Chapter 4
The (In)Vulnerability of Non-Ranked Voting Procedures to Various Paradoxes

Abstract Focusing on four procedures that do not require the voters to submit full preference rankings over candidates (Plurality Voting, Plurality with Runoff, Approval Voting, and Successive Elimination), we discuss, for each procedure, those voting paradoxes to which the procedures are immune and the reasons for this, as well as demonstrate, with the aid of illustrative examples, their vulnerability to other paradoxes.

Keywords Vulnerability to paradoxes · Non-ranked voting procedures · Proving by counterexample

4.1 The (In)Vulnerability of the Plurality Voting Procedure to Various Paradoxes

4.1.1 The Condorcet Winner, the Condorcet Loser, the Absolute Majority Loser, the Preference Inversion, and the SCC Paradoxes

The Plurality Voting procedure is vulnerable to these five paradoxes. The following example demonstrates the vulnerability of the Plurality Voting procedure to all these paradoxes simultaneously.

This chapter is partly based on Felsenthal (2012) which contains several examples devised by the second-named author.

Except for the Successive Elimination procedure all other voting procedures surveyed in this chapter are invulnerable to the Dependence on Order of Voting (DOV) Paradox (cf. Sect. 2.2.8 in Chap. 2) because under these procedures all candidates are voted upon simultaneously rather than sequentially.

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4.1.1.1 Example

Suppose there are 9 voters who must elect one out of three candidates, *a, b,* and *c,* and whose preference orderings among these candidates are as follows:

No. of voters	Preference orderings
4	$a \succ b \succ c$
3	$b \succ c \succ a$
2	$c \succ b \succ a$

Here *b* is the Condorcet Winner and *a* is not only a Condorcet Loser but also an Absolute Majority Loser. Nevertheless, if all voters vote for their top preference then *a* will be elected. Note that if *c* drops out of the race then *b* will be elected—thus demonstrating the violation of SCC. Note also that if all voters invert their preference orderings then *a* becomes an Absolute Majority Winner and hence will be elected—thus demonstrating the Preference Inversion Paradox.

4.1.2 Absolute Majority Winner Paradox

The Plurality Voting procedure is not susceptible to this paradox because, by definition, it elects the alternative which is supported by the plurality of voters. So when there is a candidate ranked first by an absolute majority, it is *a fortiori* the only one winning by a plurality of votes and hence is the Plurality Voting winner.

4.1.3 Pareto-Dominated Candidate Paradox

The Plurality Voting procedure cannot elect a Pareto-dominated candidate because it elects, by definition, the candidate who is supported by the plurality of voters. So if candidate *x* is Pareto-dominated by some other candidate *y*, then *x* cannot be ranked ahead of *y* by any voter. Hence if no voter votes for a less preferred alternative if s/he can vote for a more preferred alternative then *x* gets no votes at all, and therefore cannot be elected under the Plurality Voting procedure.

4.1.4 Lack of Monotonicity Paradox

The Plurality Voting procedure is not susceptible to lack of monotonicity since increasing candidate *x*'s support, *ceteris paribus*, will keep the number of *x*'s votes the same as originally or increase it, while no other candidate gets more votes. Thus, *x* remains the winner under the Plurality Voting procedure in a fixed electorate. In a variable electorate obtained by adding some voters ranking x first (and voting for x), the vote sums of all other candidates remain the same, but the number of x's votes increases by the number of the added voters. Hence x remains the winner under the Plurality Voting procedure with a larger margin than originally. So, both in fixed and variable electorates additional support for the winner, *ceteris paribus*, maintains its status as the winner under the Plurality Voting procedure.

4.1.5 Reinforcement Paradox

The Plurality Voting procedure is not vulnerable to the Reinforcement Paradox because it always elects the alternative which received the plurality of votes by any given electorate. Hence if two disjoint electorates—each of which awarded x the plurality of votes—are amalgamated into a single electorate then x will receive also the plurality of votes in the amalgamated electorate and hence will be elected by the Plurality Voting procedure.

4.1.6 No-Show Paradox

The Plurality Voting procedure is not vulnerable to the No-Show Paradox since the selected alternative, say x, which by definition is ranked first by the plurality of voters, can be changed to another winner, say y, only if some voters originally ranking x first, abstain. This is because the abstaining of any other voters only increases x's plurality margin. Also those originally ranking x first cannot benefit from abstaining since thereby they decrease x's plurality count, possibly even rendering x a non-winner. Thus, no voters can benefit from abstaining under the Plurality Voting procedure.

