Chapter 2
Voting Paradoxes

Abstract Voting paradoxes pertaining to the election of a single winner are introduced. The paradoxes are divided into five simple paradoxes and eight conditional ones. The simple paradoxes are paradoxes where the relevant data lead to a 'surprising' and arguably undesirable outcome, whereas the conditional paradoxes are ones where the change in one relevant datum while holding constant the other relevant data leads to a 'surprising' and arguably undesirable outcome.

Keywords Simple voting paradoxes · Conditional voting paradoxes

We define a '*voting paradox*' as an undesirable outcome that a voting procedure may produce and which may be regarded at first glance, at least by some people, as surprising or as counter-intuitive.

We distinguish between two types of voting paradoxes associated with a given voting procedure:

1. '*Simple*' or '*Straightforward*' *paradoxes*: These are paradoxes where the relevant data leads to a 'surprising' and arguably undesirable outcome. (The relevant data include, *inter alia*, the number of voters, the number of candidates, the number of candidates that must be elected, the preference ordering of every voter among the competing candidates, the amount of information voters have regarding all other voters' preference orderings, the order in which voters cast their votes if it is not simultaneous, the order in which candidates are voted upon if candidates are not voted upon simultaneously, whether voting is open or secret, and the manner in which ties are to be broken).
2. '*Conditional*' *paradoxes*: These are paradoxes where changing one relevant datum while holding constant all other relevant data leads to a 'surprising' and arguably undesirable outcome.

An array of paradoxes of one or both types are described and analyzed by McGarvey (1953), Riker (1958), Smith (1973), Fishburn (1973, 1974, 1977, 1981,

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1982), Young (1974), Niemi and Riker (1976), Doron and Kronick (1977), Doron (1979), Richelson (1975, 1978a, b, 1979, 1981), Gehrlein (1983), Fishburn and Brams (1983), Saari (1984, 1987, 1989, 1994, 2000, 2001, 2008), Niou (1987), Moulin (1988a), Merlin and Saari (1997), Brams et al. (1998), Scarsini (1998), Nurmi (1998a, b, 1999, 2004, 2007), Lepelley and Merlin (2001), Merlin et al. (2002), Merlin and Valognes (2004), Tideman (1987, 2006), Gehrlein and Lepelley (2011), among others.

2.1 Simple Paradoxes

The five best-known 'simple' paradoxes that may afflict voting procedures designed to elect one out of two or more candidates are the following:

2.1.1 *The Condorcet Winner Paradox (*Condorcet de 1785*;* Black 1958*)*

A candidate *x* is not elected despite the fact that it constitutes a 'Condorcet Winner', i.e., despite the fact that *x* is preferred by a majority of the voters over each of the other competing alternatives.

2.1.2 *The Absolute Majority Winner Paradox*

This is a special case of the Condorcet Winner Paradox. A candidate *x* may not be elected despite the fact that it is the candidate ranked first by an absolute majority of the voters.

2.1.3 *The Condorcet Loser or Borda Paradox (*Borda de 1784*;* Black 1958*)*

A candidate *x* is elected despite the fact that it constitutes a 'Condorcet Loser' i.e., despite the fact that a majority of voters prefer each of the remaining candidates to *x*. This paradox is a special case of the violation of Smith's (1973) *Condorcet Principle*. According to this principle, if it is possible to partition the set of candidates into two disjoint subsets, *A* and *B*, such that each candidate in *A* is preferred by a majority of the voters over each candidate in *B*, then no candidate in *B* ought to be elected unless all candidates in *A* are elected.

2.1.4 The Absolute Majority Loser Paradox

This is a special case of the Condorcet Loser Paradox. A candidate x may be elected despite the fact that it is ranked last by a majority of voters.

2.1.5 The Pareto (or Dominated Candidate) Paradox (Fishburn 1974)

A candidate x may be elected while candidate y may not be elected despite the fact that *all* voters prefer candidate y to x.

