Chapter 10
Missing Curious Fraction Problems

The Unknown Inheritance and the Unknown Number of Heirs

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Abstract In this paper we present a study of one of the best-known types of descriptive word fraction problems. These problems have disappeared from today's textbooks but are hugely important for developing arithmetic thinking. The aim of this paper is to examine the historical solution methods for these problems and discuss the analytical readings suggested by the authors. On the basis of this analysis we have conducted a preliminary study of the performance of 35 Spanish students who are highly trained in mathematics. Our results show that these students have a preference for algebraic reasoning, are reluctant to use arithmetic methods, and have reading comprehension difficulties that are reflected in their translations, from literal language to symbolic language, of the relationship between the parts expressed in the problem statement.

Keywords History and mathematics education · Descriptive word fraction problems · Resolution methods · Student performance

10.1 Introduction

Textbooks contain a wide range of descriptive word fraction problems whose history dates back to ancient mathematical cultures. The statements of these problems have evolved over time, adapting to social changes, mathematical developments and the predominant educational theories of the era while maintaining a common mathematical content.

These problems were essential components of the arithmetic of the past and can be found in a multitude of historical texts. Examples of these texts are *Jiuz hang*

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Suan shu, better known as the *Nine Chapters on the Mathematical Art* (ca.100 AD; Chemla and Guo 2005; Shen et al. 1999), which contains 247 Chinese mathematical problems; collections of Hindu mathematical problems, such as the Bhaskara manuscript, also known as the *Līlāvatī* (Colebroke 1817; Phadke et al. 2001); and basic texts from medieval Europe, such as the *Greek Anthology*[1] (Jacobs 1863) and the recreational mathematical collections of Bede (*De Arithmeticis Propositionibus* in 641 AD; Migne 1850) and Alcuin (*Propositiones ad acuendos juvenes* in 775 AD; Migne 1863). Descriptive word fraction problems also appear in texts that introduced the west to Islamic mathematical methods such as Fibonacci's *Liber Abaci* (Sigler 2002). Later they also appear in the first printed books on arithmetic and algebra, such as the texts in Spanish by De Ortega (1552), Silíceo (in 1513; Sánchez and Cobos 1996), Pérez de Moya (1562), and the syncopated algebra of Aurel (1552). We should also mention their presence in recreational mathematical texts, such as those by Bachet (1612), Ozanam (in 1692; Hutton 1844) and Vinot (1860) and, more recently, in popular works such as that by Swetz (2014).

However, the advent of a general public education system led to the adoption of an approach to mathematical problems that is based on the application and practice model and gives prevalence to the algebraic method over the arithmetic method. This has lowered confidence in the educational value of these problems to the extent that many have disappeared from textbooks, or appear in them merely as past time activities.

Today's basic curriculum for Spanish Primary Education explicitly states that

> Problem-solving processes are one of the main axes of mathematical activity; they constitute the cornerstone of mathematics education and as such they should be the source and main support for learning throughout this stage of education. Solving a descriptive word problem requires a multitude of basic skills, including reading, thinking, planning the solution process, establishing and reviewing strategies and procedures, modifying this plan if necessary, checking the solution, and communicating the results. (Spanish Royal Decree 126/2014; MEC 2014, p. 19386; authors' translation)

Curriculum proposals therefore consider problem solving to be a basic competence in the development of mathematical activity.

We believe that historical problem-solving methods are indispensable sources of information for mathematics education because they illustrate the reasoning the great mathematicians of the past used in their solutions to these problems. In this chapter, we compile historical evidence on the solution methods for descriptive word fraction problems and highlight certain aspects of problem solving that will enable pupils to acquire significant knowledge.

The existing literature contains numerous classifications of problems with natural numbers that follow criteria such as the mathematical structure, or the statement's syntactic characteristics, including location of the question, length and

[1]A collection originally compiled by Metrodorus, probably around the 6th century and later on greatly enriched by C. Cephalas in the 10th century and M. Planudes in the 13th century.

number of sentences, number of words, verb tenses, etc.; or semantic characteristics, which include global semantics, inclusion of superfluous information and distracters, etc. (Cerdán 2008; Goldin and McClintock 1979). However, no classification of fraction problems has been widely accepted by the research community. In textbooks, these problems appear under headings such as Methods, Rules, Contexts or Actions and Agents (see Gómez et al. 2016). Although presenting them in this way gives the problems certain recognition at least, it does not provide a sufficiently global or overall view of them. Moreover, this form of presentation is also an arbitrary one, because the same problem can be solved using different (arithmetic or algebraic) methods and because the same method can be used to solve different problems. The same occurs with the name of the problem, the context in which it is set, or the agents involved, because these say nothing about how the problem is structured, or what it contains.

To address this question, in Table 10.1 we first present a structured classification of descriptive word fraction problems. The problems are divided into categories and types according to two intrinsic variables of fraction problems: the known or unknown whole or total quantity, and the relationship between the parts (for more details see Gómez et al. 2016). This classification will be used to achieve the general objective of this paper, which is to compile a list of historical methods by analyzing each type of problem identified.

