ICME-13 Monographs

Kathleen M. Clark
Tinne Hoff Kjeldsen
Sebastian Schorcht
Constantinos Tzanakis *Editors*

Mathematics, Education and History

Towards a Harmonious Partnership

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Preface

The *13th International Congress on Mathematical Education* (**ICME-13**) took place in Hamburg, Germany, in July 2016. A major part of its scientific program consisted of *Topic Study Groups* (TSGs). These are mini-conferences designed to gather a group of the Congress participants, who are interested in a particular area of Mathematics Education. During ICME-13, there were 54 TSGs in total, one of them being **TSG 25**: ***The Role of History of Mathematics in Mathematics Education***. This TSG aimed to provide a forum for participants to share their research interests and results, as well as their teaching ideas and classroom experience in connection with the integration of the *History of Mathematics* in *Mathematics Education*. Special care was taken to have presented and promoted ideas and research results of international interest, while still paying due attention to the national aspects of research and teaching experience in this area. In total, 37 papers (regular presentations, short oral communications and posters) from 16 countries all over the world were presented during this TSG's sessions.

On the initiative of ICME-13 Organizers, a post-congress monograph series was announced, directed by ICME-13 Convenor, Prof. G. Kaiser, in order to provide the possibility of producing a monograph for each TSG that will contain elaborated and extended versions of selected contributions following a strict, peer-review procedure. Each such volume is edited by members of the Organizing Team of the corresponding TSG. Along these lines, the present volume consists of 17 papers from 9 countries that were originally presented in a shorter form during the TSG 25 sessions. It is structured in five parts that roughly correspond to the main themes of TSG 25 as they have been announced before ICME-13 and can be found in its proceedings.[1]

[1] Tzanakis, C., Wang, W., Clark, K., Kjeldsen, T. H., & Schorcht, S., (2017). Topic Study Group 25: The role of history of mathematics in mathematics education. In G. Kaiser (Ed.), *Proceedings of the 13th International Congress on Mathematical Education* (pp. 491–495). Cham: Springer.

The monograph aims to serve as a valuable contribution in exploring the role of the History of Mathematics in Mathematics Education, by making available to the international educational community reports on recent developments in this field, with special attention to relevant research results since 2000, the year of publication of a corresponding comprehensive ICMI Study.[2] Much of the work done and reported in the following chapters has been inspired by this highly collective work that has been a landmark in this area, by motivating, stimulating, orienting, encouraging, and supporting further research and its applications.

It is hoped that the effort of the authors and the editors will contribute both to have the formulation of the main issues raised in this domain of Mathematics Education sharpened and to have our understanding of them deepened.

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Tallahassee, FL, USA — Kathleen M. Clark
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[2]Fauvel, J., & van Maanen, J. (Eds.) (2000). *History in Mathematics Education: The ICMI Study*, New ICMI Study Series, vol. 6, Dordrecht: Kluwer. An extensive bibliographical survey on the developments in this domain since 2000 has been conducted by the Organizing Team of TSG 25 as a useful tool for all those interested and wanting to become informed on its main issues and looking for a concise guide of the work done. This continuously updated survey is available online at *http://www.clab.edc.uoc.gr/HPM/HPMinME-TopicalStudy-18-2-16-NewsletterVersion.pdf*, and its original form appeared in Clark, K., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang, W. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting—HPM 2016* (pp.135–179). Montpellier: IREM de Montpellier.

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Chapter 1
Introduction: Integrating History and Epistemology of Mathematics in Mathematics Education

Kathleen M. Clark, Tinne Hoff Kjeldsen, Sebastian Schorcht and Constantinos Tzanakis

Abstract This chapter serves as an introduction to the seventeen chapters of this collective volume, by providing an outline of the key points that form the core and main concern of the approaches adopted towards integrating History and Epistemology of Mathematics in Mathematics Education (the HPM domain). After a brief outline of the historical development of this domain, we address the key issues that have been central to the research in this domain and the implementation of its results in educational practice. Since these issues highlight the main points also addressed in the individual contributions to this volume, our introduction ends with a brief description of each chapter.

Keywords History and pedagogy of mathematics · History of mathematics Epistemology of mathematics · Original sources · Theoretical frameworks & constructs · Interdisciplinary teaching · Teacher education · Cultures & mathematics

The fruitful and harmonious interplay among *History*, *Education,* and *Mathematics* as three different, but complementary to each other dimensions, constitutes what is potentially interesting, stimulating and beneficial for teaching and learning both Mathematics and about Mathematics:

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- ***History*** points to the non-absolute nature of human knowledge: what is acceptable as knowledge is "time-dependent" and is potentially subject to changes. In other words, *historicity* is one of its ontological characteristics.
- ***Education*** stresses the fact that humans are different in several respects depending on age, social conditions, cultural tradition, individual characteristics, etc. In this way education helps to understand these differences and to become more tolerant towards the learners' and teachers' views, preconceptions, misconceptions and possibly idiosyncratic ways of self-expression.
- ***Mathematics*** above all sciences emphasizes most the need for logical, rational and intellectual rigor and consistency in our endeavour to understand both the mental and empirical aspects of the world around us.

Any attempt to explore the multifarious interrelations of these three dimensions, explicitly, or implicitly addresses, illuminates, and/or provides insights on one or more of the following general questions:

- Which history is suitable, pertinent, and relevant to Mathematics Education?
- Which role can the History of Mathematics play in Mathematics Education?
- To what extent has the History of Mathematics been integrated in Mathematics Education (curricula, textbooks, educational aids and resource material, teacher education)?
- How can this role be evaluated and assessed and to what extent does it contribute to the teaching and learning of mathematics?

These are the key issues explicitly addressed in and/or implicitly underlying what we detail below as the *HPM* (**H**istory—**P**edagogy—**M**athematics)[1] *perspective*.

1.1 The *HPM Perspective*

Mathematics is a human intellectual enterprise with a long history and a vivid present. Thus, mathematical knowledge is determined not only by the circumstances in which it becomes a deductively structured theory, but also by the procedures that originally led or may lead to it. Learning mathematics includes not only the "polished products" of mathematical activity, but also the understanding of implicit motivations, the sense-making actions and the reflective processes of mathematicians, which aim to the construction of meaning. Teaching mathematics should give students the opportunity to "do mathematics." In other words, although the "polished products" of mathematics form the part of mathematical knowledge that is communicated, criticized (in order to be finally accepted or rejected), and serve as the basis for new work,

[1]We use this acronym as *terminus technicus*, established within the educational community as the abbreviation of the name of the *International Study Group on the relations between the* ***H****istory and* ***P****edagogy of* ***M****athematics*, known as the ***HPM*** *Group*, one of the oldest study groups affiliated to ICMI (*International Commission on Mathematical Instruction*).

the process of producing mathematical knowledge is equally important, especially from a didactical point of view. Perceiving mathematics both as a logically structured collection of intellectual products and as processes of knowledge production should be the core of the teaching of mathematics. At the same time, it should be also central to the image of mathematics communicated to the outside world.

Along these lines, putting emphasis on integrating historical and epistemological issues in mathematics teaching and learning constitutes a possible natural way for exposing mathematics in the making that may lead to a better understanding of specific parts of mathematics and to a deeper awareness of what mathematics as a discipline is. This is important for mathematics education, helping to realize that mathematics:

- is the result of contributions from many different cultures;
- has been in constant dialogue with other scientific disciplines, philosophy, the arts and technology;
- has undergone changes over time; there have been shifting views of what mathematics is;
- has constituted a constant force for stimulating and supporting scientific, technical, artistic and social development.

This helps to improve mathematics education at all levels and to realize that although mathematics is central to our modern society and a mathematically literate citizenry is essential to a country's vitality, historical and epistemological issues of mathematics are equally important. The harmony of mathematics with other intellectual and cultural pursuits also makes the subject interesting, meaningful, and worthwhile. In this wider context, history and epistemology of mathematics have an additional important role to play in providing a fuller education of the community: not being a natural science, but a formal science closer to logic—hence to philosophy—mathematics has the ability inherent in itself to connect the humanities with the sciences. Now that societies value and want young people educated in the sciences—though it is difficult determining how to get people to "move" from humanities to the sciences—integrating history and epistemology in mathematics education can make this connection visible to students. This is most important, especially today when there is much concern about the level of mathematics that students are learning and about their decreasing interest in mathematics, at a time when the need for both technical skills and a broader education is rising.

The rationale underlying this perspective has formed the core and main concern of the approaches adopted towards integrating History and Epistemology of Mathematics in Mathematics Education (what we call the ***HPM domain***). In Sect. 1.2 we briefly outline its historical development. In Sect. 1.3 we comment on the four general questions mentioned at the beginning of this introduction, in relation to some associated key issues that have been central to the research in this domain and the implementation of its results in educational practice. As described in Sect. 1.4, these key issues highlight the main points that in one or another form are addressed in the individual contributions to this volume. At the same time they constitute a concise description of the five parts into which these contributions have been divided. Therefore, Sect. 1.4 ends with a brief description of the separate chapters in each one of these five parts.

1.2 An Outline of the Historical Development of the *HPM Domain*

Integrating the history of mathematics in mathematics education has been advocated since the second half of the 19th century, when mathematicians like De Morgan, Poincaré, Klein and others explicitly supported this path and historians like Tannery and later Loria showed an active interest on the role history of mathematics can play in mathematics education. At the beginning of the 20th century, this interest was revived as a consequence of the debates on the foundations of mathematics. Later on, history became a resource for various epistemological approaches; Bachelard's historical epistemology, Piaget's genetic epistemology, and Freudenthal's phenomenological epistemology, at the same time stimulating the formulation of specific ideas and conclusions on the learning process (Barbin and Tzanakis 2014, p. 256 and references therein; Tzanakis et al. 2000, p. 202).

This interest became stronger and more competitive in the period 1960–1980 in response to the *New Math* reform, when its proponents were strongly against "a historical conception of mathematics education," whereas for its critics, history of mathematics appeared like a "therapy against dogmatism," conceiving mathematics not only as a language, but also as a human activity. In 1969, the National Council of Teachers of Mathematics (NCTM) in USA devoted its 31st Yearbook to the history of mathematics as a teaching tool (NCTM 1969) and in the 1970s a widespread international movement began to take shape: Though the First International Congress on Mathematical Education (ICME-1) that took place in Lyon in 1969 mainly consisted of talks, the structure of ICME-2 that took place in Exeter in 1972 was made more interactive through the creation of 38 Working Groups (WGs) on main themes of mathematics education. WG 11 was entitled "Relations between the history and pedagogy of mathematics." The work of this group was continued at ICME-3 that took place in Karlsruhe in 1976. Having acknowledged the importance and the widespread interest in historical-pedagogical studies in mathematics, a resolution was forwarded to secretary of the International Commission on Mathematical Instruction (ICMI) proposing setting up a system to ensure regular sessions at future ICMEs on this theme. The ICMI Executive Committee approved the affiliation of the new Study Group, originally called "*International Study Group on Relations between History and Pedagogy of Mathematics, cooperating with the International Commission on Mathematical Instruction*." The establishment of this Group—now called the *HPM Group*[2] and the announcement of its scope in 1978 (HPM Group 1978) greatly stimulated and supported interest and educational research in this area.

Thus, during the last 40 years, integrating the history of mathematics in mathematics education has evolved into a worldwide, intensively studied area of new pedagogical practices and specific research activities and a gradually increasing

[2]cf. Footnote 1.

awareness has emerged of what was described in Sect. 1.1 as the *HPM perspective* (Fasanelli and Fauvel 2006 for a historical account and references prior to 2000; for a concise outline of later developments see Barbin 2013; Barbin and Tzanakis 2014, and references therein; Furinghetti 2012).

The rising international interest in the *HPM perspective* and the various activities of the HPM Group worldwide, led to the approval by ICMI in 1996 of launching a 4-year *ICMI Study* on the relations between the history of mathematics and mathematics education. After a *Discussion Document* written by the *Study* co-chairs (Fauvel and van Maanen 1997) and a *Study Conference* in 1998, at Luminy, France, the Study culminated in the publication of a comprehensive volume written by 62 contributors working together in 11 groups (Fauvel and van Maanen 2000). This was a landmark in establishing and making more widely visible the *HPM perspective* as a research domain in the context of mathematics education and greatly stimulated and enhanced the international interest of the educational community in this area.

Already before, but more intensively after this collective volume, research and actual implementation in education have been realized and widely communicated in various ways: Through the regular organization of conferences and meetings both at an international and regional level, including *Topic Study Groups* (TSGs) at each ICME, the *ICME Satellite Meetings of the HPM Group*, the *European Summer University on* the *History and Epistemology in Mathematics Education* (ESU), since 2009, a working group at each *Congress of the European Society for Research in Mathematics Education* (CERME), etc.; launching and establishing journals and newsletters, including the online journal *Convergence*,[3] the *Bulletin of the British Society for the History of Mathematics* (BSHM Bulletin),[4] the *HPM Newsletter*,[5] etc.; the publication of numerous collective volumes,[6] special issues of journals,[7] monographs,[8] conference proceedings[9] and individual papers in scientific journals, as well as the production of a variety of resource material and educational aids, and the writing of several doctoral theses in this domain. A comprehensive annotated survey of the work done since 2000, as well as more details on the above mentioned conferences, meetings and journals, can be found in Clark et al. (2016).

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[3]http://www.maa.org/press/periodicals/convergence/about-convergence.

[4]http://www.tandfonline.com/toc/tbsh20/current.

[5]http://www.clab.edc.uoc.gr/hpm/NewsLetters.htm.

[6]E.g. Barbin (2010, 2012, 2015), Barbin and Bénard (2007), Bekken and Mosvold (2003), Biegel et al. (2008), Boero (2007), Calinger (1996), Hanna et al. (2010), Katz (2000), Katz and Tzanakis (2011), Shell-Gellasch (2008), Shell-Gellasch and Jardine (2005, 2011), Sriraman (2012) and Swetz et al. (1995).

[7]E.g. Clark and Thoo (2014), Furinghetti et al. (2007), Karam (2015), Katz et al. (2014), Siu and Tzanakis (2004) and Stedall (2010).

[8]E.g. Filloy et al. (2008), Knoebel et al. (2007), Ostermann and Wanner (2012), Schubring (2006), Shell-Gellasch and Thoo (2015) and Stein (2010).

[9]E.g. Barbin et al. (2008, 2011a, 2015), Furinghetti et al. (2006b), Horng and Lin (2000) and Radford et al. (2016).

1.3 Comments on the General Questions and Key Issues Related to the *HPM Perspective*

From what has been presented so far, it is clear that the last two decades have generated considerable research activity related to the *HPM perspective* of great variety: doing empirical research based on actual classroom implementations; designing specific teaching units; developing various kinds of teaching aids; exploring and understanding students' response to the introduction of the history of mathematics in teaching (including teacher education); designing, applying and evaluating interdisciplinary teaching; drawing and/or criticizing parallels between the historical development and learning in a modern classroom[10]; mutually profiting from theoretical constructs and conceptual frameworks developed in the context of other disciplines, especially philosophy, epistemology and cognitive science; and evaluating the effectiveness of all this in practice.

The key issues mentioned at the beginning of this chapter permeate all these activities as recurring themes that form the *leitmotif* of the HPM domain. Below a few general ideas are outlined with reference to the literature for details.

Whether the history of mathematics is appropriate, or even relevant at all to the teaching and/or learning of mathematics, is an issue that, despite the extensive research and the many insightful and sophisticated applications in the last few decades, has not reached universal acceptance even today. In fact, a number of objections against the *HPM perspective* have been raised (Furinghetti 2012, §7; Siu 2006, pp. 268–269; Tzanakis and Thomaidis 2012, §3.4; Tzanakis et al. 2000, p. 203; cf. Panasuk and Horton 2012, p. 12):

A ***Objections of an epistemological and methodological nature***

(a) *On the nature of mathematics*

1. This is not mathematics! Teach the subject first; then its history.
2. Progress in mathematics is to make difficult problems routine, so why bother to look back?
3. What really happened can be rather tortuous. Telling it as it was can confuse rather than enlighten!

(b) *On the difficulties inherent to this approach*

1. Does it really help to read original texts, which is a very difficult and time-consuming task?
2. Is it liable to breed cultural chauvinism and parochial nationalism?

[10]The old but still discussed issue of "historical parallelism"—if and to what extent "ontogenesis recapitulates (aspects of) phylogenesis"; e.g. Furinghetti and Radford (2008), Radford et al. (2000), Schubring (2006, 2011) and Thomaidis and Tzanakis (2007).

3. Students may have an erratic historical sense of the past that makes historical contextualization of mathematics impossible without having a broader education in general history.

B ***Objections of a practical and didactical nature***

(a) *The background and attitude of teachers*

1. Lack of didactical time: no time for it in class!
2. Teachers should be well educated in history: "I am not a professional historian of mathematics. How can I be sure of the exposition's accuracy?"
3. Lack of teacher training.
4. Lack of appropriate didactical and resource material.

(b) *The background and attitude of the students*

1. They regard it as history and they dislike history class!
2. They regard it just as boring as mathematics itself.
3. They do not have enough general knowledge of culture to appreciate it.

(c) *Assessment issues*

1. How can you set questions on it in a test or exam?
2. Is there any empirical evidence that students learn better when the history of mathematics is made use of in the classroom?

Each of these objections addresses one or more of the four key issues mentioned in the beginning of this chapter. Below we comment briefly on them in the light of these objections.

1.3.1 Which History Is Suitable, Pertinent and Relevant to Mathematics Education?

This has been a permanent issue of debate among historians and mathematics educators with an interest in the HPM perspective. As early as 1984 at ICME 5, d'Ambrosio stressed the need to develop three separate histories of mathematics: history as taught in schools, history as developed through the creation of mathematics, and the history of that mathematics which is used in the street and the workplace. To deal with these differences he introduced the concept of *ethnomathematics* as compared to *learned mathematics* (Booker 1986).

In fact, implicit to the objections A(a1), A(a2), A(b1) is the idea that the term "history" is the same, whether used by historians, mathematicians, or teachers and mathematics educators. That this is not so lies at the heart of Grattan-Guinness' early refutation of some of these arguments (Grattan-Guinness 1973; see also Kjeldsen 2011a, pp. 1700–1701; Kjeldsen 2011b, pp. 166–167; Kjeldsen and

Blomhøj 2012, §3, and references therein for a different recent approach). On the other hand, it is undeniable that quite often the historical development was complicated, followed a zig-zag path, led to dead ends, included notions, methods and problems that are no longer used in mathematics as we know and work with today, etc. (A(a2), A(a3)). Thus, its integration in mathematics education on the one hand is nontrivial, and on the other hand poses the question why it must be done at all. Therefore, integrating the history of mathematics in the teaching and learning of mathematics, may force history "…to serve aims not only foreign to its own but even antithetical to them" (Fried 2011, p. 13). In other words, the danger of either unacceptably simplifying or/and distorting history to serve education as still another of its tools is real by adopting what has been called a "Whig" (approach to) history, in which "…the present is the measure of the past. Hence, what one considers significant in history is precisely what leads to something deemed significant today" (Fried 2001, p. 395).

In this connection, an important step was Grattan-Guinness' distinction between what he called *History* and *Heritage* trying to clarify existing conflicts and tensions between a mathematician's and a historian's approach to mathematical knowledge, and paying due attention to the relevance of the history of mathematics to mathematics education (Grattan-Guinness 2004a, b). In the context of the *HPM perspective*, this is a distinction close to similar ones between pairs of methodological approaches; *explicit* & *implicit* use of history, *direct* & *indirect* genetic approach, *forward* & *backward* heuristics (Tzanakis et al. 2000, pp. 209–210). Hence, this distinction is potentially of great relevance to mathematics education (Rogers 2009, 2011; Tzanakis and Thomaidis 2012), serving, among other things, to contribute towards answering the recurrent question "Why and which history is appropriate to be used for educational purposes?" (Barbin 1997).

1.3.2 Which Role Can the History of Mathematics Play in Mathematics Education?

Perhaps, this issue has been discussed and analyzed most on the basis of both a priori theoretical and epistemological arguments and of empirical research. At least implicitly, such analyses try to refute some of the above objections, especially those concerning the barriers posed by the complexity of the historical development (A (a2), A(a3), A(b1)) and/or by students' predisposition to and general knowledge of both mathematics and history as taught subjects (objections (B(b1), B(b2) and B (b3), A(b3), respectively).

It is a question that has been extensively discussed from several points of view quite early (see e.g. Grattan-Guinness 1978), and especially in relation to the appropriateness and pertinence of original historical sources in mathematics education. In this context, the history of mathematics can play three mutually complementary and supplementary roles or functions[11] (Barbin 1997, 2006; Furinghetti 2012, §5; Furinghetti et al. 2006a, pp. 1286–1287; Jahnke et al. 2000, §9.1; Jankvist 2013, §7):

A *replacement* role: Replacing mathematics as usually understood (a corpus of knowledge consisting of final results, of finished and polished intellectual products; an externally given set of techniques for solving problems given from outside; school units useful for exams etc.) by something different (not only final results, but also mental processes that may lead to them; hence perception of mathematics both as a collection of well-defined and deductively organized results, and as a vivid intellectual activity).

A *reorientation* role[12]: Changing what is (supposed to be) familiar, to something unfamiliar; thus challenging the learner's and teacher's conventional perception of mathematical knowledge as something that has always been existing in its currently established form we know it, into the deeper awareness that mathematical knowledge was an invention based on a dialectical interplay between human mind's creativity and careful intelligent (mental and/or real) experimentation; an evolving human intellectual activity.

A *cultural* role: Making possible to appreciate that the development of mathematics takes place in a specific scientific, technological or societal context at a given time and place; thus appreciating mathematical knowledge as an integral part of human intellectual history in the development of society; hence, perceiving mathematics from perspectives that lie beyond its currently established boundaries as a discipline.

Considered from the point of view of the *objective* of integrating the history of mathematics in mathematics education, there are five main areas in which the *HPM perspective* could be valuable:

- The learning of mathematics;
- The development of views on the nature of mathematics and mathematical activity;
- The didactical background of teachers and their pedagogical repertoire;
- The affective predisposition towards mathematics; and
- The appreciation of mathematics as a cultural-human endeavor.

These are analyzed in detail into more specific arguments in Tzanakis et al. (2000), §7.2, describing in this way the role of history in the educational process.

From the point of view of the *way* the history of mathematics is accommodated into this perspective, a distinction was made by Jankvist (2009b; see also Jankvist and Kjeldsen 2011); namely,

[11] In the French literature they are called respectively: *fonction vicariante, dépaysante, culturelle* (see Barbin 1997).

[12] Called by some authors as *epistemological disorientation* (cf. Guillemette's paper in this volume, Chap. 3).

- History serving as a *tool* for assisting the actual learning and teaching of mathematics; and
- History serving as a *goal* in itself for the teaching and learning of the historical development of mathematics.

A similar distinction between *history for constructing mathematical objects* and *history for reflecting on the nature of mathematics as a socio-cultural process* was made by Furinghetti (2004; 2012, §5).

In this way, a finer and more insightful categorization of the possible roles of the history of mathematics in mathematics education resulted, reflecting the variety of their possible implementations in practice.

A small selection appears below.

- Fauvel and van Maanen (2000): Chaps. 7 and 8 provide a variety of examples of possible classroom implementations, for several mathematical subjects; Chap. 9 gives examples of using original sources in the classroom and specific didactical strategies to do so.
- Katz and Tzanakis (2011): Chaps. 9, 10, 13, 14, 16 and 19, and Sriraman (2012), Chaps. 2, 7 and 14 provide particular examples, most of them emphasizing empirical results of actual implementations.
- Katz et al. (2014): Rich on recent work in the *HPM domain*, including a sufficiently comprehensive old and recent bibliography in the editors' introduction and in its 12 papers. They concern theoretical issues on the history, philosophy and epistemology of mathematics, and on empirical investigations both in school and teacher education.
- Doctoral dissertations with considerable work on both the theoretical issues of the *HPM perspective* and on empirical investigation and evaluation of actual implementations: e.g. Clark (2006), Glaubitz (2010), Guevara Casanova (2015), Jankvist (2009a), Su (2005) and van Amerom (2002).

1.3.3 To What Extent Has the History of Mathematics Been Integrated in Mathematics Education?

Considerable work has been done over the last 15 years on understanding better and formulating more sharply the methodological issues raised by the integration of the history of mathematics in mathematics education, on producing appropriate educational aids of various types (B(a4)), and on designing and implementing teaching approaches to specific subjects and instructional levels in this context, with special emphasis on teacher education (B(a2), B(a3)).

According to the classification of the various approaches to integrate the history of mathematics in teaching and learning mathematics given in Tzanakis et al. (2000), there are three broad ways that may be combined (thus complementing each other), each one emphasizing a different aim:

- *To provide direct historical information*, aiming to learn history;
- *To implement a teaching approach inspired by history* (explicitly or implicitly), aiming to learn mathematics;
- *To focus on mathematics as a discipline and the cultural and social context* in which it has been evolving, aiming to develop a deeper awareness of its evolutionary character, its epistemological characteristics, its relation to other disciplines and the influence exerted by factors both intrinsic and extrinsic to it.

From a methodological point of view, Jankvist (2009b) classified the teaching and learning approaches in three categories:

- *Illumination approaches*, in which teaching and learning is supplemented by historical information of varying size and emphasis;
- *Module approaches*, in the form of instructional units devoted to history, often based on specific cases;
- *History-based approaches*, in which history shapes the sequence and the way of presentation, often without history appearing explicitly, but rather being integrated into teaching.

Approaches may vary in size and scope, according to the specific didactical aim, the mathematical subject matter, the level and orientation of the learners (A(b1), B(b3)), the available didactical time (B(a1)), and external constraints (curriculum regulations, number of learners in a classroom etc.).

The crucial role of teachers' training for effectively following the *HPM perspective* has been stressed repeatedly (e.g. Alpaslan et al. 2014, pp. 160–162; Barbin et al. 2000, p. 70; Barbin et al. 2011b; Furinghetti 2004, p. 4; Gazit 2013, §4; Horton 2011; Huntley and Flores 2010, §1). In particular, it has been advocated that beliefs and views about mathematics and its teaching may be positively affected by history (Furinghetti 1997; Jankvist 2009b; Spies and Witzke, Chap. 14 of this volume), though skepticism has been also expressed in this connection (see Furinghetti 2007; Philippou and Christou 1998 and references therein). Similarly, it has been stressed that availability of appropriate didactical resources is equally crucial (e.g. Panasuk and Horton 2012, p. 16; Pengelley 2011, pp. 3–4; Percival 2004, p. iii; Tzanakis et al. 2000, pp. 212–213).

Though accommodating the *HPM perspective* in an essential way into the official national curricula does not seem to have attained wide applicability,[13] intensive efforts have been made to train teachers and explore changes in their attitude and/or teaching, and to design, produce and make available didactically appropriate resources, at the same time increasing the teachers' interest and participation in national and international events related to the HPM perspective. Some indicative examples include:

[13]One exception is Denmark (see Jankvist 2013, §3; Kjeldsen 2011b, §15.2; Niss and Højgaard 2011, Chap. 4). For a recent discussion and survey see Boyé et al. (2011).

Teacher education: Arcavi and Isoda (2007), Bruckheimer and Arcavi (2000), Burns (2010), Clark (2011), Liu (2003), Mosvold et al. (2014), Povey (2014), Smestad (2011) and Waldegg (2004).

Resource material and educational aids: The need for didactical resources along the lines of the *HPM perspective* has been satisfied to a considerable extent in the last 15 years, so that such material is available nowadays in a variety of forms. Some examples:

- A wide spectrum of resource material can be found in the online journal *Convergence*[14]; e.g. see the review of some examples in Beery (2015); or Clark (2009) for the detailed description of a teaching module.
- Katz and Michalowicz (2005): didactical source material in 11 modules.
- Siu (2007): a useful survey of the literature and available resources.
- Pengelley et al. (2009): Didactical material for discrete mathematics based on original sources.
- Pengelley and Laubenbacher (2014): A website with many references to published work and material available online.
- Barnett et al. (2014): Extensive information on teaching with historical sources and bibliography on its theoretical framework and available resource material.
- Books with material that can be used directly and/or inspire teaching; e.g. Barbin (2015), Demattè (2006), Shell-Gellasch and Thoo (2015) and Stein (2010).

1.3.4 How Can This Role Be Evaluated and Assessed and to What Extent It Contributes to Amend the Teaching and Learning of Mathematics?

Evaluating the effectiveness of the *HPM perspective* on improving mathematics education from the point of view of both teaching and learning mathematics is an issue clearly stressed in objections B(c1), B(c2). Those who oppose, or are reserved about the role of the history of mathematics in mathematics education rightly ask for sufficient empirical evidence about its effectiveness. Quite early it has become clear that this is a key issue (e.g. Jankvist 2007; Siu and Tzanakis 2004, p. 3), and that any such evaluation is a complex process relying more on qualitative than quantitative methodologies: to consider changes induced in teachers' own perception of mathematics; to examine how this may influence the way they teach mathematics; and to explore if and in which ways this affects students' perception and understanding of mathematics (Barbin et al. 2000, particularly Sects. 3.1 and 3.2).

Additionally, any such evaluation goes together with actual classroom implementations, in school teaching and teacher pre- and in-service education. Therefore, many, if not all, works referring to such implementations necessarily address

[14]http://www.maa.org/publications/periodicals/convergence.

evaluation issues about the effectiveness of the approach considered in each case (e.g. those listed in Sects. 1.3.2 and 1.3.3 above).

This is an area of currently active research (see e.g. Bütüner 2015a, b; Kaye 2008; Leng 2006) with no established results of universal acceptance yet, because of several reasons:

(a) Such a complex process is not expected to lead to spectacular changes in a short time interval. Preconceptions, misconceptions, predispositions either of the teachers or the students are too stable to be easily and/or quickly modified. Therefore, one should expect to see such changes after a considerable time exposure to an approach adopting the *HPM perspective*; often this time is not available.
(b) There is strong dependence on the instructional level (primary, secondary, tertiary) and orientation of the students, teacher-students included (science or humanities; elementary or secondary school teachers etc.), as well as, on their entire previous educational path, which has determined their knowledge of, attitude towards, and preconceptions about mathematics.
(c) There is influence by external "technical" factors that may favor, enhance, impede, or prevent the implementation of an approach based on the *HPM perspective*: the curriculum and the corresponding regulations; the number of students in the class (e.g. a small number facilitates group work and a teacher's effective supervision); the structure of the educational system (e.g. in a centralized system, teachers have less freedom, hence fewer possibilities to apply an innovative teaching approach not necessarily falling into the official curriculum regulations).
(d) Not all mathematical subjects are equally accessible or appropriate to be taught and/or learned in a historically motivated or driven context.

All this constitutes a complex network of factors interfering with each other, so that empirical findings of different research works are not easily comparable. Therefore, despite many thoughtfully designed and carefully applied empirical investigations, much work is still needed to evaluate the effectiveness of the role of the history of mathematics in mathematics education in an undisputable way.

1.4 Structure and Content of the Present Volume

What has been presented in Sect. 1.3 reveals that in conducting research within the HPM domain and implementing its results in educational practice, the following issues have been central:

– To perform systematically, carefully designed and applied *empirical research*, in order to examine in detail and evaluate convincingly the effectiveness of the integration of the history and epistemology in mathematics education on improving the teaching and learning of mathematics, as well as students and teachers' awareness of mathematics as a discipline and their disposition towards it.

- To put emphasis on *pre- and in-service teacher education* as a necessary prerequisite for the integration of the history and epistemology in mathematics education to be possible at all.
- To design, produce, make available and disseminate a variety of *didactical material* in the form of anthologies of original sources, annotated bibliography, description of teaching sequences or modules as a source of inspiration and/or as generic examples for classroom implementation, educational aids of various types, appropriate websites etc.
- To acquire a deeper understanding of theoretical ideas put forward in integrating history and epistemology in mathematics education and to carefully develop them into coherent *theoretical frameworks and methodological schemes* that will serve as a foundation for further research and applications.

The contributions to this volume are directly related to one or more of these central issues:

Seen as a whole, they cover all levels of education; from primary school, to tertiary education, with special focus on pre- and in-service teacher education. Additionally, in one form or another they refer to and/or are based on empirical research, in order to support, illuminate, clarify or evaluate key issues, main questions, or conjectured theses raised by the authors or in the literature on the basis of historical-epistemological or didactical-cognitive arguments.

Seen individually, each contribution's main focus and content falls in one of the five areas as detailed below, though of course, these areas are strongly interrelated:

I. Theoretical and/or conceptual frameworks for integrating history and epistemology of mathematics in mathematics education;
II. Courses and/or didactical material: Design, implementation and evaluation;
III. Empirical investigations on implementing history and epistemology in mathematics education;
IV. Original historical sources in teaching and learning of and about mathematics;
V. History and epistemology of mathematics: Interdisciplinary teaching and socio-cultural aspects.

These areas correspond to the five parts in which the remaining 17 chapters of this volume are divided. In the rest of this section, the focus of each chapter is briefly outlined, so that the reader gets a useful overview of the content of each part of the book.

Part I consists of three papers, which address theoretical issues related to the integration of the history and epistemology in mathematics education, often in connection with relevant experimental evidence and results from empirical studies.

Having as a starting point the concept of a "cognitive artifact" and the thesis that mathematical activities strongly depend on such artifacts, **M. W. Johansen** and **T. H. Kjeldsen** from Denmark discuss how case studies on exploring the historical development of important cognitive artifacts can be useful resources in mathematics education. To this end, they analyze three historical examples and develop what they call an inquiry-reflective learning environment in mathematics for the use of original sources, illustrating it by means of one of the examples.

In his paper, **D. Guillemette** from Canada attempts to clarify, enrich and deepen the role of the history of mathematics in mathematics education as an "epistemological dis-orientation" through an empirical study with pre-service (secondary school) teachers, who followed a course on the history of mathematics based on reading several selected historical texts and performing various classroom activities.

K. Clark from the USA and **I. Witzke, H. Struve,** and **G. Stoffels** from Germany report on the intensive seminar they have designed and taught to undergraduate mathematics students, which addresses the well-known problem of the transition from school to university, making aware of concept developments in the history of mathematics. With the main hypothesis that the passage from an empirical-object to a formal-abstract belief system of mathematics is a crucial obstacle for this transition and that this passage played a similar role in history, they proceed to analyze a variety of empirical data from their seminar.

The four papers of ***Part II*** refer to the design, implementation, and/or evaluation of courses and didactical material (including textbooks), in which historical and epistemological aspects have a dominant role.

K. Danielsen, **E. Gertz,** and **H. K. Sørensen** from Denmark present the development of a multi-purpose teaching material centered on original historical sources and based on an appropriately designed template, addressed to mathematics teachers of the upper-secondary school. They also report on the actual development of such material that was produced by dedicated teachers under the authors' guidance and supervision.

P. Baggett and **A. Ehrenfeucht** from the USA describe the graduate course they offer, in which students study original texts that have influenced the development of mathematics education, and prepare major projects presented in a mini conference. The authors give details about the content of the course, the students' assignments and the resources needed, as well as information about its actual implementation and evaluation.

R. Kaenders and **Y. Weiss** from Germany describe some didactical material they have developed and used with their students, in which geometrical developments and associated algebraic formulations are compared and contrasted, and the role this may play in the acquisition of a deeper conceptual understanding of the mathematics involved. The paper focuses on four projects from four different areas of mathematics, embedded in different cultural traditions.

This part ends with a paper from Germany by **S. Schorcht**, in which a great number of examples related to the history of mathematics in German mathematics textbooks for the 1st to the 7th grade are analyzed and classified in five different types of historical mathematical tasks. The theoretical and methodological framework for this classification is presented, illustrative examples are given, and the possibility to use this classification to create new tasks is further discussed.

Part III consists of three papers on particular empirical studies about the use of the history of mathematics in teaching specific mathematical subjects and/or developing students' awareness about the nature of mathematics.

Two of the papers come from Spain and concern empirical research on teaching elementary mathematics by benefiting from the use of appropriately chosen historical sources: Having as a starting point that teaching school algebra should start in relation with geometry and problem-solving, and that going from arithmetic to algebra by skipping geometry is a pedagogically and historically incorrect procedure, **I. Guevara-Casanova** and **C. Burgués-Flamarich** in their paper report on how this point of view was implemented in the classroom. More specifically, they describe their teaching approach and the results obtained using excerpts from the Chinese "*Nine Chapters*…" and Al Khwãrizmî's treatise on algebra. **M. T. Sanz** and **B. Gómez** used historically important—but nowadays forgotten—descriptive problems on fractions, in their empirical study with prospective (elementary and secondary school) mathematics teachers, as well as high school students. The problems were chosen from a variety of primary and secondary sources and the various solution methods adopted by the students are classified, aiming to assess to what extent and in which ways such problems contribute positively to the teaching of elementary mathematics.

Based on Sfard's framework of thinking as communicating and Kjeldsen's theoretical arguments on the role of the history of mathematics in illuminating meta-discursive rules of mathematical discourse, **A. Bernardes** and **T. Roque** from Brazil describe in their paper their empirical research in the context of two teaching modules on matrix algebra they designed. Specifically, they investigated how original historical sources encourage reflections about such rules and what impact these reflections may have on students' conceptions about matrices and determinants.

The significance of using original historical sources in the classroom is a subject that attracted the interest of researchers, teachers, and educators quite early, and has played a major role in the development of the HPM domain. Since then, a lot of work has been done to produce valuable resource material and to assess the role of such sources on improving the learning of mathematics. In ***Part IV***, four papers along these lines are included, reporting on research done from the elementary school to the university level.

V. Tsiapou from Greece reports on a teaching experiment with elementary school students (6th grade), using excerpts of texts from ancient Chinese mathematics ("*The Nine Chapters* ..." and Liu Hui's commentaries on it) about the calculation of the area of a circle, with a two-fold aim: to support learning a mathematical subject, and to develop adequate views about mathematics and its history.

In her paper, **C. de Varent** from France presents an empirical study with 10th grade high school students, in which an original cuneiform text from Mesopotamia was used in order to give insights and to acquire a deeper understanding of the area concept. This study also points to and stresses the various methodological and epistemological problems encountered in didactical interventions of this kind.

In their paper, **S. Spies** and **I. Witzke** from Germany consider the role of historical material to reveal the variety of individual beliefs of (pre-service) teacher-students about a particular mathematical domain. Specifically, they report on their empirical research with students—prospective mathematics teachers—in the context of a seminar on the didactics of high school calculus, using original historical sources and basing their analysis on a particular epistemologically-driven classification of the individual beliefs into six categories (or orientations as they call them).

Finally, **J. Lodder** from the USA describes the content of a university teaching module on “Networks and Spanning Trees;” one out of several curricular modules on discrete mathematics and computer science he and colleagues have developed. The author presents the pioneering research papers on which this module is based, and reports on the positive outcome of its implementation in the context of courses on combinatorics for mathematics students and on algorithm design for computer science students.

Finally, ***Part V*** includes three papers. They focus on the integration of historical and socio-cultural aspects of mathematics in the context of interdisciplinary teaching, and report on relevant empirical findings, while at the same time they rest heavily on the use of original documents.

S. Schöneburg-Lehnert from Germany reports on an interdisciplinary school project, in which students of grades 8–11 explored the interesting though elementary geometrical substratum of the “Pantograph,” a mechanical instrument invented by C. Scheiner in the early 17th century as an aid to copy pictures. The project was based on Scheiner’s original Latin text. The author describes the way it can be implemented in different grades as a stimulating module that interconnects mathematical, linguistic and handicraft issues, while giving detailed empirical data on its actual implementation.

P. Kotarinou, C. Stathopoulou, and **L. Gana** from Greece present an interdisciplinary teaching project on Hellenistic Alexandria’s mathematics, designed and implemented in a theatrical setting with/for 10th grade school students. Though teaching focused on Eratosthenes’ measurement of the earth’s circumference, it motivated discussion and exploration of other important mathematical issues. Thus, the students were involved in experiential activities, also benefiting from their own reading about history, mathematics, and mathematical literature under the supervision of their mathematics and Greek language teachers and the school librarian, who collaborated closely all along this project.

In the last chapter, **S. Lawrence** from the UK describes a project, which resulted by elaborating on the relation between mathematics and painting as a stimulating source of inspiration for teachers. The paper describes how this project was designed on the basis of famous paintings of the Renaissance, as well as pictures in medieval copies of manuscripts of Adelard of Bath. Its focus is on the analysis of the rationale underlying this project, while presenting data from the positive response of mathematics teachers after its actual implementation.

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Part I
Theoretical and/or Conceptual Frameworks for Integrating History and Epistemology of Mathematics in Mathematics Education

Chapter 2
Inquiry-Reflective Learning Environments and the Use of the History of Artifacts as a Resource in Mathematics Education

Mikkel Willum Johansen and Tinne Hoff Kjeldsen

Abstract In this paper we explore the possibility of using the historical development of cognitive artifacts as a resource in mathematics education. We present three examples where the introduction of new artifacts has played a role in the development of a mathematical theory. Furthermore, we present a methodological approach for using original sources in the classroom. The creation of an inquiry-reflective learning environment in mathematics is a significant element of this methodology. It functions as a mediating link between the theoretical analysis of sources from the past and a classroom practice where the students are invited into the workplace of past mathematicians through history. We illustrate our methodology by applying it to the use of artifacts in original sources, hereby introducing a first version of such an inquiry-reflective learning environment in mathematics through history.

Keywords Cognitive artifacts · Inquiry-based learning · Original sources
Mathematics education · History of mathematics · Reflective learning environment

2.1 Introduction

Today there is a growing awareness that human cognition cannot be understood solely in terms of mental activity taking place in isolated brains, but also involves interaction both with other humans and with various cognitive tools. Some of these ideas can be traced back to C. S. Peirce, who considered perception and

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manipulation of external tokens of iconic diagrams to play a central role in reasoning processes. In a more modern, and more general, setting the dependency on external resources has been explored within the theoretical framework of distributed cognition. Here the basic unit of analysis is taken to be the cognitive system needed to perform a specific task such as navigating a ship (Hutchins 1995; Zhang 2006). In this perspective cognition is not confined to processes taking place within a single human brain, but it can be distributed over several actors and over external objects and instruments.

A central concept in the theory of distributed cognition is the concept of cognitive artifacts, that is artifacts developed with the purpose of partaking in cognitive systems and processes. Because cognitive artifacts are artifacts, they can be seen as a culturally created resource that can be utilized by the cognitive agents that have access to the artifacts. The abacus for instance is a cognitive artifact. It was developed with a specific purpose in mind, but each individual mathematics user does not have to develop it anew as long as she has access to the culturally created cognitive resource consisting in the artifact and the instructions for its proper use.

Mathematical activities are strongly dependent on cognitive artifacts. Although humans seem to have an inborn capacity to perform certain mathematical tasks, such as comparing the relative size of two sets, this capacity is extremely limited, and consequently the distribution of cognitive tasks over external artifacts seems to be an important, if not necessary, prerequisite for most mathematical activities. Mathematical cognition is thus culturally situated in the sense that the tasks and activities a mathematician is able to perform at any point in history depends, among other things, on the cultural resources made up by the cognitive artifacts that are available to her. Focusing on this perspective of mathematical activities, the history of the development of cognitive artifacts that have facilitated the development of certain ideas and the performance of certain mathematical tasks becomes an essential part of the history of mathematics. Modern mathematics is performed in the context made up of both past mathematicians' ideas and thoughts and the cognitive artifacts they developed and have left for us to use.

The crucial role played by cognitive artifacts in the development of mathematics has been explored in several historical case studies (Carter 2010; Johansen and Kjeldsen 2015; Johansen and Misfeldt 2015; Steensen and Johansen 2016). In these case studies, it is shown that cognitive artifacts in the form of representational systems historically have played a vital part in the generalization of mathematical theories to new domains, e.g. the generalization of the operation of exponentiation from the domain of natural numbers to real numbers. Furthermore, the examples show that in some cases the content and direction of development of certain mathematical theories and of new theory development are highly influenced by the development and choice of specific representational tools.

The use of historical case studies and original sources is already a well-established theme in the mathematics education research.[1] With this chapter

[1]For an account of the development since 2000, see Clark et al. (2016).

we wish to add to this literature by discussing how case studies that explore the introduction of cognitive artifacts can be used as a resource in mathematics education. In some cases, the historical development of an artifact turned out to be a development towards the artifact currently in use, that is, towards a modern representational notation, and in other cases cognitive artifacts different from the modern notation constituted a crucial step in the development of a mathematical theory. In cases of the first type students can be invited to critically examine the reasons and motivations leading to the introduction of a particular notation. We claim that such experiences may serve to consolidate students' understanding of the artifact in question and to increase students' awareness of its importance. In cases of the second type, the encounter with the historical artifact will give the students the opportunity to experience something that seems foreign to what they already know, feel familiar with, consider as well-established or take for granted (Barbin 2011). We claim that such experiences may help the students to expand their horizon of understanding and increase their awareness of the function and importance of the cognitive artifacts they normally use (Gadamer 1975, pp. 306, 374). We have chosen three examples that illustrate various aspects of both of these two types of cases. In the first example, we explore the transition from a foreign to a familiar artifact, in the second we explore an unfamiliar artifact that was used in the development of a familiar theory and in the third example we explore the introduction of and motivation behind a familiar and taken-for-granted artifact.

Benefits of using original sources for the teaching and learning of mathematics do not materialize automatically, but need to be promoted. We have developed a methodological approach for using original sources for the teaching and learning of mathematics within an epistemological framework, where we distinguish between three types of considerations that go into the design and implementation of the learning activity; we describe these in more detail later in this chapter. The creation of an inquiry-reflective learning environment *in* mathematics is a significant element of this methodology. It functions as a mediating link between the theoretical analysis of sources from the past and a classroom practice where the students are invited into the workplace of past mathematicians through history. We introduce a first version of such an inquiry-reflective learning environment *in* mathematics for the use of artifacts in original sources, concretized through one of our three examples.

We have organized the paper in the following way: In Sect. 2.2 we present our three examples and indicate what aspects of uses of artifacts each example can be used to illustrate. In Sect. 2.3 we introduce the methodological approach we have developed for using original sources in the teaching and learning of mathematics. In Sect. 2.4 we illustrate the method by applying it to one of our three examples. In Sect. 2.5 we conclude the paper and give ideas for future research.

2.2 Cognitive Artifacts in the History of Mathematics

In the following we will present and discuss three examples where the development of suitable artifacts played a scaffolding role in the development of mathematics. The first example is the introduction of complex numbers by Girolamo Cardano[2] (see also Johansen and Kjeldsen 2015; Steensen and Johansen 2016). The second example is the formulation of the law of exponents by al-Samaw'al, and the third example is John Wallis' introduction of the number line and his attempt to give negative and complex numbers a geometric interpretation. The presentation of the examples will be rather brief and is mainly meant to illustrate the various ways in which artifacts have been used to explore mathematical tasks, develop mathematical notation and grasp mathematical concepts, and how working with these artifacts may support educational goals.

In *Ars Magna* (1545) Cardano considered several problems of the following type: To divide a given number into two parts such that the product of the parts is equal to another given number. From a modern point of view these problems can be considered as special cases of quadratic equations. Problems of this type have been known since antiquity along with algorithms for constructing solutions to them geometrically (such as Proposition VI.28 in Euclid's *The Elements*). As was then well known, the algorithm is limited since it presupposes that the square of half of the given line is greater than or equal to the given product.[3] In *Ars Magna* however, Cardano deliberately considered a case where the algorithm breaks down. In this case, a given line of length 10 must be divided into two parts such that their product is 40. Although Cardano was well aware that the "case is impossible" (Cardano 2007, p. 219), he set out to apply the standard algorithm by constructing the square of half of the given line and represented the result geometrically.

The next step in the geometric algorithm would be to subtract an area equal to the given product from the constructed square and to find the square root of the resulting area. As negative areas cannot be constructed geometrically, the algorithm cannot be completed. Cardano however responded by replacing the geometric representation with abstract algebraic symbols and then carried through with the rest of the steps in the algorithm interpreted not as geometric constructions, but as algebraic operations. This led him to the solutions $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. As the sum of these two 'numbers' is 10 and their product 40, they 'solve' the problem.

In the solution of the problem, Cardano changed cognitive artifacts. Instead of using geometric constructions (and the large catalog of standard techniques developed for using these artifacts) he began to use abstract algebraic symbols. The example illustrates how the choice of artifacts has consequences for the kind of problems one can handle and 'solve', and it illustrates the trade-offs associated with

[2]We are indebted to Professor Jesper Lützen, University of Copenhagen, for bringing this example to our attention in his talk at the Second Joint International Meeting of the Israel Mathematical Union and the American Mathematical Society, IMU-AMS in Tel Aviv, Israel, June 16–19, 2014.

[3]Or, in modern terms, that the discriminant of the resulting quadratic equation is non-negative.

Table 2.1 Excerpt of al-Samaw'al's table, adapted and slightly modernized from Berggren (1986, p. 114)

	x^{-6}	x^{-5}	x^{-4}	x^{-3}	x^{-2}	x^{-1}	x^0	x^1	x^2	x^3	x^4	x^5	x^6
	F pcc	E pmc	D pmm	C pc	B pm	A pt	0 unit	A t	B M	C C	D mm	E mc	F cc
$x = 2$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64
$x = 3$	$\frac{1}{729}$	$\frac{1}{243}$	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81	243	729

The columns are enumerated using Arabic alphabetic numerals (A = 1 etc.) but the medieval notation is abbreviated such that e.g. "mal cube" (x^5) is abbreviated by "mc" and "part of mal cube" (x^{-5}) is abbreviated "pmc". The modern notation in the top row and leftmost column was added by us. The table reproduced by Berggren has 19 columns (from x^{-9} to x^9 in modern notation)

the choice of artifacts. In the geometric representation, the problem cannot be solved. By moving to an algebraic setting, other techniques and algorithms become available and we can find 'solutions'. However, we have to give up intuitions and understandings that are specific to the conceptual framework that we already know and operate within. The algorithm becomes a set of rules we simply follow, and the solutions we find are in a sense meaningless (in the existing conceptual framework), as Cardano made explicit. So why do they have any value to us? The example makes a good starting point for discussing the different affordances various cognitive artifacts available to us may have. Why do we prefer to use geometric constructions in some cases and purely (meaningless in a concrete sense) algebraic manipulations in others? For more advanced students the example furthermore could be used to spark a discussion about the reasons for accepting a new mathematical entity. Here, the contrast between the use of complex numbers as the solution to a problem and the use of complex numbers as part of the solution process (for instance in solving cubic equations) could be introduced (see e.g. Bagni 2011, p. 51 for ideas and empirical results).

As our second example we will look at an episode in the development of the modern theory of exponentiation. As a background for the example it should be noted that in Greek mathematics, the theory of powers was limited due to the demand for geometric interpretation. Consequently, only the exponents 2 and 3, which could be interpreted as squares and cubes respectively, were considered. The only exception to this rule was Diophantus' development of a semi-symbolic notation that allowed him to consider exponents of (small) natural numbers in some settings (Heath 1921, p. 458; Katz 1998, p. 173; Thomaidis 2005). A major step towards the modern theory of exponentiation was taken in the 12th century by the Arab mathematician al-Samaw'al. Al-Samaw'al's theoretical innovation hinged on the introduction of a table of the powers of given numbers, where exponents are represented by the positions of the columns (Table 2.1).[4]

[4]The table comes from al-Samaw'al's work *Al-Bahir fi'l-Hisab* (*The Shining Book on Calculation*). As far as we know the book is not translated into English in its entirety. We here consider the original source to consist of the table and of al-Samaw'al's explanation of the law of exponentials given below.

This table allows several steps towards a modern understanding of powers. It illustrates the connection between powers with positive and negative exponents and makes it clear that any integer can be used as an exponent; one can always add more columns to the table in both directions.

The main achievement of al-Samaw'al in this connection however, was to formulate the rule of exponents. In modern notation the rule states that the product of two powers x^m and x^n with the same basis is given by x^{m+n}. Al-Samaw'al states the rule in terms of movement in the table in the following way:

> The distance of the order of the product of the two factors from the order of one of the two factors is equal to the distance of the order of the other factor from the unit. If the factors are in different directions then we count (the distance) from the order of the first factor towards the unit; but if they are in the same direction, we count away from it. (as cited in Berggren 1986, p. 114)

The word "order" here translates into "power" and is represented by the columns of the table. Al-Samaw'al thus tells us that the distance, i.e. the number of columns between the cell holding the product and the cell holding one of the factors is equal to the distance between the cell holding the other factor and the unit column. Or in modern terms (using both the modern symbolism and the modern notion of negative numbers), if x^m is located in the mth column from the unit and x^n is located in the nth column from the unit, we can find the power of the product $x^m x^n$ by going to the mth column and take n steps to the right if n is positive or n steps to the left if n is negative. From this it follows in modern terms that $x^m x^n$ equals x^{m+n}.

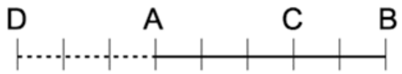
This example illustrates how the introduction of a new artifact—here, al-Samaw'al's table—can constitute an important step in the development of a mathematical theory (see Johansen and Misfeldt 2015). In an educational context the example may serve several important purposes.

Because the table differs from the modern notation and also has the 'physical' operation of 'moving' between the columns and counting 'distances', the table may expand the students' horizon of understanding and allow them to see well-known modern notation p^q in a different perspective (see Barbin 2011; Gadamer 1975, p. 306). Students may realize that the modern notation is not something given, but an invention—an artifact—just as much as al-Samaw'al's table and they may ask for its origin. The encounter with al-Samaw'al's table may also inspire the students to look for contrasts between the two notations. For instance, al-Samaw'al's notation allows only integer values of the exponents as there is no obvious way to fit, say, fractional exponents in between the columns of the table, whereas the modern notation p^q will allow q to be any type of number. This contrast may inspire questions about the modern notation. The modern notation is very permissive, but does the fact that we can express a certain power with our notation automatically make it meaningful or well-defined? Are fractional exponents for instance meaningful for all base numbers p, and what about irrational exponents?

Furthermore, this example invites for an inductive approach to teaching and learning the rule for multiplying exponentials. The artifact allows students to explore calculating with exponentials by moving back and forth in the table, and by playing around with the artifact, students can discover a pattern that can lead them to formulate the rule of exponents by themselves. This approach has successfully been tried out in a high school class by a Master's student supervised by one of us (see Svensson 2016). In this teaching experiment, two high school classes were taught the same mathematical content (involving calculation with exponents) using respectively an inductive and a deductive approach. As part of the inductive approach, students were presented with a table similar to the one used by al-Samaw'al, and by exploring the properties of this table and by expanding the table with more rows the students were able to discover the rule for multiplying exponentials as well as the related rule concerning division with powers (Svensson 2016, pp. 153–154). The study showed that student motivation as well as retention (measured six months after the teaching experiment) was better for the students who had been taught following the inductive approach (Svensson 2016, p. 135). (It should be noted that al-Samawar'al's table only constituted part of the inductive course.)

As our third example we will look at John Wallis' defense of negative and imaginary numbers from *A Treatise of Algebra, Both Historical and Practical*, chapters LXVI and LXVII.[5] In the text, Wallis begins by presenting the common idea that negative numbers are impossible because it is "Impossible, that any Quantity (though not a Supposed Square) can be *Negative* because it is not possible that any *Magnitude* can be *Less than Nothing*, or any *Number Fewer than None*" (Wallis 1685, p. 264). Wallis, however, claims that this position is wrong. When rightly understood, negative numbers denote real physical quantities and are thus not only meaningless algebraic signs. In order to bring about this change in conception Wallis (1685) asks us to consider the following thought experiment:

> Supposing a man to have advanced or moved forward, (from A to B,) 5 Yards; and then to retreat (from B to C) 2 Yards: If it be asked, how much he had Advanced (upon the whole march) when at C? or how many Yards he is now Forwarder than when he was at A? I find (by Subducting 2 from 5,) that he is Advanced 3 Yards. (Because +5 − 2 = +3.)



> But if, having Advanced 5 Yards to B, he thence Retreat 8 Yards to D; and it be then asked, How much he is Advanced when at D, or how much Forwarder than when he was at A: I say -3 Yards. (Because +5 – 8 = -3.) That is to say, he is advanced 3 Yards less than nothing. […] And consequently -3 does as truly design the point D; as +3 designed the Point C. Not forward, as was supposed, but backwards, from A. (p. 265; figure redrawn from *ibid*)

What Wallis in effect has done here is to introduce the number line. Of course, numbers and lengths had been associated before Wallis, for instance, through the use of measuring sticks, but what Wallis is doing here seems to be something far more complicated. Going through the physical activity of moving forwards and

[5]The relevant part of the text is also available in Smith (1959), pp. 46–54.

backwards which makes perfect sense in everyday life, the algebraic operations of subtraction and addition are mapped to movements left and right on the line. By introducing a point of origin (A) on the line, we would say that he has established a connection between the domain of numbers and the domain of geometry so that numbers can be understood simultaneously as algebraic entities and as points on a line. In other words, we could interpret his construction as the creation of what is known in cognitive science to be a *conceptual blend* (Lakoff and Núñez 2000, p. 278).

