Chapter 9
Geometry and Visual Reasoning

Introducing Algebraic Language in the Manner of Liu Hui and al-Khwãrizmî

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Abstract The general aim of this chapter is to identify the potential learning opportunities provided by the introduction of historical geometric diagrams into student tasks. To this end, we examine some problem sets for secondary education students concerning situations to be solved with diagrams in which right triangles or solving second-degree equations are involved. In all cases the objective is that students should transfer linguistically expressed reasoning (second-degree algebraic expressions) to reasoning with visual diagrams (figures with squares and rectangles) that are the geometric interpretation of the second-degree algebraic expressions. The research is therefore focused on students' learning process, and specifically, the results they achieve by the use of these diagrams.

Keywords Teaching and learning of algebra · Geometry-algebra connection Visualization · Historical context · al-Khwãrizmî · Liu Hui

9.1 Geometrical and Historical Diagrams

The purpose of introducing diagrams is to connect symbolic algebraic thought with visual thought regarding geometrical shapes. Historians, educators and many specialists in the teaching of mathematics advocate the connection between seemingly

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different themes as one of the important mathematical processes (Burgués 2008; Burgués and Sarramona 2013; Fauvel and van Maanen 2000; Giaquinto 2007; Jankvist 2009; Katz and Barton 2007; NCTM 2000; Niss 2002; Niss and Højgaard 2011).

Katz and Barton (2007) describe the various stages in the history of the construction of algebra which, according to these authors, lead to implications in teaching and learning. The learning of algebra should begin with the close relation between geometry and problem solving. The introduction of the new mathematical topics should be discussed orally in class with groups of students as a whole. Katz poses the following question: Why not begin algebra by thinking about geometrical figures? The results of this research clearly support the introduction of this approach.

In the history of mathematics, Radford and Puig (2007) associate algebra with geometry in at least two instances in which algebraic reasoning (reasoning about unknowns) is associated with geometric reasoning: some of the problems on cuneiform tablets of Mesopotamian scribes (1900–1600 BCE), and the calculations by al-Khwãrizmî (9th century) in the study and classification of first- and second-degree equations in *Hisāb al-ğabr wa'l-muqābala*. This latter instance provides the source for the second activity set out in this research. Radford and Guérette (2000) and Siu (2000) also endorse visual reasoning and the use of historical texts. In their study, Radford and Guérette address the Babylonian Geometric Method, while Siu, giving *Nine Chapters on Mathematical Procedures* (1st century) as an example, claims that problems in ancient Chinese mathematics also provide evidence in support of proofs[1] through drawings, analogies, generic examples and algorithmic calculations. According to Siu, this can all be of great educational value to complementing and supplementing the teaching of mathematics, with emphasis placed on traditional deductive logical thinking.

In the past, the first author has used historically-based activities in the classroom to provide students with the context in which the mathematics they are studying has been developed, and also to introduce them to alternative ways of thinking about or reasoning in mathematics (Guevara 2009; Guevara et al. 2006), but without ever conducting a systematic collection and analysis of data. The resources used consist of historical diagrams taken from Arabic (9th century) and Chinese (1st century) cultures. It should be mentioned that other authors have also considered such diagrams to be very helpful in the teaching and learning of algebra (Demattè 2010; Puig 2008–11; Siu 2000).

With regard to the use of diagrams, Barwise and Etchemendy (1996) argue that inference and reasoning not only occur in sentences with linguistic expressions, but also with the use of diagrams and charts, and that the use of diagrams is a historical legacy. They relate the use of diagrams with geometry and cite as examples the countless proofs of the Pythagorean Theorem that are found throughout history and in different cultures around the world.

In short, the diagrams used in the research are geometrical and historical; geometrical because they are geometric figures with letters and numbers indicating

[1] In the sense of explanatory notes, which serve to convince and enlighten (Siu 2000, p. 161).

areas or lengths, and historical because they come from the history of mathematics and students use them to solve classical problems usually solved only with algebraic language.

The problems posed to students correspond to Chap. 9 of the *Nine Chapters*; specifically, problems 4 through 12 in the version of Chemla and Guo (2005). What we wish to emphasize is the justification of the calculation procedure in the classical text (1st century) with geometric figures that Liu Hui conducted in the year 263 AD. These figures were described by him but did not appear properly in the text until centuries later (13th century; Fig. 9.1).

The material for the unit of solving quadratic equations has likewise been prepared. In this case, it is based on the justification of a geometric equation with squares and rectangles, in accordance with Rosen's (1831) edition (Fig. 9.2).

From these historical problems we have drawn two different topics that are appropriate for use in the secondary education curriculum: (I) *The resolution of problems of right triangles with diagrams: The Pythagorean Theorem in ancient China*; (II) *Solving equations by completing geometrical squares: Quadratic equations*.

Fig. 9.1 *Nine Chapters* Edit. of Bao Huanzhi (1213) in Chemla and Guo (2005), p. 695

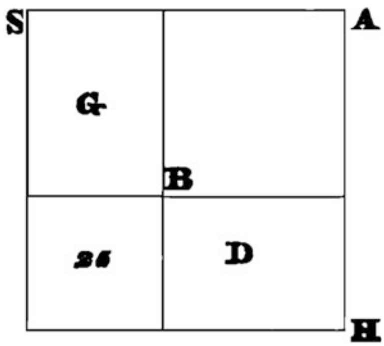

Fig. 9.2 Al-Khwãrizmî (original 813), *The Treatise of Algebra* (al-Khwãrizmî 1831, p. 16)

By means of visual reasoning concerning the areas of squares and rectangles, students solve problems that are usually solved using either the formula associated with the Pythagorean Theorem or quadratic equations. They are also cognizant of the historical contexts in which this occurred: in ancient China and within the Arabic culture. The students in this study are in the 9th grade (14–15-year-olds) in a high school at Badalona, Barcelona (Spain). The characteristics of students and how they work in the classroom are described in greater detail in Sect. 9.4.

9.2 Resolution of Problems of Right Triangles with Diagrams

In secondary education, the classical problems of right triangles are normally posed by giving two sides of the triangle and then asking students to solve for the third side. In problems from Chap. 9 of the *Nine Chapters* the situation is more difficult, because in most of the problems the data consist of one side of the triangle and the difference or the sum of the other two.

In this situation, when the relationship between the data and the unknown is expressed by an algebraic equation, some ability in solving equations is required in order to solve the problem. This difficulty can be overcome by introducing diagrams, on the basis of which students are able to argue visually. Then they follow the procedure for calculating the solution to the problem by manipulating geometric shapes that are introduced after analyzing the data of the problem (see Figs. 9.14 and 9.15).

9.2.1 Solving Right Triangle Problems with Diagrams in the Manner of Liu Hui

Take as an example Liu Hui's explanation of the solution to problem 6[2] in Chap. 9 of *Nine Chapters*, as given in Chemla and Guo (2005, p. 711). In order to solve the case of a right triangle in which one of the perpendicular sides and the difference between the hypotenuse and the other perpendicular sides are known, Liu Hui states the following:

> Here take half the side of the pond, 5 *chi*, as *gou*, the depth of the water as the *gu*, and the length of the reed as the hypotenuse. Obtain the *gu* and the hypotenuse from the *gou* and the difference between the *gu* and the hypotenuse. Therefore, square the *gou* for the area of

2"Given a reed at the center of a pond 1 *zhang* square and which is 1 *chi* high above the water. When it is drawn to the bank, it is just within reach. Tell: the depth of the water and the length of the reed. Answer: The water is 1 *zhang* 2 *chi* deep and the reed 1 *zhang* 3 *chi* tall" (Dauben 2007, p. 286).

> the gnomon. The height above the water is the difference between the *gu* and the hypotenuse. Subtract the square of this difference from that of the area of the gnomon; take the remainder. Let the difference between the width of the gnomon and the depth of the water be the *gu*. Therefore construct [a rectangle] with a width of 2 *chi*, twice the height above the water. Its length is the depth of the water to be found. (Dauben 2007, p. 287).

