The Non-equilibrium Nature of Active Motion

Francisco J. Sevilla

Abstract In this contribution chapter, the non-equilibrium nature of active motion is explored in the framework of the Generalized Langevin Equation. The persistence effects that distinguish active motion, observed in a variety of biological organisms and man-made colloidal particles, from the passive one, are put in correspondence with the memory function that characterizes the retarded dissipative effects in the equation. The non-equilibrium aspects of this approach rely on the relaxation of the fluctuation-dissipation relation, that couples the memory function with the autocorrelation function of the fluctuating force in order to describe the equilibrium. In the case of freely diffusing active particles, the Fokker-Planck equation is derived and an effective temperature can be identified if the total overlap between the deterministic solutions of the Generalized Langevin Equation at two times, weighted by the noise correlation function, exists and is finite. Active motion confined by the harmonic, external potential is analyzed on the same framework leading to analogous conclusions.

Keywords Active motion · Generalized Langevin equation · Confined active particles · Fluctuation-dissipation theorem · Persistent Brownian motion

1 Introduction

The systems in out-of-equilibrium conditions are ubiquitous in nature and have been the subject of intense study in many fields of knowledge during the last two centuries, at least. Among those systems, the biological ones are the most representatives of non-equilibrium situations, which despite the most obvious non-equilibrium feature, *life*, other non-equilibrium aspects of biological organisms have

F. J. Sevilla (✉)
Instituto de Fisica, Universidad Nacional Autónoma de México, Apdo. Postal 20-364, 01000, CD.MX., Mexico City, Mexico
e-mail: fjsevilla@fisica.unam.mx

L. Olivares-Quiroz, O. Resendis-Antonio (eds.), *Quantitative Models for Microscopic to Macroscopic Biological Macromolecules and Tissues*,
https://doi.org/10.1007/978-3-319-73975-5_4

received considerable attention, for instance: biological micromotors or molecular motors, which are able to travel along polymer filaments inside a cell [1]; motile organisms, like bacteria, that employs diverse motility patterns to traverse complex habitats [2, 3], for example *E. coli* which performs the so-called *run-and-tumble* dynamics [4] as pattern of motion. More recent advances have allowed the design of artificial particles that take advantage of different physical and/or chemical mechanisms, to self-induced motion that mimics that of biological organisms [5], as such, the design of artificial micromotors, which use locally supplied fuels, to autonomously deliver and release therapeutic payloads and manipulate cells [6].

All these examples have in common that the mobile entities involved, either biological or man-made, are able to develop their own motion by using the locally available energy from the environment and transform it, by complex internal mechanisms (micromotor or microengine) or by ingeniously self-phoretic mechanisms, into self-locomotion [5, 7, 8]. These particles are called *self-propelled* or *active particles*, in contrast to the motion of a pollen grain in water (effect observed by the botanist Robert Brown) that moves passively due to the myriads of collisions with the molecules of the embedding liquid. Systems composed of a collection of self-propelled or active particles receive nowadays the name of *active matter* and are the subject of intense research, mainly because the non-equilibrium features exhibited in the collective and single particle behavior give the possibility to discover new physics, and as consequence, their potential applications.

The intrinsic non-equilibrium aspects of active matter have attracted the attention on these systems, and a front of intense research in different disciplines, mainly in statistical physics and biology, is still growing [5, 7, 8]. From the point of view of non-equilibrium statistical physics, active matter has become a well-defined class of non-equilibrium systems and a fertile field of research, that has allowed the rapid theoretical advancements in the field. Succinctly, active matter escapes from a description of equilibrium since, at microscopic scales, detailed balance between injection and dissipation of energy is not satisfied, which leads inexorably to the production of entropy.

One aspect to highlight in regard to the motility of active particles is that active motion is persistent, that is to say, the particles approximately retain the state of motion for a characteristic finite time scale, called the persistence time. This feature is indeed observed in the patterns of motion of different biological organisms and in the motility behavior of some artificial particles. For instance, the run-and-tumble pattern of motion alternates periods of time for which *E. coli* moves almost with constant speed and in a straight line along a randomly chosen direction, with short periods of time for which the particle is almost in rest tumbling. On a statistical description of this process, run-and-tumble dynamics can be characterized by finite time scale of persistence, which makes motility behavior strongly correlated in time, making non-equilibrium signatures conspicuously observable.

On the other hand, a finite persistence time scale can be induced on micron silica spheres through a diffusion-phoretic mechanism (Janus particles). Such mechanism confer out-of-equilibrium fluctuations on a region of the particle surface (by coating one of the particle hemispheres with a suitable metal for instance), breaking, locally, the symmetry of the effects of fluctuations on the particle [9, 10]. This breaking of symmetry causes an overdamped description of the translational degree of freedom (particle usually diffuses in aqueous solutions), and an underdamped description of rotational diffusion that leads to the appearance of persistence effects. The pattern of motion generated by this process is called *active Brownian motion*.

Many of the accomplished advancements in the understanding of active matter partly rely on the intuition built from equilibrium systems [11–13]. In reference [14] the authors provide arguments that show that the transition to *collective motion* in the Vicsek model [15]—a system of active particles under the influence of mutual motion alignment and non-thermal angular noise—can be best explained as a liquid-gas transition rather than an order-disorder one. Indeed, the inhomogeneous phases (smectic arrangements of traveling ordered bands surrounded by particles that form a gaseous-like phase) observed at intermediate noise and density are reminiscent of the coexistence liquid-gas phases in the transition.

Other non-equilibrium aspects exhibited in systems of interacting active particles refer to the observed *motility induced phase separation* (MIPS) [11, 13, 16–19], that corresponds to the coexistence of liquid-gas phases of repulsive self-propelled particles where no collective motion emerges. Attempts to give a description of MIPS in terms of equilibrium concepts have appeared, however these are limited and debated [16]. In some models that exhibit MIPS [19], time-reversal symmetry is preserved if the persistence time is small but finite, this endows such systems with an effective fluctuation-dissipation theorem akin to that of thermal equilibrium systems. In systems of spherical active Brownian particles it has been possible to derive an expression for the equation of state of the mechanical pressure of the system as function of particle density. Previous attempts identified the contribution to the pressure exerted by a suspension of active particles that originates upon the notion that an active body would swim away in space unless confined by boundaries [11]. This observation allowed the authors to find a non-equilibrium equation of state whose phase diagrams resemble a van der Waals loop from equilibrium gas-liquid coexistence.

Another equilibrium concept that has resulted valuable in the description of out-of-equilibrium systems is that one of *effective temperature* [20–22], in particular in systems of active particles [23–30]. In general, the possibility of defining an effective temperature relies on the fulfillment of a non-thermal fluctuation-dissipation relation. This is the case for time scales larger than the persistence one, since in this regime the motion of free active particles is well characterized by an effective diffusion coefficient, and can be interpreted as the motion of a passive Brownian particle diffusing in a *fictitious* environment at the effective temperature.

The out-of-equilibrium nature of systems of active particles is revealed markedly when the system is under confinement. Particularly, in the regime for which the characteristic length scale of confinement is smaller, or of the order, of the persistence length scale. Under such conditions, the zero-current probability distribution of non-interacting active particles deviates conspicuously from the equilibrium distribution of Boltzmann and Gibbs. Such effects have been certainly predicted by theory in models of active particles under the confinement of an external potential [31, 32] and observed experimentally, for instance, in acoustically confined active Brownian particles [33], in confined worker termites [34], and in passive Brownian particles swimming within an active Bath [35].

The non-Boltzmann-Gibbs distributions that correspond to the stationary distributions of run-and-tumble particles, that move with constant speed under the confinement of an external potential, can be understood as the distributions of passive Brownian particles diffusing under the influence of the external potential and of a fictitious inhomogeneous thermal bath (Sevilla, Vasquez-Arzola, Puga-Cital, 2017, unpublished), where the precise spatial dependence of the effective temperature profile can be put in an exact manner in correspondence with the trapping potential. In contrast, models that consider the case for which particle speed fluctuates can lead to Boltzmann-Gibbs-like distributions at least when the particles are confined by an external harmonic potential, as has been shown in a one-dimensional model of active motion [29]. For this particular model of active particles (also known as active-Ornstein-Uhlenbeck particles), the existence of a uniform effective temperature is shown.

The existence of many different theoretical frameworks that consider persistence in their own formulation makes worthy their study and analysis in relation with concepts of non-equilibrium statistical mechanics, partly because some aspects may well correspond to qualitatively distinct phenomena, as has been described in the last paragraph, but mainly because many of these frameworks incorporate straightforwardly the important effects of correlations that leads, for instance, to a proper description of anomalous diffusion. Among these frameworks, we can cite: the *persistent random walks*, recurrently used in biology [36] and studied intensively in statistical physics towards the last decade of the last century [37, 38]; the *continuous-time random walk* [39] which endows random walk with correlations in continuous time; and the *generalized Langevin Equation* [40], which endows the standard equation of Langevin of Brownian motion with finite time correlations. The generalized Langevin equation usually models systems in equilibrium with a viscoelastic bath, which describe the equilibrium of the retarded effects in the viscous drag term of the equation and correlated noises.

A straightforward way to model the active motion of Brownian-like particles, is to include a non-linear dissipative term in the Langevin equation, i.e., by the introduction of a non-linear friction term that keeps the speed of the particles almost constant in time [41, 42], as approximately occurs in many active systems. The non-equilibrium nature of these systems, when particles move under the influence of an external potential, has been analyzed by the calculation of the entropy production [43]. However, given the non-linear nature of the equations involved, no exact expressions for the stationary distributions of the particle positions exist.

In this contribution, we explore the non-equilibrium nature of confined active motion in an external potential, by making a correspondence of the persistence effects of active motion with the memory function that characterizes the retarded dissipative effects in the linear generalized Langevin equation. The non-equilibrium aspects of this approach rely on the relaxation of the fluctuation-dissipation relation, which describes equilibrium if and only if the memory function in the retarded friction term is proportional to the autocorrelation function of the fluctuating force in the equation.

