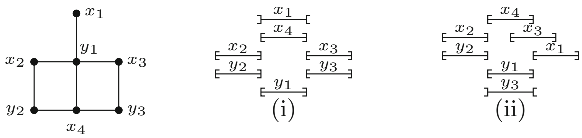

Fig. 4. The bipartite tent H_3 and its two $\mathcal{U}$-intersection representation.

entire model about a point on the real line, translation of the entire model, and relabeling of some intervals).

Therefore the class of $\mathcal{U}^{\pm}$-bigraphs is a strict superclass of the class of unit interval bigraphs. We have characterized these class of bigraphs in [4] (Fig. 5).

For an $\mathcal{I}^{++}$-bigraph if two vertices u and v are copies then they belong to the same partite set. And in the $\mathcal{I}^{++}$-interval representation we can take same interval for these two vertices. Thus we consider that our bigraphs are copy free.

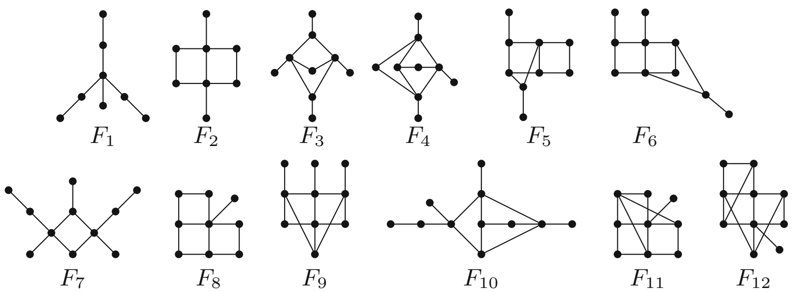

Fig. 5. Forbidden induced subgraphs of $\mathcal{U}^{\pm}$-bigraphs.

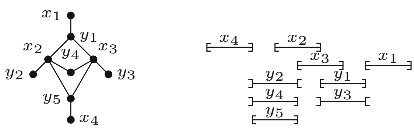

Fig. 6. The bigraph F_3 and its $\mathcal{U}$-intersection representation.

Theorem 4 **([4]).** *For a copy-free bipartite graph B, the following statements are equivalent.*

(i) *B is a $\{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}\}$-free interval bigraph.*
(ii) *B is an almost proper interval bigraph.*
(iii) *B is a $\mathcal{U}^{++} \cup \mathcal{U}^{--}$-bigraph.*

3 Forbidden Induced Subgraphs of Mixed Unit Interval Bigraphs

It can be observed that the bigraphs F_2, F_4, F_5, F_8, F_9, F_{10}, F_{11} and F_{12} have no $\mathcal{U}$-intersection representation. In this section we shall give some other forbidden induced subgraphs of mixed unit interval bigraphs (Figs. 7, 8, 9, 10, 11 and 12).

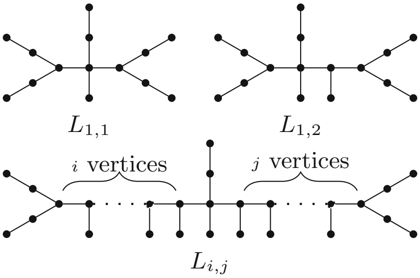

Fig. 7. The class $\mathcal{L}$

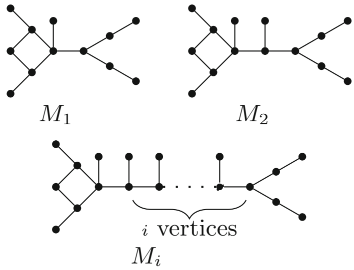

Fig. 8. The class $\mathcal{M}$.

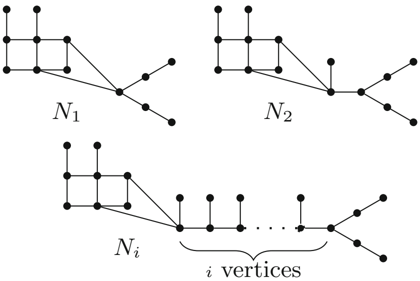

Fig. 9. The class $\mathcal{N}$.

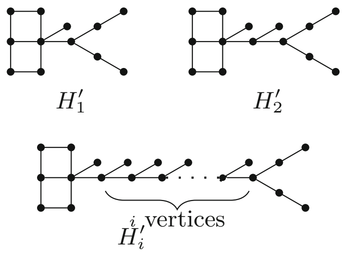

Fig. 10. The class $\mathcal{H}'$.