This type of reasoning with areas rather than calculating with algebraic symbols is what we believe students should be encouraged to do. We may imagine the right triangle devised by Liu Hui as that described by Dauben (2007, p. 284) in Fig. 9.3, even though Liu Hui himself did not include any figures in his explanation. These geometrical shapes have been studied and transcribed by historians such as Cullen (1996, pp. 206–217), Chemla and Guo (2005, pp. 673–683) and Dauben (2007,

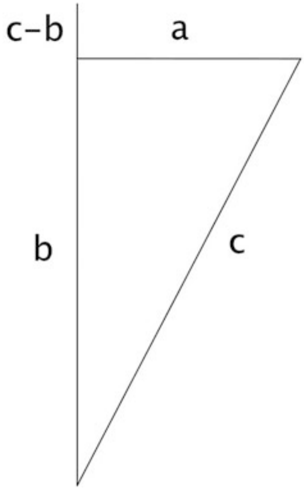

Fig. 9.3 Problem 9.6

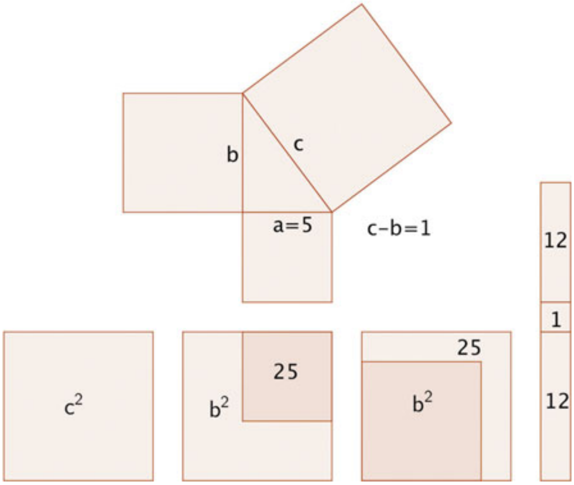

Fig. 9.4 The calculation of b and c from a and $c - b$

pp. 223–287). These demonstrations have been chosen in order to introduce the diagrams used in solving problems involving right triangles.

Figure 9.4 shows that when a, b and c are the sides of a right triangle, the large square on side c is the same as the sum of the squares on sides a and b. Since in ancient mathematics the concept of area is not explicit, neither in Greek nor in Chinese mathematics, we must now address the area of the three squares. Figure 9.4 also shows what happens with the area of the squares, the gnomon and the final rectangle.

In case of problem 9.6, $a = 5$ and $c - b = 1$, and we can solve b and c with geometrical and visual reasoning just as Liu Hui does[3] (see the previous quotation from Dauben 2007, p. 287):

> By performing the multiplication of the base (gou) by itself, we first show the area of the gnomon

$a = 5$ is one of the sides of the right triangle

$a^2 = 25$ is the area of the square, but also the area of the gnomon. Figure 9.4 shows three squares with the same area, the first one c^2, the second one gnomon (b^2) + 25, and the third one b^2 + gnomon (a^2), so this last gnomon must be 25.

Liu Hui goes on to state that:

> What extends above the water is the difference between the height and the hypotenuse;

in this case $c - b = 1$.

> One subtracts the square of this difference from the area of the gnomon and only then does one divide. The difference is the width of the area of the gnomon; the depth of the water is the height.

$25 - 1 = 24$; $24/2 = 12$ is the area of a rectangle with base $c - b = 1$ and altitude b; so, $b = 12/1$ and $c = 12 + 1 = 13$.

This is so because the gnomon becomes a rectangle of base $c - b$ and altitude $c + b$. This entire explanation becomes shorter with a figure and the comparison of area, as shown in Fig. 9.4.

In algebraic terms, the situation is as follows:

$a^2 + b^2 = c^2, \quad 25 + b^2 = (b+1)^2 = b^2 + 2 \cdot 1 \cdot b + 1, \quad 25 - 1 = 2 \cdot 1 \cdot b,$ and finally $b = (25-1)/2, \quad c = (25-1)/2 + 1.$

[3] 1 *zhang* = 10 *chi* = 100 *cun* (see Chemla and Guo 2005, inside cover).

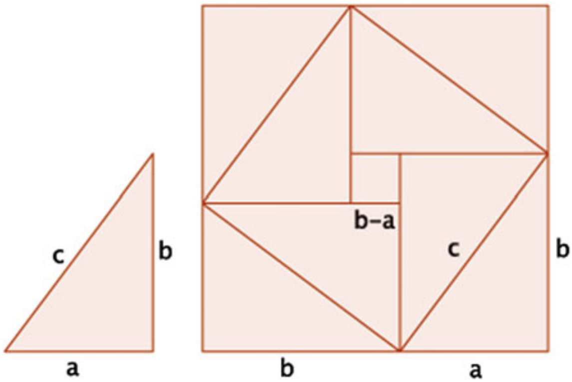

Fig. 9.5 The initial triangle and the first fundamental figure

9.2.2 The Fundamental Figures

The use of visual reasoning for solving problems of right triangles is based on diagrams described by Liu Hui and analyzed by the historians Cullen (1996, pp. 206–217), Chemla and Guo (2005, pp. 673–683) and Dauben (2007, pp. 223, 287). Chemla and Guo called these diagrams *fundamental figures*.

The First Fundamental Figure

The first fundamental figure (Fig. 9.5) is a square of side $a + b$ (base and altitude of the initial triangle). It contains the square of side c (hypotenuse of the triangle). This triangle is inside the square $a + b$ in a manner that determines four right triangles of sides a, b, c, and a square $(b - a)$.

If the hypotenuse (c) and the difference of the perpendicular sides $(b - a)$ are known, the first fundamental figure serves to calculate $(a + b)$, the side of the big square. Once $a + b$ is known, and taking into account that $(b - a)$ is known, a and b are calculated by adding or subtracting two numbers and dividing by two, because: $(a + b) + (b - a) = 2b$, and also $(a + b) - (b - a) = 2a$.

Liu Hui used this argument to solve problem 11 in Chap. 9.[4] We translate his arguments into diagrams with two different explanations in the same way that Liu Hui himself did (Figs. 9.6 and 9.7).

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[4]"Let us assume that we have a single-leaf door whose height exceeds its width by 6 *chi* 8 *cun* and whose two [opposite] corners are separated one from the other by exactly 1 *zhang*. We ask what is the value of the height and the width of the door, respectively. Answer: The width measures 2 *chi* 8 *cun*; the height measures 9 *chi* 6 *cun*" (Chemla and Guo 2005, p. 717; authors' translation).