1.1 The Generalized Langevin Equation: Diffusion of a Free Particle in an Equilibrium Bath at Temperature T

The Generalized Langevin Equation, that describes the kinematic state of a Brownian particle of mass m in one dimension, are

$$m\frac{d}{dt}v(t) = -\gamma \int_{t_0}^{t} ds\, \phi(t-s)v(s) + \xi(t), \tag{1a}$$

$$\frac{d}{dt}x(t) = v(t), \tag{1b}$$

where $x(t)$ and $v(t)$ denote the position and velocity of the particle, respectively. The retarded effects in the dissipation term are encoded in the memory function $\phi(t)$ which has units of $[\text{time}]^{-1}$, while the coefficient γ, the Stokes dragging coefficient, is used as a scale of the net friction force. In the cases of interest, the memory function vanishes after some characteristic time scale that can well be assumed smaller than t_0, therefore $t - t_0$ is the time span for which the memory effects are important and thus, any effect prior to time t_0 can be neglected. In the of-equilibrium processes where the effects of ageing are not important, of particular physical relevance are those cases for which Eq. (1) describe a stationary process, i.e., a process whose statistical properties are invariant under temporal translations, in such a case, t_0 can be set to zero without loss of generality. For simplicity the *noise* term, $\xi(t)$, is assumed to be stationary and Gaussian with vanishing average $\langle \xi(t) \rangle = 0$ and autocorrelation function

$$\langle \xi(t)\xi(s) \rangle = \langle \xi^2 \rangle_{\text{eq}} \Gamma(|t-s|), \tag{2}$$

with $\langle \xi^2 \rangle_{\text{eq}}$ is the parameter that characterizes the fluctuations of thermal equilibrium at temperature T, and $\Gamma(t)$ is a dimensionless function of time.

If in addition to the physical assumptions framed in Eq. (1), it is supposed that the effects of the correlated thermal fluctuations over the particle—induced by a surrounding thermal bath at equilibrium and characterized by the temperature T—are balanced by the dissipation term, then the stochastic process $v(t)$ reaches a stationary regime for which the distribution of velocities, $P_{\text{eq}}(v)$, corresponds to that of equilibrium. Under these conditions, one can safely assume that equipartition is valid for the kinetic energy of the Brownian particle and an explicit relation among the functions $\phi(t)$ and $\Gamma(t)$ can be obtained. Such relation has been called the fluctuation-dissipation relation [40]. In contrast, since the particle diffuses freely, the stochastic process $x(t)$ will not reach a stationary regime. Indeed, the distribution width of the particle positions, measured by the distribution variance, is expected to grow with time indefinitely. Thus, in regards to the joint dynamics of both processes, $v(t)$ and $x(t)$, we say that the system equilibrates partially.

Though formal derivations of the fluctuation-dissipation relation are known [44], the following derivation is heuristically straightforward. Consider the solution of Eq. (1a), given explicitly by

$$v(t) = v(0)\Phi(t) + \frac{1}{m}\int_0^t ds\, \Phi(t-s)\xi(s), \tag{3}$$

where $v(0)$ is the initial velocity and $\Phi(t)$ is the function that satisfies the equation

$$m\frac{d\Phi(t)}{dt} + \gamma\int_0^t ds\, \phi(t-s)\Phi(t) = 0, \tag{4}$$

which in the Laplace domain has the explicit dependence on $\widetilde{\phi}(\epsilon)$ through the relation

$$\widetilde{\Phi}(\epsilon) = \left[\epsilon + \frac{\gamma}{m}\widetilde{\phi}(\epsilon)\right]^{-1}. \tag{5}$$

A symbol with tilde, $\widetilde{f}(\epsilon)$, denotes the Laplace transform of $f(t)$ given by

$$\widetilde{f}(\epsilon) = \mathcal{L}\{f(t);\epsilon\} = \int_0^\infty dt\, e^{-\epsilon t} f(t) \tag{6}$$

with ϵ the Laplace variable, a complex number.

The assumption that the distribution of velocities of the Brownian particle reaches a stationary regime characterized by thermal equilibrium implies that an equipartition-like theorem applies for the Brownian particle velocity, i.e.

$$\frac{m}{2}\langle v^2(t)\rangle = \frac{k_B T}{2}, \tag{7}$$

where k_B is the Boltzmann constant. If the square of (3) is taken and then multiplied by $m/2$ we have that the ensemble average over all possible realizations of the random force $\xi(t)$ gives

$$\frac{m}{2}\langle v^2(t)\rangle = \frac{m}{2}v^2(0)\Phi^2(t) + \frac{\langle\xi^2\rangle_{\mathrm{eq}}}{m}\int_0^t ds_1\int_0^{s_1} ds_2\, \Phi(s_1)\Phi(s_2)\Gamma(s_1-s_2), \tag{8}$$

where we used the fact that $\langle\xi(t)\rangle = 0$ and the explicit dependence on the initial conditions $v(0)$ is shown. However, if the initial velocities of the Brownian particle were distributed according to thermal equilibrium, we must have that

$$\frac{m}{2}\overline{v^2(0)} = \frac{m}{2}\langle v^2(t)\rangle = \frac{k_B T}{2}, \tag{9}$$

must be satisfied, where the $\overline{v^2(0)}$ denotes the ensemble average over initial velocities. We have straightforwardly that

$$k_B T\left[1-\Phi^2(t)\right] = \frac{\langle\xi^2\rangle_{\rm eq}}{m}\int_0^t ds_1 \int_0^{s_1} ds_2\, \Phi(s_1)\Phi(s_2)\Gamma(s_1-s_2), \tag{10}$$

where use of the relation (2) has been made. Now take the time derivative on both sides of the last expression, and after making some rearrangements we get

$$-\frac{2mk_BT}{\langle\xi^2\rangle_{\rm eq}}\frac{d\Phi(t)}{dt} = \int_0^t ds_2\, \Phi(s_2)\Gamma(t-s_2), \tag{11}$$

whose Laplace transform is given by

$$\frac{2mk_BT}{\langle\xi^2\rangle_{\rm eq}}[\Phi(0)-\epsilon\widetilde{\Phi}(\epsilon)] = \widetilde{\Phi}(\epsilon)\widetilde{\Gamma}(\epsilon). \tag{12}$$

We must assume that $\phi(t)$ is such that $\lim_{\epsilon\to\infty}\widetilde{\phi}(\epsilon)/\epsilon$ goes to zero, then, by the *Tauberian* theorems we have that $\Phi(0) = 1$ and thus

$$\frac{2mk_BT}{\langle\xi^2\rangle_{\rm eq}}\left[\frac{1}{\widetilde{\Phi}(\epsilon)}-\epsilon\right] = \widetilde{\Gamma}(\epsilon). \tag{13}$$

After substitution of (5) we finally get

$$\widetilde{\Gamma}(\epsilon) = \frac{2k_BT}{\langle\xi^2\rangle_{\rm eq}}\gamma\widetilde{\phi}(\epsilon) \tag{14}$$

or by recalling that $\langle\xi^2\rangle_{\rm eq}\Gamma(t) = \langle\xi(s)\xi(s+t)\rangle$ we get

$$\langle\xi(s)\xi(s+t)\rangle = 2k_BT\gamma\phi(t). \tag{15}$$

This is the fluctuation-dissipation relation [40], that establishes the equilibrium temperature T as the proportionality factor between the autocorrelation function of the thermal force (*internal noise* [45–47]) and the function that characterizes the retarded effect of the dragging force. Such a relation guarantees the reach of equilibrium for the process $v(t)$ at the temperature of the bath T, for arbitrary time dependence of the noise autocorrelation function as long as this decays to zero with time.

$P(v,t)$ and Its Associated Fokker-Planck Equation

As a consequence of the fluctuation-dissipation relation (15), the stationary distribution of the particle velocities is given by the Maxwell distribution at the fluid temperature T, but evidently with the mass m of the Brownian particle.

The conditional probability distribution, $P(v, t|v(0))$, of a particle having the velocity v at time t given that had velocity $v(0)$ at time $t = 0$ is defined by

$$P(v, t|v(0)) \equiv \langle \delta[v - v(t)] \rangle \,, \tag{16}$$

where $v(t)$ is the solution of Eq. (1a) given by (3), and the ensemble average $\langle \, \rangle$ is made over the noise realizations of $\xi(t)$ with fixed initial conditions. We have then that

$$P(v, t|v(0)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ik[v - v(0)\Phi(t)]} \left\langle e^{i\frac{k}{m}\int_0^t ds\, \Phi(t-s)\xi(s)} \right\rangle . \tag{17}$$

The quantity

$$\left\langle e^{i\frac{k}{m}\int_0^t ds\, \Phi(t-s)\xi(s)} \right\rangle \tag{18}$$

is the characteristic functional of the stochastic process variable $\xi(t)$, since it has been assumed to be Gaussian has the explicit expression [48]

$$\exp\left\{ -\frac{k^2\langle \xi^2 \rangle_{\text{eq}}}{m^2} \int_0^t ds_1 \int_0^{s_1} ds_2 \, \Phi(s_1)\Phi(s_2)\Gamma(s_1 - s_2) \right\}$$

and therefore we have that

$$P(v, t|v(0)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ik[v - v(0)\Phi(t)]} e^{-k^2 \frac{\langle \xi^2 \rangle_{\text{eq}}}{m^2} \int_0^t ds_1 \int_0^{s_1} ds_2 \, \Phi(s_1)\Phi(s_2)\Gamma(s_1 - s_2)} .$$

Last expression can be evaluated explicitly and gives

$$P(v, t|v(0)) = \frac{\exp\left\{ -\dfrac{[v - v(0)\Phi(t)]^2}{2\langle [v - v(0)\Phi(t)]^2 \rangle} \right\}}{\sqrt{2\pi \langle [v - v(0)\Phi(t)]^2 \rangle}} , \tag{19}$$

where we have used (3) and (15) to write explicitly in terms of $\phi(t)$

$$\left\langle [v - v(0)\Phi(t)]^2 \right\rangle = 2\frac{\gamma\, k_B T}{m^2} \int_0^t ds_1 \int_0^{s_1} ds_2 \, \Phi(s_1)\Phi(s_2)\phi(s_1 - s_2). \tag{20}$$

In the stationary regime $t \to \infty$, we have $\Phi(t) \to 0$ and $\langle [v - v(0)\Phi(t)]^2 \rangle \to \langle v^2(\infty) \rangle = k_B T/m$ and therefore

$$P_{\text{st}}(v) = \frac{1}{\sqrt{2\pi k_B T/m}} \exp\left\{ -\frac{mv^2}{2k_B T} \right\} , \tag{21}$$

which corresponds to the Maxwell distribution of velocities at the bath's temperature T.