In this chapter, we will focus on a particular type of descriptive problem that involves an unknown whole and related parts. We present the methods that have been used in textbooks to solve this type of problem and the analytical readings that have been used to support these methods. By analytical reading we mean the reduction of the statement to a list of quantities and a list of relationships between these quantities (see Gómez 2003; Puig 2003). Then, we use this information to conduct a pilot study to investigate the extent to which these methods and readings are reflected in students' performance.

The rest of the chapter is organized as follows. First, we present a sub-classification of the problems that contain an unknown whole and related parts in order to contextualize the specific type of problem analyzed in this paper. We then examine this type of problem based on the various analytical readings and problem-solving methods. Finally, we present the results of our pilot study of students who attempted to solve these problems.

10.2 Study Problem

Gómez et al. (2016) present a subdivision in which the problems with unknown wholes and related parts are divided into four groups (see Table 10.2).

In this chapter we focus on the fourth type of problem illustrated in the above subdivision. As Singmaster (1998) pointed out, this type of problem first appeared in Fibonacci's *Liber Abaci* (1202). Singmaster (1998) calls it *the problem of inheritance, with the ith son getting* 1 + 1/7 *of the rest and all getting the same*

Table 10.1 Classification of the descriptive word fraction problems

Known whole		Unknown whole	
Non-related parts	Related parts	Non-related parts	Related parts
Dying man	The cloth	Lotuses	The eggs (Passing through tax guards)
"*A dying man left 6,000 escudos to be distributed in the following way: "I wish," he said, "that half be given to the Jacobite monastery; a third to the convent of Saint Augustine; a quarter to the monastery of the Friars Minor; and a fifth to the Order of the Carmelites." Question: if the whole is 6,000 escudos, how many escudos will each monastery receive?*"[a] (Silíceo 1513; see Sánchez and Cobos 1996, p. 266 Authors' translation)	"*A certain man buys 4 pieces of cloth for 80 bezants. He buys the first for a certain price, and he buys another for 2/3 the price of the first. He truly buys the third for ¾ the price of the second. Moreover, the fourth he buys for 4/5 the price of the third. It is sought how much each piece is worth.*"[b] (Fibonacci 1202; see Sigler 2002, pp. 274–275)	"*From a bunch of lotuses, 1/3 are offered to Lord Siva, 1/5 to Lord Visnu, 1/6 to the Sun, and 1/4 to the goddess. The remaining 6 were offered to the guru. Find quickly the number of lotuses in the bunch.*"[c] (Bhaskara 1150; see Phadke et al. 2001, pp. 57–58)	"*A country-woman carrying eggs to a garrison, where she had three guards to pass, sold at the first, half the number she had and half an egg more; at the second, the half of what remained, and half an egg more; and at the third, the half of the remainder, and half an egg more; when she arrived at the market place, she had three dozen still to sell. How was this possible without breaking any of the eggs?*"[d] (Ozanam 1692; see Hutton 1844, pp. 207–208)

[a]*Dying Man Solution*: "For the Jacobite monastery, half of the 6,000 escudos, i.e. 3,000 escudos; for the convent of St. Augustine, a third of the 6,000 escudos, i.e., 2,000 escudos; for the monastery of the Friars Minor, a quarter of the 6,000 escudos, i.e. 1,500 escudos; and for the Order of the Carmelites, a fifth of the 6,000 escudos, i.e. 1,200 escudos. All of these parts add up 7,700 escudos but this is not possible because the man only has 6,000 escudos. The divisor is considered 7,700 and the multiplier is the money that must be distributed, i.e. 6,000 escudos. Therefore, each part is multiplied by this ratio. Then, if the part for the Jacobites is multiplied by the multiplier and divided by the divisor, they receive 2,337 escudos, 23 .duodenos and 2 turonos and 1400/7700 turonos. This is therefore the amount that corresponds to the Jacobite Monastery. For other cases we proceed in the same way to obtain the amount corresponding to each. Note that 1 escudo = 35 duodenos and that 1 duodeno = 12 turonos."

[b]*The Cloth Solution* (the modern notation for fractions is used): "You put it that the first piece is worth 60 bezants, because 60 is the least common multiple of the 5 and 4 and 3. Therefore, if the first is worth 60, then the second, worth $\frac{2}{3}$ of it, is worth 40 bezants, and the third is worth 30 bezants, that is $\frac{3}{4}$ the price of the second. The fourth truly is worth 24 bezants, that is $\frac{4}{5}$ of 30. Afterwards you add the 60, and the 40, and the 30, and the 24, namely the put prices of the abovesaid four pieces; there will be 154 that should be 80; you say, I put 60 for the price of the first piece and 154 bezants result as the sum of the four pieces; what shall I put so that the sum of the pieces is 80 bezants? You multiply the 60 by the 80; there will be 4800 that you divide with the rule for 154, that is $\frac{0}{11}+\frac{0}{77}+\frac{1}{154}$, the quotient will be $31+\frac{1}{11}+\frac{6}{77}$ bezants. And this is the value of the first piece. Also in order to have the price of the second, you multiply the 40 by the 80, and you divide again with the $\frac{0}{11}+\frac{0}{77}+\frac{1}{154}$; the quotient will be $15+\frac{6}{11}+\frac{3}{77}$ bezants for the price; at last, so that you know the price of the fourth, you multiply the 24 by the 80, and you divide with the $\frac{0}{11}+\frac{0}{77}+\frac{1}{154}$; the quotient will be $12+\frac{5}{11}+\frac{1}{77}$ bezants for the price, and you realize that in each of the above written four products a ½ is canceled."