Liu Hui stated: "Let us say that the width of the door is the base (*gou*), its height is the height (*gu*), the distance between the two corners, 1 *zhang*, is the hypotenuse, and that the height exceeds the width by 6 *chi* and 8 *cun*, which is the difference between the base (*gou*) and the height (*gu*). Their positions are established from the figure. The square of the hypotenuse covers exactly 10,000 cun. If this is doubled, and one subtracts the square of the difference between the base and the height, and if by extraction one divides this by the square root, what one obtains is the value of the

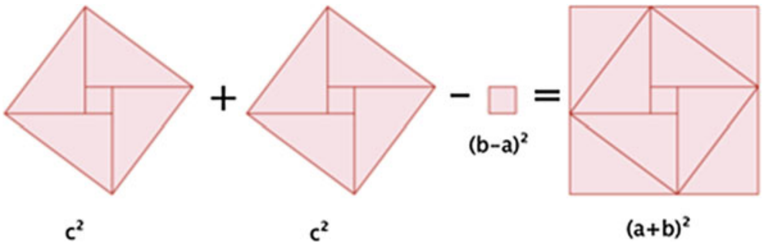


Fig. 9.6 The calculation of $a + b$ from c and $b - a$

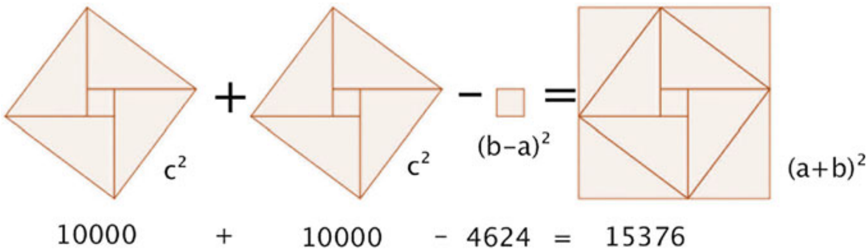


Fig. 9.7 Problem 9.11, and the first fundamental figure (Solution of Problem 9.11 based on Liu Hui's comment)

With the data from problem 9.11 (see Footnotes 3 and 4), we have: $100^2 + 100^2 - 68^2 = 10{,}000 + 10{,}000 - 4624 = 15{,}376$ and its square root (equal to 124) is $a + b$.

$(a+b) - (b-a) = 2a \rightarrow a = (124 - 68)/2 = 56/2 = 28$ and $a + (b-a) = b = 28 + 68 = 96$.

The Second Fundamental Figure

The second fundamental figure is the one described at the beginning of Sect. 9.2.1 in order to exemplify Liu Hui's explanation. Figure 9.8 shows the second fundamental figure with algebraic labels. Given this situation, when a and $(c - b)$ are known, one can calculate b using the area a^2 corresponding to a square, a gnomon and a rectangle, as in the sequence of diagrams in Fig. 9.8, because the base $(c - b)$ is known and one can calculate the height $(b + c)$ or $b + (c - b) + b$.

sum of the height and the width. If the difference is subtracted from the sum and one takes half of this, that gives the width of the door. If one adds to this the value of how much one exceeds the other, this will give the height of the door" (Chemla and Guo 2005, p. 719; authors' translation).

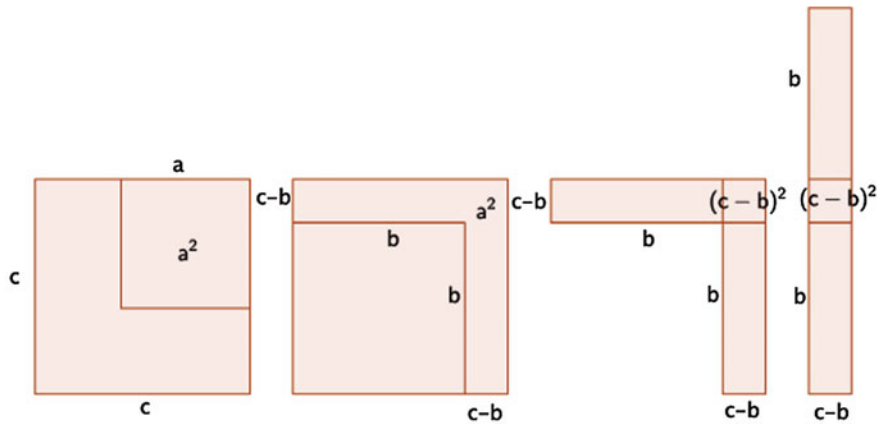


Fig. 9.8 The second fundamental figure with algebraic labels

9.3 Solving Equations by Completing Geometrical Squares

In secondary education, solving quadratic equations is a compulsory topic in the mathematics curricula. In this proposal, our students learn to solve equations in two different grades:

- 9th grade (14–15-year-olds): completing geometrical squares
- 10th grade (15–16-year-olds): solving equations $ax^2 + bx + c = 0$ with the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The aim of this topic was to lead students to use geometrical reasoning in the manner of al-Khwãrizmî, who justified the rules of computational algorithms with geometrical rules of transformations of figures and areas. While al-Khwãrizmî used rhetorical algebra, we use geometrical rules of transformations of figures and areas to introduce symbolic computation with algebra.

To this end, the process consists of four steps:

(a) The equation is translated into geometrical figures (squares and rectangles). In Sect. 9.3.2 we explain the translation from geometrical figures to symbolic algebra.
(b) The figures are manipulated and transformed (changing their lengths and areas).
(c) New lengths and new areas are deduced from the figures.

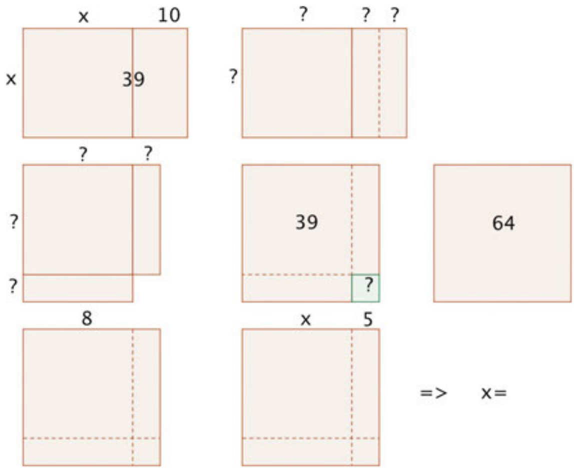


Fig. 9.9 Process for solving the equation by completing a geometric square

(d) The students interpret the figures and the measures and the solution of equation follows from this interpretation (the solution is one of the measures of the last figure obtained in the transformations).

Figure 9.9 shows the idea of the transformation of figures and the deductions of new lengths and new areas.

Radford and Guérette (2000) presented a teaching sequence whose purpose is to induce students to reinvent the formula for solving the general quadratic equation. Their teaching sequence was centered on the resolution of geometrical problems regarding rectangles by using an elegant and visual method developed by Babylonian scribes during the first half of the second millennium BC, and on the resolution of many problems found in a medieval book, the *Liber Mesuratinonum* by Abû Bekr (probably 9th century), as well as on al-Khwãrizmî's *Al-Jabr* (9th century). Their goal was achieved through a progressive itinerary that starts with the use of manipulatives and evolves through an investigative problem-solving process combining both numerical and geometrical experiences. Instead of introducing students to modern algebraic symbolism from the start—an approach that often discourages many of them—algebraic symbols are only introduced at the end, after the students have truly understood the geometric methods.

In this sense, our two proposals (the resolution of second-degree equations, and problems with rectangular triangles) are aimed at teaching students to develop visual geometric reasoning with diagrams. We encouraged the students to use the diagrams on their own initiative and leave the reinvention of the formula for solving the general quadratic equation for the following year. The point of departure in our research is not the resolution of a geometrical problem, but rather the resolution of a second-degree equation that has been transformed into a problem of calculating areas and lengths related to squares and rectangles. The aim is for students to move from algebraic expressions to geometry, to find solutions in the geometric field and then do the final reading in algebraic terms, thereby avoiding calculations with algebraic expressions.

9.3.1 Solving Equations in the Manner of al-Khwãrizmî

In his *Treatise of Algebra,* al-Khwãrizmî classified second-degree equations with positive coefficients and admitting only positive solutions, into six different types. He did not use algebraic symbols to denote the unknown coefficients and the equations could only be solved with what is known as rhetorical algebra.

For each of the six cases, al-Khwãrizmî justified the proposed algorithms with areas of squares and rectangles. From one of these cases, the solution process proposed to the students in learning activities has been taken in order to solve quadratic equations by completing a geometric square.