4.1.7 Truncation Paradox

This paradox is irrelevant to the Plurality Voting procedure because under this procedure voters do not rank-order the alternatives.

4.1.8 Twin Paradox

The Plurality Voting procedure is invulnerable to the Twin Paradox. On the contrary, the more voters having the same preferences will vote for the same alternative, the more likely will this alternative be selected by the Plurality Voting procedure.

4.2 The (In)Vulnerability of the Plurality with Runoff Procedure to Various Paradoxes

4.2.1 The Condorcet Winner, Lack of Monotonicity, and the SCC Paradoxes

Example 4.2.1.1 below demonstrates the vulnerability of the Plurality with Runoff procedure to these three paradoxes.

4.2.1.1 Example

Suppose there are 43 voters whose preference orderings among three candidates, *a, b*, and *c*, are as follows:

No. of voters	Preference orderings
7	$a \succ b \succ c$
9	$a \succ c \succ b$
14	$b \succ c \succ a$
13	$c \succ a \succ b$

Here the pairwise majority comparisons yield that the social preference ordering is $c \succ a \succ b$, i.e., *c* is the Condorcet Winner. But if all voters vote sincerely then under the Plurality with Runoff procedure *c* will be eliminated in the first round and *a* will beat *b* in the second round thus becoming the ultimate winner. Note that if *b* would have withdrawn from the race prior to the first round then, *ceteris paribus*, *c* would have been elected already in the first round, thereby demonstrating this procedure's vulnerability to SCC.

Now suppose that, *ceteris paribus*, five of the 14 voters whose preference ordering is $b \succ c \succ a$ (who are not very happy with the prospect that *a* may be elected) change it to $a \succ b \succ c$ thereby *increasing a*'s support. As a result of this change *b* (rather than *c*) will be eliminated in the first round, and *c* (the Condorcet Winner) will beat *a* in the second round—thereby demonstrating the vulnerability of the Plurality with Runoff procedure to non-monotonicity. Notice the other bizarre effect of the preference change of the five voters: the candidate whom they now rank last in their preferences becomes the winner even though it wasn't one before the change.

4.2.2 Absolute Majority Winner Paradox

The Plurality with Runoff procedure is not vulnerable to this paradox because, by definition, if there exists an alternative which is supported by an absolute majority of the voters then this alternative is elected under the Plurality with Runoff procedure.

4.2.3 Condorcet Loser and Absolute Majority Paradoxes

Under the Plurality with Runoff procedure an alternative which is a Condorcet Loser or an Absolute Majority Loser may receive the plurality (but not the majority) of votes in the first round, and thus be one of the two alternatives which may compete in the second stage. However, it cannot win in the second stage because, by definition, the majority of voters will prefer the other alternative.

4.2.4 Pareto-Dominated Candidate Paradox

The Plurality with Runoff procedure cannot elect a Pareto-dominated alternative, because if all voters prefer alternative w to alternative z then alternative z cannot constitute the top preference of any voter and therefore cannot obtain any votes in any stage of the Plurality with Runoff procedure.

4.2.5 Truncation Paradox

This paradox is irrelevant to the Plurality with Runoff procedure in its first version in which voters may go once or twice to the balloting box—voting for just one candidate in each of these times without ranking the candidates.

The Plurality with Runoff procedure in its second version is also not vulnerable to this paradox. To understand why this is so suppose that x and y are the two alternatives which received more votes than each of the remaining alternatives in the first count, and that a voter who prefers x to y contemplates whether it would be worthwhile for him/her to truncate from his/her preference ordering x, y or both. Since there are no more than two counting rounds, the potential truncating voter realizes that s/he cannot change the second-round contestants by truncating any of the candidates whom s/he does not list first in his/her preference ordering. Since s/he prefers x to y, s/he can by truncating x only make x's chances worse or at best maintain its status. To wit, if the truncation involves both x and y, s/he might actually bring about the victory of y (his/her less preferred alternative). If s/he truncates y (and some other alternatives, but not x), s/he does not change the second round outcome. The same is of course true if s/he truncates other alternatives but not x and y. So, the occurrence of the Truncation Paradox is not possible.