[c]*Lotuses Solution*: "Suppose the total number of lotuses is 1. Then the number of lotuses left is $1-\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{4}\right)=1-\frac{20+12+10+15}{60}=1-\frac{57}{60}=1-\frac{19}{20}=\frac{1}{20}$. So [if $\frac{1}{20}$ is 6] the total number of lotuses is $\frac{6\times 1}{\frac{1}{20}}=120$."

[d]*The Eggs (Passing through Tax Guards) Solution*: "It would appear, at first view, that this problem is impossible to solve, for how can half an egg be sold without breaking it? It becomes clearer, however, when we consider that by taking the greater half of an odd number we take exactly half of that odd number $+\frac{1}{2}$. We can find, therefore, that before she passed the last guard the woman had 73 eggs remaining, for by selling 37 of them to that guard, which is half of $73+\frac{1}{2}$, she would have 36 remaining. Similarly, before reaching the second guard she had 147; and before reaching the first, she had 295."

Table 10.2 Classification of descriptive word fraction problems with unknown wholes and related parts

Unknown whole and related parts			
Problems involving removing fractions "from the remainder" of the unknown whole. The final amount resulting from this process is known	Problems involving removing a fixed number and a fraction from the remainder of an unknown whole. The final amount resulting from this process is known	Problems involving removing from and replacing to an unknown whole. The final amount resulting from this process is known	Problems involving removing fractions and integers from an unknown whole. The distribution is known to be equitable
Lapis-lazuli	The eggs (Passing through tax guards)	Wine and water	Ducats
"*An unknown quantity of lapis-lazuli loses one-third, one-fourth, and one-fifth, and the remainder after the three-fold operation on the original quantity is twenty-seven. State what the total was, O wise one, and also tell me the loss.*"[a] Bakhshali (ca. 300 AD; see Sarasvati and Jyotishmati 1979, p. 97)	This example is included in Table 10.1	"*A man has a full glass of wine and drinks a quarter of it. He then fills the glass with water and drinks a third of it. He fills the glass again with water and drinks half of it. Finally, he fills the glass with water again and drinks it all. How much wine has he drunk each time and how much water has he drunk in total?*"[b] (Bruño 1940, p. 125; Authors' translation)	"*A sick man makes his will and his property is divided between his children. The distribution is equitable. The father dies. The eldest son has 1 ducat, and 1/10 of the remainder. To the second son, he leaves 2 ducats and 1/10 of the remainder. To the third son, he leaves 3 ducats and 1/10 of the remainder. Successively, each son has one ducat more than the other and 1/10 of the remainder. In this way, the will of the father is satisfied because he leaves an equitable will. How many ducats and how many children has he got?*" (Aurel 1552, fol. 92; Authors' translation; solution in §10.2(a))

[a] *Lapis-lazuli Solution*: "$\frac{2}{3}\frac{3}{4}\frac{4}{5}=\frac{2}{5}$; [this is what remains] $\left(1-\frac{2}{5}\right)=\frac{3}{5}$; [this is what is lost] 27 $\div\ \frac{2}{5}$ = 67½ and 67½ – 27 = 40½ is the loss."

[b] *Wine and Water Solution*: "The first time, he drinks $\frac{1}{4}$; the remainder is $\frac{3}{4}$ of the wine. The second time, he drinks $\frac{1}{3}$ of $\frac{3}{4}$, i.e. $\frac{3}{12}=\frac{1}{4}$ of wine, and $\frac{1}{4}\cdot\frac{1}{3}=\frac{1}{12}$ of water. After the second time, $1-\frac{1}{2}=\frac{1}{2}$ [he has just drunk $\frac{2}{4}=\frac{1}{2}$] of wine and as he drinks $\frac{1}{2}$, he drinks $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$ of wine in the third time, and $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$ of water. The last time, he drinks the rest of the wine, i.e. $\frac{1}{4}$, which is the last remainder, and $1-\frac{1}{4}=\frac{3}{4}$ of water. Solution: He has drunk $\frac{1}{4}$ of wine every time; in total, 1 glass of wine and $\frac{1}{12}$ of a glass of water."

amount. This is a descriptive word problem that comprises several stages in which the whole is unknown, the parts are related by an additive complement, and the distribution is equitable.