If transcribed with algebraic symbols, the equation is as follows: $x^2 + 10x = 39$. Figure 9.2 shows the geometric basis of his reasoning. Figure 9.9 contains the proposed activity with squares and rectangles visualized as provided to the students in *Solving equations by completing geometrical squares: Quadratic equation.*

9.3.2 Visualization of the Terms of an Equation in a Geometrical Sense

Figure 9.10 shows the visualization of the first-degree terms as used with students. Here, (a) x is the length of a segment; (b) two segments of length x together, one

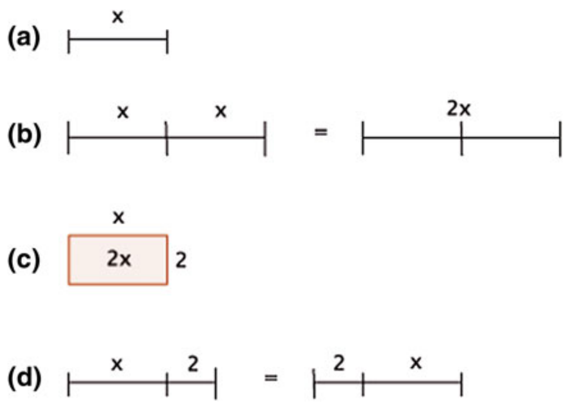


Fig. 9.10 Visualization of the first-degree terms

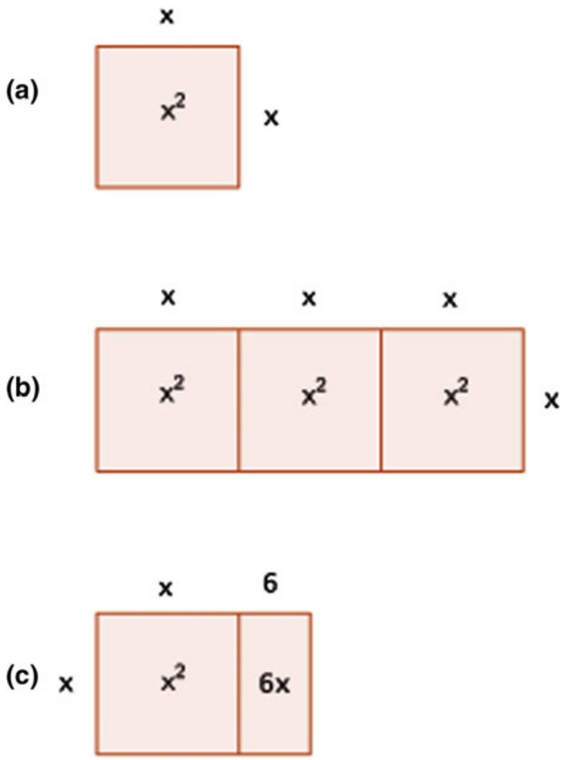


Fig. 9.11 Visualization of the second-degree terms

after the other, have length $2x$; (c) but also $2x$ could be the area of a rectangle with base x and altitude 2; (d) $x + 2$ is the same as $2 + x$, because it is the length of the same segment.

Figure 9.11 shows the visualization of the second degree terms: (a) x^2 is the area of square of side x: (b) $3x^2$ is the area of a rectangle which contains exactly three squares of area x^2 and is the same as $x^2 + x^2 + x^2$; (c) finally, $x^2 + 6x$ is a rectangle formed by a square x^2 and another rectangle $6x$.

When students more or less understand this relationship between algebraic terms and the length and area of squares and rectangles, they are able to solve quadratic equations by completing geometric squares.

9.4 Students Solve Problems in the Manner of Liu Hui and al-Khwãrizmî

9.4.1 The Students Involved in the Research and Classroom Activities

The population under study consisted of a group of 21 students of the 9th grade (3rd ESO[5]; 14–15-year-olds) of the Institute Badalona VII Secondary School during the academic year 2009–2010, and of which the first author of the study was their teacher of mathematics. The two topics analyzed—solving problems of right triangles and solving second degree equations—form part of the 9th Grade ESO curriculum. The first topic was taught during the first quarter and the second topic during the third quarter.

With regard to the characterization of these 21 students and their learning of mathematics, the class was a heterogeneous and diverse group. In general terms, they can be divided into four different subgroups of 8, 7, 2 and 4 students, respectively, each one with a different profile. The first subgroup of eight students possessed the algebraic skills introduced during their previous year (2nd ESO); students were able to solve simple first-degree equations and had sufficient understanding of the basic rules for isolating the unknowns. The second subgroup of seven students had not acquired sufficient algebraic reasoning, and one might even say that their use of letters (x) for identifying unknowns constituted an excessive degree of abstraction for their level of reasoning. The third subgroup of two students, while failing to possess the required level of the symbolic language of algebra corresponding to Spanish Secondary School 1st Grade (at age 12), were sufficiently familiar with numerical calculation but were somewhat unreceptive to the introduction of visual and geometrical reasoning. The fourth subgroup of four students had not acquired the level required at the 2nd ESO, but were able to work with numbers, although possibly without a fully developed understanding of the operations. One of the students in this subgroup, who faced serious reading difficulties, had handwriting that was only partially legible and experienced learning problems in all subjects. The students belonging to this last subgroup possibly lacked the sufficient basic knowledge for undertaking the activities, even geometrically, with the use of diagrams, as well as algebraically for solving of equations.

Throughout the course, the students worked together in groups of three or four without textbooks, and for each topic they were provided with a work dossier. The work groups were assigned by the teacher and were heterogeneous in terms of students' level and capabilities, so that in each group there was at least one student belonging to the first profile, as described in the previous paragraph, and one from the second profile. The 6 (2 + 4) students with the lowest levels of competence were also distributed in different groups.

[5]Secondary School Compulsory Education (in Spain).

Dossiers were also provided for the two topics on which the research is based. In the case of the right triangles, the dossier was called "Pythagoras' Theorem in Ancient China." It contained nine problems from Chap. 9 of the *Nine Chapters* (4, 5, 6, 7, 8, 9, 10, 11 and 12), in the Chinese and Catalan versions based on the text by Chemla and Guo (2005, pp. 711–721). First of all, the students were shown the procedure of "base & height" (*Gou* and *gu*), using paper cut-outs, before going on to familiarize themselves with the first and second figures. Then they were requested to solve the problems, deciding beforehand which of the two figures was appropriate for each problem. It was necessary to realize first that the first figure served to demonstrate the base-height procedure (*Gou gu*) and to solve problems 4, 5, 11 and 12; while the second figure was used to solve problems 6, 7, 8, 9 and 10. A script was also included to help students to look for information about Liu Hui and his historical context (cf. Sect. 9.7). The activities comprised ten teaching sessions of 55 min each and an eleventh session for the test, which included three problems, two of which are analyzed in this chapter.

The dossier for working in the manner of al-Khwarizimi was entitled "Equations of the Second Degree." It began with one of the problems on the measurement of rectangles from the cuneiform tablet YBC 4663 (ca. 1800 BC), inviting resolution by trial and error. It was followed by a series of incomplete equations of second degree, such as $ax^2 = c$ and $ax^2 + bx = 0$, which the students had to solve by both algebraic methods (the previous year they had worked on first-degree equations), and geometric methods, transforming the equations into additions and subtractions of areas of squares and rectangles. In the second part of the dossier, the resolution of second-degree equations in al-Khwãrizimî's manner was introduced, with the completion of geometric squares, as seen in Fig. 9.9. Solving the general second-degree equation $ax^2 + bx + c = 0$ was performed in all cases by geometric procedures, with the deliberate omission of the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ until the following school year, in order to make the students solve the equations at all times by completion of the geometric squares. A script was also included to enable the students to look for information about al-Khwãrizmî and his historical context. The activities comprised eight teaching hours of 55 min each, and a ninth session for the test, which included three problems, two of which are analyzed in this chapter.