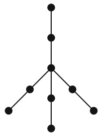

Fig. 11. The bigraph B_1

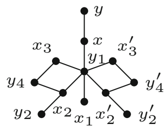

Fig. 12. The bigraph B_2

Lemma 5. *The bigraph F_1 has unique $\mathcal{U}$-intersection upto trivial modifications.*

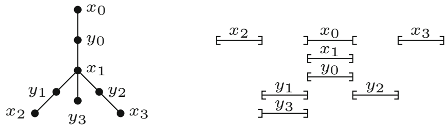

Fig. 13. The graph F_1 and its $\mathcal{U}$-representation.

Proof. The proof follows from the Proposition 5 of [4]. The bigraph $F_1 - y_3$ is the bipartite claw (H_1). Thus from that Proposition, H_1 has a unique $\mathcal{U}$-intersection as shown in Fig. 2. Since the vertex y_3 is adjacent to x_1 only, so we can take $I(y_3)$ as in Fig. 13 to get the $\mathcal{U}$-intersection representation of F_1. Again $I(y_3)$ can be taken as closed-open copy of $I(y_2)$ and make some trivial modifications to get the same representation of F_1 as earlier. This completes the proof. □

From the above Lemma we have the following corollary.

Corollary. *The bigraph B_1 is the minimal forbidden induced subgraph of the class of $\mathcal{U}$-intersection bigraphs (Fig. 14).*

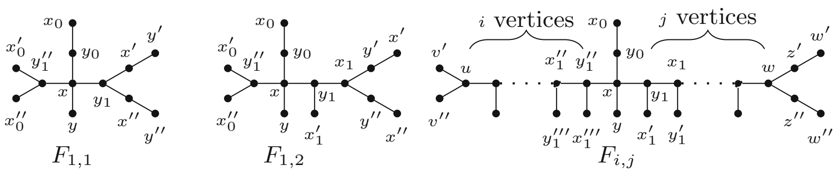

Fig. 14. The class $\mathcal{F}'$ of bigraphs

In the bigraph $F_{i,j}$, if i is even then $u \in X$ and $v', v'' \in Y$; and if i is odd then $u \in Y$ and $v', v'' \in X$. Similarly if j is even then $w \in X$ and if j is odd then $w \in Y$ and w, z are vertices of different partite sets. The two vertices v' and v'' are the *special vertices* of $F_{i,j}$. In the next lemma, we show that the bigraph $F_{i,j}$ has a unique $\mathcal{U}$-intersection representation upto trivial modifications.

Lemma 6. *Let* $i, j \in \mathbb{N}$

(a) A $\mathcal{U}$*-intersection representation* $I : V(F_{i,j}) \to \mathcal{U}$ *of* $F_{i,j}$*, where* $I(V(F_{i,j}))$ *consists of the following intervals*
- $I(x) = I(y_0) = [0, 1]$, $I(x_0) = (0, 1)$, $I(y) = (-1, 0]$
- $I(y_k) = [2(k-1)+1, 2(k-1)+2]$, $I(x_k) = [2(k-1)+2, 2(k-1)+3]$, $(k \geq 1)$
- $I(x'_k) = (2(k-1)+1, 2(k-1)+2)$, $I(y'_k) = (2(k-1)+2, 2(k-1)+3)$, $(k \geq 1)$
- $I(y''_k) = [-2(k-1)-1, -2(k-1)]$, $I(x''_k) = [-2(k-1)-2, -2(k-1)-1]$, $(k \geq 1)$
- $I(x'''_k) = (-2(k-1)-2, -2(k-1)-1]$, $I(y'''_k) = (-2k-1, -2k]$, $(k \geq 1)$
- $I(w) = [j, j+1]$, $I(z'') = I(w'') = (j, j+1)$, $I(z') = [j+1, j+2]$, $I(w') = [j+2, j+3]$ or $[\mathrm{j}+2, \mathrm{j}+3)$ *and*
- $I(u) = [-i, -i+1]$, $I(v') = I(v'') = [-i-1, -i]$ or $(-\mathrm{i}-1, -\mathrm{i}]$

is unique upto trivial modifications.