From an educational point of view, this example is interesting for several reasons. Simply realizing that a familiar artifact such as the number line is not something naturally given, but an artifact that was introduced and used by a specific person (Wallis) at a specific time (late 17th century) in order to grasp “negative squares and imaginary roots” which had shown up in connection with “the Solution of some Quadratick and Cubick Equations,” as Wallis phrased it, can be used to illustrate the evolving nature of mathematics to the students: mathematical knowledge, ideas and artifacts can be criticized and new ideas and artifacts can be introduced as a response to shortcomings of existing ways of doing and understanding mathematics. Furthermore, the number line is a central artifact in modern primary school mathematics and it may be valuable for the students to get an explicit introduction to an authentic historic context in which the idea and motivation behind it surfaced, and to make this motivation an object for students’ discussions and reflections. Also, the example addresses a cognitive conflict the students themselves may have experienced at some point in their education. On the one hand numbers are considered and often introduced to children as cardinals; we ascribe numbers to sets of objects and to other quantities. But, as Wallis noted, quantities are positive, so how can we make sense of negative numbers? Wallis’ text gives one interpretation: In order to make sense of negative numbers we will have to use another part of our basic experience as source domain for our interpretation. Instead of understanding numbers as quantities and connecting them to our experiences of handling the cardinality of small sets of objects, we can think of numbers as locations on a directed path with a given starting point (the origin). The number line understood as an actual physical representation is used to facilitate and support this shift in interpretation. In this way, this example illustrates how cognitive artifacts in the form of representations can facilitate the intuitive understanding of mathematical entities with an analogy that forms a bridge between a mathematical domain and a domain of everyday experiences (in this case an analogy between numbers and movement on a path). Furthermore, and more importantly, the example illustrates how different artifacts can support analogies with different domains of everyday experiences, and that these differences in the analogies can have important mathematical consequences (here, whether or not negative numbers are acceptable mathematical entities). This aspect of the example can be strengthened for instance by contrasting the number line with the Pythagorean dot-notation for numbers and ask the students to investigate what parts of their everyday experience the different artifacts activate, and to explore the consequences entailed by such shifts of interpretation (e.g. Steensen and Johansen

2016). This might illustrate for the students that, in Wallis' text, the negative numbers in a sense are part and parcel of the artifact. Negative numbers do not make sense in the Pythagorean dot-notation, so that an artifact cannot just be replaced with another artifact.

So far, the Wallis original source has been analyzed mainly with respect to its potential to strengthen students' understanding of a well-known artifact, the number line. The example, however, can also be used to introduce students to creative inventive processes in mathematics. As previously noted, Wallis not only wanted to defend negative, but also imaginary numbers. He did so by extending the reasoning beyond the number line from one to two dimensions. So, instead of considering movement backwards and forwards on a directed path, Wallis asked us to imagine that we have a given amount of land and that the sea may either add to or subtract from it (Wallis 1685, p. 265). If the reasoning behind the number line is applied to this case the given amount of land is the origin, and land added to that can be counted as positive area, whereas land subtracted from the origin must be seen as negative area. And clearly, just as the positive area can be arranged into a square that has a root, so must the negative area; consequently, we will have to accept imaginary numbers.

This extension of the reasoning behind the number line may be an interesting challenge for students who have been taught to accept negative numbers, but have not yet been taught imaginary numbers. On the one hand, if Wallis' reasoning is accepted, the rule of thumb that one cannot take the square root of negative numbers must be rejected; on the other hand, if Wallis' reasoning concerning imaginary numbers is rejected, so must the reasoning behind negative numbers and consequently they should be rejected as well. In other words, Wallis presented negative and imaginary numbers as two sides of the same coin and the students can be invited to challenge this reasoning or to be challenged by it.

However, it must be emphasized at this point that there is at least one clear difference between the case of negative numbers and the case of imaginary numbers in Wallis' text. With the number line Wallis was able to give a clear geometric construction of negative numbers, whereas he did not have a similar construction for imaginary numbers. In other words, he lacked an artifact similar to the number line, and in the rest of the text he tried to provide one by combining the number line with the catalogue of by-then standard geometric constructions. He took a step towards the construction of the modern complex plane by realizing that imaginary and complex numbers must be displaced from the number line (Wallis 1685, p. 267), but apart from that, his attempts to provide a convincing construction of imaginary numbers must be regarded as a failure. The modern complex plane is an impressive cognitive artifact that both allows reasoning with complex numbers and provides an intuitive geometric construction of them.[6] Wallis' failed attempt illustrates how difficult it is to create such artifacts; although his intuitive reasoning

[6]Caspar Wessel's work *On the analytic representation of direction* is another source that could be used in this context.

can easily be expanded from one to two dimensions (that is from movement on a path to areas) the corresponding cognitive artifact is much harder to expand in a similar way. Therefore, Wallis' 'failure' can be didactically beneficial for the students. When reading Wallis' text students who are familiar with the complex plane will be given a chance not only to see the complex plane as an artifact but also to appreciate the ingenuity of its construction. Students who are not familiar with the complex plane can be invited to make the expansion of the number line from one to two dimensions themselves.

2.3 A Methodology for Using Original Sources in Math Education

Thus far, we have identified and analyzed the use of cognitive artifacts in three original sources from the history of mathematics, and we have made some claims regarding the benefits of using these original sources for the teaching and learning of mathematics. However, these benefits do not necessarily materialize by having the students read the relevant original sources. We need to develop a mediating link that can connect the theoretical considerations with the practice of teaching and learning. In general, what we are dealing with are questions of *why* and *how* to use original sources in the classroom for the teaching and learning *of* and *about* mathematics (see Barnett et al. 2014). In the present context, we propose to use history for students' learning of some aspects of mathematics and for them to develop some understanding of the nature of mathematics. Even though our focus here is on the *use* of artifacts in sources from the past and not on the historical development per se, it is important to be conscious about the underlying conception of history beneath one's particular approaches to history in using original sources in the mathematics classroom. As explained in the introduction, we consider mathematical cognition to be culturally situated and cognitive artifacts as cultural resources. We are looking at mathematicians' strategies and techniques in the production of mathematical knowledge available to us in original sources from the perspective of the significance of cognitive artifacts. This is consistent with a multiple perspective approach to history of practices of mathematics where past mathematicians' development of mathematics in time and place is looked upon from various perspectives depending on the historical and/or philosophical questions under consideration (see Kjeldsen 2012).

In science teaching, invoking such epistemological insights is often attempted through inquiry-based teaching in which students become engaged in activities that to some extent mimic what scientists do when they produce new knowledge. However, it is extremely difficult to create learning situations in mathematics teaching in which students can get first-hand experiences with mathematical research; this is probably the reason why inquiry-based teaching in mathematics seems to be almost entirely directed towards applications of mathematics and mathematical modeling. However, we argue that the history of mathematics can serve as a means for implementing an inquiry-reflective learning environment where students can gain insight into authentic research processes related to the creation of new mathematics, that is accessible to them, and which are still relevant for today's research. On the other hand, as has been emphasized by Abd-El-Khalick (2013) for science teaching:

> It now is well understood and documented that while inquiry might serve as an ideal context for helping students and teachers develop informed NOS [nature of science] views, it does not follow that engagement with inquiry would necessarily result in improved understandings. […] research also shows that engagement with HPS [history and/or philosophy of science], *absent* [of] critical and structured reflection, also is not likely to achieve desired NOS [Nature of Science] understandings. (pp. 2089–2090)

The key words here are "critical and structured reflections" the presence of which, according to Abd-El-Khalick (2013, p. 2090), make inquiry and history of science "ideal contexts for teaching and learning about NOS." He found that what he calls an 'explicit-reflective framework' is needed in order to integrate nature of science with science education in such a way that students will develop informed NOS understandings. By 'explicit', he refers to specific NOS learning outcomes that should be part of the curriculum, whereas 'reflective' refers to instructions for how students can be encouraged to reflect upon their experiences with learning science from within an epistemological framework.

Analyses of intentionally designed implementations of history and philosophy of mathematics in Danish upper secondary school have shown that, if issues of the historical development of mathematics and the nature of mathematics are made explicit objects of students' reflections, learning possibilities such as the ones above can be fulfilled in practice (see e.g. Jankvist 2010; Kjeldsen and Petersen 2014).

Here we focus on the history of cognitive artifacts as a resource for mathematics teaching and learning, but other aspects of the history of mathematics and original sources as a resource for promoting inquiry-reflective teaching *in* mathematics could be the focus (Kjeldsen 2016). For instance, students could study the original sources from the perspective of the use of such artifacts in the production of mathematical knowledge with the purpose of becoming aware that choices regarding techniques, interpretations and mathematical contexts are made in mathematical research processes—and that these choices have an impact on our understanding and our development of mathematical knowledge. Such insights will enhance the students' mathematical competency. In our three examples, the students' ability to follow mathematical reasoning and understand the use of representations was especially relevant; in other cases, sources can be used to improve the students' ability to follow deductive proofs. Such competencies are generally intrinsic in teaching and learning mathematics. Thus, historical sources and artifacts from the past can be used for the teaching and learning of mathematics whether the development of students' historical awareness is part of the curriculum or not.

What we propose as a mediating link between the theoretical analysis of sources from the past and classroom practice (see Fig. 2.1) is the creation of an

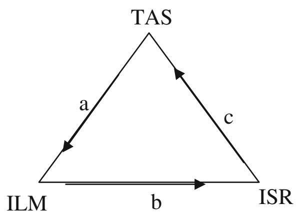


Fig. 2.1 Methodological triangle for using original sources in mathematics education

inquiry-reflective learning environment in which the students are invited into the workplace of past mathematicians through history. We are developing a methodological approach for using original sources for the teaching and learning of mathematics within an epistemological framework, where we distinguish between three types of considerations that go into design and implementation:

(1) **T**heoretical **A**nalysis of historical **S**ources (TAS) from the perspective of aspects of the nature of mathematics and historical insights and awareness;
(2) Creation and framing of an **I**nquiry-reflective **L**earning environment *in* **M**athematics (ILM); and
(3) **I**nstructions for practice promoting **S**tudents' situated **R**eflections (ISR).

Three processes combine the three considerations (see Fig. 2.1):

(a) designation of which aspects of mathematical research practices the teaching episode should mimic, i.e. which part of a mathematical research 'workplace' should the students be invited into;
(b) design of teaching material that can promote students to reflect upon the aspects chosen for inquiry; and
(c) evaluation of the development of students' informed conception of the aspects of the nature of mathematics, historical insights and awareness with respect to the results of the theoretical analyses of the sources.

2.4 Putting the Method to Use: An Example

In the present chapter, the idea is to use sources from history of mathematics to make students aware of the function, importance and limitations of cognitive artifacts in the development of mathematics. As we claimed in the introduction, all of our three examples can be used to consolidate students' understanding of the value of a particular notation, to understand the reasons underlying its creation, and to make students experience something foreign, though each example puts a different emphasis on each one of these three aspects. In this section we will use the Cardano example to illustrate our methodological approach.

To set up an inquiry-reflective learning environment *in* mathematics, it should be decided which of the aspects of the mathematical research processes, that have been identified in the source, should be used to develop students' understanding of how mathematical knowledge is generated (process (a)). Cardano introduced the algebraic notation of the square root of a negative number in his exploration of the 'impossible' case of dividing a line of length 10 in two parts such that their product is 40. Analyzed with respect to research processes in mathematics, Cardano's text illustrates at least two 'strategies' in mathematical research: (1) to try out things and methods even though they have no solid mathematical foundation (hereby introducing and operating with the square root of a negative number), and (2) to use a new method to investigate a known problem (here by switching domain from geometry to algebra).

The cognitive artifact, the notion Cardano introduced, made it possible for him to operate with square roots of negative numbers and enlarge the types of solvable problems, whatever 'solvable' then might mean, by moving the method of solution from the geometrical to the algebraic domain. For students, this will illustrate the power of the modern, algebraic notation. The other part of Cardano's approach, the geometric conception that Cardano moved away from, is something foreign for most students in secondary schools of today, as these students will perceive the problems as equations (not as geometrical entities) and solve them algebraically. This 'foreign' geometrical method of constructing solutions shows the students that what can be considered as a solution (i.e. what it means to solve a problem) depends on the domain of inquiry, the artifact in use, and the cultural and historical contexts. The students finally may come to understand that the different artifacts have different advantages and disadvantages. The geometrical constructions speak to our intuition whereas the algebraic symbols support a general technique that is not limited to specific cases such as the geometrical procedure. Hence, by studying Cardano's text, the students should be able to: (1) identify the artifact and domains that Cardano created and worked with in the source and compare it to our current practice, and (2) discuss the function and importance of the cognitive artifact for Cardano's mathematical activities and argumentation in the source and compare it to our current practice. If this can be accomplished, the students will have been in an inquiry-reflective learning situation, in which they have gained insights into how mathematical knowledge evolve and have experienced the significance of cognitive artifacts in mathematics (process (c)).

As discussed in Sect. 2.3, in order for students to develop such informed conceptions of the nature of mathematics and research processes, the reading of the sources must be supported didactically. This can be done by designing a set of instructions directed towards promoting the students' situated reflections (process (b)). In the present case, the task is to encourage students to reflect upon their experiences with learning mathematics using the cognitive artifact in the source, and the significance of the artifact for producing and establishing mathematical knowledge. This can be done by preparing worksheets for the students,

Worksheet for the text of Cardano

Read Cardano's text, "Rule II" (part of *Ars Magna* Chapter XXXVII: On the Rule for Postulating a Negative), and pay special attention to the 'tools' and domains Cardano evoked in order to explain his rule. The text is available in an English translati on (Cardano2007, pp. 219–220) where modern symbolism has been used instead of Cardano's original notation (e.g. 5p:Rm:15 is written as $5+\sqrt{-15}$). His original notation can be seen in the original Latin text which is reproduced in the source book by Struik (1969, p. 68).

1. Which mathematical task/example was Cardano using to explain his "Rule II"?
2. Analyze Cardano's demonstration of the rule.
   a. How did he proceed in order to demonstrate his rule?
   b. Which tool/artifact did he use to represent (parts of) the algorithm he used in the beginning of the text?
   c. Which domain of mathematics was he working within?
   d. How is this connected to Cardano's statement about reaching a "true understanding"?
   e. At some point Cardano asks us to operate with the square root of negative 15. What is the problem with that?
   f. Why did he introduce this 'imagined' number/this cognitive artifact?
   g. What was the benefit of introducing this cognitive artifact?
3. Cardano claimed in the presentation of the rule, that the example he used was an impossible case.
   a. Why was it an impossible case?
   b. How is this impossibility reflected in/connected to the artifact and domain?
   c. In what way did Cardano's solution depend on his introduction of the cognitive artifact of the square root of negative numbers?
4. How would we formulate Cardano's example today? To which domain of mathematics would we say it belongs?
5. Compare and contrast Cardano's way of representing the mathematical task of the text with our current way of representing such tasks.
6. What are the benefits/disadvantages of Cardano's way of dealing with such tasks?
7. What are the benefits/disadvantages of invoking the cognitive artifact of the square root of negative numbers?
8. What problems did the artifact solve?
9. What problems did the artifact create?
10. How are we dealing with problems such as the one Cardano attacked in our current practice?
11. What are the benefits/disadvantages of how we usually deal with such tasks today?
12. Compare and contrast Cardano's solution algorithm with our current one.

Fig. 2.2 Worksheet for the text of Cardano

to guide them through their work with the sources, and in which requirements for essay writing and/or group and project work are stated, depending on the choice of student activities and pedagogical choices. Figure 2.2 displays such a worksheet. It is designed to bring students' attention to Cardano's introduction and use of the cognitive artifact, and to have them reflect upon its function in Cardano's mathematical activities as explained in the source—that is, to scaffold the students' learning towards the development of the desired informed conception of the nature of mathematics and mathematical research processes that were identified in process (a).

2.5 Concluding Remarks and Future Research

We have illustrated how our methodological approach (depicted in Fig. 2.1) makes it possible to use historical sources as a way to teach students aspects of the nature of mathematics and some specific research strategies that have been used in authentic inquiries *in* mathematics. A significant element of this methodology is the creation of an inquiry-reflective learning environment *in* mathematics, that can function as a mediating link between the theoretical analysis of sources from the past and a classroom practice where the students are invited into the workplace of past mathematicians through history and historical sources. Through the analyses of our three examples, and the explicit design of how to use our methodology for using original sources in the case of Cardano's text, we have exemplified how cognitive artifacts can be used as a resource in mathematics education to make some aspects of the nature of mathematics and of research strategies in authentic inquiries *in* mathematics explicit objects of students' reflections within this methodology.

The three examples we have presented here show that mathematical theories are culturally and historically situated as a result of their interrelation with the development of specific cognitive artifacts. This may serve as an important correction and expansion of a naïve conception of mathematics as a contextless and purely deductive discipline. And yet the situatedness of mathematical knowledge hardly exhausts all that mathematics is. The examples, rather, illustrate only one aspect of the nature of mathematics. This naturally leads to the question: what are the other (important) aspects and how could they be taught? The two authors of this chapter aim to address these questions in forthcoming research where we will (a) attempt to give a more precise and detailed description of the nature of mathematics in relation to teaching and (b) describe how the different aspects of this nature can be taught using the methodology described above.

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Chapter 3
History of Mathematics and Teachers' Education: On Otherness and Empathy

David Guillemette

Abstract In this chapter, I develop some major points from the results of an empirical study searching to describe the *dépaysement épistémologique* (epistemological disorientation) lived by six secondary school pre-service teachers taking part of a history of mathematics course. Following a phenomenological approach, a description of the lived experience of the participants engaged in the reading of historical texts was produced. This description takes the form of a polyphonic narration (in a Bakhtinian dialogical perspective) that carries a plurality of points of view responding to each other. Our reading of this narration leads us to important reflections about otherness and empathy concerning the role of history of mathematics in the context of teachers' training.

Keywords History of mathematics · Teachers' education · Mathematics education Dépaysement épistémologique · Empathy

3.1 Introduction

In this chapter, I will quickly summarize an empirical study that has been implemented during winter 2013 searching to describe lived experience of six secondary school pre-service teachers taking part in a history of mathematics course.[1] After the presentation of the research problem, I will develop the epistemological background of the study, particularly by describing Emmanuel Levinas' and Mikhail Bakhtin's epistemology and human nature perspectives, which will lead us to the research objective. After a quick presentation of the context of the study and the methodology that was deployed, I will focus and develop, specifically for this

[1]For a more in-depth presentation of this study see Guillemette (2015a, in press).

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contribution, on two major points that arose during the analysis: the notions of otherness and empathy.

As we will see, the phenomenological analyses that have been conducted show, on the one hand, that the students engaged in the reading of seven historical texts in mathematics experienced empathic relation related to the mathematicians, which the authors of these historical texts perceived in their own mathematical, historical and cultural background. On the other hand, analyses show that this empathic relation is also oriented toward their future pupils confronted with the learning of mathematics, and these pupils are also perceived as subjects grounded in their own mathematical, historical and cultural background trying to understand an encoded object of culture. These elements bring new insights concerning the role of history of mathematics in mathematics education, especially concerning mathematics teachers' education.

3.2 The Role of History of Mathematics in the Context of Teachers' Training

3.2.1 The Argument of dépaysement épistémologique

For decades, many researchers have explored the contribution of the study of the history of mathematics in the context of teachers' education. A more and more recurring theme is that of *dépaysement épistémologique*[2] (epistemological disorientation) (Barbin 1997, 2006; Jahnke et al. 2000). Indeed, researchers point out that the history of mathematics stuns and shakes customary prospects of students on the discipline by highlighting its historical and cultural dimension (Barbin 1997). Overall, the study of history, above all the reading of historical texts, would bring a critical look at social and cultural aspect of mathematics and push future teachers to reconsider their view on the discipline and the classroom.

That said, numerous considerations about this *dépaysement épistémologique* argument does not seem to have been the subject of a systematic field's research that truly gives voice to the actors in the training environments (Jankvist 2007). However, many theoretical discourses have tried to think about the phenomenon in their own epistemological perspective. Major instances are the French historical epistemology (e.g. Barbin 1997, 2006), humanism related to the concept of *self-knowledge* (e.g. Fried 2007) and sociocultural approaches to mathematics education (e.g. Guillemette 2015b; Radford et al. 2007; Roth and Radford 2011). For a more detailed analysis of these perspectives and an outline of dialogue between them, see (Guillemette 2016).

[2]*Dépaysement épistémologique* is a French term that has been sometimes translated by *reorientation* in English-language literature.

3.2.2 Research Problems

As it is now classically seen in the domain, theoretical and empirical studies seem to walk side by side, both types of research having difficulties to stimulate each other (Gulikers and Blom 2001). On the one hand, we can find interesting theoretical studies that bring brilliant conceptualization regarding history of mathematics in mathematics education. On the other hand, we can find empirical studies[3] that seem to work on their own, without taking into consideration the theoretical development in the field, and very few of them seem to present a complete research device and a framework for data analyses (Guillemette 2011). The study evoked here has tried, in its manner, to fill that gap between those two parts of research by going in the field and taking account of theoretical consideration and bringing back new idea from its empirical movement to enrich these theoretical reflections.

Another crucial aspect of the recent movement in the field is the need to better reflect the contribution of the use of history in motivational and affective terms. Some researchers (e.g. Barbin 2012; Fried 2014; Guillemette 2016) have highlighted a particular need to support research through the production of theoretical and conceptual frameworks enabling these elements to be considered more thoroughly and more closely articulated with clear and solid epistemological foundations.

This need has become conspicuous by the appearance in research of what is called by many the "motivational theme" (Fried 2014; Fried et al. 2016; Gulikers and Blom 2001). This expression refers to a form of leitmotif encountered in a large number of studies that emphasize an "affective gain" or "motivational gain" associated with the introduction of history and cultural elements in the classroom. This instrumentalization of the historical and cultural dimension of mathematics for the realization of objectives such as the motivation of learners or the dynamization of the mathematics classroom makes it difficult for the field of research in its quest for theoretical and conceptual frameworks. Indeed, as Fried et al. (2016) point out:

> Assume that history truly has this motivational effect on students and that, further, it is not merely the effect of novelty, that is, it persists even after it has become routine. For the "motivational theme" to be the basis of a theoretical framework for teaching history of mathematics in the mathematics classroom, one would have to ask what is it about history that touches students and moves them? What part of their intellectual lives is touched by history? Asking these questions is essential to the question of a theoretical framework; however, if we obtain an answer to them or even if we manage to bring out the force of them, it will be something quite different from a statement of the form, "history of mathematics increases students' motivation". The affective gain will, in this regard, be a kind of epiphenomenon only. (p. 212)

[3]As shown by Jankvist (2007) the number of empirical studies has increased since the last ICMI study on history of mathematics in mathematics education (Fauvel and van Maanen 2000). But empirical study is still underrepresented in the field confronted with theoretical study, historical analysis related to education motivation or presentations of classroom propositions based on the history of mathematics.

In other words, the purely instrumental role of the historico-cultural dimension of mathematics, seeking to attain other goals than the historico-cultural dimension of mathematics itself in the classroom, hampers the development of theoretical and conceptual frameworks that are truly anchored in this dimension. Although, these different objectives (motivational and emotional gains, classroom dynamization, positive relation to discipline, etc.) are important and not insignificant, they just cannot be used as a basis for research since they could not lead to a reflection where the historico-cultural dimension of mathematics makes sense for learners, where the very nature of mathematical activity is questioned.

The idea here is, not to disqualify any discourse that refers to this "motivational theme" in research or to develop a catalogue of research or classroom practices and options, or to target an ideal approach. Rather, this study tries in his very own elaboration and in its conclusion to propose answers in the form of openings to theoretical or conceptual wording that allow to think more precisely about these questions, avoiding and going beyond, the "motivational theme".

3.3 Theoretical Orientations and Research Objective

The study is rooted in the theory of objectification (Radford 2011, 2013; Roth and Radford 2011), an emerging sociocultural theory in mathematics education that problematizes teaching and learning from the notion of Alterity conceived by Mikhail Bakhtin and Emmanuel Levinas. This perspective brings much importance for both history and affect in the teaching and learning of mathematics. History in the context of pre-service teacher training is perceived here as a place where students can encounter other voices and way-of-being-in-mathematics, coming from the past (Radford et al. 2007).

3.3.1 A Levinasian Perspective

More precisely, the notion of Alterity is taken, as Levinas put it in his multiple phenomenological essays, as the central and the core of human being. In his theses, Levinas overturned the traditional—from Plato to Heidegger—ontological way of thinking human being. Indeed, for Levinas, the philosophical inquiry on human being does not begin with and capitalize on the nature of human being (ontological perspective) but on his relation to the Other (ethical perspective). In other words, ethic here is not taken in as a "satellite" element of human existing; it is rather the central and the determinant field of reflections. With Levinas, ethic is the *philosophie première* instead of, classically, ontology. The complex and profound philosophical reflections from Levinas provide basic elements and language to the theory of objectivation. It provides, above all, a subject perceived primly and fundamentally in his ethical aspect, in its relation to the Other (peoples, objects, concepts, ideas, history, sciences… everything that has a meaning).

In his latest work Levinas, in response to the famous critique of his early works by Jacques Derrida in *Violence et métaphysique*, develop a notion of the subject literally constituted of the Others. He then quit his traditional phenomenological investigations on Alterity, based on the tradition inherited from his masters Husserl and Heidegger, to open a very new perspective on human being which he no longer perceived as an isolated subject being beset by phenomena, an ipseity thrown in the reality as would have said Heidegger, but an ethical subject constituted and revealing himself, in the relation to the Other.

3.3.2 A Bakhtinian Perspective

Concerning Bakhtin in his famous dialogical critic, the notion of Alterity is incorporated in his sociolinguistical way of thinking human being. Just like Levinas, Bakhtin tried to think human being by developing a fundamental reflection on his relation to the Other, particularly here with other human being. Consequently, in a very in-depth Marx lecture, his reflection is directed less to the nature of human being than to his cultural productions and its essentiality in the social.

More specifically, Bakhtin is doing so by putting in the front row linguistic, social, aesthetic and ideological aspects of human life. In his perspective, no act of meaning can be taken individually. It has to be taken with consideration to the acts of meaning to which it responds and those that are made possible by it in a certain sphere of communication, a certain ideological horizon and a certain aesthetic space, three intimately related elements in his dialogical critic. Again, these conceptual and philosophical considerations provide to the theory of objectivation conceptual elements to think about learning as a social process of critical encounters of other voices, aesthetic spaces and ideological horizons by the means of social interaction and artefacts (Radford 2011, 2013).

3.3.3 Research Objective

Back to history in mathematics education, from the perspective of the theory of objectivation, the dialogue that emerges from the encounters with the history of mathematics brings a particular experience of otherness that could be related to Barbin's concept of *dépaysement épistémologique*. Indeed, the notion of Alterity, philosophically and theoretically developed above, can help us to think about/of the phenomenon. And in this way to elaborate on its social, cultural and, of course, historical aspects, so that it can help us to build a consistent and coherent methodological approach for a refined empirical investigation.

Inhabited by those different theoretical discourses on history and the research problem quickly summarized above, this study has given itself the objective to describe the *dépaysement épistémologique* experienced by future secondary mathematics teachers through training activities that involve history of mathematics, particularly the reading of historical texts.

After a brief presentation of the context of the study and the methodology that has been deployed, instead of giving here an exhaustive description of the phenomena, I will focus, as said in the introduction, on two interrelated elements that have been major points of the results: the notion of otherness and the notion of empathy.

3.4 Context and Elements of Method: An Adapted Phenomenological Approach

3.4.1 Setting the Scene

For this study, a phenomenological approach in human sciences was adopted and adapted to the "dialogical Bakhtinian perspective" carried by the theory of objectification. These choices help us to understand a priori the phenomenon of *dépaysement épistémologique* and to highlight our own reflection as researchers on the role of history in mathematics education. Phenomenology and dialogism, that are in the core of the theory of objectivation, help us to develop a consistent methodological framework with its epistemological background. Concerning the phenomenological approach (Van Manen 1994), it aims to describe the intimate and subjective experience of the participants and to clarify the meaning of their experiences. Concerning Bakhtin's dialogical perspective (Bakhtin 1929/1977, 2003), it emphasizes that a scientific or literary work must be "polyphonic." That is to say, providing a plurality of discourses and understandings of the world, which are echoing and responding to each other. In such polyphonic work, reality loses its static aspect and its naturalism. It allows readers to grasp a lively world showing tendencies and anticipations instead of linear and sterile discourse on reality. Inhabited by this comprehensive and critical perspective in human sciences, this study proposes a description of the lived experience of *dépaysement épistémologique* that takes the form of a polyphonic narration.

The selection of the six participants for the study was done on a voluntary basis from among future secondary school teachers enrolled in a history of mathematics course offered at the *Université du Québec à Montréal* in Canada during winter 2013. Seven activities consisting in the reading of historical texts were experienced:

- A'hmosè: *Rhind Papyrus*, problem 24
- Euclid: *Elements*, proposition 14, book 2
- Archimedes: *The Quadrature of the Parabola*
- Al-Khwãrizmî: *The Compendious Book on Calculation by Completion and Balancing* (*Al-kitāb al-mukhtaṣar fī ḥisāb al-ğabr wa'l-muqābala*), types 4 and 5
- Chuquet: *Tripartys en sciences des nombres*, problem 166

– Roberval: *Observations sur la composition des mouvements et sur le moyen de trouver les touchantes des lignes courbes*, problem 1
– Fermat: *Méthode pour la recherche du minimum et du maximum*, problems 1 to 5

These reading activities were conducted following Fried's (2007, 2008) recommendations. For this author, just like many others, the reading of historical texts appears to be the preferred approach when using history of mathematics in order to create this *dépaysement épistémologique*, the very meeting with the mathematicians from the past.

But Fried stresses that this reading should not be done in any way. He emphasizes that, on the one hand, the historian's goal is to immerse himself or herself in the mathematician's era, to perceive the idiosyncrasies of the latter and to situate his or her work in a continuum of mathematical development related to the social and historical background. On the other hand, the mathematician's view tries to decode the obsolete symbols, to restore them to the modern language and to grasp, essentially, the mathematical aspect of the author's words. He describes as "diachronic" the reading of the historian and as "synchronic" the reading of the mathematician, terms borrowed from the famous Swiss linguist, Ferdinand de Saussure.

For Fried, the synchronous reading of mathematical texts is too often reinforced by teachers and mathematicians. Also, the role of the teacher would be to switch constantly with the learner between these two visions. The continuous back and forth between the two perspectives on the text would allow the learner to develop a certain awareness of his own conceptions of mathematics, his personal understandings and the possibility for him to confront them constructively with those of the students other.

3.4.2 Data Sources and Analyses

Video recordings of classroom activities, individual interviews and group interview were conducted and provide the study data. For video recordings, analysis allowed us to describe the learning process that took place in the classroom. For individual interviews, processing and analysis of data (Lamarre 2004; Van Manen 1994) led to specific descriptions of the epistemological disorientation lived by each participant. The polyphonic novel was then constructed from extracts of the interview group and enhanced from video recordings and individual interviews analyses.

3.4.3 The Construction of a Polyphonic Novel

More precisely, in order to obtain this polyphonic novel, a first step was to construct the transcript of the group interview with care. Then, several attentive readings of the transcript were made. These readings have revealed some extracts of dialogue containing rich and profound reflections in relation to the lived experience of the participants. Twelve extracts of the transcript were selected. Thereafter, a careful reading of each of these extracts was made again and a list of various topics, thematics, reflections or statements was created for each of these extracts. The twelve extracts were then systematically treated individually. For each of them, four writing phases succeeded each other.

The first step of writing was to rework the raw extract from the transcription of the dialogue. The dialogue was then shaped so as to make it more readable with the addition of paragraphs and spacing. Also, the author of each excerpt was more clearly identified without duplication. The first traces of narration then appeared with the addition of "incised" phrases like; "she said lightly," "revived Aliocha squirming on his chair" or "I thought." These phrases highlighted the ways of being and attitudes of the characters/participants.

The second writing step was to complete the extract, with the addition of information on the participants. These additions allowed to "defend" each participant in the dialogue and to refine and highlight their thoughts and appreciative orientations. Taking the form of paragraphs inserted into the dialogue, these additions allow us to position ourselves as author/researcher as the agent of the participants, as their spokesman. These intercessions were both fueled and justified by the descriptions of reading activities and the specific descriptions of the experience of the participants obtained during previous phases of analysis.

In the third step of writing, personal reflections were added so as to be heard more as an author/researcher in the narrative. Usually at the beginning of the extract, one or more paragraphs were added. These provided space to express our thoughts that were emerging at the time of writing.

The fourth and final step of writing was to refine the narrative by emphasizing the theme of the extract and the polyphonic style exercised.

These four writing steps were repeated for each of the twelve extracts released initially. These were then combined to form the final polyphonic novel describing the *dépaysement épistémologique* experienced by future teachers of mathematics. This narration of the collective experience takes its density from fine description of each character/participant from previous analyses. It has led to the emergence of tensions, viewpoints moving away and approaching each other, viewpoints that overlap and influence each other, a sort of siphonophore, both singular and plural.

The description provides multiple looks, which, in tension, carries fruitful discourses on the lived experience of participants. As Bakhtin put it in its dialogical critic explained above, it is in the tension between discourses coming from different spheres of communication, different ideological horizons and different aesthetic spaces that one could grasp the reality of human life.

Globally, the form of a polyphonic narration for this description is a methodological response to an epistemological challenge that underpinned this research. Indeed, this discursive strategy allows the production of a description that, first, can respect the phenomenological requirement and stringency to keep alive the subjectivity of the participants without objectivizing it in any manner and, second, embrace a conception of teaching and learning in mathematics education, which claims that learning is necessarily "learning-with-others" (Radford 2011, 2013).[4]

3.5 A Comprehensive and Interpretative Paradigm

Before going any further, it seems important to mention that this qualitative research is obviously anchored in an interpretative/comprehensive paradigm. It means that it is rather exploratory than "confirmatory." That is to say that it fails completely and in advance, in the project of formulating generalizable, predictable or falsifiable statements. These are not the objective here. The study brings a description of a specific, contextualized and particular event that can be generalized with huge difficulties and dangers. That said, the description obtained can provide space for reflection concerning theoretical and/or conceptual frameworks for integrating history in mathematics education and concerning practices in this context.

This is why the study cannot provide any clue concerning the way one could provoke "systematically" *dépaysement épistémologique* in his classroom, and above all, in the same way that happened in this particular study or in any "positive" way. This study has much more humble objectives. It is simply based on a fact that something happens when pre-service teachers are engaged in the reading of historical text; some have already thought about it and have brought theoretical considerations calling this phenomenon *dépaysement épistémologique*. This research tried, in an empirical manner, not to "confirm" or "infirm" these theoretical considerations, but to enrich and deepen them by a reflection that is "co-emerging" from the contact with the participants.

On the contrary, a systematic method or a pedagogical recipe that provokes *dépaysement épistémologique* in the classroom could not draw on and recognize the situated, contextualized and organic aspects of a classroom. Searching for such elements could only lead to disembodied, procedural and superficial ways of being in the classroom.

This said, the next remarks and commentaries constituted a specific, but privileged reading of the polyphonic narration. But now, *ad rem*.

[4]The entire description, and also the phenomenological descriptions (French version), can be obtained from the author.

3.6 On Otherness and Empathy

3.6.1 Making Sense of the Lived Experience of the Participants

The description obtained points out two major interrelated elements: the experience of otherness as well as empathy. It reveals that the future teachers make serious efforts to understand the texts without uprooting them from the context in which they were produced. This interpretative work is hindered by numerous difficulties associated with various elements; language, notations, implicit theorems, odd styles, unknown definitions, unusual arguments, unusual typography, etc. Literally, the students "suffer" the texts. In the context of teachers' training, these reading activities appear as precarious and risky hermeneutics exercises. Indeed, the experience of otherness in mathematics seems brutal from a cognitive and emotional point of view.

More specifically, their testimonies are quite clear concerning the adversity that they have lived. "Suffering the text" means that their difficulties in the interpretation of the texts and the misunderstanding of the author's culture and way of doing mathematics often lead to frustration and to a sentiment of being awkward and clumsy. Those elements appear clearly and recurrently in the specific description of each participant and were one of the main themes that comes through the polyphonic novel. This supports Barbin's (2006) discourse on *dépaysement épistémologique* which is there associated to a sentiment that one can feel in a foreign country or in a foreign context: disoriented and confused.

That said, in this otherness experience related to what we can call *dépaysement épistémologique*, the students can sometimes answer violently. In this section, I will explore the notion of "empathy" and "violence" as it is developed in phenomenological literature in order to clarify our position concerning the lived experience of *dépaysemnent épistémologique* and to make sense of the participants' testimonies. A few excerpts will be presented and commented afterwards.

Phenomenologically, as Levinas (1971/2010, 1979/2011) put it, violence is a "thematization of the Other," a reification of the Other, a way to make the Other a Mine. This is where the concepts of otherness and empathy join together. Again with Levinas, and also with Bakhtin (1986/2003), empathy can be understood as an effort of the establishment of a nonviolent relation with the Other. Within the experience of otherness, empathy is this modality of being tends to keep the Other's subjectivity free and alive, to keep it mysterious and indefinite. Levinas developed that this nonviolent and empathic relation "[is] not an idyllic and harmonious relation of communion, nor a sympathy that, by putting ourselves in its place, we recognize him as similar, but external to us, the relation with the Other is a relation with a Mystery" (1979/2011, p. 63; author's translation).

Classically, empathy is defined as the ability of putting ourselves intuitively "in the place of the Other," to feel the same way as the other or to identify ourselves to the other. Thus, empathy implies a particular relation with the other.

Furthermore, empathy is not an experience received in passivity. It is a possible answer to the experience of otherness. Bakhtin notices about this the following:

> I actively identify with individuality and consequently, not a single moment do I get lost nor do I lose ground outside of it. It is not the object, which, in an unexpected way, takes possession of me, passive, but it is me who identify actively to it. The act of empathy is my act and that is where its productivity and newness lies. (1986/2003, pp. 35–36; author's translation)

In this way, if the empathic movement towards the Other is a participative act, if it is actively inaugurated in the peculiarity of any moment of the human existence and if it is not passively received, it can then be requested and encouraged, and it can be maintained.

Back to the participants' lived experience, violence, as developed above, occurs during the reading of historical texts. Indeed, the subjectivity of the authors is arduously preserved, as it is possible to read it in the specific description and in the polyphonic novel. The students hardly maintain an empathic relation with the authors. In the suffering of the *dépaysement épistémologique*, the answer is often violent. This violent answer is the disappearance of the empathic relation. The authors are dispossessed of their lively peculiarity; they are transformed, summarized and reified. There is a violence of the synchronization thinking.

Indeed, students have a strong tendency or a natural propensity to simply translate and report on the author's mathematical activities in modern language. In this manner, the author is hardly considered in his mathematical and in his socio-historical context. Then, the authors see themselves dispossessed of their uniqueness; they are often translated and summarized in modern mathematical language.

Furthermore, concerning teachers' training, the need to keep students maintaining an empathic, nonviolent relation with the author seems crucial now. Consequently, it seems necessary to accompany future teachers in the *dépaysement épistémologique*, so that they maintain a nonviolent relation with the authors. The reason is, on the one hand, as the description obtained shows it, this empathy allows the welcoming of the mathematician and implies an epistemological reflection on mathematics and a new relationship with the discipline; on the other hand, it appears that this empathy can move towards the classroom. Indeed, through an empathic answer to the otherness experience that characterizes the *dépaysement épistémologique*, another which is the students' subjectivity reveals itself. Here students direct their reflections on their awkward experiences towards their future mathematics classrooms.

This small excerpt of the narration can illustrate a few reflections towards the future mathematics classroom of the participants:

– What fascinated me is the way to see things. I was like, "Wow!," began Katia.
  She was referring to Archimedes whose style and efforts had particularly impressed her. In this text, Archimedes uses with virtuosity principles of physics to solve mathematical problems. She saw then a completely original thought, and was profoundly shaken by it.

– And, indeed, she continued, what we said earlier about all the reasoning that we imposed to students… well not imposed, but…
– Guided, proposed Martha.
– They are clearly imposed, threw Grouchenka abruptly.
– Personally, continued Katia, that course has allowed me to be more open to different possibilities of mathematical reasoning. When I'll be confronted with a student saying: "Yes, no… but I did the same with the center of gravity of a triangle…" I' m not going to say: "No, you're wrong!", I'll be like: "Wait a minute!" You know, I'm going to check.

Actually, the polyphonic narration shows that the students perceived their future pupils as also being confronted with mathematical texts to be interpreted. Aware of the possibility of engaging in violence with the authors in reaction to the *dépaysement épistémologique*, the students give themselves new responsibilities concerning the mathematical activity which will take place in their classrooms, that of welcoming their learners and their reasoning in a nonviolent way, as the next excerpt shows:

– Like I told you before, I thought, "Oh, my God, my students, when I'm teaching to them, that's how they feel!" I mean… you see it in their face… continued Ninotchka which seemed illuminated. Because it touches you, not to understand, for every pupil, when you're not able to make the connection, it's infuriating and it's frustrating. That is what I get, for me anyway, from these experiences. I really see the class and the different types of pupils in my class, and I say, "OK, yes, he probably felt this when I taught him that…" Then, you know, you can play on these feelings, too. It helps here to debate together; I think it helps in order to understand.
– That's it! Katia agreed so much that she was not able to speak.
– Then it helps to have a better background to be able to work and then help them make these links, ended Ninotchka.
– Aliocha, who proudly smiled and nodded, was looking at the group.

The readings can thus support and encourage this participative act among the students that is the empathic movement towards the Other in the classroom.

More specifically, an empathic response to the experience of otherness, which is intrinsically linked to the *dépaysement épistémologique*, seems to lead to a valorization of creativity, marginality and originality in the mathematical activity. Indeed, in the *dépaysement épistémologique* the pre-service teachers make this perspective of mathematics "in the making," fragile and precarious. Fragility of mathematics, adversity in the meeting of the authors and empathy are three central and interconnected themes in the polyphonic narration.

That is why I would like to suggest that the readings of historical texts, through the *dépaysement épistémologique* they generate, support a nonviolent mathematical education. Furthermore, the polyphonic narration leads to predict that this nonviolent mathematical education can take place in the future students' classrooms.[5]

[5]For a more in-depth discussion about this question of nonviolent mathematics education see Guillemette (2015a, in press).

Thus, the meeting with the author perceived in his socio-historical and mathematical context is not done easily. This empathic reading of the historical text demands a considerable effort for students and these difficulties could be explained by the academic culture of mathematics and the context of teacher training that makes the students animated by a pragmatic concern for the development of directly useful teaching tools.

The difficulties therefore seem not solely to come from the teacher or trainer who would guide the learners in a sterile translation process. Of course, the teacher or trainer can only accompany the learners in their search for meaning, which cannot be done without the support of their academic knowledge, their academic experience and their other mathematical background. In this perspective, we feel some resistance from the students to deploy an empathic reading.

3.6.2 Dépaysement épistémologique*: Empathy and Self-knowledge*

Cognitive empathy in turn makes possible the moral emotions of sympathy and compassion, in which we feel genuine concern for the other. These emotions require certain cognitive abilities and a well developed, "sense of self." The results of the study about empathy join up with the theoretical point of view of Barbin (1997, 2012) and Michael Fried (2001, 2007, 2008).

Indeed, from a practical point of view, history may allow prospective teachers to "understand the difficulties of students who are not like teachers, in a well-known country" and "better hear their questions or to better interpret their mistakes" (Barbin 2012, p. 548). More specifically, the "exotic aspects" of the history of mathematics can be a good way to "start thinking about the content taught and programs," "to sketch students' answers and questions about the status of mathematical knowledge," to "avoid the fake concrete-abstract debate and finally allow the teachers to change the way they teach, but also their educational relations" (Barbin 1997, p. 24; author's translation).

As Fried (2007, 2008) mentions, history of mathematics, in general, should be playing a central role in this quest of self-knowledge. For him it is a special contact with the history of the discipline that could make emerge in the learner some awareness of his own ideas, his individuality and his ability to confront constructively with those of others. Fried considers history, when it is taken as history and not as a means for something else, to be able to contribute to the personal growth of individuals through the discovery of their own individuality. This individuality would not lead to a form of isolation, but on the contrary, on the openness and the possibility to the exchange with the others and to understand the others. In this sense, mathematics education, through history, must aim at mutual enrichment between knowledge, self-knowledge and knowledge of others.

Indeed, Fried stressed that the back-and-forth movement between the current understanding of mathematical objects and understandings from other eras brings learning to a deeper understanding of himself: "a movement towards self-knowledge, a knowledge of ourselves as a kind of creature who does mathematics, a kind of mathematical being" (Fried 2007, p. 218). He proposes that this self-knowledge, that is to say, the knowledge of ourselves as a "mathematical-being," should be the primary objective that must give itself all forms of mathematics education based on the history of the discipline. Fried does not hesitate to emphasize the background of his thinking around these considerations by stating that: "[Education], in general, is directed towards the whole human being, and, accordingly, mathematics education, as opposed to, say, professional mathematical training, ought to contribute to students' growing into whole human beings" (id. p. 219).

In this perspective, the experience associated with the encounter with history of mathematics would be accompanied by an awareness and a growing movement. This would be a personal experience involving relation to ourselves (introspective element) through the history of mathematics, experience that supports the movement of growth, which is that of the learner. This "humanistic" perspective on the history of mathematics is also present, and developed, in many other speculative works in the field (e.g. Bidwell 1993; Brown 1996; Tang 2007).

3.6.3 **Dépaysement épistémologique***: Empathy and Learning-to-Listen*

It has been claimed that one of the main roles of the history of mathematics is to "disorient." Indeed, history of mathematics, in a classroom or teachers' training context, surprises and astonishes with the diversity of the mathematical activity across cultures and the history of societies, which involves many considerations as to the form and use of mathematical objects. For many, these experiences could lead to a more cultural understanding of mathematics and invite to a historical-anthropological reflection on mathematical activity by repositioning the discipline as "human activity" (D'Enfer et al. 2012).

As Barbin put it many times, history could bring a "culture shock" in "immediately immersing the history of mathematics in history itself" (2012, p. 552; author's translation). Therefore, the objective is not to read historical texts simply related to our (modern) knowledge, but rather in the context of the one who wrote them. This is when history can become a source of "epistemological astonishment" by questioning knowledge and procedures typically taken as "self-evident" (ibid.). For Barbin, as history invites the learners (especially here the pre-service teachers) to stand out in that tone of "epistemological astonishment," it may also invite them to investigate the following questions: "Why contemporaries did not understand such a novelty?" and "Why students do not understand?" (ibid.).

The question of understanding the students is not new, and our reflection on empathy join here the theoretical discourses and the position of Jahnke (1994, 2014) around the idea of learning-to-listen with the history of mathematics. For Jahnke—starting from a hermeneutic approach that can be related to the work of the German philosopher, Hans-Georg Gadamer—the reading of a historical text in mathematics brings two interrelated forms of reflections.

First, there is the experience of "dissonance" or "alienation," just like the feeling of being in a foreign country. The students learn something about their own mathematics by experiencing and "reflecting on the contrast between modern concepts and their historical counterparts" (Fried et al. 2016, p. 218). And the point of the "hermeneutic circle"—a concept borrowed from Gadamer—understood as a process in which a hypothesis is put up related to what the student is confronted, tested against the source by confronting it with other parts of the text, modified, tested again and so on, and so on, until the reader arrives at a satisfying result after a kind of saturation of meaning. This reflection goes in both directions, so that the students deepen both their understanding of history and of their own set of modern conceptualizations regarding mathematics and mathematical objects.

Second and equally important for Jahnke, is the fact that in reading a source (modern) mathematics itself is applied as a tool (Jankvist 2009). The task is now to think about the situation of the mathematicians living in the past. This task requires being able to argue from the assumptions of these persons, to use their symbols and methods. This poses completely new demands on the students' abilities in their mathematical activities. Thus, "reading a source deepens the mathematical understanding on both levels, on that of doing mathematics and on that of reflecting about mathematics" (Fried et al. 2016, p. 219).

According to the hermeneutics perspective, a text consists in the merging of different horizons, the horizon of the reader and the horizon of the author. This means of course, that different readers embedded in their different backgrounds arrive at different interpretations. The texts here are the problems and the things that students are confronted with. As Jahnke put it, "this might cause a feeling of participation or solidarity" (ibid., p. 220), a feeling of being with the others, in a community. This can join up with the result of our study concerning empathy, because empathy, as we have seen with Bakhtin, is not received in passivity, but could be heard as a participative act.

Jahnke goes deeper by claiming that this feeling of participation or solidarity can engage prospective teachers to a more attentive relation to their future pupils. Jahnke summon the thesis of Arcavi and Isoda (2007) in order to support his point of view:

> History of mathematics can provide many solution approaches (to problems) which are very different from what is common nowadays. Such solution processes may conceal the thinking behind them. Thus, one has to engage in a 'deciphering' exercise in order to understand what was done, what could have been the reasoning behind it and what is the mathematical substrate that makes an unusual method/approach valid and possibly general. Engaging in such an exercise bears some similarities to the process of grasping what lies behind our students' thinking and actions. We do not claim that there may be parallels

> between the mathematics underlying primary sources and that of our students. What we do claim is that experiencing the process of understanding the mathematical approach of a primary historical source can be a sound preparation towards listening to students. (pp. 115–116, as cited in Fried et al. 2016, p. 221)

Back to the result of our study, it is possible to ask what is happening precisely during this process of "hermeneutic circle"? What lies beneath this preparation towards the listening to students by the means of history? What is the very lived experience, in a phenomenological sense, that is related to this phenomenon? We can argue now that one of the main lived experiences here is one of empathy, an empathy directed at the author of the historical text or the primary sources. What I do claim is that history can be a preparation towards listening to students, but also a preparation in order to look at the pupils and perceive them as confronted with the learning of mathematics and also perceived as subjects grounded in their own mathematical, historical and cultural background trying to understand an encoded object of culture.

3.7 Conclusion and Research Perspectives

To summarize, in this contribution I have tried to develop two interrelated major points of the results of an empirical study that has given itself the objective to describe the *dépaysement épistémologique* (epistemological disorientation) lived by six secondary school pre-service teachers taking part in a history of mathematics course. Trying to play the role of Hermes, the study tried to fill the gap or to create a dialogue between empirical and theoretical studies in the field. Following a phenomenological approach in human sciences, and borrowing important concepts from the Bakhtinian dialogical perspective, a polyphonic narration has been constructed in order to grasp the lived experience of the participants engaged in the reading of historical texts.

This description of *dépaysement épistémologique* leads, as we have seen to important reflections about the two notions of otherness and empathy concerning the role of history of mathematics in the context of teachers' training. Otherness and empathy characterize the lived experience of the student, but, above all, these experiences have a particular meaning in the context of teachers' education. Indeed, for the pre-service teachers these experiences are related to their future pupils confronted with the learning of mathematics that are perceived, just as the mathematicians—authors of the historical texts that have been encountered—subjects grounded in their own mathematical, historical and cultural background trying to understand an encoded object of culture.

Moreover, I provided explicit links between these observations and remarks on otherness and empathy and different theoretical discourses in the field, and in particular theoretical discourses grounded in a different paradigm or epistemology: Évelyne Barbin's notion of *dépaysement épistémologique* grounded in a historical epistemology, Michael Fried's notion of self-knowledge, grounded in a humanist perspective and Hans Niels Jahnke's discourse grounded in a hermeneutic perspective.

Despite its limited scope, this study raises decisive questions concerning history of mathematics in mathematics education. First, how to accompany students in the difficult experience of *dépaysement épistémologique*? It might be interesting to rethink the role of the teachers' educators in the context of reading historical texts. Indeed, it seems important to clarify different ways of intervening in the classroom to accompany the students, but also to highlight different ways of conducting the readings.

Another important question is, how can we help the students maintain an empathic relationship with the author? A more refined investigation of the modality of the apparition of this empathy appears now crucial.

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Chapter 4
Addressing the Transition from School to University

Evolution of a Seminar Emphasizing Historical Sources and Student Reflections

Ingo Witzke, Kathleen M. Clark, Horst Struve and Gero Stoffels

Abstract Beginning in 2015 we designed and taught an intensive seminar, which addressed the transition from school to university by making students aware of concept changes in the history of geometry. This paper focuses on the design of a pilot study and its development into a semester-long seminar. We use the case of one participant, Inga, to highlight the data and results emerging from the pilot seminar. Inga's case indicated that the seminar raised explicit awareness of the transition from school to university mathematics and was worth expanding into a semester-long seminar. Additionally, students' experiences in this seminar can also support their transition back to school from university as teachers, paving the way for teacher students to overcome Klein's well-known double discontinuity.

Keywords Transition from school to university · Beliefs · History of geometry Case study

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4.1 Introduction

The first ideas for the seminar and its possible basic structure were introduced by Witzke in (2015), and which were prompted by experiences of an exploratory study with 250 German pre-service mathematics teachers that he conducted in 2013. Compared with the results of other studies (e.g. Gruenwald et al. 2004; Hoyles et al. 2001) Witzke concluded from the empirical data that the pre-service teachers clearly distinguished between school and university mathematics regarding the *nature of mathematics*. The exploratory study showed the necessity for further research regarding questions about the ways in which beliefs on the nature of mathematics held by students play a crucial role for mastering the transition from school to university. The research-seminar, "Addressing the Transition Problem from School to University Mathematics" (which we refer to as the ÜberPro Seminar, from the German, **Über**gangs**Pro**blematik, "transition problem"), was designed to serve these needs. It was developed, implemented, and revised by Clark, Stoffels, Struve, and Witzke several times since 2015. A deeper look at the activities and the content of the pilot seminar is available (Witzke et al. 2016).

Based on the needs for further research, we developed a hypothesis for designing and evaluating the seminar based on a theoretical framework, which is focused on the learners' own experiences with mathematics and the beliefs about mathematics brought forth by their experiences. Of course, not only university students develop beliefs about mathematics. If we focus on the strong connection of mathematical knowledge and beliefs about mathematics, people who have done mathematics in the past have created their own experiences while shaping their beliefs about mathematics.

Based on these preliminary considerations our hypothesis for describing the transition from school to university and for supporting students during their individual transition is:

> The change from an empirical-object oriented to a formal-abstract belief system of mathematics constitutes a crucial obstacle for the transition from school to university.
>
> And, on epistemological grounds, similar changes regarding different natures of mathematics can be described for the history of mathematics. The explicit analysis of the historical genesis provides support for students dealing with their individual transition processes.

We clarify two major assumptions made in this hypothesis:

1. The transition problem cannot be easily "smoothed out" (as discussed in Gueudet et al. 2016) and probably should not be smoothed out, because it gives the opportunity to reflect on one's own beliefs, knowledge, and affect in mathematics during the transition (cf. Sierpinska 1987).
2. With respect to our theoretical framework, especially regarding the concept of subjective domains of experience (Bauersfeld 1983), students can overcome the transition only on their own—with impulses from historical sources or the course instructor.

4.2 Theoretical Framework

The transition problem from school to university has many complex facets and an important one concerns necessary changes in beliefs of learners. Recent literature and results of empirical surveys broach the issue of beliefs/belief systems about mathematics (Grigutsch et al. 1998; Liljedahl et al. 2007; Rolka 2006). Schoenfeld (1985), a strong advocate for the importance of beliefs regarding mathematical behavior, introduced the term “empirical belief system” to describe an archetype of pupils relying solely on argumentation based on empirical objects (e.g. precisely drawn figures in geometry).

Following this approach, we focused on belief systems concerning the ontology of mathematical objects. For our study we extended Schoenfeld’s terminology by distinguishing between an empirical-object oriented belief system (empirical bonding is necessary) for school and a formal-abstract belief system (logical consistency only) as they are held by many professional researchers in mathematics (Davis et al. 2012; Hempel 1945; Tall 2013). A deeper epistemological perspective on mathematics shows that there are more than just these two views on mathematics and mathematical entities. Girnat (2011) identified on the basis of categories from possible ontologies of mathematical objects four views on mathematics: the formalistic, the idealistic, the rationalistic, and the empiristic view on mathematical objects. In our work we are interested in the views on mathematics that draw a distinction regarding the intended applications of a mathematical theory. The formalistic view on mathematics is distinct from the other views on mathematics because of its conscious loss of intended applications. We believe that pupils who want to be successful in university mathematics need to advance to a formal-abstract belief system, in the sense that they are aware of the conscious loss of intended applications in the communication of mathematics. However, this does not mean that modern mathematics cannot be applied in real contexts.

In addition, our hypothesis claims that the necessary development of belief systems during the transition from school to university has, on epistemological grounds, paralleled the historical development of nature of mathematics. Witzke (2009) and Burscheid and Struve (2001) showed that the work of relevant historical figures (e.g. mathematicians) could be reconstructed as empirical theory. Such figures reconstructed, for example, the calculus of Leibniz as an empirical theory by the means of structuralism, a concept of philosophy of science (Balzer et al. 1987; Burscheid and Struve 2001). Finally, it was Hilbert who cut mathematics’ “bond to reality” with the famous work, *Foundations of Geometry*, in 1899 (cf. Freudenthal 1961), giving post-Hilbertian mathematicians the opportunity to do mathematics in a formalistic way—disregarding questions of empirical truth.

4.3 The Pilot Seminar (Spring 2015)

On the basis of the hypothesis formulated above, we designed the ÜberPro Seminar, which was organized as a three-day intensive course, comprising 18 h of instructional time with 20 pre-service teachers. This pilot seminar was taught and evaluated for the first time in spring 2015 in Germany.