In both cases, the work dossier made reference to historical contexts; ancient China, in the first instance, and the ancient Arabic context in the second. The mathematical knowledge of the students constituted the explicit point of departure, as well as the measurement of areas of squares and rectangles, solving equations of the second degree by completing geometric squares, and techniques for identifying unknowns in equations of the first degree. This approach generates learning by comparison and uses what has worked in other situations in order to arrive at new situations that require new tools. These new tools are diagrams or figures with written information that act as a support for performing calculations with which students are unfamiliar by means of algebraic manipulation. In this way, by using geometric transformations, a new figure is arrived at on which the solution can now be read.

During the test on “Pythagoras’ Theorem in Ancient China” the students worked in groups; nine groups of two students and one group of three. So, we analyze ten different solutions of problem 1 and ten of problem 2. On the other hand, during the test on “Equations of the Second Degree” the 21 students worked individually. Therefore, we analyze 21 examples of the solution of one equation, and 21 different examples of the other.

9.4.2 Solving Problems of Right Triangles with Diagrams: Student Work Examples

We illustrate the process of problem solving with diagrams by using two problems taken from the test; one solved with the second fundamental figure and the other with the first fundamental figure. Note that the problems in this final test consisted of instances relevant to the 21st century in order to show that mathematics is not only a tool for solving old historical problems.

Statement of Problem 1:

> An antenna tensioner, suspended from the top of the antenna without being tightened, exceeds the height of the antenna by 6 m. When tightened on the ground, it lies 16 m from the base of the antenna. What is the height of the antenna and how long is the cable tensioner?

The process of problem solving with diagrams consists of four steps: construction, processing, interpretation and reading. We can follow the process in Fig. 9.8 with an example of one student (from the pair of students 1 and 4).

(a) The student constructs a diagram of data, a triangle with one side known, the relationship of the other two sides, and the unknown sides (Fig. 9.12a).
(b) The student decides which of the two calculation diagrams corresponds to the problem to be solved (Fig. 9.12b).
(c) The student constructs a calculation diagram and writes the known data (Fig. 9.12c).
(d) In order to be able to read the solution to the problem on the last diagram (Fig. 9.12d), the student transforms this first diagram into as many other successive diagrams as he/she wishes.

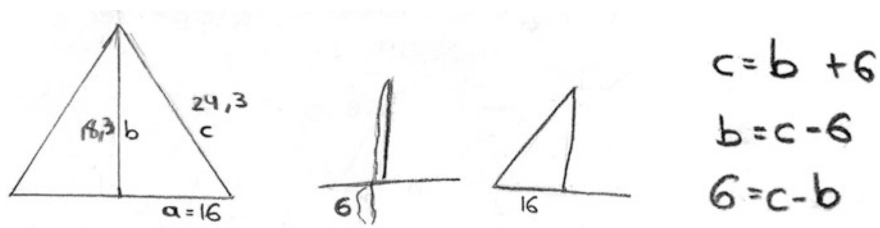


Fig. 9.12a The diagram of data and the relationship of the other two sides

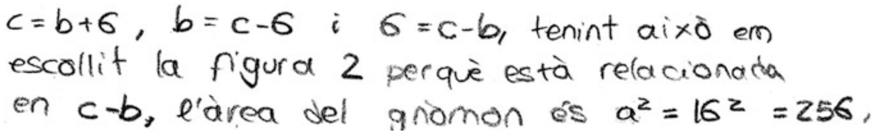


Fig. 9.12b Student's explanation of choice ($c = b + 6$, $b = c - 6$ and $6 = c - b$. With this knowledge, we choose Fig. 9.8 because it is related to $c - b$, the area of the gnomon is $a^2 = 16^2 = 256$)

Fig. 9.12c .

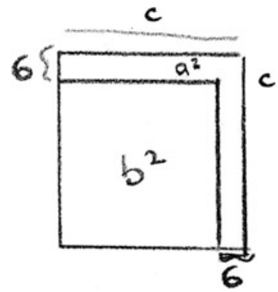
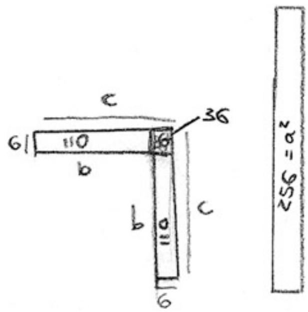


Fig. 9.12d .



Statement of Problem 2:

When it is said that a screen measures 26 inches, it means that the diagonal of the rectangle is 26. If the width of the screen exceeds its height by 14 inches, calculate the sides of the screen.

Again, problem solving with diagrams consists of four steps: construction, processing, interpretation and reading. We can follow this process with an example from one of the students who produced Fig. 9.12. Figures 9.13 and 9.14 correspond to this same student's group (comprising two students).

In this case, the student solved the problem using a different interpretation of Fig. 9.5 (note that there is a slight mistake in the calculation: 676 + 480 = 1156, and not 1176, as she computed).

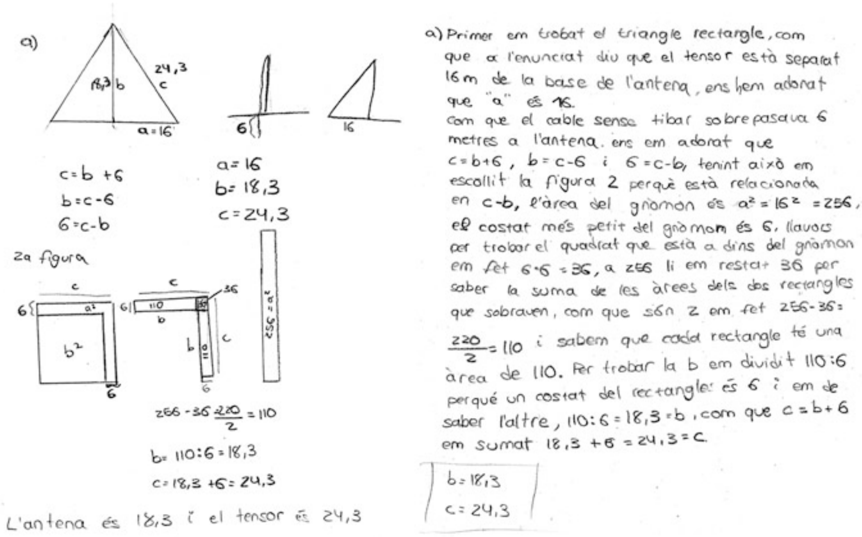


Fig. 9.12 The solution of one student and her explanations (First, we find the right triangle, so as in the statement the tensioner is said to be 16 m from the base of the antenna, we know that it is 16. Since the untightened cable exceeds the height of the antenna by 6 m, we know that $c = b + 6$, $b = c - 6$ and $6 = c - b$. With this knowledge, we choose Fig. 9.8 because it is related to $c - b$, the area of the gnomon is $a^2 = 16^2 = 256$, the shortest side of the gnomon is 6, so in order to find the square inside the gnomon we do $6 \cdot 6 = 36$. We subtract 36 from 256 in order to find the areas of the two remaining rectangles, and since there are two we do $256 - 36 = 220$ and $220/2 = 110$, and so we know that each rectangle has an area of 110. In order to find b, we divide 110 by 6, because one side of the rectangle is 6. Since we have to find the other, $110/6 = 18.3 = b$, and since $c = b + 6$ we add $18.3 + 6 = 24.\ 3 = c$)

9.4.3 *Solving Second-Degree Equations: Student Work Examples*

We asked students to solve different kinds of equations in the same manner as al-Khwãrizmî, both arithmetically and geometrically, so that they associate x as a length, $2x$ as a length, or an area, and x^2 as an area. Examples of the first types of these are: $x^2 = 25$; $3x^2 = 12$; $x^2 = 20$; $2x^2 = 12$; $x^2 - 81 = 0$; $x^2 - 24 = 0$. Students then solve $x^2 - 5x = 0$; $x^2 + 5x = 0$; $2x^2 - 8x = 0$; $3x^2 + 81x = 0$.