For the purposes that will be clear in Sect. 2, we present a derivation of the Fokker-Planck equation that corresponds to the linear Generalized Langevin Equation (1) of the process $v(t)$, where $\xi(t)$ is a Gaussian stochastic process with vanishing average and autocorrelation function (2). The derivation of this Fokker-Planck has been considered by several authors before [49–52], here we present a heuristic, yet rigorous, derivation of such equation.

First notice that for the Gaussian white noise, i.e. $\phi(t) = \delta(t)$, and therefore $\Phi(t) = e^{-\gamma t/m}$, the relation (15) turns (1) into the Ornstein-Uhlenbeck process for which the conditional probability distribution, $P(v, t|v(0))$ is given by well-known result

$$P(v, t|v(0)) = \frac{\exp\left\{-\dfrac{m[v - v(0)e^{-\gamma t/m}]^2}{2k_BT(1 - e^{-2\gamma t})}\right\}}{\sqrt{2\pi k_BT(1 - e^{-2\gamma t})/m}}, \tag{22}$$

which satisfies the Fokker-Planck equation [53]

$$\frac{\partial}{\partial t}P(v, t|v(0)) = \frac{\partial}{\partial v}\left[\frac{\gamma}{m}vP(v, t|v(0))\right] + \frac{\gamma}{m^2}k_BT\frac{\partial^2}{\partial v^2}P(v, t|v(0)). \tag{23}$$

Given the linearity of the Langevin equation (1a), it is tempting to propose as a suitable anzats, the following Fokker-Planck equation

$$\frac{\partial}{\partial t}P(v, t|v(0)) = \frac{\partial}{\partial v}\left[\frac{\gamma}{m}\chi_1(t)vP(v, t|v(0))\right] + \frac{\gamma}{m^2}k_BT\chi_2(t)\frac{\partial^2}{\partial v^2}P(v, t|v(0)), \tag{24}$$

where $\chi_1(t)$ and $\chi_2(t)$ are functions to determine with the initial condition $P(v|v(0)) = \delta[v-v(0)]$. The function $\chi_1(t)$ in the first term in the right-hand side of Eq. (24) accounts for the retarded effects of the friction force in (1a), characterized by $\phi(t)$, while $\chi_2(t)$ in the second term represents the effects of the correlations of the internal noise that arise from the fluctuation-dissipation relation (15). In such a case, a connection between $\chi_1(t)$ and $\chi_2(t)$ is expected.

The solution to (24) can be found by the use of Fourier transform which turns the second order partial differential equation (24) into the first order one

$$\frac{\partial}{\partial t}\hat{P}(k, t|v(0)) = -\frac{\gamma}{m}\chi_1(t)\,k\frac{\partial}{\partial k}\hat{P}(k, t|v(0)) - \frac{\gamma}{m^2}k_BT\,\chi_2(t)\,k^2\,\hat{P}(k, t|v(0)), \tag{25}$$

where a function with hat, $\hat{f}(k)$, denotes its Fourier transform

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx\, e^{ikx}f(x), \tag{26}$$

with Fourier variable k.

By the use of the well-known method of the characteristics, we have to solve the following pair of ordinary differential equations

$$\frac{d}{dt}k(t) = \frac{\gamma}{m}\chi_1(t)\,k(t),$$
$$\frac{d}{dt}\hat{P}(t) = -\frac{\gamma}{m^2}k_BT\chi_2(t)\,k^2\,\hat{P}(t),$$

where $\hat{P}(t) = \hat{P}[k(t), t|v(0)]$ denotes the solution of (25) along the characteristic trajectory $k(t)$. The solutions to the last equations are given, respectively, by

$$k(t) = k(0)e^{(\gamma/m)\int_0^t ds\,\chi_1(s)}, \tag{27a}$$

$$\hat{P}(t) = \hat{P}(0)e^{-(\gamma/m^2)k_BT\int_0^t ds\,\chi_2(s)\,k^2(s)}. \tag{27b}$$

From this, the solution in Fourier space is explicitly written as

$$\hat{P}(k, t|v(0)) = \hat{P}\left(ke^{-(\gamma/m)\int_0^t ds\,\chi_1(s)}|v(0)\right)$$
$$\exp\left\{-(\gamma/m^2)k_BT\,k^2\int_0^t ds\,\chi_2(s)e^{-2(\gamma/m)\int_s^t ds'\,\chi_1(s')}\right\}, \tag{28}$$

where the first factor corresponds to the Fourier transform of the initial distribution $P(v|v(0)) = \delta[v - v(0)]$. After inversion of the Fourier transform we have that

$$P(v, t|v(0)) = \frac{\exp\left\{-\dfrac{\left[v - v(0)e^{-(\gamma/m)\int_0^t ds\,\chi_1(s)}\right]^2}{4(\gamma/m^2)k_BT\int_0^t ds\,\chi_2(s)e^{-2(\gamma/m)\int_s^t ds'\chi_1(s')}}\right\}}{\sqrt{4\pi(\gamma/m^2)k_BT\int_0^t ds\,\chi_2(s)e^{-2(\gamma/m)\int_s^t ds'\chi_1(s')}}}. \tag{29}$$

The functions $\chi_1(t)$ and $\chi_2(t)$ are determined by comparison of expression (29) with (19), we have, respectively, that

$$e^{-(\gamma/m)\int_0^t ds\,\chi_1(s)} = \Phi(t), \tag{30a}$$

$$2\frac{\gamma}{m^2}k_BT\int_0^t ds\,\chi_2(s)e^{-2(\gamma/m)\int_s^t ds'\,\chi_1(s')} = \left\langle[v - v(0)\Phi(t)]^2\right\rangle. \tag{30b}$$

The explicit relation of $\chi_1(t)$ with the memory function $\phi(t)$ is indirect via the Laplace transform

$$\mathcal{L}\left\{e^{-(\gamma/m)\int_0^t ds\,\chi_1(s)}; \epsilon\right\} = \frac{1}{\epsilon + \frac{\gamma}{m}\widetilde{\phi}(\epsilon)}, \tag{31}$$

where we have used that $\widetilde{\Phi}(\epsilon) = [\epsilon + \frac{\gamma}{m}\widetilde{\phi}(\epsilon)]^{-1}$. In terms of $\Phi(t)$ we have

$$\frac{\gamma}{m}\chi_1(t) = \frac{d}{dt}\ln \Phi(t)^{-1}. \tag{32}$$

To determine $\chi_2(t)$, notice that the integral in left hand side of Eq. (30b) can be written in terms of $\Phi(t)$ as

$$\int_0^t ds\, \chi_2(s)\frac{\Phi^2(t)}{\Phi^2(s)} \tag{33}$$

thus rearranging (30b) we have that

$$\int_0^t ds\, \frac{\chi_2(s)}{\Phi^2(s)} = \frac{1}{\Phi^2(t)}\int_0^t ds_1 \int_0^{s_1} ds_2\, \Phi(s_1)\Phi(s_2)\phi(s_1 - s_2) \tag{34}$$

and therefore, after taking the derivative with respect to time

$$\chi_2(t) = \left[\frac{d}{dt}\ln \Phi^{-2}(t)\right]\int_0^t ds_1 \int_0^{s_1} ds_2\, \Phi(s_1)\Phi(s_2)\phi(s_1 - s_2) + \Phi(t)\int_0^t ds\, \phi(t-s)\Phi(s). \tag{35}$$

With expressions (32) and (35), the Fokker-Planck equation corresponding to the generalized Langevin equation (1a) is determined.

The Mean Square Displacement

As it was commented above, the stochastic process $x(t)$ does not reach a stationary regime in the framework of the generalized Langevin equation (1) and therefore the distribution of the particles positions $P(x, t)$ is not stationary, in fact, it is expected the distribution to become broader as time is passing on. The scaling of how fast the distribution becomes broader depends exclusively on the time dependence of the noise autocorrelation function. This fact has allowed the use of (1) [45, 54] to give an alternative to *fractional diffusion equations* [55] and other mathematical frameworks to describe anomalous diffusion, by properly choosing the time dependence of $\phi(t)$ [or equivalently of $\Gamma(t)$ if (15) holds]. The mean squared displacement (second moment of the probability distribution of positions) is the quantity commonly used to describe anomalous diffusion.

From the explicit expression for $v(t)$, given in Eq. (3), the particle position is

$$x(t) = x(0) + v(0)\Phi_I(t) + \frac{1}{m}\int_0^t ds\, \Phi_I(t-s)\xi(s),$$

where

$$\Phi_I(t) \equiv \int_0^t ds\, \Phi(s). \tag{36}$$

By noting that $\langle x(t)\rangle = x(0) + v(0)\Phi_I(t)$ we have that the variance of the position or mean squared displacement with respect to the average trajectory, namely

$$\sigma_{xx}(t) \equiv \left\langle [x(t) - \langle x(t)\rangle]^2 \right\rangle, \tag{37}$$

can be written as

$$\sigma_{xx}(t) = \frac{4k_B T}{m} \int_0^t ds\, \Phi_I(s)\, [1 - \Phi(s)]\,. \tag{38}$$

The asymptotic behavior of the memory function $\phi(t)$ determines the long-time regime of the mean squared displacement (38), through Eqs. (5) and (36). A sufficient condition for the standard linear dependence on time (the so-called normal diffusion) to emerge is that as ϵ goes asymptotically to zero, then $\epsilon\widetilde{\phi}(\epsilon) \sim \epsilon$, for which the corresponding asymptotic behavior of $\Phi(t)$ and $\Phi_I(t)$ is obtained from $\widetilde{\Phi}(\epsilon) \sim C$ and $\widetilde{\Phi}_I(\epsilon) \sim \tau\epsilon^{-1}$, by the Tauberian theorems respectively, where C is a dimensionless constant and τ a constant with units of time, then we have

$$\sigma_{xx}(t) \sim \frac{4k_B T\tau}{m}\, t. \tag{39}$$

Examples Consider as a simple example the case of a memory that decays exponentially fast with time, i.e., $\phi(t) = \alpha e^{-\alpha t}$.