(b) $L_{i,j}$ *is a minimal forbidden induced subgraph for the class of* $\mathcal{U}$*-bigraphs.*

Proof. (a) It can be easily observed that every $F_{i,j}$ contains F_1 as an induced subgraph. Consider $F_{1,1}$, it contains F_1 as an induced subgraph, where $V(F_1) = \{x_0, y_0, x, y, y_1, y''_1, x'_0, x'\}$. Without loss of generality we consider the $\mathcal{U}$-intersection of F_1 for the vertices $x_0, y_0, x, y, y_1, y''_1, x'_0, x'$ as follows: $I(x) = I(y_0) = [0, 1]$, $I(x_0) = (0, 1)$, $I(y_1) = [1, 2]$, $I(x') = [2, 3]$, $I(y'') = [-1, 0]$, $I(y) = (-1, 0]$, $I(x'_0) = [-2, -1]$ or $(-2, -1]$. Next we take $I(x'') = I(y'') = (1, 2)$, $I(y') = [3, 4]$ or $[3, 4)$ and $I(x''_0) = I(x'_0)$. Now consider $F_{i,j}$ and let $j = 2n$. Then $w = x_n$, and the path $y_1, x_1, y_2, x_2, \ldots, y_k, x_k, \ldots, y_n, x_n$ has the representation $I(y_1) = [1, 2]$, $I(x_1) = [2, 3]$, $I(y_2) = [3, 4]$, $I(x_2) = [4, 5]$, and by induction $I(y_k) = [2(k-1)+1, 2(k-1)+2]$ and $I(x_k) = [2(k-1)+2, 2(k-1)+3]$. Then we have $I(x_n) = [2(n-1)+2, 2(n-1)+3] = [2n, 2n+1]$. In the other case, if $j = 2n+1$, then $w = y_{n+1}$ and we have $I(y_{n+1}) = [2(n+1-1)+1, 2n+2] = [2n+1, 2n+2]$. Thus $I(w) = [j, j+1]$, $I(z') = [j+1, j+2]$, $I(w') = [j+2, j+3]$ or $[j+2, j+3)$. As z'' is adjacent to w only in this path and w'' is adjacent to z'' so we take $I(z'') = I(w'') = (j, j+1)$. As the vertices x_0, y_0, x, y belong to $F_{i,j}$, we take the interval representation of these vertices as before (i.e. as in the case of $F_{1,1}$).

Next x'_k is adjacent to y_k and y'_k is adjacent to x_k only so we take $I(x'_k)$ as the open copy of $I(y_k)$ and $I(y'_k)$ as the open copy of $I(x_k)$. Again consider the path $x, y''_1, x''_1, \ldots, y''_k, x''_k, \ldots, u$. As $I(x) = [0, 1]$ we take the interval representation of $I(y''_1) = [-1, 0]$, $I(x''_1) = [-2, -1]$, $I(y''_2) = [-3, -2]$, $I(x''_2) = [-4, -3]$, and by induction $I(y''_k) = [-2(k-1)-1, -2(k-1)]$ and $I(x''_k) = [-2(k-1)-2, -2(k-1)-1]$. If $i = 2m$, then $u = x_m$; so $I(u) = I(x_m) = [-2(m-1)-2, -2(m-1)-1] = [-2m, -2m+1]$. And if $i = 2m+1$, then $u = y_{m+1}$, so $I(u) = [-2(m+1-1)-1, -2(m+1-1)] = [-2m-1, -2m]$. Thus $I(u) = [-i, -i+1]$ and $I(v'_0)$ and $I(v''_0) = [-i-1, -i]$ or $(-i-1, -i]$. Finally, x'''_k is adjacent to y''_k only and y'''_k is adjacent to x''_k only. So we take $I(x'''_k)$ as the open-closed copy of $I(x''_k)$ and $I(y'''_k)$ as the open-closed copy of $I(y''_{k+1})$. Which completes the proof of (a).

(b) $L_{i,j}$ is obtained from $F_{i,j}$ by adjoining two distinct vertices u' and u'' with v' and v'', where u' is adjacent to v' and u'' is adjacent to v''. Thus from $\mathcal{U}$-intersection representation of $F_{i,j}$, it follows that $L_{i,j}$ is the minimal forbidden induced subgraph of $\mathcal{U}$-bigraphs. □

As before the two vertices v' and v'' are the special vertices of M_i'. In the following lemma we show that the bigraph M_i' has a unique $\mathcal{U}$-intersection representation upto trivial modifications.

Lemma 7. *Let* $i \in \mathbb{N}$

(a) A $\mathcal{U}$*-intersection representation* $I : V(M_i') \to \mathcal{U}$ *of* M_i'*, where* $I(V(M_i'))$ *consists of the following intervals*
- $I(x_2'') = I(y_4'') = [0,1]$, $I(x_3'') = I(y_1) = [1,2]$, $I(y_2'') = [-1,0]$ *or* $(-1,0]$, $I(y_3'') = (1,2)$
- $I(x_k) = [2k, 2k+1]$, $I(x_k') = [2k, 2k+1)$, $(k \geq 1)$
- $I(y_k) = [2k-1, 2k]$, $I(y_{k-1}') = [2k-1, 2k)$, $(k \geq 2)$ *and*
- $I(u) = [i+1, i+2]$, $I(v') = I(v'') = [i+2, i+3]$ *or* $[i+2, i+3)$

is unique upto trivial modifications.