The statement for this problem, which Fibonacci called *The Bequest of a Man's Fortune*, is:

> A certain man coming to the end of his life said beforehand to his eldest child, My movable goods you will divide among you thus: you will take one bezant and one seventh of all remaining; to another child he truly said, And you will take 2 bezants and a seventh part of the remaining. And thus he said to all his children in order, giving each one more than the preceding, and by steps always a seventh of the remaining; the last child had that which was left. It happened however that each child had of his father's property equally under the aforesaid conditions. It is sought how many children there were and how much was the fortune. (Sigler 2002, p. 399)

The analytical readings for this type of problem found in historical textbooks focus on three fundamental relationships, all of which are equivalent:

(a) all the children have the same amount;
(b) the difference between what two children receive is zero; and
(c) the difference between the amount remaining before the last distribution and what the last child receives is zero. We now present three problems to illustrate these three analytical readings.

(a) **All sons receive the same amount**

An algebraic approach to this problem is found in the syncopated algebra by Aurel (1552):

Problem: A sick man makes his will and determines that his property be divided equally among his sons so that each receives the same amount. On the death of the father, the eldest son receives one ducat and $\frac{1}{10}$ of the remainder. The second son receives 2 ducats and $\frac{1}{10}$ of the remainder. The third son receives 3 ducats and $\frac{1}{10}$ of the remainder. In this way, each son receives one ducat more than the previous one plus $\frac{1}{10}$ of the remainder. In this way, the sick man's wish is fulfilled because all sons receive the same number of ducats. How many ducats did the father leave and how many children did he have?

Solution. The man left x ducats. The eldest son received 1 ducat, thus leaving $x-1$ ducats, and $\frac{1}{10}$ of the remainder is $\frac{x-1}{10}$, which added to 1 ducat means that the eldest son received $\frac{x+9}{10}$ ducats. Taking these ducats from the x initial ducats leaves $\frac{9x-9}{10}$ ducats for the remaining children. Of these ducats, the second child receives 2 ducats, leaving $\frac{9x-29}{10}$ ducats, $\frac{1}{10}$ of which is $\frac{9x-29}{100}$, which when added to the 2 ducats already received by the second son makes $\frac{9x+171}{100}$ ducats in total for the second son. Since both sons inherited the same amount, the number of ducats for the first son must equal the number of ducats for the second son. I say, therefore, that the $\frac{x+9}{10}$ ducats received by the first son are equal to the $\frac{9x+171}{100}$ received by the second son. Reducing this equation to integers (cross-multiplying) leaves $100x + 900 = 90x + 1710$. Solving this equation leaves $x = 81$, which is the number of ducats left by the father. To find how many sons he had, find how much each son received. Taking 1 ducat from 81 leaves 80, 1/10 of which is 8. If we add 1 ducat to 8 ducats, we get 9 ducats in total for the first son, which is the same number received by all sons (fol. 92; authors' translation).

As we can see in the text, Aurel uses relationship (a), which allows him to formulate the equation: $\frac{x+9}{10} = \frac{9x+171}{100}$. In the transcription of the solution to this problem, for greater clarity, we have replaced the cossic symbols with current algebraic symbols.

(b) **The difference between what two sons receive is zero**

The following example, taken from Euler (1822/1770), also uses the algebraic method but that time in Cartesian form (Descartes 1701; Descartes wrote in that book what one needs to do to translate a problem into equations and Polya (1966) rewrote the Cartesian rules to show it as rules to solve problems with algebraic signs).

Problem: A father leaves at his death several children, who share his property in the following manner: namely, the first receives a hundred pounds, and the tenth part of the remainder; the second receives two hundred pounds and the tenth part of the remainder; the third takes three pounds and the tenth part of what remains and the fourth takes four hundred pounds and the tenth part of what remains; and so on. And it is found the property has thus been divided equally among all the children. Required is how much it was, how many children there were, and how much each received?

Solution. Let us suppose that the father's total fortune amounts to z pounds and that each son will receive the same equal share, which we will call x. The number of children will therefore be $\frac{z}{x}$. Now let us solve the problem.

Sum or inheritance to be divided	Order of sons	Share for each son	Differences
z	1st	$x = 100 + \frac{z-100}{10}$	
$z - x$	2nd	$x = 200 + \frac{z-x-200}{10}$	$100 - \frac{x+100}{10} = 0$
$z - 2x$	3rd	$x = 300 + \frac{z-2x-300}{10}$	$100 - \frac{x+100}{10} = 0$
$z - 3x$	4th	$x = 400 + \frac{z-3x-400}{10}$	$100 - \frac{x+100}{10} = 0$
$z - 4x$	5th	$x = 500 + \frac{z-4x-500}{10}$	$100 - \frac{x+100}{10} = 0$
$z - 5x$	6th	$x = 600 + \frac{z-5x-600}{10}$	And so on

We have included the differences between successive shares in the final column. These are obtained from each share minus its preceding share. Since all shares are equal, this difference is equal to zero. By solving the equation $100 - \frac{x+100}{10} = 0$, we obtain $x = 900$.

We now know, therefore, that each son will receive 900 pounds. So, by taking any of the formulas from the third column we obtain $x = 100 + \frac{z-100}{10}$. Therefore, $z = 8100$ pounds and the number of sons is 8100/900 = 9 (Euler 1822/1770, p. 173).

In this case, Euler uses relationship (b) to propose the equal relationships reflected in the fourth column of the above solution.