4.2.6 Reinforcement Paradox

Example 4.2.6.1 demonstrates the vulnerability of the Plurality with Runoff procedure to the Reinforcement Paradox.

4.2.6.1 Example

Suppose there are two districts, I and II. In district I there are 17 voters whose preference orderings among three candidates, *a, b*, and *c*, are as follows:

No. of voters	Preference orderings
4	$a \succ b \succ c$
1	$b \succ a \succ c$
5	$b \succ c \succ a$
6	$c \succ a \succ b$
1	$c \succ b \succ a$

and in district II there are 15 voters whose preference orderings among the three candidates are as follows:

No. of voters	Preference orderings
6	$a \succ c \succ b$
8	$b \succ c \succ a$
1	$c \succ a \succ b$

If all voters vote sincerely then no candidate is ranked first by an absolute majority of the voters in district I. Consequently candidate *a* is deleted from the race after the first round and candidate *b* beats candidate *c* in this district in the second round.

In district II candidate *b*, who is ranked first by an absolute majority of voters, is elected in the first round.

However if, *ceteris paribus*, the two districts are amalgamated into a single district, we obtain the following distribution of preference orderings of the 32 voters:

No. of voters	Preference orderings
4	$a \succ b \succ c$
6	$a \succ c \succ b$
1	$b \succ a \succ c$
13	$b \succ c \succ a$
7	$c \succ a \succ b$
1	$c \succ b \succ a$

If all voters vote sincerely then no candidate is ranked first by an absolute majority of the voters. Consequently *c* is deleted after the first round and *a* beats *b* and is elected in the second round—thus demonstrating the susceptibility of the Plurality with Runoff procedure to the Reinforcement Paradox.

4.2.7 No Show and Twin Paradoxes

Example 4.2.7.1 demonstrates the vulnerability of the Plurality with Runoff procedure to the No-Show and to the Twin Paradoxes.

4.2.7.1 Example

Suppose there are 11 voters whose preference orderings among three candidates, *a*, *b*, and *c*, are as follows:

No. of voters	Preference orderings
4	$a \succ b \succ c$
3	$b \succ c \succ a$
1	$c \succ a \succ b$
3	$c \succ b \succ a$

If all voters vote sincerely then no candidate is ranked first by an absolute majority of the voters. Consequently *b* is deleted after the first round and *c* beats *a* in the second round and is elected. Since the election of *c* is the worst outcome for the voters whose preference ordering is $a \succ b \succ c$, suppose that, *ceteris paribus*, two of them decide not to participate in the election (No-Show). We thus obtain the following distribution of preference orderings:

No. of voters	Preference orderings
2	$a \succ b \succ c$
3	$b \succ c \succ a$
1	$c \succ a \succ b$
3	$c \succ b \succ a$

Here *a* (rather than *b*) is eliminated in the first round, and *b* beats *c* in the second round. Thus the $a \succ b \succ c$ voters obtained, *ceteris paribus*, a better outcome when two of them did not participate in the election than when all of them participated in the election thereby demonstrating the vulnerability of the Plurality with Runoff procedure to the No-Show Paradox.

This example demonstrates also the vulnerability of the Plurality with Runoff procedure to the of the Twin Paradox. Suppose that, *ceteris paribus*, there are originally only two voters with preference ordering $a \succ b \succ c$. One would expect these voters to welcome other “twin” voters having identical preference ordering to theirs thereby presumably giving an increased weight to their common preference ordering. Yet as we saw, the addition of these twins to the electorate results in the election of *c*, their worst preference—thereby demonstrating the vulnerability of the Plurality with Runoff procedure to the Twin Paradox.

4.2.8 Preference Inversion Paradox

Example 4.2.8.1 demonstrates the vulnerability of the Plurality with Runoff procedure to the Preference Inversion Paradox.

4.2.8.1 Example

Suppose there are 11 voters whose preference orderings among three candidates, *a*, *b*, and *c*, are as follows:

No. of voters	Preference orderings
5	$a \succ b \succ c$
4	$b \succ c \succ a$
2	$c \succ a \succ b$

If all voters vote sincerely for their top preference in the first round, then *c* will be eliminated at the end of the first round and thereafter *a* will beat *b* in the second round. However, if all voters invert their preference orderings then *b* will be eliminated at the end of the first round and *a* will beat *c* in the second round—thus demonstrating the vulnerability of the Plurality with Runoff procedure to the Preference Inversion Paradox.