(b) M_i *is a minimal forbidden induced subgraph for the class of* $\mathcal{U}$*-bigraphs (Fig. 15).*

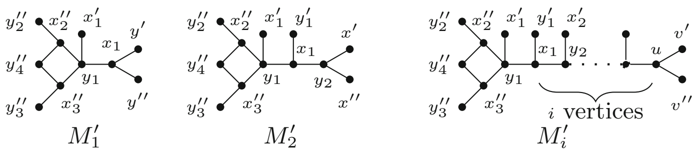

Fig. 15. The class $\mathcal{M}'$ of bigraphs.

Proof. (a) It can be easily observed that the bigraph M_i' contains H_2 as an induced subgraph, where $V(H_2) = \{x_1, y_1, x_2'', y_2'', x_3'', y_3'', y_4''\}$. From Proposition 6 of [4] we take the intervals corresponding to these vertices as $I(x_2'') = I(y_4'') = [0,1]$, $I(y_2'') = [-1,0]$ or $(-1,0]$, $I(x_3'') = I(y_1) = [1,2]$, $I(y_3') = (1,2)$, $I(x_1) = [2,3]$. As x_1' is adjacent to y_1 only we must take $I(x_1') = [2,3)$. In M_i' consider the path $P : y_1, x_1, y_2, x_2, \ldots, y_k, x_k, \ldots, u$. If $i = 2n$, then $u = y_{n+1}$ and if $i = 2n+1$, then $u = x_{n+1}$. Now intervals corresponding to the vertices y_2, x_2, y_3, x_3 are $I(y_2) = [3,4]$, $I(x_2) = [4,5]$, $I(y_3) = [5,6]$, $I(x_3) = [6,7]$. Thus by induction, $I(y_k) = [2k-1, 2k]$ and $I(x_k) = [2k, 2k+1]$. Hence $I(y_{n+1}) = [2n+1, 2n+2]$ and $I(x_{n+1}) = [2n+2, 2n+3]$.

Thus $I(u) = [i+1, i+2]$ and $I(v)$ and $I(v'') = [i+2, i+3]$ or $[i+2, i+3)$. Now x_k' is adjacent to y_k only so we take $I(x_k')$ as the closed-open copy of $I(x_k)$. Again y_{k-1}' is adjacent to x_{k-1} only, and as y_k is adjacent to x_{k-1} in P. So we take $I(y_{k-1}')$ as the closed-open copy of $I(y_k)$, $(k \geq 2)$. This completes the proof of (a).

(b) From the above representation of M_i' it follows that M_i has no $\mathcal{U}$-intersection representation and hence M_i is a minimal forbidden induced subgraph for the class of $\mathcal{U}$-bigraphs. □

The vertices v' and v'' are the special vertices of N_i'. In the next lemma we show that the bigraph N_i' has a unique $\mathcal{U}$-intersection representation upto trivial modifications.

Lemma 8. *Let $i \in \mathbb{N}$.*

(a) A $\mathcal{U}$-intersection representation $I : V(N_i') \to \mathcal{U}$ of N_i', where $I(V(N_i'))$ consists of the following intervals
- $I(x_2'') = I(y_2'') = [-2, -1],\ I(x_4'') = I(y_1'') = [-1, 0],\ I(x_1'') = (-1, 0)$
- $I(x_3'') = [0, 1],\ I(y_3'') = [0, 1)$
- $I(y_4'') = [-3, -2]$ *or* $(-3, 2]$
- $I(y_k) = [2k-2, 2k-1],\ I(x_k) = [2k-1, 2k],\ (k \geq 1)$
- $I(x_k') = [2k-1, 2k),\ (k \geq 1)$ *and* $I(y_k') = [2k-2, 2k-1),\ (k \geq 2)$ *and*
- $I(u) = [i-1, i],\ I(v')$ *and* $I(v'') = [i, i+1]$ *or* $[i, i+1)$

is unique upto trivial modifications.

(b) N_i is a minimal forbidden induced subgraph for the class of $\mathcal{U}$-bigraphs (Fig. 16).