(c) **The difference between the amount remaining before the final distribution and what the last son receives is zero**

The following quick solution is the one we previously mentioned from Fibonacci.

> For the seventh that he gave each child you keep 7 from which you subtract 1; there remains 6 and this many were his children, and the 6 multiplied by itself makes 36, and this was the number of bezants. (Sigler 2002, p. 399)

Because the explanation is regulated, i.e. based on an unexplained rule, there is nothing in the text to help us ascertain which reading analysis was used to support the solution. However, it may be possible that Fibonacci used an arithmetic method based on factorization and proportion. To explain this method, we will use symbolic language.

Let C be the final remainder of bezants and $1 \cdot n + \frac{1}{7}(C - 1 \cdot n), n \geq 2$ be the amount received by the final son. Relationship (c) is then written as:

$$C - \left(1 \cdot n + \frac{1}{7}(C - 1 \cdot n)\right) = 0, \tag{10.1}$$

where n is the number of sons. From Eq. 10.1 we obtain Eq. 10.2:

$$7C - C - 7n - n = 0 \rightarrow (7 - 1)C = (7 - 1)n \rightarrow 6 \cdot C = 6 \cdot n \tag{10.2}$$

From Eq. 10.2 we deduce that $C = n$, i.e. the number of children is equal to the final remainder and, according to Eq. 10.1, this is equal to the amount received by the final son. Therefore, since the distribution is equitable, each son receives this same amount and the inheritance (which we can call H) will be equal to n^2 (the number of sons by n bezants for each son).

All we need now is to find the value of n, which can be obtained, for example, from the equation corresponding to the amount received by the first son:

$$1 + \frac{1}{7}(n^2 - 1) = n \rightarrow n = 6. \tag{10.3}$$

From 10.3 we find that $n = 6$. We also find that this result could also have been obtained from Eq. 10.2 by using factorization and proportion as we stated before.

Another example of this method of factorization and proportion is the following problem extracted from Puig (1715).

Problem: A sick merchant makes his testament, in which he leaves a certain amount of his property to each of his sons. He determines that the eldest son will receive a sixth of his property plus 300 ducats, the second son will receive a sixth of the remainder plus 600 ducats, the third son will receive a sixth of the new remainder plus 900 ducats, and so on for the next sons, giving each one a sixth part of the new remainder plus 300 ducats more than the preceding one. On the father's death, the property was divided and it was found that all the sons received the same amount. How many sons did the father have, how much property did he leave, and how much did each son receive?

Solution. Subtract the numerator from the denominator, i.e. 1 from 6, to leave 5, which is how many sons the father left. Then multiply the 300 ducats, which is the number of ducats more that are successively given to each son, by 6, the denominator, to give 1800 ducats, which is the total amount given to each son. Multiply this amount by 5 to find the value of the property left by the father. Try it and you will find this is true (Puig 1715, p. 209; authors' translation).

Using symbolic language to follow the reasoning shown earlier, relationship (c) is written as follows:

$$C - \left(300n + \frac{1}{6}C\right) = 0, \tag{10.4}$$

where $300n + \frac{1}{6}C, n \geq 2$ is the amount received by the youngest son, n is the number of sons, and C is the final remainder.

From Eq. 10.4 we obtain:

$$6C - C = 6 \times 300n \rightarrow (6-1)C = 6 \times 300n, \tag{10.5}$$

The number of children and the amount inherited must be whole numbers. If we observe the equality and the above explanation in terms of factorization and proportion, then $\frac{6 \times 300}{6-1} = \frac{C}{n}$, which shows that $n = 6 - 1$ and that $C = 6 \times 300$. This is a possible solution and may be the idea that was applied by Fibonacci.

In conclusion, we have found three analytical readings for the same inheritance problem in which the whole is unknown, the parts are related by an additive complement, and the distribution is equitable. We have also observed two methods for solving the problem: the regulated arithmetic method and the algebraic method using syncopated algebra and Cartesian algebra.

10.3 Pilot Study

For our pilot study, we chose a similar but more intuitive statement to that of Fibonacci, which is taken from Tahan (1993), who calls it *The Raja's Pearls*.

> A rajah on his death left to his daughters a certain number of pearls with instructions that they be divided up in the following way: his eldest daughter was to have one pearl and a seventh of those that were left. His second daughter was to have two pearls and a seventh of those that were left. His third daughter was to have three pearls and a seventh of those that were left. And so on. The youngest daughters went before the judge complaining that this complicated system was extremely unfair. The judge, who as tradition has it, was skilled in

> solving problems, replied at once that the claimants were mistaken, that the proposed division was just, and that each of the daughters would receive the same number of pearls. How many pearls were there? How many daughters had the rajah? (Tahan 1993, p. 76)

The study included the following participants: 27 future high school mathematics teachers (hereafter, HSMT), two future primary school teachers (hereafter, PST), and six high school students (hereafter, HSS). With this non-homogeneous sample, our aim was to observe the participants' skills in using problem-solving methods and analytical readings at each of the levels of mathematical knowledge. The following are the results of a pilot study and provide us with just a first view. For more significant conclusions a larger sample should be used in future studies.