The aim is for them to relate the terms of an equation to length and area through the visualization of the terms of an equation in a geometrical sense, as explained in Sect. 9.3.2. in which the equation $x^2 + 6x = 40$ appears. The expected learning outcomes are shown in Fig. 9.15. We do not have the solution when we draw the figure; the idea is that figures do not have to be realistic.

Figure 9.15 shows the four steps of the process: cutting the initial rectangle, moving a piece of the rectangle to obtain a square (or more or less a square), in order to complete the new figure and arrive at a whole square.

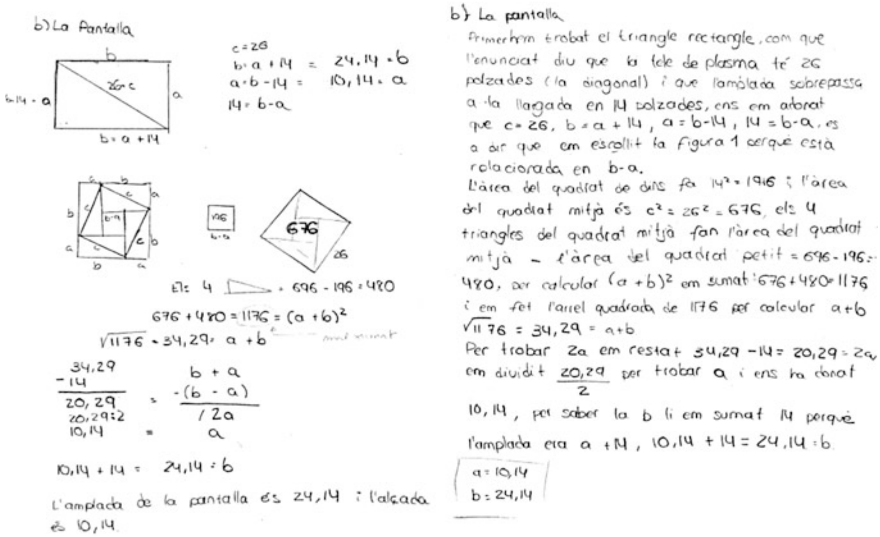


Fig. 9.13 A student's solution of the problem using the first fundamental figure, and student's explanation (The screen: First we find the right triangle, and since in the statement it says that the plasma screen measures 26 in. (diagonally) and that the width exceeds the length by 14 in., we then have that $c = 26$, $b = a + 14$, $a = b - 14$, $14 = b - a$, that is, we choose Fig. 9.1 because it is related to $b - a$. The area of the inner square is $14^2 = 196$ and the area of the middle square is $c^2 = 26^2 = 276$. The four triangles in the middle square are the middle square minus the small square = $696 - 196 = 480$. To calculate $(a + b)^2$ we add $676 + 480 = 1176$ and we take the square root of 1176 in order to calculate $a + b$, $\sqrt{1176} = 34.29 = a + b$. To find $2a$, we subtract $34.29 - 14 = 20.29 = 2a$. Then we divide 20.29/2 to find a, which gives 10.14, and to find b we add 14, because the width was $a + 14$, $10.14 + 14 = 24.14 = b$)

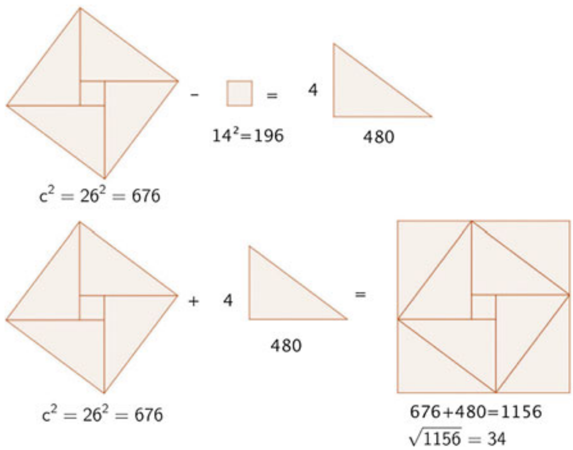


Fig. 9.14 Solving the problem with another interpretation of the relationship in the first fundamental figure

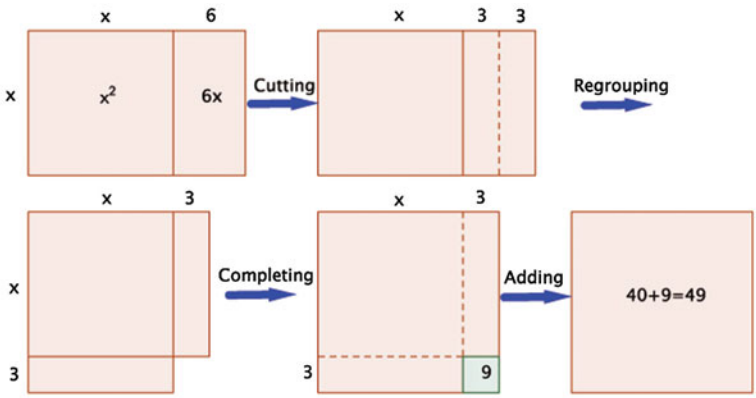


Fig. 9.15 Solving $x^2 + 6x = 40$ geometrically

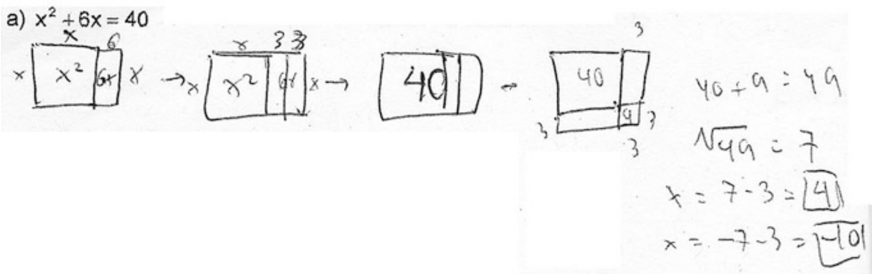


Fig. 9.16 A student (number 13) solving $x^2 + 6x = 40$ geometrically

Figure 9.16 shows the process of solving an equation ($x^2 + 6x = 40$) geometrically with an example by a student (belonging the second subgroup of seven students described in Sect. 9.4.1). He begins with a square x^2; he adds the rectangle $6x$; he cuts the rectangle into two pieces; he moves one of the rectangles, and finally he completes the figure with a small square of area 9.

9.5 The Process of Problem Solving with Diagrams

Solving right triangles with diagrams and solving equations by completing geometric squares provide two topics for secondary education taken from the history of mathematics, but from the mathematical point of view they possess some similar characteristics. For the first topic, while solving the problem it is necessary to give the geometrical meaning of algebraic terms. In the second, the solution involves working with calculation diagrams and the transformations of these diagrams. The difference resides in the point of departure: In right triangles the context is geometrical, and the data are measurements of a right triangle. This consists of diagrams of data expressing algebraic relations between the data. Afterwards, we return to geometry to solve the problem with a calculation diagram. In solving equations by completing geometrical squares, the context is algebraic, the data are equations, and we then move to geometry to find the solution with a calculation diagram.

The diagrams employed in this research follow the classification of Barwise and Etchemendy (1996) and the nomenclature of Mason et al. (2005) and Giardino (2009, 2014). From the analyses of these authors, we have adopted two features associated with the diagrams thus introduced and analyzed in this work: (i) the expressive efficacy of the diagram, that is, the ability to express semantic properties and computational efficiency, and (ii) the ability to infer new information. Two types of these diagrams have been identified to differentiate their role in the problem-solving process; one type is denoted as data diagrams and the other as calculation diagrams, and both have expressive efficacy. A data diagram expresses semantic properties between data, and helps to choose the correct calculation diagram when solving right triangles. A calculation diagram has expressive efficacy for solving right triangles and second-degree equations because they help to calculate the solution of the problem.