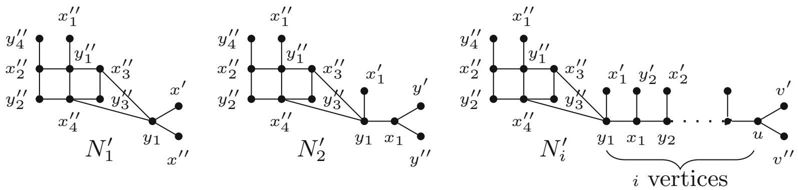

Fig. 16. The class $\mathcal{N}'$ of bigraphs.

Proof. (a) It is easy to obeserve that each of the bigraph N_i' contains the graph $H_3 + y_4''$ as an induced subgraph, where $V(H_3 + y_4'') = \{x_1'', y_1'', x_2'', y_2'', x_3'', y_3'', x_4'', y_4''\}$. Without loss of generality we take the following representation of $H_3 + y_4''$. Where $I(y_2'') = I(x_2'') = [-2, -1]$, $I(x_4'') = I(y_1'') = [-1, 0]$, $I(x_1'') = (-1, 0)$, $I(x_3'') = [0, 1]$ and $I(y_3'') = [0, 1)$. As y_1 is adjacent to x_3'' and x_4'' and x_1 is adjacent to y_1 so we take $I(y_1) = [0, 1]$ and $I(x_1) = [1, 2]$. Again as y_2 is adjacent to x_1, x_2 is adjacent to y_2 and so on; we can take the interval representation of the path $y_1, x_1, y_2, x_2, \ldots, y_k, x_k, \ldots, u$ where $I(y_1) = [0, 1]$, $I(x_1) = [1, 2]$, $I(y_2) = [2, 3]$, $I(x_2) = [3, 4]$. And by induction, we have $I(y_k) = [2k-2, 2k-1]$ and $I(x_k) = [2k-1, 2k]$. Now if $i = 2m$, then $u = x_m$. Then $I(x_m) = [2m-1, 2m] = [i-1, i]$. In the other case, if $i = 2m-1$, then $u = y_m$. And $I(y_m) = [2m-2, 2m-1] = [i-1, i]$. As v and v' are adjacent to u, we take $I(v) = I(v') = [i, i+1]$ or $[i, i+1)$. Again as x_k' is adjacent to y_k only also y_k is adjacent to x_k we take $I(x_k') = [2k-1, 2k)$, $(k \geq 1)$. As y_k' is adjacent to only x_{k-1} also x_{k-1} is adjacent to y_k, we take $I(y'_k) = [2k-2, 2k-1)$, $(k \geq 2)$. This completes the proof of (a).

(b) From the representation of N'_i it follows that N_i is the minimal forbidden induced subgraphs for $\mathcal{U}$-bigraphs. □

The two vertices v' and v'' are the special vertices of H''_i. In the next lemma we show that the bigraph H''_i has a unique $\mathcal{U}$-intersection representation upto trivial modifications.

Lemma 9. *Let $i \in \mathbb{N}$.*

(a) A $\mathcal{U}$-intersection representation $I : V(H''_i) \to \mathcal{U}$ of H''_i, where $I(V(H''_i))$ consists of the following intervals
- $I(x''_2) = [-1, 0]$ *or* $(-1, 0]$, $I(y''_2) = [-1, 0]$ *or* $(-1, 0]$
- $I(x''_4) = I(y''_1) = [0, 1]$, $I(y''_3) = (0, 1)$, $I(x''_3) = [\frac{1}{2}, \frac{3}{2}]$
- $I(x_1) = [1, 2]$, $I(x''_1) = [1, 2)$
- $I(x_k) = [2k-1, 2k]$, $I(y_k) = [2k, 2k+1]$, $(k \geq 1)$
- $I(y'_k) = [2k, 2k+1)$, $I(x'_k) = [2k+1, 2k+2)$, $(k \geq 1)$
- $I(u) = [i, i+1]$, $I(v')$ *and* $I(v'') = [i+1, i+2]$ *or* $[i+1, i+2)$

is unique upto trivial modifications.

(b) H'_i is a minimal forbidden induced subgraph for $\mathcal{U}$-bigraph (Fig. 17).

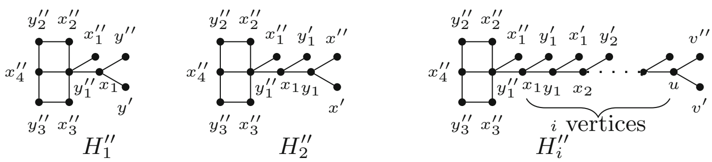

Fig. 17. The class $\mathcal{H}''$ of bigraphs.