The aim of the seminar was to make students aware of the changes regarding the nature of mathematics from school to university (by discussing transcripts, textbooks, standards, historical sources, etc). The conceptual design of the course drew upon positive experience with explicit approaches regarding changes in the belief systems of students from science education (Abd-El-Khalick and Lederman 2001). Geometry was used as the topic of the seminar's mathematical content. That is, with geometry we discussed a mathematical field that served as a prototype of empirical-object oriented mathematics and developed into what is referred to today as formal-abstract mathematics. Questions connected to this development are: Why did this change in thinking about mathematics happen? What were reasons for this development? How does it affect us today? Throughout the study, we remained connected to the idea that students would link their individual learning biography to the emergence of different views of mathematics present in the historical development of geometry. Moreover, we posited that history would help to explain the "abstraction shock" encountered by so many students when facing university mathematics by the means of its historical origin.

4.3.1 Overview of the Pilot Seminar's Content

The seminar was organized in four parts.[1] Part I was designed to raise attention to the importance of beliefs about and philosophies of mathematics. We began with individual reflections and work with authentic material such as transcripts from Schoenfeld's (1985) research that clearly showed the meaning and relevance of the concept of an empirical belief system. Afterwards students compared different types of textbooks: university course textbooks, school textbooks, and historical textbooks.

Part II focused on a historical case study, in which geometry from Euclid to Hilbert was investigated. The overall aim of the historical case study used as the content of the pilot seminar was to make students aware of how the nature of mathematics changed throughout history. In doing so, readings from numerous sources were used in the seminar. Students worked with examples from Euclid's *Elements* (in particular, the Pythagorean Theorem), projective geometry, and non-Euclidean geometry. Regarding our theoretical framework, we endeavored to

[1]Additional details about seminar content for the pilot seminar are provided in Witzke et al. (2016). Similarly, content for the expanded seminar is displayed in Fig. 4.1.

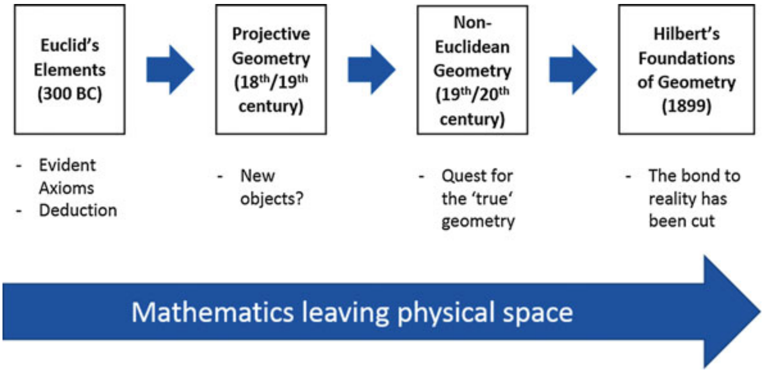

Fig. 4.1 The historical and philosophical development of mathematics along four key historical points in the development of geometry (cf. Witzke et al. 2016)

make explicit how geometry—which for hundreds of years seemed to be the prototype of empirical mathematics, describing physical space[2]—developed into the prototype of a *formalistic* mathematics as formulated in Hilbert's *Foundations of Geometry* in 1899 (Fig. 4.1).

The question, "What characterizes modern formalistic mathematics?" guided Part III of the seminar. Hilbert actually gave an answer to this problem—not only in a philosophical and programmatic way but also by formulating a geometry "exempla trahunt" (Freudenthal 1961, p. 24). Mathematics was seen for ages as the natural description of physical space, and after Hilbert can be characterized in a formalistic sense and characterized by a modern axiomatic structure. The established axioms are fully detached and independent from the empirical world, which leads to an absolute notion of truth: mathematical certainty in the sense of consistency. Thus, with Hilbert the bond of geometry to reality is cut. This came to life in the seminar when students read Hilbert's *Foundations of Geometry* (1899/1902) in detail.

The final session, Part IV of the seminar, entailed a whole-group discussion (facilitated by the first author) in which we sought to connect insights gained from the historical perspectives with the individual participants' mathematical

[2]Struve reconstructed Euclid's geometry as empirical theory using the methods of structuralism (Balzer et al. 1987) in his foundations of a didactics of geometry (Struve 1990). A hint for favoring an empirical interpretation is given in the way of defining the concepts. Thus, there is no need, for example, to think about "infinite lines." The lines need only to be potentially infinite by requiring the construction for extending them. Another argument for this reconstruction is Euclid's (via Heath) formulation of the parallel postulate: *"That if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are angles less than the two right angles"* (Heath and Heiberg 1926). Davis and Hersh gave a similar description of Euclidean geometry: "*Since the Greeks, geometry has had a dual aspect. It is claimed to be an accurate description of the space in which we live and it is also an intellectual discipline, a deductive structure*" (2012, p. 241).

biographies. We first reminded students about the preliminary discussions regarding different personal belief systems that occurred in the first session of the seminar. The intention was that the transparency on the historical problems that led to a modern abstract understanding of mathematics can therefore lead to an understanding of what happens if students live through this revolution on epistemological grounds as individuals, thus opening differentiated views on the transition problem.

4.3.2 Overview of the Pilot Seminar's Instructional Methods

The pilot seminar used various methods, especially for structuring the students' work phases. We used common instructional methods, i.e. jigsaw-group work, written discussions between students, and Think-Pair-Share, but we also employed non-traditional methods like using Metaplan™ cards and gallery walks, which included inspiring pictures and quotations to provide focus for the next part of the seminar or to summarize the ideas of the previous session.

In addition to these methods for structuring student work sessions we used two methods that we called "the big scroll" and coffee breaks. The "big scroll" was a large brown package paper roll that we used for capturing main ideas and findings of the seminar after and during the seminar work. During the three-day seminar, the scroll was visible in front of the students and could be used to look something up or to reconstruct the learning path of the whole group. The idea of capturing the learning path of the group was a fruitful idea, especially for showing the growth of knowledge in the group and the diversity of thoughts and beliefs that manifested during the seminar. The "big scroll" method had some disadvantages during the seminar. These include the handling of the scroll during the sessions, loss of session time used to write down and discuss in which ways general ideas should be preserved, and more importantly, the lack of individual reflection, especially through students' formulating their own thoughts. In the subsequent semester-long seminar we decided to implement reflection journals (c.f. Sect. 4.7), which prompted students to engage in a reflection process and draw connections between the single sessions of the full-semester seminar.

The coffee breaks played several roles in the pilot seminar. One obvious role was that they provided a clear break for recreation between the different parts of the intensive seminar. Another role was to prepare the participants for the next seminar session, through the use of posted pictures in the coffee break room. The pictures helped to pique students' curiosity about new topics. The coffee breaks provided important opportunities to create an open and positive environment for the participants, seminar facilitators, and observers, because this was the key for getting to know the students' personal struggles with their own transitions from school to university. Many students mentioned the openness and positive atmosphere among all members of the group working together in the seminar and the coffee breaks for getting know each other much better. Of course, the implementation of coffee breaks in a seminar is more reasonable in an intensive three-day seminar, and we did not use them in the semester-long seminar, which was structured in 1.5-h sessions.

4.4 Methods

As previously stated, the primary objective of the entire project was to design a seminar to address and support university mathematics students' transition from school to university mathematics. And, an important first step was to design and implement a pilot seminar to investigate the possibilities and outcomes. That is, we conjectured when we entered this work that the pilot investigation would lead us to a full-semester seminar. Consequently, the methods described here for the pilot seminar also correspond to the semester-long seminar; however, we describe differences in implementation as needed.

We defined several research questions that would enable us to characterize the ways in which students recognize and describe the transition from their own perspective. In particular, we were also interested in the ways in which the historical content played a role in students' recognition and description of the transition problem. In order to understand the results of students confronting their mathematical belief system(s) in this manner and using this intervention, we formulated several research questions deducted from our research hypothesis:

1. In what ways do students recognize and articulate an abstract/empirical or knowledge gap from school mathematics to university mathematics? And, how do they articulate their experience with navigating this gap?
2. In what ways do students use the historical development for explaining the transition problem?

For the purpose of examining what students recognized and described about school and university mathematics, we also found it necessary to determine the beliefs held by the students regarding mathematics:

3. What are the beliefs of mathematics held by students (pre-service mathematics teachers at university) prior to and after the ÜberPro Seminar?

To achieve our research goals, we conducted an intrinsic (multiple) case study (cf. Stake 1994), with the intent to provide insight of a particular case of university students' views of mathematics, while in the throes of the transition from school mathematics to university mathematics. The study was conducted at the University of Siegen. Twenty students (12 female; 8 male) participated in the pilot seminar, and although an elective course, it was one of several seminar options that students were required to take (i.e. they needed to take at least one). The age of the student participants ranged from 19 to 26 years, with an average age of 23. With respect to their time at university, students had been at university between three and 13 semesters; however, two students who took the seminar for "fun" were actually advanced master's students, and had been at university either 10 or 13 semesters. Accounting for these two students, the median time at university was five semesters.

Of the 20 seminar participants, we selected a subset of eight students to investigate as case studies. Students were identified as a potential case study respondent by either (a) interesting pre-survey responses that prompted one or more of the authors to want to know more about the particular student's views of mathematics or (b) their participation level during whole group discussions on the seminar content. Of the eight students identified, one student was outside of the focus population (one of the advanced master's students); that is, we wanted to investigate students' views on mathematics and their recognition of the transition problem earlier in their university mathematics career and not at the end of it. Another one of the eight selected students possessed an incomplete data profile. Finally, we determined six participants for whom to construct case studies, and the six represent the overall demographics of the seminar population, including gender, age, and number of semesters completed at university.

4.4.1 Data Sources

Several data sources were used to inform the case study, including pre- and post-surveys, a semi-structured interview (recorded and transcribed), and a final essay. The pre-survey was composed of four parts: background data (eight items), conceptions of mathematics (eight items), content (specifically related to the seminar content, including proof and argumentation in mathematics in general, and in geometry specifically (10 items)), and a small subset (20 total) of attitudes and beliefs items (Grigutsch et al. 1998). After a five-minute welcome message to the students, the pre-survey was distributed to students at the beginning of the three-day ÜberPro Seminar.

The post-survey included items from only two of the four parts of the pre-survey, since it was not necessary to collect background data again, nor did we feel that a three-day time lapse would yield remarkable changes in the subset of beliefs items. The final post-survey included four of the eight pre-survey items on conceptions of mathematics and six of the 10 pre-survey items on content. The post-survey was distributed to students at the end of the ÜberPro Seminar.

In addition to pre- and post-survey responses, a student participant's data profile also included an audio recorded and transcribed semi-structured interview, and the final essay (assigned as part of the seminar). Finally, we audio recorded and transcribed comments during a "summary discussion" at the end of the seminar, and this provided an additional data source for some of the case study constructions, as in the case of Inga, which we describe later in this chapter.

After all of the data were translated from German to English, the texts of these sources were analyzed using aspects of grounded theory. The analytic process began with open coding, defined by Strauss and Corbin (1998) as one in "which concepts are identified and their properties and dimensions are discovered in data" (p. 101). To identify initial codes, we used line-by-line analysis of each student's pre-survey, summary discussion comments, post-survey, semi-structured interview, and final essay. We began the process by first identifying initial codes from students' written responses on the eight conceptions of mathematics items in the pre-survey. Next, we used the initial codes in our review of the interview transcripts and the students' written essays. In our review, we coded for the two overarching domains from our first research question. That is, we searched for the ways in which students described their recognition and how they articulated an abstract/empirical or knowledge gap in the transition from school to university mathematics. In our analysis, we were able to collapse these codes into several categories (which we called "dimensions") for which to capture the ways in which Inga was able to recognize and articulate the abstract/empirical or knowledge gap present in the transition from school to university mathematics in general, and specifically for and from her own experience.

4.5 The Case of Inga

We found Inga as a future teacher to be an interesting case for several reasons. As a fifth semester university student in mathematics and French (as her second subject), she had quite a bit to say about the transition problem. We completed several passes through all of Inga's data and decided to construct her case to address the research questions outlined above. Inga's case was also interesting because of the level of emotionality in many of her interview responses, as well as during the summary discussion. For this paper, we provide only an English translation of Inga's responses and essay excerpts. The phases of analysis (with data sources) and the development of the resulting coding dimensions (general, self, historical, and praxis) for the two domains (recognition and articulation) addressed in the first research question are shown in Fig. 4.2.

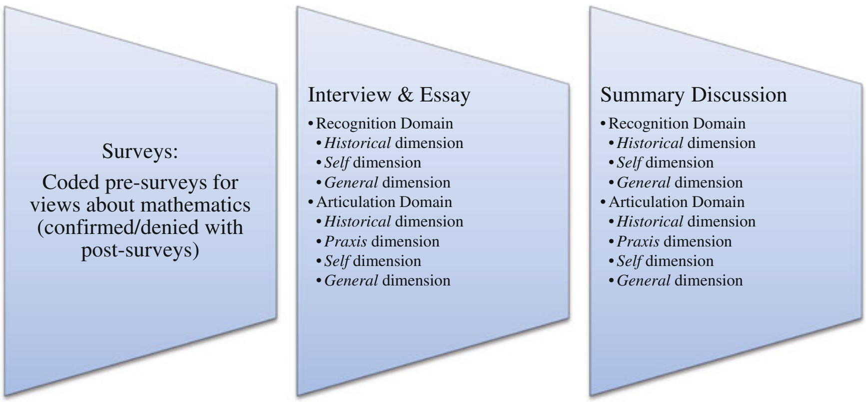

Fig. 4.2 Data sources, with accompanying coded domains and dimensions

We defined the four dimensions as:

General dimension: the student describes aspects of the transition from school to university in terms of the features one might identify as being part of the content or instructional practice at either school or university.

Self dimension: the student describes aspects of the transition in terms of their own, personal experience.

Historical dimension: the student's response includes features or examples from the historical development of mathematics in general (or geometry, in particular).

Praxis dimension: the student's response includes references to actual practices that have been (or that they perceive will be) problematic in perpetuating the transition problem.

4.5.1 Before the Seminar: Inga's Views of Mathematics and Recognition of the Transition Problem

On the pre-survey Inga's view of school and university mathematics were distinguished by a clear dichotomy. In response to the question, "Are there differences between school and university mathematics? And if so, name them and explain your answer." Inga drew the following (Fig. 4.3).

Additionally, Inga recognized that a gap does occur between school and university, and for her, this recognition is oriented along the *self* dimension:

> I find it unfortunate that the gap between school mathematics and university math is so large. I had after my LK (*Leistungskurs*; in English, "advanced course") felt that I'm good at math. At university the feeling has become quite quickly to the contrary. (Inga, pre-survey)

School Mathematics	**University Mathematics**
Computing, often according to the book	Not formulaic; independent solving strategies are required
(Almost) exclusively computing → only a bit of "why" asked	Many proofs + arguments
Tangible context / Application → related to everyday life and conceivable	Theoretical and very abstract

Fig. 4.3 Inga's identified differences between school and university mathematics

4.5.2 During the Seminar: Articulation for a Specific Application

We also examined the summary discussion transcripts for ways in which Inga recognized or further articulated the transition problem. Although the seminar was only three days in length, we would find it encouraging if students were able to articulate specific aspects or potential outcomes of the transition problem after only three days' of exploring different views and beliefs about mathematics and the accompanying historical genesis. In the discussion Inga shared:

> A few days ago I did [an] oral didactics exam for which I learnt a lot and read a few books, which all stated that everything should be made as graphical as possible and you need to neglect the abstract ideas, all while using realistic examples. The problem concerning the limit was introduced there… now I think that this graphical introduction is one of the largest problems concerning the transition to university. …I don't understand that why these didactic books all refer to this introduction as flawless and superb despite being the source of the problem. I think that's very controversial.

Thus, at some point Inga was able to articulate potential repercussions associated with the transition problem. In this way, Inga was able to move from recognizing the existence of the transition problem from her own learning biography (i.e. the *self* dimension) to articulating the transition in terms of future teaching practice (i.e. the *praxis* dimension).

4.5.3 After the Seminar: Inga's Recognition and Articulation Along Different Dimensions

Finally, Inga's interview and essay revealed her evolving consideration of the transition problem—within both the recognition and articulation domains. In particular, we provide one example of recognition (from the interview), and two articulation examples (from the final essay). In both domains, Inga was able to refer to the historical genesis that framed the content of the seminar.

Recognition (*historical* dimension):

> And I think it is important, that this change [e.g. the transition from school to university mathematics] does not only happen to us, but rather you can see it also in earlier times, when you are looking in history. And that then many other people probably had the same problems, or even bigger problems…

Articulation (*self* dimension):

> Nevertheless, I did not know before our seminar what the actual causes of my initial, sudden difficulties in mathematics were. Originally this fact – that I suddenly had difficulty in math – gnawed at my very self-confidence…However, in the seminar I realized that this altered image of mathematics plays a major role in the transition between school and university.

Articulation (*historical* and *praxis* dimensions):

> …compare this [modern mathematician's view] with the idea after Hilbert, which allows other geometries to be true, even though they contradict the original Euclidean geometry first. This fact is responsible for ensuring that the mediated image of mathematics in school falters and must be changed. [One] is now on his own [at university] to figure out that mathematics not only possesses one truth, but under different circumstances, even contradictory statements can be perceived as quite correct, because mathematics is a science, something true, once it can be logically deduced. This is an abstract and very theoretical requirement that students must tackle in order to deal with the university mathematics.

4.6 Inga's Confrontation of the Transition Problem: Influenced by History?

Although we were interested in capturing Inga's overall experiences with and impression of the ÜberPro Seminar, we were also particularly interested in the ways in which the historical content impacted how Inga discussed her personal confrontation of the transition problem. Thus, a specific sub-question we investigated related to the first research question was:

> 1a. In what ways do students refer to history in their description and reflection of their individual problem with the transition (from school to university mathematics)?

And, with respect to the second research question, we asked:

> 2a. In what ways do students deal with (e.g. confront, agree with, reflect upon) the research hypothesis?

4.6.1 History of Mathematics Informing Inga's Transition

As we have previously described, Inga's case was of interest to us because she articulated an almost palpable frustration with the transition from school to university mathematics, and she did so in different ways. In one way, Inga used the content of the seminar to compare what had occurred historically with her own transition. When asked to describe in her words during the interview what was the message of the ÜberPro seminar, Inga stated:

> Actually I would say we talked a lot on geometry; on different geometries. I would say, that the message was, that there was a change of view also in the history of mathematics. But I must say I recognized this personally at the very end of the seminar…

When we prompted Inga further about the potential support for the hypothesis for the seminar, she observed:

> We…sat there in [the coffee] breaks and in the evening and thought about how this is connected to the main theme [transition problem]. And then you figured out, that the change [that took place] in history also takes place in the transition from school to university. That was something that you become aware of.

During the interview, we explicitly asked Inga about the connection between what had taken place in history and the transition from school to university. However, even before the interview, Inga was aware of this connection and she addressed this in her essay. She wrote:

> During the seminar for me it became clear that this changed picture of mathematics plays a crucial role for the transition from school to university. This finding was further supported by a look into the history of mathematics. We have especially concentrated on geometry, because there you can also find a significant change of beliefs.

After Inga provided an overview about the historical development of geometry in her essay, she then connected views about mathematics at school and university to historical views on geometry.

> But how can the different beliefs on geometry and Hilbert's statement be connected with the transition problem?
>
> Especially the visual, tactile geometry of Euclid can be described metaphorically by the mathematical view, which is transmitted at school. […] When students decide to study mathematics at university, they will be confronted with a different view on mathematics. [And this is] similar to the notions of Hilbert.

Thus, Inga identified parallels between the historical development of mathematics (in this instance for the seminar, the historical development of geometry) with her own transition; that is, a change of view was evident in both.

4.6.2 The Research Hypothesis Realized: Supporting Inga in Her Transition

There are two key elements within our research hypothesis. First, we claim the change from an empirical-object oriented belief system to a formal-abstract belief system constitutes a crucial obstacle in the transition from school to university. Second, we state that changes regarding the differing natures of mathematics can be described in similar ways for the history of mathematics. And, related to this, explicitly analyzing the historical genesis provides support for students as they deal with, navigate, or journey through their individual transition.

Thus, we began this research with the idea that we would make the transition problem explicit to students and that we would do so using the historical development of geometry as the vehicle to portray a change in nature of mathematics (or, in this case, geometry as a specific example). However, in doing so we were unsure about how students would deal with the key premise of the hypothesis, which was that the seminar would support the students in their transition.

We anticipated that a three-day seminar would not serve to substantially change students' views on mathematics. Nonetheless, we questioned Inga to determine how she currently assessed her view on mathematics and she stated:

> Actually I think not. For my view on mathematics, I had in school a more application-oriented view, which we get taught. How should I explore another one? And at university, first I could not deal with mathematics and then I have understood mathematics as a language, I also study a language, because this is a contained system and so I could compare mathematics to it very well and at least accept it [mathematics]. And this doesn't change; that there are two different views, which changed in time for me, and now all in all my view doesn't change.

Yet the belief change Inga did experience was from an empirical view of mathematics to a formal view of mathematics, as she described in her essay:

> Meanwhile for me, mathematics is characterized as a language, because it is an enclosed system of different rules, formulas, and abstract signs. This picture about mathematics I achieved firstly at university, because I was firmly convinced that mathematics has a strong relationship to the everyday world and that usual situations can be explained by appropriate mathematical models. Foremost, I was curious that mathematics was almost everywhere. Sadly, it was not possible to hold this view on mathematics because of my course of [mathematical] studies, because the contents for learning became more abstract and theoretical.

Inga's written description of her change in beliefs was echoed during her interview (which we have previously mentioned), in which Inga declared that, "… in the seminar I realized that this altered image of mathematics plays a major role in the transition between school and university."

And, finally, Inga's case is exemplary in that she claims to support her future students in a way similar to what we intended in the seminar. In her essay, Inga wrote:

> I think it is thoroughly appropriate to unveil different views on mathematics to students at school. I hope that I can convey a broadly diverse view on mathematics while I am a teacher, so that my future students have the opportunity to get to know different beliefs, and so that they are protected – in case of studying mathematics at university – against the abstraction shock, which I suffered myself.

4.7 Rethinking the Seminar: Converting the Pilot Seminar into a Semester-Long Seminar

Our experiences from the pilot seminar, the accompanying research, and students' feedback gave us some ideas about what should be changed in a second seminar trial (cf. Fig. 4.4). The main ideas and aims of the seminar remained the same as in the pilot seminar since the research hypothesis was supported by the empirical data of the pilot seminar (cf. Sect. 4.6). The major changes on the seminar content and structure were the expansion of the intensive three-day seminar to a semester-long seminar, consisting of 11, 1.5-h sessions (Sect. 4.7.1), instructional changes (Sect. 4.7.2), and the inclusion of requiring reflection journals from the seminar participants (Sect. 4.7.3).

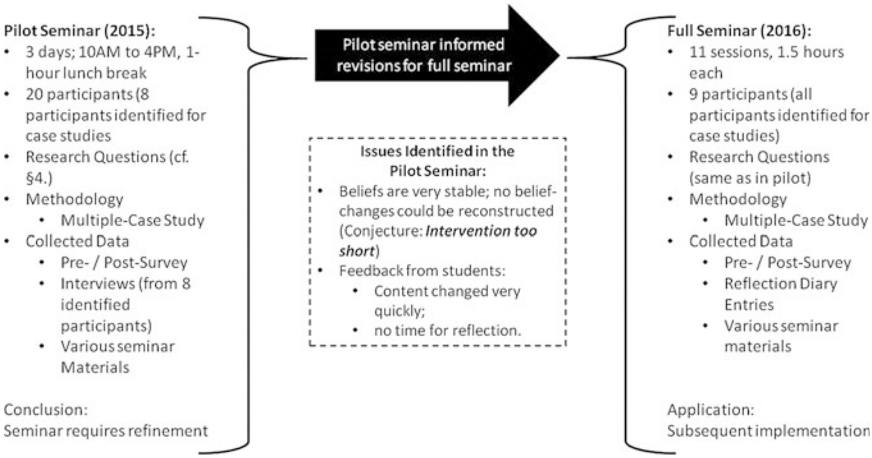

Fig. 4.4 Schematic summarizing the need for modifying pilot seminar into the full seminar

4.7.1 Content Changes Implemented for the Semester-Long Seminar

The expansion of the seminar was motivated by students' feedback about the three-day seminar. Students stated that the seminar was very intense and thus difficult to remain aware of the research hypothesis while exploring the views on mathematics in the history of mathematics. Instead, the seminar content was restructured by dividing the original four parts of the seminar into 11 sessions. This enabled us to strengthen the focus of each topic and to give sufficient time for the students to reflect. Another change implemented in the semester-long seminar was to repeatedly remind students of the research hypothesis, for the purpose of making the students aware of the parallels between the historical development of mathematics discussed during the seminar and their individual transition process. The agenda of the seminar in summer 2016 is shown in Table 4.1.

In addition to extending the number of seminar sessions (originally taking place over three days, to comprising a semester-long seminar), we also modified the content of the seminar.

The first change included the insertion of one session on proofs of the Pythagorean Theorem. The pilot seminar included only a motivation of Euclid's *Elements* as a piece of mathematics par excellence, as it is described by historians and philosophers of mathematics, the popular media, and mathematicians themselves. In addition to this motivation, we also focused on the question in which ways the theorems in Book 1 of Euclid's *Elements* are connected and can be traced down to the axioms and postulates. Thus, after session 4 we inserted a new session (now session 5) in which students explored several proofs (e.g. Göpel, Bhaskara, Perigal, Garfield, Epstein and Nielsen, Bride's Chair) of the Pythagorean theorem.

Table 4.1 Agenda of the ÜberPro geometry seminar in summer 2016

Session	Topic	Part of the seminar
	ÜberPro geometry: pre-questionnaire	
1	Organization and overview	I
2	What is mathematics? What are beliefs about mathematics?	I
3	Between the poles of empirical-concrete versus formal-abstract	I
4	A paradigmatic example of the axiomatic structure of a mathematical theory: Euclid's *Elements*	II
5	The Pythagorean theorem: typical proofs in the classroom	II
6	Projective geometry: from grabbing to seeing	II
7	Projective geometry at school?	II
8	Proving the parallel postulate	II
9	Non-Euclidean geometries and the angle sum of triangles	II
10	Paving the way to formal mathematics	III
11	Debate and final discussion	IV
	ÜberPro geometry: post-questionnaire	

There were two reasons for this addition. On the one hand the topic is strongly connected to session 4, because the Pythagorean Theorem can be seen as the climax of Book 1 of Euclid's *Elements*. On the other hand, the Pythagorean Theorem is a prominent topic in elementary geometry, even when proving this theorem is no longer mandatory in school instruction (e.g. Kultusministerkonferenz 2003).

The most important argument for adding this session is that we were able to discuss students' understanding of what comprises mathematical proof. We planned two activities in the session to encourage reflection about this important concept. The first activity included two videos. The first video showed an experiment in which water flows from two cuboid canisters with square base areas into another cuboid canister, whose base area was a square with the area equal to the sum of the areas of the other two bases. In the second video, two solid cuboids and another cuboid with similar properties of their shapes were weighed against each other using a beam balance. After a short discussion of the benefits for motivation and empirical verification of the Pythagorean Theorem in the real world as well as the risks of these examples (e.g. that students could understand them as proper mathematical proofs), students prepared a short presentation in small groups (two to three students each). The presentation included one of six different proofs of the Pythagorean Theorem based on a worksheet for Grade 8 (QUA-LIS 2003). Afterwards, students highlighted the differences among the proofs in their reflection journals.

The second change in creating the semester-long seminar involved the sources, materials, and literature used during the seminar for the topic of non-Euclidean geometry. In the pilot seminar, we used various textual sources, which described the myth of Gauss' experiment, for proving empirically that the interior sum of a triangle is 180°, and excerpts from geometry scripts dealing with hyperbolic and elliptic geometries. For a first understanding of the different geometries and their particular geometrical properties we used real models in the form of a Styrofoam sphere and a paper structure, where the students experienced different interior sums of triangles drawn on these objects. Still the students stated that thinking about these different geometries was difficult for them. We decided in the subsequent seminar to use a more coherent source. Our choice was "Nicht-Euklidische Geometrie" (van Gulik-Gulikers 2010), which is one chapter of an original Dutch book for upper secondary classes about geometry and is designed for initiating autonomous project work with students. The chapter begins with a recapitulation of the axiomatic structure of non-Euclidean geometries and develops from this the historical question about the truth of the Parallel Postulate, which leads to elliptic and hyperbolic geometries. The described historical development and materials about both geometries are accompanied with (reflective) tasks, which should be solved by the students on their own. Inspired by the way in which the text promoted working on these geometries, we used models of the geometry including solid spheres and GeoGebra tasks for understanding the different geometries. In the seminar we discussed the texts and the solutions of the tasks, especially regarding similarities and differences between Euclidean and non-Euclidean geometries. From our perspective, the students were able to delve more deeply into non-Euclidean geometries. This can be seen in the following excerpts from the reflection journals after the last session of the non-Euclidean geometries.

Questions for reflection:
1. What are similarities and differences between the non-Euclidean, Euclidean, and projective geometries?
2. What do you think? How can the knowledge addressed in today's session help you to overcome the transition from school to university?

One example of a student answer:
1. The different geometries have in common the names they use. But what the meaning of a line or a point is, is very different. And therefore the constructs based on these fundamental objects as well as their relations are different as well. The interior angle sum of a triangle might not be 180° at all, but can be smaller or greater, depending on the geometry, than 180°. Also the understanding of parallel lines changes, because the basic notions are changing. There can be one or more or no parallels depending on the point of view.
2. I believe that the knowledge on further forms/kinds of geometries can change the understanding on mathematics in the way that you change the perspective from a static view on mathematics to a more flexible one. In mathematics you deal with closed systems, that means you have to know continuously in which system you are, and what are the foundations of this system. You need to have the mental ability to be flexible to draw a distinction and work in different systems. In the sense of the transition problem this example can give the opportunity that mathematics does not have to be seen as one construct, but more as a collection of constructs which should be handled flexibly and in which you can independently and freely move between and work within them.

4.7.2 Instructional Changes Implemented for the Semester-Long Seminar

In the semester-long seminar we incorporated significant refinements regarding the instructional methods that went beyond the implementation of participants' reflection journals and use of new media, i.e. GeoGebra. In addition to learning new content and participants reflecting on their own beliefs, we wanted to give the students the opportunity to regulate their learning processes more freely and to acquire a toolbox of teaching methods throughout the seminar. Therefore, we used the learning management system Moodle™ for sharing materials, providing web links or resources that described the teaching method used in the seminar, giving opportunities to discuss and prepare via Moodle™ communication forums and sharing modules for student interaction, and providing the opportunity to write their reflection journals online. All in all, we tried to make the use of different methods and their benefits for different seminar sessions explicit and accessible.

We closed the pilot seminar with a summary session, which included a presentation of the seminar topics. In session 11 of the semester-long seminar we used a debate with the topic, "What is geometry and what is it for?". For the debate, groups of two to three students prepared for the role of a moderator, selecting from David Hilbert, Moritz Pasch, Gottlob Frege, and Hermann von Helmholtz. The groups had one week to prepare for the debate, after which one student from each group represented the role of the historical figure in the debate. In the students' short evaluation feedback about whether this method should be used in future seminars they made clear that even if they were only observers of the debate, they felt more aware of the differences among the different views on geometry, which were embodied by the four historical roles. When offering improvements about this particular session for future seminar implementation, students commented that they wished they had more time to prepare for the debate.[3]

4.7.3 Implementation of Reflection Journals

Implementing reflection journals at school and in university is not a new idea. In Germany, the most popular grounding for the use of learning diaries or reflection journals in mathematics classes is found in the work of Gallin and Ruf (1991). Motzer (2007) provided an example of the use of learning diaries to accompany a number theory course at university. Written reflection journals are not only a profitable opportunity for learning and reflection for learners, but also for teachers. When students share their journals with the teacher, he or she can in turn improve their teaching by gaining deeper insight into what students already know or feel.

[3]During the first version of the semester-long seminar, students prepared for the debate in groups, and this took place between sessions 10 and 11.

Furthermore, for researchers, students' reflection journals are a rich resource for several reasons. Our work is informed by the reflection journals in three ways.

First, the reflection journals provide an evaluation of the seminar, including whether the materials are beneficial for the students' understanding, as well as transporting the views about mathematics held by the authors. Second, the reflection journals are a much more individual enterprise than the "big scroll" used in the pilot seminar. Third, the reflection journals provided important evidence concerning whether the seminar was really giving students *"…the explicit analysis of the historical genesis"* and whether it was providing *"support for students dealing with their individual transition processes"* (research hypothesis for the seminar).

The students were able to complete their reflection journals online via the journal module in Moodle™. Alternatively, they could complete their journal entries outside of the online environment using pen and paper if they preferred to handwrite their journals. The reflection journals played a major role in the semester-long seminar as well as for the research on the evolution of students' beliefs during the semester.

The reflection journals seem to be an important data source from the semester-long seminar. The analysis of students' journal entries (one excerpt is shown in Sect. 4.7.1) from two semester-long seminars (summer 2016 and summer 2017) will further inform our research on the transition problem, and which will inform future publications.

4.8 Implications

Describing the case of Inga enabled us to assert that students participating in the three-day ÜberPro Seminar were able to recognize and articulate the transition problem from school to university mathematics. Furthermore, as in the case of Inga, the seminar's goal of raising explicit awareness of the problem was of particular value for her (*self* dimension). There is evidence to indicate that the historical content of the seminar can play an important role in university mathematics students' (and pre-service mathematics teachers') ability to navigate the crucial transition (*historical* dimension). Implementing the initial three-day seminar was vital in providing evidence to pursue a longer intervention with prospective mathematics teachers at university.

This research study was initially started to empirically test the theoretical framework developed in the working groups of two of the authors (first and third), and with additional empirical research (second and fourth authors), now exhibits benefits of this approach for understanding the transition problem. The project data indicate on the one hand that students' belief systems are quite stable and need further long-term measures. On the other hand, the case of Inga shows that the idea of an intervention seminar using epistemological parallels regarding historical developments and individual learning biographies is a useful tool to reflect on the transition problem, giving a jump-start to an evolutionary process regarding adequate belief systems about mathematics. Additionally, the seminar provided participants (university mathematics teacher-students) with the ability to formulate their own point of view regarding their understanding of mathematics—which is an absolute necessity for respecting and developing others, particularly when preparing to teach mathematics. And, as we witnessed with the case of Inga, she discussed her intent to share the different views about mathematics with her future students.

As a result of analysis of the three-day seminar piloted in early 2015, the first and fourth authors implemented a semester-long seminar in summer 2016, and the first, second, and fourth authors will implement a semester-long seminar in summer 2017. As we have described here, modifications were made to expand the three-day seminar into a semester-long seminar. Implementing these modifications while keeping the historical content essentially the same allows for additional time to be spent on the four historical points of view in the development of geometry (cf. Sect. 4.3.1). This in turn enables longer engagement with seminar tasks and supporting more substantial small group and whole group discussions. Additionally, during the semester-long seminar, students were required to keep reflection journals by responding to weekly reflection questions. The research analysis of the 2016 implementation will focus on the particular elements from the historical content that students used to recognize and articulate the transition problem. In particular, we plan to extend cases like Inga's to include commentary of the historical content from the intervention on students' individual transition process.

We anticipate that the semester-long seminar will enable us to identify deeper insights about the diversification of students' belief systems and their awareness about their own beliefs, and to support university students in their articulation and awareness of these diverse beliefs. In particular, we aim to assist in overcoming the double discontinuity that Klein (2004, p. 1) identified, by highlighting the importance of views on mathematics for mathematical learning, which can also survive in the mathematics teacher candidates well beyond the second transition of becoming mathematics teachers at school:

> The young university student found himself, at the outset, confronted with problems which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time-honored way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching. (Klein 2004, p. 1)

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Part II
Courses and/or Didactical Material: Design, Implementation and Evaluation

Chapter 5
Facilitating Source-Centered History of Mathematics

Developing Materials for Danish Upper Secondary Mathematics Education

Kristian Danielsen, Emilie Gertz and Henrik Kragh Sørensen

Abstract We present and discuss initiatives to develop source-centered teaching materials in history of mathematics for upper secondary education, aiming at meeting the objective of the Danish curriculum to make history of mathematics relevant. To this end we present the design template for such multi-purpose materials we developed, which allows devising materials neither too superficial nor too specialized, and we address the constraints on and affordances of historical sources in adapting to teaching objectives. It includes differentiation and scalability for using historical sources, and provides opportunity for interdisciplinary teaching, another requirement for Danish upper secondary education. We also report on (i) the recent application of our design approach to develop such source-centered materials in collaboration with small groups of dedicated teachers, and (ii) students' positive response to the inquiry-driven teaching based on this material.

Keywords Teaching with sources · Interdisciplinary teaching · Logistic growth
Developing teaching materials

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5.1 Introduction

Teaching a subject—such as mathematics—involves teaching a set of core topics, a set of competencies, and an outlook on that subject, including its identity, methods, historical development, and relation to other subjects in the curriculum and its societal role. Different curricula put different emphases on these components, and in some respects, the Danish case is a special one for the role the curriculum attributes to history of mathematics and philosophical issues about the identity of mathematics and the processes involved in mathematical modeling.

The use of history of mathematics in teaching mathematics in upper secondary schools provides a potential to have students engage with sources in order to address questions of 'why', 'how', 'when' and 'where' which are otherwise rare in the mathematics classroom. This potential can be exploited by an inquiry-based, source-centered approach to teaching history of mathematics, but this raises two demands: Suited materials must be available and teachers must have relevant training in teaching with history of mathematics.

In this chapter, we present and discuss initiatives to develop source-centered teaching materials in history of mathematics aimed at upper secondary mathematics teaching in the Danish school system. We present a design template for such multi-purpose materials that includes differentiation and scalability for using historical sources and provides opportunity for interdisciplinary teaching which is also a requirement for Danish upper secondary education. This approach allows us to devise materials that are neither too superficial nor too specialized. In particular, we address the constraints on and affordances of historical sources in adapting to teaching objectives. We also report on a project with a small group of dedicated teachers aiming at facilitating the development of source-centered materials.

5.2 History of Mathematics in Danish Upper Secondary Schools

The largest sector of Danish upper secondary education (16–19-year-old students) is the *general gymnasium*, STX. This education provides a general education in a broad range of subjects, including mathematics. In the STX, mathematics is taught at the mandatory one-year C-level aimed at providing general, critical education for citizenship, at the two-year B-level, adding focus on mathematics applied in the sciences, and at the three-year A-level, further adding grounding for the sciences. The education also includes a large project (SRP) in the final year, combining two subjects. Until 2016, a mandatory interdisciplinary component (AT) focused on the methods of different disciplines was also taught. This component has since been abolished and some of its perspectives moved into the large project where philosophical considerations will form part of the evaluation criteria, and interdisciplinary reflections on methodology remain central.

In STX, history of mathematics is—and has been for decades—a mandatory part of teaching mathematics (see Kjeldsen and Carter 2014). For instance, at all levels of the STX, the supplementary material (25 h per year) has to include topics in history of mathematics. These topics can, at least in part, be involved in addressing the objective to enable the students to "demonstrate knowledge about mathematical methods, the application of mathematics, and *examples of the interaction between mathematics and other developments in science and culture*" (*Læreplan Matematik C*, translation by the authors, emphasis added). This quote is taken from the STX-C level, but the role for history of mathematics is even stronger at STX-B and STX-A levels, where the ministerial recommendations for teachers also stress history of mathematics as a way for teachers to collaborate with colleagues in other subjects, particularly history, to seek out opportunities for interdisciplinary teaching. The recommendations now (2017) also explicitly mention working with sources as a means to stimulate inquiry and curiosity in the students (*Matematik A/B/C, stx: Vejledning*, p. 26). Yet, no specific suggestions are made concerning what history of mathematics to teach and how to teach it.

Thus, the curriculum and ministerial recommendations stress including history of mathematics as part of teaching *about* mathematics and to enable interdisciplinary teaching. The underlying reasoning is not only that history of mathematics can be important in and by itself, but that history of mathematics contributes to the students' understanding *about* mathematics and its social and scientific use (see also Jankvist 2009).

Although teaching of history of mathematics is mandatory in upper secondary schools, studies have shown that mathematics teachers experience challenges in implementing it; these challenges span from insufficient confidence and competence in reading historical sources to lack of well-designed, relevant mono- or interdisciplinary teaching situations integrating historical sources (see e.g. Johannesen 2015; Jørgensen 2013). Additionally, teachers are under constant time pressure, and since no fixed recommendations exist on *which* topics to address from the history of mathematics, it is left to the individual teacher to identify and screen suggestions. Participants in our workshops confirm this and stress that collecting materials, and in particular identifying good sources, are difficult tasks for teachers to perform. These concerns call for activities on two fronts: Providing teachers with the means and confidence to use sources to teach history of mathematics is not easily addressed in traditional written communication but must, we believe, be taught either in university level courses on history of mathematics or in in-service training of more experienced teachers (see also Clark 2014). And providing relevant, easy-to-use and quality-tested teaching materials and devising interesting teaching situations in which to use them is a challenge that should be approached through integrative collaborations between academic historians of mathematics and motivated teachers. Thus, two challenges arise: how to educate the teachers so that they can utilize these materials and how to devise and develop suitable materials for classroom usage. In the following, we present and discuss our framework for selecting and developing source-centered materials.

5.3 The Potentials of Source-Centered History of Mathematics

To meet the objective of the Danish curriculum to make history of mathematics relevant and enable a variety of interdisciplinary collaborations, we have developed a source-centered framework for presenting authentic history of mathematics relevant to the educational situation and teaching objectives. Our combined backgrounds as professional historians of mathematics and mathematics teachers at the upper secondary level integrate the main concern of such an endeavor, namely to balance the needs incurred by teaching a specific curriculum while not losing sight of historical complexity (see Fried 2001). A central challenge for a source-centered approach to teaching with history of mathematics thus lies in *identifying* and *making accessible* sources that can address relevant teaching objectives.

Recently, the use of primary sources in teaching mathematics has been addressed from an educational perspective (see e.g. Jahnke et al. 2000; Jankvist 2014; Pengelley 2011). Our approach consists in integrating reflections on teaching objectives with efforts to identify sources and develop materials to achieve relevant teaching of history of mathematics. Thus, this integration should evolve into an extended dialectic process, optimally undertaken in collaborations such as ours between (academic) historians of mathematics and devoted mathematics teachers.

Among the main characteristic constraints of a potential source to take into account are its length, language and level.

The source can obviously not be too long, since students will not be able to work their way through it in a reasonable amount of time. On the other hand, the source also cannot be too short, as a mere snippet will not facilitate the range of perspectives intended. Thus, what constitutes the entity of a source needs to be negotiated; it could be an entire (short) research article, a chapter or section from a monograph or article, a few graphical illustrations, a set of data, or a series of images of mathematics and mathematicians. Importantly, the central source should be complemented by other sources from its contexts of production and influence. In order for the central source to be accessible to students in the Danish context, it must be in either English or Danish. In reality, given the literacy of Danish students in English, even English texts must probably be translated, not least if they use archaic language or orthography. Providing such a translation is not, in itself, a restriction on the source but rather on the expertise required for the development of suitable materials. Ideally, original mathematical notation is to be preserved as it provides an opportunity for students to practice reading authentic, foreign mathematics and develop their skills at manipulating mathematical symbols and relations (see Kjeldsen and Petersen 2014). An interesting perspective could be added to the teaching objectives by including (extracts from) the original text in facsimile, using original orthography and print.

Probably the most difficult constraint to meet is matching the mathematical level of the central source to the desired teaching objectives. Again, this is confirmed by participants in our workshops who also emphasize that the use for mono- and cross-disciplinary purposes may raise different concerns about the appropriate level of technical mathematics in the source. As one of the participants explained about locating historical sources: "It is difficult to find anything that contains enough mathematical content without it being too difficult. And finding interdisciplinary connections provides another difficult challenge." Thus, the suitable source need not be immediately accessible to the students, but it cannot be too far beyond their mathematical level and it will require scaffolding for the students to engage with it. Thus, not all sources are usable in upper secondary mathematics education as they may well involve techniques or topics beyond reach of the curriculum. However, by broadening the search beyond the original formulations of new ideas and looking for historical didactical texts (such as expositions or textbooks), some otherwise inaccessible topics may still become open to the source-centered approach. Thus, the initial search for sources, which could start with standard collections, is likely to benefit from engaging with the expertise of research historians of mathematics.

These main constraints are to be balanced against the affordances that the source provides for attaining relevant teaching objectives (see Fig. 5.1). In other words, the source can allow for interesting perspectives in at least the following three main ways: the content and its presentation, itself, can be an attraction for using the source, the source can feature in an interesting internal context such as the development of an important theory, or the source can be embedded in an interesting

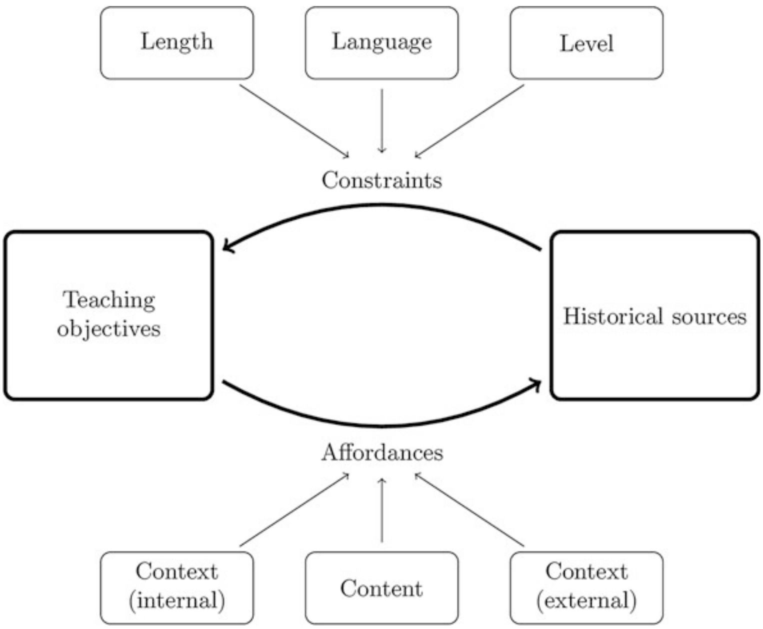

Fig. 5.1 Dialectic of affordances and constraints between sources and teaching objectives. In the process of developing materials (see Fig. 5.3), this corresponds to tasks 1 (source) and 4 (teaching objectives)

external context such as being the solution to a problem of societal or scientific importance.

The relative importance of these affordances depends on the teaching objectives stipulated by the curriculum and the teaching situation. It will be an extra attraction if the content would fall within the core curriculum or at least be in direct continuation of topics already familiar to the students. The teaching objectives might include motivating new topics by situating their invention or cultivation within the internal or external contexts in which they arose or were developed. Moreover, the potentials for integrated teaching or interdisciplinary collaborations with other subjects such as history, Danish, languages and classical culture may be attractive affordances.

Ideally, the dialectic process of searching for sources and expounding the teaching objectives they can address will lead to a central source that can be adapted for multiple purposes (see Fig. 5.2). Among the specific purposes for using the source-centered approach are (i) serving as (relatively short) introductions and motivations of new topics within the mathematics teaching, (ii) grounding more extensive teaching of a mathematical topic in a (longer) inquiry-driven approach to reading a historical source, (iii) the integration with another subject (in particular history or foreign languages), (iv) providing students with the foundations for independent projects (in the Danish STX in particular the cross-disciplinary SRP), (v) forming the foundation or perspective of an interdisciplinary project (such as the SRP), or even (vi) providing the inspiration and content for study trips abroad (which are semi-mandatory in the Danish STX). Naturally, not all these objectives are realistic to achieve to the same extent for every source or material, but keeping them in mind and addressing them as they become relevant is a key design feature of our approach. In order to provide multi-purpose materials, which allow for a wide variety of such teaching objectives, we address our materials at the teacher, not the students, and expect the teachers to provide the transformation into teaching the specific objectives that they have in mind.

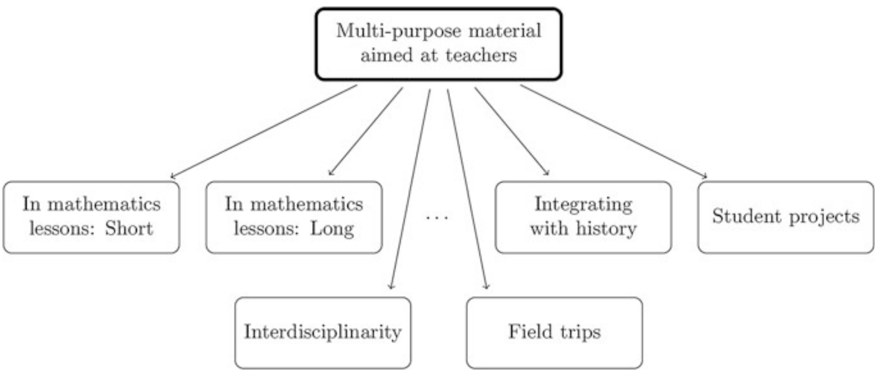

Fig. 5.2 Multi-purpose material aimed at teachers who will transpose it for specific use

Obviously, focusing on a single source may impose restrictions on the learning outcomes to be gained. There are (important) aspects of the historical development of mathematics that are not accessible from a strict focus on just one instance in the past. Yet, by narrowing the focus to one such instance and providing the relevant contextualization, we gain authenticity and access to equally important learning objectives about the motivation, use and changing nature of mathematics. Thus, by showing authentic differences in mathematical thought and text, we can at least identify different points on a time-spectrum of mathematical development. If the historical processes in-between these points are the desired teaching outcomes, the source will need to be properly contextualized in these respects and, preferably, supplemented by other authentic materials. Thus, in a sense, our tradeoff is one between authenticity on one side and historical continuity on the other side. As such, the balance is not particular to upper secondary history of mathematics, but is inherent in history teaching more generally.

5.4 Developing Materials and Introducing Teachers to Sources

In developing pilot materials directed at teachers, we have adopted a common template:

1. An *introduction* addressed to the teacher.
2. A short, engaging *narrative* situating the source by introducing central actors, places, developments and perspectives. To a limited extent, this narrative can include historical dramatization. It, too, is intended for the teacher, but it could also prove relevant (and readable) to students.
3. The *source* in Danish translation (if necessary) and with as much preservation of mathematical notation as possible.
4. A detailed, step-by-step *elaboration and commentary* of the source, including suggested exercises for students and hints and solutions for the teacher. The purpose of these exercises is to engage students to work with technical, conceptual and contextual aspects necessary for understanding the source. Thus, this part will consist both of historical contextualization (4a) and elaboration of technical mathematics (4b), providing a mixture of ordinary explanation and exercises for guiding students' inquiries.
5. A series of suggestions for addressing various *teaching objectives* based on the source. These suggestions range in perspective and in scope from short, motivating and contextualizing one-lesson introductions (*tall*), over in-depth approaches to the relevant mathematical topics through the source spanning multiple lessons (*grande*) to suggestions for interdisciplinary teaching and student projects (*venti*).

Furthermore, a set of resources that include PDF files of the central source, our suggested exercises, and examples of adaptation to teaching situations provided by teachers who have used the material are made available online.[1]

The set of pilot materials presently includes one on logistic growth based on Pierre-François Verhulst's (1804–1849) original source from 1838 and one on Hero of Alexandria's (c. 10AD–c. 70AD) procedure for computing the area of a triangle based on his *Metrica* (Danielsen and Sørensen 2014, 2016). Additional material on the introduction of Hindu-Arabic numerals in the Icelandic thirteenth-century manuscript *Hauksbok* is in preparation (Jensen et al., in preparation; see also Bjarnadóttir and Halldórsson 2011), and two further materials have been planned. These sources are all chosen because they offer a combination of important historical (and philosophical) reflection about mathematics with a mathematical content that is accessible and interesting. At the same time, they open possibilities for interdisciplinary teaching with history, biology, classical culture, etc. Thus, Verhulst (Danielsen and Sørensen 2014) offers a way of discussing the development of mathematics for societal problems and an important insight into mathematical modeling. At the same time, logistic growth is something students need to study, anyway. The case of Hero's formula (Danielsen and Sørensen 2016) offers an opportunity to supplement the standard picture of Greek mathematics in the Euclidean tradition with a mathematician quite explicit about the role of mathematics and computation, and students will be surprised to find that a "formula" was described in such verbose text. And materials about Arabic arithmetic (Jensen et al., in preparation) may provide an occasion for showing that mathematics was part of a great cultural exchange, while at the same time challenging students to understand relatively simple mathematics presented in unfamiliar forms. Thus, our criteria for choosing and developing these materials were precisely combinations of historical and mathematical constraints and affordances (for a suggestion for developing historical awareness, see also Kjeldsen 2010, 2011).

The design of the template was based on a variety of academic and practical input. From related discussions about teaching Nature of Science (NOS), we were inspired by Douglas Allchin to engage students with an introductory narrative to which they could relate (point No. 2 above; see e.g. Allchin 2013). Since the ambition was to support student inquiry, it was required to provide teachers with a provisional background (point No. 4 above). The extent of the contextualization and commentary was mainly determined by drawing on the teaching experience of members of the group. In order to make the introduction into actual teaching practices feasible, we felt a need to provide examples and suggestions of practical use (point No. 5 above). These considerations were largely validated by the feedback that we have received (see below).

A specific role for the contextualization is to provide a background for an understanding of the mathematical source in a nuanced way that does not, in particular, reduce to anachronisms or mere translations into modern notation and

[1]Available at http://www.matematikhistorie.dk/; accessed August 7, 2017.

terminology (see also Fried 2014). Since a main purpose of using history of mathematics in this way is to provide a broad conception *about* mathematics, it is necessary to address both similarities and differences between historical and contemporary mathematical practice. Thus, although the sources may be chosen for their relevance to the contemporary curriculum, they need to be situated in their historical contexts for this higher-order learning objective to be feasible. This objective can be brought to the students by explicitly engaging them with translation and interpretation of concepts, terms, and notations that are foreign to them (see also Danielsen and Sørensen 2016).

Our framework and reflections were developed as we produced the first material on logistic growth, and subsequently refined in working with the latter materials. The case of logistic growth was initially chosen because it offered a number of attractive affordances (see also Danielsen and Sørensen 2015). For example, the original source has a suitable length (approx. 5 pages). Its language was rather non-technical and could be translated from the original French into Danish with minimal effort. Its mathematical content formed part of the core curriculum in STX-A, whence the teacher could tag on a historical component to existing teaching at little extra cost in the form of teaching time. And, the reflections of Verhulst in the midst of his modeling process opened for some very interesting perspectives on mathematical models that were well known in the philosophical literature but were difficult to explain in teaching situations by showing rather than by merely telling.

The produced material on logistic growth was aimed at teachers but began by situating the source through an introduction that was also meant to be readable by students. Subsequently, historical and mathematical elaboration and context was provided to allow the teacher to quickly orient herself in preparing for teaching. This led up to suggestions for *tall*, *grande* and *venti* adaptations to teaching in an open-ended fashion so as to not instruct but inspire teachers. For instance, the source could be read as homework for an initial discussion about the problem of modeling population growth (*tall*). Or the source could be worked through in more detail (*grande*), leading to the necessity of learning about solutions of differential equations by separation of variables. Or, by opening up discussion about the national importance of demographic data, the source could form part of an interdisciplinary sequence about the birth of nation states in the nineteenth century (*venti*).

The material on logistic growth was distributed to all members of the organization of upper secondary mathematics teachers (LMFK), and the reception has been favorable. Teachers who have used the material report that it was easy to use and that it worked very well to engage students, and they found multiple uses for it in teaching, along each of the *tall*, *grande* and *venti* suggestions. Many teachers followed our suggestions, but they also appropriated the material for other purposes; for instance, the source could be used in interdisciplinary teaching about climate and population modeling. Teachers who used the material also reported that efforts to transpose the material into tasks suited for students were supported by the materials we developed for teachers.

This latter point accords well with our approach, which is to develop materials addressed at teachers and providing them with sufficient resources (historical, mathematical and educational) that the inevitable transposition for specific use was made feasible. Specific teaching objectives and desired uses of the material span a wide variety, and teachers and classes also vary, raising the issue of adaptation and transposition for use (see Fig. 5.2). However, inspiration for the variety of uses and suggestions for the necessary transpositions are, we believe, among the core benefits of carefully crafted teaching materials in history of mathematics. Thus, the ambition of our approach is to limit the need for vast specialization in the history of mathematics on the part of the end-user teacher, while instead depending on her qualifications in adopting a wide-ranging, multi-purpose material to her specific teaching style and objective, or to specific learning challenges among her pupils.

5.5 Working with Teachers to Develop More Materials

In spring 2016, we set up an intervention, supervising a group of nine dedicated teachers to develop new source-centered teaching materials. The format included online collaboration, a one-day seminar, supervision and discussion, and eventually presentation at an international workshop in August 2016 and the preparation for publishing the materials and making them accessible through LMFK.