In the limit of infinitely rapid decay, i.e., $\alpha \to \infty$ we have that $\phi(t) = \delta(t)$ and we recover the standard Langevin equation

$$m\frac{dv(t)}{dt} + \gamma v(t) = \xi(t), \tag{40}$$

for the so-called Ornstein-Uhlenbeck process, which gives the well-known result

$$\langle [x(t) - x_0]^2\rangle = \frac{m}{\gamma}\, \overline{v_0^2} \left[2t - \frac{m}{\gamma}(1 - e^{-2\frac{\gamma}{m}t})\right], \tag{41}$$

from which the ballistic result, $\overline{v_0^2}\, t^2$, is obtained when $\frac{\gamma}{m}t \ll 1$, and the standard diffusion result, $\frac{m}{\gamma}\langle v_0^2\rangle t$, is recovered in the regime $\frac{\gamma}{m}t \gg 1$.

For finite α we have that

$$\widetilde{\phi}(\epsilon) = \frac{\alpha}{\epsilon + \alpha}, \tag{42a}$$

$$\widetilde{\Phi}(\epsilon) = \frac{\epsilon + \alpha}{\epsilon(\epsilon + \alpha) + \alpha\frac{\gamma}{m}}. \tag{42b}$$

Power Law Memory

Let's consider the case $\widetilde{\phi}(\epsilon) = \frac{1}{\epsilon^{\alpha}}$, and therefore $\widetilde{\Phi}(\epsilon) = \frac{\epsilon^{\alpha}}{\epsilon^{\alpha+1}+(\gamma/m)^{\alpha+1}}$ which corresponds, in time-domain, to $E_{\alpha,1}[-(\frac{\gamma}{m}t)^{\alpha}]$, where $E_{\alpha,1}(z)$ is the Mittag-Leffler function. It is straightforward to check that then $\Phi_I(t) = tE_{\alpha,2}[-(\frac{\gamma}{m}t)^{\alpha}]$. By considering the asymptotic behavior of the Mittag-Leffler function we have

$$\sigma_{xx}(t) \sim 2\langle v^2(0)t^{2-\alpha}. \tag{43}$$

1.2 Diffusion of a Free Particle Under External Noise: The Out-of-Equilibrium Case (The Effective Temperature)

In the previous section, the partial equilibration of a freely diffusing Brownian particle was considered in the context of the fluctuation-dissipation relation. The equilibrium distribution of velocities requires the satisfaction of this relation and implies basically that the time scale involved in the persistence effects of the particle motion, taken into account by the memory function of the retarded friction term, is the same as the time scale of the correlations of fluctuation of the random force.

We consider now the case when the fluctuation-dissipation relation (15) does not hold, therefore the time dependence of the memory function $\phi(t)$ in the retarded term in Eq. (1a) is independent of the autocorrelation function of the fluctuating force. Under this circumstance, the fluctuating force $\xi(t)$ is called *external noise* [47] and is assumed a Gaussian process with autocorrelation function

$$\langle \xi(t)\xi(s)\rangle = \langle \xi^2\rangle \Gamma(|t-s|), \tag{44}$$

where $\langle \xi^2\rangle$ is a factor that characterizes the intensity of *non-thermal* fluctuations.

The retarded dissipative force is now decoupled from the fluctuating one, and the effects of the first one, do not necessarily balance those of the other, therefore, in the general case, the system will not reach the equilibrium state [45, 54], not even the equilibrium stationarity of the process $v(t)$, as it does in the case described when the fluctuation-dissipation relation holds. This, however, does not preclude the possibility that the process $v(t)$ reaches a non-equilibrium stationary regime. In consequence, there has been fairly interest in systems out of equilibrium and in the search of quantities that could serve as a measure or indicators of a "distance" from equilibrium [19, 22]. Though it is well-known that the rate of entropy production is a clear indicator of the out-of-equilibrium nature of a system, a more straightforward concept to describe this condition is the *effective temperature*. This is not directly built from the concept of temperature in equilibrium thermodynamics, but on a generalization of the fluctuation-dissipation relation [22].

Last considerations make the conditional probability density, given by (19), still valid, however we must have now that

$$\left\langle [v - v(0)\Phi(t)]^2 \right\rangle = \frac{\left\langle \xi^2 \right\rangle}{m^2} \int_0^t ds_1 \int_0^{s_1} ds_2\, \Phi(s_1)\Phi(s_2)\Gamma(s_1 - s_2), \tag{45}$$

where the left-hand side in the last expression depends implicitly on $\phi(t)$ and $\Gamma(t)$, through $\Phi(t)$ and the ensemble average over noise realizations, respectively. The right-hand side on the other hand makes explicit the appearance of $\Gamma(t)$, and as before, $\phi(t)$ appears implicitly through $\Phi(t)$.

If a non-equilibrium stationarity of the process $v(t)$ is expected, it must happen that

$$\left\langle [v - v(0)\Phi(t)]^2 \right\rangle \underset{t\to\infty}{\longrightarrow} \frac{k_B T_{\text{eff}}}{m}, \tag{46}$$

and an effective temperature T_{eff} can be defined if and only if

$$\kappa \equiv \lim_{t\to\infty} \int_0^t ds_1 \int_0^{s_1} ds_2\, \Phi(s_1)\Phi(s_2)\Gamma(s_1 - s_2) \tag{47}$$

exists. If that is the case, then T_{eff} can be related to the intensity of noise through

$$T_{\text{eff}} = \frac{\left\langle \xi^2 \right\rangle}{k_B m} \kappa. \tag{48}$$

Thus, the existence of an effective temperature requires that the time dependence of $\phi(t)$ and $\Gamma(t)$ be such that κ remains finite and constant.

A plausible physical situation corresponds to the case of external noise exponentially correlated,

$$\langle \xi(t)\xi(s)\rangle = \langle \xi^2 \rangle e^{-(|t-s|)/\tau_{\text{cor}}}, \tag{49}$$

with correlation time τ_{cor} and a memory function that decays exponentially as

$$\phi(t) = \tau_{\text{pers}}^{-1} e^{-t/\tau_{\text{pers}}}, \tag{50}$$

where τ_{pers} denotes the persistence time. In such a case we have, after use of the Tauberian theorem, that

$$\kappa = \tau_{\text{cor}} \frac{m}{\gamma} \tag{51}$$

and therefore, the prescription (48) gives

$$T_{\mathrm{eff}} = \frac{\langle \xi^2 \rangle \tau_{\mathrm{cor}}}{k_B \gamma}, \tag{52}$$

which is independent of the persistence time τ_{pers}.

2 Generalized Langevin Equation for Brownian Motion Confined by an External Potential

We now turn to the case of our interest, which corresponds to that one for which both processes, $v(t)$ and $x(t)$, attain a stationary state under out-of-equilibrium conditions, and we focus on the non-equilibrium nature of the zero-flux stationary distributions of the particle positions $P_{\mathrm{st}}(x)$. Stationary solutions for the process $x(t)$ are expected if the Brownian particle is confined either by hard walls or by the external potential $U(x)$. For this case the generalized Langevin equation is given by [40, 56]

$$m\frac{d}{dt}v(t) = -\frac{\partial}{\partial x}U(x) - \gamma \int_0^t ds\, \phi(t-s)v(s) + \xi(t), \tag{53a}$$

$$\frac{d}{dt}x(t) = v(t). \tag{53b}$$

In the physically plausible cases for which the fluctuation-dissipation relation (15) holds, the stochastic processes $v(t)$ and $x(t)$ can be treated as statistically independent, and the joint equilibrium distribution $P_{\mathrm{eq}}(x, v)$ can be factorized as the product of their corresponding equilibrium distribution of Maxwell $P_{\mathrm{eq}}(v)$, Eq. (21), and

$$P_{\mathrm{eq}}(x) = \mathcal{Z}^{-1} e^{-U(x)/k_B T} \tag{54}$$

of Boltzmann-Gibbs, both characterized by the equilibrium temperature T. Evidently the probability distribution (54) has local maxima at the minima of $U(x)$, i.e., particles accumulate around the stable states of $U(x)$. Such conclusions can be reached from the zero-flux stationary solution of the corresponding Fokker-Planck-Kramers equation associated to Eq. (53).

Even though there is no general solution to (53) for arbitrary $U(x)$—due to the non-linear character of the equation—there are solutions for the case of the harmonic potential $\frac{1}{2}m\omega^2 x^2$ which makes Eq. (53) linear in $v(t)$ and $x(t)$. Due to this simplification, this case has been treated almost exhaustively.

In the following section we revisit the procedure to obtain the equilibrium distribution of a Brownian particle trapped in arbitrary external potential $U(x)$ subject to Gaussian white noise and under the assumption that the fluctuation-dissipation relation (15) holds.