Proof. (a) Every H''_i contains H_3 as an induced subgraph, where $V(H_3)$ is $\{x''_2, y''_2, x''_3, y''_3, x''_4, y''_1, x_1\}$. Consider the following representation of H_3, where $I(x''_2) = I(y''_2) = [-1, 0]$ or $(-1, 0]$, $I(x''_4) = I(y''_1) = [0, 1]$, $I(y''_3) = (0, 1)$, $I(x''_3) = [\frac{1}{2}, \frac{3}{2}]$, $I(x_1) = [1, 2]$. Next, consider the interval representation of the path $y''_1, x_1, y_1, x_2, y_2, \ldots, x_k, y_k, \ldots, u$, where $I(x_1) = [1, 2]$, $I(y_1) = [2, 3]$, $I(x_2) = [3, 4]$, $I(y_2) = [4, 5]$. And by induction $I(x_k) = [2k-1, 2k]$, $I(y_k) = [2k, 2k+1]$, $(k \geq 1)$. For the ith vertex u, if $i = 2m$ then $u \in Y$ and $u = y_m$, and $I(u) = [2m, 2m+1] = [i, i+1]$. Again if $i = 2m+1$, $u \in X$ and $u = x_{m+1}$, then $I(u) = [2(m+1)-1, 2(m+1)] = [2m+1, 2m+1+1] = [i, i+1]$. Consequently $I(v')$ and $I(v'')$ are $[i+1, i+2]$ or $[i+1, I+2)$. Again x''_1 is adjacent to y''_1 only so $I(x''_1) = [1, 2)$. Similarly y'_k is adjacent to x_k only so $I(y'_k) = [2k, 2k+1)$ as y_k is also adjacent to x_k. Next x'_k is adjacent to y_k and x_{k+1} is also adjacent to y_k, so $I(x'_k) = [2k+1, 2k+2)$, $(k \geq 1)$, this completes the proof of (a).

(b) From the $\mathcal{U}$-representation of H''_i it follows that H'_i is minimal induced forbidden subgraph for $\mathcal{U}$-bigraphs. □

Lemma 10. *The bigraph B_2 is minimal forbidden induced subgraph for $\mathcal{U}$-bigra-phs.*

Proof. B_2 contains H_1 as an induced subgraph, where $V(H_1) = \{y, x, y_1, x_2, y_2, x'_2, y'_2\}$. As H_1 has a unique $\mathcal{U}^{\pm}$-representation upto trivial modifications, we take the following representation of it, $I(x) = I(y_1) = [1,2]$, $I(y) = (1,2)$, $I(x_2) = [0,1]$, $I(x'_2) = [2,3]$ and $I(y_2) = [-1,0]$ or $(-1,0]$, $I(y'_2) = [3,4]$ or $[3,4)$. As x_3 is adjacent to y_1 but not to y_2 we take $I(x_3) = (0,1]$. Similarly $I(x'_3) = [2,3)$. Again as y_4 is adjacent to x_2 and x_3 only so we may take $I(y_4) = (0,1)$. Similarly $I(y'_4) = (2,3)$. Now it is not possible to give an interval representation for the vertex x_1.

Also it may be noted that B_2 contains $H_2 - y_3$ as an induced subgraph, where $V(H_2 - y_3) = \{x_1, y_1, x_2, y_2, x_3, y_4\}$. Consider the representation of $H_2 - y_3$, where $I(x_3) = [1,2]$, $I(x_1) = (1,2)$ or $(1,2]$, $I(y_1) = [1,2]$, $I(x_2) = [0,1]$, $I(y_4) = [0,1]$, $I(y_2) = [-1,0]$ or $(-1,0]$. Now x'_2 and x'_3 are adjacent to y_1 and y'_2 is adjacent to x'_2. So we take $I(x'_2) = [2,3]$, $I(x'_3) = [2,3)$, $I(y'_2) = [3,4]$ or $[3,4)$. Again y'_4 is adjacent to x'_2 and x'_3 only so $I(y'_4) = (2,3)$. As x is adjacent to y_1 only so we take $I(x) = (1,2)$. But now, it is not possible to give an interval representation for the vertex y. Also it can be verified that for other representation of $H_2 - y_3$, it is not possible to give an interval representation of B_2. This completes the proof of the lemma. □

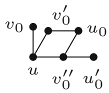

Fig. 18. The bigraph B_0

Lemma 11. *In the bigraphs $F_{i,j}$, M'_i, N'_i, H''_i if we have the bigraph B_0 containing u as an induced subgraph (the vertices v' and v'' are absent) then the resulting bigraphs are still minimal forbidden induced subgraphs for $\mathcal{U}$-bigraphs (u and v are vertices of different partite sets) (Fig. 18).*