4.3 The (In)Vulnerability of the Approval Voting Procedure to Various Paradoxes

4.3.1 The Condorcet Winner Paradox

Example 4.3.1.1 demonstrates the vulnerability of the Approval Voting procedure to the Condorcet Winner Paradox.

4.3.1.1 Example

This example is adapted from Felsenthal and Maoz (1988, p. 123, Example 2). Suppose there are 49 voters whose preference orderings among three candidates, *a, b,* and *c*, are as follows:

No. of voters	Preference orderings
18	$(a) \succ b \succ c$
6	$(b \succ c) \succ a$
8	$(b \succ a) \succ c$
2	$(c \succ a) \succ b$
15	$(c) \succ b \succ a$

The social preference ordering is $b \succ a \succ c$, i.e., *b* is the Condorcet Winner. However, if all voters approve (and vote for) the candidates denoted between parentheses then *a* would get the largest number of approval votes (28) and will thus be elected.

4.3.2 Absolute Majority Winner Paradox

Example 4.3.2.1 demonstrates the vulnerability of the Approval Voting procedure to the Absolute Majority Winner Paradox.

4.3.2.1 Example

Suppose there are 100 voters whose preference orderings among three candidates, *a, b,* and *c*, are as follows:

No. of voters	Preference orderings
99	$a \succ b \succ c$
1	$b \succ c \succ a$

The social preference ordering is $a \succ b \succ c$, i.e., *a* is the Condorcet Winner who is ranked first by an absolute majority of the voters. However, if only one candidate must be elected and if each voter approves (and votes for) his/her top two preferences, then *b* will be elected despite the fact that *a* is ranked first by an absolute majority of the voters.

4.3.3 Condorcet Loser, Absolute Majority Loser, and Preference Inversion Paradoxes

Example 4.3.3.1 demonstrates the vulnerability of the Approval Voting procedure to the Absolute Majority Loser and to the Condorcet Loser Paradoxes.

4.3.3.1 Example

Suppose there are 15 voters whose preference orderings among three candidates, *a, b,* and *c*, are as follows:

No. of voters	Preference orderings
6	$(a) \succ b \succ c$
4	$(b) \succ c \succ a$
1	$(c \succ a) \succ b$
4	$(c) \succ b \succ a$

The social preference ordering is $b \succ c \succ a$, i.e., a is not only the Condorcet Loser but also the Absolute Majority Loser because this candidate is ranked last by an absolute majority of the voters. However, if only one candidate must be elected and if all voters approve (and vote for) the candidate(s) denoted between parentheses then a will be elected.

This example can also be used to demonstrate the susceptibility of the Approval Voting procedure to the Preference Inversion Paradox. If in the above example all voters invert their preference ordering and decide to vote, as before, i.e., either only for their top preference or for their top two preferences, then we obtain the following distribution of votes:

No. of voters	Preference orderings
6	$(c) \succ b \succ a$
4	$(a) \succ c \succ b$
1	$(b \succ a) \succ c$
4	$(a) \succ b \succ c$

Here a is not only the Condorcet Winner but also the Absolute Majority Winner and is elected—thereby demonstrating the susceptibility of Approval Voting to the Preference Inversion Paradox.

4.3.4 The Pareto-Dominated Candidate Paradox

Example 4.3.4.1 demonstrates the vulnerability of the Approval Voting procedure to the Pareto-Dominated Candidate Paradox.

4.3.4.1 Example

Suppose there are three voters whose preference orderings among three candidates, *a, b* and *c*, are as follows:

No. of voters	Preference orderings
2	$a \succ b \succ c$
1	$c \succ a \succ b$

The social preference ordering is $a \succ b \succ c$, i.e., *a* is the Condorcet Winner. However, if the first two voters with identical preferences approve (and vote for) their top two preferences, while the third voter approves (and votes for) only his/her top ranked candidate, then a tie would occur between the number of votes (2) obtained by candidates *a* and *b*, and if this tie were to be broken randomly then there is a 0.5 probability that *b* would be elected. So if *b* were to be elected it would demonstrate not only that the Condorcet Winner (*a*) was not elected but also that a Pareto-dominated candidate can be elected under the Approval Voting procedure. (Note that *all* voters prefer *a* to *b*).