We began by setting the scene by introducing the notion of source-centered history of mathematics, as we perceive it, and circulating a description of the objectives and an introduction to working with sources. In particular, we structured the development process according to the diagram in Fig. 5.3. This process diagram

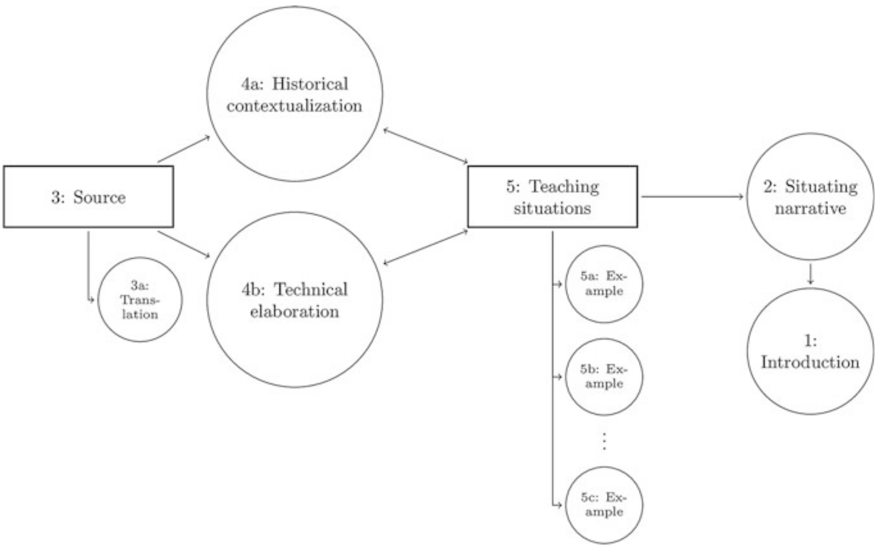

Fig. 5.3 Process diagram for developing source-centered materials, indicating the temporal relation and interdependency of tasks and eventual chapters in the design template. The boxes correspond to the dialectic process in Fig. 5.1

shows the reverse order in which the elements of the general design template are likely to be achievable for the teachers who work to produce materials: Once a source has been chosen through the dialectic process (blocks 3 and 5; see also Fig. 5.2), the mapping of the educational potentials and the specific demands raised by the source influence the substantiation of the contextualization and elaboration (tasks 4a and 4b, respectively). The translation of the source (3a) and the development of specific teaching situations (5a, 5b, etc.) are tasks which can run in parallel, once the general structure is set. Finally, the introductory elements (2 and 1) are likely to be the last to be produced. Thus, this process diagram served to transform the general design template into concrete and prioritized tasks for teachers to pursue.

Based on the methodological reflections discussed above, we identified three topics with associated central sources that address different dimensions of our design concerns. These topics are:

A. Edmund Halley's (1656–1742) estimates of mortality (1693) and his use of it for computing annuities on lives. In the 1690s, quantitative data on mortality rates were a novel phenomenon, and Halley made innovative uses of data obtained from Breslau to describe a general model for taking out annuities on lives. Based on surveys of the age distribution of the population of Breslau, Halley made generalizations to mortality rates in the United Kingdom. The mathematical content of the source is fairly basic, although it can lead on to more advanced conditional probabilities. Reading from an old, authentic text and puzzling over aspects of computational methods and historical context can inspire students to engage with mathematics in new ways. And the situation of Halley's interest in the problem merits historical attention, as the prevailing national model for life insurances at the time was about to ruin the scheme.
B. John Snow's (1813–1858) innovative use of graphical data representation (1855) in determining the cause of the cholera epidemic in London. During the nineteenth century, Europe was hit repeatedly by cholera epidemics. It was generally believed that cholera was caused by miasma, but John Snow was convinced that the cause was water pollution. When London's Soho suffered from an outbreak of cholera in 1854, Snow collected data about the deaths in the district, in particular, dates and addresses. Snow represented his data in a so-called ghost map which he used to convince the authorities that pollution of a particular water pump in Broad Street had caused the outbreak. The mathematical content is simple and is suited for STX-C. The students also have to work with different types of historical sources to complement Snow's own description of the procedure, namely graphical representations of data. Thus, understanding the mathematical source is valuable in training mathematical skills and in deepening and broadening the historical analyses of an interesting episode.
C. Augustin-Louis Cauchy's (1789–1857) redefinition of the integral (1823) by what is still known as the Cauchy-integral based on limits. Since the invention of the calculus, integrals had been defined as anti-derivatives. Yet, Cauchy

inverted the relation between anti-derivatives and area, defining the integral as the area under the curve and proving the Fundamental Theorem of Calculus. In his definitions and his proof that the theorem holds for any continuous function, Cauchy employed arguments that have since become standard practice in rigorous arguments in analysis. The conceptual mathematical content is thus more advanced in this source, and its use is aimed at STX-B and A. The mathematical terminology and symbolism in the source is very similar to the modern textbooks, but there are differences, and the students have to be attentive to the small differences when they work with the source.

Thus, these topics and the identified central sources all invite different teaching objectives and provide different affordances. For instance, topics B and C both address core curriculum stipulations but in different ways: Integral calculus is a mandatory part of STX-A, and working with graphical data representation and training students to read information off various graphs is mandatory at STX-C. Similarly, topics A and B invite collaborations with history, English, or even social science or biology, whereas topic C could be related both to important internal contexts within mathematics and the role of mass education in shaping the discipline in the wake of the French Revolution.

In groups, and under supervision, the teachers began in January 2016 to research and develop appropriations of the source along the methodological lines discussed above. Obviously, as developing such appropriations is quite time consuming, the groups would gradually choose to focus their attention on one or more of the objectives while continuing to keep the multi-purpose material approach in mind. During an intense and productive one-day seminar in April 2016, we met with the participants to assist them in their efforts. This occasion also marked the beginning of the collaborative work of each group, and quite a lot of coordination was required as the teachers were geographically dispersed at different schools. It was clear that they found the process both stimulating and challenging, as it required them to work beyond their ordinary teaching practices. To address such challenges, we provided them with a process diagram (Fig. 5.3) and a general outlook and methodology for developing source-centered materials. These, they reported, were clarifying and helped them structure their work and their intended product. This pointed to another recurring logistic challenge, namely to stimulate discussion and communication between the participating teachers and us when we could not meet in person.

During the summer, the groups of teachers worked on their projects, especially towards presenting their products at the conference "History of Mathematics in Education: An Anglo-Danish Collaboration" which we organized together with Sue Pope and under the auspices of the Danish LMFK and the British Association of Teachers of Mathematics (ATM) in Bath in August 2016. There, the groups of Danish teachers were able to present their materials in workshop format and receive constructive feedback from the participants, both Danish and British. Although differences between the national curricula and teaching formats and traditions pointed to some of the specificity of the Danish context, the experience of presenting their work and discussing it proved very useful for the groups.

While they were developing their materials, the groups were also able to test some of it in their own teaching. These hands-on experiences and the feedback they provided to the materials were also essential. We followed and observed some of the lessons taught to first-year students based on the material on topic B (John Snow), and we interviewed a focus group of students. When asked whether the approach through historical sources made a difference to their experience, the students responded positively to the inquiry-driven involvement:

> Student 1: I think it is more motivating, when we are to do something, to know what it can be used for. [...] When we are to investigate something that was relevant in the past, and we can see that it is still useful and has been used, I find that it becomes much more interesting.

Thus, the source-driven approach was seen as adding authenticity and relevance to the problems treated. Some also contrasted this approach to their ordinary mathematics lessons:

> Student 2: Personally, I like to do the kinds of mathematics, where you just have to sit down with some exercises and solve them. But this [the source-centered approach] shows that this kind of mathematics can be used in specific connections with medicine.

They were also explicitly reflecting on the methodological differences in history and mathematics:

> Student 3: I think that you learn history differently when you combine it with mathematics. In a history lesson, if we were to learn about cholera, we would probably only be taught how it worked. Now that we have worked with it, we know what the causes were and how many died and so on.
>
> Student 2 [continuing from student 3]: And how he found out. Mathematics makes it much more manageable.

In observations of the teaching, we also experienced that the confrontation with original sources prompted students to reflect on their practices in new ways. It has been argued (Kjeldsen and Blomhøj 2012; Kjeldsen and Petersen 2014) that such confrontation can lead to 'commognitive conflicts' in the sense of Anna Sfard and thus conceptual learning in mathematics (Sfard 2010). We saw that when asked to work with authentic data from Snow's observations, students were forced to develop suitable means of entering data into their spreadsheets. This is likely to happen whenever the data are not spoon-fed from the teacher or textbook in ordinary notation and format. But the use of authentic, historical sources is certainly one way to achieve this. And compared to historically inspired problems, it provides a platform for addressing the past that goes well beyond translating it into modern terms.

The other groups report similar experiences from their use of the materials that are being developed. In summary, these experiences point to the usefulness of a source-centered approach both for mathematical learning and for interdisciplinary teaching in history and mathematics.

After the conference and their calibration with input from their teaching, the groups have revised their material and submitted them to a volume that we are currently editing, to be published by the LMFK (Danielsen and Sørensen 2018) and made widely available for other teachers to use. In their final form, the three materials exhibit the variety of voices, experiences and concerns of the teachers involved. They all follow the design template and the source-centered approach by including the source, its contextualization and elaboration, and suggestions for a variety of uses in teaching. Although the teachers found the process both stimulating and worthwhile, they also experienced that working with sources and developing new teaching uses in this way was extremely time-consuming. Thus, some teachers found themselves forced to withdraw from the final phase of development, and some additional editing was required from us.

5.6 Conclusion

Integrating historical perspectives in mathematics teaching has both a long history and a specific place in the curriculum of Danish upper secondary education (STX). We have developed a design template for a source-centered approach to using history of mathematics in both mathematics-specific and interdisciplinary teaching. The approach is inquiry-based and centered on the contextualization and elaboration of well-chosen historical sources. These sources are to be identified in a dialectic process with their educational potentials in mind. This process demands an academic overview of the possible sources and experiences with the concrete day-to-day constraints of teachers. Developing such sources for use in teaching is thus a candidate for collaboration with teachers, and we have developed and tested a format in which teachers are guided to produce materials. This development process was generally successful, but is not likely to scale easily as it is time-consuming for teachers and requires well-motived teachers with competencies in both mathematics and other disciplines such as languages or history. On the other hand, the use of such materials is likely to scale more easily as more materials become available.

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Chapter 6
Involving Students in Original Research with Primary Sources

A Graduate Course in the History of Mathematics Education

Patricia Baggett and Andrzej Ehrenfeucht

Abstract We describe the structure and content of a graduate mathematics course, *History and Theories of Mathematics Education*, which focuses mostly on the history of mathematics education in colonial America and the US, including different authors' opinions about the purpose/methods of mathematics education. Students study original antiquarian books and read articles by writers who have influenced the development of mathematics education, preparing a major final project that they present at a conference at the course's end, open to faculty, students and guests. Our aim in designing and implementing this course is to use original sources in the history of mathematics education (rather than the history of mathematics) to allow each student to carry out his/her own individual research in the history of the issues described in these sources, and to report publicly on these results. We give details on the actual implementation of this course and its evaluation by the students enrolled in it.

Keywords Undergraduate and graduate mathematics · Student research
History of mathematics education · Original sources

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6.1 Introduction

Using primary original sources in university mathematics courses is a topic that has been studied by many authors. Theoretical, motivational, educational and other aspects have been examined. In this article we describe the structure and content of a graduate course in mathematics education, taught in a mathematics department, that uses historical sources in mathematics education. The course has several characteristics that are different from courses that have been discussed in the literature. The main difference is that the historical sources in mathematics education, rather than in pure mathematics, allow each student to carry out his/her own unique individual research in the history of the issues described in them. The goal of the research is that the student may present his/her results at a local or national conference, or even publish them.

When an investigation is in the history of mathematics, the main questions are about the development of new mathematical knowledge at a particular time. But in the history of mathematics education, the questions are how mathematics was taught and used at a particular time in a given historical and social environment. Students are able to investigate such questions when they have adequate historical and cultural backgrounds, which the course intends to provide.

6.2 The Course, The Students and Their Backgrounds

The one-semester course, offered for three hours of graduate credit, has been taught eight times. It is the only course in mathematics education history in the Department of Mathematical Sciences. It is an elective course; there are no qualifying exams based on it. The only prerequisite is graduate standing (in any discipline) or permission from the instructor. All students who enroll have completed at least two semesters of calculus, and almost all have taken a modern algebra and analysis course. Attendees have been multinational; thus far, students from Mexico, Jordan, Lebanon, Saudi Arabia, Nepal, China, Ghana, Cameroon, Nigeria, Guam and of course, the United States, have enrolled. Some are teaching or plan to teach in a high school or community college. Those seeking a Ph.D. (in mathematics or other fields) are primarily interested in a university position, but not necessarily a position in the history of mathematics education. Only two students who have taken the course have gone on to complete Ph.D.s in the history of mathematics education; they are now employed as assistant professors in the US.

6.3 Resources That Are Needed

The historical resources that students study are mostly old textbooks relating to mathematics/mathematics education that were actually used in the past, either from the instructor's private collection, or digitized online (Google Books, Hathi-Trust, Openlibrary.org, archive.org, etc.). Other materials that students access come from a course pack of readings, discussed below, and from Interlibrary Loan, WorldCat, GoogleScholar, etc.

At the first class session, we bring in scores of actual books, mostly school textbooks, from a private antiquarian collection. Books are spread out on tables and students are invited to handle them. Students pick up the books—they turn their pages. They notice their fragility, their tattered and discolored pages, their small formats. They are allowed to take home a book that they find interesting, and at the next class we discuss what they have found. Students thus begin their work with books that were actually used in the past.[1]

6.4 Students' Assignments

Students have three essential assignments:

- Preparing and delivering a book report on an antiquarian book;
- Turning in answers to questions from background readings; and
- Preparing and delivering a final project suitable for a conference.

In order to do the assignments, students need to get background knowledge about the place and the time period in which the books they are studying were written. They need to be familiar with the methodology of historical research in general, and how to ask and answer relevant questions and to justify their final conclusions. Finally, they need to compress their findings into a 15 to 20 min conference talk.

6.4.1 Book Reports

Old mathematics textbooks provide a richness of educational and historical information. To quote John Dewey (1938), "Books, especially textbooks, are the chief representatives of the lore and wisdom of the past" (p. 18).

[1]This aspect of studying an actual book (rather than a digital copy) is under-appreciated.

In our course, a student's first assignment is to prepare and deliver a book report. Students most often choose an antiquarian textbook, one that was actually used for instruction, that is available online and/or from antiquarian mathematics books in the instructor's private collection, written in English and published in the United States or England. However, one may instead choose a book, modern or historical, from another culture or nation (international students often make such choices). The student gives an in-depth talk about the book in the class.

Such an assignment is new for almost every student, and they are rather lost about how to proceed. Many students do not know how to use Google Books and similar internet sources. The first thing a student needs to do is read the whole book from cover to cover. This is something that many have not done since high school.

A student also needs to decide what to cover in his/her report, and what to leave out. The instructor may help by asking leading questions such as:

What do you find interesting or weird in your book?
For whom was the book intended?
What do you know about the author?
What topics were included? How were they sequenced?
Did the author describe his/her reasons for teaching mathematics or his/her views on the psychology of learning?
How was the mathematics taught? Was there any advice to the teacher or schoolmaster about how to teach?
Is there any mathematical terminology that seems strange? Any unusual mathematical symbols or abbreviations?
What algorithms were taught?
What about weights and measures?
What currency was used at the time the book was written?
If there are word problems, what content do they have?
Is your author's content original? Or do you find content that is essentially identical in earlier books? Or, can you find parts of your book's content in later books?
Were answers to problems and methods of solutions provided in the book? Or were they in a separate "key"? Did pupils work problems directly in the book itself?
What changes do you see in the mathematics presented in schools today and when your book was printed, in terms of content, sequencing, illustrations, …?
Look for concepts in your book that are not in modern books, or whose meaning has changed from when the book was written to today, e.g. diagonal scale, St. Andrew's Cross, aliquot part, surd, wrangler, zero, proof, …

Such questions allow the student to find interesting features of the book, and topics that match the interests of the class. Most students also need help in selecting the most relevant aspects of the book that they need to concentrate on in their reports. Often the student is asked to practice his/her talk in front of the instructor, before giving it in front of the class. Presentations are given in PowerPoint, and at least one session of the class is devoted to instruction in its features and use, since many have no experience with it. And there is one rule: You do not have to cover everything in your book. However, when you give your book report, the class may ask you about anything that you include in your presentation. Thus, never cut and paste into your report anything that you do not understand.

Some books that have been reviewed, which are also listed in the references, are:

(1) *The Nine Chapters on the Mathematical Art* (10th–2nd century BCE) (used in class in Chinese and in English translation).
(2) *Principles of Hindu Reckoning* (Kushyar Ibn Labban). A Translation with Introduction and Notes, 1965.
(3) *The Treviso Arithmetic* (Arte dell'Abbaco 1478) (used in English translation by D. E. Smith and Frank Swetz).
(4) Adam Riese, *Rechenung nach der lenge auff den Linihen und Feder* (1550) (English translation of selected parts).
(5) *Sumario Compendioso of Brother Juan Diez* (1556) (in Spanish and English translation by D. E. Smith).
(6) Robert Recorde, *The Grounde of Artes* (1632).
(7) Edward Cocker's *Arithmetick* (1715) (a book studied by Benjamin Franklin).
(8) John Ward, *Young Mathematician's Guide* (1771) (used as a textbook at Harvard as early as 1726).
(9) Nicholas Pike, *A New and Complete System of Arithmetic Composed for the Citizens of the United States* (1788).
(10) Stephen Pike, *The teacher's assistant or a System of practical arithmetic* (1811).
(11) William Hawney and Thomas Keith, *Hawney's Complete Measurer* (1813).
(12) Jeremiah Day (president of Yale University), *An Introduction to Algebra* (1834).
(13) Isaac A. Clark, *Prussian Calculator* (1846).
(14) John B. Jones, *Elementary Arithmetic in Cherokee and English* (1870).

Students wrote their opinions about the book reports in an anonymous survey at the end of a recent course. Here, we provide three student opinions:

- The book reports were an excellent way to get acquainted with one another as well as the scope and goals of the class overall.
- I found the book report worthwhile in the sense [that] it not only makes us prepared for the final project but it also gives a deep idea about how a particular subject was taught in the past in America.
- I loved the book reports! I liked diving into an old text and presenting it to the class. It was a nice way to practice this type of investigative research.

Students who discovered something in their book that led to their final project seemed to have an advantage in preparing their projects. Several commented on this fact in their surveys (see Appendix).

6.4.2 Background Readings

Students need to acquire a lot of background information about the American Colonies, the Revolutionary War, philosophies of teaching/learning, the beginning of high schools and other topics that come out from reading old texts. There is a basic course pack of readings (online and on paper) for the class (see references). During the semester, additional readings are added to match the students' interests and books and projects that they are planning. Especially when we have multinational students, we will read one or more articles related to educational issues in different nations and cultures (e.g. the *Sumario Compendioso*, mentioned above, or parts of Bhaskara II's *Lilavati*, in both English and Sanskrit, or *Principles of Hindu Reckoning* (Kushyar Ibn Labban), written in Arabic with English translation).

We include many maps with the readings, including, for example, the 1507 Waldseemüller map,[2] which is the first map on which the word "America" appears, as well as the 13 British Colonies, the Louisiana Purchase, divisions during the US Civil War and the locations of "land grant" colleges.

The articles that are assigned, all in English, are particularly difficult for international students who are not fluent in English and not familiar with American history. But these students comment that as the course progresses their English improves, and reading the articles and writing about them become easier. When we discuss education in the United States, we cannot assume that students from different countries are familiar with it. Selections from Hillway (1964) are especially helpful, as are modern articles about the No Child Left Behind Act, the Common Core Standards Initiative (CCSSI) (2010) and the Every Student Succeeds Act. Similarly, some topics can be about mathematics education in other countries (e.g. China, Mexico, Nepal). Readings should provide enough context so that the other elements of the course make sense.

At each class meeting students are given reading assignments and reading questions for topics discussed in class. Answers are to be turned in at the next class, and we discuss them as soon after as possible.

A small sample of readings, also listed in the references, include:

- Toby Lester (2009). "Putting America on the Map," in *Smithsonian Magazine.*
- John Dewey (1915). *The School and Society* (selected parts).
- Morris Kline (1974). *Why Johnny Can't Add* (critique of the "New Math").
- Hans Freudenthal (1981). "Major Problems of Mathematics Education," in *Educational Studies in Mathematics*.
- B. F. Skinner (1984). "The Shame of American Education," in *American Psychologist*.
- George Stanic (1986). "Mental Discipline Theory and Mathematics Education," in *For the Learning of Mathematics*.

[2]Available at https://www.wdl.org/en/item/369/view/1/1/. Accessed August 7, 2017.

- Anna Sfard (2012). "Why Mathematics? What Mathematics?" in *The Mathematics Educator*.

Additionally, sample reading questions for the above articles included:

- According to the article by Lester, How did America get its name?
- What did John Dewey (1859–1952) mean by "progressive education"?
- Why does Morris Kline say "Johnny can't add"?
- Give at least two major problems of mathematics education discussed by Freudenthal, and his suggestions of how to fix them.
- What does Skinner say is "the shame of American education", and what does he propose to do about it?
- What does Stanic say is the "basic metaphor" for mental discipline theory? Does the theory "require" that mathematics be the necessary topic of study? Briefly, what caused the decline of mental discipline theory?
- Anna Sfard gives her opinions about why to study mathematics and what mathematics to study. What are her opinions? Do you agree with them? If not, why not?

Students' opinions of the readings, from an anonymous survey at the end of the course (see Appendix), were generally positive:

- The readings were diverse, appropriate, and thought provoking. The questions were often demanding and helped to better digest the material and read more closely.
- The reading questions were definitely worthwhile. They helped me to comprehend the readings.
- The readings were appropriate as were the reading questions. I think it would have been nice to have more time to discuss them. Either a two-semester course, longer class meetings, or allotting ten-fifteen minutes at the beginning of each class would be helpful.

6.4.3 Final Projects

The essential aspect of any research is the choice of the questions that the researcher intends to answer. Students find that it is the most difficult part of their task. Often, from their book reports, they find topics that they want to know more about. However, determining a specific question to delve into is complex.

1. *The choice of questions*

As we discussed in the introduction, work in the history of mathematics education is different from work in the history of mathematics. All questions concerning education are always situated in a broad social, economical and cultural context. Thus, students cannot concentrate only on mathematical features of the text, and at the same time they must choose a very narrowly defined question in order to be able to investigate it during one semester.

Even a simple question such as, “Why was mathematics taught?” can result in answers belonging to

- psychology: “It develops a student’s mind”;
- sociology: “It is needed in everyday situations”;
- politics: “It is essential for national security”;

and many others.

Any question that is investigated has to be of interest to the student, but also to the instructor and to a more general audience. In the course, the emphasis is on mathematics education in the 18th and 19th centuries in the United States, but the range of research questions has been much broader.

For example, one student investigated the history of mathematics education in Guam; another student provided an English translation of a short, but very interesting, 15th-century Arabic poem in algebra that had never been translated before (see the seventh project described below).

As mentioned above, sometimes, as a result of having given a book report, a student will notice something of interest in the book that the student still has questions about. This can lead to the student’s final project. However, the student’s choice must be within the reach of his/her knowledge. Some students may need to restrict their choices to topics in elementary mathematics and not higher mathematics. The instructor is available and willing to suggest topics if the student has difficulty (more about this below).

Basic guidelines for the final project are:

- What question(s) will you try to answer?
- How will you try to answer it/them?
- What resources do you have?
- Trace your questions with respect to time and geographical location.
- Justify your answer(s) using historical sources (say something new about something old).
- Do you have enough background knowledge to handle your topic?
- A historical analysis tries to make sense of something. What was it? Why was it important? When did it start, and why? Why did it persist? (E.g. was it related to some other thing that existed and disappeared?) For how long did it exist? When did it stop, and why? Did anyone else write about it?

2. *Preliminary analysis*

The goal of the preliminary analysis is to make a plan of work for the rest of the semester. Each student is responsible for his/her own work, but the plan of work must be approved by the instructor and is usually discussed with the rest of the class.

Usually the student needs to do a considerable amount of background reading specific to the topic being investigated. The previous assignments (book report and readings) prepare students to do this reading without the instructor's supervision.

Some students who took this course before are invited by the instructor to show their projects in class, and to describe how they did their work. This is a very important element of the whole course, not only because it helps current students plan their research, but also because it shows them how a final presentation should look.

Each student designs, prepares, and delivers a 15 to 20 min talk at a mini-conference at the end of the semester, open to faculty, students and guests. This final project requires extensive background work and should be suitable for a national conference.

6.5 Examples of Final Projects

Below are several examples of book reports that have led to final projects. The first six listed resulted in publications (two; one in a journal and one in a book) and acceptances/presentations to national conferences (four). The seventh project below, which did not begin with a book report, also resulted in a presentation at a national conference.

6.5.1 *Kristina Leifeste Brantley,* Mathematics Graduate Student

Book report: *Hawney's Complete Measurer* (1813) describes an unusual measuring instrument, a diagonal scale (Fig. 6.1). She decided to investigate this instrument, and her exploration culminated in her final project, "A forgotten contrivance: A study of the diagonal scale and its appearance in mathematics texts from 1714 to the

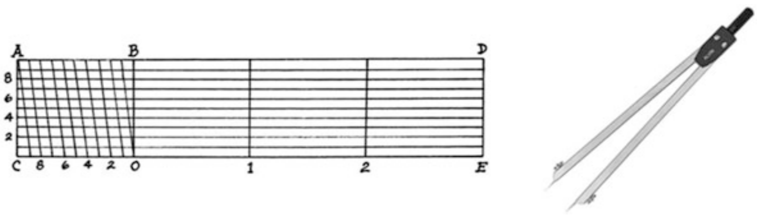

Fig. 6.1 A diagonal scale and a two-point compass, allowing measurements from one to 1/10 to 1/100 unit

present." She expanded it into a master's thesis in mathematics, and an article with the same title was published (Leifeste 2015).

6.5.2 *Valeria Aguirre-Holguin,* Interdisciplinary Mathematics and Education Graduate Student

Book report: Stephen Pike's 1811 *The Teacher's Assistant*. She learned that Abraham Lincoln (the 16th President of the United States) studied the book when he was a teenager (Fig. 6.2). For her final project she investigated this and as a result became co-author of a chapter in Ellerton and Clements (2014).

6.5.3 *Lokendra Paudel,* Mathematics Graduate Student

Book report: *An introduction to algebra: Being the first part of a course of mathematics: adapted to the method of instruction in the American Colleges* by Jeremiah Day (1823). Day's preface states that he used as his sources Newton, Maclaurin, Saunderson, Simpson, Euler, Emerson, Lacroix, and others. Mr. Paudel matched parts of his text to selections from algebra books by some of these authors, and to some others (see Fig. 6.3 for an example). But in many cases, as was noted by Cajori (1890), Day's content is simplified. Mr. Paudel discussed the contents of Day's book and the matches he found, together with the content (especially definitions and axioms) that Day seems to have created on his own. He also discussed why Day's book made such a unique contribution to algebra in the 19th century United States. His paper was accepted for presentation at the Joint Mathematics Meetings in Seattle, WA in January 2016, but due to extenuating circumstances it was not presented.

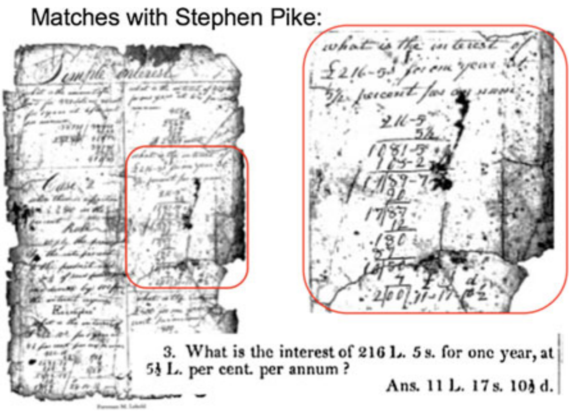

Fig. 6.2 Possible matches of Lincoln's copy book with Steven Pike (1811)

Elements of Algebra: **Nicholas Saunderson (1761)** 160. What two numbers are those whose sum is a and the sum of their cubes b? An example of the foregoing canon: What two numbers are those whose sum is 7 and the sum of their cubes 133?	**Jeremiah Day (1823)** Prob. 8. What two numbers are those whose sum is 6 and the sum of their cubes 72? Ans 2 and 4
Elements of Algebra: **Silvestre Francois Laçroix (1818)** Let there be the general equation $x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} \ldots + Tx + U = 0$	**Jeremiah Day (1823)** 503.*c.* An equation of the *m*th degree consists of x^m, the several inferior powers of x with their co-efficients, and one term in which x is not contained. If A, B, C, ... T, be put for the several co-efficients, and U for the last term, then $x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3} \ldots . +Tx+U=0$. will be a general expression for an equation of any degree.

Fig. 6.3 Possible matches of Day's 1823 *Algebra* with Saunderson (1761) and Laçroix (1818)

6.5.4 Crystal Montana

Crystal Montana, a master's student in Curriculum and Instruction, reviewed Warren Colburn's 1825 *First Lessons in Arithmetic on the Plan of Pestalozzi*, for her book report. She noticed the "plates" that he used in early editions of his book and how clearly they were used to explain fractions (Figs. 6.4 and 6.5). Then, for her project, she compared the book's information on fractions to the material on fractions in a 2012 Common Core State Standards mathematics text, *GoMath*! She presented her project at the Joint Mathematics Meetings in Seattle, WA in January 2016.[3]

6.5.5 Jill Duke

Jill Duke, an engineer working at NASA (National Aeronautics and Space Agency) who took the course because the title caught her eye, reviewed for her book report Catharine Beecher's *Arithmetic Simplified* (1832; Fig. 6.6), which is possibly the first arithmetic text by a female author in the United States.

Ms. Duke's final project was titled "*Arithmetic Simplified* (1832): The Story Behind Catharine Beecher's Most Unrecognized Work." It delved into Ms. Beecher's life story and her motivations for writing the book. Her project was presented at the Joint Mathematics Meetings in Seattle, WA in January 2016.[4]

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[3]The abstract is at http://jointmathematicsmeetings.org/amsmtgs/2181_abstracts/1116-c1-1172.pdf. Accessed August 7, 2017.

[4]The abstract may be found at http://jointmathematicsmeetings.org/amsmtgs/2181_abstracts/1116-d1-1287.pdf. Accessed August 7, 2017.

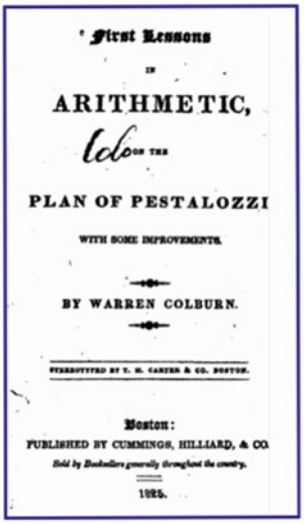

First Lessons
IN
ARITHMETIC,
ON THE
PLAN OF PESTALOZZI
WITH SOME IMPROVEMENTS.
BY WARREN COLBURN.
STEREOTYPED BY T. H. CARTER & CO. BOSTON.
Boston:
PUBLISHED BY CUMMINGS, HILLIARD, & CO.
Sold by Booksellers generally throughout the country.
1825.

Fig. 6.4 Colburn's *First Lessons in Arithmetic* (1825)

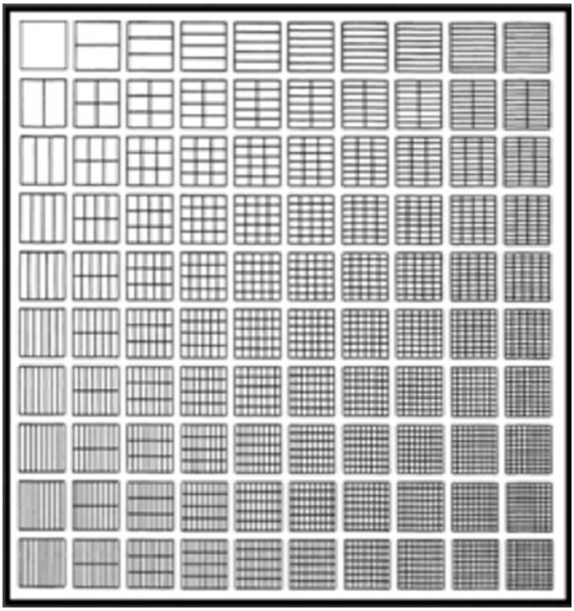
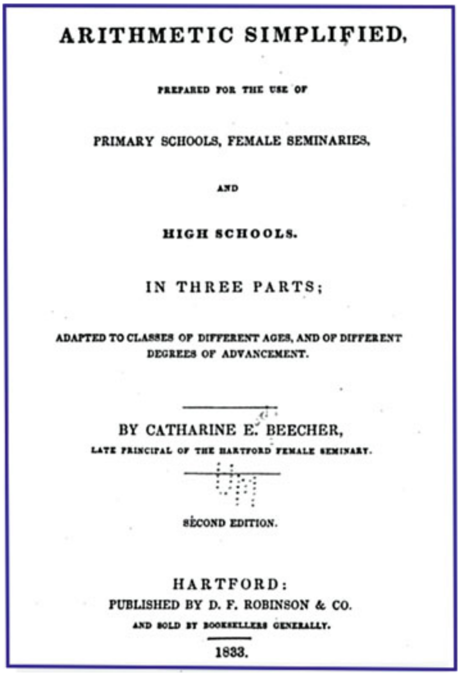


Fig. 6.5 One of Colburn's plates

6.5.6 *Meredith Anderson*

Meredith Anderson, a doctoral student in mathematics (now an assistant professor of mathematics at a university in Colorado), reviewed *The Public School Euclid & Algebra* (1897; Fig. 6.7), a book authorized for use in the Public Schools of


ARITHMETIC SIMPLIFIED,

PREPARED FOR THE USE OF

PRIMARY SCHOOLS, FEMALE SEMINARIES,

AND

HIGH SCHOOLS.

IN THREE PARTS;

ADAPTED TO CLASSES OF DIFFERENT AGES, AND OF DIFFERENT DEGREES OF ADVANCEMENT.

BY CATHARINE E. BEECHER,

LATE PRINCIPAL OF THE HARTFORD FEMALE SEMINARY.

SECOND EDITION.

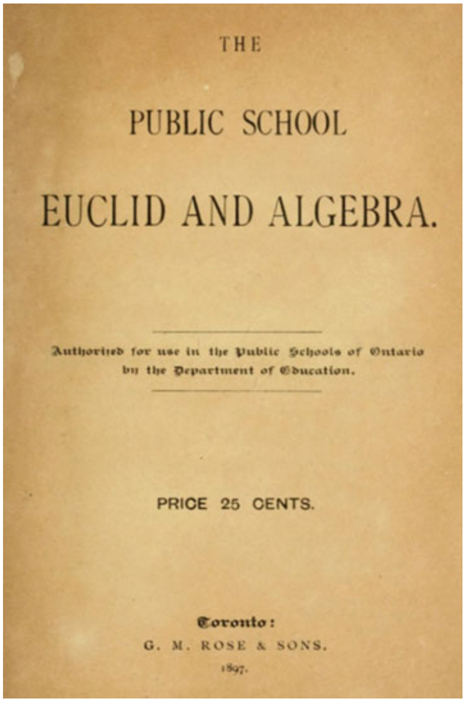
HARTFORD:

PUBLISHED BY D. F. ROBINSON & CO.

AND SOLD BY BOOKSELLERS GENERALLY.

1833.

Fig. 6.6 Beecher's *Arithmetic Simplified* (1833). It was prepared for, among others, female seminaries. In an 1835 version, her name on the title page was changed to "An experienced teacher", presumably to hide her gender


THE

PUBLIC SCHOOL

EUCLID AND ALGEBRA.

Authorized for use in the Public Schools of Ontario by the Department of Education.

PRICE 25 CENTS.

Toronto:

G. M. ROSE & SONS.

1897.

Fig. 6.7 *Public School Euclid and Algebra*, a Canadian text

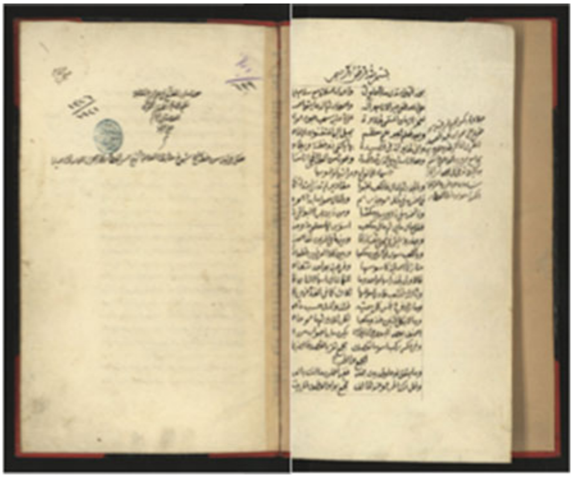


Fig. 6.8 First part of 1402 poem in Arabic, *Al Mkni fi'l-jabr wa'l-muqābala*

Ontario. She then found the book *Euclid and his Modern Rivals* (first edition 1879) by Charles Dodgson (also known as (AKA) Lewis Carroll, who wrote *Alice in Wonderland* and *Through the Looking Glass*). It is a rousing and humorous defense of using Euclid's *Elements* as the main geometry textbook in schools. Her final project was titled "How Charles Dodgson AKA Lewis Carroll Would Evaluate a 19th century Canadian Geometry Text." She revised her paper and gave it the title "The Trend Away from Euclid: A Glimpse Through the Looking Glass" and discussed how it was that Dodgson lost the battle, and the teaching of geometry in schools was forever changed. She presented her paper at MathFest in Columbus, OH in August 2016.[5]

6.5.7 Ishraq Al-Awamleh

Ishraq Al-Awamleh, a first-year mathematics graduate student from Jordan, discovered on the internet a 1402 manuscript in Arabic, with a 58-line poem, *Al Mkni fi'l-jabr wa'l-muqābala*, (*Exposition of Algebraic Operations*), by Ibn Al-Ha'im (Figs. 6.8 and 6.9). It is a poetic abstraction of the algebraic rules of that time, based on the six canonical forms of al-Khwãrizmî. It had never before been translated into English. She translated part of it for the mini-conference in May 2016, and she presented part of the poem in translation with commentary at MathFest in Columbus, OH in August 2016.[6] Ms. Al-Awamle continued an extra semester in an

[5]Abstract located at http://www.maa.org/sites/default/files/pdf/mathfest/info/MF2016AbstractBook_0.pdf. Accessed August 7, 2017.

[6]Abstract available at http://www.maa.org/sites/default/files/pdf/mathfest/info/MF2016AbstractBook_0.pdf. Accessed August 7, 2017.

Fig. 6.9 Second part of the poem

independent study course, working on the poem, and she presented a complete English translation at MathFest in Chicago, IL in July 1017.[7]

6.5.8 *A Closer Look into the Project by Jill Duke (Project 5 Above), Titled* **Arithmetic Simplified (1832): The Story Behind Catharine Beecher's Most Unrecognized Work**

Each project has its own unique backstory. To give the reader at least an idea about what actually goes into its preparation, we describe in more detail the project on Catharine Beecher and her book.[8]

The author, Jill Duke, first reviewed Catharine Beecher's 1832 book, using an original copy from the instructor's collection and an electronic copy easily available from Google Books. In her report she described the book's contents and gave a brief biography of Ms. Beecher (1800–1878), noting that, as far as Ms. Duke could find, Ms. Beecher's book might be the first arithmetic book written by a woman in the United States.

For her project, Ms. Duke decided to pursue Ms. Beecher's background, especially in mathematics, and how she came to write an arithmetic book.

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[7]Abstract at https://www.maa.org/sites/default/files/pdf/mathfest/2017/AbstractBookFINAL.pdf. Accessed August 26, 2017.

[8]For more information about her project, the answers to the questions above, and her PowerPoint presentation, Ms. Duke can be reached at jrduke@nmsu.edu.

Below we give a brief outline of some of the questions and issues that Ms. Duke addressed and answered about Catharine Beecher.

She found that she was the sister of Harriet Beecher Stowe, who wrote the anti-slavery novel, *Uncle Tom's Cabin*. In fact, at age 22 Ms. Beecher was engaged to marry a professor at Yale University, but he died in a shipwreck before the wedding. She decided she would need to care for herself and become educated,[9] and help other women to do so as well, which is how she decided to write an arithmetic book, and why was it used in female seminaries (which she founded). In connection with what mathematics did she learn, Ms. Duke found in Beecher's memoirs that she studied Nathan Daboll, Jeremiah Day, a little logic, and some geometry.

Moreover, Ms. Duke explored further issues related to Ms. Beecher, who lived until 1878: What really motivated her to learn mathematics and with whom did she study? Was the content of her book original, or did she copy much of it from others? Why there were three different editions of her book, and why, in the 1835 edition, was her name on the title page changed to "AN EXPERIENCED TEACHER"? What pedagogy was in her book? What was her philosophy of education, and especially of the education of women? And more generally, what was her life story after the publication of her arithmetic books?

Ms. Duke cited some of Ms. Beecher's many writings, such as textbooks, advice books, pamphlets, newspaper articles and essays, as well as information by others such as Burstyn (1947).

6.6 Students' Statements About What They Will Do Next with Their Projects

Students have commented on the anonymous questionnaires completed at the end of the course about the "next steps" they will take with their final projects. A sample from these comments are:

- I am very interested in developing a full-blown argument and paper in this field.
- I would very much like to submit my project to a conference, though I will have quite a bit of work to do on it first.
- I think my project can definitely be expanded. The time spent in class is a great starting point for me to take this research further.

[9]The following is a characteristic quote from Ms. Duke's presentation: "In 1870 at age 70, Catharine enrolled in a course at Cornell University. She was told that Cornell had no courses open to women. She replied, 'Oh that is quite all right…, in fact I prefer to take it with men'. When offered to have a suitable place found for her to stay while attending the university, she announced that a room in one of the dormitories would be satisfactory. She said of the exclusively male residence, '…and as for those young men, who are of appropriate age to be my grandsons, they will not trouble me in the least'."

- I was really challenged in this class, and I succeeded in making a project that I am proud of. I never thought I could do it!
- The only thing that might prevent me from presenting at a conference [besides getting my paper accepted] is enough financial resources to attend a conference.
- I do not have the time now to attend a conference because I just started in my job. Probably I can submit an expanded version of it, especially if I am able to find more information.

Several students indicated that the course should be two semesters long, so that they could develop their projects more fully.

6.7 Adjustments We Have Made to Try to Improve the Course

The course continues to evolve (the course was offered most recently in Spring 2017).[10] As a result, we continue to make adjustments, such as:

a. Adapt the readings and reading questions to students who are currently taking the course, their choices of books for their book reports, their projects and their interests. We have had book reports and projects involving materials in not just English, but Latin, Greek, Hindi, Sanskrit, Nepali, Chinese, German, Italian, Spanish, Arabic, Cherokee (a Native American language) and Chamarro (the native language of Guam).
b. Guide students in how to do a book report. Meet with them outside of class if they need help.
c. Help students learn to use InterLibrary Loan (ILL), Google Scholar and JStor (online database for journal articles), and other internet resources such as multilingual dictionaries.
d. Book reports and projects are accompanied by slides in PowerPoint, and this software is new for some students. Offer instruction outside of class in making simple PowerPoint presentations.
e. Provide a great deal of assistance with projects. Require an out of class meeting at least once a week with the instructor to discuss where individual students are in the project process.
f. Invite others who previously took the class to present their book reports and their projects (this is especially helpful and was mentioned as a high point by several on course evaluations).
g. Require that students present their draft projects in class, for others to critique and question.

[10] More information about the course can be found at https://www.math.nmsu.edu/~breakingaway/Syllabus_562. Accessed August 7, 2017.

h. Have at least one run-through of the entire mini-conference beforehand, using both a timer and laser pointer.
i. If a student wishes to go further with his/her project after the class is over, offer independent study to continue the work; for example, to prepare an article for submission, if the student desires it.

6.8 Final Remarks

In this course we gave each student a chance to do his/her own individual research. Students were given help from the instructor and from other students, but each student was fully responsible for his/her project, and took credit for its success or blame for its failure.

Students have found it very difficult to prepare and present a final presentation. The student must practice his/her talk many times: alone, in front of the instructor, and in front of the class, revising it as needed. The last few weeks are typically devoted to students' practicing and refining their presentations. During practice sessions, class members are urged to ask questions, and the speaker needs to learn how to handle them. Keeping the talk within a strict time limit, typically 15 min, is essential; students have usually never had to operate with such a constraint. They have also rarely before had to stand in front of an audience and give a professionally acceptable talk; this is often their first such experience. Based on comments on course evaluations, students indicate that they gain considerable confidence by organizing and presenting such a talk.

Completing a project is not a requirement for passing the course, but it is a requirement for receiving a course grade of "A". The number of successful completions of projects surprised us (all but about 10 of over 80 students). In comparison to other graduate courses in mathematics, this course is usually not mathematically difficult, but it is very time consuming and requires sustained effort through the whole semester. Such effort is possible only when students are truly interested in the topic they investigate.

It is clear why a student interested in mathematics education may want to take such a course. However, why students working toward a Ph.D. in pure mathematics might want to take such a time-consuming course requires some explanation. The content of most of the Ph.D. dissertations in pure mathematics is so specialized that it is inaccessible not only to students, but also to colleagues working in the same department. However, historical research in mathematics education is accessible and interesting for the whole mathematical community. After the course is completed, students have a presentation that they can modify or give at a moment's notice, for example, at a job interview or to those with less specialized mathematical backgrounds. Several students indicated on course evaluations that they thought, with some work, they could teach a course in the history of mathematics education.

In this course, students do research under supervision. They do not create new mathematics. Their investigation produces the answer(s) to some question, and the answer(s) was/were not known before. During the period of the course they formulate a (small) question related to the history of mathematics or mathematics education as used in society. They present it in a format that would be of high enough quality to be accepted to a national conference, and that would be understandable to anyone interested in mathematics education.

Appendix

Below is the anonymous questionnaire given at the end of the course. It is not meant as a course evaluation, but it addresses some specific parts of the course and what changes the students see as desirable.

Anonymous questionnaire for Math 562: *History and Theories of Mathematics Education*

1. Please comment on the readings that were handed out. You could address some of these questions:
   Do you think they were appropriate? Do you think having reading questions to answer about them was worthwhile? How could we arrange the class so that there would be more time for discussing them? (One possibility would be to make this course two semesters long.) Are there any topics that you wish we would have had readings about that we missed?
2. Do you think that you will carry through on your project and submit it to a conference? Why or why not? Do you think you will take it farther, e.g. perhaps submit an expanded version of it to a journal?
3. Comment about the book reports. Do you think they were worthwhile?
4. Did you find project presentations by former students useful or helpful? Please comment!
5. Any additional comments are welcome.

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Chapter 7
Algebra Without Context Is Empty, Visualizations Without Concepts Are Blind

Rainer Kaenders and Ysette Weiss

Abstract In the acquisition and formalization of mathematical concepts, the transition between algebraic and geometric representations and the use of different modes of representation contextualizes abstract algebra. Regrettably, the role of geometry is often limited to the visualization of algebraic facts and figurative memory aids. Such visualizations are blind for the underlying concepts, since transitions between concepts in different representations assume the existence of symbols, language, rules and operations in both systems. The history of mathematics offers contexts to develop geometrical language and intuition in areas currently being taught in school in a purely algebraic fashion. The example of the determination of zeros of polynomials shows how reflecting on posing a problem in ancient Greek mathematics, engineering mathematics (19th century) and paper folding (beginning of the 20th century) can help to develop geometrical concepts, language and intuition stemming from an algebraic context.

Keywords Engineering · Greek mathematics · Algebraic/geometric representations
Pictorial/symbolic visualizations · Horner's scheme · Lill's method
Paper folding · Historical context

The title is an allusion to Immanuel Kant's famous quote: "*Gedanken ohne Inhalt sind leer, Anschauungen ohne Begriffe sind blind*"—Thoughts, without the content are empty, perception without a concept is blind.

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7.1 Introduction

Concept development in high school mathematics—in particular A-level subjects in many countries—is characterized by predominant algebraization. Definitions and formulas are often introduced according to the *Rule of Three (Regula de Tri) of mathematics education*: *That is what it is called; that is how it goes; that is how it is* (Jahnke 2012; authors' translation). The derivation of formulas is carried out, if at all, by rearranging algebraic terms and solving equations. The high degree of abstraction and technical complexity of algebraic symbolic language gives students few opportunities to question the underlying rules, to introduce their own situated notations and notions reflecting an individual understanding of a problem and its context, or to develop their own mathematical questions.

It seems that, in the modern teaching of mathematics, the training of pattern detection of types of problems and the matching of a type to its solution scheme increasingly dominates. Linear systems of equations, the application of the solution formula for quadratic equations, or the calculation of extreme values of a function using derivatives are just such trained solution schemes.

Because elementary geometry does not allow for such reductions to algorithmic solution schemes, we regard geometrical contexts as a means to teach abstract modern concepts without hindering autonomous thinking and experimental discovery. For this, we need geometrical concepts, which can lead to variants of typical algebraic and analytic school subjects. A brief glance at the presentation of such subject in school mathematics textbooks seems to indicate no need to add more geometry: they are full of images and visualizations. One can hardly find problems without seeing corresponding sketches of function graphs or vectors as arrows in Cartesian coordinates.

However, do these pictures allow us to see geometrical concept development? Are visualizations geometrical symbols? In most cases the answer is no. Even in some geometric proofs, visualizations are only figurative presentations of algebraic structures, because of the identification of length, angles and areas with numbers and reasoning in the language of term manipulations related to algebraic and arithmetical objects and operations. In these approaches, the development of a geometrical symbolic language that formalizes geometrical operations among geometrical objects is hardly found. In particular, for the calculation of extreme values and zeros of polynomial functions in senior classes, the geometrical representations serve only as visualizations and figurative notations of algebraic and analytic structures; a geometrical concept development is lacking.

In this chapter, we use the history of mathematics as a tool (Jankvist 2009) to contrast different geometrical concept developments and an algebraic approach. The aforementioned difference between the visualizations of an algebraic object and its geometrical concept development is illustrated by means of an example. We introduce geometrical presentations to an algebraically formulated problem with the help of four projects from four areas of mathematics that appeared within different cultural traditions and contexts for concept development: algebra, ancient geometry, the engineering mathematics of the 19th century and geometrical constructions by paper folding. These areas have different backgrounds and motivations for their existence (mathematical engineering, investigative–problem orientated, mathematical structuring, etc.). Every project can be varied and adapted corresponding to the available time, the existing knowledge, experiences and interests of the students—be it in high school or university. The projects discussed in the next sections are:

1. The Horner scheme (Sect. 7.2)
2. Lill's method and geometrical solutions of polynomial equations (Sect. 7.3)
3. The mean proportional and doubling the square and the cube in ancient Greek mathematics (Sect. 7.4)
4. Solving quadratic and cubic equations by paper folding (Sect. 7.5).

The detailed algebraic presentation of the problem in the next section serves both to familiarize the reader with the mathematical question and its elegant algebraic solution, as well as to draw attention to definitions, descriptions, notions, and proofs that are characteristic of the algebraic approach. The latter is important in order to contrast the algebraic context with different geometrical contextualizations of the problem in the projects discussed in the subsequent sections. We refer to materials that have been used and further developed in several national and international workshops for students in upper secondary school and in university education for teachers, as well as in workshops for mathematics teachers.

7.2 Horner's Scheme

The first project deals with a typical algebraic procedure. The starting point is a polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with real coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ and $a_n \neq 0$. For this situation we introduce the following computational scheme. Choose some real number x_0. Write the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ in a row, define b_n to be a_n and write it in a row below the first one such that some space between the rows remains. Then proceed as follows. Write $x_0 b_n$ in the second column between the two first rows. Let b_{n-1} be the sum of the numbers in the second column, i.e. $b_{n-1} := a_{n-1} + x_0 b_n$ and write it in the last row, second column. Now we go on like this. In the i.e. k-th column we write $x_0 b_{n-k-1}$ and add the two numbers in that column, i.e. $b_{n-k-1} := a_{n-k-1} + x_0 b_{n-k}$.

$$\begin{array}{cccccc} a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ \downarrow & + & + & \cdots & + & + \\ & x_0 b_n & x_0 b_{n-1} & \cdots & x_0 b_2 & x_0 b_1 \\ b_n = a_n & b_{n-1} = a_{n-1} + x_0 b_n & b_{n-2} = a_{n-2} + x_0 b_{n-1} & \cdots & b_1 = a_1 + x_0 b_2 & b_0 = a_0 + x_0 b_1 \end{array}$$

The algorithm is called *Horner's scheme* after the British mathematician William George Horner (1786–1837). Via some suitable examples, the students conjecture that the number b_0 is nothing but the value $P(x_0)$, which can alternatively be computed by inserting x_0 into the polynomial. Asking how many elementary operations (multiplication and addition) are needed to calculate the value draws attention to the algebraic algorithmic structure of the problem. The first challenge is to prove that $b_0 = P(x_0)$. The proof is done in a purely algebraic fashion by skillful factoring.

As a next step, by means of many concrete suitable examples we discover the special meaning of the row of numbers $b_n, b_{n-1}, b_{n-2}, b_{n-3}, \ldots, b_2, b_1$ where x_0 is a zero of the polynomial $P(x)$. It is the row of coefficients of the polynomial $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1$ with the unique property that $P(x) = (x - x_0)Q(x)$. The students prove this by expanding the polynomial expression $(x - x_0)(b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1)$ and comparing the result coefficientwise with $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

With $P(x) = (x - x_0)Q(x)$, we can express the derivative of P as $P'(x) = Q(x) + (x - x_0)Q'(x)$. Therefore, the derivative in x_0 is just $Q(x_0)$. Since the students already know how to evaluate $P(x_0)$ from the numbers $a_n, a_{n-1}, \ldots, a_1, a_0$, they can do the same with $Q(x_0)$ from the row $b_n, b_{n-1}, \ldots, b_1$.

The procedure and the methods applied are typical of school-level algebra: the introduction of suitable notations and term transformations according to algebraic rules like commutativity, associativity and distributivity. The students argue algebraically, and the proof is done purely symbolically.

At the end of the "algebra" workshop, a historical excursion is worthwhile since we can see how algebraic language developed. For instance, Horner (1819) wrote his methods with coefficients named $\alpha, \beta, \gamma, \delta, \varepsilon, \eta, \ldots$, which gives a clear limit in expressing the nature of the recursion. The Chinese roots (Wang and Needham 1954) of Horner's scheme would give even more insight into the early development of algebra. In order to contrast algebraic language with geometrical concept development we do not present the scheme in Horner's notation (Horner 1819) but in modern notation based on modern index usage and set theory.

In all our workshops, students managed to show that the calculations with Horner's scheme lead to the evaluation of a function's value in a given point. The algebraic formalization and the algebraic proof of the connection between Horner's scheme and the factorized polynomial and the derivative of the polynomial required additional instructions depending on the mathematical skills and experiences of the participants of the groups.

7.3 Lill's Method

In the realm of engineering mathematics, the former captain of the Austrian-Hungarian Army (*Genie-Akademie*) Eduard Lill (1830–1900) invented, as engineer of the *Nordwestbahn*, a method to geometrically compute polynomials (see ÖBLBD 1971, p. 214ff for more biographical data). Nowadays it is known as *the method of Captain Lill* (see Hull 2011; Kalman 2008; Klein 1897, p. 267; Lill 1867; Anonyme 1868; Tabachnikov 2017). This provides the second historical excursion, which sheds some light on the industrial revolution and how its engineers replaced computations by drawings. In this workshop, we explain the method as a geometrical algorithm via the example of a polynomial with only positive coefficients. Consider the example $2x^5 + 6x^4 + 7x^3 + 5x^2 + 4x + 1 = 0$. In order to show the general nature of the method we denote the polynomial by

$$P(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0.$$

One can argue in an analogous way for polynomials of any degree. We draw a polygonal line by first drawing a line segment of the length of the first coefficient $a_5 = 2$. Then we turn left through 90° and draw a line segment of the length of the second coefficient $a_4 = 6$. Then take a left turn through 90° at each vertex and draw the next line segment of length $a_3 = 7$. Continue that way up to the last line segment of length $a_0 = 1$ (see Fig. 7.1). By this procedure, we obtain a polygonal line $OP_5P_4P_3P_2P_1P_0$.

Now we try to inscribe a polygonal line $OQ_4Q_3Q_2Q_1Q_0$ with the same starting point O and endpoint P_0 and with five subsequently rectangular line segments (the grey ones in Fig. 7.1). Each of the vertices should lie on a different line segment of the initial polygonal line and we take a left turn through 90° at each vertex when we run through this polygonal line from the start to the end. If we are able to find such a polygonal line, then we consider $t = \tan \varphi$, the slope of the first line segment. Then we claim that $x_0 = -t$ is a zero of the polynomial.

We demonstrate the idea of the proof in the concrete case of the considered example of degree 5 with positive coefficients in terms of Fig. 7.1. The first observation is the similarity of the right-angled triangles with the grey segments of the new polygonal line as hypotenuses. The second observation is that if our equation would be just a linear one, $a_0 + a_1x = 0$, the negative slope of the (only) grey segment $-\tan \alpha = -a_0/a_1$ would be the solution of the linear equation.