10.4 Results

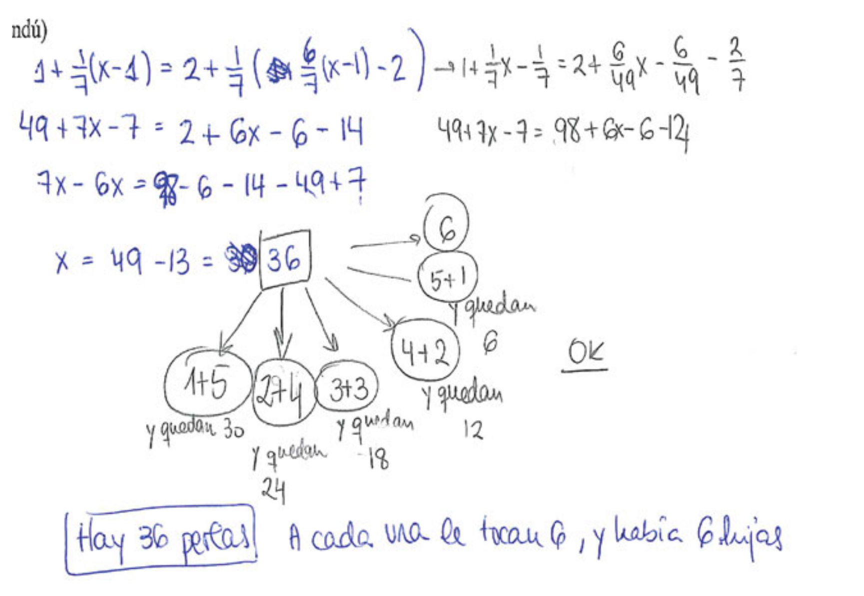
The problem was solved algebraically by 12 students (11 HSMT and 1 HSS) using the fundamental relationship (a). As an example, Fig. 10.1 shows the solution produced by one of these students. This student obtained the number of pearls received by the first daughter $\left(1 + \frac{1}{7}(x - 1)\right)$ and the number of pearls received by the second daughter, $\left(2 + \frac{1}{7}\left(\frac{6}{7}(x - 1) - 2\right)\right)$ and then formulated the equation by equating the two expressions.

Two students (HSMT) solved the problem using an arithmetic method and relationship (c). To do so, they assumed that the total number of pearls minus one had to be a multiple of 7 and then worked backwards to solve the problem by trial and error (Fig. 10.2).

The rest of the students were unable to solve the problem due to one of two reasons:

1. They were unable to translate some of the expressions in the statement into symbolic language. For example:
   (a) the characteristic expression for this type of problem: "of those that were left" (see Fig. 10.3).
   (b) the expression "a pearl and a seventh of those that were left" (see Fig. 10.4).
2. They had problems working out the fractions.
   This student started with the number of pearls corresponding to the first daughter but did not transcribe the expression "of those that were left" correctly, writing $1 + \frac{1}{7}x$, instead of $1 + \frac{1}{7}(x - 1)$. From this point on, the solution is incorrect and the errors accumulate in the subsequent steps. Although the student equates what the first daughter receives with what the second daughter receives, the problem is now impossible to solve and the student expresses this fact.

In the first line in Fig. 10.4 we can see that the student transcribes the share of the pearls that should be inherited by all the daughters. The student expresses the



$$1+\frac{1}{7}(x-1)=2+\frac{1}{7}\left(\frac{6}{7}(x-1)-2\right)$$

x = the number of pearls at the beginning

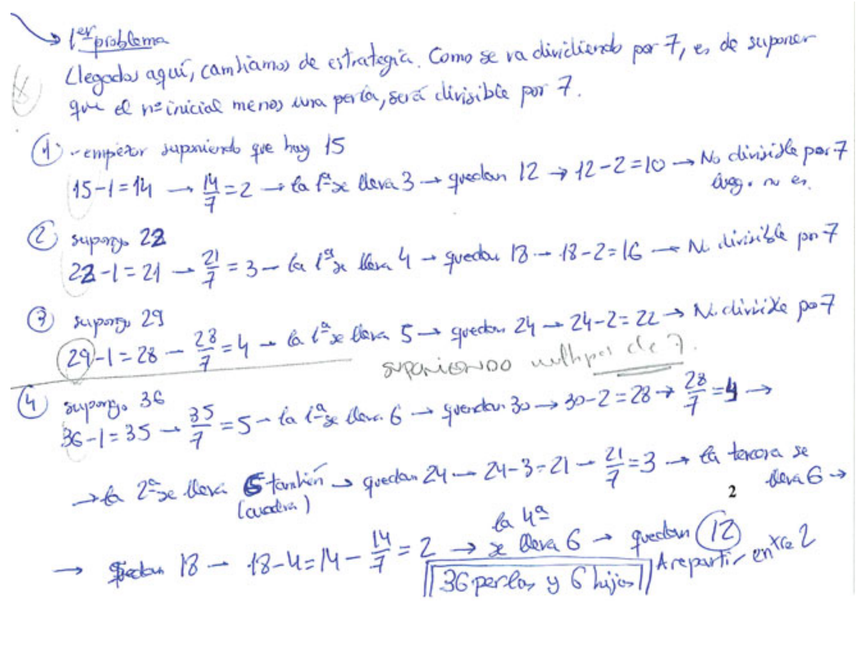
Solution. *There were 36 pearls and 6 daughters with 6 pearls each.*