2.1 Fokker-Planck-Kramers Equation for a Trapped Brownian Particle in an External Potential $U(x)$: The Equilibrium Distribution for Gaussian White Noise

In the case when the fluctuation-dissipation relation (15) holds, and the internal noise corresponds to Gaussian white noise, we have that Eq. (53) reduce to the Markovian Langevin equation

$$m\frac{d}{dt}v(t) = -\gamma v(t) - \frac{\partial}{\partial x}U(x) + \xi(t), \tag{55a}$$

$$\frac{d}{dt}x(t) = v(t). \tag{55b}$$

We are interested in the long-time regime where the system reaches the expected equilibrium regime. Thus one may avoid the currents due to the effects of the initial data. The probability distribution $P(x, v, t)$ of finding a Brownian particle at position x, moving with velocity v at time t satisfies the so-called Fokker-Planck-Kramers equation

$$\begin{aligned}\frac{\partial}{\partial t}P(x, v, t) + \frac{\partial}{\partial x}vP(x, v, t) - \frac{1}{m}\frac{\partial}{\partial v}\left[\frac{\partial}{\partial x}U(x)P(x, v, t)\right]\\ = \frac{\gamma}{m}\frac{\partial}{\partial v}\left[v + \frac{k_BT}{m}\frac{\partial}{\partial v}\right]P(x, v, t).\end{aligned} \tag{56}$$

Under the assumption that $P(x, v, t)$ is normalizable, then Eq. (56) can be written as a continuity equation, namely

$$\frac{\partial}{\partial t}P(x, v, t) + \nabla_{x,v} \cdot \boldsymbol{J}(x, v, t) = 0, \tag{57}$$

where

$$\nabla_{x,v} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial v}\right) \tag{58}$$

and $\boldsymbol{J}(x, v, t)$ is a two-dimensional vector that denotes the total probability current which can be decomposed as the sum of two contributions [53]: the deterministic one

$$\boldsymbol{J}_{\text{det}}(x, v, t) = \left[vP(x, v, t), -\frac{1}{m}\frac{\partial}{\partial x}U(x)\, P(x, v, t)\right]; \tag{59}$$

and the irreversible one

$$\boldsymbol{J}_{\text{irr}}(x, v, t) = \left[0, -\frac{\gamma}{m}\left(v + \frac{k_BT}{m}\frac{\partial}{\partial v}\right)P(x, v, t)\right]. \tag{60}$$

The stationary solution is obtained as a consequence of imposing the so-called *detailed balance conditions*, namely,

$$\boldsymbol{J}_{\rm irr} = (0, 0), \tag{61a}$$

$$\nabla_{x,v} \cdot \boldsymbol{J}_{\rm det} = 0, \tag{61b}$$

which lead to the equilibrium solutions $P_{\rm eq}(x, v)$. The first condition, which establishes the vanishing of the current due to balance of dissipation and thermal fluctuations, implies that

$$\left[v + \frac{k_B T}{m}\frac{\partial}{\partial v}\right] P_{\rm eq}(x, v) = 0, \tag{62}$$

whose solution can be obtained straightforwardly as

$$P_{\rm eq}(x, v) = P_{\rm eq}(x) e^{-mv^2/2k_B T}, \tag{63}$$

where $P_{\rm eq}(x)$ is a function that depends only on x and is determined by the other detailed balance condition (61b). After substitution of the last expression into (61b) we have

$$\frac{\partial}{\partial x} P_{\rm eq}(x) + \frac{1}{k_B T}\left[\frac{\partial}{\partial x} U(x)\right] P_{\rm eq}(x) = 0, \tag{64}$$

whose solution corresponds exactly to the Boltzmann-Gibbs weight

$$P_{\rm eq}(x) = \mathcal{Z}^{-1} e^{-U(x)/k_B T}, \tag{65}$$

where the constant $\mathcal{Z}$ is found from the normalization condition for $P_{\rm eq}(x, v)$, i.e.

$$\mathcal{Z} = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dx \, P_{\rm eq}(x, v). \tag{66}$$

Thus, the equilibrium distribution is given by the equilibrium distribution of Maxwell and Boltzmann-Gibbs

$$P_{\rm eq}(v, x) = \mathcal{Z}^{-1} \, e^{-U(x)/k_B T} e^{-mv^2/2k_B T}. \tag{67}$$

3 Non-Markovian Trapped Brownian Motion in an External Potential: The Harmonic Oscillator

We consider the generalized Langevin equations (53) for a Brownian particle diffusing under the influence of the harmonic potential (non-Markovian Brownian harmonic oscillator)

$$U(x) = \frac{1}{2} m\omega^2 x^2. \tag{68}$$

It has been argued that the stationary distribution that corresponds to equilibrium, which in the particular case considered in this section is given by

$$P_{\rm eq}(v,x) = \mathcal{Z}^{-1}\, e^{-m\omega^2x^2/2k_BT} e^{-mv^2/2k_BT}, \tag{69}$$

is a consequence of the fluctuation-dissipation relation, independently of the specific time dependence of the noise autocorrelation function. This can be proved explicitly for this particular case as follows in the next section.

Due to its linear character, this linear process has been studied comprehensively in the case of internal and external noise as well [46, 47, 57–60], where attention has been paid mainly to the mean values, variances, and velocity autocorrelation function.

3.1 Fluctuation-Dissipation Relation and the Equilibrium Distribution

Consider the non-Markovian Brownian harmonic oscillator

$$m\frac{d}{dt}v(t) = -\gamma \int_{t_0}^{t} ds\, \phi(t-s)v(s) - m\omega^2 x(t) + \xi(t), \tag{70a}$$

$$\frac{d}{dt}x(t) = v(t), \tag{70b}$$

where we have kept explicitly the characteristic time t_0.

As before, our purpose is to compute the quantity $\langle \xi(t_0+t)\xi(t_0)\rangle$. From (70) we have that $\xi(t_0+t)$, $\xi(t_0)$ are given, respectively, by

$$\begin{aligned}\xi(t_0+t) &= m\dot{v}(t_0+t) + \gamma \int_{t_0}^{t_0+t} ds\phi(t_0+t-s)v(s) + m\omega^2 x(t_0+t),\\ \xi(t_0) &= m\dot{v}(t_0) + m\omega^2 x(t_0),\end{aligned} \tag{71}$$

where the dotted symbols denote the change in time of the corresponding symbols without dot.

Besides the assumptions made in the last case we will also assume that $\langle \xi(t)x(t_0)\rangle = 0$ for $t \geq 0$, and that $x(t)$ is also stationary. Since the Laplace transforms of $x(t)$ and $v(t)$ are required to carry out the calculation, we simply give the corresponding expressions, explicitly

$$\widetilde{x}(\epsilon) = \frac{\widetilde{\xi}(\epsilon) + mv(t_0) + x(t_0)[m\epsilon + \gamma\widetilde{\phi}(\epsilon)]}{\epsilon[m\epsilon + \gamma\widetilde{\phi}(\epsilon)] + m\omega^2}, \tag{72}$$

$$\widetilde{v}(\epsilon) = \frac{\epsilon[\widetilde{\xi}(\epsilon) + mv(t_0)] - x(t_0)m\omega^2}{\epsilon[m\epsilon + \gamma\widetilde{\phi}(\epsilon)] + m\omega^2}. \tag{73}$$

Thus,

$$\int_0^\infty dt\, e^{-\epsilon t}\langle \xi(t_0+t)\xi(t_0)\rangle = \int_0^\infty dt\, e^{-\epsilon t}\Bigg[m^2\langle \dot{v}(t_0+t)\dot{v}(t_0)\rangle + m^2\omega^4\langle x(t_0+t)x(t_0)\rangle$$
$$+m^2\omega^2[\langle \dot{v}(t_0+t)x(t_0)\rangle + \langle x(t_0+t)\dot{v}(t_0)\rangle]$$
$$+m\omega^2\gamma\int_{t_0}^{t_0+t} ds\phi(t_0+t_s)\langle v(s)x(t_0)\rangle$$
$$+m\gamma\int_{t_0}^{t_0+t} ds\phi(t_0+t_s)\langle v(s)\dot{v}(t_0)\rangle\Bigg]$$

We compute each term by following the same procedure as before. For the first term we have

$$\int_0^\infty dt\, e^{-\epsilon t} m^2\langle \dot{v}(t_0+t)\dot{v}(t_0)\rangle = m^2\epsilon\int_0^\infty dt\, e^{-\epsilon t}\langle v(t_0+t)\dot{v}(t_0)\rangle - m^2\langle v(t_0)\dot{v}(t_0)\rangle$$
$$= -m^2\epsilon\int_0^\infty dt\, e^{-\epsilon t}\langle \dot{v}(t_0+t)v(t_0)\rangle + m^2\omega^2\langle v(t_0)x(t_0)\rangle,$$

where we have used that $v(t)$ is stationary, therefore $\langle v(t_0+t)\dot{v}(t_0)\rangle = -\langle \dot{v}(t_0+t)v(t_0)\rangle$, and since $\langle v(t_0)\xi(t)\rangle = 0$ that $\langle v(t_0)\dot{v}(t_0)\rangle = -\omega^2\langle v(t_0)x(t_0)\rangle$. To simplify the analysis consider that $\langle v(t_0)x(t_0)\rangle = 0$. With this we get

$$\int_0^\infty dt\, e^{-\epsilon t} m^2\langle \dot{v}(t_0+t)\dot{v}(t_0)\rangle = -m^2\epsilon^2\langle \widetilde{v}(\epsilon)v(t_0)\rangle + m^2\epsilon\langle v^2(t_0)\rangle.$$

The quantity $\langle \widetilde{v}(\epsilon)v(t_0)\rangle$ is computed by using (73), we get

$$\langle \widetilde{v}(\epsilon)v(t_0)\rangle = \frac{\epsilon\langle \widetilde{\xi}(\epsilon)v(t_0)\rangle + m\epsilon\langle v^2(t_0)\rangle - m\omega^2\langle x(t_0)v(t_0)\rangle}{\epsilon[m\epsilon + \gamma\widetilde{\phi}(\epsilon)] + m\omega^2},$$

but since $\langle \xi(t)v(t_0)\rangle = 0$, and the assumption $\langle v(t_0)x(t_0)\rangle = 0$, we simply get that

$$\int_0^\infty dt\, e^{-\epsilon t} m^2\langle \dot{v}(t_0+t)\dot{v}(t_0)\rangle = m^2\epsilon\frac{\epsilon\gamma\widetilde{\phi}(\epsilon) + m\omega^2}{\epsilon[m\epsilon + \gamma\widetilde{\phi}(\epsilon)] + m\omega^2}\langle v^2(t_0)\rangle. \tag{74}$$