4.3.5 *Lack of Monotonicity Paradox*

The Approval Voting procedure is not vulnerable to lack of monotonicity for the same reasons that the Plurality Voting procedure is not vulnerable to this paradox assuming that the improvement of a candidate's position does not change its approvability status, i.e., the candidates approved of initially will remain approved after the improvement and the same holds for disapproved candidates.

4.3.6 *Reinforcement Paradox*

The Approval Voting procedure is not vulnerable to the Reinforcement Paradox for the same reasons that the Plurality Voting procedure is not vulnerable to this paradox assuming that the improvement of a candidate's position does not change its approvability status, i.e., the candidates approved of initially will remain approved after the improvement and the same holds for disapproved candidates.

4.3.7 *No-Show Paradox*

The Approval Voting procedure is not vulnerable to the No-Show Paradox for the same reasons that the Plurality Voting procedure is not vulnerable to this paradox assuming that the improvement of a candidate's position does not change its approvability status, i.e., the candidates approved of initially will remain approved after the improvement and the same holds for disapproved candidates.

4.3.8 Twin Paradox

The Approval Voting procedure is not vulnerable to the Twin Paradox for the same reasons that the Plurality Voting procedure is not vulnerable to this paradox assuming that the improvement of a candidate's position does not change its approvability status, i.e., the candidates approved of initially will remain approved after the improvement and the same holds for disapproved candidates.

4.3.9 Truncation Paradox

Example 4.3.9.1 demonstrates the vulnerability of the Approval Voting procedure to the Truncation Paradox.

4.3.9.1 Example

Suppose there are 100 voters whose preference orderings among three candidates, *a, b,* and *c*, are as follows:

No. of voters	Preference orderings
99	$(a \succ b) \succ c$
1	$(b) \succ a \succ c$

If all voters approve and vote for the candidates listed between parentheses then *b* will be elected. However, by truncating their ballots to include only *a*, the 99 voters (or any proper subset of them consisting of at least 2 voters) will get their most favorite candidate elected thus demonstrating the vulnerability of the Approval Voting procedure to the Truncation Paradox.

4.3.10 Remark

Note that under Approval Voting voters may benefit not only by curtailing their ballots of some of the candidates they approve, but also by adding to their ballots some of the candidates whom they disapprove. To see this consider again the first part of Example 4.3.3.1. In this example *a* was elected. But since *a* constitutes the last preference of the group of 4 voters who approved and voted only for *b*, as well as of the group of 4 voters who approved and voted for only for *c*, any of these groups would be better off if, *ceteris paribus*, they would vote also for their second-ranked (disapproved) candidate.

4.3.11 The SCC Paradox

The Approval Voting procedure satisfies SCC if voters are assumed to approve and vote for precisely the same available candidates in all subsets of the candidates. This conclusion follows from the fact that the approval tallies of the candidates remain the same in all subsets. Hence the winners remain winners in all subsets they are members of.[1]

4.4 The (In)Vulnerability of the Successive Elimination Procedure to Various Paradoxes

4.4.1 Absolute Majority Winner and Condorcet Winner Paradoxes

If voters are assumed to vote sincerely in each voting round then it follows that a Condorcet Winner—and *a fortiori* an Absolute Majority Winner—will always be elected under the Successive Elimination procedure because these alternatives will always beat any other alternative pitted against them in any voting round in which they are first included through the last voting round.

4.4.2 Absolute Majority Loser and Condorcet Loser Paradoxes

If voters are assumed to vote sincerely in each voting round then it follows that a Condorcet Loser—and *a fortiori* an Absolute Majority Loser—cannot be elected

[1]However, it may be rational sometime, under the Approval Voting procedure too, to vote for previously unapproved candidate(s) rather than to keep voting only for previously approved candidate(s) or abstain when some previously approved (and losing) candidate(s) is (are) no longer available. Thus, for example, it would be beneficial for the 15 $(c) \succ b \succ a$ voters in Example 4.3.1.1 to vote for their previously unapproved candidate (*b*)—and thereby obtain, *ceteris paribus*, the election of *b,* than to abstain and obtain the election of *a*. In such eventualities the Approval Voting procedure too may be considered as being vulnerable to the SCC Paradox.

under the Successive Elimination procedure because these alternatives will always be beaten by every other alternative pitted against them in any voting round in which they are first included through the last voting round.