Now, back to the 5th degree equation. When a_5 is the first short leg of the first right-angled triangle, a smart choice to denote the second short leg is $-ta_5$. Now we express the first short leg of the second triangle by $a_4 - ta_5$ and the second leg by $t(a_4 - ta_5)$ using the similarity of the triangles. By continuing this iteration procedure, we get

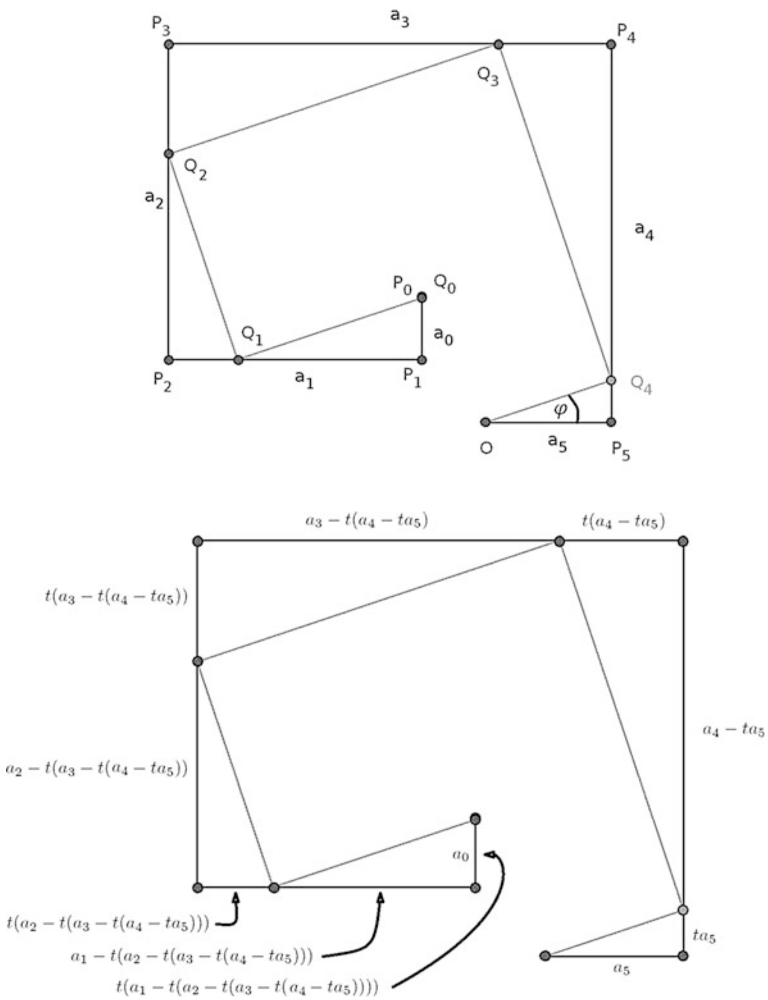

Fig. 7.1 Lill polygonal line of $2x^5 + 6x^4 + 7x^3 + 5x^2 + 4x + 1 = 0$

$$a_0 = t(a_1 - t(a_2 - t(a_3 - t(a_4 - ta_5)))) \quad \text{or}$$
$$a_0 - t(a_1 - t(a_2 - t(a_3 - t(a_4 - ta_5)))) = 0$$

Clearly $x_0 = -t$ is a root of the polynomial $P(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

At this stage, most of the participants of the workshop observed the analogy to Horner's scheme. In addition, the negative value of the root appears plausible from the algebraic point of view because of the choice of positive coefficients.

By slightly modifying the above procedure to find a root x_0 of a polynomial $P(x)$ of degree n, one can also get the value of $P(x_1)$ for any value x_1. For this, we choose

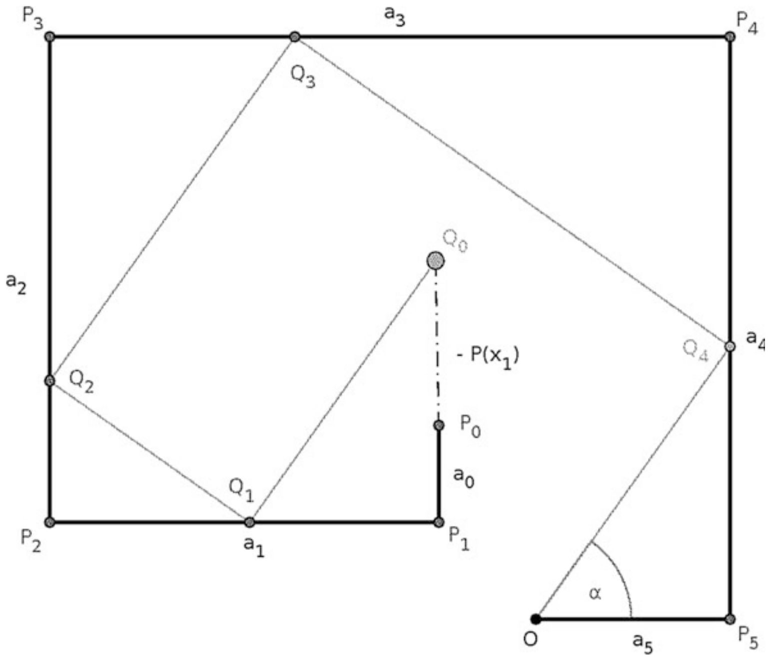

Fig. 7.2 Geometrical determination of $P(x_1) = 2x_1^5 + 6x_1^4 + 7x_1^3 + 5x_1^2 + 4x_1 + 1$ for an $x_1 \in \mathbb{R}$

an angle α with $x_1 = -\tan\alpha$ and draw a polygonal line $OQ_4Q_3Q_2Q_1Q_0$ that starts out in O with angle α (Fig. 7.2).

With $c := P(x_1)$ we know that $x_1 = -\tan\alpha$ is a zero of the polynomial

$$P(x) - c = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + (a_0 - c)$$

and therefore, the polygonal line $OQ_{n-1}Q_{n-2}\ldots Q_1Q_0$ fits exactly in the Lill polygon of $P(x) - c$. So our $P(x_1) = c$ appears as the difference between the constant values of the polynomials $P(x)$ and $P(x) - c$. In the diagram, we identify it as the oriented distance between P_0 and Q_0 (i.e. $P(x_1)$ has the same sign as the constant term a_0 if and only if Q_0 lies strictly between P_1 and P_0).

Now let us try to find the geometrical interpretation of other studied applications of Horner's scheme for a general polynomial $P(x) = \sum_{k=0}^{n} a_kx^k$ of degree n. For some real number x_1, the polynomial division with remainder of $P(x)$ by $(x - x_1)$ yields $P(x) = Q(x)(x - x_1) + c$ with $c = P(x_1)$ and we obtain for the derivative $P'(x) = Q'(x)(x - x_1) + Q(x)$. By inserting x_1, we find $P'(x_1) = Q(x_1)$. But can we find this polynomial $P(x)$ back in the Lill polygon? Can we determine for an arbitrary x_1 the derivative $P'(x_1) = Q(x_1)$? The answer is astonishing.

When α is chosen such that $x_1 = -\tan\alpha$, we draw a Lill polygonal line $OQ_{\text{n}-1}Q_{\text{n}-2}\cdots Q_2Q_1Q_0$ that starts out in O with angle α and ends somewhere on the straight line corresponding to a_0. Then this polygonal line can be interpreted as

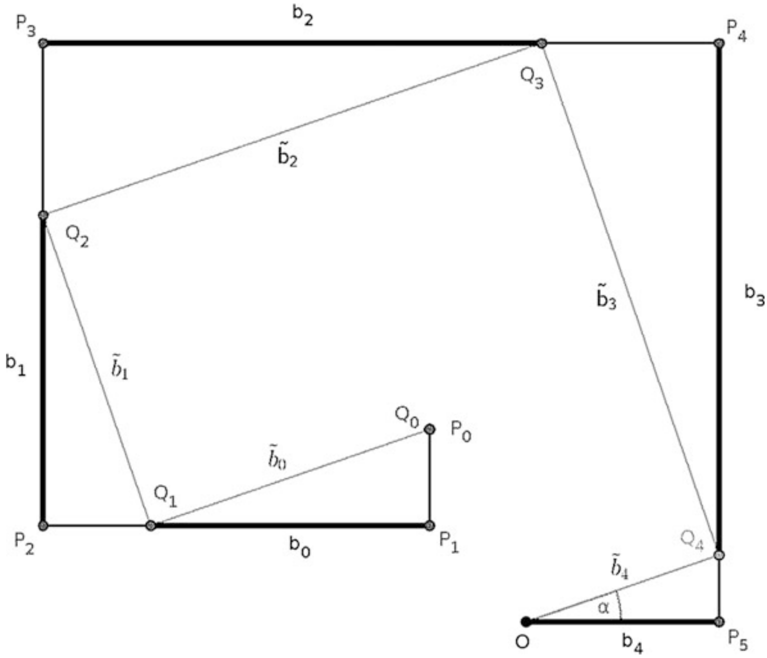

Fig. 7.3 Finding the coefficients of $Q(x) = \sum_{k=0}^{n-1} b_k x^k$ in the Lill polygon

a Lill polygon of some polynomial $\tilde{Q}(x) = \sum_{k=0}^{n} \tilde{b}_k x^k$ itself. We claim that this polynomial is in fact our polynomial $Q(x)$ up to a positive factor bigger than one.

To show this, we denote the coefficients of $Q(x)$ by $b_0, \ldots, b_{n-1}$, i.e. $Q(x) = \sum_{k=0}^{n-1} b_k x^k$. From Horner's scheme we know that $a_n = b_{n-1}$ and $a_k = b_{k-1} - tb_k$ for $k = 1, \ldots, n-1$. In our polygon, we find the b_k as the first short leg of the right-angled triangles lying on the lines segment of a_k (Fig. 7.3).

Using once more the similarity of the triangles and that $\cos\alpha = b_k / \tilde{b}_k$, or the Pythagorean Theorem, we get $\tilde{b}_k = \sqrt{1+x_1^2} \cdot b_k$; or, in other words, we obtain $\tilde{Q}(x) = \sqrt{1+x_1^2} \cdot Q(x)$.

Now it remains to identify $P'(x_1) = Q(x_1)$ in the Lill picture. Since $\tilde{Q}(x)$ is already represented by a Lill polygon and since we know how to use such a Lill polygon to evaluate a function, we can easily identify $\tilde{Q}(x_1)$. For this, we draw a Lill polygon $OR_{n-1}R_{n-2}\ldots R_2R_1R_0$ into $OQ_{n-1}Q_{n-2}\ldots Q_1Q_0$ that starts out in O with angle α (with respect to OQ_{n-1}) and ends somewhere on the straight line corresponding to Q_1Q_0 (see Fig. 7.4). Then the oriented distance R_0Q_0 is $\tilde{Q}(x_1)$. Let L be the pedal point of R_0 onto P_2P_1. Again, by similarity, we see that $R_0Q_0 = \sqrt{1+x_1^2} \cdot LP_1$ and therefore, the oriented (in the sense explained above) distance LP_1 is the value $\tilde{Q}(x_1) \Big/ \sqrt{1+x_1^2} = P'(x_1)$ for which we are looking.

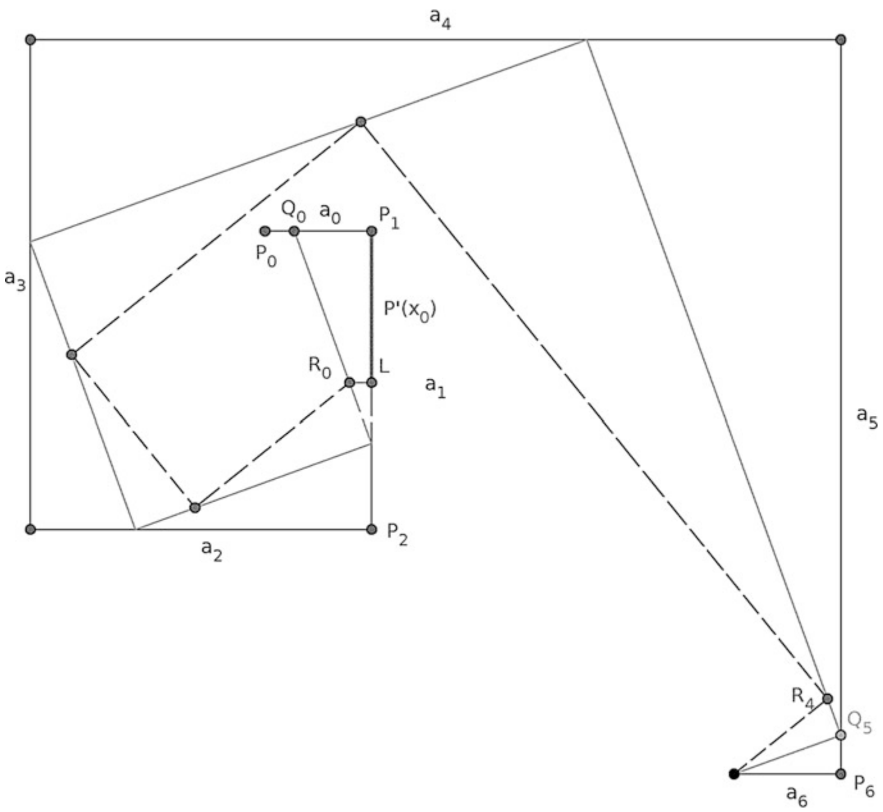

Fig. 7.4 Computation (for $n = 6$) of the derivative with Lill polygon as $P'(x_1) = LP_1$

The method even works for polynomials with negative and with vanishing coefficients. Here we first need a rule of thumb to be able to draw our generalized polygon lines. For this we start out with a general polynomial

$$P(x) = a_n x^n + \cdots + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$$

Now, starting in a horizontal direction, we draw our starting polygonal line with oriented segments to the right when all coefficients a_k are positive, like the spiral in Fig. 7.5. When one of the coefficients a_k is negative, we look at the spiral and draw a respective oriented line segment in the oriented polygonal line in just the opposite direction. Therefore, the direction of the k-th line segments is independent of the direction of neighboring line segments.

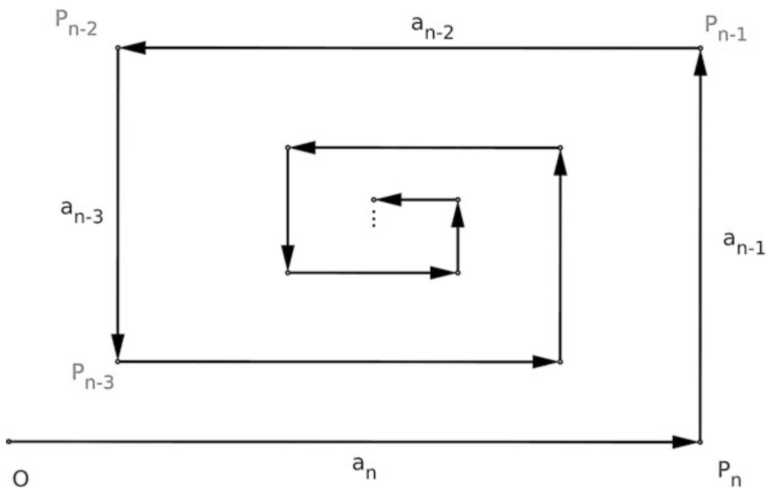

Fig. 7.5 Spiral as a rule of thumb

Let us view an example. We consider a polynomial equation of degree 4 of the form

$$P(x) = a_4x^4 - a_3x^3 - a_2x^2 - a_1x + a_0 = 0,$$

with coefficients $a_4, a_3, a_2, a_1, a_0 \geq 0$. We draw our spiral and apply our rule of thumb to find the directions in which we have to find our polygonal line (Fig. 7.6).

A polygonal line that will lead to the solution can also start with a negative ratio $-t$, i.e. we do not turn from O to the left, but to the right. Thus, we proceed as follows. When we come from the line that carries a_k, we look for the line that is formed by a_{k+1} (Fig. 7.7).

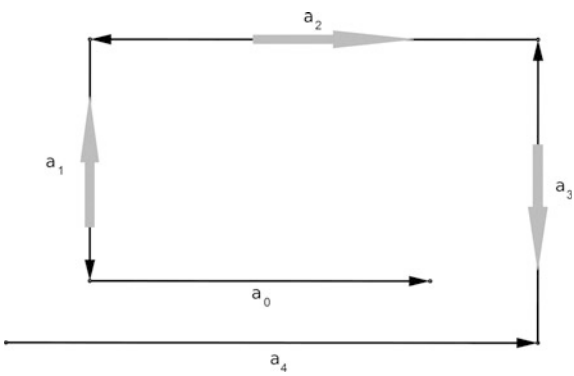

Fig. 7.6 Rule of thumb for $P(x) = a_4x^4 - a_3x^3 - a_2x^2 - a_1x + a_0 = 0$

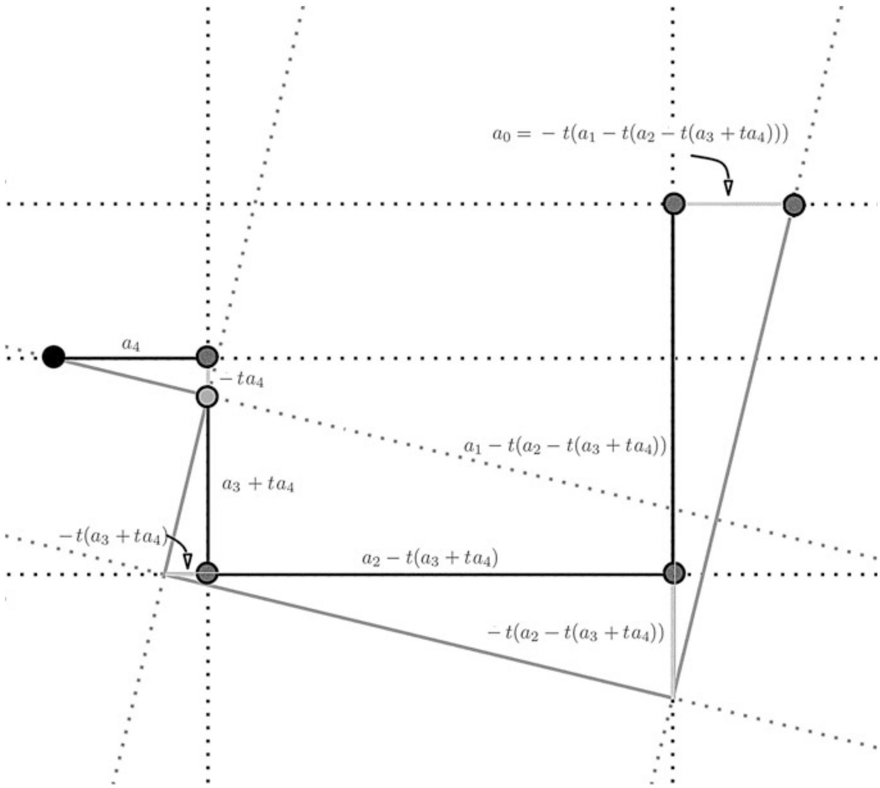

Fig. 7.7 A more general equation of degree 4 and an attempt to find a solution

If we do so, we end up with $a_0 = -t(a_1 - t(a_2 - t(a_3 + ta_4)))$. Again, we set $x_0 = -t$ and get the equation $a_0 = x_0(a_1 + x_0(a_2 + x_0(a_3 - x_0 a_4)))$ or likewise

$$P(x) = a_4 x_0^4 - a_3 x_0^3 - a_2 x_0^2 - a_1 x_0 + a_0 = 0.$$

Before we considered vanishing coefficients, we looked with the students at still another example and considered a polynomial equation of degree 6 of the form

$$P(x) = a_6 x_0^6 - a_5 x_0^5 + a_4 x_0^4 - a_3 x_0^3 - a_2 x_0^2 + a_1 x_0 + a_0 = 0$$

where the numbers $a_6, a_5, a_4, a_3, a_2, a_1$ and a_0 are all positive. The corresponding polygonal line can be found in an analogous way, as we see in Fig. 7.8.

The method works even for vanishing coefficients. For a coefficient $a_k = 0$, we imagine a line segment of length zero lying on a straight line orthogonal to the straight line corresponding to the coefficient a_{k-1}, which might be zero, too. Thus, in the Lill polygon $OP_nP_{n-1} \ldots P_3P_2P_1P_0$, it is possible that consecutive points coincide. However, the first line segment OP_{n} is not just a point since the

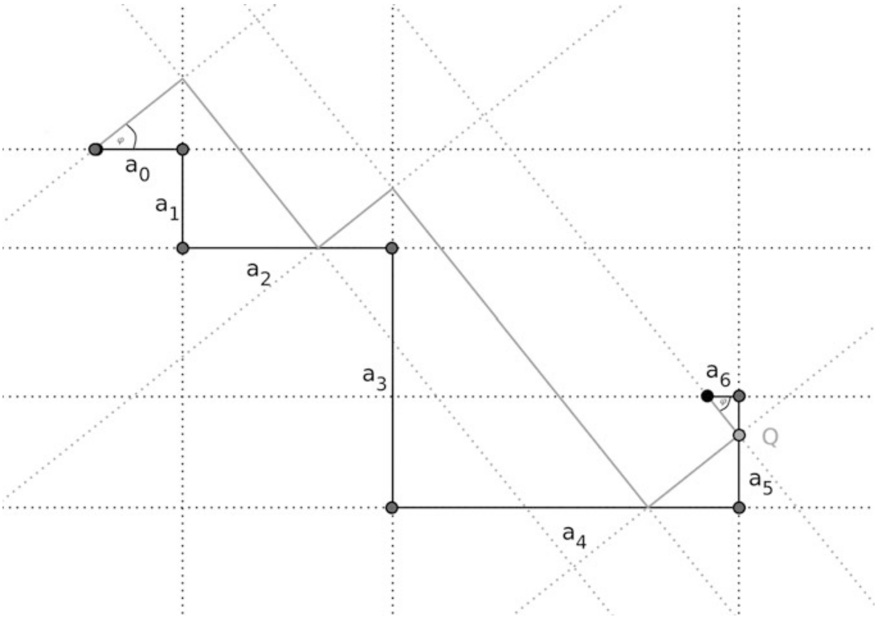

Fig. 7.8 A more general equation of degree 6 and an attempt to find a solution

polynomial is supposed to be of degree n. To any subsequent pair P_kP_{k-1} for $k = n, n-1, \ldots, 2, 1$, corresponds a straight line through P_{k-1} that is orthogonal to the respective straight line that corresponds to $P_{k+1}P_k$, where we set $P_{n+1} = O$. This gives a recursive procedure to associate a straight line to any P_kP_{k-1}, which we (by abuse of language) also denote by P_kP_{k-1}.

For instance, the equations $x^k - a = 0$ for any positive number a, and in particular the quadratic and cubic equations, can be solved that way. This approach gives precisely the same constructions and figurative presentations for the n-th roots of a as the ancients used by employing mean proportionals.

In this method, we reason mostly geometrically. The pictures in the Lill method are not just visualizations of Horner's scheme. We create and manipulate these pictures without explicit recourse to algebra or arithmetic. The method is not restricted to these pictures; for instance, one can also vary the angles (Fig. 7.9).

After some examples, the participants of the workshops varied polygonal lines without referring to algebraic equations and searched for conditions, which allow finding inscribed polygonal lines in geometrical terms. For quadratic equations, they constructed the inscribed polygonal line, i.e. its real solutions using Thales' theorem.

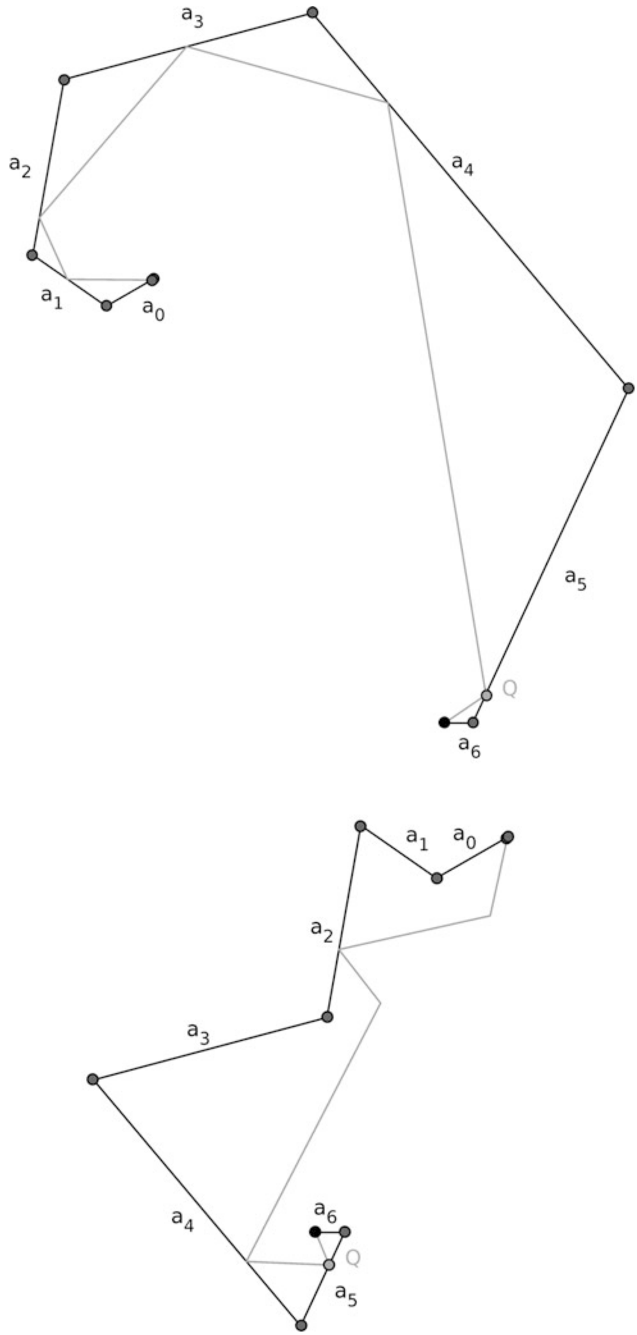

Fig. 7.9 The Lill polygonal lines are not restricted to polygonal lines with right angles

7.4 Mean Proportionals

In the third workshop, we also employ historical excursions. Additionally, we repeat topics from the school curriculum for 14/15-year-old students—such as the *Height on Hypotenuse Theorem*. First, we study how algebraic operations can be constructed geometrically (Courant and Robbins 1941, p. 120ff). In order to introduce geometrical notions and notations, the students develop geometrical versions of addition, subtraction, multiplication and division. In doing so we emphasize that the latter constructions rely on the choice of a unit length. This excursion goes back to the 17th century, especially to Fermat and Descartes.

Then we go even further back in history. The ancients [e.g. Euclid: *Elements*, Book VI, §8; Book II, §14 (implicitly)] already knew that if in a right triangle ABC the height from the vertex of the right angle is drawn, a little miracle occurs. All of a sudden, three similar triangles are formed; AHC, CHB and ABC (Fig. 7.10); with the height h being the *mean proportional* of u and v.

This leads us to the ratios $u/h = h/v$, which enables us to geometrically construct square roots of a line segment of lengths u, when we choose v to be 1. We can iterate the method of finding mean proportionals in two ways, as indicated in Fig. 7.11. These methods can explicitly be found in connection with the duplication of the cube.

By both constructions, we find mean proportionals of order two h_1, h_2, i.e. $u/h_1 = h_1/h_2 = h_2/v$ or $v = h_1^3/u^2$. When we choose $u = 1$ and $v = a$ then h_1 is the third root of a line segment of length a. Conversely, with $u = a$ and $v = 1$ we find h_2 to be the third root of a line segment of length a. The construction to the left in Fig. 7.11 is due to Plato (Herrmann 1927, p. 43ff).

The construction to the right in Fig. 7.11 is ascribed to Eratosthenes in connection with his *mesolabium* (Burton 2011, pp. 184–185), which is sketched in Fig. 7.12. There we consider three congruent rectangular triangles $\Delta_1 = A_1B_1C_1$, as well as $\Delta_2 = A_2B_2C_2$ and $\Delta_3 = A_3B_3C_3$, where for $i = 1, 2, 3$ the hypotenuses c_i and the short sides a_i and b_i are such that the sides a_i, b_i, c_i of Δ_i are oriented counterclockwise and that the three short sides b_i are always lying on a fixed line g.

Now we slide the triangles $\Delta_1, \Delta_2, \Delta_3$ along this line g, such that the three intersection points A_1 (the common vertex of c_1 and b_1), F (the intersection point of a_1 with c_2), and G (the intersection point of a_2 with c_3), are all lying on a common

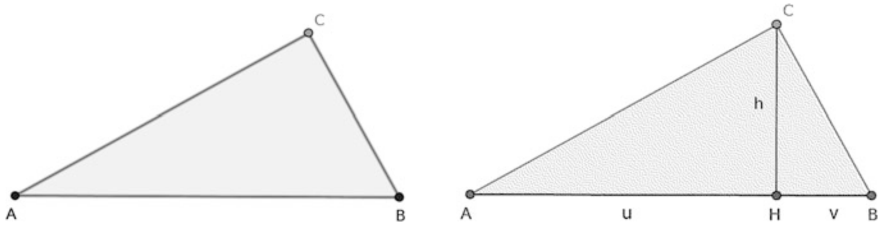

Fig. 7.10 Construction of an altitude in a right triangle

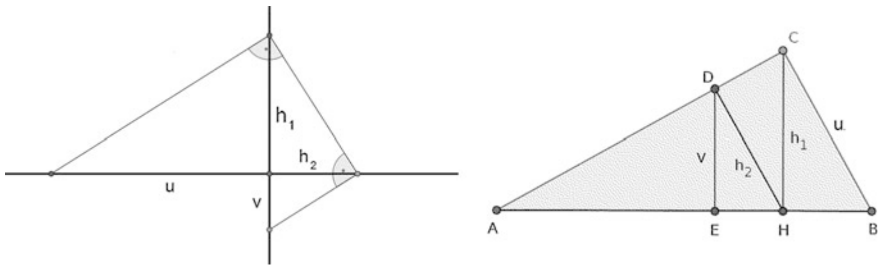

Fig. 7.11 Mean proportionals of order two following Plato (left) or Eratosthenes (right)

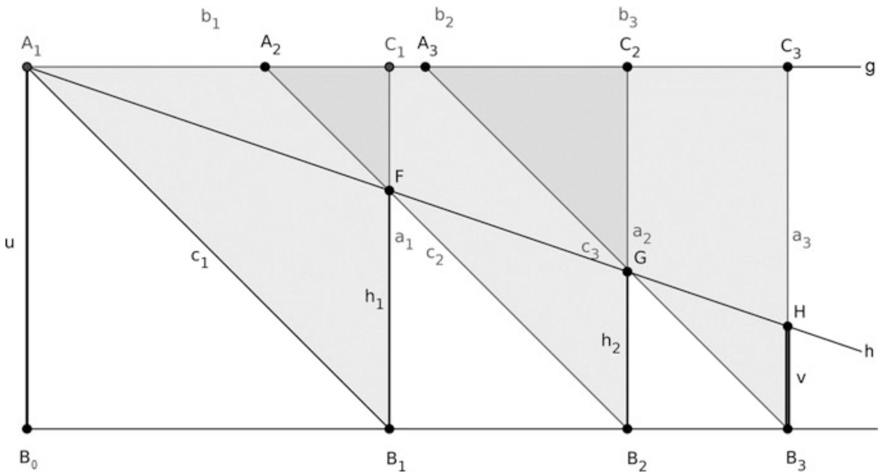

Fig. 7.12 Mesolabium of Eratosthenes

line h. Denote by h_1 the distance between F and B_1 (the common vertex of c_1 and a_1) and let h_2 be the distance between G and B_2 (the common vertex of c_2 and a_2) (see Fig. 7.12). If u is the distance from A_1 to B_0, and v the distance from H (the intersection point of a_3 with h) and B_3 (the common vertex of a_3 and c_3), then, by the similarity of the three trapezia $B_0B_1FA_1$, B_1B_2GF and B_2B_3HG, it follows that:

$$u/h_1 = h_1/h_2 = h_2/v$$

and $h_1 = \sqrt[3]{vu^2}$. Starting with $u = 1$ and $v = a$ we can reverse the construction and find h_1 to be the third root of a. There is also a mesolabium of Descartes (1954, p. 46), whose principle is similar to the Eratosthenes' mesolabium, but which will not be discussed here.

The mesolabium and Plato's construction give rise to the invention of drawing and construction devices for getting roots of line segments geometrically. Knowing the basic geometrical constructions, such devices can be invented by the students themselves.

The construction of mean proportionals of order two or higher cannot be accomplished by straightedge and compass anymore, as was shown by Pierre Laurent Wantzel in 1837 (see Courant and Robbins 1941, p. 134ff). However, geometrical language does not have the expressive power (cf. Kvasz 2000, p. 11ff) to find this result by geometrical means. Here the geometric language is not a visualization of the algebraic one, simply because in ancient times the latter did not exist. We rely only on the notions of length and the ratio of line segments. This workshop is related to topics from school mathematics, which are, according to the German curriculum, taught to 14- and 15-year-old students. The participants of our workshops knew from school that the height to the hypotenuse of a rectangular triangle has the property of dividing this triangle into two smaller triangles, which are both similar to the original one. However, they did not establish the connection between Lill's method for the equation $x^2 - a = 0$ and the construction of the geometric mean before the third workshop.

7.4.1 *Paper Folding Constructions*

Another way to promote geometrical concepts and intuition are paper-folding constructions, also called "origami" due to their origin in the Japanese culture going back to the 6th century (Kasahara 2004). Paper folding construction as a mathematical problem-solving approach is quite a modern development in mathematics, as well as in mathematics education. Paper folding introduces new geometrically meaningful objects and transformations. Generating a straight line by paper folding is associated with the reflection of the (paper) plane leaving every point of the crease line invariant. It allows hands-on experiences related to angles, triangles, quadrilaterals, polygons, congruence and similarity from the perspective of symmetry. The first person to discover the full power of paper folding as a geometric construction tool was Margharita Piazolla Beloch in the 1930s (Beloch 1936). The principles of paper folding the so-called Huzita-Hatori axioms or Huzita-Justin axioms describe the operations that can be made when folding a piece of paper as mathematical operations. The six axioms were, as the names suggest, rediscovered several times between 1986 and 1995. The idea that mathematics can be lost or forgotten is for most of the students unusual and strange. In the mathematical theory of paper folding, constructions are made by sequences of basic moves, which can be classified by enumerating all the allowed ways of folding: a single straight crease line can be made by aligning given points or lines to other points or lines already made on the paper (Hull 2011). For clarity, we use drawn lines and points for images and pre-images of the reflections, crease lines are associated with symmetry axes of reflections. We give some of the basic moves M1 to M6. The first move M1 connects two given points by a fold that passes through both of them, i.e. the crease line is the straight line defined by the two points.

The second move, M2, is the reflection of the plane folding a point P_1 onto a point P_2; in other words, it is the construction of the symmetry axis of the line segment P_1P_2. The third move M3 is the reflection of the plane leaving a given point P and a given line g invariant (Fig. 7.13). These moves can also be made by straightedge and compass. The students are asked to construct an equilateral triangle and an angle bisector of two lines by folding. In contrast to straightedge and compass constructions—where the notion of length and size dominates the description and notations—the basic notion of folding constructions are *reflections* and their invariant *symmetry axes*.

The next exercise is to fold a given point P to a given line g in different ways, introducing the corresponding move M4. The result is the construction of the envelope of a parabola defined by its focus P and its directrix g (see Fig. 7.14).

The pointwise construction of a parabola can also be made by straightedge and compass. The students know parabolas as graphs of quadratic functions. The appearance of the well-known shape in the context of square roots leads to questions related to the introduction of suitable Cartesian coordinates and the relation between the locus and the graph and the construction of square roots by folding.

The fifth move M5 allows the construction of the square root shown in Fig. 7.15 (left) by paper folding: Move M2 defines the center P_3 of P_1P_2 by folding P_1 onto P_2. The move M5 is a fold that places P_1 onto g and passes through P_3 (Fig. 7.15, right).

The next move M6 folds (if possible) two given points on two given lines. This construction cannot be done by straightedge and compass. The well-defined description of the move, the folding of the needed reflection and its pictorial and symbolic presentation, as well as its composition with other moves, lead to a new, more powerful geometric language related to transformations.

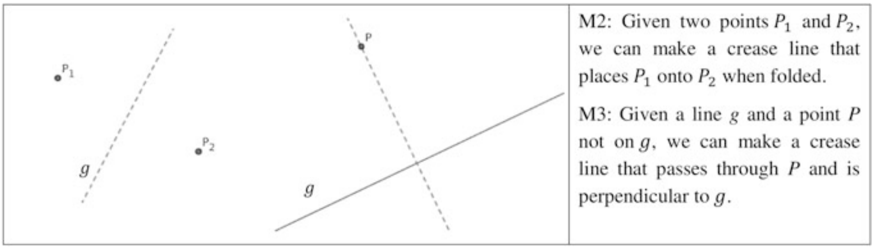

Fig. 7.13 The two basic folding moves

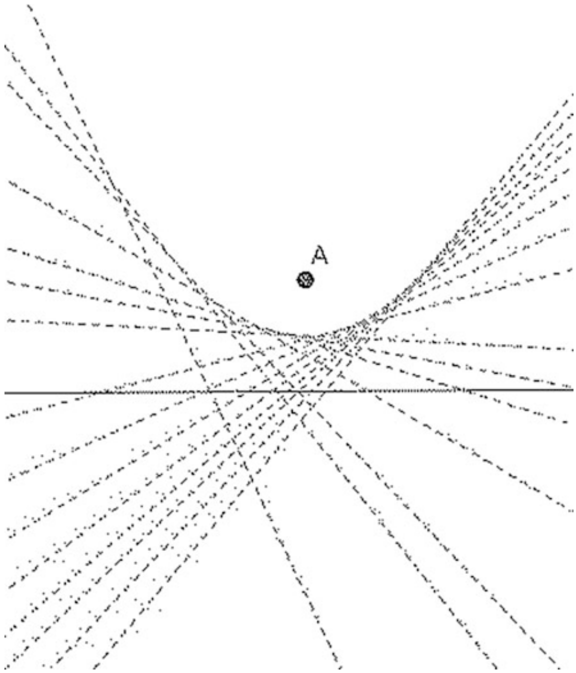

Fig. 7.14 Folding a parabola as an envelope by move M4

In order to construct cubic roots we introduce Beloch's square (Hull 2011, p. 309) as a sequence of moves. This part of the workshop is led by guided instruction. Squared paper is used for the folding construction of the cubic root of two (Fig. 7.16).

Note that we also encounter here the same pictorial representations that we previously discovered in the compass and straightedge context as well in the context of polygonal lines.

Suitable secondary literature about paper folding as a mathematical method for the students is Felix Klein's short historical and mathematical outline of approaches to the geometrical solutions of quadratic and cubic equations, including paper folding (Klein 1897, p. 42), as well as his book, *Elementary Mathematics from an Advanced Standpoint—Geometry* (Klein 1926; in the English edition: Klein 2016, pp. 280–281), where he also discusses Lill's method and quotes it as well known.

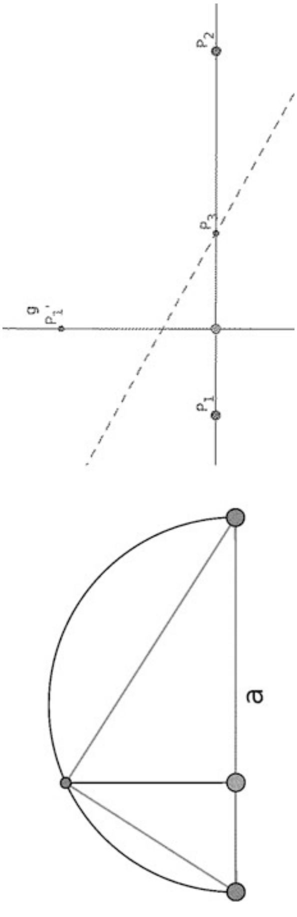

Fig. 7.15 Constructing and folding the square root of a positive number *a*, by move M5

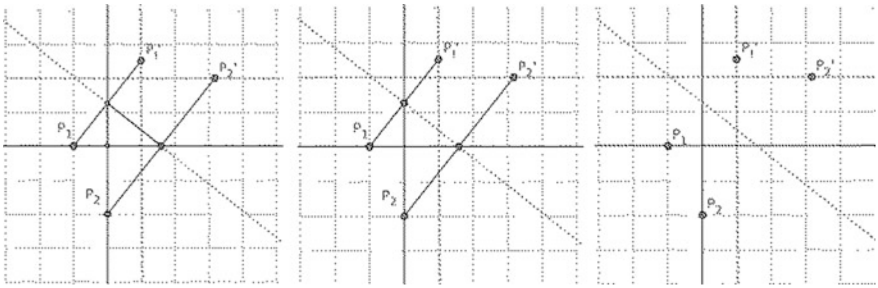

Fig. 7.16 Constructing $\sqrt[3]{2}$ by folding

7.5 Conclusion

In our project, we use history as a tool to foster mathematical understanding. Treated as an algorithm, the algebraic setup is the easiest context for the students to grasp (according to our experience in the workshops and seminars).

Nevertheless, it is also a context, which does not allow one to easily vary the problem and to ask individual questions. This is due to the underlying theoretical and abstract structure of the algebraic setup. In the algebraic context, appropriately given variations are variations of the involved objects (numbers, degree, ring…) and operations (addition, multiplication). The latter deal with algebraic structures, which are visually unintuitive, however.

Canonical visualizations of the algebraic concept of values of polynomials and their zeros in school would be presentations as a graph of the "polynomial function." The geometrical operations for graphs known to the students are changes of the coordinate system (e.g. rescaling of the axes, shift of the origin). The latter are not visualizations of operations related to Horner's scheme.

In our approach, the history of mathematics is used by the teacher as a source of inspiration to prepare geometrical contextualizations, and as a form of discovery learning by the students during and after the classroom project in their individual work. For the latter, suitable secondary literature is an essential requirement. From our experience, the students consider the algebraic and different geometrical contextualizations to be the same idea, which can be interpreted differently. We are aware that this is a Whiggish approach to history without historical methodology. Nevertheless, our experience shows that the feeling of understanding such deeper interdependences provides motivation to become seriously engaged in the history of mathematics. This holds for both the teacher, as well as the students. In this way, the history of mathematics becomes both the object of interest and the goal of the activity.

Participants in our workshops were able to pose independent research questions in geometrical terms with geometrical meaning. For instance, they studied the folding procedures of enveloping curves and searched for interpretations of binomial coefficients or Pythagorean triples in the realm of Lill's method. We interpret this as an indication of their developing a geometrical language and perspective and the ability to switch between algebraic and geometrical contexts.

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Chapter 8
History of Mathematics in German Mathematics Textbooks

Typology of Tasks

Sebastian Schorcht

Abstract Student textbooks and the tasks that they feature play a crucial role in mathematical lessons (Hiebert et al. 2003). While there have been several international studies on the history of mathematics in textbooks (Lakoma 2000; Shen et al. 2013; Smestad 2000a; Smestad 2000b; Smestad 2002; Xenofontos and Papadopoulos 2015), a comparable German study has yet to be published. This study analyzes and classifies one hundred and fifty-one tasks associated with the history of mathematics in mathematics textbooks and groups each task into one of four dimensions: "connection between the present and the past," "evolution of mathematics over time," "people throughout mathematics history" and "the aims and purposes of mathematics." The results show five types of tasks: informative present, acting present, informative past, acting past and personalization type.

Keywords History of mathematics · Typology construction · Textbooks
Mathematics education · General mathematics · General education

8.1 Introduction

The benefits of the history of mathematics and the role it plays within mathematics education have been debated for more than two hundred years. Important persons such as Lindner (1808), De Morgan (1865), Toeplitz (1927), Klein (1933), Schubring (1978), Fauvel and van Maanen (2000), Sriraman (2012), and Matthews (2014) have weighed in on this debate and their contributions are valuable in terms of understanding the potential of the history of mathematics in mathematics education and what it can accomplish. Some have offered that it might motivate the student of mathematics (Lindner 1808) or provide real evidence that mathematics has undergone an evolution (Jankvist 2009)—an evolution that has been greatly influenced by diversity across mathematical cultures. The themes of these debates

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are also diverse, from the Mayans' concept of numbers, to Al-Khwãrizmî's al-jabr, to the creation of measures—such as a cubit—by authorities. There is also an acknowledgement that mankind has had an effect on the development of mathematics and that mathematics has evolved into an intellectual tool (Jankvist 2009). Jahnke and Habdank-Eichelsbacher (1999) offer a more mathematical focus in their writings about mathematical languages. They believe the student learns more about mathematical languages and has a better understanding of them when applying the history of mathematics in mathematics education. As these scholars have shown, there are diverse and varying opinions as to how to incorporate the history of mathematics into the education of mathematics.

Defining qualitative and meaningful tasks about the history of mathematics in mathematics education requires both historical and mathematical skills. Tasks about the history of mathematics in mathematics education are "defined as a classroom activity, the purpose of which is to focus the student's attention on a particular mathematical idea" (Stein et al. 1996) or a particular period in history. Defining new and suitable tasks and providing examples presents a difficult challenge for the teacher who needs advice about how to use the history of mathematics in mathematics education, advice that has largely been lacking. As a consequence, there is a reliance and dependence on suitable tasks that already exist.

Many references to the history of mathematics in mathematics education can be found in professional teaching journals or collective volumes (Biegel et al. 2008; Fauvel and van Maanen 2000; Jahnke and Habdank-Eichelsbacher 1999; Jahnke et al. 1991, 1999, 2000, 2008; Winter et al. 1986). There is an abundance of elaborate tasks to be found in thematic journals, while student textbooks also offer examples of tasks within the history of mathematics; however, there has never been a classification of the types of tasks found within German textbooks.

The tasks that appear in textbooks are primarily adopted in schools and have already been referred to in international research. For example, Shen et al. (2013) studied mathematical textbooks from China, Singapore, USA and France. Lakoma (2000) researched Polish textbooks to describe how the history of mathematics can be used. Smestad (2000a, b, 2002) analyzed tasks about the history of mathematics and collected themes used in Norwegians textbooks. Thomaidis and Tzanakis (2010) described different kinds of tasks, from factual information to mathematical activity to historical events, and Xenofontos and Papadopoulos (2015) discovered tasks that inform students about the history of mathematics, as well as tasks that call upon students to act.

However, there have been no such studies in Germany. This chapter attempts to redress this by posing the following question of the History and Pedagogy of Mathematics community (mentioned in the introduction of this book): To what extent has the history of mathematics been integrated into mathematics education, and in particular, what does the history of mathematics look like in German textbooks?

8.2 Framework for Integrating the History of Mathematics

This paper presents theories about using the history of mathematics in mathematics education from a genetic view, a view by historical didactics and a view by general education, or the 'Allgemeinbildung' of mathematics education. In a comprehensive summary of the ideas for example from Toeplitz (1927), Jankvist (2009), Heymann (1996), and Radbruch (1997) or Wille (2001) four possible dimensions can be extrapolated to describe types of tasks:

(1) Connection between the present and the past (genetic explanation);
(2) Presentation of the evolution of mathematics over time (historicity);
(3) Historical people in mathematics history (identity); and
(4) Focus on the aims and the purpose of mathematics (orientation).

8.2.1 Genetic Explanation

The dimension of genetic explanation focuses on two attributes: *linkage to the present* and *remain in the past*. The former refers to a linkage between historical mathematical tasks and the present environment of the student; the latter refers to tasks about the history of mathematics where the aforementioned linkage does not exist. Table 8.1 presents all attributes associated with each of the four dimensions.

Hug (1985, p. 71), a researcher on history of education, distinguishes three possible linkages. Firstly, former events describe the history of the present and

Table 8.1 Dimensions and attributes used for the study of tasks about history of mathematics in mathematics textbooks

Dimension	Attributes
Genetic explanation	Linkage to the present
	Remain in past times
Historicity	Mathematics as product
	Mathematics in evolution
Identity	Personalization
	Personification
Orientation	Historical information
	Mathematical information
	Historical acting
	Mathematical acting
	Historical orientation
	Mathematical orientation

contain the conditions, causes and prerequisites of present circumstances. The present relates back to former progress or decisions and both have an effect on the present; as a consequence the present depends on its past and is based on former traditions. The main aspect in this linkage is the evolution of traditions.

Secondly, history-based structures and events are models that help to explain a modern situation. The example of history could be paradigmatic for a modern situation. It allows a deeper understanding of this situation because there are hints and explanations in the historical situation that help in the understanding of modern problems or facts. The main aspect here is the explanation of a current situation.

Thirdly, different historical situations could be compared in the contrast or the alternative, thereby demonstrating the salient facts and opening the student's mind to recognizing changeable facts in the present. Therefore, relativity and the commonality of situations are visible. The student might be able to assess present situations by analyzing the relatives and the restrictions. The main aspect in Hug's third linkage is identification of the alternatives.

Rüsen (2001, p. 83) offers a sophisticated explanation for this linkage. In his opinion, historical artifacts that occur in the past also occur in the present, but differently. These artifacts could be a promise of what the future holds: a valuable inheritance, or a fact on which hopes are pinned for changing circumstances. Teaching the understanding of unfamiliar attitudes is the important role of this linkage.

The different nature of the past should not be interpreted as unprogressive pre-history of the present, but rather offer the student the opportunity to assess and reflect on present situations. These situations should provide an orientation and meaning. Experiences with alterity in tasks are a consequence of contrasts and alternatives to the present situation.

To summarize, the history of mathematics offers a historical, genetic view of mathematics. Present mathematical activity affords a historical, genetic meaning through a linkage to the past. If children were to understand current mathematics as a product of evolution, they would need a linkage between a current mathematics and its past. Linkages are prepared by genuine, historical views on particular topics and therefore create accessions to history, for example, as in the case of the linkages by Hug (1985) or Rüsen (2001). People have questions about history. They want to explain traditions or present situations as well as discover new alternatives. The origins of historical studies are always based on questions from the point of view of the present (Rüsen 2001; Bergmann 2008). Therefore, tasks about the history of mathematics in mathematics textbooks can produce linkages between the present and the past and answer actual questions posed by the student. Tasks without linkages are collections of anecdotes and the history of mathematics provides an avenue into mathematical topics. Thus, tasks offer a *linkage to the present* or they *remain in the past.*

8.2.2 Historicity

The dimension of historicity focuses on two attributes: *mathematics as product* and *mathematics in evolution*. Mathematics is a product and a result of progress or evolution. On the one hand, tasks about the history of mathematics can present an evolution of mathematics, while on the other hand, tasks support the perception of mathematics as a product. These two views are described below.

Jankvist (2009, pp. 21–26) compares two possible goals about the history of mathematics in mathematics education: "history-as-a-tool" and "history-as-a-goal." History-as-a-tool refers to the knowledge about the inner issues of mathematics. That is, knowledge about types of numbers, their interdependency and their cardinality. History-as-a-goal refers to the knowledge about the outer issues, such as how mathematics evolves over time. Jankvist agrees with Hersh (1997) about the concept of the inner issues and outer issues of mathematics. Similarly, Furinghetti (2004, p. 2) writes there are two possible goals: (a) "History for reflecting on the nature of mathematics as socio-cultural process" and (b) "History for constructing mathematical objects." The former contains the outer issues of mathematics where the focus is on history-as-a-goal. The latter contains the inner issues and focuses on history-as-a-tool.

History-as-a-goal centers on mathematics as a process in progress. If the focus is on unfamiliar mathematical procedures, courses will discuss this mathematical process. Tasks that lack focus on unfamiliar mathematics are restricted to current mathematics, thus shifting the main focus to mathematics as a product. As a consequence, progress is not in the discussion within mathematical learning; essentially, mathematics has not been exposed to changes, but rather has been "discovered" or "developed." For example, theorems may be presented as a product. They are "discovered" and integrated into mathematical concepts. In this view, other historical aspects are neglected. Contemporary mathematics is presented as an advanced solution, a product without changes in the past, present and hence the future.

The progress of mathematics is apparent if a view on changeable facts exists (Jahnke et al. 2000, p. 292). Analyzing the progress within education presents the advantages and disadvantages of mathematical actions. For example, algorithms for addition have experienced diverse changes over time. By demonstrating these changes, the student realizes the efficiency of contemporary procedures and understands there is more than one approach to numerical addition.

To summarize, tasks about the history of mathematics in mathematics education have to present the progress in mathematics and according to Jankvist (2009): "[Mathematics] is a discipline that has undergone an evolution and not something that has appeared out of thin air" (p. 22). The aim of research on history in education is to ameliorate historical awareness for the student, while the goal of the history of mathematics in education is to promote awareness of the historicity of mathematics. In accordance with the point of view about historical awareness expressed by the German researcher Schieder (1974, p. 78), we can say that being aware of the historicity of mathematics means the permanent awareness that mathematics exists in time, and therefore has an origin and a future. Mathematics is not unconditionally stable or static. Thus, tasks about the history of mathematics present *mathematics as a product* or *mathematics in evolution.*

8.2.3 Identity

The dimension of identity focuses on two attributes: *personalization* and *personification*. Personalization represents prominent, historical figures (from now on referred to as 'celebrities') who have played an important role in the history of mathematics and include such personalities like Thales, Diophantus or Euler. Personification represents trans-regional, ordinary individuals (from now on referred to as 'persons') whose approach to mathematics is one akin to that of a manufacturer, craftsman or merchant.

Presenting celebrities who have influenced mathematical development confers a "human face" on mathematics (Jankvist 2009). Studies on the history of education, including those of Bergmann (2008) and Sauer (2009), embrace similar concepts of associating famous historical figures with mathematics. This teaching principle is known as the personalization and personification of history. In the last century, people who acted in history are celebrities of history (Sauer 2009, p. 85); whereas in the present day, history of education juxtaposes the concept of historical celebrity with the abstract concept of common day experience and attitude. Bergmann (2008, p. 158) and Sauer (2009, pp. 85–88) argue for a balance between personalization and personification. Personalization, means celebrities are the sole participants in history and seemingly it is only those whose actions in the present influence what is later viewed as history. The result, according to Sauer, is a servile spirit because celebrities became a leitmotif for all people. In contrast, personification has the objective of depersonalizing historical events. History should refer to a historical living environment that compares favorably to that in which the present day student exists. In this way, history refers to the student's own living environment. Furthermore, Bergmann (2008, p. 159) urgently insists on the balance between personalization and personification. An imbalance between personification and personalization leads to a unilateral representation of history.

Regarding tasks about the history of mathematics and personalization, these refer to biographical information about important mathematicians. Epple (2000, p. 135) writes about these kinds of tasks and ventures, stating that there is no canon of topics, only preferences for teaching about some mathematicians while disregarding others. This subjective assessment concerning topics can never be clarified for education courses. Epple refers to Nietzsche (1874/2009, p. 27), who called this type of history "monumentalistische […] Art der Historie," or, a monumentalism type of history, which according to Nietzsche lauds bygone celebrities while at the same time criticizes present day celebrities. Even Nickel (2013, p. 260) criticizes the history of biographies and refers to them as a "caricature." He writes that mathematics would be performed by intellectual, unattainable giants whose intimidating shadow allows no likelihood of one's own mathematical activities. The presumption is that one's own mathematical activities are irrelevant compared to the major discoveries of deceased mathematicians. Nickel (2013, p. 260) presents mathematicians as "joviale Karikatur," or jovial caricature, meaning that the discoveries of mathematicians are analyzed within contemporary knowledge and interpreted as a preliminary stage to present-day mathematics. This view is known as "Whig" history (Fried 2001, p. 395). Opposing a monumentalism type of history, as well as caricature and jovial caricature, requires personification, or rather, persons and their tasks about the history of mathematics in mathematics education. Therefore, the history of mathematics in mathematics education proves to the student that not only celebrities influence mathematics, but also persons.

In summary, one of the objectives in the study of the history of mathematics is to expose human influence on mathematics (Epple 2000; Jankvist 2009). The influence of celebrities, or persons should be a topic associated with the tasks about the history of mathematics, but the balance between these two types of representations is also significant (Bergmann 2008, p. 159). Thus, tasks about the history of mathematics present celebrities (*personalization*) as well as persons (*personification*).

8.2.4 Orientation

The dimension of orientation focuses on six attributes: *historical information*, *mathematical information*, *historical acting*, *mathematical acting*, *historical orientation* and *mathematical orientation*. Tasks about the history of mathematics can provide information inciting students into action and introducing "orientational knowledge." This requires some explanation.

Available knowledge is knowledge; "I know that I don't know," Radbruch (1997; author's translation). Available knowledge can be remembered or transferred and is objective. Therefore, it describes mathematical acting, which Fischer (2006, p. 86) concludes as the presentation, calculation and interpretation of abstracts. Mathematical acting in tasks about the history of mathematics should enrich possible presentations, stimulate mathematical calculations or practice mathematical interpretations.

Orientational knowledge is knowledge; "I don't know that I know" (Radbruch 1997; author's translation). It is non-quantifiable and comprises experiences, dispositions and intuitions. Radbruch distinguishes available and orientational knowledge as follows:

- "Life in awareness becomes accessible to reality by orientational knowledge, and only in a small part by available knowledge."
- Orientational knowledge fundamentally precedes every available knowledge in time. In other words: every available knowledge has its conceptual roots in orientational knowledge" (Radbruch 1997, p. 7; author's translation).

Thus it appears that orientational knowledge benefits in two possible situations. Firstly, orientational knowledge precedes available knowledge. People first stimulate orientational knowledge through experiences and dispositions. Secondly, once a person resolves a problem, they retain knowledge they can use in a similar situation. Experiences become goods that can be transported to others. At the same time, available knowledge loses its connection to orientational knowledge and is independent from context.