Proof. In the $\mathcal{U}$-intersection representation of any of the bigraphs $F_{i,j}$, M'_i, N'_i or H''_i, let the interval corresponding to u is $I(u) = [a, a+1]$. As v'_0 and v''_0 are adjacent to u, u_0 is adjacent to v'_0, v''_0 and u'_0 is adjacent to v''_0 only, we take intervals corresponding to these vertices as follows: $I(v''_0) = [a+1, a+2]$, $I(v'_0) = [a+1, a+2)$, $I(u_0) = (a+1, a+2)$ and $I(u'_0) = [a+2, a+3]$ or $[a+2, a+3)$. Now the interval representation of v_0 is not possible as there exists an interval $I(u') = [a, a+1)$ in the interval representation of each of the bigraphs $F_{i,j}$, M'_i, N'_i, H''_i. This completes the proof of the lemma. □

4 A Conjecture for Mixed Unit Interval Bigraphs

In the previous Section we have seen that the bigraphs $F_2, F_4, F_5, F_8, F_9, F_{10}, F_{11}$, F_{12}, B_1 and B_2 are minimal forbidden induced subgraphs for $\mathcal{U}$-bigraphs. Also several infinite families of bigraphs, namely, $\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{H}'$ that are the forbidden families of $\mathcal{U}$-bigraphs. Next, we observe that in the bigraph $F_{i,j}$, the vertex u is adjacent to two special vertices v' and v''. Now if there exist two distinct vertices u' and u'' such that u' is adjacent to v' and u'' is adjacent to v'' we have the bigraph $L_{i,j}$. Similar observation can be made for the bigraphs M_i, N_i and H_i''. Also in the Lemma 11, we proved that for the bigraphs $F_{i,j}, M_i', N_i', H_i''$ where vertices v' and v'' are deleted, if we have the bigraph B_0 as an induced subgraph containing the vertex u, then the resulting graph is also a forbidden induced subgraph for $\mathcal{U}$-bigraphs. These results inspire us to pose a conjecture. But before that we introduce a new definition. For notational convenience we write $l(I(v)) = l(v)$ and $r(I(v)) = r(v)$

A bigraph $B = (X, Y, E)$ is a *mixed proper interval bigraph* if it has an $\mathcal{I}$-intersection representation $I : V(B) \to \mathcal{I}$ such that

(i) for two distinct vertices u and v of B with $I(u), I(v) \in \mathcal{I}^{++}$, $I(u) \not\subset I(v)$ and $I(v) \not\subset I(u)$, and
(ii) for every vertex u of B with $I(u) \notin \mathcal{I}^{++}$, there is a vertex v of B with $I(v) \in \mathcal{I}^{++}$, $l(u) = l(v)$ and $r(u) = r(v)$, that is no closed interval is properly contained in another closed interval and for any non closed interval, there is a closed interval with same end point.

Let $\mathcal{B}'$ be the class of bigraphs, where $\mathcal{B}' = \mathcal{F}' \cup \mathcal{M}' \cup \mathcal{N}' \cup \mathcal{H}''$. We are now in a position to phrase our conjecture.

Conjecture 12. *For a bigraph B, the following statements are equivalent.*

(a)
- *B is $\{B_1, B_2, F_2, F_4, F_5, F_8, F_9, F_{10}, F_{11}, F_{12}\} \cup \mathcal{L} \cup \mathcal{M} \cup \mathcal{N} \cup \mathcal{H}'$-free interval bigraph and*
- *for every induced subgraph H of B that is isomorphic to one of the bigraphs of the class $\mathcal{B}'$ and any vertex $u^* \in V(B) \setminus V(H)$ is such that u^* is adjacent to exactly one of the special vertices of H,*
- *if $H' = H \setminus \{v', v''\}$ then $H' \cup B_0$ is not an induced subgraph of B.*

(b) B is a mixed proper interval bigraph.
(c) B is a mixed unit interval bigraph.

In the last two results we verify the conjecture partly and we leave open the problem of finding the complete list of forbidden bigraphs of mixed unit interval bigraphs.