4.4.3 Pareto-Dominated Candidate, SCC, No-Show, and Dependence on Order of Voting (DOV) Paradoxes

Example 4.4.3.1 demonstrates the vulnerability of the Successive Elimination procedure to the election of a Pareto-dominated candidate. A necessary condition for this to happen is that the social preference ordering is cyclical and there are at least four candidates (Fishburn 1982, p. 131).

4.4.3.1 Example

Suppose there are 11 voters whose preference orderings among four candidates, *a, b, c,* and *d*, are as follows:

No. of voters	Preference orderings
3	$a \succ b \succ c \succ d$
2	$c \succ a \succ b \succ d$
1	$c \succ d \succ a \succ b$
5	$d \succ a \succ b \succ c$

Thus the social preference ordering is cyclical ($b \succ c \succ d \succ a \succ b$). Suppose further that all the voters always vote sincerely for their preferred candidate in each round, and that the order in which the divisions are carried out is as follows:

In round 1: *d* against *a*;
In round 2: the winner of round 1 against *c*;
In round 3: the winner of round 2 against *b*;

Given this order *d* beats *a* (6:5) in the first round, *c* beats *d* (6:5) in the second round, and *b* beats *c* (8:3) in the third round and becomes the ultimate winner. Note, however, that *b* is a Pareto-dominated candidate because *all* the voters prefer *a* to *b*.

This example can also be used to demonstrate the vulnerability of the Successive Elimination procedure to SCC.

If, *ceteris paribus*, *d* is deleted, then in the first round *a* will beat *c* (8:3), and in the second round *a* will beat *b* (11:0) and thus *a* will become the ultimate winner—in violation of SCC.

Similarly, this example can also be used to demonstrate the vulnerability of the Successive Elimination procedure to the No-Show Paradox.

If *ceteris paribus*, two of the voters whose top preference is *d* decide not to participate, then *a* becomes the Condorcet Winner and hence will be elected under the Successive Elimination procedure. Note that this outcome is preferred over the election of *b* by the two $d \succ a \succ b \succ c$ voters who decided not to participate—thus demonstrating the vulnerability of the Successive Elimination procedure to the No-Show Paradox.

This example can also be used to demonstrate the vulnerability of the Successive Elimination procedure to the order in which the alternatives are voted upon (Dependence on Order of Voting Paradox) when the social preference ordering is cyclical.

Given the above preference orderings of the 11 voters, if the order of the divisions in each round were changed such that:

In round 1: *a* against *b*;
In round 2: the winner of round 1 against *c*;
In round 3: the winner of round 2 against *d*;
Then in the first round *a* would beat *b* (11:0), in the second round *a* would also beat *c* (8:3), but in the third round *d* would beat *a* (6:5) and become the ultimate winner.

4.4.4 *Lack of Monotonicity Paradox*

Assuming that voters vote sincerely in each voting round and that the order in which the alternatives are voted upon stays the same, it follows that the Successive Elimination procedure is invulnerable to monotonicity failure in *fixed electorates*. This is so because if alternative *x* is the ultimate winner when the number of voters is fixed then *x* will *a fortiori* continue to be the winner if some voters who previously voted against *x* will change their minds and vote for *x*. However, the Successive Elimination procedure may be susceptible to lack of monotonicity in *variable electorates*. For examples of such monotonicity failure see Felsenthal and Nurmi (2016, 2017).

4.4.5 *Reinforcement Paradox*

Example 4.4.5.1 demonstrates the vulnerability of the Successive Elimination procedure to the Reinforcement Paradox.

4.4.5.1 Example

Suppose there are two districts, I and II. In district I there are three voters whose preference orderings among four candidates are as follows:

No. of voters	Preference orderings
1	$a \succ b \succ d \succ c$
1	$b \succ d \succ c \succ a$
1	$d \succ c \succ a \succ b$

and in district II there are four voters whose preference orderings among the four candidates are as follows:

No. of voters	Preference orderings
3	$c \succ d \succ b \succ a$
1	$d \succ a \succ b \succ c$

If the order of divisions in each district is:

b versus *d* in round 1;
winner of 1st round against *a* in round 2;
winner of 2nd round against *c* in round 3;

then in each district *c* will be the ultimate winner.
However if, *ceteris paribus*, the two districts are amalgamated into a single district of seven voters, then *d* becomes the Condorcet Winner and will therefore be elected under the Successive Elimination procedure—in violation of the Reinforcement postulate.