Giardino (2009) identifies two kinds of elements in the sequence of problem solving with calculation diagrams: the diagrams and the actions involved. The actions are four-fold: construction, processing, interpretation and reading. In Sect. 9.2, we have described the process with an example of right triangles, while in Sect. 9.3 we have described it with an example of quadratic equations. Figures 9.17 and 9.18 show the different steps in the process and the connections between them.

Figure 9.17 should be read from top to bottom and from left to right. The rectangles contain data and diagrams. Arrows indicate actions, in accordance with Giardino (2009). The data diagram is constructed from the data of the problem. Decide first what calculation diagram to use by looking for the relations in the data

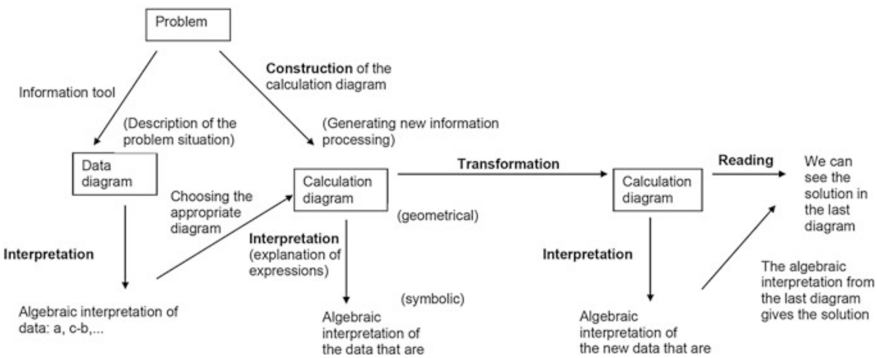


Fig. 9.17 The process of problem solving with diagrams

diagram. The calculation diagram is then constructed. Each diagram has a corresponding interpretation in algebraic terms (bottom of Fig. 9.17). The first calculation diagram contains different transformations and processing, in accordance with Giardino (2009). Finally, the solution is obtained through the last calculation diagram.

Following the diagram in Fig. 9.17 for the antenna problem (Fig. 9.12), we arrive at the diagram in Fig. 9.18.

9.6 Organization of the Conclusions in Accordance with the Four Characteristics of Problem-Solving Diagrams

The four actions in problem solving with diagrams (*construction*, *processing*, *interpretation* and *reading*) can be combined into three procedures: *translation*, *transformation* and *diagrammatic reasoning*. We summarize the resulting conclusions in Sects. 9.6.1–9.6.3 and in more general terms in Sect. 9.6.4 about the advantages of using diagrams. More specifically, *construction* and *reading* form part of problem solving with diagrams and the end of the process with the solution. We combine these two processes into one, namely *translation*. Processing involves the *transformation* of diagrams. Interpretation connects geometry with algebra and explains reasoning to students; this is *diagrammatic reasoning*. Each one of these procedures leads to several conclusions drawn from our empirical research, some of which are briefly outlined in the next subsections.

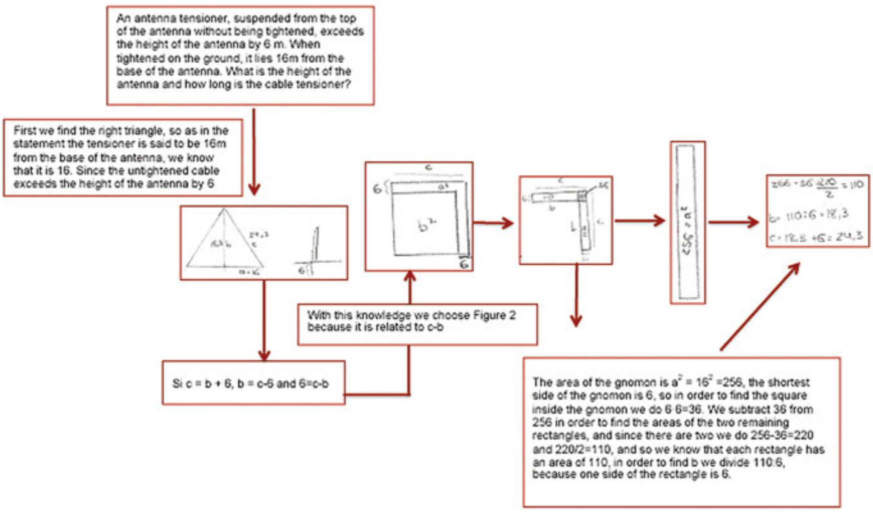


Fig. 9.18 The process of solving the antenna problem

9.6.1 Translation

The first procedure is the translation of the problem into the language of diagrams; that is, the study of the equivalence established by students between algebraic language and the geometric representation when they construct the first diagram with the data of the problem, and likewise, the reinterpretation of the final diagram in terms of the problem posed.

Of the various conclusions concerning *translation* (Guevara 2015, pp. 428–435), we emphasize the most important one: the students associate terms of first degree with unit coefficient (x) to lengths; the first degree with other coefficients ($6x$) to areas; the second degree also to areas; the numerical values with lengths or areas. Figure 9.16 shows the equivalences that one student established between the terms $6x$, x, 40 and the sides and areas of the figure, with the use of corresponding labels. He places it within, or outside the figure, according to whether it represents an area, or a length.

Table 9.1 shows how students related the labels to length or area. On the whole, the labels are related correctly, and in three cases the error is not transferred to the equation-solving process.

9.6.2 The Transformation of Diagrams

The second procedure is the transformation of diagrams; that is, a description of the diagram transformation process used by students in problem-solving. Several conclusions (Guevara 2015, pp. 435–445) are crucial for the series of transformation diagrams when solving the problem and its predictability (the number of diagrams containing the number associated with a problem; the number of diagrams for a given student to solve a particular problem; the structure and direction of the

Table 9.1 Students' labels when solving $x^2 + 6x = 40$

Label	Number of students	Related to length/area	Ratio correct/errors
x	14	Length	8/6
x^2	4	Area	2/2
6	10	Length: 8; area: 2[a]	9/1
$6x$	6	Area: 5; length: 1[b]	4/2
40	19	Area	12/7
3	20	Length	15/5
9	20	Area	14/6
49	7	Area	4/3
7	2	Length	2/0

[a]Two students solved the equation correctly
[b]The student solved the equation correctly

changes), but the most important one seems to be the thread that guides the diagram transformation process of the areas in the figures (squares and rectangles).

Students have identified the key areas for constructing the first calculation diagram: 256 for the first right triangle problem; 676 and 196 in the second problem; and 40 for the quadratic equations. Figures 9.12 and 9.20 contain the solution obtained by two students using the second fundamental figure to solve a right triangle problem. The thread that guides the process of transformation is the area 256. Figure 9.13 contains the solution obtained by a student using the first fundamental figure to solve a right triangle problem. The thread that guides the process of transformation is the two areas 676 and 196. In Figs. 9.16, 9.19 and 9.21, the problem involves the solution of a second degree equation. This time the guide is the area that equals 40.

9.6.3 Diagrammatic Reasoning

The third procedure is diagrammatic reasoning; that is, identifying the key elements of diagrams in problem solving and what they represent in the reasoning followed by the students.

Mancosu (2001, 2005) distinguishes between visualization and diagrammatic reasoning. He uses visualization as a discovery tool, in the same way as Giaquinto (1992). On the other hand, he uses diagrammatic reasoning as a demonstration tool, in the same way as Barwise and Etchemendy (1996).

One conclusion to be drawn is that when solving the problem with the use of diagrams, each student has a key diagram and this key diagram is the same for all students. In the case of equations, for example, Fig. 9.16 shows the key diagram, and the final square with two squares and two rectangles inside it. However, we would like to emphasize another conclusion as the most important; namely, that the essential labels used by the students are the numerical labels in preference to algebraic ones, without which the diagram does not help to solve the problem.