Proposition 13. *The implication $(c) \Rightarrow (a)$ of Conjecture 12 is true.*

Proof. Let B be a $\mathcal{U}$-bigraph, and let I be a $\mathcal{U}$-intersection representation of B. Then obviously B is $F_2, F_4, F_5, F_8, F_9, F_{10}, F_{11}, F_{12}$-free interval bigraphs. Also corollary of Lemmas 5 and 10 imply that B is B_1 and B_2-free. And from Lemmata 6, 7, 8 and 9, B is $\mathcal{L} \cup \mathcal{M} \cup \mathcal{N} \cup \mathcal{H}'$-free interval bigraph

Now the H be an induced subgraph of B that is isomorphic to any bigraph of the class $\mathcal{B}'$. Let the vertices in H be denoted as in the definition of the bigraphs in the class $\mathcal{B}'$. Then the two pendant vertices v' and v'' are special vertices which are adjacent to u. And $v', v'', u \in V(H)$. Let $u^* \in V(B) \setminus V(H)$ be such that u^* is adjacent to v' but not to v''. By Lemmata 6, 7, 8 and 9 we may assume that $I(v') = [a, a+1]$ and $I(v'') = [a, a+1)$, where $a \in \mathbb{R}$. $r(I(v)) \leq a$ for any $v \in V(H) \setminus \{v', v''\}$. Thus $I(u^*)$ can be taken as any of intervals $[a+1, a+2]$ or $[a+1, a+2)$ and this implies that u^* is adjacent to v' only. From Lemma 11, it follows that $H' \cup B_0$ is forbidden induced subgraph of B. This completes the proof. □

Theorem 14. *A bigraph is a mixed proper interval bigraph if and only if it is a $\mathcal{U}$-bigraph; that is; statements* (b) *and* (c) *of Conjecture 12 are equivalent.*

Proof. The 'only if' part of the proof is similar to the proof of Theorem 8 in [6]. For the sake of completeness, we give here details.

Let B be a mixed proper interval bigraph and I be a mixed proper interval representation of B. Let V_1 denote the set of vertices u of B such that $I(u) \in \mathcal{U}^{++}$. By the definition of mixed proper interval bigraphs, the subgraph $B[V_1]$ induced by the vertex set V_1 is a proper interval bigraph. And the interval representation of $B[V_1]$ is given by the corresponding intervals of I. Bogart and West [1], gave a constructive method how a proper interval representation I_1 produces to a unit interval representation I_2 gradually converting the intervals into unit intervals by means of successive contraction, dilations and translation. In this procedure it may be noted that two intervals intersect at a single point in I_1 if and only if the corresponding intervals intersect at a single point in I_2. And two intervals intersect more than one point in I_1 if and only if corresponding intervals intersects more than one point in I_2. Also two intervals do not intersect in I_1 if and only if they do not intersect in I_2. This implies that reinserting the mixed intervals corresponding to the vertices in $V(B) \setminus V_1$ as mixed copies of the corresponding closed intervals results in a $\mathcal{U}$-intersection representation of B, and this completes the 'only if' part.

For the 'if' part let B be a $\mathcal{U}$-bigraph and $I : V(B) \to \mathcal{U}$-intersection representation of B. Let $I(u)$ be an open interval of I. As $I(u)$ is forced to be open there must exist v_1 and v_2, such that $I(v_1)$ and $I(v_2)$ are closed and $r(v_1) = l(u)$, $l(v_2) = r(u)$. Now $I(v_1)$ and $I(v_2)$ must not be moved to the left and the right respectively as then $I(u)$ can be made to a closed interval, so there exist u' such that $I(u')$ is closed and $l(u) = l(u')$, $r(u) = r(u')$, u and v vertices of different partite sets.

Next, let $I(u)$ be an open-closed interval (i.e. $l(u)$ is open and $r(u)$ is closed). By the similar reason there exists a closed interval $I(v)$ such that $l(u) = r(v)$. Since B is connected we have $I(u')$ intersecting $I(v)$. Now $I(u') \in \mathcal{U}^{++}$ and $l(u') = l(u)$ and $r(u') = r(u)$, otherwise $I(v)$ can be moved to the left and may makes $l(u)$ closed. Thus for every $I(z) \notin \mathcal{U}^{++}$, $z \in V(B)$, we have closed interval with the same end points as of $I(z)$. This completes the proof. □

5 Conclusion

In this paper we provide some forbidden subgraphs and four infinite families of forbidden subgraphs of mixed unit interval bigraphs. We also put forward a conjecture and hope that this will motivate to give a complete characterization of the class of mixed unit interval bigraphs in terms of forbidden induced subgraphs. In an earlier paper [4] we give the forbidden subgraph characterization of unit interval bigraphs of open and closed intervals, but the forbidden subgraph characterization of interval bigraphs is an interesting open problem. In [5] Das et al. have made considerable progress to solve it. The complexity of the only known recognition of interval bigraphs given by Müller [14] is very high. The problem of finding a recognition algorithm for interval graphs (or bigraphs) of open and closed intervals is still open. However, in a very recent paper [17] Talon and Kratochvíl have given a quadratic-time algorithm to recognize the class of mixed unit interval graphs.

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