There are two kinds of orientational knowledge: one based on discovery, and the other based on experience. If a new way to generate energy in a factory is developed, this is an example of orientational knowledge through discovery. Experience over many years is orientational knowledge through experience. The former occurs as an event at a specific moment in time, whereas the latter occurs over a period of time. The more experiences, the more visible are the boundaries and possibilities of techniques. Both the progress of, and the social experiences with, available knowledge severs the connection between available and orientational knowledge. In the transfer of experiences and dispositions, available knowledge loses its connection to context-dependent experiences and dispositions. The aim is to reconstruct this moment of separation and in this context Radbruch states (1997, p. 7; author's translation): "It follows from there that every available knowledge is at its greatest power when it is accompanied by orientational knowledge. Consequently, everything shared and transferred, and everything taught about available knowledge, has to be linked with orientational knowledge."

Orientational knowledge in connection with available knowledge would be visible by analyzing the progress from pre-scientific acts to transferable scientific descriptions (Radbruch 1997, p. 8). The moment of separation is a historical moment. The aims and the purposes of mathematical decisions and experiences with mathematics can also be historical. Therefore, the point is that the progress of mathematics enriches the teaching of available knowledge through its orientational knowledge. Orientational knowledge refers to the source and origin of mathematical decisions.

To summarize, besides teaching available knowledge and presenting information in tasks, the history of mathematics is supposed to initiate orientational knowledge in order to provide insight into the aims and purposes of mathematical acts and to initiate a discussion in mathematics education about mathematical changes. Heymann (1996) offers a similar meaning about the history of mathematics in mathematics education when he writes about adapting existing culture and acting creatively within cultural coherence. The aim of teaching orientational knowledge is to become aware about the aims and the purposes of mathematics. In conclusion, tasks about the history of mathematics can provide information such as *historical information* or *mathematical information*, available knowledge such as *historical acting* or *mathematical acting,* and *orientational knowledge* such as *historical orientation* or *mathematical orientation.*

The eventual outcomes of the four dimensions are twelve attributes. They characterize possible attributes to describe tasks about the history of mathematics. Those tasks may, or may not, have a linkage between the present and the past. They could present the evolution of mathematics, or present-day mathematics as a product. Tasks could also show how historical celebrities and/or persons inform the student about history as well as mathematics, ask for available knowledge of mathematics or history and enrich that knowledge with orientational knowledge about mathematics or history. These attributes are included in those mentioned above.

This completes the discussion concerning the study's attributes. The next section concerns the study's source material and how it was analyzed.

8.3 Method and Material of the Study

Classifying the tasks of the history of mathematics in German textbooks can be determined through the Qualitative Content Analysis by Mayring (2008, pp. 89, 91), particularly the typological structuring of the foundations of the study presented. He advocates a systematic procedure in ten steps. In the first step, units of analyses have to be extracted from the material. In the second step, possible dimensions of the history of mathematics in mathematics education are determined. In the third and fourth steps, Mayring proposes a determination of attributes by a system of categories. In the fifth to seventh steps, the examples have to be described and extracted by the researcher, which may result in changes to the attributes. Attributes have to be adjusted to the object of the investigation. The list of attributes given in Table 8.1 is the final list. The eighth step determines attribution by extrema, theoretical interest and empirical frequency. Consequently, this study uses the *formal concept analysis* developed by the mathematicians Ganter et al. (2005). This method guarantees the mapping of multiple answers in a linear diagram, also known as a Hasse diagram. The procedure matches all attributes, arranges all tasks in an order using binary relations, and presents conceptual hierarchies graphically. Types are extracted out of this ordered set of tasks and their attributes. Finally, the ninth step determines prototypes, while the tenth step delivers an exact description of these prototypes.

One hundred and fifty-one tasks appear in 41 German textbooks that constitute 12 series for students in the first through seventh grades. Examples of the history of mathematics in these textbooks were sorted in this study using Jankvist's approach that integrates the history of mathematics (Jankvist 2009) with Rezat's structure of levels in German mathematics textbooks (Rezat 2009). These two views on tasks are now explained as they offer the opportunity to extract tasks about the history of mathematics from the textbooks that were analyzed.

Jankvist (2009, p. 26) describes three approaches to integrate the history of mathematics: "illumination approaches," "modules approaches" and "history-based approaches." Illumination approaches are succinct text such as names, biographical information, and deeds or epilogues as narrative stories. The possibility of more time to learn and to teach the history of mathematics in tasks is found in the modules approaches (for example, in projects and exercises that are orientated in a curriculum). According to Jankvist, these approaches range from two hours to a school year or an academic year. History-based approaches use the history of mathematics to structure school courses and are not necessarily made explicit to the student. Jankvist (2009, p. 256) is a researcher who wrote about approaches on how to integrate the history of mathematics—approaches that can be followed in any study of this nature. At the same time, he warns about unreflecting assumption because characterization depends on the object of investigation (Jankvist 2009, p. 20). Therefore, the characterization by Jankvist must be adapted for this study on German tasks in mathematics textbooks. At this point, the study uses Rezat's structure of levels in German textbooks.

Textbooks have different structural levels. Rezat notes there are three levels in German mathematic textbooks: the book level, the chapter level and the lesson level (Rezat 2013, p. 661). These levels hold an essential place within the textbook as they are similar to Jankvist's illumination approaches and modules approaches. Each level comprises several blocks. The chapter level is characterized by blocks such as introductory pages, activities, lessons and topical pages. Learning the blocks within the chapter level requires two or three teaching hours (Valverde et al. 2002, p. 139). In this way, lessons are similar to Jankvist's modules approaches, or in terms of Tzanakis et al. (2000, p. 214), they could be research projects based on historical texts, worksheets or primary sources. By contrast, Jankvist's illumination approaches are found in the lesson level in which the blocks require up to one hour to learn. Tzanakis et al. (2000, p. 214) refer to these blocks as historical snippets and may include introductory tasks or activities, expositions, tasks and problems.

Tasks are extracted and identified in response to key questions where each question assigns tasks to a specific block. For example, tasks are reasoned as exercises if the response is in the affirmative to the key question "is there an explicit call for action within the historic mathematical content?" Tasks about the history of mathematics exist at all levels. This chapter and its study focus on the levels of chapter and lesson and as such, books about the history of mathematics are not addressed. The source material for the investigation is first through seventh grade German mathematics textbooks wherein examples of the history of mathematics are distinguished in blocks at the chapter and lesson levels. One hundred and fifty-one tasks are classified through responses to key questions. Table 8.2 illustrates the distribution of tasks associated with the history of mathematics and mathematics textbooks across the school grades. The preponderance of tasks occurs in the fifth grade because the subject of Roman numerals appears primarily in fifth grade German mathematics textbooks. Each federal state in Germany determines its educational requirements. Roman numerals are included in the curriculum of federal states such as Saxony (Sächsisches Staatsministerium für Kultus 2004, p. 21), while other states may adopt textbooks that include subjects similar to that of Roman numerals.

Table 8.2 also illustrates the number of different textbooks that include tasks about the history of mathematics. Five textbooks are from the primary school level and seven textbooks are from the secondary school level. The grades taught at German primary and secondary schools vary from state to state. Primary school comprises the first four grades in some states, or the first six grades in others. Secondary schools include up to the ninth grade in some states and up to the tenth

Table 8.2 Distribution of tasks associated with the history of mathematics and mathematics textbooks across scholastic grades

Grade	Number of tasks about the history of mathematics	Number of textbooks with tasks about the history of mathematics
1st	4	1
2nd	2	1
3rd	8	4
4th	16	5
5th	68	7
6th	31	6
7th	22	7

grade in others. Therefore, the selection of the source material used as input to the study are those used at primary schools and any transition grade into secondary school, which is state dependent. All textbooks are used in schools and reflect a progression of different themes, or have a fragmented content coverage (Valverde et al. 2002, pp. 63–73). The selection of books depends on the diversity of secondary schools. Germany has three different schools at the secondary level with different educational levels. Therefore, there are books from each of these three different levels. Furthermore, the selection depends on the diversity of publishers in Germany. There are books from the publishers Cornelsen, Duden, Ernst Klett, Schroedel and Westermann. This diversity guarantees a broad analysis of school textbooks. There is no textbook used in the source material that has specific content on the History of Mathematics.

8.4 Analysis of Tasks About History of Mathematics

The tasks were classified in accordance with a set of key questions that are answered either in the affirmative or negative and notated on a decision graph, or flow chart. This process is based on Ott (2016), who used this type of analysis to classify the quality of student representation. For example, if the answer to a question belonging to the dimension of genetic explanation is affirmative, a *linkage to the present* is notated on the chart; if the answer is in the negative, *remain in the past* is notated. Once all tasks were notated on the chart, each task was evaluated against the twelve attributes in Table 8.1 to determine which attributes exist for that task. This combination of dimension, task and its apparent attributes is the basis for the study's formal concept analysis.

For example, Fig. 8.1 defines a task that informs the student about the history of length measurement (Kliemann et al. 2006, p. 56; author's translation):

Das Längenmaß in der Geschichte

Das Metermaß ist zwar schon 200 Jahre alt, aber dennoch sind in vielen Bereichen die alten Maße recht beharrlich.
Moderne ICE-Züge fahren auf einer Spurweite von 4 **Fuß** und 8,5 **Zoll** (1435 mm), Piloten fliegen ihre Jets in 10 000 **Fuß** Höhe,

16 Fuß = 1 Rute

Schiffe fahren in **Knoten** – Seemeilen (1852 m) pro Stunde, Felgendurchmesser beim Fahrrad und beim Auto werden in Zoll angegeben.

Diese Maße waren allerdings selten einheitlich, so schwankte die Länge des Fußes deutlich zwischen 25 cm und 35 cm.
Das heute noch gebräuchliche Fuß wurde vor 1000 Jahren von König Edgar festgelegt: „36 der Länge nach aneinander gelegte Gerstenkörner aus der Mitte der Ähre.“ Ein Fuß – englisch: 1 Foot – beträgt heute 30,48 cm.

„4 Fuß = 1 m“

Fig. 8.1 Task of informative present type in Kliemann et al. (2006, p. 56). © Ernst Klett Verlag GmbH

> The meter unit of length dates back two hundred years. However, older units of length have endured in many areas. Modern ICE trains run on tracks of 4 **feet** gauge and 8.5 **inches** wide (1,435 mm), pilots fly airplanes at altitudes of 10,000 **feet**, ships travel in **knots** – a sea mile is 1,852 meters per hour, and a car or bicycle's wheel rim diameter is measured in inches. These measurements, however, were not uniformly consistent and therefore the length of one foot varied considerably - anywhere between 25 cm and 35 cm. A thousand years ago, the unit of measure we know today as the foot was established by King Edgar: 36 barleycorns laid end to end. One foot – English: 1 foot averages 30.48 cm.

The decision graph supports the classification of this task within each of the four dimensions. The first dimension of genetic explanation shows a *linkage to the present* in the task because contemporary measurements occur as a historical genetic consequence caused by the inaccuracy of ancient measurements. However, these ancient units of measure are still studied by the student of today, where the diameter of a bicycle wheel rim is measured in inches and an airplane's altitude is measured in feet. Therefore, the task includes *mathematics in evolution* within the second dimension of historicity. Figure 8.1 illustrates how units of measure have undergone change over a thousand years, as explained by Kliemann et al. above. The third dimension of identity acknowledges King Edgar as a celebrity within the history of mathematics because he is supposed to be the first person to define the foot as a unit of length. Although there is a photograph in the task of people producing a rod and a meter, the children in Fig. 8.1 are used simply as a means for creating the "foot" as a standardized unit for measuring length. They do not create mathematics in the same way King Edgar is supposed to have invented the foot as a unit of length measurement. Therefore, the task is an example of personalization and not personification. The fourth dimension of orientation focuses on mathematical and historical information, which are evident in the example above. Kliemann et al.'s task merely provides information and does not call for any mathematical or historical action. The possible aims and purposes of mathematics are described in the sentence: "These measurements, however, were not uniformly consistent and therefore the length of one foot varied considerably—anywhere between 25 cm and 35 cm." Thus, one objective of a mathematical measurement was to create consistency as it applies to the unit of measure. The student had to guess in respect to the purpose or objective of the measurement, as well as to whether the measurement was consistent. However, people needed consistent measurement for fair comparison and trade and measurements varied depending on the city, principality and country. In conclusion, there is mathematical orientation. Moreover, the task refers to historical artifacts in the present that lead to discussions about history. Therefore, access to, or entrance into, the history of mathematics is an authentic one and the task has historical orientation.

In summary, the task in Fig. 8.1 contains a *linkage to the present*, illustrates *mathematics in evolution*, has *personalization*, and provides *mathematical* and *historical information* as well as *mathematical* and *historical orientation*. This specific combination is the basis for concept lattices by Ganter et al. (2005). The formal concept consists of the set of tasks and the set of attributes. These two sets are ordered by an incidence relation. Using the tools of formal concept analysis, the study puts in subsets similar combinations of attributes. Every task within a subset shares common combinations of attributes with other tasks within the subset. Out of these subsets, the types are extracted as the supremum of a subset. In conclusion, a type has a specific combination of attributes that characterize all tasks in this type. Each task in a type can be different, but all have a certain, common combination. The next section characterizes the types of tasks found in German mathematics textbooks.

8.5 Research Results

The study establishes the following five types of historical mathematical tasks in German textbooks for the student in grades one through seven:

1. informative present;
2. acting present;
3. informative past;
4. acting past; and
5. personalization type.

These types classify the tasks discovered in the history of mathematics. Each type is characterized by a prototype. There are always tasks in a type that possess an additional attribute. Finally, all tasks in a type have a specific combination of attributes. Table 8.3 illustrates each combination of attributes for each type.

Tasks about the history of mathematics that belong to the informative present type are those tasks that inform the student about mathematics and build a linkage to present-day mathematics, allowing the student to connect today's mathematics with its past. Out of the 151 tasks, 103 belong to this type and have a linkage to the present and provide historical information such as that given in Fig. 8.1.

Tasks belonging to the acting present type (77 out of 151) contain linkages to present and require independent mathematical acting by the student. All tasks of this type comprise these two attributes such as the task in Fig. 8.2.

Figure 8.2 illustrates a call for action: "arrange by value" (Böttner et al. 2008, p. 30; author's translation). The student interprets the numerical symbols and arranges them in accordance with their value. The symbols are Western Arabic and Roman numerals. The Western Arabic numerals appear both in the decimal and binary positional systems. By comparing the present mathematical acting, such as the Western Arabic numerals, and the past mathematical acting such as the Roman

Table 8.3 Combination of attributes per type found in the tasks that were examined

Types of tasks	Combination of attributes within all tasks of one type
Informative present type	Linkage to the present
	Historical information
Acting present type	Linkage to the present
	Mathematical acting
Informative past type	Remain in the past
	Historical information
Acting past type	Remain in the past
	Historical information
	Mathematical acting
Personalization type	Remain in the past
	Personalization
	Historical information

Fig. 8.2 Task of acting present type in Böttner et al. (2008, p. 30). © Ernst Klett Verlag GmbH

numerals, the student must address alterity through the two different representations. This comparison results in an identification of alternatives and provides a linkage to the present. Mathematics appears as a product because there is no explicit reference to the evolution of numbers. This evolution is explained four pages before and the task only refers to this page. Nevertheless, the tasks are analyzed separately and Roman numerals occur only as a possibility of numerical symbols like those that appear on church clocks and for page numbers in a preface. The task furthers mathematics as a product. There are neither celebrities nor persons and this is why personalization and personification do not occur in the task. There is no historical or mathematical information that the student could use to translate into other number systems. The call for action only affects mathematical acting, but historical acting is missing because a call for historical interpretation of the original sources does not exist. The same is true in the case of mathematical or historical orientation. The task does not refer to the aims and the purposes of Roman or Western Arabic numerals in the same way the task does not refer to an original source. Consequently, there is a means to enter into historical discussions. This task symbolizes the acting present type. Each task within this type is similar to the task in Fig. 8.2, but each task could possess several of the attributes that appear in Table 8.3.

Tasks of the informative past type (38 out of 151) inform the student about the history of mathematics without a linkage to present-day mathematics. All tasks of this type remain in the past but require historical information.

Tasks of the acting past type (28 out of 151) direct students to independent mathematical acting—but without a linkage to present-day mathematics. All tasks of this type remain in the past, offer historical information and require mathematical acting by the student.

Tasks of the personalization type (28 out of 151) only use personalization to inform about the history of mathematics. All tasks of this type remain in the past, prefer personalizations and possess historical information. For the most part, persons are the focus of interest as is the case of the task that appears in Fig. 8.3.

Figure 8.3 provides biographical information about Sofia Kowalewskaja (Wittmann and Müller 2011, p. 115; author's translation):

> Sofia Kowalewskaja was born in Moscow in 1850. Her parents and teacher recognized Sofia's mathematical talents at an early age. Since girls could not attend the university in Russia, she went to Germany when she was 20 years old. However, there were also many obstacles for women in Germany. Sofia did not allow herself to be discouraged and continued her struggle for equal rights. In 1884, she became the first female professor of mathematics in Sweden.

After this informative section, there is a section that requires the student to act by looking for biographical information. Three questions prompt the student into a mathematical operation: "(a) How old was Sofia Kowalewskaja?", "(b) At what age did she become a professor?" and "(c) In which year did she move from Russia to Germany?"

A connection between the present and the past does not exist. The task deals with historical biographical information. The student has an opportunity to enrich

Fig. 8.3 Task of the personalization type in Wittmann and Müller (2011, p. 115). © Ernst Klett Verlag GmbH

their mind with a deeper awareness of mathematical concepts. A genetic-historical understanding of present mathematical acts is not included in this task. Therefore, an evolution of mathematics over time is not present in this task. One reason for this is the lack of any linkage to the present; another is the missing mathematical concept. The task does not deal with a mathematical concept and therefore an evolution of mathematics cannot be illustrated.

In conclusion, the tasks remain in the past. In regard to the meaning of personalization in mathematics history, the task presents Sofia Kowalewskaja (Fig. 8.3). She influenced mathematics because she was a researcher. Thus, the operation within mathematics is an attribute of celebrities. This characteristic of dimension identity is emblematic for tasks of the personalization type. The aims and the purposes of mathematics do not exist in the task and the focus is on historical information.

The types of tasks that are found within German mathematics textbooks can serve as a tool to create new tasks systematically. Each type is classified by one or more attributes. The types can be modified if these attributes are known. As tasks have evolved naturally and culturally over time, the examples analyzed in the study could be expanded to fit normative claims. Normative claims are extracted by the four dimensions used in this study. Hence, tasks associated with the history of mathematics in textbooks should question the student about present time, demonstrate the evolution of mathematics through a perspective of change, present public personalities of mathematics, as well as trans-regional, ordinary people who influenced the evolution of mathematics and enrich the student's knowledge with the aims and purposes of mathematics. This possibility to change tasks empowers the teacher to adjust tasks for their lessons; the teacher can employ tasks about the history of mathematics more often.

For example, the task in Fig. 8.1 has a linkage to the present, demonstrates the evolution of mathematics over time and provides orientational knowledge. The task has all the attributes associated with a good task should have. The only dimension that changes is that of identity. In the top right corner of the task in Fig. 8.1, Köbel (1536) draws an illustration of sixteen people placing their toes on the heels of the person standing next to them. A study about Köbel extends the field of people who create mathematics and as a consequence, the student learns about people and may easily identify themselves with people of their socio-economic level. The task is extended through the attribute of personalization. Hopefully, more historical tasks will be used in school courses once teachers are able to change tasks if required—as discussed in the example above.

8.6 Conclusion

While there have been international studies on tasks about the history of mathematics, there has been a lack of research regarding tasks in German mathematics textbooks. This study attempts to redress this imbalance and offers possibilities for the teacher on how to transform tasks in mathematics textbooks.

The results of this study are similar to those of international studies. For example, the study by Thomaidis and Tzanakis (2010) of Greek tasks has similar attributes to this study and both comprise factual information and mathematical activity. This study and that presented by Xenofontos and Papadopoulos (2015) discover tasks that inform the student or call them into an action. Though more attributes are used in the study presented here, those presented by Thomaidis and Tzanakis or Xenofontos and Papadopoulos are not dissimilar and are definitive for the typology of tasks in German mathematics textbooks.

The dimensions used in this study are conducive to generating a typology of tasks and future research may conclude if there are more than four dimensions and associated attributes. Tasks were extracted from the textbooks as part of this study using key questions on types of blocks (Jankvist 2009; Rezat 2009). Both of these studies help to identify tasks about the history of mathematics and provide valuable definitions for introductory pages, activities, lessons and topical pages and historical snippets (for example, introductory tasks or activities, expositions, tasks and problems). After extracting tasks, Mayring (2008), along with Ganter et al. (2005), provide a typology that takes into account multiple answers in relation to attributes. A typology of tasks is realized by analyzing the linear diagram in which whole sets of tasks and attributes are grouped into subsets and each subset has a supremum. These suprema are the basis for the typology of tasks.

Another benefit of this study is that it enables the teacher to make their own changes to tasks in textbooks. Even though a historical study for the teacher in mathematics education is a challenge, they can themselves evaluate tasks according to a detailed analysis about history of mathematics. They only have to know the normative claims: linkage to the present, mathematics in evolution, balance between personalization and personification, and mathematical orientation. Future research must verify these normative claims and may discover new ones. In conclusion, the aim is to empower the teacher to use history of mathematics in mathematics education through a tool of evaluating tasks.

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Part III
Empirical Investigations on Implementing History and Epistemology in Mathematics Education

Chapter 9
Geometry and Visual Reasoning

Introducing Algebraic Language in the Manner of Liu Hui and al-Khwãrizmî

Iolanda Guevara-Casanova and Carme Burgués-Flamarich

Abstract The general aim of this chapter is to identify the potential learning opportunities provided by the introduction of historical geometric diagrams into student tasks. To this end, we examine some problem sets for secondary education students concerning situations to be solved with diagrams in which right triangles or solving second-degree equations are involved. In all cases the objective is that students should transfer linguistically expressed reasoning (second-degree algebraic expressions) to reasoning with visual diagrams (figures with squares and rectangles) that are the geometric interpretation of the second-degree algebraic expressions. The research is therefore focused on students' learning process, and specifically, the results they achieve by the use of these diagrams.

Keywords Teaching and learning of algebra · Geometry-algebra connection Visualization · Historical context · al-Khwãrizmî · Liu Hui

9.1 Geometrical and Historical Diagrams

The purpose of introducing diagrams is to connect symbolic algebraic thought with visual thought regarding geometrical shapes. Historians, educators and many specialists in the teaching of mathematics advocate the connection between seemingly

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This chapter is based on the thesis written by the first author under the supervision of the second, and presented at the University of Barcelona in June 2015, with the title: "The use of historical contexts in the secondary school mathematics classroom: The specific case of visualization in the connection between geometry and algebra."

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different themes as one of the important mathematical processes (Burgués 2008; Burgués and Sarramona 2013; Fauvel and van Maanen 2000; Giaquinto 2007; Jankvist 2009; Katz and Barton 2007; NCTM 2000; Niss 2002; Niss and Højgaard 2011).

Katz and Barton (2007) describe the various stages in the history of the construction of algebra which, according to these authors, lead to implications in teaching and learning. The learning of algebra should begin with the close relation between geometry and problem solving. The introduction of the new mathematical topics should be discussed orally in class with groups of students as a whole. Katz poses the following question: Why not begin algebra by thinking about geometrical figures? The results of this research clearly support the introduction of this approach.

In the history of mathematics, Radford and Puig (2007) associate algebra with geometry in at least two instances in which algebraic reasoning (reasoning about unknowns) is associated with geometric reasoning: some of the problems on cuneiform tablets of Mesopotamian scribes (1900–1600 BCE), and the calculations by al-Khwãrizmî (9th century) in the study and classification of first- and second-degree equations in *Hisāb al-ğabr wa'l-muqābala*. This latter instance provides the source for the second activity set out in this research. Radford and Guérette (2000) and Siu (2000) also endorse visual reasoning and the use of historical texts. In their study, Radford and Guérette address the Babylonian Geometric Method, while Siu, giving *Nine Chapters on Mathematical Procedures* (1st century) as an example, claims that problems in ancient Chinese mathematics also provide evidence in support of proofs[1] through drawings, analogies, generic examples and algorithmic calculations. According to Siu, this can all be of great educational value to complementing and supplementing the teaching of mathematics, with emphasis placed on traditional deductive logical thinking.

In the past, the first author has used historically-based activities in the classroom to provide students with the context in which the mathematics they are studying has been developed, and also to introduce them to alternative ways of thinking about or reasoning in mathematics (Guevara 2009; Guevara et al. 2006), but without ever conducting a systematic collection and analysis of data. The resources used consist of historical diagrams taken from Arabic (9th century) and Chinese (1st century) cultures. It should be mentioned that other authors have also considered such diagrams to be very helpful in the teaching and learning of algebra (Demattè 2010; Puig 2008–11; Siu 2000).

With regard to the use of diagrams, Barwise and Etchemendy (1996) argue that inference and reasoning not only occur in sentences with linguistic expressions, but also with the use of diagrams and charts, and that the use of diagrams is a historical legacy. They relate the use of diagrams with geometry and cite as examples the countless proofs of the Pythagorean Theorem that are found throughout history and in different cultures around the world.

In short, the diagrams used in the research are geometrical and historical; geometrical because they are geometric figures with letters and numbers indicating

[1]In the sense of explanatory notes, which serve to convince and enlighten (Siu 2000, p. 161).

areas or lengths, and historical because they come from the history of mathematics and students use them to solve classical problems usually solved only with algebraic language.

The problems posed to students correspond to Chap. 9 of the *Nine Chapters*; specifically, problems 4 through 12 in the version of Chemla and Guo (2005). What we wish to emphasize is the justification of the calculation procedure in the classical text (1st century) with geometric figures that Liu Hui conducted in the year 263 AD. These figures were described by him but did not appear properly in the text until centuries later (13th century; Fig. 9.1).

The material for the unit of solving quadratic equations has likewise been prepared. In this case, it is based on the justification of a geometric equation with squares and rectangles, in accordance with Rosen's (1831) edition (Fig. 9.2).

From these historical problems we have drawn two different topics that are appropriate for use in the secondary education curriculum: (I) *The resolution of problems of right triangles with diagrams: The Pythagorean Theorem in ancient China*; (II) *Solving equations by completing geometrical squares: Quadratic equations*.

Fig. 9.1 *Nine Chapters* Edit. of Bao Huanzhi (1213) in Chemla and Guo (2005), p. 695

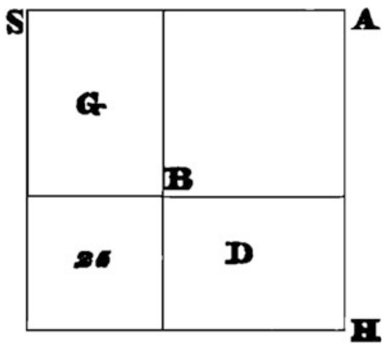


Fig. 9.2 Al-Khwãrizmî (original 813), *The Treatise of Algebra* (al-Khwãrizmî 1831, p. 16)

By means of visual reasoning concerning the areas of squares and rectangles, students solve problems that are usually solved using either the formula associated with the Pythagorean Theorem or quadratic equations. They are also cognizant of the historical contexts in which this occurred: in ancient China and within the Arabic culture. The students in this study are in the 9th grade (14–15-year-olds) in a high school at Badalona, Barcelona (Spain). The characteristics of students and how they work in the classroom are described in greater detail in Sect. 9.4.

9.2 Resolution of Problems of Right Triangles with Diagrams

In secondary education, the classical problems of right triangles are normally posed by giving two sides of the triangle and then asking students to solve for the third side. In problems from Chap. 9 of the *Nine Chapters* the situation is more difficult, because in most of the problems the data consist of one side of the triangle and the difference or the sum of the other two.

In this situation, when the relationship between the data and the unknown is expressed by an algebraic equation, some ability in solving equations is required in order to solve the problem. This difficulty can be overcome by introducing diagrams, on the basis of which students are able to argue visually. Then they follow the procedure for calculating the solution to the problem by manipulating geometric shapes that are introduced after analyzing the data of the problem (see Figs. 9.14 and 9.15).

9.2.1 Solving Right Triangle Problems with Diagrams in the Manner of Liu Hui

Take as an example Liu Hui's explanation of the solution to problem 6[2] in Chap. 9 of *Nine Chapters*, as given in Chemla and Guo (2005, p. 711). In order to solve the case of a right triangle in which one of the perpendicular sides and the difference between the hypotenuse and the other perpendicular sides are known, Liu Hui states the following:

> Here take half the side of the pond, 5 *chi*, as *gou*, the depth of the water as the *gu*, and the length of the reed as the hypotenuse. Obtain the *gu* and the hypotenuse from the *gou* and the difference between the *gu* and the hypotenuse. Therefore, square the *gou* for the area of

[2]"Given a reed at the center of a pond 1 *zhang* square and which is 1 *chi* high above the water. When it is drawn to the bank, it is just within reach. Tell: the depth of the water and the length of the reed. Answer: The water is 1 *zhang* 2 *chi* deep and the reed 1 *zhang* 3 *chi* tall" (Dauben 2007, p. 286).

> the gnomon. The height above the water is the difference between the *gu* and the hypotenuse. Subtract the square of this difference from that of the area of the gnomon; take the remainder. Let the difference between the width of the gnomon and the depth of the water be the *gu*. Therefore construct [a rectangle] with a width of 2 *chi*, twice the height above the water. Its length is the depth of the water to be found. (Dauben 2007, p. 287).

This type of reasoning with areas rather than calculating with algebraic symbols is what we believe students should be encouraged to do. We may imagine the right triangle devised by Liu Hui as that described by Dauben (2007, p. 284) in Fig. 9.3, even though Liu Hui himself did not include any figures in his explanation. These geometrical shapes have been studied and transcribed by historians such as Cullen (1996, pp. 206–217), Chemla and Guo (2005, pp. 673–683) and Dauben (2007,

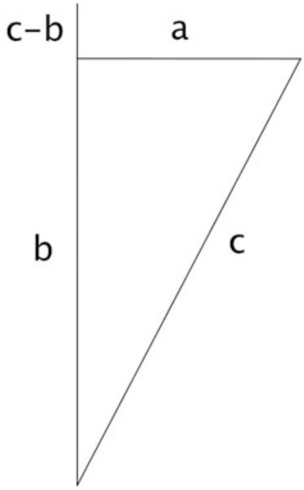


Fig. 9.3 Problem 9.6

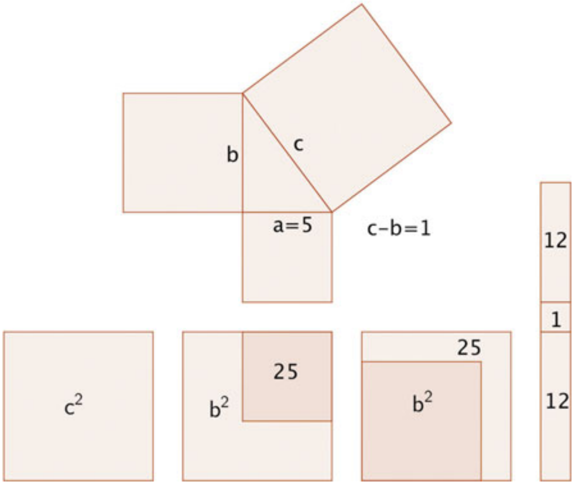


Fig. 9.4 The calculation of b and c from a and $c - b$

pp. 223–287). These demonstrations have been chosen in order to introduce the diagrams used in solving problems involving right triangles.

Figure 9.4 shows that when a, b and c are the sides of a right triangle, the large square on side c is the same as the sum of the squares on sides a and b. Since in ancient mathematics the concept of area is not explicit, neither in Greek nor in Chinese mathematics, we must now address the area of the three squares. Figure 9.4 also shows what happens with the area of the squares, the gnomon and the final rectangle.

In case of problem 9.6, $a = 5$ and $c - b = 1$, and we can solve b and c with geometrical and visual reasoning just as Liu Hui does[3] (see the previous quotation from Dauben 2007, p. 287):

> By performing the multiplication of the base (gou) by itself, we first show the area of the gnomon

$a = 5$ is one of the sides of the right triangle

$a^2 = 25$ is the area of the square, but also the area of the gnomon. Figure 9.4 shows three squares with the same area, the first one c^2, the second one gnomon (b^2) + 25, and the third one b^2 + gnomon (a^2), so this last gnomon must be 25.

Liu Hui goes on to state that:

> What extends above the water is the difference between the height and the hypotenuse;

in this case $c - b = 1$.

> One subtracts the square of this difference from the area of the gnomon and only then does one divide. The difference is the width of the area of the gnomon; the depth of the water is the height.

$25 - 1 = 24$; $24/2 = 12$ is the area of a rectangle with base $c - b = 1$ and altitude b; so, $b = 12/1$ and $c = 12 + 1 = 13$.

This is so because the gnomon becomes a rectangle of base $c - b$ and altitude $c + b$. This entire explanation becomes shorter with a figure and the comparison of area, as shown in Fig. 9.4.

In algebraic terms, the situation is as follows:

$$a^2 + b^2 = c^2, \quad 25 + b^2 = (b+1)^2 = b^2 + 2 \cdot 1 \cdot b + 1, \quad 25 - 1 = 2 \cdot 1 \cdot b, \quad \text{and finally}$$
$$b = (25 - 1)/2, \quad c = (25 - 1)/2 + 1.$$

[3] 1 *zhang* = 10 *chi* = 100 *cun* (see Chemla and Guo 2005, inside cover).

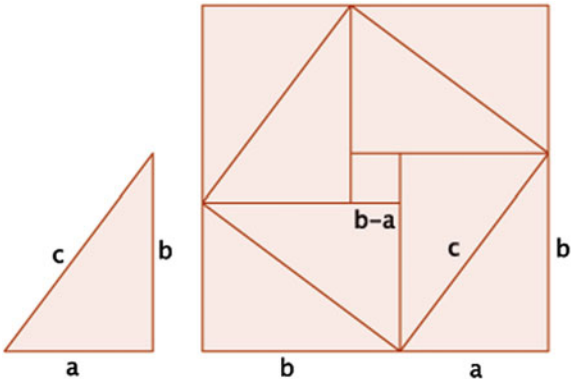


Fig. 9.5 The initial triangle and the first fundamental figure

9.2.2 The Fundamental Figures

The use of visual reasoning for solving problems of right triangles is based on diagrams described by Liu Hui and analyzed by the historians Cullen (1996, pp. 206–217), Chemla and Guo (2005, pp. 673–683) and Dauben (2007, pp. 223, 287). Chemla and Guo called these diagrams *fundamental figures*.

The First Fundamental Figure

The first fundamental figure (Fig. 9.5) is a square of side $a + b$ (base and altitude of the initial triangle). It contains the square of side c (hypotenuse of the triangle). This triangle is inside the square $a + b$ in a manner that determines four right triangles of sides a, b, c, and a square $(b - a)$.

If the hypotenuse (c) and the difference of the perpendicular sides $(b - a)$ are known, the first fundamental figure serves to calculate $(a + b)$, the side of the big square. Once $a + b$ is known, and taking into account that $(b - a)$ is known, a and b are calculated by adding or subtracting two numbers and dividing by two, because: $(a + b) + (b - a) = 2b$, and also $(a + b) - (b - a) = 2a$.

Liu Hui used this argument to solve problem 11 in Chap. 9.[4] We translate his arguments into diagrams with two different explanations in the same way that Liu Hui himself did (Figs. 9.6 and 9.7).

---

[4]"Let us assume that we have a single-leaf door whose height exceeds its width by 6 *chi* 8 *cun* and whose two [opposite] corners are separated one from the other by exactly 1 *zhang*. We ask what is the value of the height and the width of the door, respectively. Answer: The width measures 2 *chi* 8 *cun*; the height measures 9 *chi* 6 *cun*" (Chemla and Guo 2005, p. 717; authors' translation).

Liu Hui stated: "Let us say that the width of the door is the base (*gou*), its height is the height (*gu*), the distance between the two corners, 1 *zhang*, is the hypotenuse, and that the height exceeds the width by 6 *chi* and 8 *cun*, which is the difference between the base (*gou*) and the height (*gu*). Their positions are established from the figure. The square of the hypotenuse covers exactly 10,000 cun. If this is doubled, and one subtracts the square of the difference between the base and the height, and if by extraction one divides this by the square root, what one obtains is the value of the

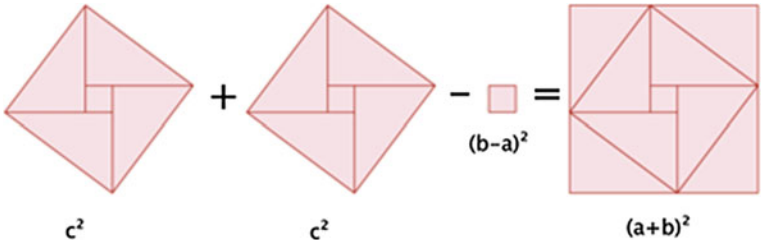


Fig. 9.6 The calculation of $a + b$ from c and $b - a$

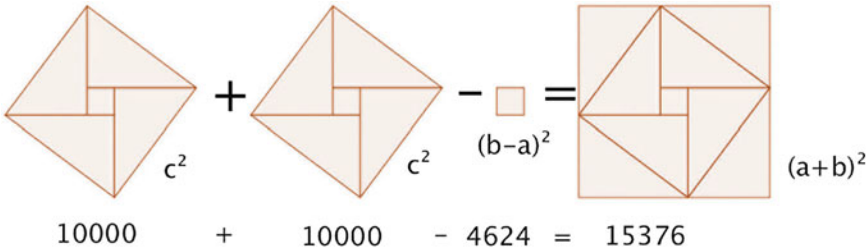


Fig. 9.7 Problem 9.11, and the first fundamental figure (Solution of Problem 9.11 based on Liu Hui's comment)

With the data from problem 9.11 (see Footnotes 3 and 4), we have: $100^2 + 100^2 - 68^2 = 10{,}000 + 10{,}000 - 4624 = 15{,}376$ and its square root (equal to 124) is $a + b$.

$$(a + b) - (b - a) = 2a \rightarrow a = (124 - 68)/2 = 56/2 = 28 \text{ and}$$
$$a + (b - a) = b = 28 + 68 = 96.$$

The Second Fundamental Figure

The second fundamental figure is the one described at the beginning of Sect. 9.2.1 in order to exemplify Liu Hui's explanation. Figure 9.8 shows the second fundamental figure with algebraic labels. Given this situation, when a and $(c - b)$ are known, one can calculate b using the area a^2 corresponding to a square, a gnomon and a rectangle, as in the sequence of diagrams in Fig. 9.8, because the base $(c - b)$ is known and one can calculate the height $(b + c)$ or $b + (c - b) + b$.

sum of the height and the width. If the difference is subtracted from the sum and one takes half of this, that gives the width of the door. If one adds to this the value of how much one exceeds the other, this will give the height of the door" (Chemla and Guo 2005, p. 719; authors' translation).

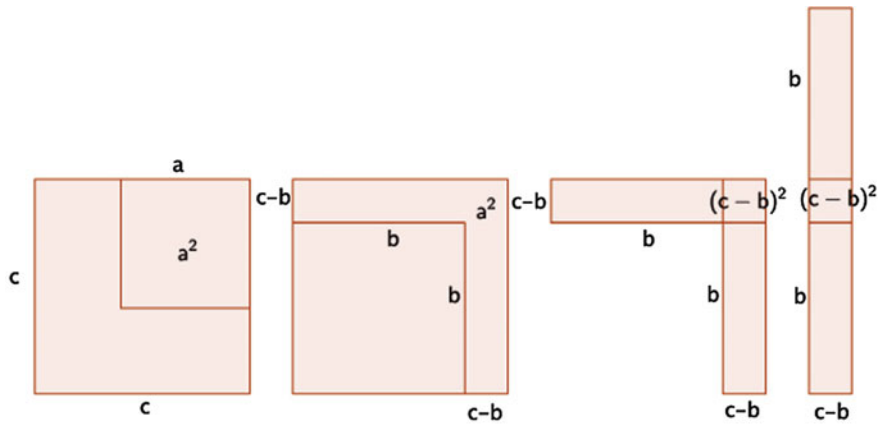


Fig. 9.8 The second fundamental figure with algebraic labels

9.3 Solving Equations by Completing Geometrical Squares

In secondary education, solving quadratic equations is a compulsory topic in the mathematics curricula. In this proposal, our students learn to solve equations in two different grades:

- 9th grade (14–15-year-olds): completing geometrical squares
- 10th grade (15–16-year-olds): solving equations $ax^2 + bx + c = 0$ with the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The aim of this topic was to lead students to use geometrical reasoning in the manner of al-Khwãrizmî, who justified the rules of computational algorithms with geometrical rules of transformations of figures and areas. While al-Khwãrizmî used rhetorical algebra, we use geometrical rules of transformations of figures and areas to introduce symbolic computation with algebra.

To this end, the process consists of four steps:

(a) The equation is translated into geometrical figures (squares and rectangles). In Sect. 9.3.2 we explain the translation from geometrical figures to symbolic algebra.
(b) The figures are manipulated and transformed (changing their lengths and areas).
(c) New lengths and new areas are deduced from the figures.

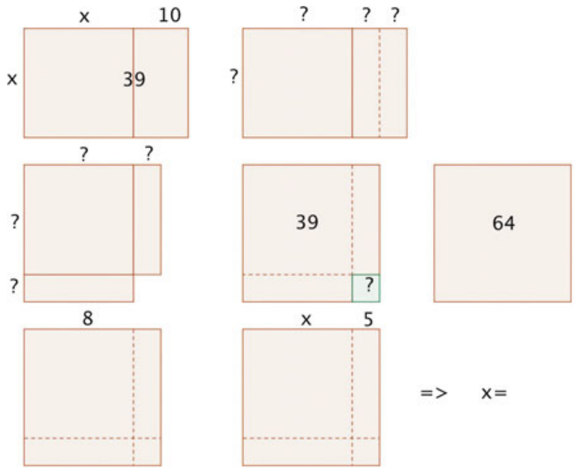


Fig. 9.9 Process for solving the equation by completing a geometric square

(d) The students interpret the figures and the measures and the solution of equation follows from this interpretation (the solution is one of the measures of the last figure obtained in the transformations).

Figure 9.9 shows the idea of the transformation of figures and the deductions of new lengths and new areas.

Radford and Guérette (2000) presented a teaching sequence whose purpose is to induce students to reinvent the formula for solving the general quadratic equation. Their teaching sequence was centered on the resolution of geometrical problems regarding rectangles by using an elegant and visual method developed by Babylonian scribes during the first half of the second millennium BC, and on the resolution of many problems found in a medieval book, the *Liber Mesuratinonum* by Abû Bekr (probably 9th century), as well as on al-Khwãrizmî's *Al-Jabr* (9th century). Their goal was achieved through a progressive itinerary that starts with the use of manipulatives and evolves through an investigative problem-solving process combining both numerical and geometrical experiences. Instead of introducing students to modern algebraic symbolism from the start—an approach that often discourages many of them—algebraic symbols are only introduced at the end, after the students have truly understood the geometric methods.

In this sense, our two proposals (the resolution of second-degree equations, and problems with rectangular triangles) are aimed at teaching students to develop visual geometric reasoning with diagrams. We encouraged the students to use the diagrams on their own initiative and leave the reinvention of the formula for solving the general quadratic equation for the following year. The point of departure in our research is not the resolution of a geometrical problem, but rather the resolution of a second-degree equation that has been transformed into a problem of calculating areas and lengths related to squares and rectangles. The aim is for students to move from algebraic expressions to geometry, to find solutions in the geometric field and then do the final reading in algebraic terms, thereby avoiding calculations with algebraic expressions.

9.3.1 Solving Equations in the Manner of al-Khwãrizmî

In his *Treatise of Algebra,* al-Khwãrizmî classified second-degree equations with positive coefficients and admitting only positive solutions, into six different types. He did not use algebraic symbols to denote the unknown coefficients and the equations could only be solved with what is known as rhetorical algebra.

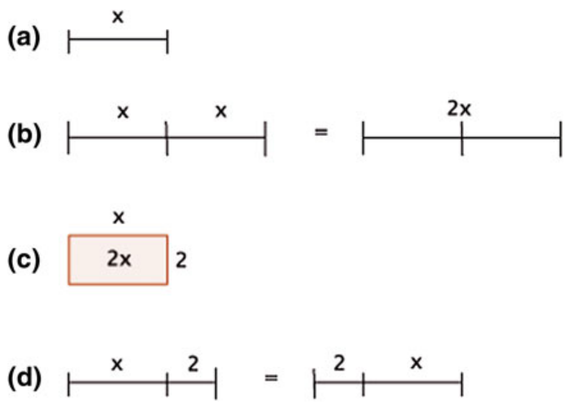
For each of the six cases, al-Khwãrizmî justified the proposed algorithms with areas of squares and rectangles. From one of these cases, the solution process proposed to the students in learning activities has been taken in order to solve quadratic equations by completing a geometric square.

If transcribed with algebraic symbols, the equation is as follows: $x^2 + 10x = 39$. Figure 9.2 shows the geometric basis of his reasoning. Figure 9.9 contains the proposed activity with squares and rectangles visualized as provided to the students in *Solving equations by completing geometrical squares: Quadratic equation.*

9.3.2 Visualization of the Terms of an Equation in a Geometrical Sense

Figure 9.10 shows the visualization of the first-degree terms as used with students. Here, (a) x is the length of a segment; (b) two segments of length x together, one

Fig. 9.10 Visualization of the first-degree terms



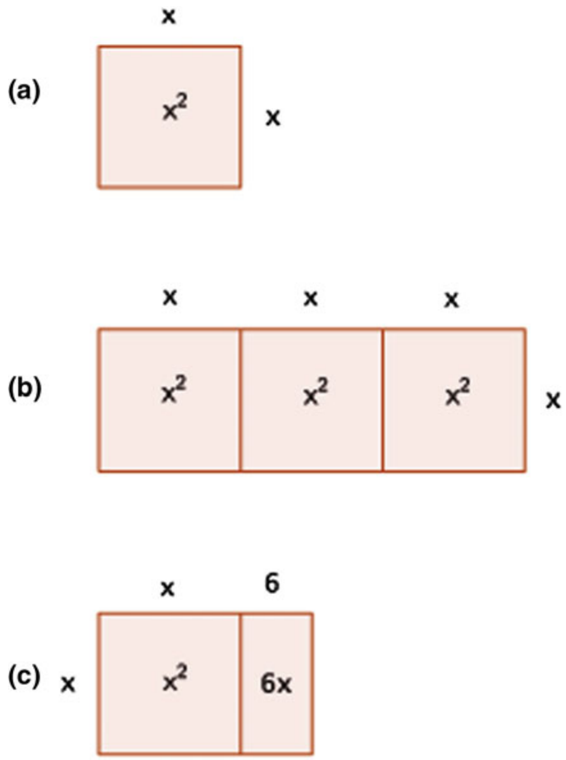


Fig. 9.11 Visualization of the second-degree terms

after the other, have length $2x$; (c) but also $2x$ could be the area of a rectangle with base x and altitude 2; (d) $x + 2$ is the same as $2 + x$, because it is the length of the same segment.

Figure 9.11 shows the visualization of the second degree terms: (a) x^2 is the area of square of side x: (b) $3x^2$ is the area of a rectangle which contains exactly three squares of area x^2 and is the same as $x^2 + x^2 + x^2$; (c) finally, $x^2 + 6x$ is a rectangle formed by a square x^2 and another rectangle $6x$.

When students more or less understand this relationship between algebraic terms and the length and area of squares and rectangles, they are able to solve quadratic equations by completing geometric squares.

9.4 Students Solve Problems in the Manner of Liu Hui and al-Khwãrizmî

9.4.1 The Students Involved in the Research and Classroom Activities

The population under study consisted of a group of 21 students of the 9th grade (3rd ESO[5]; 14–15-year-olds) of the Institute Badalona VII Secondary School during the academic year 2009–2010, and of which the first author of the study was their teacher of mathematics. The two topics analyzed—solving problems of right triangles and solving second degree equations—form part of the 9th Grade ESO curriculum. The first topic was taught during the first quarter and the second topic during the third quarter.

With regard to the characterization of these 21 students and their learning of mathematics, the class was a heterogeneous and diverse group. In general terms, they can be divided into four different subgroups of 8, 7, 2 and 4 students, respectively, each one with a different profile. The first subgroup of eight students possessed the algebraic skills introduced during their previous year (2nd ESO); students were able to solve simple first-degree equations and had sufficient understanding of the basic rules for isolating the unknowns. The second subgroup of seven students had not acquired sufficient algebraic reasoning, and one might even say that their use of letters (x) for identifying unknowns constituted an excessive degree of abstraction for their level of reasoning. The third subgroup of two students, while failing to possess the required level of the symbolic language of algebra corresponding to Spanish Secondary School 1st Grade (at age 12), were sufficiently familiar with numerical calculation but were somewhat unreceptive to the introduction of visual and geometrical reasoning. The fourth subgroup of four students had not acquired the level required at the 2nd ESO, but were able to work with numbers, although possibly without a fully developed understanding of the operations. One of the students in this subgroup, who faced serious reading difficulties, had handwriting that was only partially legible and experienced learning problems in all subjects. The students belonging to this last subgroup possibly lacked the sufficient basic knowledge for undertaking the activities, even geometrically, with the use of diagrams, as well as algebraically for solving of equations.

Throughout the course, the students worked together in groups of three or four without textbooks, and for each topic they were provided with a work dossier. The work groups were assigned by the teacher and were heterogeneous in terms of students' level and capabilities, so that in each group there was at least one student belonging to the first profile, as described in the previous paragraph, and one from the second profile. The 6 (2 + 4) students with the lowest levels of competence were also distributed in different groups.

[5]Secondary School Compulsory Education (in Spain).

Dossiers were also provided for the two topics on which the research is based. In the case of the right triangles, the dossier was called "Pythagoras' Theorem in Ancient China." It contained nine problems from Chap. 9 of the *Nine Chapters* (4, 5, 6, 7, 8, 9, 10, 11 and 12), in the Chinese and Catalan versions based on the text by Chemla and Guo (2005, pp. 711–721). First of all, the students were shown the procedure of "base & height" (*Gou* and *gu*), using paper cut-outs, before going on to familiarize themselves with the first and second figures. Then they were requested to solve the problems, deciding beforehand which of the two figures was appropriate for each problem. It was necessary to realize first that the first figure served to demonstrate the base-height procedure (*Gou gu*) and to solve problems 4, 5, 11 and 12; while the second figure was used to solve problems 6, 7, 8, 9 and 10. A script was also included to help students to look for information about Liu Hui and his historical context (cf. Sect. 9.7). The activities comprised ten teaching sessions of 55 min each and an eleventh session for the test, which included three problems, two of which are analyzed in this chapter.

The dossier for working in the manner of al-Khwarizimi was entitled "Equations of the Second Degree." It began with one of the problems on the measurement of rectangles from the cuneiform tablet YBC 4663 (ca. 1800 BC), inviting resolution by trial and error. It was followed by a series of incomplete equations of second degree, such as $ax^2 = c$ and $ax^2 + bx = 0$, which the students had to solve by both algebraic methods (the previous year they had worked on first-degree equations), and geometric methods, transforming the equations into additions and subtractions of areas of squares and rectangles. In the second part of the dossier, the resolution of second-degree equations in al-Khwãrizimî's manner was introduced, with the completion of geometric squares, as seen in Fig. 9.9. Solving the general second-degree equation $ax^2 + bx + c = 0$ was performed in all cases by geometric procedures, with the deliberate omission of the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ until the following school year, in order to make the students solve the equations at all times by completion of the geometric squares. A script was also included to enable the students to look for information about al-Khwãrizmî and his historical context. The activities comprised eight teaching hours of 55 min each, and a ninth session for the test, which included three problems, two of which are analyzed in this chapter.

In both cases, the work dossier made reference to historical contexts; ancient China, in the first instance, and the ancient Arabic context in the second. The mathematical knowledge of the students constituted the explicit point of departure, as well as the measurement of areas of squares and rectangles, solving equations of the second degree by completing geometric squares, and techniques for identifying unknowns in equations of the first degree. This approach generates learning by comparison and uses what has worked in other situations in order to arrive at new situations that require new tools. These new tools are diagrams or figures with written information that act as a support for performing calculations with which students are unfamiliar by means of algebraic manipulation. In this way, by using geometric transformations, a new figure is arrived at on which the solution can now be read.

During the test on "Pythagoras' Theorem in Ancient China" the students worked in groups; nine groups of two students and one group of three. So, we analyze ten different solutions of problem 1 and ten of problem 2. On the other hand, during the test on "Equations of the Second Degree" the 21 students worked individually. Therefore, we analyze 21 examples of the solution of one equation, and 21 different examples of the other.

9.4.2 Solving Problems of Right Triangles with Diagrams: Student Work Examples

We illustrate the process of problem solving with diagrams by using two problems taken from the test; one solved with the second fundamental figure and the other with the first fundamental figure. Note that the problems in this final test consisted of instances relevant to the 21st century in order to show that mathematics is not only a tool for solving old historical problems.

Statement of Problem 1:

> An antenna tensioner, suspended from the top of the antenna without being tightened, exceeds the height of the antenna by 6 m. When tightened on the ground, it lies 16 m from the base of the antenna. What is the height of the antenna and how long is the cable tensioner?

The process of problem solving with diagrams consists of four steps: construction, processing, interpretation and reading. We can follow the process in Fig. 9.8 with an example of one student (from the pair of students 1 and 4).

(a) The student constructs a diagram of data, a triangle with one side known, the relationship of the other two sides, and the unknown sides (Fig. 9.12a).
(b) The student decides which of the two calculation diagrams corresponds to the problem to be solved (Fig. 9.12b).
(c) The student constructs a calculation diagram and writes the known data (Fig. 9.12c).
(d) In order to be able to read the solution to the problem on the last diagram (Fig. 9.12d), the student transforms this first diagram into as many other successive diagrams as he/she wishes.

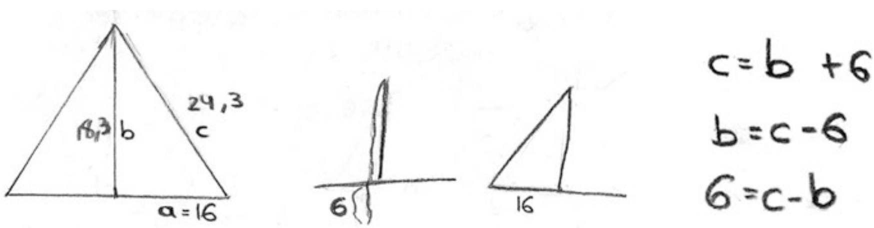


Fig. 9.12a The diagram of data and the relationship of the other two sides

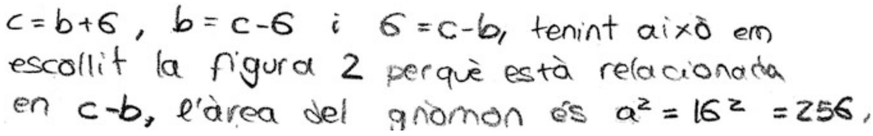


Fig. 9.12b Student's explanation of choice ($c = b + 6$, $b = c - 6$ and $6 = c - b$. With this knowledge, we choose Fig. 9.8 because it is related to $c - b$, the area of the gnomon is $a^2 = 16^2 = 256$)

Fig. 9.12c .

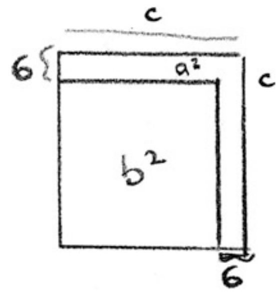
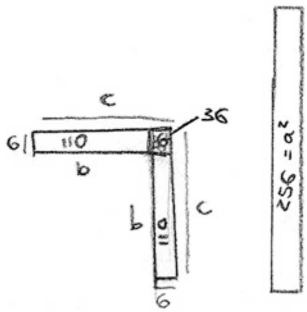


Fig. 9.12d .



Statement of Problem 2:

When it is said that a screen measures 26 inches, it means that the diagonal of the rectangle is 26. If the width of the screen exceeds its height by 14 inches, calculate the sides of the screen.

Again, problem solving with diagrams consists of four steps: construction, processing, interpretation and reading. We can follow this process with an example from one of the students who produced Fig. 9.12. Figures 9.13 and 9.14 correspond to this same student's group (comprising two students).

In this case, the student solved the problem using a different interpretation of Fig. 9.5 (note that there is a slight mistake in the calculation: 676 + 480 = 1156, and not 1176, as she computed).