In Figs. 9.16, 9.19 and 9.21, different types of labels are used, while in Fig. 9.19 one may see that in the process followed by the student for the solution to the equation $x^2 + 6x = 40$ no algebraic labels are used.

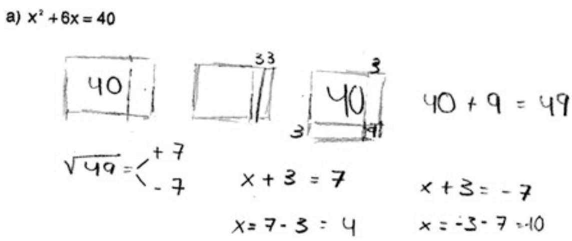


Fig. 9.19 Solution without algebraic labels (student 1)

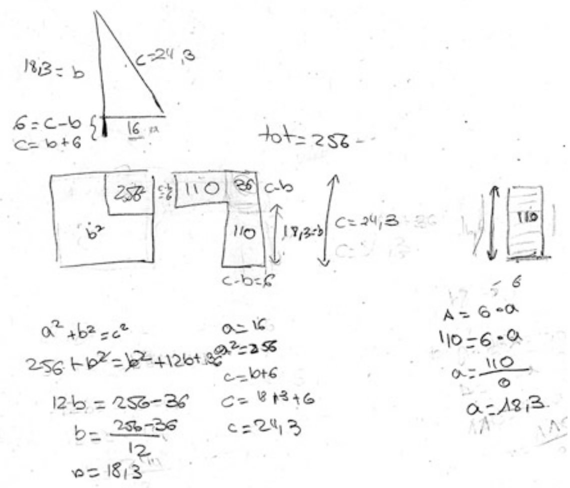


Fig. 9.20 Geometrical and algebraic solution (student 6)

Table 9.1 shows the labels used by students, the majority of whom used the numerical labels 40, 3 and 9, which are indeed the most essential labels for solving the equation correctly.

9.6.4 The Advantages of Using Diagrams

Finally, in a more general sense, four conclusions regarding the advantages of using diagrams to solve problems in secondary school concur with the ultimate aim of this study.

(1) Students mainly chose the geometric method of solving problems. However, some students used both geometrical and algebraic methods, as may be seen in Fig. 9.18.
(2) The second conclusion concerns effectiveness: visual diagrams enable students to be more effective. Not only did more students solve the problem in this way, they also solved it better. Analysis of the 21 results achieved by students with right triangles and the quadratic equation supports this assertion. In the case of the quadratic equation, they were unable to use the algebraic method because they were not familiar with it, as explained in Sect. 9.3.
(3) Regarding the advantages of using diagrams (Guevara 2015, pp. 452–458), we would like to emphasize the following as the most notable one: manipulating diagrams with ease is necessary to distinguish between the concept of perimeter and area, and also between measuring a length and a surface. Figure 9.21 shows one student's solution, who added values for length (6) and area (9), thus preventing herself from getting the final result correctly.

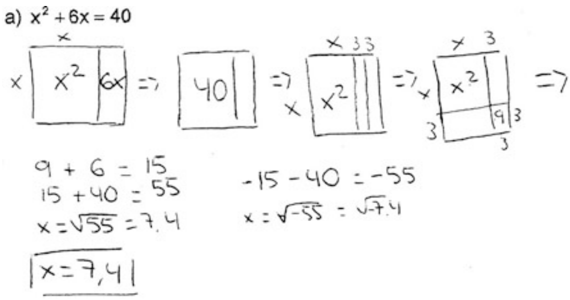


Fig. 9.21 Sum of areas and lengths in the solution by one student (17)

(4) The last conclusion we mention is related to the third one above: In order to connect geometry and algebra, one must determine how many figures are contained within a given one (visual perception) and also associate the data on algebraic relations, with lengths and areas.

In the present research, unlike that by Radford and Guérette (2000) mentioned above, the objective is not to arrive at algebraic formulas, but rather for students to acquire knowledge at the geometric stage with the help of diagrams. To this end, the students' own productions are analyzed and the construction of knowledge with the support of the diagrams is identified.

9.7 Using History to Teach Mathematics and the Teaching and Learning of Algebra

Activities based on the analysis of historical texts as part of the curriculum contribute to the improvement of the students' overall training by providing them additional knowledge about the social and scientific context of the periods involved, because in both cases the dossier included a script containing information to enable them to understand the context of the historical character being studied.

Students thereby acquire a vision of mathematics not as a final product but as a science that has been developed on the basis of seeking answers to questions that humankind has been asking about the world around us throughout history.

As an example, in the case of "The Pythagorean Theorem in ancient China," students were provided with a script containing the following questions (cf. Sect. 9.4.1):

ANCIENT CHINESE MATHEMATICS

1. Who was the first Chinese emperor? What dynasty did he found? In what period?
2. What were the public buildings in that time? For what purpose were they built?

3. What are the two oldest Chinese mathematical texts for which documentary evidence exists? What type of mathematics do they contain? To whom were they addressed?
4. What has been the most important classical text for many centuries in Chinese mathematics?
5. Situate this classical text: Title; author; period; to whom it is addressed; describe briefly the contents of the different chapters. (Guevara 2015, p. 492)

A detailed analysis of students' outcomes and the conclusions drawn from the learning process using this resource indicates that this way of introducing algebra in relation to geometrical interpretation may be profitably applied. Through geometry, both operations with numbers (arithmetic) and operations with letters (algebra) express the results of measuring lengths and areas and in this way they acquire the same meaning.

Given the results obtained from the analysis of student activities and the conclusions reached thereby, it can be stated that the teaching of algebra in the first year should go hand in hand with visual arguments and the use of diagrams. In other words, the introduction of algebra, besides being just a generalization of arithmetic where the rules of operations with numbers are generalized to rules with letters, should also have a visual component which gives the geometrical interpretation of algebraic formulas. With this paradigm, and depending on the situation, linear expressions can be interpreted as areas or lengths of segments, while quadratic expressions can be interpreted as areas. All operations and the rules for operating with letters have their interpretation in the geometric model. In this way, the properties of operations are not justified solely via general syntactic rules on symbols, but rather have an equivalent in the geometric model.

This claim is supported by the results obtained in the present research: In the case of "The Pythagorean Theorem in ancient China," from the 10 outcomes analyzed,[6] six students tackled the problem with geometrical reasoning using diagrams, three treated it using algebraic expressions and one was unable to do anything. Four students out of the first six obtained the solution using diagrams, and one student from the second three reached the solution with algebraic reasoning. In the case of the second-degree equations, we analyzed 21 outcomes, in all of which diagrams were used because the students did not know the algebraic form for solving second-degree equations, and 15 out of these 21 obtained the solution correctly.

As Katz and Barton (2007) state: "An historical view places number and geometry on at least equal footing in mathematical development, and highlights the powerful interrelationship between the two" (p. 198). In this sense, we believe that moving from arithmetic to algebra by skipping geometry may be regarded as a pedagogical and historical error. Though this is an approach that fits well to the 17th century, when the force of the new symbolic language replaced visual geometric reasoning, it may be considered not appropriate in the 21st century. Nevertheless,

[6]Recall that 21 students performed the activities in pairs.

for many centuries, in the absence of formal algebra and with only the four basic arithmetic operations available, humanity was able to solve problems that we now solve with equations. And this cannot be ignored altogether, in view of the fact that a significant number of students exist who are unable to solve problems just because they do not thoroughly understand the rules of this language, and thus may be regarded as mathematically illiterate. We think that there is sufficient ground to believe that, when beginning to learn algebra, it is necessary to return to the reasoning used by ancient mathematicians, who calculated on the basis of geometric models to justify the validity of their operations. It is our contention that these geometric models can be used to help our students to understand algebra.

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