Similarly,

$$m^2\omega^2\int_0^\infty dt\, e^{-\epsilon t}\langle \dot{v}(t_0+t)x(t_0)\rangle = m^2\omega^2\epsilon\langle \widetilde{v}(\epsilon)x(t_0)\rangle$$
$$= -\frac{m^3\omega^4\epsilon}{\epsilon[m\epsilon + \gamma\widetilde{\phi}(\epsilon)] + m\omega^2}\langle x^2(t_0)\rangle; \tag{75}$$

$$
\begin{aligned}
m\gamma\widetilde{\phi}(\epsilon)\int_0^\infty dt\, e^{-\epsilon t}\langle v(t_0+t)\dot{v}(t_0)\rangle &= -m\gamma\widetilde{\phi}(\epsilon)\int_0^\infty dt\, e^{-\epsilon t}\langle \dot{v}(t_0+t)v(t_0)\rangle \\
&= -m\gamma\widetilde{\phi}(\epsilon)\epsilon\langle\widetilde{v}(\epsilon)v(t_0)\rangle + m\gamma\widetilde{\phi}(\epsilon)\langle v^2(t_0)\rangle \\
&= m\gamma\widetilde{\phi}(\epsilon)\frac{\epsilon\gamma\widetilde{\phi}(\epsilon)+m\omega^2}{\epsilon[m\epsilon+\gamma\widetilde{\phi}(\epsilon)]+m\omega^2}\langle v^2(t_0)\rangle;
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
m\omega^2\gamma\widetilde{\phi}(\epsilon)\int_0^\infty dt\, e^{-\epsilon t}\langle v(t_0+t)x(t_0)\rangle &= m\omega^2\gamma\widetilde{\phi}(\epsilon)\langle\widetilde{v}(\epsilon)x(t_0)\rangle \\
&= -\frac{m^2\omega^4\gamma\widetilde{\phi}(\epsilon)}{\epsilon[m\epsilon+\gamma\widetilde{\phi}(\epsilon)]+m\omega^2}\langle x^2(t_0)\rangle.
\end{aligned}
\tag{77}
$$

To compute $m^2\omega^2\int_0^\infty dt\ \ e^{-\epsilon t}\langle x(t_0+t)\dot{v}(t_0)\rangle$ we use that $x(t_0+t)=x(t_0)+\int_{t_0}^{t_0+t}ds\ v(s)$ and since $\langle x(t_0)\xi(t)\rangle=0$, that $\langle x(t_0)\dot{v}(t_0)\rangle=-\omega^2\langle x^2(t_0)\rangle$, thus

$$
\begin{aligned}
&m^2\omega^2\int_0^\infty dt\, e^{-\epsilon t}\langle x(t_0+t)\dot{v}(t_0)\rangle \\
&= -\frac{m^2\omega^4}{\epsilon}\langle x^2(t_0)\rangle + m^2\omega^2\int_0^\infty dt\, e^{-\epsilon t}\int_0^t ds\langle v(t_0+s)\dot{v}(t_0)\rangle \\
&= -\frac{m^2\omega^4}{\epsilon}\langle x^2(t_0)\rangle + \frac{m^2\omega^2}{\epsilon}\int_0^\infty dt\, e^{-\epsilon t}\langle v(t_0+t)\dot{v}(t_0)\rangle \\
&= -\frac{m^2\omega^4}{\epsilon}\langle x^2(t_0)\rangle - \frac{m^2\omega^2}{\epsilon}\int_0^\infty dt\, e^{-\epsilon t}\langle \dot{v}(t_0+t)v(t_0)\rangle \\
&= -\frac{m^2\omega^4}{\epsilon}\langle x^2(t_0)\rangle - \frac{m^2\omega^2}{\epsilon}\left[\epsilon\langle\widetilde{v}(\epsilon)v(t_0)\rangle-\langle v^2(t_0)\rangle\right] \\
&= -\frac{m^2\omega^4}{\epsilon}\langle x^2(t_0)\rangle + \frac{m^2\omega^2}{\epsilon[m\epsilon+\gamma\widetilde{\phi}(\epsilon)]+m\omega^2}\left[\gamma\widetilde{\phi}(\epsilon)+\frac{m\omega^2}{\epsilon}\right]\langle v^2(t_0)\rangle;
\end{aligned}
\tag{78}
$$

$$
m^2\omega^4\int_0^\infty dt\, e^{-\epsilon t}\langle x(t_0+t)x(t_0)\rangle = \frac{m^2\omega^4[m\epsilon+\gamma\phi(\epsilon)]}{\epsilon[m\epsilon+\gamma\widetilde{\phi}(\epsilon)]+m\omega^2}\langle x^2(t_0)\rangle. \tag{79}
$$

By adding (74) to (79) we get

$$
\int_0^\infty dt\, e^{-\epsilon t}\langle\xi(t_0+t)\xi(t_0)\rangle = m\gamma\widetilde{\phi}(\epsilon)\langle v^2(t_0)\rangle + \frac{m^2\omega^2}{\epsilon}\left[\langle v^2(t_0)\rangle-\omega^2\langle x^2(t_0)\rangle\right]. \tag{80}
$$

Finally by inverting the expression (80) we get

$$
\langle\xi(t_0+t)\xi(t_0)\rangle = m\gamma\phi(t)\langle v^2(t_0)\rangle + m^2\omega^2\left[\langle v^2(t_0)\rangle-\omega^2\langle x^2(t_0)\rangle\right]. \tag{81}
$$

This is the *fluctuation-dissipation* theorem for the non-Markovian Brownian harmonic oscillator.

If the initial conditions correspond to those in thermodynamic equilibrium, equipartition theorem assures that $m\langle v^2(t_0)\rangle = k_B T$ and $m\omega^2\langle x^2(t_0)\rangle = k_B T$, therefore (81) reduces to the previous one $\langle \xi(t_0+t)\xi(t_0)\rangle = m\gamma\phi(t)\langle v^2(t_0)\rangle$.

The Mean Squared Displacement for Internal Noise

The formal solution to the generalized Langevin equations (70) can be computed straightforwardly by use of the Laplace transform, and is explicitly given by

$$\widetilde{x}(\epsilon) = x(0)\widetilde{\Phi}_{II}(\epsilon) + v(0)\widetilde{\Phi}_{\mathrm{p}}(\epsilon) + \frac{1}{m}\widetilde{\xi}(\epsilon)\widetilde{\Phi}_{\mathrm{p}}(\epsilon), \tag{82a}$$

$$\widetilde{v}(\epsilon) = v(0)\widetilde{\Phi}(\epsilon)\left[1-\omega^2\widetilde{\Phi}_{\mathrm{p}}(\epsilon)\right] - x(0)\omega^2\widetilde{\Phi}_{\mathrm{p}}(\epsilon) + \frac{1}{m}\widetilde{\xi}(\epsilon)\widetilde{\Phi}(\epsilon)\left[1-\omega^2\widetilde{\Phi}_{\mathrm{p}}(\epsilon)\right], \tag{82b}$$

where we have assumed without loss of generality that $t_0 = 0$ in Eq. (70). The function $\widetilde{\Phi}(\epsilon)$ is given by expression (5), $\widetilde{\Phi}_{II}(\epsilon)$ explicitly by

$$\widetilde{\Phi}_{II}(\epsilon) = \left[\epsilon + \omega^2\widetilde{\Phi}(\epsilon)\right]^{-1}, \tag{83}$$

and

$$\begin{aligned}\widetilde{\Phi}_{\mathrm{p}}(\epsilon) &= \widetilde{\Phi}(\epsilon)\widetilde{\Phi}_{II}(\epsilon)\\ &= \left\{\epsilon\left[\epsilon + \frac{\gamma}{m}\widetilde{\phi}(\epsilon)\right] + \omega^2\right\}^{-1}.\end{aligned} \tag{84}$$

From expressions (82), the mean values $\langle x(t)\rangle$ and $\langle v(t)\rangle$ are

$$\langle x(t)\rangle = x(0)\Phi_{II}(t) + v(0)\Phi_{\mathrm{p}}(t), \tag{85a}$$

$$\langle v(t)\rangle = v(0)\left[\Phi(t) - \omega^2\int_0^t ds\,\Phi(t)\Phi_{\mathrm{p}}(t-s)\right] - x(0)\omega^2\Phi_{\mathrm{p}}(t), \tag{85b}$$

respectively, and the respective mean squared displacement, with respect to the average, is given by

$$\begin{aligned}\langle[x(t)-\langle x(t)\rangle]^2\rangle &= \frac{1}{m^2}\int_0^t ds_1\,\Phi_{\mathrm{p}}(t-s_1)\int_0^t ds_2\,\Phi_{\mathrm{p}}(t-s_2)\langle\xi(s_1)\xi(s_2)\rangle\\ &= 2\frac{\langle\xi^2\rangle}{m^2}\int_0^t ds_1\,\Phi_{\mathrm{p}}(s_1)\int_0^{s_1} ds_2\,\Phi_{\mathrm{p}}(s_2)\Gamma(s_1-s_2).\end{aligned} \tag{86}$$

Define $\sigma_{xx}^2(t) \equiv \langle [x(t) - \langle x(t)\rangle]^2\rangle$, then by differentiating expression (86) with respect to time we get

$$\frac{m^2}{2\langle \xi^2\rangle}\dot{\sigma}_{xx}^2 = \Phi_{\mathrm{p}}(t)\int_0^t ds\, \Phi_{\mathrm{p}}(s)\Gamma(t-s), \tag{87}$$

and therefore that

$$\mathcal{L}\left\{\frac{m^2\dot{\sigma}_{xx}^2}{2\langle \xi^2\rangle \Phi_{\mathrm{p}}(t)}\right\} = \widetilde{\Phi}_{\mathrm{p}}(\epsilon)\widetilde{\Gamma}(\epsilon). \tag{88}$$

If the fluctuation-dissipation relation (15) holds, we get

$$\mathcal{L}\left\{\frac{m^2\dot{\sigma}_{xx}^2}{2\Phi_{\mathrm{p}}(t)}\right\} = 2k_BT\gamma\widetilde{\Phi}_{\mathrm{p}}(\epsilon)\widetilde{\phi}(\epsilon). \tag{89}$$