2.2 Conditional Paradoxes

The eight best-known 'conditional' paradoxes that may afflict voting procedures for electing a single candidate are the following:

2.2.1 Additional Support (or Lack of Monotonicity or Negative Responsiveness) Paradox (Smith 1973; Fishburn 1974a; Fishburn and Brams 1983)

If candidate x is elected under a given distribution of voters' preferences among the competing candidates, it is possible that, *ceteris paribus*, x may not be elected if some voter(s) *increase*(*s*) *his/her* (*their*) *support for* x by moving x to a higher position in his/her (their) preference ordering. Alternatively, if candidate x is not elected under a given distribution of voters' preferences among the competing candidates, it is possible that, *ceteris paribus*, x will be elected if some voter(s) *decrease*(*s*) his/her (their) support for x by moving x to a lower position in his/her (their) preference ordering.[1]

[1]Another version of the non-monotonicity paradox (which is not demonstrated in this book) is a situation where x is elected in a given electorate but may not be elected if, *ceteris paribus*, additional voters join the electorate who rank x at the top of their preference ordering, or, alternatively, a situation where x is not elected in a given electorate but may be elected if, *ceteris paribus*, additional voters join the electorate who rank x at the bottom of their preference ordering. For this version of non-monotonicity see Felsenthal and Nurmi (2016, 2017).

2.2.2 *Reinforcement (or Inconsistency or Multiple Districts) Paradox (***Young 1974***)*

If x is elected in each of several disjoint electorates, it is possible that, *ceteris paribus*, x will not be elected if all electorates are combined into a single electorate.

2.2.3 *Truncation Paradox (***Brams 1982***;* **Fishburn and Brams 1983***)*

A voter may obtain a more preferable outcome if, *ceteris paribus*, s/he lists in his/her ballot only part of his/her (sincere) preference ordering instead of listing his/her entire preference ordering among all the competing candidates.

2.2.4 *No-Show Paradox (***Fishburn and Brams 1983***;* **Ray 1986***;* **Moulin 1988b***;* **Holzman 1988/1989***;* **Perez 1995***)*

This is an extreme version of the Truncation Paradox. A voter may obtain a more preferable outcome if s/he decides not to participate in an election than, *ceteris paribus*, if s/he decides to participate in the election and vote sincerely for his/her top preference(s). A particular version of this paradox is stated thus: "The addition of identical ballots with candidate x ranked last may change the winner from another candidate to x." (cf. Fishburn and Brams 1983, p. 207).

2.2.5 *Twin Paradox (***Moulin 1988b***)*

This is a special version of the No-Show Paradox. Two voters having the same preference ordering may obtain a preferable outcome if, *ceteris paribus*, one of them decides not to participate in the election while the other votes sincerely.

2.2.6 *Violation of the Subset Choice Condition (SCC) (***Fishburn 1974b, *c*; 1977***)*

SCC requires that when there are at least three candidates and candidate x is the unique winner, then x must not become a loser whenever any of the original losers is removed and all other things remain the same. In the context of individual choice theory SCC is known as Chernoff's condition (1954, p. 429, postulate 4) which states that if an alternative x chosen from a set T is an element of a subset S of T, then x must be chosen also from S. This principle is called 'heritage' by Aizerman and Malishevski (1981, p. 1033) and 'property alpha' by Sen (1970, p. 17).

2.2.7 Preference Inversion Paradox

If the individual preferences of each voter are inverted it is possible that, *ceteris paribus,* the (unique) original winner will still win.

2.2.8 Dependence on Order of Voting (DOV) Paradox (Farquharson 1969)

If the voting on the competing candidates is conducted sequentially rather than simultaneously, it is possible that candidate x will be elected under a particular sequence but not, *ceteris paribus*, under an alternative sequence.

Exercises

Problem 2.1 Consider the following profile of five voters among five alternatives, a–e:

No. of voters	Preference orderings
1	$a \succ e \succ c \succ b \succ d$
1	$b \succ a \succ d \succ e \succ c$
1	$c \succ a \succ e \succ b \succ d$
1	$a \succ b \succ d \succ e \succ c$
1	$d \succ c \succ a \succ b \succ e$

Are there Pareto-dominated candidates, i.e., such candidates that are ranked lower than some other candidate by *all* voters?

Problem 2.2 What is the largest margin of victory in pairwise comparisons in the above profile?

Problem 2.3 Is there a Condorcet Winner in the above profile? Is there a Condorcet Loser?

Problem 2.4 If there is a Condorcet Winner in the above profile, would s/he gain the plurality of votes (and therefore be elected) if all voters would vote for their top 3 preferences?

Answers to Exercises

Problem 2.1 Candidate e is Pareto-dominated by candidate a. None of the other candidates can be Pareto-dominated since there is at least one voter that ranks them first.

Problem 2.2 5–0 (a vs. e)

Problem 2.3 Yes, a is the Condorcet Winner. No, there isn't a Condorcet Loser since all candidates defeat at least one other candidate in pairwise contests by a majority of votes.

Problem 2.4 Yes, the Condorcet Winner (a) would be elected since a would receive more votes (5) than any of the other alternatives.

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