4.4.6 Twin Paradox

Example 4.4.6.1 demonstrates the vulnerability of the Successive Elimination procedure to the Twin Paradox.

4.4.6.1 Example

This example is due to Moulin (1988, p. 54). Suppose there are six voters whose preference orderings among three candidates, *a, b,* and *c,* are as follows:

No. of voters	Preference orderings
2	$a \succ b \succ c$
2	$b \succ c \succ a$
1	$c \succ a \succ b$
1	$c \succ b \succ a$

Suppose further that the order in which the divisions are conducted is as follows:

a versus b in round 1; winner of round 1 versus c in round 2; and that if there is a tie between two candidates in any of the divisions it is broken lexicographically, i.e., in favor of the candidate who is denoted by the letter that is closer to the beginning of the alphabet.

Accordingly, there is a tie between a and b in the first round which is broken in favor of a, and in the second round c beats a and becomes the ultimate winner.

In view of this result one could expect that, *ceteris paribus*, the single $c \succ b \succ a$ voter should welcome if an additional "twin" voter would join the electorate thereby providing more weight to their common preferences. However, an addition of a second $c \succ b \succ a$ voter would result, *ceteris paribus*, in a net loss to the first $c \succ b \succ a$ voter because b would become the Condorcet Winner and hence also the ultimate winner under the Successive Elimination procedure—thus demonstrating the vulnerability of the Successive Elimination procedure to the Twin Paradox.

4.4.7 Truncation Paradox

Example 4.4.7.1 demonstrates the vulnerability of the Successive Elimination procedure to the Truncation Paradox.

4.4.7.1 Example

Suppose there are six voters with the following preference orderings:

No. of voters	Preference orderings
1	$a \succ b \succ c \succ d$
1	$c \succ b \succ a \succ d$
2	$c \succ d \succ b \succ a$
2	$d \succ a \succ b \succ c$

Suppose further that the order in which the divisions are conducted is as follows:

First round: b versus c;
Second round: winner of 1st round versus d;
Third round: winner of 2nd round versus a;

Additionally, suppose that if a tie occurs between two candidates it is broken in favor of the one denoted by a letter closer to the beginning of the alphabet.

Accordingly, in the first round there is a tie between b and c which is broken in favor of b. In the second round d beats b, and in the third round d beats a and hence becomes the ultimate winner. This is of course a very bad outcome for the single voter whose preference ordering is $a \succ b \succ c \succ d$. So suppose that, *ceteris paribus*, this voter would truncate his/her preferences between *b, c,* and *d*, and indicate just his/her top preference, *a*, i.e., this voter will participate only in the third round in which *a* will compete against the winner from the second round. As a result of such truncation *c* would beat *b* in the first round, *c* would beat also *d* in the second round, but in the third round there would be a tie between *a* and *c*—which will be broken in favor of *a*, a much better result for the $a \succ b \succ c \succ d$ voter, thus demonstrating the vulnerability of the Successive Elimination procedure to the Truncation Paradox.

4.4.8 Preference Inversion Paradox

Example 4.4.8.1 demonstrates the vulnerability of Successive Elimination procedure to the Preference Inversion Paradox.

4.4.8.1 Example

Suppose that nine voters have to elect one out of three candidates, *a*, *b*, or *c*, under the Successive Elimination procedure and that their preference orderings among these candidates are as follows:

No. of voters	Preference orderings
4	$a \succ b \succ c$
3	$b \succ c \succ a$
2	$c \succ a \succ b$

Suppose further that the order of voting is as follows:

First round: *a* versus *b;*
Second round: winner of 1st round versus *c*.

Accordingly, in the first round *a* will beat *b*, and in the second round *c* will beat *a* and will become the ultimate winner. *Ceteris paribus*, *c* will also be elected if each of the three voters inverts his/her preference ordering.

A summary of the performance of the above four non-ranked systems with respect to the 13 paradoxes is presented in Table 4.1.