The product $\widetilde{\Phi}_{\mathrm{p}}(\epsilon)\widetilde{\phi}(\epsilon)$ in the right-hand side of the last expression can be computed straightforwardly from (84) in terms of $\widetilde{\Phi}_{\mathrm{p}}(\epsilon)$ alone as

$$\widetilde{\Phi}_{\mathrm{p}}(\epsilon)\widetilde{\phi}(\epsilon) = \frac{m}{\gamma}\left[\frac{1}{\epsilon} - \frac{\widetilde{\Phi}_{\mathrm{p}}(\epsilon)}{\epsilon}\left(\epsilon^2 + \omega^2\right)\right], \tag{90}$$

then after inversion of the Laplace transform in (87) and rearranging the resulting terms we get

$$\dot{\sigma}_{xx}^2 = \frac{4k_BT}{m}\left[\Phi_{\mathrm{p}}(t) - \Phi_{\mathrm{p}}(t)\frac{d}{dt}\Phi_{\mathrm{p}}(t) - \omega^2\Phi_{\mathrm{p}}(t)\int_0^t ds\, \Phi_{\mathrm{p}}(s)\right], \tag{91}$$

where we have used that $\Phi(0) = 0$ as this can be proved by taking the limit $\epsilon\widetilde{\Phi}(\epsilon)$ as ϵ goes to ∞. Therefore after integration of last expression we finally have

$$\langle [x(t) - \langle x(t)\rangle]^2\rangle = \frac{4k_BT}{m}\left[\int_0^t ds\, \Phi_{\mathrm{p}}(s) - \frac{1}{2}\Phi_{\mathrm{p}}^2(t) - \frac{\omega^2}{2}\left(\int_0^t ds\, \Phi(s)\right)^2\right]. \tag{92}$$

Let us discuss the case of algebraically correlated noise, specifically the *continuous time fractional Gaussian noise* (ctfGn) $\langle \xi(t)\xi(s)\rangle = \langle v_0^2\rangle_{\mathrm{eq}}\frac{\gamma^{2H}}{m^{2H-2}}2H(2H-1)|t-s|^{2H-2}$. The long-time limit may be found with the help of the Tauberian theorem. In this particular case we have

$$\Phi(t) = \frac{1}{m}\mathcal{L}^{-1}\left\{\frac{1}{\epsilon^2 + \epsilon^{2-2H}\gamma_{2H} + \omega^2}\right\}, \tag{93}$$

where we have defined $\gamma_{2H} \equiv \left(\frac{\gamma}{m}\right)^{2H} \Gamma(2H+1)$. Therefore

$$\begin{aligned}
\langle [x(t) - \langle x(t)\rangle]^2\rangle \xrightarrow[t\to\infty]{} & \, 2\langle V_0^2\rangle_{\mathrm{eq}} m \lim_{\epsilon\to 0} \epsilon \left[\frac{1}{\epsilon m[\epsilon^2 + \epsilon^{2-2H}\gamma_{2H} + \omega^2]}\right] \\
& -2\langle V_0^2\rangle_{\mathrm{eq}} m \left[\frac{m}{2}\left(\lim_{t\to\infty}\Phi(t)\right)^2 + \frac{m\omega^2}{2}\left[\lim_{t\to\infty}\int_0^t ds\,\Phi(s)\right]^2\right] \\
= & \, 2\frac{\langle V_0^2\rangle_{\mathrm{eq}}}{\omega^2} - 2\langle V_0^2\rangle_{\mathrm{eq}} m \left[\frac{m}{2}\left(\lim_{\epsilon\to 0}\frac{\epsilon}{m[\epsilon^2 + \epsilon^{2-2H}\gamma_{2H} + \omega^2]}\right)^2\right. \\
& \left. + \frac{m\omega^2}{2}\left[\lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon m[\epsilon^2 + \epsilon^{2-2H}\gamma_{2H} + \omega^2]}\right]^2\right] \\
= & \, 2\frac{\langle V_0^2\rangle_{\mathrm{eq}}}{\omega^2} - 2\langle V_0^2\rangle_{\mathrm{eq}} m \left[\frac{m\omega^2}{2}\left[\frac{1}{m\omega^2}\right]^2\right] \\
= & \, 2\frac{\langle V_0^2\rangle_{\mathrm{eq}}}{\omega^2} - \frac{\langle V_0^2\rangle_{\mathrm{eq}}}{\omega^2} \\
= & \, \frac{\langle V_0^2\rangle_{\mathrm{eq}}}{\omega^2} \\
= & \, \frac{k_B T}{m\omega^2}.
\end{aligned} \tag{94}$$

We used the fact the $1 < 2H < 2$ and that the equilibrium distribution is given by the Maxwell distribution.

3.2 The Brownian Harmonic Oscillator: The Out-of-Equilibrium Case (The Effective Temperature)

We discussed in the previous section that the Boltzmann-Gibbs factor,

$$e^{-U(x)/k_B T}, \tag{95}$$

($e^{-m\omega^2 x^2/2k_B T}$ for the Brownian harmonic oscillator) is an explicit consequence of the balance of the effects of the retarded dragging force and the correlated fluctuating noise, which makes the irreversible component of the probability current to vanish, at least for the case of Gaussian white noise.

Motivated by the observation that the persistent motion is the cause for which the distribution of positions of confined active particles does not correspond to the one of Boltzmann-Gibbs, we explore the consequences of relaxing the fluctuation dissipation relation in Eq. (70), on the stationary distribution of positions $P_{\mathrm{st}}(x)$.

That is to say, we ask the question: What are the corresponding effects on $P_{\mathrm{st}}(x)$ when the dissipative effects are decoupled from the fluctuating ones?

The method followed in Sect. 2.1 does not apply directly since the Fokker-Planck-Kramers equation corresponding to Eqs. (70) involves time dependent transport coefficients as the ones obtained for Fokker-Planck-Kramers equation that corresponds to the generalized Langevin equation of the free Brownian particle [see Eq. (24)].

Our starting point is, instead, the conditional probability density,

$$P(x,t|x(0),v(0)) = \langle \delta[x-x(t)]\rangle, \tag{96}$$

of finding a particle located at position x at time t, given that it was located at $x(0)$ moving with velocity $v(0)$ at time $t=0$, where the explicit dependence of $x(t)$ on the external noise $\xi(t)$ is given by the Laplace inversion of (82b), namely

$$x(t) = x(0)\Phi_{II}(t) + v(0)\Phi_{\mathrm{p}}(t) + \frac{1}{m}\int_0^t ds\, \Phi_{\mathrm{p}}(t-s)\xi(s). \tag{97}$$

The linear nature of Eqs. (70) ensures that $x(t)$ is a Gaussian process. We have then that

$$P(x,t|x(0),v(0)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik[x-x(0)\Phi_{II}(t)-v(0)\Phi_{\mathrm{p}}(t)]}\left\langle e^{i\frac{k}{m}\int_0^t ds\, \Phi_{\mathrm{p}}(t-s)\xi(s)}\right\rangle, \tag{98}$$

where

$$\left\langle e^{i\frac{k}{m}\int_0^t ds\, \Phi_{\mathrm{p}}(t-s)\xi(s)}\right\rangle \tag{99}$$

is the characteristic functional of the stochastic process $\xi(t)$, which now has the explicit expression [48]

$$\exp\left\{-\frac{k^2\langle\xi^2\rangle}{m^2}\int_0^t ds_1 \int_0^{s_1} ds_2\, \Phi_{\mathrm{p}}(s_1)\Phi_{\mathrm{p}}(s_2)\Gamma(s_1-s_2)\right\}, \tag{100}$$

since the process $\xi(t)$ has been assumed Gaussian. We have that

$$P(x,t|x(0),v(0)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik[x-x(0)\Phi_{II}(t)-v(0)\Phi_{\mathrm{p}}(t)]} \\ \times e^{-k^2\frac{\langle\xi^2\rangle}{m^2}\int_0^t ds_1\int_0^{s_1} ds_2\, \Phi_{\mathrm{p}}(s_1)\Phi_{\mathrm{p}}(s_2)\Gamma(s_1-s_2)} \tag{101}$$

from which, we obtain

$$P(x,t|x(0),v(0)) = \frac{\exp\left\{-\dfrac{\left[x-x(0)\Phi_{II}(t)-v(0)\Phi_{\mathrm{p}}(t)\right]^2}{2\langle[x-x(0)\Phi_{II}(t)-v(0)\Phi_{\mathrm{p}}(t)]^2\rangle}\right\}}{\sqrt{2\pi\langle[x-x(0)\Phi_{II}(t)-v(0)\Phi_{\mathrm{p}}(t)]^2\rangle}}. \tag{102}$$

The quantity $\left\langle \left[x - x(0)\Phi_{II}(t) - v(0)\Phi_{\rm p}(t)\right]^2 \right\rangle$ is given explicitly in (86). In this case, the probability density of positions attains a stationary form, $P_{\rm st}(x)$ if and only if, in the asymptotic regime

$$\bar{\kappa} \equiv \lim_{t\to\infty} \frac{\langle \xi^2 \rangle}{m^2} \int_0^t ds_1\, \Phi_{\rm p}(s_1) \int_0^{s_1} ds_2\, \Phi_{\rm p}(s_2)\Gamma(s_1 - s_2) \tag{103}$$

is finite.

For the exponentially correlated external noise (49), and the exponentially decaying memory function (50), we have that

$$P_{\rm st}(x) = \mathcal{Z}^{-1} \exp\left\{ -\frac{m\omega^2 x^2}{2\langle \xi^2 \rangle (\tau_{\rm cor}/m\omega)} \right\}, \tag{104}$$

from which the effective temperature can be recognized to be

$$T_{\rm eff} = \frac{\langle \xi^2 \rangle\, \tau_{\rm cor}}{k_B\, m\omega}. \tag{105}$$

4 Conclusions and Final Remarks

We have investigated the effects of persistent motion on the stationary distribution of positions of trapped active particles. The persistence of motion has been taken into account within the theoretical framework of the generalized Langevin equation, more precisely, we have assumed that the time dependence of the memory function, that appears in the non-Markovian dissipative force, describes persistent motion. We have shown that the intrinsic non-equilibrium aspects of active motion can be incorporated into such description, when unbinding the dissipative dynamics from the fluctuating one, both binded in equilibrium by the fluctuation-dissipation relation. Thus, no connection between the autocorrelation of noise and the memory function is assumed. We found that the probability density of the particle positions is akin to the Boltzmann distribution, but with an effective temperature, as occurs in some other models of active motion, particularly in the model of Szamel of Ornstein-Uhlenbeck active particles [29].