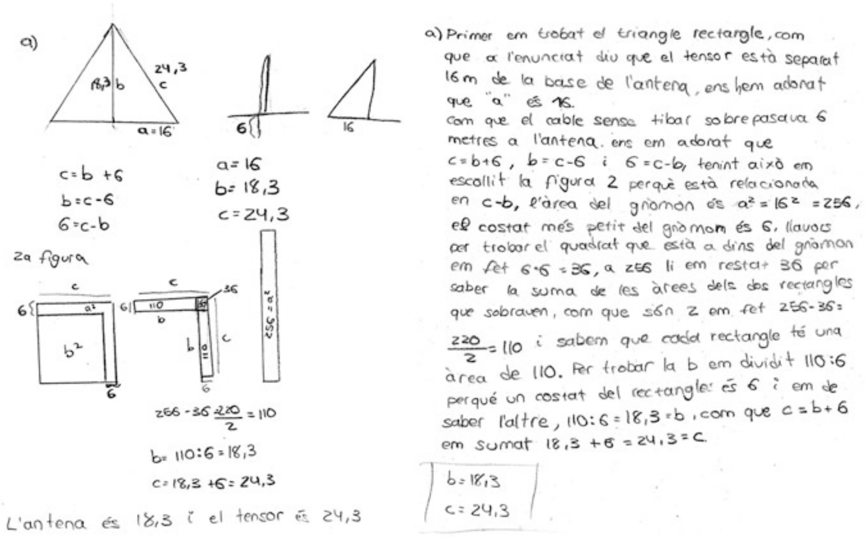


Fig. 9.12 The solution of one student and her explanations (First, we find the right triangle, so as in the statement the tensioner is said to be 16 m from the base of the antenna, we know that it is 16. Since the untightened cable exceeds the height of the antenna by 6 m, we know that $c = b + 6$, $b = c - 6$ and $6 = c - b$. With this knowledge, we choose Fig. 9.8 because it is related to $c - b$, the area of the gnomon is $a^2 = 16^2 = 256$, the shortest side of the gnomon is 6, so in order to find the square inside the gnomon we do $6 \cdot 6 = 36$. We subtract 36 from 256 in order to find the areas of the two remaining rectangles, and since there are two we do $256 - 36 = 220$ and $220/2 = 110$, and so we know that each rectangle has an area of 110. In order to find b, we divide 110 by 6, because one side of the rectangle is 6. Since we have to find the other, $110/6 = 18.3 = b$, and since $c = b + 6$ we add $18.3 + 6 = 24.\ 3 = c$)

9.4.3 Solving Second-Degree Equations: Student Work Examples

We asked students to solve different kinds of equations in the same manner as al-Khwãrizmî, both arithmetically and geometrically, so that they associate x as a length, $2x$ as a length, or an area, and x^2 as an area. Examples of the first types of these are: $x^2 = 25$; $3x^2 = 12$; $x^2 = 20$; $2x^2 = 12$; $x^2 - 81 = 0$; $x^2 - 24 = 0$. Students then solve $x^2 - 5x = 0$; $x^2 + 5x = 0$; $2x^2 - 8x = 0$; $3x^2 + 81x = 0$.

The aim is for them to relate the terms of an equation to length and area through the visualization of the terms of an equation in a geometrical sense, as explained in Sect. 9.3.2. in which the equation $x^2 + 6x = 40$ appears. The expected learning outcomes are shown in Fig. 9.15. We do not have the solution when we draw the figure; the idea is that figures do not have to be realistic.

Figure 9.15 shows the four steps of the process: cutting the initial rectangle, moving a piece of the rectangle to obtain a square (or more or less a square), in order to complete the new figure and arrive at a whole square.

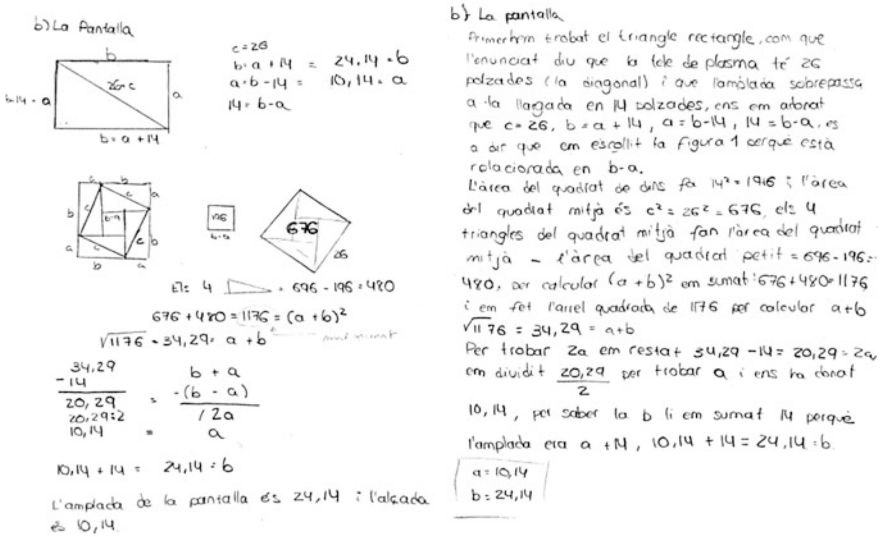
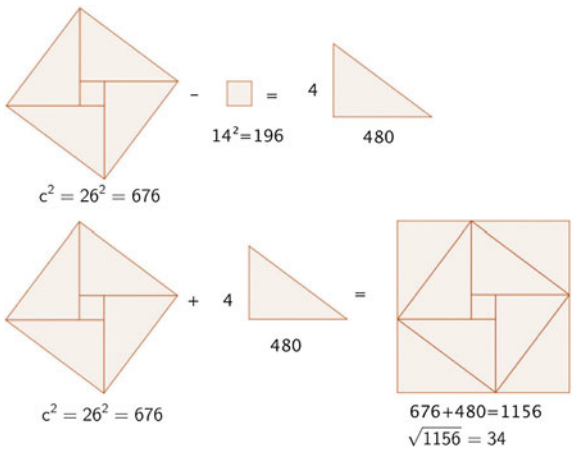


Fig. 9.13 A student's solution of the problem using the first fundamental figure, and student's explanation (The screen: First we find the right triangle, and since in the statement it says that the plasma screen measures 26 in. (diagonally) and that the width exceeds the length by 14 in., we then have that $c = 26$, $b = a + 14$, $a = b - 14$, $14 = b - a$, that is, we choose Fig. 9.1 because it is related to $b - a$. The area of the inner square is $14^2 = 196$ and the area of the middle square is $c^2 = 26^2 = 276$. The four triangles in the middle square are the middle square minus the small square = 696 − 196 = 480. To calculate $(a + b)^2$ we add 676 + 480 = 1176 and we take the square root of 1176 in order to calculate $a + b$, $\sqrt{1176} = 34.29 = a + b$. To find $2a$, we subtract 34.29 − 14 = 20.29 = $2a$. Then we divide 20.29/2 to find a, which gives 10.14, and to find b we add 14, because the width was a + 14, 10.14 + 14 = 24.14 = b)

Fig. 9.14 Solving the problem with another interpretation of the relationship in the first fundamental figure



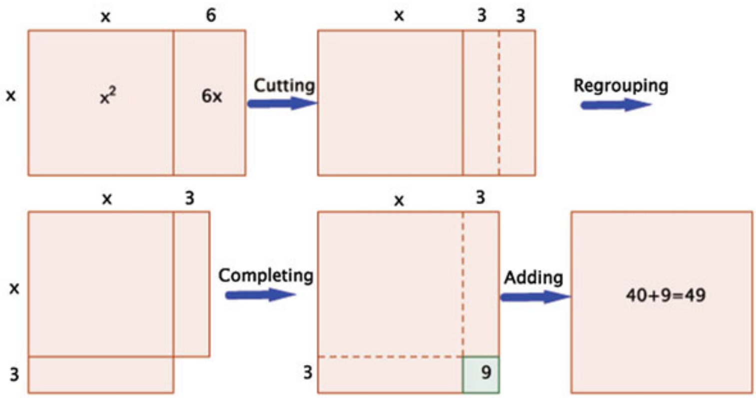


Fig. 9.15 Solving $x^2 + 6x = 40$ geometrically

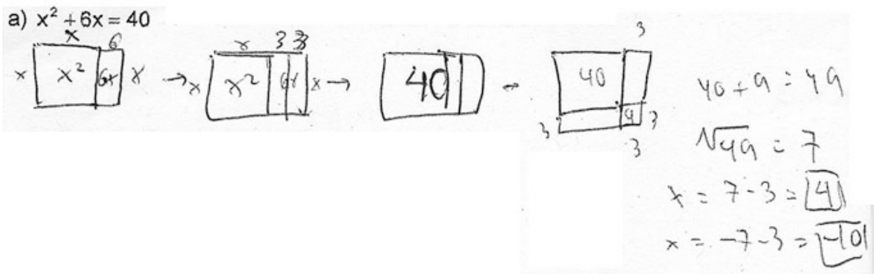


Fig. 9.16 A student (number 13) solving $x^2 + 6x = 40$ geometrically

Figure 9.16 shows the process of solving an equation ($x^2 + 6x = 40$) geometrically with an example by a student (belonging the second subgroup of seven students described in Sect. 9.4.1). He begins with a square x^2; he adds the rectangle $6x$; he cuts the rectangle into two pieces; he moves one of the rectangles, and finally he completes the figure with a small square of area 9.

9.5 The Process of Problem Solving with Diagrams

Solving right triangles with diagrams and solving equations by completing geometric squares provide two topics for secondary education taken from the history of mathematics, but from the mathematical point of view they possess some similar characteristics. For the first topic, while solving the problem it is necessary to give the geometrical meaning of algebraic terms. In the second, the solution involves working with calculation diagrams and the transformations of these diagrams. The difference resides in the point of departure: In right triangles the context is geometrical, and the data are measurements of a right triangle. This consists of diagrams of data expressing algebraic relations between the data. Afterwards, we return to geometry to solve the problem with a calculation diagram. In solving equations by completing geometrical squares, the context is algebraic, the data are equations, and we then move to geometry to find the solution with a calculation diagram.

The diagrams employed in this research follow the classification of Barwise and Etchemendy (1996) and the nomenclature of Mason et al. (2005) and Giardino (2009, 2014). From the analyses of these authors, we have adopted two features associated with the diagrams thus introduced and analyzed in this work: (i) the expressive efficacy of the diagram, that is, the ability to express semantic properties and computational efficiency, and (ii) the ability to infer new information. Two types of these diagrams have been identified to differentiate their role in the problem-solving process; one type is denoted as data diagrams and the other as calculation diagrams, and both have expressive efficacy. A data diagram expresses semantic properties between data, and helps to choose the correct calculation diagram when solving right triangles. A calculation diagram has expressive efficacy for solving right triangles and second-degree equations because they help to calculate the solution of the problem.

Giardino (2009) identifies two kinds of elements in the sequence of problem solving with calculation diagrams: the diagrams and the actions involved. The actions are four-fold: construction, processing, interpretation and reading. In Sect. 9.2, we have described the process with an example of right triangles, while in Sect. 9.3 we have described it with an example of quadratic equations. Figures 9.17 and 9.18 show the different steps in the process and the connections between them.

Figure 9.17 should be read from top to bottom and from left to right. The rectangles contain data and diagrams. Arrows indicate actions, in accordance with Giardino (2009). The data diagram is constructed from the data of the problem. Decide first what calculation diagram to use by looking for the relations in the data

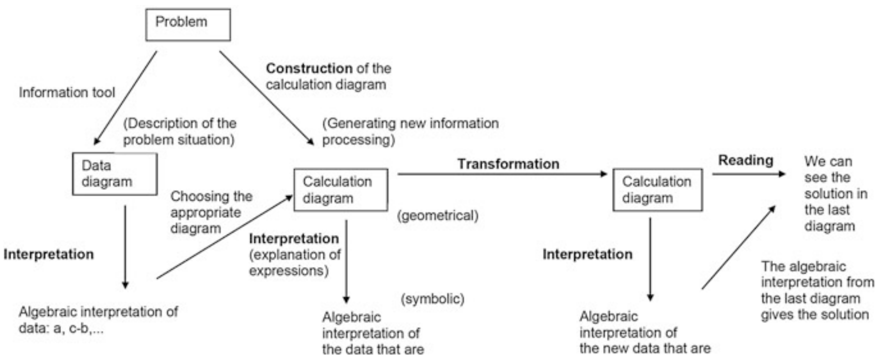


Fig. 9.17 The process of problem solving with diagrams

diagram. The calculation diagram is then constructed. Each diagram has a corresponding interpretation in algebraic terms (bottom of Fig. 9.17). The first calculation diagram contains different transformations and processing, in accordance with Giardino (2009). Finally, the solution is obtained through the last calculation diagram.

Following the diagram in Fig. 9.17 for the antenna problem (Fig. 9.12), we arrive at the diagram in Fig. 9.18.

9.6 Organization of the Conclusions in Accordance with the Four Characteristics of Problem-Solving Diagrams

The four actions in problem solving with diagrams (*construction*, *processing*, *interpretation* and *reading*) can be combined into three procedures: *translation*, *transformation* and *diagrammatic reasoning*. We summarize the resulting conclusions in Sects. 9.6.1–9.6.3 and in more general terms in Sect. 9.6.4 about the advantages of using diagrams. More specifically, *construction* and *reading* form part of problem solving with diagrams and the end of the process with the solution. We combine these two processes into one, namely *translation*. Processing involves the *transformation* of diagrams. Interpretation connects geometry with algebra and explains reasoning to students; this is *diagrammatic reasoning*. Each one of these procedures leads to several conclusions drawn from our empirical research, some of which are briefly outlined in the next subsections.

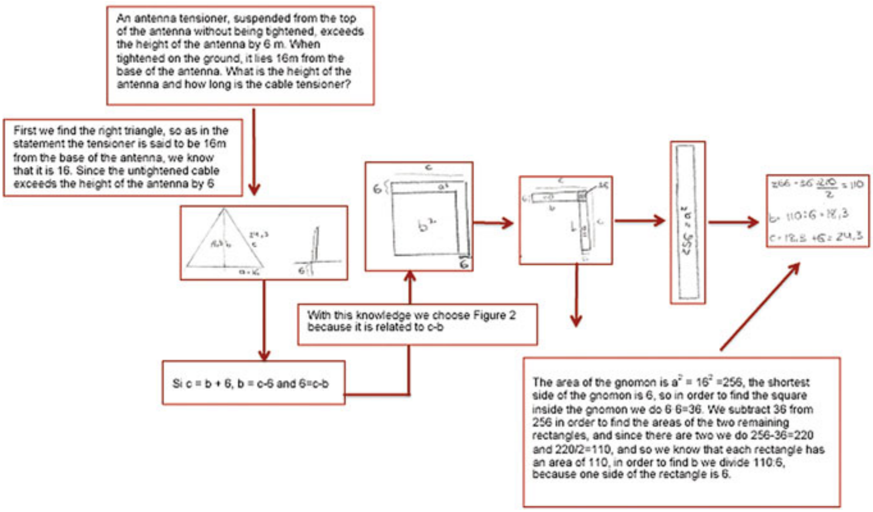


Fig. 9.18 The process of solving the antenna problem

9.6.1 Translation

The first procedure is the translation of the problem into the language of diagrams; that is, the study of the equivalence established by students between algebraic language and the geometric representation when they construct the first diagram with the data of the problem, and likewise, the reinterpretation of the final diagram in terms of the problem posed.

Of the various conclusions concerning *translation* (Guevara 2015, pp. 428–435), we emphasize the most important one: the students associate terms of first degree with unit coefficient (x) to lengths; the first degree with other coefficients ($6x$) to areas; the second degree also to areas; the numerical values with lengths or areas. Figure 9.16 shows the equivalences that one student established between the terms $6x$, x, 40 and the sides and areas of the figure, with the use of corresponding labels. He places it within, or outside the figure, according to whether it represents an area, or a length.

Table 9.1 shows how students related the labels to length or area. On the whole, the labels are related correctly, and in three cases the error is not transferred to the equation-solving process.

9.6.2 The Transformation of Diagrams

The second procedure is the transformation of diagrams; that is, a description of the diagram transformation process used by students in problem-solving. Several conclusions (Guevara 2015, pp. 435–445) are crucial for the series of transformation diagrams when solving the problem and its predictability (the number of diagrams containing the number associated with a problem; the number of diagrams for a given student to solve a particular problem; the structure and direction of the

Table 9.1 Students' labels when solving $x^2 + 6x = 40$

Label	Number of students	Related to length/area	Ratio correct/errors
x	14	Length	8/6
x^2	4	Area	2/2
6	10	Length: 8; area: 2[a]	9/1
$6x$	6	Area: 5; length: 1[b]	4/2
40	19	Area	12/7
3	20	Length	15/5
9	20	Area	14/6
49	7	Area	4/3
7	2	Length	2/0

[a]Two students solved the equation correctly
[b]The student solved the equation correctly

changes), but the most important one seems to be the thread that guides the diagram transformation process of the areas in the figures (squares and rectangles).

Students have identified the key areas for constructing the first calculation diagram: 256 for the first right triangle problem; 676 and 196 in the second problem; and 40 for the quadratic equations. Figures 9.12 and 9.20 contain the solution obtained by two students using the second fundamental figure to solve a right triangle problem. The thread that guides the process of transformation is the area 256. Figure 9.13 contains the solution obtained by a student using the first fundamental figure to solve a right triangle problem. The thread that guides the process of transformation is the two areas 676 and 196. In Figs. 9.16, 9.19 and 9.21, the problem involves the solution of a second degree equation. This time the guide is the area that equals 40.

9.6.3 *Diagrammatic Reasoning*

The third procedure is diagrammatic reasoning; that is, identifying the key elements of diagrams in problem solving and what they represent in the reasoning followed by the students.

Mancosu (2001, 2005) distinguishes between visualization and diagrammatic reasoning. He uses visualization as a discovery tool, in the same way as Giaquinto (1992). On the other hand, he uses diagrammatic reasoning as a demonstration tool, in the same way as Barwise and Etchemendy (1996).

One conclusion to be drawn is that when solving the problem with the use of diagrams, each student has a key diagram and this key diagram is the same for all students. In the case of equations, for example, Fig. 9.16 shows the key diagram, and the final square with two squares and two rectangles inside it. However, we would like to emphasize another conclusion as the most important; namely, that the essential labels used by the students are the numerical labels in preference to algebraic ones, without which the diagram does not help to solve the problem.

In Figs. 9.16, 9.19 and 9.21, different types of labels are used, while in Fig. 9.19 one may see that in the process followed by the student for the solution to the equation $x^2 + 6x = 40$ no algebraic labels are used.

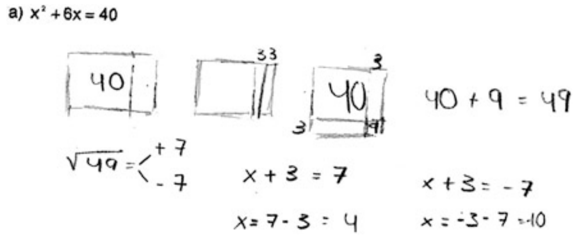


Fig. 9.19 Solution without algebraic labels (student 1)

Fig. 9.20 Geometrical and algebraic solution (student 6)

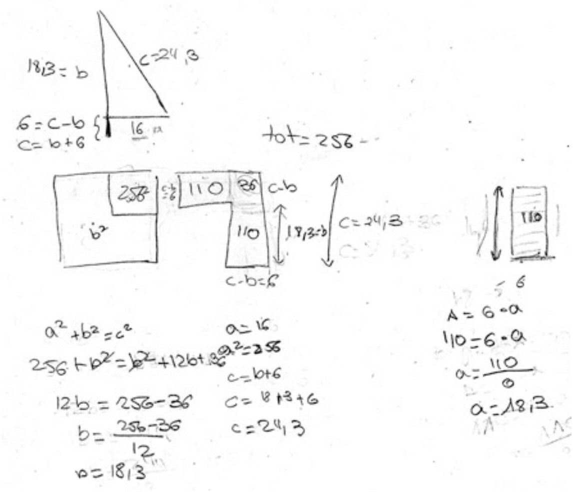


Table 9.1 shows the labels used by students, the majority of whom used the numerical labels 40, 3 and 9, which are indeed the most essential labels for solving the equation correctly.

9.6.4 The Advantages of Using Diagrams

Finally, in a more general sense, four conclusions regarding the advantages of using diagrams to solve problems in secondary school concur with the ultimate aim of this study.

(1) Students mainly chose the geometric method of solving problems. However, some students used both geometrical and algebraic methods, as may be seen in Fig. 9.18.

(2) The second conclusion concerns effectiveness: visual diagrams enable students to be more effective. Not only did more students solve the problem in this way, they also solved it better. Analysis of the 21 results achieved by students with right triangles and the quadratic equation supports this assertion. In the case of the quadratic equation, they were unable to use the algebraic method because they were not familiar with it, as explained in Sect. 9.3.

(3) Regarding the advantages of using diagrams (Guevara 2015, pp. 452–458), we would like to emphasize the following as the most notable one: manipulating diagrams with ease is necessary to distinguish between the concept of perimeter and area, and also between measuring a length and a surface. Figure 9.21 shows one student's solution, who added values for length (6) and area (9), thus preventing herself from getting the final result correctly.

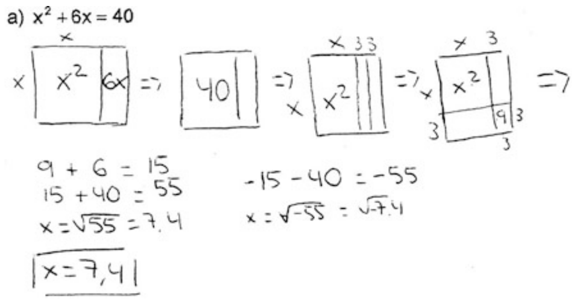


Fig. 9.21 Sum of areas and lengths in the solution by one student (17)

(4) The last conclusion we mention is related to the third one above: In order to connect geometry and algebra, one must determine how many figures are contained within a given one (visual perception) and also associate the data on algebraic relations, with lengths and areas.

In the present research, unlike that by Radford and Guérette (2000) mentioned above, the objective is not to arrive at algebraic formulas, but rather for students to acquire knowledge at the geometric stage with the help of diagrams. To this end, the students' own productions are analyzed and the construction of knowledge with the support of the diagrams is identified.

9.7 Using History to Teach Mathematics and the Teaching and Learning of Algebra

Activities based on the analysis of historical texts as part of the curriculum contribute to the improvement of the students' overall training by providing them additional knowledge about the social and scientific context of the periods involved, because in both cases the dossier included a script containing information to enable them to understand the context of the historical character being studied.

Students thereby acquire a vision of mathematics not as a final product but as a science that has been developed on the basis of seeking answers to questions that humankind has been asking about the world around us throughout history.

As an example, in the case of "The Pythagorean Theorem in ancient China," students were provided with a script containing the following questions (cf. Sect. 9.4.1):

ANCIENT CHINESE MATHEMATICS

1. Who was the first Chinese emperor? What dynasty did he found? In what period?
2. What were the public buildings in that time? For what purpose were they built?

3. What are the two oldest Chinese mathematical texts for which documentary evidence exists? What type of mathematics do they contain? To whom were they addressed?
4. What has been the most important classical text for many centuries in Chinese mathematics?
5. Situate this classical text: Title; author; period; to whom it is addressed; describe briefly the contents of the different chapters. (Guevara 2015, p. 492)

A detailed analysis of students' outcomes and the conclusions drawn from the learning process using this resource indicates that this way of introducing algebra in relation to geometrical interpretation may be profitably applied. Through geometry, both operations with numbers (arithmetic) and operations with letters (algebra) express the results of measuring lengths and areas and in this way they acquire the same meaning.

Given the results obtained from the analysis of student activities and the conclusions reached thereby, it can be stated that the teaching of algebra in the first year should go hand in hand with visual arguments and the use of diagrams. In other words, the introduction of algebra, besides being just a generalization of arithmetic where the rules of operations with numbers are generalized to rules with letters, should also have a visual component which gives the geometrical interpretation of algebraic formulas. With this paradigm, and depending on the situation, linear expressions can be interpreted as areas or lengths of segments, while quadratic expressions can be interpreted as areas. All operations and the rules for operating with letters have their interpretation in the geometric model. In this way, the properties of operations are not justified solely via general syntactic rules on symbols, but rather have an equivalent in the geometric model.

This claim is supported by the results obtained in the present research: In the case of "The Pythagorean Theorem in ancient China," from the 10 outcomes analyzed,[6] six students tackled the problem with geometrical reasoning using diagrams, three treated it using algebraic expressions and one was unable to do anything. Four students out of the first six obtained the solution using diagrams, and one student from the second three reached the solution with algebraic reasoning. In the case of the second-degree equations, we analyzed 21 outcomes, in all of which diagrams were used because the students did not know the algebraic form for solving second-degree equations, and 15 out of these 21 obtained the solution correctly.

As Katz and Barton (2007) state: "An historical view places number and geometry on at least equal footing in mathematical development, and highlights the powerful interrelationship between the two" (p. 198). In this sense, we believe that moving from arithmetic to algebra by skipping geometry may be regarded as a pedagogical and historical error. Though this is an approach that fits well to the 17th century, when the force of the new symbolic language replaced visual geometric reasoning, it may be considered not appropriate in the 21st century. Nevertheless,

[6]Recall that 21 students performed the activities in pairs.

for many centuries, in the absence of formal algebra and with only the four basic arithmetic operations available, humanity was able to solve problems that we now solve with equations. And this cannot be ignored altogether, in view of the fact that a significant number of students exist who are unable to solve problems just because they do not thoroughly understand the rules of this language, and thus may be regarded as mathematically illiterate. We think that there is sufficient ground to believe that, when beginning to learn algebra, it is necessary to return to the reasoning used by ancient mathematicians, who calculated on the basis of geometric models to justify the validity of their operations. It is our contention that these geometric models can be used to help our students to understand algebra.

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Chapter 10
Missing Curious Fraction Problems

The Unknown Inheritance and the Unknown Number of Heirs

Maria T. Sanz and Bernardo Gómez

Abstract In this paper we present a study of one of the best-known types of descriptive word fraction problems. These problems have disappeared from today's textbooks but are hugely important for developing arithmetic thinking. The aim of this paper is to examine the historical solution methods for these problems and discuss the analytical readings suggested by the authors. On the basis of this analysis we have conducted a preliminary study of the performance of 35 Spanish students who are highly trained in mathematics. Our results show that these students have a preference for algebraic reasoning, are reluctant to use arithmetic methods, and have reading comprehension difficulties that are reflected in their translations, from literal language to symbolic language, of the relationship between the parts expressed in the problem statement.

Keywords History and mathematics education · Descriptive word fraction problems · Resolution methods · Student performance

10.1 Introduction

Textbooks contain a wide range of descriptive word fraction problems whose history dates back to ancient mathematical cultures. The statements of these problems have evolved over time, adapting to social changes, mathematical developments and the predominant educational theories of the era while maintaining a common mathematical content.

These problems were essential components of the arithmetic of the past and can be found in a multitude of historical texts. Examples of these texts are *Jiuz hang*

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Suan shu, better known as the *Nine Chapters on the Mathematical Art* (ca.100 AD; Chemla and Guo 2005; Shen et al. 1999), which contains 247 Chinese mathematical problems; collections of Hindu mathematical problems, such as the Bhaskara manuscript, also known as the *Līlāvatī* (Colebroke 1817; Phadke et al. 2001); and basic texts from medieval Europe, such as the *Greek Anthology*[1] (Jacobs 1863) and the recreational mathematical collections of Bede (*De Arithmeticis Propositionibus* in 641 AD; Migne 1850) and Alcuin (*Propositiones ad acuendos juvenes* in 775 AD; Migne 1863). Descriptive word fraction problems also appear in texts that introduced the west to Islamic mathematical methods such as Fibonacci's *Liber Abaci* (Sigler 2002). Later they also appear in the first printed books on arithmetic and algebra, such as the texts in Spanish by De Ortega (1552), Silíceo (in 1513; Sánchez and Cobos 1996), Pérez de Moya (1562), and the syncopated algebra of Aurel (1552). We should also mention their presence in recreational mathematical texts, such as those by Bachet (1612), Ozanam (in 1692; Hutton 1844) and Vinot (1860) and, more recently, in popular works such as that by Swetz (2014).

However, the advent of a general public education system led to the adoption of an approach to mathematical problems that is based on the application and practice model and gives prevalence to the algebraic method over the arithmetic method. This has lowered confidence in the educational value of these problems to the extent that many have disappeared from textbooks, or appear in them merely as past time activities.

Today's basic curriculum for Spanish Primary Education explicitly states that

> Problem-solving processes are one of the main axes of mathematical activity; they constitute the cornerstone of mathematics education and as such they should be the source and main support for learning throughout this stage of education. Solving a descriptive word problem requires a multitude of basic skills, including reading, thinking, planning the solution process, establishing and reviewing strategies and procedures, modifying this plan if necessary, checking the solution, and communicating the results. (Spanish Royal Decree 126/2014; MEC 2014, p. 19386; authors' translation)

Curriculum proposals therefore consider problem solving to be a basic competence in the development of mathematical activity.

We believe that historical problem-solving methods are indispensable sources of information for mathematics education because they illustrate the reasoning the great mathematicians of the past used in their solutions to these problems. In this chapter, we compile historical evidence on the solution methods for descriptive word fraction problems and highlight certain aspects of problem solving that will enable pupils to acquire significant knowledge.

The existing literature contains numerous classifications of problems with natural numbers that follow criteria such as the mathematical structure, or the statement's syntactic characteristics, including location of the question, length and

[1]A collection originally compiled by Metrodorus, probably around the 6th century and later on greatly enriched by C. Cephalas in the 10th century and M. Planudes in the 13th century.

number of sentences, number of words, verb tenses, etc.; or semantic characteristics, which include global semantics, inclusion of superfluous information and distracters, etc. (Cerdán 2008; Goldin and McClintock 1979). However, no classification of fraction problems has been widely accepted by the research community. In textbooks, these problems appear under headings such as Methods, Rules, Contexts or Actions and Agents (see Gómez et al. 2016). Although presenting them in this way gives the problems certain recognition at least, it does not provide a sufficiently global or overall view of them. Moreover, this form of presentation is also an arbitrary one, because the same problem can be solved using different (arithmetic or algebraic) methods and because the same method can be used to solve different problems. The same occurs with the name of the problem, the context in which it is set, or the agents involved, because these say nothing about how the problem is structured, or what it contains.

To address this question, in Table 10.1 we first present a structured classification of descriptive word fraction problems. The problems are divided into categories and types according to two intrinsic variables of fraction problems: the known or unknown whole or total quantity, and the relationship between the parts (for more details see Gómez et al. 2016). This classification will be used to achieve the general objective of this paper, which is to compile a list of historical methods by analyzing each type of problem identified.

In this chapter, we will focus on a particular type of descriptive problem that involves an unknown whole and related parts. We present the methods that have been used in textbooks to solve this type of problem and the analytical readings that have been used to support these methods. By analytical reading we mean the reduction of the statement to a list of quantities and a list of relationships between these quantities (see Gómez 2003; Puig 2003). Then, we use this information to conduct a pilot study to investigate the extent to which these methods and readings are reflected in students' performance.

The rest of the chapter is organized as follows. First, we present a sub-classification of the problems that contain an unknown whole and related parts in order to contextualize the specific type of problem analyzed in this paper. We then examine this type of problem based on the various analytical readings and problem-solving methods. Finally, we present the results of our pilot study of students who attempted to solve these problems.

10.2 Study Problem

Gómez et al. (2016) present a subdivision in which the problems with unknown wholes and related parts are divided into four groups (see Table 10.2).

In this chapter we focus on the fourth type of problem illustrated in the above subdivision. As Singmaster (1998) pointed out, this type of problem first appeared in Fibonacci's *Liber Abaci* (1202). Singmaster (1998) calls it *the problem of inheritance, with the ith son getting* 1 + 1/7 *of the rest and all getting the same*

Table 10.1 Classification of the descriptive word fraction problems

Known whole		Unknown whole	
Non-related parts	Related parts	Non-related parts	Related parts
Dying man	The cloth	Lotuses	The eggs (Passing through tax guards)
"*A dying man left 6,000 escudos to be distributed in the following way: "I wish," he said, "that half be given to the Jacobite monastery; a third to the convent of Saint Augustine; a quarter to the monastery of the Friars Minor; and a fifth to the Order of the Carmelites." Question: if the whole is 6,000 escudos, how many escudos will each monastery receive?*"[a] (Silíceo 1513; see Sánchez and Cobos 1996, p. 266 Authors' translation)	"*A certain man buys 4 pieces of cloth for 80 bezants. He buys the first for a certain price, and he buys another for 2/3 the price of the first. He truly buys the third for ¾ the price of the second. Moreover, the fourth he buys for 4/5 the price of the third. It is sought how much each piece is worth.*"[b] (Fibonacci 1202; see Sigler 2002, pp. 274–275)	"*From a bunch of lotuses, 1/3 are offered to Lord Siva, 1/5 to Lord Visnu, 1/6 to the Sun, and 1/4 to the goddess. The remaining 6 were offered to the guru. Find quickly the number of lotuses in the bunch.*"[c] (Bhaskara 1150; see Phadke et al. 2001, pp. 57–58)	"*A country-woman carrying eggs to a garrison, where she had three guards to pass, sold at the first, half the number she had and half an egg more; at the second, the half of what remained, and half an egg more; and at the third, the half of the remainder, and half an egg more; when she arrived at the market place, she had three dozen still to sell. How was this possible without breaking any of the eggs?*"[d] (Ozanam 1692; see Hutton 1844, pp. 207–208)

[a]*Dying Man Solution*: "For the Jacobite monastery, half of the 6,000 escudos, i.e. 3,000 escudos; for the convent of St. Augustine, a third of the 6,000 escudos, i.e., 2,000 escudos; for the monastery of the Friars Minor, a quarter of the 6,000 escudos, i.e. 1,500 escudos; and for the Order of the Carmelites, a fifth of the 6,000 escudos, i.e. 1,200 escudos. All of these parts add up 7,700 escudos but this is not possible because the man only has 6,000 escudos. The divisor is considered 7,700 and the multiplier is the money that must be distributed, i.e. 6,000 escudos. Therefore, each part is multiplied by this ratio. Then, if the part for the Jacobites is multiplied by the multiplier and divided by the divisor, they receive 2,337 escudos, 23 .duodenos and 2 turonos and 1400/7700 turonos. This is therefore the amount that corresponds to the Jacobite Monastery. For other cases we proceed in the same way to obtain the amount corresponding to each. Note that 1 escudo = 35 duodenos and that 1 duodeno = 12 turonos."

[b]*The Cloth Solution* (the modern notation for fractions is used): "You put it that the first piece is worth 60 bezants, because 60 is the least common multiple of the 5 and 4 and 3. Therefore, if the first is worth 60, then the second, worth $\frac{2}{3}$ of it, is worth 40 bezants, and the third is worth 30 bezants, that is $\frac{3}{4}$ the price of the second. The fourth truly is worth 24 bezants, that is $\frac{4}{5}$ of 30. Afterwards you add the 60, and the 40, and the 30, and the 24, namely the put prices of the abovesaid four pieces; there will be 154 that should be 80; you say, I put 60 for the price of the first piece and 154 bezants result as the sum of the four pieces; what shall I put so that the sum of the pieces is 80 bezants? You multiply the 60 by the 80; there will be 4800 that you divide with the rule for 154, that is $\frac{0}{11}+\frac{0}{77}+\frac{1}{154}$, the quotient will be $31+\frac{1}{11}+\frac{6}{77}$ bezants. And this is the value of the first piece. Also in order to have the price of the second, you multiply the 40 by the 80, and you divide again with the $\frac{0}{11}+\frac{0}{77}+\frac{1}{154}$; the quotient will be $15+\frac{6}{11}+\frac{3}{77}$ bezants for the price; at last, so that you know the price of the fourth, you multiply the 24 by the 80, and you divide with the $\frac{0}{11}+\frac{0}{77}+\frac{1}{154}$; the quotient will be $12+\frac{5}{11}+\frac{1}{77}$ bezants for the price, and you realize that in each of the above written four products a ½ is canceled."

[c]*Lotuses Solution*: "Suppose the total number of lotuses is 1. Then the number of lotuses left is $1-\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{4}\right)=1-\frac{20+12+10+15}{60}=1-\frac{57}{60}=1-\frac{19}{20}=\frac{1}{20}$. So [if $\frac{1}{20}$ is 6] the total number of lotuses is $\frac{6\times 1}{\frac{1}{20}}=120$."

[d]*The Eggs (Passing through Tax Guards) Solution*: "It would appear, at first view, that this problem is impossible to solve, for how can half an egg be sold without breaking it? It becomes clearer, however, when we consider that by taking the greater half of an odd number we take exactly half of that odd number $+\frac{1}{2}$. We can find, therefore, that before she passed the last guard the woman had 73 eggs remaining, for by selling 37 of them to that guard, which is half of $73+\frac{1}{2}$, she would have 36 remaining. Similarly, before reaching the second guard she had 147; and before reaching the first, she had 295."

Table 10.2 Classification of descriptive word fraction problems with unknown wholes and related parts

Unknown whole and related parts			
Problems involving removing fractions "from the remainder" of the unknown whole. The final amount resulting from this process is known	Problems involving removing a fixed number and a fraction from the remainder of an unknown whole. The final amount resulting from this process is known	Problems involving removing from and replacing to an unknown whole. The final amount resulting from this process is known	Problems involving removing fractions and integers from an unknown whole. The distribution is known to be equitable
Lapis-lazuli	The eggs (Passing through tax guards)	Wine and water	Ducats
"*An unknown quantity of lapis-lazuli loses one-third, one-fourth, and one-fifth, and the remainder after the three-fold operation on the original quantity is twenty-seven. State what the total was, O wise one, and also tell me the loss.*"[a] Bakhshali (ca. 300 AD; see Sarasvati and Jyotishmati 1979, p. 97)	This example is included in Table 10.1	"*A man has a full glass of wine and drinks a quarter of it. He then fills the glass with water and drinks a third of it. He fills the glass again with water and drinks half of it. Finally, he fills the glass with water again and drinks it all. How much wine has he drunk each time and how much water has he drunk in total?*"[b] (Bruño 1940, p. 125; Authors' translation)	"*A sick man makes his will and his property is divided between his children. The distribution is equitable. The father dies. The eldest son has 1 ducat, and 1/10 of the remainder. To the second son, he leaves 2 ducats and 1/10 of the remainder. To the third son, he leaves 3 ducats and 1/10 of the remainder. Successively, each son has one ducat more than the other and 1/10 of the remainder. In this way, the will of the father is satisfied because he leaves an equitable will. How many ducats and how many children has he got?*" (Aurel 1552, fol. 92; Authors' translation; solution in §10.2(a))

[a]*Lapis-lazuli Solution*: "$\frac{2}{3}\frac{3}{4}\frac{4}{5} = \frac{2}{5}$; [this is what remains] $\left(1 - \frac{2}{5}\right) = \frac{3}{5}$; [this is what is lost] 27 ÷ $\frac{2}{5}$ = 67½ and 67½ – 27 = 40½ is the loss."

[b]*Wine and Water Solution*: "The first time, he drinks $\frac{1}{4}$; the remainder is $\frac{3}{4}$ of the wine. The second time, he drinks $\frac{1}{3}$ of $\frac{3}{4}$, i.e. $\frac{3}{12} = \frac{1}{4}$ of wine, and $\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$ of water. After the second time, $1 - \frac{1}{2} = \frac{1}{2}$ [he has just drunk $\frac{2}{4} = \frac{1}{2}$] of wine and as he drinks $\frac{1}{2}$, he drinks $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ of wine in the third time, and $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ of water. The last time, he drinks the rest of the wine, i.e. $\frac{1}{4}$, which is the last remainder, and $1 - \frac{1}{4} = \frac{3}{4}$ of water. Solution: He has drunk $\frac{1}{4}$ of wine every time; in total, 1 glass of wine and $\frac{1}{12}$ of a glass of water."

amount. This is a descriptive word problem that comprises several stages in which the whole is unknown, the parts are related by an additive complement, and the distribution is equitable.

The statement for this problem, which Fibonacci called *The Bequest of a Man's Fortune*, is:

> A certain man coming to the end of his life said beforehand to his eldest child, My movable goods you will divide among you thus: you will take one bezant and one seventh of all remaining; to another child he truly said, And you will take 2 bezants and a seventh part of the remaining. And thus he said to all his children in order, giving each one more than the preceding, and by steps always a seventh of the remaining; the last child had that which was left. It happened however that each child had of his father's property equally under the aforesaid conditions. It is sought how many children there were and how much was the fortune. (Sigler 2002, p. 399)

The analytical readings for this type of problem found in historical textbooks focus on three fundamental relationships, all of which are equivalent:

(a) all the children have the same amount;
(b) the difference between what two children receive is zero; and
(c) the difference between the amount remaining before the last distribution and what the last child receives is zero. We now present three problems to illustrate these three analytical readings.

(a) **All sons receive the same amount**

An algebraic approach to this problem is found in the syncopated algebra by Aurel (1552):

Problem: A sick man makes his will and determines that his property be divided equally among his sons so that each receives the same amount. On the death of the father, the eldest son receives one ducat and $\frac{1}{10}$ of the remainder. The second son receives 2 ducats and $\frac{1}{10}$ of the remainder. The third son receives 3 ducats and $\frac{1}{10}$ of the remainder. In this way, each son receives one ducat more than the previous one plus $\frac{1}{10}$ of the remainder. In this way, the sick man's wish is fulfilled because all sons receive the same number of ducats. How many ducats did the father leave and how many children did he have?

Solution. The man left x ducats. The eldest son received 1 ducat, thus leaving $x - 1$ ducats, and $\frac{1}{10}$ of the remainder is $\frac{x-1}{10}$, which added to 1 ducat means that the eldest son received $\frac{x+9}{10}$ ducats. Taking these ducats from the x initial ducats leaves $\frac{9x-9}{10}$ ducats for the remaining children. Of these ducats, the second child receives 2 ducats, leaving $\frac{9x-29}{10}$ ducats, $\frac{1}{10}$ of which is $\frac{9x-29}{100}$, which when added to the 2 ducats already received by the second son makes $\frac{9x+171}{100}$ ducats in total for the second son. Since both sons inherited the same amount, the number of ducats for the first son must equal the number of ducats for the second son. I say, therefore, that the $\frac{x+9}{10}$ ducats received by the first son are equal to the $\frac{9x+171}{100}$ received by the second son. Reducing this equation to integers (cross-multiplying) leaves $100x + 900 = 90x + 1710$. Solving this equation leaves $x = 81$, which is the number of ducats left by the father. To find how many sons he had, find how much each son received. Taking 1 ducat from 81 leaves 80, 1/10 of which is 8. If we add 1 ducat to 8 ducats, we get 9 ducats in total for the first son, which is the same number received by all sons (fol. 92; authors' translation).

As we can see in the text, Aurel uses relationship (a), which allows him to formulate the equation: $\frac{x+9}{10} = \frac{9x+171}{100}$. In the transcription of the solution to this problem, for greater clarity, we have replaced the cossic symbols with current algebraic symbols.

(b) **The difference between what two sons receive is zero**

The following example, taken from Euler (1822/1770), also uses the algebraic method but that time in Cartesian form (Descartes 1701; Descartes wrote in that book what one needs to do to translate a problem into equations and Polya (1966) rewrote the Cartesian rules to show it as rules to solve problems with algebraic signs).

Problem: A father leaves at his death several children, who share his property in the following manner: namely, the first receives a hundred pounds, and the tenth part of the remainder; the second receives two hundred pounds and the tenth part of the remainder; the third takes three pounds and the tenth part of what remains and the fourth takes four hundred pounds and the tenth part of what remains; and so on. And it is found the property has thus been divided equally among all the children. Required is how much it was, how many children there were, and how much each received?

Solution. Let us suppose that the father's total fortune amounts to z pounds and that each son will receive the same equal share, which we will call x. The number of children will therefore be $\frac{z}{x}$. Now let us solve the problem.

Sum or inheritance to be divided	Order of sons	Share for each son	Differences
z	1st	$x = 100 + \frac{z-100}{10}$	
$z - x$	2nd	$x = 200 + \frac{z-x-200}{10}$	$100 - \frac{x+100}{10} = 0$
$z - 2x$	3rd	$x = 300 + \frac{z-2x-300}{10}$	$100 - \frac{x+100}{10} = 0$
$z - 3x$	4th	$x = 400 + \frac{z-3x-400}{10}$	$100 - \frac{x+100}{10} = 0$
$z - 4x$	5th	$x = 500 + \frac{z-4x-500}{10}$	$100 - \frac{x+100}{10} = 0$
$z - 5x$	6th	$x = 600 + \frac{z-5x-600}{10}$	And so on

We have included the differences between successive shares in the final column. These are obtained from each share minus its preceding share. Since all shares are equal, this difference is equal to zero. By solving the equation $100 - \frac{x+100}{10} = 0$, we obtain $x = 900$.

We now know, therefore, that each son will receive 900 pounds. So, by taking any of the formulas from the third column we obtain $x = 100 + \frac{z-100}{10}$. Therefore, $z = 8100$ pounds and the number of sons is 8100/900 = 9 (Euler 1822/1770, p. 173).

In this case, Euler uses relationship (b) to propose the equal relationships reflected in the fourth column of the above solution.

(c) **The difference between the amount remaining before the final distribution and what the last son receives is zero**

The following quick solution is the one we previously mentioned from Fibonacci.

> For the seventh that he gave each child you keep 7 from which you subtract 1; there remains 6 and this many were his children, and the 6 multiplied by itself makes 36, and this was the number of bezants. (Sigler 2002, p. 399)

Because the explanation is regulated, i.e. based on an unexplained rule, there is nothing in the text to help us ascertain which reading analysis was used to support the solution. However, it may be possible that Fibonacci used an arithmetic method based on factorization and proportion. To explain this method, we will use symbolic language.

Let C be the final remainder of bezants and $1 \cdot n + \frac{1}{7}(C - 1 \cdot n), n \geq 2$ be the amount received by the final son. Relationship (c) is then written as:

$$C - \left(1 \cdot n + \frac{1}{7}(C - 1 \cdot n)\right) = 0, \quad (10.1)$$

where n is the number of sons. From Eq. 10.1 we obtain Eq. 10.2:

$$7C - C - 7n - n = 0 \rightarrow (7 - 1)C = (7 - 1)n \rightarrow 6 \cdot C = 6 \cdot n \quad (10.2)$$

From Eq. 10.2 we deduce that $C = n$, i.e. the number of children is equal to the final remainder and, according to Eq. 10.1, this is equal to the amount received by the final son. Therefore, since the distribution is equitable, each son receives this same amount and the inheritance (which we can call H) will be equal to n^2 (the number of sons by n bezants for each son).

All we need now is to find the value of n, which can be obtained, for example, from the equation corresponding to the amount received by the first son:

$$1 + \frac{1}{7}(n^2 - 1) = n \rightarrow n = 6. \quad (10.3)$$

From 10.3 we find that $n = 6$. We also find that this result could also have been obtained from Eq. 10.2 by using factorization and proportion as we stated before.

Another example of this method of factorization and proportion is the following problem extracted from Puig (1715).

Problem: A sick merchant makes his testament, in which he leaves a certain amount of his property to each of his sons. He determines that the eldest son will receive a sixth of his property plus 300 ducats, the second son will receive a sixth of the remainder plus 600 ducats, the third son will receive a sixth of the new remainder plus 900 ducats, and so on for the next sons, giving each one a sixth part of the new remainder plus 300 ducats more than the preceding one. On the father's death, the property was divided and it was found that all the sons received the same amount. How many sons did the father have, how much property did he leave, and how much did each son receive?

Solution. Subtract the numerator from the denominator, i.e. 1 from 6, to leave 5, which is how many sons the father left. Then multiply the 300 ducats, which is the number of ducats more that are successively given to each son, by 6, the denominator, to give 1800 ducats, which is the total amount given to each son. Multiply this amount by 5 to find the value of the property left by the father. Try it and you will find this is true (Puig 1715, p. 209; authors' translation).

Using symbolic language to follow the reasoning shown earlier, relationship (c) is written as follows:

$$C - \left(300n + \frac{1}{6}C\right) = 0, \tag{10.4}$$

where $300n + \frac{1}{6}C, n \geq 2$ is the amount received by the youngest son, n is the number of sons, and C is the final remainder.

From Eq. 10.4 we obtain:

$$6C - C = 6 \times 300n \rightarrow (6-1)C = 6 \times 300n, \tag{10.5}$$

The number of children and the amount inherited must be whole numbers. If we observe the equality and the above explanation in terms of factorization and proportion, then $\frac{6 \times 300}{6-1} = \frac{C}{n}$, which shows that $n = 6 - 1$ and that $C = 6 \times 300$. This is a possible solution and may be the idea that was applied by Fibonacci.

In conclusion, we have found three analytical readings for the same inheritance problem in which the whole is unknown, the parts are related by an additive complement, and the distribution is equitable. We have also observed two methods for solving the problem: the regulated arithmetic method and the algebraic method using syncopated algebra and Cartesian algebra.

10.3 Pilot Study

For our pilot study, we chose a similar but more intuitive statement to that of Fibonacci, which is taken from Tahan (1993), who calls it *The Raja's Pearls*.

> A rajah on his death left to his daughters a certain number of pearls with instructions that they be divided up in the following way: his eldest daughter was to have one pearl and a seventh of those that were left. His second daughter was to have two pearls and a seventh of those that were left. His third daughter was to have three pearls and a seventh of those that were left. And so on. The youngest daughters went before the judge complaining that this complicated system was extremely unfair. The judge, who as tradition has it, was skilled in

solving problems, replied at once that the claimants were mistaken, that the proposed division was just, and that each of the daughters would receive the same number of pearls. How many pearls were there? How many daughters had the rajah? (Tahan 1993, p. 76)

The study included the following participants: 27 future high school mathematics teachers (hereafter, HSMT), two future primary school teachers (hereafter, PST), and six high school students (hereafter, HSS). With this non-homogeneous sample, our aim was to observe the participants' skills in using problem-solving methods and analytical readings at each of the levels of mathematical knowledge. The following are the results of a pilot study and provide us with just a first view. For more significant conclusions a larger sample should be used in future studies.

10.4 Results

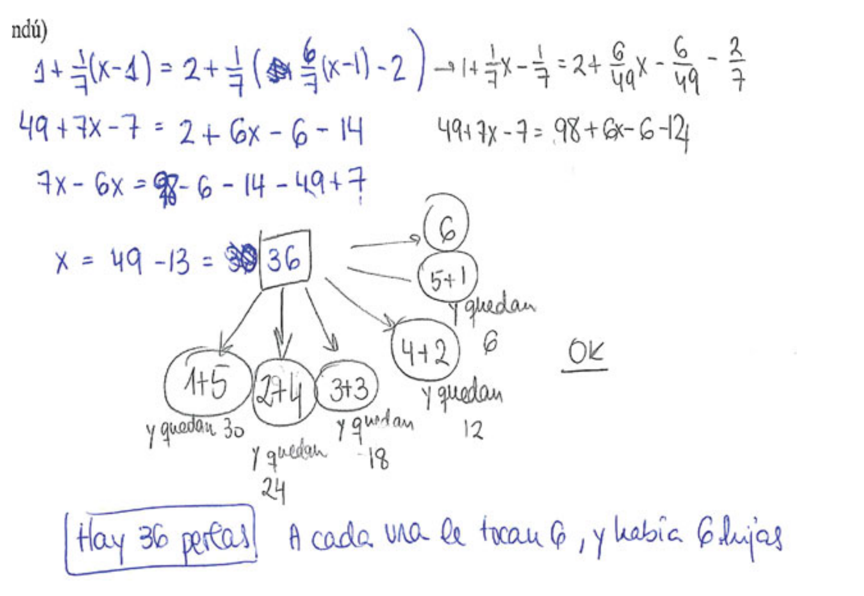
The problem was solved algebraically by 12 students (11 HSMT and 1 HSS) using the fundamental relationship (a). As an example, Fig. 10.1 shows the solution produced by one of these students. This student obtained the number of pearls received by the first daughter $\left(1 + \frac{1}{7}(x-1)\right)$ and the number of pearls received by the second daughter, $\left(2 + \frac{1}{7}\left(\frac{6}{7}(x-1) - 2\right)\right)$ and then formulated the equation by equating the two expressions.

Two students (HSMT) solved the problem using an arithmetic method and relationship (c). To do so, they assumed that the total number of pearls minus one had to be a multiple of 7 and then worked backwards to solve the problem by trial and error (Fig. 10.2).

The rest of the students were unable to solve the problem due to one of two reasons:

1. They were unable to translate some of the expressions in the statement into symbolic language. For example:
   (a) the characteristic expression for this type of problem: "of those that were left" (see Fig. 10.3).
   (b) the expression "a pearl and a seventh of those that were left" (see Fig. 10.4).
2. They had problems working out the fractions.
   This student started with the number of pearls corresponding to the first daughter but did not transcribe the expression "of those that were left" correctly, writing $1 + \frac{1}{7}x$, instead of $1 + \frac{1}{7}(x-1)$. From this point on, the solution is incorrect and the errors accumulate in the subsequent steps. Although the student equates what the first daughter receives with what the second daughter receives, the problem is now impossible to solve and the student expresses this fact.

In the first line in Fig. 10.4 we can see that the student transcribes the share of the pearls that should be inherited by all the daughters. The student expresses the



$$1 + \frac{1}{7}(x - 1) = 2 + \frac{1}{7}\left(\frac{6}{7}(x - 1) - 2\right)$$

x = the number of pearls at the beginning

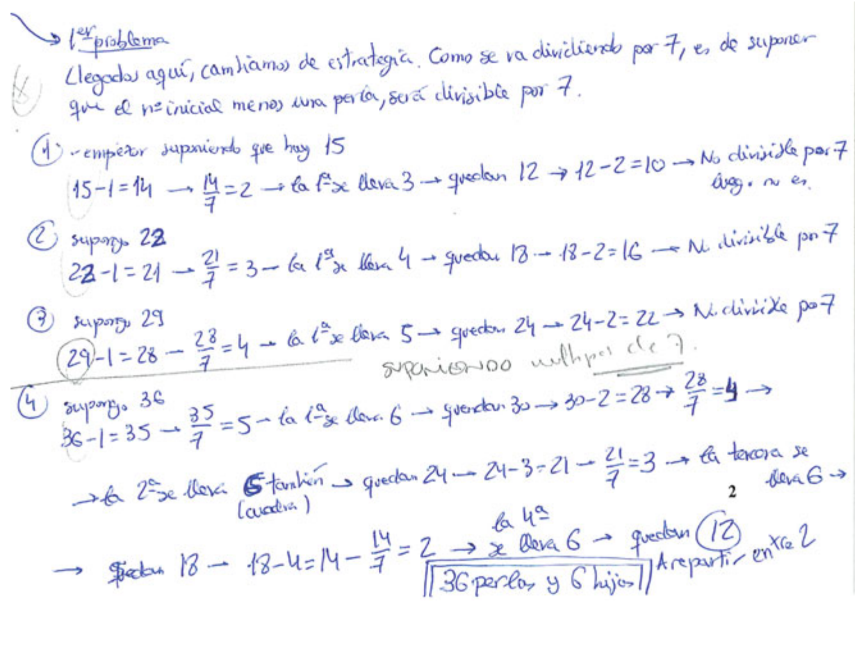
Solution. *There were 36 pearls and 6 daughters with 6 pearls each.*

Fig. 10.1 The correct solution, for which the student used the algebraic method and assumed that all the daughters received the same amount, and its translation. A literal translation is given in italics and clarifications are given in non-italics

number of pearls received by the first daughter correctly, $1 + \frac{1}{7}(x - 1)$, but then makes an error when expressing the number of pearls received by the second daughter. We can see how the student is unable to correctly transcribe the expression "two pearls and a seventh of those that were left" symbolically, writing $2 + \frac{1}{7}\left(\frac{1}{7}(x - 1)\right)$ instead of $2 + \frac{1}{7}\left(\frac{6}{7}(x - 1) - 2\right)$. After this, the solution makes no sense.

10.5 Final Remarks

We have conducted a historical-epistemological study and a pilot study with students on descriptive word fraction inheritance problems in which the distribution is equitable, the whole is unknown, and the relationship between the parts is based on an additive complement.



We imagine that the total amount minus one is a number divisible by seven. Start by assuming 15 – 1 = 14. 14/7 = 2. The first daughter has 3 pearls; the remainder is 12, 12 – 2 = 10, which is not divisible by seven.

The students test with different numbers until they reach 36 pearls:

36 – 1 = 35, 37/5 = 5. The first daughter has 6 pearls, the remainder is 30.

30 – 2 = 28, 28/7 = 4. The second daughter has 6 pearls, the remainder is 24.

24 – 3 = 21, 21/7 = 3. The third daughter has 6 pearls, the remainder is 18.

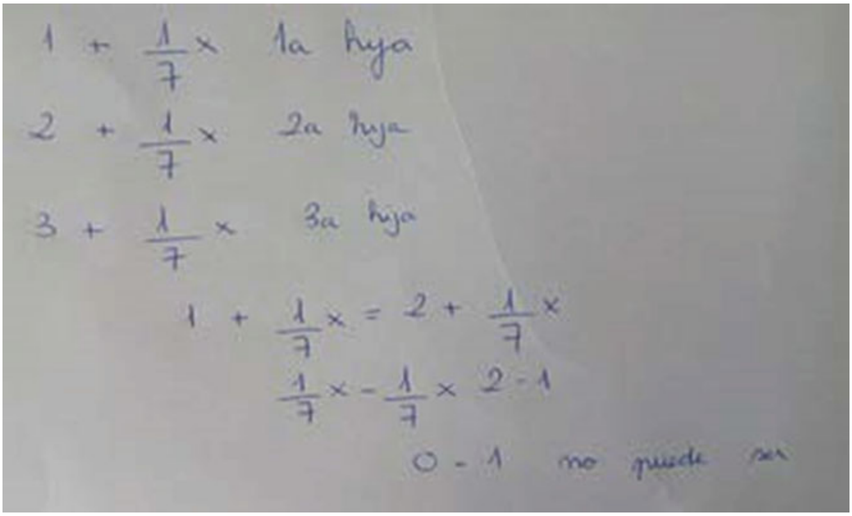
18 – 4 = 14, 14/7 = 2. The fourth daughter has 6 pearls, the remainder is 12.