Fig. 10.1 The correct solution, for which the student used the algebraic method and assumed that all the daughters received the same amount, and its translation. A literal translation is given in italics and clarifications are given in non-italics

number of pearls received by the first daughter correctly, $1+\frac{1}{7}(x-1)$, but then makes an error when expressing the number of pearls received by the second daughter. We can see how the student is unable to correctly transcribe the expression "two pearls and a seventh of those that were left" symbolically, writing $2+\frac{1}{7}\left(\frac{1}{7}(x-1)\right)$ instead of $2+\frac{1}{7}\left(\frac{6}{7}(x-1)-2\right)$. After this, the solution makes no sense.

10.5 Final Remarks

We have conducted a historical-epistemological study and a pilot study with students on descriptive word fraction inheritance problems in which the distribution is equitable, the whole is unknown, and the relationship between the parts is based on an additive complement.


1er problema
Llegados aquí, cambiamos de estrategia. Como se va dividiendo por 7, es de suponer que el nº inicial menos una perla, será divisible por 7.
(1) - empiezo suponiendo que hay 15
15 - 1 = 14 → 14/7 = 2 → la 1ª se lleva 3 → quedan 12 → 12 - 2 = 10 → No divisible por 7 luego no es.
(2) supongo 22
22 - 1 = 21 → 21/7 = 3 → la 1ª se lleva 4 → quedan 18 → 18 - 2 = 16 → No divisible por 7
(3) supongo 29
29 - 1 = 28 → 28/7 = 4 → la 1ª se lleva 5 → quedan 24 → 24 - 2 = 22 → No divisible por 7
suponiendo múltiplos de 7.
(4) supongo 36
36 - 1 = 35 → 35/7 = 5 → la 1ª se lleva 6 → quedan 30 → 30 - 2 = 28 → 28/7 = 4 →
→ la 2ª se lleva 6 también (cuadra) → quedan 24 → 24 - 3 = 21 → 21/7 = 3 → la tercera se lleva 6 →
→ quedan 18 → 18 - 4 = 14 → 14/7 = 2 → la 4ª se lleva 6 → quedan 12 A repartir entre 2
36 perlas y 6 hijos

We imagine that the total amount minus one is a number divisible by seven. Start by assuming 15 – 1 = 14. 14/7 = 2. The first daughter has 3 pearls; the remainder is 12, 12 – 2 = 10, which is not divisible by seven.

The students test with different numbers until they reach 36 pearls:

36 – 1 = 35, 37/5 = 5. The first daughter has 6 pearls, the remainder is 30.

30 – 2 = 28, 28/7 = 4. The second daughter has 6 pearls, the remainder is 24.

24 – 3 = 21, 21/7 = 3. The third daughter has 6 pearls, the remainder is 18.

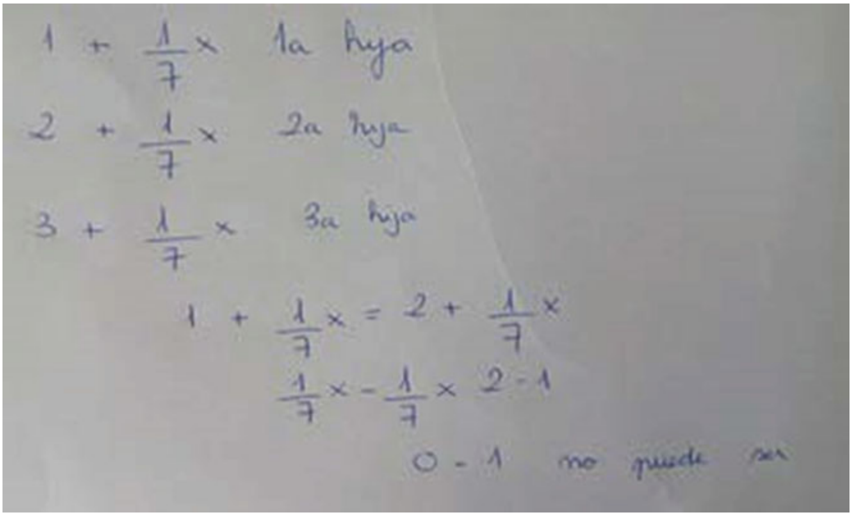
18 – 4 = 14, 14/7 = 2. The fourth daughter has 6 pearls, the remainder is 12.

12 – 5 = 7, 7/7 = 1. The fifth daughter has 6 pearls, the remainder is 6.

The sixth daughter has the final 6 pearls.