To the author's knowledge, this is the first time the intrinsic non-equilibrium aspects of active motion have been considered within the framework of the generalized Langevin equation, and certainly, there are general aspects, and particular ones as well, still to be investigated. For instance, a derivation of a Fokker-Planck-Kramers equation for arbitrary trapping external potential is still missing in the literature and the generalization of the present analysis (70) to the case of non-Gaussian noise is worthy to be pursued.

Acknowledgements The author kindly acknowledges the support from grant UNAM-DGAPA-PAPIIT-IN114717.

References

1. Reimann P (2002) Phys Rep 361:57. ISSN 0370-1573. http://www.sciencedirect.com/science/article/pii/S0370157301000813
2. Taktikos J, Stark H, Zaburdaev V (2014) PLoS One 8:1. http://dx.doi.org/10.1371%2Fjournal.pone.0081936
3. Taute KM, Gude S, Tans SJ, Shimizu TS (2015) Nat Commun 6:8776. https://www.nature.com/articles/ncomms9776#supplementary-information
4. Schnitzer MJ (1993) Phys Rev E 48:2553. http://link.aps.org/doi/10.1103/PhysRevE.48.2553
5. Bechinger C, Di Leonardo R, Löwen H, Reichhardt C, Volpe G, Volpe G (2016) Rev Mod Phys 88:045006. https://link.aps.org/doi/10.1103/RevModPhys.88.045006
6. Gao W, Dong R, Thamphiwatana S, Li J, Gao W, Zhang L, Wang J (2015) ACS Nano 9:117. pMID: 25549040, http://dx.doi.org/10.1021/nn507097k
7. Ramaswamy S (2010) Annu Rev Condens Matter Phys 1:323. https://doi.org/10.1146/annurev-conmatphys-070909-104101
8. Marchetti MC, Joanny JF, Ramaswamy S, Liverpool TB, Prost J, Rao M, Simha RA (2013) Rev Mod Phys 85:1143. https://link.aps.org/doi/10.1103/RevModPhys.85.1143
9. Howse JR, Jones RAL, Ryan AJ, Gough T, Vafabakhsh R, Golestanian R (2007) Phys Rev Lett 99:048102
10. Buttinoni I, Volpe G, Kümmel F, Volpe G, Bechinger C (2012) J Phys Condens Matter 24:284129. http://stacks.iop.org/0953-8984/24/i=28/a=284129
11. Takatori SC, Yan W, Brady JF (2014) Phys Rev Lett 113:028103. http://link.aps.org/doi/10.1103/PhysRevLett.113.028103
12. Ginot F, Theurkauff I, Levis D, Ybert C, Bocquet L, Berthier L, Cottin-Bizonne C (2015) Phys Rev X 5:011004. https://link.aps.org/doi/10.1103/PhysRevX.5.011004
13. Takatori SC, Brady JF (2015) Phys Rev E 91:032117. http://link.aps.org/doi/10.1103/PhysRevE.91.032117
14. Solon AP, Chaté H, Tailleur J (2015) Phys Rev Lett 114:068101. http://link.aps.org/doi/10.1103/PhysRevLett.114.068101
15. Vicsek T, Czirók A, Ben-Jacob E, Cohen I, Shochet O (1995) Phys Rev Lett 75:1226
16. Speck T, Bialké J, Menzel AM, Löwen H (2014) Phys Rev Lett 112:218304. https://link.aps.org/doi/10.1103/PhysRevLett.112.218304
17. Solon AP, Stenhammar J, Wittkowski R, Kardar M, Kafri Y, Cates ME, Tailleur J (2015) Phys Rev Lett 114:198301. https://link.aps.org/doi/10.1103/PhysRevLett.114.198301
18. Cates ME Tailleur J (2015) Annu Rev Condens Matter Phys 6:219. https://doi.org/10.1146/annurev-conmatphys-031214-014710
19. Fodor E, Nardini C, Cates ME, Tailleur J, Visco P, van Wijland F (2016) Phys Rev Lett 117:038103. https://link.aps.org/doi/10.1103/PhysRevLett.117.038103
20. Oukris H, Israeloff NE (2010) Nat Phys 6:135. ISSN 1745-2473. https://doi.org/10.1038/nphys1482
21. Colombani J, Petit L, Ybert C, Barentin C (2011) Phys Rev Lett 107:130601. http://link.aps.org/doi/10.1103/PhysRevLett.107.130601
22. Dieterich E, Camunas-Soler J, Ribezzi-Crivellari M, Seifert U, Ritort F (2015) Nat Phys 11:971
23. Loi D, Mossa S, Cugliandolo LF (2008) Phys Rev E 77:051111. http://link.aps.org/doi/10.1103/PhysRevE.77.051111
24. Tailleur J Cates ME (2009) EPL (Europhys Lett) 86:60002. http://stacks.iop.org/0295-5075/86/i=6/a=60002
25. Palacci J, Cottin-Bizonne C, Ybert C, Bocquet L (2010) Phys Rev Lett 105:088304
26. Enculescu M Stark H (2011) Phys Rev Lett 107:058301. http://link.aps.org/doi/10.1103/PhysRevLett.107.058301
27. Ben-Isaac E, Park Y, Popescu G, Brown FLH, Gov NS, Shokef Y (2011) Phys Rev Lett 106:238103. http://link.aps.org/doi/10.1103/PhysRevLett.106.238103

28. Loi D, Mossa S, Cugliandolo LF (2011) Soft Matter 7:3726. http://dx.doi.org/10.1039/C0SM01484B
29. Szamel G (2014) Phys Rev E 90:012111. http://link.aps.org/doi/10.1103/PhysRevE.90.012111.
30. Levis D Berthier L (2015) EPL (Europhys Lett) 111:60006. http://stacks.iop.org/0295-5075/111/i=6/a=60006
31. Pototsky A, Stark H (2012) EPL 98:50004
32. Solon AP, Cates ME, Tailleur J (2015) Eur Phys J Spec Top 224:1231. ISSN 1951-6401. http://dx.doi.org/10.1140/epjst/e2015-02457-0
33. Takatori SC, De Dier R, Vermant J, Brady JF (2016) Nat Commun 7:10694. http://www.nature.com/articles/ncomms10694#supplementary-information
34. Miramontes O, DeSouza O, Paiva LR, Marins A, Orozco S (2014) PLoS One 9:e111183
35. Argun A, Moradi A-R, Pinçe E, Bagci GB, Imparato A, Volpe G (2016) Phys Rev E 94:062150. https://link.aps.org/doi/10.1103/PhysRevE.94.062150
36. Codling EA, Plank MJ, Benhamou S (2008) J R Soc Interface 5:813
37. Masoliver J, Lindenberg K, Weiss GH (1989) Physica A 157:891
38. Weiss GH, Rubin RJ (1983) Adv Chem Phys 52:363
39. Masoliver J, Lindenberg K (2017) Eur Phys J B 90:107. ISSN 1434-6036. https://doi.org/10.1140/epjb/e2017-80123-7
40. Kubo R (1966) Rep Prog Phys 29:255. http://stacks.iop.org/0034-4885/29/i=1/a=306
41. Mikhailov AS, Zanette DH (1999) Phys Rev E 60:4571. https://link.aps.org/doi/10.1103/PhysRevE.60.4571
42. Erdmann U, Ebeling W, Schimansky-Geier L, Schweitzer F (2000) Eur Phys J B 15:105. ISSN 1434-6036. http://dx.doi.org/10.1007/s100510051104
43. Chaudhuri D (2014) Phys Rev E 90:022131. https://link.aps.org/doi/10.1103/PhysRevE.90.022131
44. Marconi UMB, Puglisi A, Rondoni L, Vulpiani A (2008) Phys Rep 461:111. ISSN 0370-1573. http://www.sciencedirect.com/science/article/pii/S0370157308000768
45. Wang K, Tokuyama M (1999) Physica A 265:341. ISSN 0378-4371. http://www.sciencedirect.com/science/article/pii/S037843719800644X
46. Despósito MA, Viñales AD (2008) Phys Rev E 77:031123. https://link.aps.org/doi/10.1103/PhysRevE.77.031123
47. Sandev T, Metzler R, Tomovski Ž (2014) J Math Phys 55:023301. http://dx.doi.org/10.1063/1.4863478
48. Kubo R, Toda M, Hashitsume N (1991) Statistical physics II: nonequilibrium statistical mechanics. Springer series in solid-state sciences, 2nd edn., vol 31. Springer, Berlin. ISBN 978-3-540-53833-2, 978-3-642-58244-8
49. Adelman SA (1976) J Chem Phys 64:124. http://dx.doi.org/10.1063/1.431961
50. Fox RF (1977) J Math Phys 18:2331. http://dx.doi.org/10.1063/1.523242
51. Fox RF (1978) Phys Rep 48:179. ISSN 0370-1573. http://www.sciencedirect.com/science/article/pii/037015737890145X
52. Miguel MS, Sancho JM (1980) J Stat Phys 22:605. ISSN 1572-9613. https://doi.org/10.1007/BF01011341
53. Hannes Risken TF (1996) The Fokker-Planck equation: methods of solution and applications. Springer series in synergetics, 2nd edn. Springer, Berlin. ISBN 9783540504986, 3540504982, 9780387504988, 0387504982
54. Viñales AD, Despósito MA (2007) Phys Rev E 75:042102. https://link.aps.org/doi/10.1103/PhysRevE.75.042102
55. Metzler R, Klafter J (2000) Phys Rep 339:1. ISSN 0370-1573. http://www.sciencedirect.com/science/article/pii/S0370157300000703
56. Zwanzig R (1973) J Stat Phys 9:215. ISSN 1572-9613. https://doi.org/10.1007/BF01008729
57. Viñales AD, Despósito MA (2006) Phys Rev E 73:016111. https://link.aps.org/doi/10.1103/PhysRevE.73.016111

58. Viñales AD, Wang KG, Despósito MA (2009) Phys Rev E 80:011101. https://link.aps.org/doi/10.1103/PhysRevE.80.011101
59. Camargo RF, de Oliveira EC, Vaz J Jr (2009) J Math Phys 50:123518. https://doi.org/10.1063/1.3269587
60. Sandev T, Tomovski Ž (2010) Phys Scr 82:065001. http://stacks.iop.org/1402-4896/82/i=6/a=065001