Table 4.1 (In)Vulnerability of non-ranked voting procedures to 13 voting paradoxes

Paradox	Procedure			
	Plurality	Plurality with Runoff	Approval Voting	Successive Elimination
Condorcet Winner Paradox	+	+	+	–
Absolute Majority Winner Paradox	–	–	⊕	–
Condorcet Loser Paradox	⊕	–	⊕	–
Absolute Majority Loser Paradox	⊕	–	⊕	–
Pareto Dominated Candidate	–	–	⊕	⊕
Lack of Monotonicity	–	⊕	–	–
Reinforcement	–	+	–	+
No-Show	–	+	–	+
Twin	–	+	–	+
Truncation	–	–	+	+
Subset Choice Condition (SCC)	+	+	–	+
Preference Inversion	+	+	+	+
Dependence on Order of Voting (DOV)	–	–	–	+
Total ⊕ signs	2	1	4	1
Total +& ⊕ signs	5	7	7	8

Notes
A + sign indicates that a procedure is vulnerable to the specified paradox;
A ⊕ sign indicates that a procedure is vulnerable to the specified paradox which seems to us an especially intolerable paradox;
A – sign indicates that a procedure is invulnerable to the specified paradox;
It is assumed that all voters have linear preference orderings among all competing candidates.

Exercises

Problem 4.1 Consider the following 11-voter, 3-alternative profile.

No. of voters	Preference orderings
5	$a \succ c \succ b$
4	$b \succ c \succ a$
2	$c \succ b \succ a$

Determine (i) the Plurality Voting winner; (ii) the Plurality with Runoff winner, (iii) the Condorcet Winner if one exists, (iv) the Approval Voting winner if the last 2 voters approve of (and vote for) their top two preferences whereas the remaining nine voters approve of (and vote for) only their top preference, and (v) the winner under the Successive Elimination procedure regardless of the order in which the alternatives are voted upon.

Problem 4.2 Construct a preference profile over 4 alternatives so that there is no Condorcet Winner.

Problem 4.3 Construct a 3-alternative, 5-voter example where the Plurality with Runoff procedure fails to elect the Condorcet Winner.

Problem 4.4 Construct a 3-alternative profile where the Plurality Voting procedure, Plurality with Runoff procedure and the Borda count each result in a different choice.

Problem 4.5 Consider the following profile containing all six possible strict rankings (no ties) over three alternatives *a, b, c*.

No. of voters	Preference orderings
$v(1)$	$a \succ b \succ c$
$v(2)$	$a \succ c \succ b$
$v(3)$	$b \succ a \succ c$
$v(4)$	$b \succ c \succ a$
$v(5)$	$c \succ a \succ b$
$v(6)$	$c \succ b \succ a$

For any profile of strict preferences $v(j)$, $j = 1, \ldots, 6$ denotes the number of individuals having the j'th ranking. Express in terms of values of $v(j)$ a sufficient condition for c being the unanimity winner, i.e., ranked first by all individuals.

Express in terms of inequalities those profiles where a is the Plurality Voting winner.

Express in terms of inequalities those profiles where a is the Condorcet Winner.

When would alternative c be unanimously elected?

Answers to Exercises

Problem 4.1 The Plurality Voting winner is *a;* the Plurality with Runoff winner is *b*; the Condorcet Winner is *c*; the Approval Voting winner is *b*; the Successive Elimination winner is *c* (the Condorcet Winner).

Problem 4.2 A 4-alternative profile where there is no Condorcet Winner is, for example, the following profile:

No. of voters	Preference orderings
5	$a \succ c \succ b \succ d$
4	$b \succ d \succ a \succ c$
2	$c \succ d \succ a \succ b$

Here the social preference ordering is cyclical ($b \succ d \succ a \succ c \succ b$).

Problem 4.3 Here is an example:

No. of voters	Preference orderings
2	$a \succ c \succ b$
2	$b \succ c \succ a$
1	$c \succ b \succ a$

Here *c* is the Condorcet Winner but will be eliminated at the end of the first counting round and then *b* will beat *a* in the second round to become the ultimate winner.

Problem 4.4 Here is an example:

No. of voters	Preference orderings
5	$a \succ c \succ b$
4	$b \succ c \succ a$
2	$c \succ b \succ a$

Here *a* is the Plurality Voting winner, *b* is the winner in the Plurality with Runoff procedure, and *c* is the Borda winner.

Problem 4.5 Alternative *a* is the Plurality Voting winner if $v(1) + v(2) > v(3) + v(4)$ and $v(1) + v(2) > v(5) + v(6)$.

Alternative *a* is the Condorcet Winner if $v(1) + v(2) + v(5) > v(3) + v(4) + v(6)$ and $v(1) + v(2) + v(3) > v(4) + v(5) + v(6)$.

Alternative *c* is the unanimity winner if $v(5) + v(6)$ equals the total number of voters.

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