Fig. 10.2 A correct solution, for which the students used the arithmetic method and assumed that the difference between the amount remaining before the final distribution and what the last daughter receives is zero, and its translation. Literal translations are given in italics and clarifications are given in non-italics

In our historical-epistemological study, we observed the use of regulated arithmetic methods as well as syncopated algebra and Cartesian algebra. We also observed three equivalent analytical readings: when all heirs are considered to receive the same inheritance; when the difference between what two heirs receive is



$1 + \frac{1}{7}x$ *1st daughter*
$2 + \frac{1}{7}x$ *2nd daughter*
$3 + \frac{1}{7}x$ *3rd daughter*

$$1 + \frac{1}{7}x = 2 + \frac{1}{7}x$$

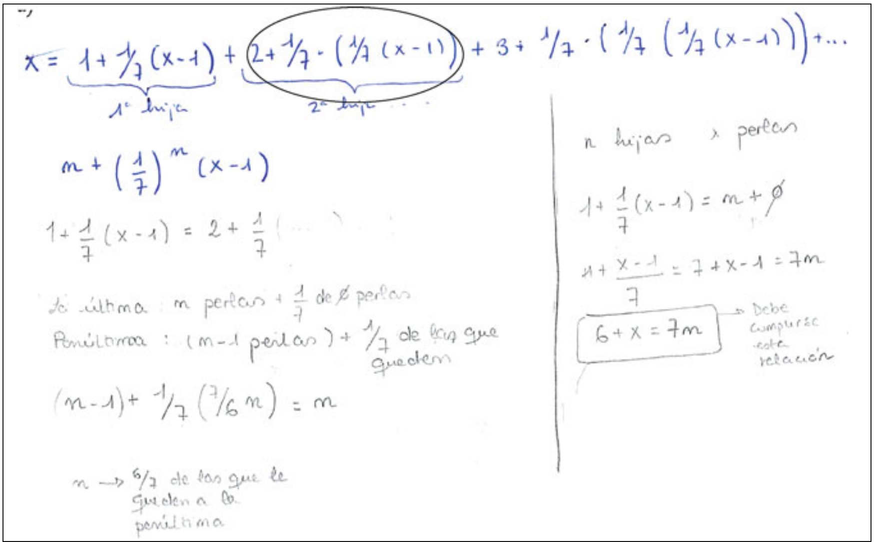
$$\frac{1}{7}x - \frac{1}{7}x = 2 - 1$$

$0 = 1$, *it is impossible.*

Fig. 10.3 An incorrect solution because of the incorrect translation of the expression "the remainder." Literal translations are given in italics and clarifications are given in non-italics

considered to be zero; and when the difference between the final remainder and the final amount inherited is considered to be zero.

The results of our pilot study were as expected. On the one hand, thirteen of the future high school teachers solved the problem correctly. Most of these students used the Cartesian method and the analytical reading that identified that both heirs would receive the same amount. We should stress that only two of these students solved the problem through arithmetical reasoning, and this was by trial and error and the inverse method. On the other hand, future primary school teachers were not able to solve the problem, though we knew from their curriculum that they had



x = the number of the pearls at the beginning

In this case, we observe two problems.

Firstly, the student calculates the first part for the first daughter correctly but not the second part for the second daughter.

Secondly, the student does not show the equation.

The student said: *The last daughter:*

n pearls+1/7 ∅ pearls [∅ meaning "nothing"]

The penultimate daughter:

(n−1 pearls) +1/7 the last remainder.

$$(n-1)+\frac{1}{7}\left(\frac{7}{6}n\right)=n$$

n→6/7 of the remaining at the penultimate.

n daughters, x pearls

$$1+\frac{1}{7}(x-1)=n+\emptyset$$

$$1+\frac{x-1}{7}=7+x-1=7n$$

$6+x=n$ *This condition must be satisfied.*

Fig. 10.4 An incorrect solution because of the incorrect translation of the expression "*a pearl and a seventh of those that were left*." A literal translation is given in italics and clarifications are given in non-italics

received training in elementary algebra during their secondary school education. However, in this research we had only two such teachers and with such a restricted sample no conclusions can be drawn. However, this situation could be an indication of what is expected if a larger sample were used in future research. We also found that many students had problems transcribing the expression "of those that were left." This highlights the constructively interfering complementary roles of the literary and symbolic languages.

Our study also showed that the Cartesian method is the one that is most used by today's students, since we found no evidence that arithmetic reasoning was used to solve the problem. The problems presented in this chapter, which seem to have been lost from the educational record, are by themselves a rich source of knowledge. As such, we found that they have helped to communicate mathematical applications, techniques, approaches, methods and reasoning, and that the historical sources illuminate solutions of the historical authors. Thus, it may be useful to teach how to use these problems as an object of study, rather than to find their solution as the by-product of another branch of learning; namely, algebra.

This information may also prove useful for research on numerical and algebraic thinking, since it provides a historical framework for studying classical problems, not as individual components of a mathematical content but as elements related to the roots of mathematics and to analysis of its evolution.

Finally, the challenge for teachers and researchers is to keep this wealth of knowledge alive, preventing it from being forgotten. They must take into account the aims of the curriculum, which considers that problem solving is a basic competence to be adapted in education in accordance to the students' background.

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