12 – 5 = 7, 7/7 = 1. The fifth daughter has 6 pearls, the remainder is 6.

The sixth daughter has the final 6 pearls.

Fig. 10.2 A correct solution, for which the students used the arithmetic method and assumed that the difference between the amount remaining before the final distribution and what the last daughter receives is zero, and its translation. Literal translations are given in italics and clarifications are given in non-italics

In our historical-epistemological study, we observed the use of regulated arithmetic methods as well as syncopated algebra and Cartesian algebra. We also observed three equivalent analytical readings: when all heirs are considered to receive the same inheritance; when the difference between what two heirs receive is



$1 + \frac{1}{7}x$ *1st daughter*

$2 + \frac{1}{7}x$ *2nd daughter*

$3 + \frac{1}{7}x$ *3rd daughter*

$$1 + \frac{1}{7}x = 2 + \frac{1}{7}x$$

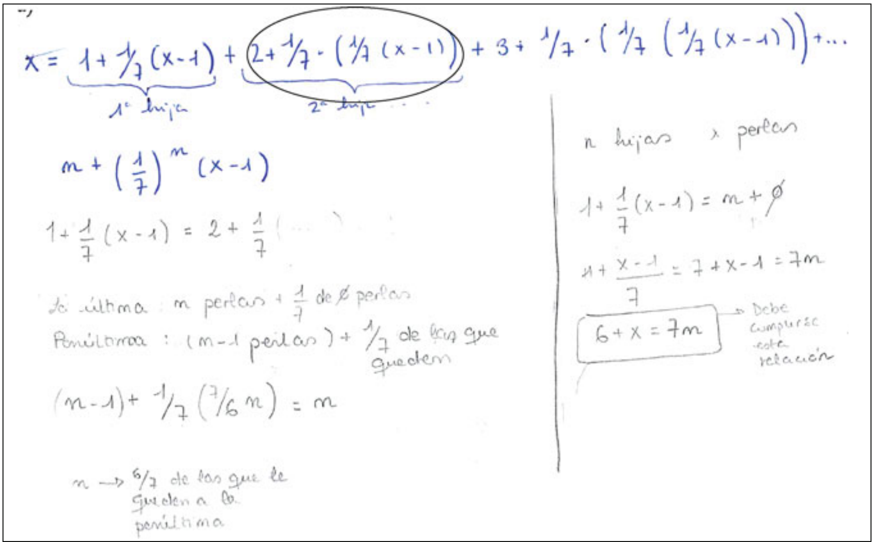
$$\frac{1}{7}x - \frac{1}{7}x = 2 - 1$$

$0 = 1$, *it is impossible.*

Fig. 10.3 An incorrect solution because of the incorrect translation of the expression "the remainder." Literal translations are given in italics and clarifications are given in non-italics

considered to be zero; and when the difference between the final remainder and the final amount inherited is considered to be zero.

The results of our pilot study were as expected. On the one hand, thirteen of the future high school teachers solved the problem correctly. Most of these students used the Cartesian method and the analytical reading that identified that both heirs would receive the same amount. We should stress that only two of these students solved the problem through arithmetical reasoning, and this was by trial and error and the inverse method. On the other hand, future primary school teachers were not able to solve the problem, though we knew from their curriculum that they had



x = the number of the pearls at the beginning

In this case, we observe two problems.

Firstly, the student calculates the first part for the first daughter correctly but not the second part for the second daughter.

Secondly, the student does not show the equation.

The student said: *The last daughter:*

n pearls+1/7 Ø pearls [Ø meaning "nothing"]

The penultimate daughter:

(n−1 pearls) +1/7 the last remainder.

$$(n-1)+\frac{1}{7}\left(\frac{7}{6}n\right)=n$$

n→6/7 of the remaining at the penultimate.

n daughters, x pearls

$$1+\frac{1}{7}(x-1)=n+\emptyset$$

$$1+\frac{x-1}{7}=7+x-1=7n$$

$6+x=n$ *This condition must be satisfied.*

Fig. 10.4 An incorrect solution because of the incorrect translation of the expression "*a pearl and a seventh of those that were left*." A literal translation is given in italics and clarifications are given in non-italics

received training in elementary algebra during their secondary school education. However, in this research we had only two such teachers and with such a restricted sample no conclusions can be drawn. However, this situation could be an indication of what is expected if a larger sample were used in future research. We also found that many students had problems transcribing the expression "of those that were left." This highlights the constructively interfering complementary roles of the literary and symbolic languages.

Our study also showed that the Cartesian method is the one that is most used by today's students, since we found no evidence that arithmetic reasoning was used to solve the problem. The problems presented in this chapter, which seem to have been lost from the educational record, are by themselves a rich source of knowledge. As such, we found that they have helped to communicate mathematical applications, techniques, approaches, methods and reasoning, and that the historical sources illuminate solutions of the historical authors. Thus, it may be useful to teach how to use these problems as an object of study, rather than to find their solution as the by-product of another branch of learning; namely, algebra.

This information may also prove useful for research on numerical and algebraic thinking, since it provides a historical framework for studying classical problems, not as individual components of a mathematical content but as elements related to the roots of mathematics and to analysis of its evolution.

Finally, the challenge for teachers and researchers is to keep this wealth of knowledge alive, preventing it from being forgotten. They must take into account the aims of the curriculum, which considers that problem solving is a basic competence to be adapted in education in accordance to the students' background.

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Chapter 11
History of Matrices

Commognitive Conflicts and Reflections on Metadiscursive Rules

Aline Bernardes and Tatiana Roque

Abstract This chapter contains a teaching proposal based on the history of matrices inspired by the conceptual and methodological framework introduced by Kjeldsen (2011) to integrate history into the teaching of mathematics. Kjeldsen's conceptual framework is based on Sfard's (2008) theory of thinking as communicating. Our goal is to create conflicting situations in which students are encouraged to reflect upon the metadiscursive rules related to matrices and determinants, comparing them with those found in some historical writings. Two teaching modules were created, dealing with two episodes in the history of matrices, based on the works of the mathematicians Sylvester and Cayley, and on the historical interpretation of Brechenmacher (2006). Two field studies were conducted with undergraduate mathematics students, from two universities in Rio de Janeiro. In this chapter we also explain how some historical metadiscursive rules were identified.

Keywords History of matrices · Teaching of matrices · Metadiscursive rules
Commognitive conflicts

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11.1 The Context of the Work

The main motivation of this study was the teaching of matrices in undergraduate Linear Algebra courses for future teachers. Many Linear Algebra courses in Brazil, and their corresponding textbooks, start with the concept of the matrix as a stand-alone mathematical object. Both the definition of matrix and its operations are often presented without problematizing why this object must be defined and why the operations are done the way they are. In this kind of approach, the teaching is often guided by procedures. The same focus on procedures marks the teaching of determinants.

In the context of Linear Algebra courses, students tend to better grasp the concept of matrices with the study of linear transformations, where they are seen as objects that act geometrically on vectors. But oddly, when asked why matrix multiplication is defined in terms of inner products between rows and columns of the matrices, students nearing graduation from university, and even some high school teachers, usually do not give an adequate answer.

Mathematical concepts are often presented in the teaching of mathematics as ready and finished products—that is, as static entities. The problems and demands that drove the emergence and development of the concepts, as well as the factors that determined their choice as objects to be studied, are not taken into account in the training of mathematics teachers.

In this sense, the history of mathematics may contribute to the understanding of certain subtleties inherent to the development of the concept being studied: what led to its current definition, for what ends it was created, etc. Let us take the example of matrices, which were historically introduced *after* determinants. This means, in particular, that determinants have not always been computed with matrices. Thus, the history of mathematics may play an important role in the teacher training as a way to reveal the paths taken by the development of the concepts, which are hidden in their modern presentation.

This study used historical sources on matrices and elaborated a teaching proposal exploring two mathematical practices, by the English mathematicians Arthur Cayley (1821–1895) and James Joseph Sylvester (1814–1897), respectively, in which matrices appeared as a useful representation (Brechenmacher 2006). The practices show that this notion was not proposed immediately as a mathematical object and the way that matrices and determinants were understood and used has changed over time.

The conceptual framework is based on the perspective of mathematics as a discourse, and the notions of *commognitive conflicts* and *metadiscursive rules* proposed by Sfard (2008), and on the theoretical argument introduced by Kjeldsen (2011), which gives to the history of mathematics a new role in the learning of mathematics grounded in Sfard's theory. Teaching and learning situations were therefore developed with the aim of creating commognitive conflicts, in which students were encouraged to reflect upon the metadiscursive rules that define their actions when dealing with matrices and determinants, after comparing them with the rules that appear in some historical writings.

This chapter intends to contribute to the discussions related to the theme of "theoretical and/or conceptual frameworks for integrating history in mathematics education." We will present an example of an empirical study using Kjeldsen's theoretical argument applied to the use of history in the teaching of Linear Algebra, in particular, of matrices and determinants.

A pilot study was first carried out to test two teaching modules (Bernardes and Roque 2014). After that, two additional field studies were done with three research goals: (i) to investigate how historical sources encourage reflections about metarules related to matrices and determinants; (ii) to investigate how reflections about metarules impact students' conceptions about matrices and determinants; and (iii) to investigate the development of a historical consciousness in the students. The investigation related here is part of the research of a doctoral thesis by the first author (Bernardes 2016). This paper focuses on the results related to the first research goal above.

The next section introduces briefly the main concepts of Sfard's theory which were used in the work, and explains how Kjeldsen has proposed using this theory to integrate history with the teaching of mathematics. Then, two episodes from the history of matrices are presented—inspired by the work of Brechenmacher (2006)—along with an explanation on how four metadiscursive rules were identified in the historical sources of Sylvester and Cayley. After that, a brief description is given of the field studies carried out and of the teaching material. The results related to the first research goal above are presented and discussed, before the presentation of some conclusions drawn from the investigation.

11.2 Theoretical Framework

The theoretical framework of the research draws inspiration from the works of Kjeldsen and collaborators (Kjeldsen 2011; Kjeldsen and Blomhøj 2012; Kjeldsen and Petersen 2014). In the first paper, Kjeldsen introduced a theoretical argument to integrate history in the teaching of mathematics based on Sfard's theory of thinking as communication (Sfard 2008).

Sfard couches her theory within a participative perspective, in opposition to an acquisitionist one, and emphasizes the social, cultural and historical aspects of human development by shifting the focus of learning from the individual to the collective and to human activity. According to Sfard, humans are social beings "engaged in collective activities from the day they are born and throughout their lives" (Sfard 2008, p. 79).

Through this perspective, Sfard defines communication as a standardized collective activity, whose standards can be described as the result of processes governed by rules. In this way, there are different ways of communicating and they are called discourses. Mathematics is therefore seen as a type of discourse; chemistry, physics and history are also examples of discourses, and so on. Learning mathematics requires becoming part of the discourse of mathematics and being capable of individualizing it. In other words, it means "becoming able to have mathematical communication not only with others, but also with oneself" (Sfard 2007, p. 575). Sfard furthermore posits that learning mathematics is equivalent to modifying and extending one's own discourse.

There are two types of rules that shape the standards of communication in the discourses: object-level rules and metadiscursive rules. Object-level rules are defined as "narratives about regularities in the behavior of objects of the discourse," while metadiscursive rules, or metarules, "define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard 2008, p. 201). In the case of mathematical discourse, object-level rules relate to the properties of mathematical objects. For example, in Euclidean geometry, the interior angles of a triangle always add up to 180°. In physics, Newton's laws of gravity and of motion are examples of object-level rules. Metarules are related to the actions of discursants, or rather, the way that they interpret the content of the discourse. They are generally implicit in the discourse and are made manifest when we judge, for example, that a specific description could be considered as a definition, or if a proof can be accepted as correct.

Metarules govern "when to do what and how to do it" (Sfard 2008, p. 208). So, they affect the way in which participants of a discourse interpret its content. Learning of mathematics is thus the development of appropriate metarules. On the other hand, as these rules are contingent and tacit (Sfard 2008, pp. 203, 206), participants do not observe them in a conscious and natural way. For this reason, it is unlikely that participants can learn metarules by themselves.

Kjeldsen (2011) argued that the history of mathematics plays a fundamental role in order to "illuminate metadiscursive rules" of the mathematical discourse. They are historically established and may thus be treated as the object-level of a historical discourse. In this way, metadiscursive rules stop being tacit and can be made explicit objects of reflection.

The idea is then to promote situations in which students are encouraged to investigate the development of mathematical practices, through the use of historical sources, and to understand the vision mathematicians had about their own practices, as well as how they viewed and used their objects of study, and how they formulated and substantiated their mathematical narratives. Doing so, students can have contact with discourses governed by metarules that are different from the modern ones and different from their own metarules.

Kjeldsen's theoretical argument rests on the concept of commognitive conflict, defined as "a situation in which communication is hindered by the fact that different discursants are acting according to different metarules" (Sfard 2007, p. 576). The use of historical sources can lead to these conflicts, since history offers several discourses governed by distinct metarules.

The research done in this study used Kjeldsen's theoretical argument as a starting point and was greatly inspired by her works; but, it concerns a different mathematical subject that has yet to be widely explored in the history-based teaching experiments reported in the literature. This study was interested in using historical sources on the history of matrices to show how the metarules regarding matrices and determinants have changed over time. In this way, (future or current) teachers—the investigatory subjects of our research—can make their own metarules about matrices and determinants explicit.

11.3 The Historical Part and How the Metarules Were Identified

11.3.1 Two Episodes from the History of Matrices

This section sets forth some historical aspects of matrices based on the ideas discussed in the article *Les Matrices: formes de représentation et pratiques opératoires* (1850–1930) (Brechenmacher 2006) and on some original sources from the English mathematicians James Joseph Sylvester and Arthur Cayley. This section is aimed at providing a basis for introducing the metarules identified in the sources, which were explored in some of the teaching experiments.

Brechenmacher's historical interpretation was chosen because his works are based on the conception of mathematics as a practice, or series of practices, and because he takes into account the social, cultural and intellectual factors within which mathematical activities are developed. It is important to mention that this kind of historical approach facilitates the identification of metarules. In contrast, an anachronistic historical approach can hinder identification of metarules.

(a) **Sylvester's episode**

Between 1850 and 1851, Sylvester published a series of articles analyzing the nature of points of intersection (real/complex, finite/infinite) of two conics and the types of contacts between two conics and two quadrics. The Sylvester's research episode was limited in accordance with the articles (Sylvester 1850a, b, 1851a), focusing on *the problem of classifying the types of contacts between two conics*, which it will be referred to moving forward as the *problem of contacts*.

The term "contact" was employed to denote the points of intersection where two conics are tangent to each other. There are four types of contacts that can be characterized by the multiplicity[1] (2, 3, or 4) of the point(s) of intersection where the conics are tangent: simple contact, diploidal contact, proximal contact and confluent contact (Figs. 11.1 and 11.2).

The main mathematical tool used by Sylvester in order to solve the problem of contacts was the notion of determinant. However, he did not compute determinants

[1]The term "multiplicity," used in reference to the points of intersection of the conics, refers to the algebraic concept of the *index of intersection*, which generalizes the intuitive notion of counting the number of times that two algebraic curves intersect at a point.

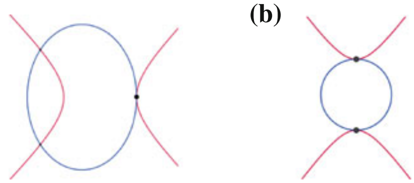

Fig. 11.1 **a** Simple contact: one double intersection point; **b** Diploidal contact: two double intersection points

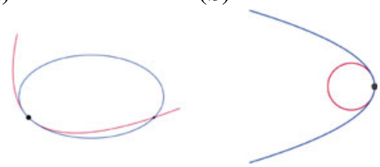

Fig. 11.2 **a** Proximal contact: one triple intersection point; **b** Confluent contact: one quadruple intersection point

of matrices. This last notion was introduced later. In order to classify the type of contact between two conics, Sylvester analyzed the multiplicity of the roots of the equation $\det(U + \mu V) = 0$, U and V being homogeneous polynomials of degree 2 in three variables. To clarify:

$$U : ax^2 + by^2 + cz^2 + 2a'xy + 2b'xz + 2c'yz = 0,$$
$$V : \alpha x^2 + \beta y^2 + \gamma z^2 + 2\alpha' xy + 2\beta' xz + 2\gamma' yz = 0,$$

and the coefficients are real numbers. So, Sylvester computed determinants of (homogeneous) polynomial functions. He did not present a definition for determinants in his works about the contact problem. The equality $\det(U + \mu V) = 0$ yields a cubic polynomial equation. In some examples, he presented this equation directly, but there are representations for determinants using a table form.

When the roots of $\det(U + \mu V) = 0$ are distinct, there are no points of contact. In the case of double or triple roots (with algebraic multiplicity 2 or 3, respectively), there are points of contact (Figs. 11.1 and 11.2). Knowing of the existence of four types of contact, the analysis of the multiplicity of roots of $\det(U + \mu V) = 0$ proved to be an insufficient criterion for considering all cases. Then, Sylvester introduced the notion of minor determinants:

> Imagine any determinant set out under the form of a square array of terms. This square may be considered as divisible into lines and columns. Now conceive any one line and any one column to be struck out, we get in this way a square, (…) and by varying in every possible manner the selection of the line and column excluded, we obtain, (…), n^2 such minor squares, each of which will represent what I term a First Minor Determinant relative to the principal or complete determinant (…). (Sylvester 1850a, p. 147)

Following that, properties of the minor determinants were stated and applied in order to classify the type of contact between two conics. Sylvester developed a practice that consisted of comparing the common factors in the polynomial development of the complete determinant $\det(U + \mu V)$ and of the first minor determinants.

The practice of classifying the types of contacts between two conics was extended to investigate the intersections between two quadrics (represented by the second-degree homogenous equations in four variables) and, more broadly, between two quadratic forms (in n variables). The generalization of the technique for extracting systems of minor determinants was based on a representation in the form of a rectangular table, which Sylvester dubbed a *matrix* (Brechenmacher 2006):

> (…) we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p and selecting at will p lines and p columns, the square corresponding to which we may be termed determinants of the pth order. (Sylvester 1850a, p. 150)

In this quote, Sylvester made explicit his understanding of a matrix as a source of minor determinants, concisely called by Brechenmacher (2006) as "*mère de mineurs*" (p. 15).

(b) **Cayley's episode**

The generalization of the practice of extracting minor determinant systems from determinants of any order created the problem of enumerating these systems, which attracted Cayley's attention to the notion of matrix. Eight years later, Cayley published a text in which he defined the matrix operations and stated their properties (Cayley 1858). Matrices arise, according to the author, naturally from "an abbreviated notation" for linear systems of equations. Cayley also indicated the use of matrices as a notation for bilinear and quadratic forms in other works, but the 1858 memoir places more emphasis on the association between matrices and linear systems (what he called a set of linear equations).

The definitions of matrix operations were based on the relationship between matrices and systems of equations. The multiplication of matrices, for example, was defined through a "composition between linear systems." It is interesting to note that, although Cayley initially referred to a system of equations, upon explaining the notation, he referred to a set of linear functions. He apparently did not find it necessary to distinguish between systems of linear equations and linear transformations,[2] as is done today.

One issue raised by Brechenmacher (2006) relates to the "remarkable theorem," around which the theory of matrices was developed, and which was announced on the first page of the memoir: "I obtain the remarkable theorem that any matrix

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[2]It is worthwhile to note that the expression "linear transformations" was often used by Cayley, despite his not having used it in the 1858 memoir. Some articles are dedicated entirely to linear transformations; for example, Cayley (1845).

whatever satisfies an algebraical equation of its own order, the coefficient of the highest power being unity, and those of the other powers [are] functions of the terms of the matrix, the last coefficient being in fact the determinant" (Cayley 1858, p. 17).

More than half of the memoir was dedicated to applications of the remarkable theorem, which led Cayley to pose the problem of expressing the powers and roots of a matrix. To solve this problem, he developed a practice based on *a dual interpretation of the notion of a matrix*: sometimes a matrix was interpreted as a *system of numbers*, sometimes as a *single quantity* or as a number. Soon after defining the operation of multiplying a matrix by a number, Cayley introduced the notion of the matrix considered as a *single quantity*. In modern[3] notation:

$$m = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}$$

According to Cayley, "the matrix on the right-hand side is said to be the single quantity m considered as *involving the matrix unity*" (Cayley 1858, p. 20, italics in the original). In current terms, Cayley sometimes used the same symbol m to denote both a number and a square matrix ($m\mathrm{I}_\mathrm{n}$). The dual interpretation of a matrix was expressed in a slightly different way in the proof of the "remarkable theorem," which was done for a particular case with square matrices of order 2:

"Imagine a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and form the determinant

$$\begin{vmatrix} a - M & b \\ c & d - M \end{vmatrix}$$

the developed expression of this determinant is

$$M^2 - (a + d)M^1 + (ad - bc)M^0; "$$

(Cayley 1858, p. 23).

Even so, he justified the procedure based on his notion of a matrix as a "single quantity." In the sequence of the proof, he explained that:

[3]To keep the presentation simpler, Caley's original notation for a matrix (a combination of parentheses and vertical lines) is not used here (though his notation for determinants using vertical lines is identical to the modern one).

$$\begin{pmatrix} a-M & b \\ c & d-M \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - M\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the "original matrix, decreased by the same matrix considered as a simple quantity involving the matrix unity" (Cayley 1858, p. 24).

As it can be seen above, the notion of a matrix changed and acquired a new identity with Cayley's memoir. Matrices were no longer considered merely as generators of minor determinant systems, but began to be associated with the laws of a symbolic calculation and to a "remarkable theorem" (Brechenmacher 2006).

11.3.2 Metarules in the Discourses of Sylvester and Cayley

Four metarules were identified in the discourses of Sylvester and Cayley, which were to be explored in the teaching proposal. An interest in rules expressing actions, or influencing concepts in relation to determinants and matrices and that are clearly different from the current metarules is what determined the choice of these four metarules over so many others implicit in these mathematicians' discourses. The path traced to identify the metarules is based on Kjeldsen's theoretical argument, which affirms that metarules implicit in the mathematical discourse become explicit in the historical discourse. So, the metarules were first identified in the historical interpretation of Brechenmacher (2006). An analysis of some primary sources (Cayley 1858; Sylvester 1850a, 1851a, b) was then done in an effort to understand where the identified metarules would fall, in relation to the underlying routines in Sylvester and Cayley's practices.

As seen in the historical section on Sylvester's practices, the original aspect of his work was his computation of determinants to solve the problem of identifying the types of contacts between two conics or two quadrics. Brechenmacher (2006) expressed a fundamental idea underlying Sylvester's practice when he says[4]:

> These successive publications make it possible to follow the progressive elaboration of a method which is characterized by a translation of geometric or analytical properties in the context of the calculation of determinants. (Brechenmacher 2006, p. 10; authors' translation)

It was also noted that Sylvester spoke of *determinants of functions*, with the functions being homogeneous polynomials of degree 2, which represent conics. Thus, the manner in which this mathematician utilized determinants led to the consideration of the following metarule: *Determinants are tools used to investigate*

[4]In the original: "Ces publications successives permettent de suivre l'élaboration progressive d'une méthode qui se caractérise par une traduction de propriétés géométriques ou analytiques dans le cadre du calcul des déterminants."

the geometric properties of curves, and are calculated using functions (*homogeneous polynomials of degree* 2).

A huge difference is immediately noticeable between this metarule and the current ones, since a determinant is currently defined in the context of Linear Algebra as a function that associates square matrices to numbers and is seen as a property of square matrices. This was not previously the case, since the notion of matrices emerged after determinants.

By introducing the term "matrix," Sylvester makes explicit his conception of matrices as a source of minor determinants (Sylvester 1850a, p. 150), or *the mother of minors*, according to Brechenmacher (2006). This vision was reinforced in another paper dedicated to stating the properties of minor determinants:

> I have in previous papers defined a "Matrix" as a rectangular array of terms, out of which different systems of determinants may be engendered, as from the womb of a common parent; these cognate determinants being by no means isolated in their relations to one another […]. (Sylvester 1851b, p. 302)

Such a conception of the notion of a matrix is directly linked to Sylvester's action of supporting his practice in a representation with the form of a table to extract the minor determinants and to formulate the narratives about these new objects; that is, to state their properties. This action led to the consideration of the following metarule: *Matrix as the mother of minor determinants: the use of matrices as a representation connected to the technique of generating a system of minors.* First of all, this metadiscursive rule highlights the moment at which matrices were introduced and shows how this notion was used before the mathematical object which we now know today was constituted.

In Cayley's memoir, the interpretation of the matrix as "a notation that is very convenient for the theory of linear equations" (Cayley 1855, p. 282; authors' translation) substantially influenced the way in which operations with matrices were defined. Cayley used an association between linear systems and matrices as the basis for introducing the addition of matrices, the multiplication of one matrix by a "single quantity" (number), the multiplication (or composition) of matrices, as well as the definitions of matrix zero (null matrix) and of matrix unity (identity matrix). The use of this relationship led to consideration of the following metarule: *A matrix is a convenient notation to use when working with linear equation systems.*

Currently, operations with matrices are defined in an abstract way, without problematizing the reason for, or the origin of, such definitions. Reflecting upon this metarule and its role in the theory of matrices exposed by Cayley allows for an understanding of why there is a specific rule for multiplying matrices.

Another aspect of Cayley's practice is the dual interpretation of a matrix, namely, sometimes a matrix was used as a system of numbers and sometimes as a "single quantity" or a number. In the routine of constructing the proof of the remarkable theorem, Cayley alternated between the two interpretations above. Relying on the dual interpretation about what a matrix is to Cayley, the last metarule considered is: *Dual interpretation of the notion of matrix: a matrix is sometimes considered as a single quantity (number) and sometimes as a multiple quantity (system of numbers).*

The metarule above differs substantially from current ones, according to which a matrix in general is not seen as a quantity or number, but rather as a table of numbers or, from a more formal point of view, as a function. Mathematically, the association between a scalar matrix mI_n and a real or complex number m is correct. While complex numbers can also be identified with a 2×2 real matrix of a special form, it is not possible to have such an identification between a matrix and a number in a more general context (without assuming restrictions on the matrix entries).

The four (historical) metarules stated above were explored through *historical activities* in two teaching modules in which the works of Sylvester and Cayley were presented in an abbreviated manner. The next section describes how teaching modules were designed and tested in two field studies.

11.4 The Empirical Part

11.4.1 Two Field Studies

Two experiments were carried out with nine undergraduate mathematics students (prospective teachers), from two universities in Rio de Janeiro. A mini-course under the topic of "Different roles of the notion of matrix in two episodes of the history of matrices" was offered for both groups of volunteers. The participants had each taken at least one course in Linear Algebra. The goal was not to use the history of mathematics to introduce the concept of matrix. Students were selected who had already taken a preliminary course in Linear Algebra and who had learned about matrices. Besides matrices, the participants had learned about determinants, linear systems, rank and linear transformations.

The work was done with the two groups above separately. The mini-course had a duration of six sessions lasting three hours each. The first and the last meetings were set aside for carrying out interviews with the participants.

The teaching module on Sylvester was developed during two sessions: First, there was an introduction to the mathematician Sylvester and to the problem of classifying the types of contacts between two conics, discussing the concepts of projective geometry necessary to accompany Sylvester's mathematical practice. The concepts of projective geometry were introduced using modern language. However, the solution of the problem was presented in a way that sought to follow the original notation whenever possible and without using matrices, so that the participants could dive into the thinking of the time. These choices were not an imposition; on a few occasions matrices and modern notation were used to facilitate understanding. During the next session, the participants worked on the historical activities in groups.

The teaching module on Cayley was also developed during two sessions. First, a guided study was conducted on the introductory pages to Cayley's (1858) memoir, in order to understand how and for what Cayley used matrices, and how he defined the operations with matrices, especially matrix multiplication. During the next session, the participants discussed the historical activities in groups.

The following instruments were used to generate data in order to investigate how historical sources encourage reflections about metarules: (i) record of written answers for the historical activities in both modules, and (ii) audio recordings of the discussions when participants carried out the historical activities for both modules. The audio of the discussions was fully transcribed for data analysis.

11.4.2 The Teaching Modules

The first teaching module was entitled: "How matrices appeared in Sylvester's study of conics." The geometric context, in which the term "matrix" was proposed by Sylvester, was introduced along with an explanation of how he solved the problem of the classification of the types of contact between two conics using determinants. Original excerpts were used as much as possible; however, for the sake of clarity, modern definitions and illustrations were used occasionally. Some concepts from projective geometry were necessary, like homogeneous coordinates, projective points, and projective conics. Mathematical exercises were proposed to help the students understand the mathematics of Sylvester's practice. After introducing these notions, this teaching module presented a summary of the practice developed by Sylvester in order to solve the problem of contacts. In the end, the students had to discuss historical questions in groups.

The goal of the historical activities was to elicit a discussion among the students concerning the historical metarules and, therefore, to encourage reflections and discussions about their own metarules related to matrices and determinants. The historical activities of the teaching module about Sylvester were:

- Summarize how Sylvester classified the types of contacts between two conics U and V.
- Sylvester uses various mathematical concepts/tools in the practice he developed to solve the problem of classifying the types of contact between two conics. To understand the role of each one of these in his research, identify which played the role of inducing new knowledge (object(s) of investigation) and which helped to supply answers to the given problem (techniques). The object of Sylvester's investigation is: the classification of the types of contacts between two conics.
- List all the mathematical concepts/tools that constitute the techniques used by Sylvester, according to the text.
- Describe the difference between how Sylvester used determinants and how we use them today. See Extract IV.

- Explain what a first minor determinant is according to the definition presented by Sylvester in Extract I. What is a second minor determinant? Finally, what is a minor determinant of order *r*?
- Why did Sylvester have to introduce the minor determinants?
- Based on Extracts II and III, explain what a matrix was and what role this notion fulfilled for Sylvester.
- Compare the definition of matrix presented by Sylvester in Extract II with the definition that is used nowadays. Note at least one similarity and at least one difference.

Extracts I, II, III and IV mentioned above are excerpts of Sylvester's articles and available in the teaching module. Extract III, for instance, shows how Sylvester interpreted matrices in the episode considered[5]; that is, as the mother of minors.

The second teaching module was entitled "Cayley and symbolic calculus with matrices." It began by presenting a translation of one part of the 1858 memoir (Cayley 1858). Initially, the students read the translation, and then a discussion was held on how Cayley introduced the matrix operations as well as the reasons why he used symbolic calculus with matrices. Afterwards, the following historical activities were proposed in order to give the students the opportunity to reflect on the metarules:

- What is the object of Cayley's investigation according to what you saw in this module? List the techniques used by Cayley in the part of the memoir you studied.
- Compare the description of a matrix presented by Cayley (see the first page of the translated memoir) with the current definition. Do you see similarities? If yes, what are they? Do you see differences? If yes, what are they?
- Discuss the way Cayley established the rules for the laws of addition, of multiplication by a single quantity, and multiplication or compounding of two matrices. Compare that with the method that Linear Algebra textbooks present for operations with matrices.
- Explain what Cayley meant to say by "a matrix considered as a single quantity involving a matrix unity" (see item 10 in the excerpts).
- State, in your own words, the "remarkable theorem" that Cayley mentioned in the first page of his memoir and presented in items 21, 22 and 23 of the memoir.
- The proof of the theorem for matrices of order 2, in item 21, uses the following determinant:

$$\begin{vmatrix} a-M, & b \\ c, & d-M \end{vmatrix}$$

[5]For this study, Sylvester's episode was delimited by his publications on the problem of contacts; some of them are Sylvester (1850a, b, 1851a).

whose expression is $M^2 - (a+d)M^1 + (ad - bc)M^0$. Would Cayley's proof be accepted as correct nowadays? Explain.

- Compare the way that Sylvester used determinants—according to the first module—and the way Cayley used determinants—according to this module.
- What is a matrix for you? What was a matrix for Sylvester? What was a matrix for Cayley?
- Compare the description of a matrix presented with the current definition. Can you see any similarities? If yes, what are they? Can you see any differences? If yes, what are they?

The participants worked in groups while answering the historical activities. An audio recording was made of the participants' discussions while they did the activities.

11.5 Analysis and Discussion

11.5.1 Metarules

In order to investigate how historical sources encourage reflections about metarules, the participant discussions—transcribed from the audio recordings—were analyzed. Some questions in the historical activities were more specific in order to encourage discussion about historical metarules, and those questions were selected as a starting point.

Although the two empirical studies were conducted separately, we analyzed all the data together. In the first part of the analysis, we sought to locate discussions about the historical metarules, and discussions in which participants laid out their metarules regarding matrices and determinants.

Most of the groups had intense discussions on the historical metarules. In addition to reflections on the historical metarules, which is in itself a result, three metarules in the participants' discourse were found. To illustrate how the reflections on metarules were identified the dialogue below shows a discussion of one group about how Sylvester dealt with and computed determinants:

Yhedi: So, the determinant associated to the matrix. We are the ones doing it the other way around, **we associate the matrix with a real number, which is the determinant**.

Maria: Ah, ok. I got it. [Pause]

Maria: The way it's used nowadays is different. Because nowadays, from the matrix you get a determinant, and with him [Sylvester] it was the opposite, from the determinant he asked to find a matrix and associate it to this determinant. **And for him this determinant was the determinant of a function, nowadays the determinant is [calculated] from a matrix**.

In the dialogue above, the participants discussed ideas related to the first historical metarule (identified in the Sylvester episode), according to which *determinants were computed using functions* (homogeneous polynomials that represent the conics). The discussion about Sylvester's practice led the participants to make their own metarules explicit, according to which *determinants are properties of a [square] matrix,* in contrast to what governed Sylvester's discourse. In this way, the study provided opportunities for the students to perceive that the metadiscursive rules and the object-level ones change over time.

A second metarule was detected during the Cayley teaching module activities. Discussions were found indicating that some participants were guided by a metarule according to which *the concept of a matrix is described based on the characteristics of its representation in the form of a table.*

Francisca:	What's a matrix for you? A table, or rather, a matrix is a vector where I can place various data and manipulate them the way I want.
Raelo:	Ok, for me the…. But, hold on, there are three [participants].
Francisca:	We have to come to a consensus.
Mathematician:	I think that it's an arrangement of numbers.
	[…]
Raelo:	Let A be **a square matrix of two columns** … You said it's an arrangement of numbers.
Mathematician:	An arrangement of real or complex numbers **arranged in rows and columns**.

Sylvester and Cayley also described matrices based on their representation; namely as an "arrangement of terms" in Sylvester's words, or as a "set of quantities arranged in the form of a square" in Cayley's words. Nevertheless, there is a big difference in the understanding of the concept by Sylvester, Cayley and the participants. For those mathematicians, the notion of matrix was identified with their practices: the mother of minors by Sylvester and a convenient notation for linear systems by Cayley. It is clear that the concept of a matrix had a meaning or a sense for them.

It is curious that the participants did not mention connections between matrices and other concepts from Linear Algebra, such as systems of linear equations or linear transformations in the dialogue above. To answer the question, "What is a matrix for you?," they based themselves in the common elements between the historical descriptions of matrices by Sylvester and Cayley, and the current definition of a matrix. Usually, Linear Algebra textbooks define matrix as a table or a rectangular grouping of numbers; that is, the concept of matrix is also associated with its representation in the form of a table.

A third metarule was identified in the discourse of some participants during the Cayley teaching module activities. The excerpt below illustrates a discussion about the validity of Cayley's proof of the "remarkable theorem," which was done by Cayley in a particular case with square matrices of order 2 (see *Cayley's episode*: part (b) of Sect. 11.3.1):

Mario: This only applies to a particular case and is extended to…
João: Only a particular case?
Mario: Accept that it is true… I don't think so. Letter b.
João: **We consider this to be only an example**. [Pause]
Mario: **It would be considered a specific case or an example**.

The participants concluded that the proof would not be accepted as correct because it was constructed for a matrix of order 2; that is, only a specific case is considered in the proof. The conversation above suggests that Mario and João needed to use a metarule for guidance according to which *proofs that are based only on particular cases are not valid*. Such a metarule is in accordance with the rules for proofs in mathematics, but the group did not notice a bigger problem with the proof presented: the dual interpretation of a matrix applied to a full matrix. More specifically, the notion of matrix as a single quantity was applied to a full matrix. Cayley started the proof by forming the determinant $\begin{vmatrix} a-M & b \\ c & d-M \end{vmatrix}$ from the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The determinant resulted in the expression $M^2 - (a+d)M^1 + (ad-bc)M^0$. So, Cayley sometimes used the symbol M as a matrix and sometimes as a number.

11.5.2 Commognitive Conflicts

In relation to the commognitive conflicts, an effort was made to locate discussions that showed their manifestation, i.e. conflicting narratives in which the discussants (the participants and the historical sources) were guided by different metarules.

The historical metarule in Cayley's episode about the dual interpretation of the notion of matrix produced commognitive conflicts in all the groups. The excerpt below shows one group's discussion, which took place when the participants were trying to understand the symbolic calculation carried out by Cayley in the proof presented on the "remarkable theorem" in the memoir:

Fernando: What a trip. It's really crazy because look what he does next. He takes the big M, which is the full matrix. This is as much a matrix as it is a number.
Yhedi: No.
Fernando: But here, look, he's operating with numbers. Here he's operating with numbers. Here, this here is a number. Except that this is a matrix.
Yhedi: But when he uses the matrix as a number, he's doing M times identity.
Fernando: Huh?
Yhedi: When he treats the matrix like a number, it's the number times identity.
Fernando: But this matrix here, man?
[At this point, they asked for the researcher's help.]

In the excerpt above, the participants were puzzled by the determinant formed and by the symbolic calculation, where the notion of matrix as a single quantity is applied to a full matrix. The commognitive conflict in the last dialogue arose from the differences in the metarules behind Cayley's discourse and the participants' discourse. Cayley was guided by a metarule that allowed for a dual interpretation of the notion of a matrix as sometimes being a single quantity (i.e. a number), and sometimes being a "system of numbers." Today, this would not be accepted because a matrix is not seen as a quantity.

One case of commognitive conflict was chosen for description in this text, whose manifestation occurred in an explicit way; specifically, it is possible to see divergent discourses between the discursants. However, this did not occur in all cases. There was a situation where the participants noticed that the discourse was governed by metarules that are no longer fashionable. In this case, no occurrence was found of conflicting narratives expressed as a result of a lack of understanding, or through disagreements in relation to the historical source. The participants knew that they were dealing with historical sources and therefore, that they would be confronted with very different ideas.

11.6 Closing Remarks

Comparing the results of the study with those of Kjeldsen and Petersen (2014), the findings about metarules and commognitive conflicts have confirmed the potential of historical sources to promote reflections on metarules. Furthermore, the analysis showed that it is possible to diagnose participants' metarules in teaching situations where historical sources are investigated. On the other hand, one difference is that this study did not detect participants' metarules that are at odds with the discourse of modern textbooks or with the discourse passed along by the educational system. However, the discourse of students engaged in activities of an exclusively mathematical nature was not analyzed in this study as it was done for the aforementioned researchers.

The task of detecting commognitive conflicts was not easy, as noted by Sfard (2008, p. 256). Initially the hope was to find conflicts in the common acceptance of a word, that is, the expectation was to find some instances of difficulty or moments of surprise or doubt in discussions. However, there was a case of commognitive conflict where the occurrence of conflicting narratives was not observed. The participants knew that they were dealing with historical sources and perceived that they were facing a discourse governed by metarules that are no longer operating.

Some methodological options were important to show the practices of Sylvester and Cayley; for instance, to make several excerpts available from the original sources, to maintain their original notations and to show how they defined their objects and argued their proofs. All these contributed to elicit reflections about metarules. As Kjeldsen (2011) noted, historical texts play the role of interlocutors, as discursants acting in accordance with specific metarules.

In addition, metadiscursive rules in the mathematical discourse become object-level rules in the historical discourse (Kjeldsen 2011); hence, what is implicit in one discourse (in this case, the mathematical discourse) can become explicit in another (historical) discourse. Therefore, all the support given to investigating the sources, with the summary presented in the first module and the researcher's oral presentations, played a role in bringing the metarules to the object level, i.e. making them explicit objects of discussion. Historical activities also played an important role in guiding the discussions toward the historical metarules, and in getting participants to explain their own metarules.

As the study by Kjeldsen and Blomhøj (2012) indicated, the analysis showed that the use of original sources guided by historical activities spurred not only the learning of metadiscursive rules, but also an opportunity to discuss object-level rules. Similarly to the conclusion reached by Kjeldsen and Petersen (2014), the study showed that students went through a learning situation that was not familiar to them. They had the opportunity to discuss the notion of matrix and related concepts in a non-operational way; that is, at a more conceptual level, outside of the pattern of applying readymade techniques to solve mathematical problems. The historical activities led the students to reflect upon what the matrix object is, and this was not a trivial task for them.

Reflecting on the findings and on the study as a whole, we have questioned whether it is appropriate to start a Linear Algebra course with the concept of a matrix as an object in itself. Historically, the notion of the matrix was the last to emerge; i.e. matrices were introduced after determinants, linear systems, linear transformations and quadratic forms. They arose first as a technique, before being constituted as a mathematical object. Sylvester's and Cayley's episodes showed that their introduction and development were motivated by the necessity of a representation in a table form. So one could argue that the concept of matrix should only be introduced when there is a need for representation in a matrix—for example, during the study of linear systems—as well as the introduction of the multiplication of matrices alongside the composition of linear transformations. In this way, students might make more sense of matrices and their operations and might learn them in a more problematized way.

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Part IV
Original Historical Sources in Teaching and Learning of and About Mathematics

Chapter 12
Liu Hui Shares His Views with Young Students

Vasiliki Tsiapou

Abstract This study reports on the integration of original sources from the history of Chinese mathematics in a class of Greek sixth-grade students. The aims were: to (1) support students' learning of mathematical concepts and processes, and (2) provide opportunities for students to develop adequate views about mathematics. Here, the focus is placed on the second aim and, in particular, on the design of the teaching sequences and the analysis of specific activities accompanied by dialogues from the actual teaching. I present relevant results from the post-questionnaire and, in the discussion, I include reasons for the integration of original sources in elementary school with references to benefits and obstacles from the intervention.

Keywords Original sources · Diachronic reading · Elementary students' mathematics-related beliefs · Sociocultural approach · Anchoring

12.1 Introduction

The study described here integrated original sources from the history of mathematics in an elementary school setting, an educational level less investigated, and responded to calls for more empirical studies that take into account mathematics, history and didactics (e.g., Siu and Tzanakis 2004). The basic assumption was that a detailed description of the design, implementation, and didactical material would provide insight on the transferability or the adaptation in other classrooms. The objectives for students were taken from the study of Tzanakis et al. (2000):

- For the learning of mathematics: to develop reasoning and proving abilities in order to facilitate the transition to the more theoretical geometry of middle school;

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- For developing adequate views about mathematics: to appreciate the value of explaining and justifying; to become aware of the impact of external factors on the evolution of mathematics.

Given that the focus here has been placed on the second objective, I briefly discuss students' views (beliefs), a topic of interest in the last decades. Schoenfeld (1985) defined beliefs as one's mathematical worldview, the perspective one takes to approach mathematics and mathematical tasks. Students' mathematics-related beliefs have been studied from different perspectives. Op't Eynde et al. (2002), for example, categorized beliefs regarding mathematics education, the self and the social context. Jankvist (2015) discussed a missing dimension: beliefs about mathematics as a discipline in order to embrace views about the nature of mathematics as a whole. Reviews of studies (e.g. Muis 2004) that have investigated students' beliefs have shown that many students believe that mathematics is an unchanging and, accordingly, an ahistorical subject; others believe that it is just a collection of rules and procedures. Ernest (1989) called the first views *Platonist* and the second ones *Instrumental*. However, original sources have the potential to promote a dynamic, problem-solving view of mathematics (Ernest 1989), in which mathematics is regarded a historically shaped and evolving discipline, a process of inquiry and a cultural product.

12.2 Theoretical Perspectives

12.2.1 Learning with Artifacts and Signs: The Case of Original Sources

Compatible with the dynamic view of mathematics are socio-cultural theories according to which learning is a communal event where the learner gives meaning to the world and reasons mathematically through the prism of his/her culture (Radford 2010). Meaning is a double-sided construct; on the one hand, it is subjective, linked to the individual's personal history and experience, and, on the other hand, it is a cultural construct endowed with the values and the theoretical content of the culture(s). While participating in social activities, in order to give meaning to these concepts and values, we resort to artifacts and signs[1] (language, systems of representation and also gestures) (Radford 2006). Both artifacts and signs embed the experience of cognitive activity, and styles of inquiry of the culture. The knowledge embedded in the artifacts was highlighted by Bartolini Bussi and Mariotti (2008). They identify *primary artifacts* as the physical (e.g. compass) and

[1]The use of the term *sign* is inspired by Peirce. "A sign is in a conjoint relation to the thing denoted and to the mind"… "The sign is related to its object only in consequence of a mental association, and depends upon a habit" (Hartshorne and Weiss 1933, as cited in Bartolini Bussi and Mariotti 2008, p. 779).

software tools, and *secondary artifacts* as the written texts. Original sources, under this perspective, are secondary artifacts. They have the potential to become *semiotic tools* (Bartolini Bussi and Mariotti 2008) as long as the voice of the author finds a space to interact with the students' voice and communicate mathematical and/or meta-mathematical meanings. The semiotic process starts when students produce signs and interpret (give meaning to) the signs of the others while interacting with the various artifacts and the social environment of the class. The teacher's role is to link the students' subjective meanings with the cultural meanings embedded in the original source-artifact.

12.2.2 Approach to History and Implications on Teaching

What approach to history can a teacher adopt when integrating original sources in order to develop dynamic views, and what are the implications on teaching? Jahnke (1994) and Jahnke et al. (2000) argue that the reading of an original source requires a hermeneutic (interpretive) effort, a constant change between the modern reader's views (synchronic) and those of the author of the source (diachronic). With the synchronic reading, students reflect on their own views on mathematics and mathematical activity, while, with the diachronic one, they become able to understand how the synchronous and the diachronous culture are intertwined. An idea connected with the reading of an original source is that "it may influence the students on their meta-cognitive level and contribute to their ability to reflect on mathematics" (Jahnke et al. 2000, p. 317). Methodologically, one way of framing such reflections is Jankvist's (2011) *anchoring* approach in which reflections on meta-mathematical ideas are anchored to related mathematical ideas. In this way "concrete historical examples which illustrate somewhat general features, may help students to learn about the historical development of mathematics, even though they may only be exposed to one historical case" (Jankvist and Kjeldsen 2011, pp. 850–851). Whilst anchoring is a framework for designing historical activities and studying the students' related meta-mathematical views, when one comes to the teaching process, they need to analyze how to approach the view of the author of the source. The teacher's role is critical, and Arcavi and Isoda (2007) proposed various hermeneutic tools: parsing of the text for local understanding, posing questions to adopt the writers' perspective, pasting the pieces for an understanding of the whole, etc.

12.3 The Original Sources and an Overview of the Intervention

(i) The first source is problem 1.32 of the Mathematical Canon *Jiu Zhang Suan Shu* (JZSS; 1st century BCE or CE). The Canon offered four methods for the area of a circular field, but, in the data, the ratio of the circumference to the diameter was taken as 3 (Lay-Yong and Tian-Se 1986).

(ii) The second source belongs to Liu Hui (3rd century CE), who commented on problem 1.32. Liu Hui proved the correctness of the first algorithm and estimated π as 3.1416. Liu Hui inscribed double-sided polygons commencing from a regular hexagon, and, employing a circle dissection technique, he reached the circle's area: ½ Circumference · ½ Diameter (Lay-Yong and Tian-Se 1986).

(iii) The third source is in fact an excerpt of the second source (Siu 1993). At the end of his proof, Liu Hui comments on the wrong ratio, highlighting both the impact of culture on mathematics and the value of explaining and justifying for trustful knowledge.

> However, those who transmit this method of calculation to the next generation never bother to examine it thoroughly but merely repeat what they learned from their predecessors, thus passing on the error. Without a clear explanation and definite justification, it is very difficult to separate truth from fallacy. (pp. 348–349)

(iv) The fourth source is also an excerpt from Liu Hui's preface of his commentary on JZSS in which he reveals pedagogical considerations for the reader (Siu 1993).

> […] If we elucidate by prose and illustrate by pictures, then we may be able to attain conciseness as well as comprehensiveness, clarity as well as rigor. (p. 355)

12.3.1 Participants and Settings

This intervention was implemented in the 2014–2015 school year. The participants were fourteen sixth-grade students of a public urban primary school in Thessaloniki, which served as a convenience sample because the school is the teacher-researcher's workspace. The intervention was held mainly during the project development hours and, occasionally, during mathematics and ICT class hours. In order to detect the students' views, I administered a pre-questionnaire. After the intervention, I administered a relevant post-questionnaire. Other data, for triangulation purposes, were audio recordings, worksheets, material produced by students and individual student journal writing.

12.3.2 An Overview of the Intervention

The intervention consisted of three parts:

1. Introduction: Using an indirect strategy in the form of presentation (Jahnke et al. 2000), students became familiar with the author's historical context. They also

found differences and similarities between the Roman Empire (and the following Byzantine one) and the Chinese Han Dynasty in educational and administrative issues. In addition, they created hypothetical stories, in order to reconstruct Liu Hui's life based on the information.
2. Analysis: We dealt with the mathematical content of the first and the second sources. I parsed them into small excerpts using the "guided reading" strategy, a combination of group work and whole-class exploration together with a series of tasks (Laubenbacher et al. 1994). The excerpts were not modified, but I replaced or modified certain methods of computation.
3. Synthesis: Connections were made with the Introduction and Analysis parts, with the third and fourth sources, and with information from secondary literature.

During the Analysis part, the students were involved in mathematical activities and reflected on their culture's values, mainly with the second source. Consequently, the reading could be seen as a synchronic one. In the Synthesis, the target was the interaction between the synchronous and diachronous cultures. In Sects. 12.4 and 12.5, I focus on the design and aspects of the intervention, in order to show how I addressed this issue in relation to the second objective.

12.4 Synthesis Part: Research Questions and Ways to Answer

Here, I formulate a research question (R.Q.) analyzed in two sub-questions. First, research question 1 was formulated as: How and under what conditions does the integration of original sources from the history of mathematics in the teaching contribute to the development of students' views regarding:

- R.Q.1a: the impact of cultural circumstances on the evolution of mathematics?
- R.Q.1b: the value of explaining and justifying based on mathematical arguments?

In order to answer the questions, I present the components and processes of the design of the activities in the Synthesis part, using Arcavi's and Isoda's (2007, p. 122) metaphor "pasting pieces together towards a global understanding of the whole" (Fig. 12.1). If we view the development of dynamic views about mathematics as a whole, the pieces (the components of the whole) would be Liu Hui's third and fourth sources, relevant secondary literature, and activities from the Introduction and Analysis parts. In order to paste the pieces, I anchored the students' mathematical knowledge and meanings developed in the Analysis part to (a) the historical knowledge of the Introduction part; and (b) the meta-mathematical meanings of the third and fourth sources and secondary literature that uncovered the author's reasoning and context. As for the conditions, I considered it important to (a) use hermeneutic tools (Arcavi and Isoda 2007), and (b) evoke norms (Op't

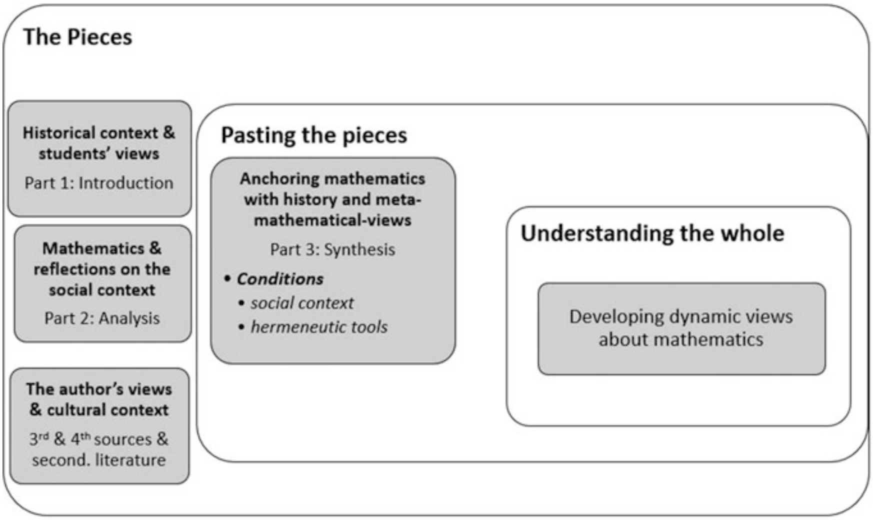

Fig. 12.1 Synthesis part: the design for developing students' meta-mathematical views

Eynde et al. 2002) established in the previous parts, such as justifying under shared reasoning processes. Such norms are values of the synchronous culture, but, at the same time, they are diachronic, since Liu Hui expressed relevant viewpoints in the sources. I conjectured that by prompting students to reflect on the values they would appreciate them as diachronic after reading the sources.

12.5 Activities for Developing Meta-mathematical Views

12.5.1 The Cards of Philosophy Activity

The activities during the Synthesis part mainly revolved around *The Cards of Philosophy* (a name given by a student, which I found interesting to adopt). I created the cards because I did not want to subject students directly to the information of secondary literature, as it would be a challenging task. The historical and philosophical context of the cards dates back to the Han Dynasty. The school of Confucianism influenced society and consequently the work of mathematicians, which was framed by the effort of preserving earlier wisdom rather than testing and surpassing it (Lloyd 1990). For the consolidation of a single world view, the scholars' role was crucial. On the contrary, during the Three Kingdom's period, which followed the collapse of the Han dynasty, political unrest and weakening of Confucianism enhanced uninhibited thinking (Siu 2008). "Mathematics entered its theoretical phase and importance was attached to proofs in their own right and not

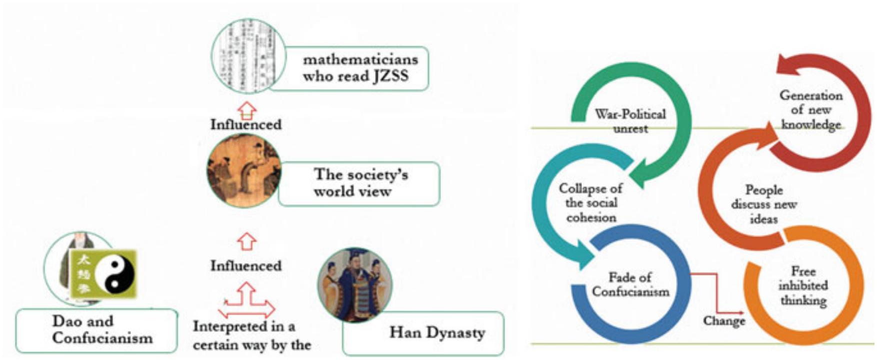

Fig. 12.2 Cultural impact on mathematics: on the left before, and on the right during, Liu Hui's era

only for practical purposes" (Martzloff 1997, p. 14). In this context, Liu Hui felt free to test the knowledge of the past (Fig. 12.2).

In order to show the evolution of students' subjective meanings, in Sect. 12.5.2, I present a dialogue from the Introduction part and, in Sect. 12.5.3, I compare it with a dialogue from the Synthesis part which took place within the Cards of Philosophy activity.

12.5.2 Examples from Teaching: Introduction Part

During the Introduction part, the students were asked to compare the presentation format of JZSS, '*problem—answer—methods*,' with the format in their mathematics textbook: '*problem—solution with explanation—answer*.' Some of the students pointed out the absence of reasoning behind the formulas in JZSS, but others rejected the idea offering arguments against it. The dialogue below shows how students justified their thinking.[2]

1. Te: What is missing from the ancient text?
2. M: The reasoning.
3. K: If the book says so, why do we care?
4. Y: The farmers and the excavators do not need explanations. They need to learn the methods for their daily duties.
5. M: But, in this way, they do not know why they do what they do.
6. Y: What is the use of knowing why? Farmers want to measure the field and then plant. They are not students.

[2]Within transcript excerpts, the abbreviation "Te" is used for the teacher (author) and single initials are used for individual students.

7. Te: Why should students know [the reasoning behind a formula]?
8. Y: Students have to know, especially those that will follow studies in mathematics and physics.

For K and Y, knowing the reasoning behind the formulas is unnecessary, since, in K's view (line 3), a mathematics book is the source of unquestionable truths (a Platonist view) and, in Y's view (lines 4 and 6), mathematics is procedures for direct implementation. Even if Y holds this view only as far as workers are concerned, when referring to students (line 6), he is either directed from a utility perspective (instrumental views) or he considers that knowing the reasoning behind formulas is a necessary evil. In the journal writing after the lesson, both students K and Y responded according to the aforesaid views when asked to position themselves in a hypothetical scenario: *Imagine you are a merchant in ancient China who reads the JZSS's methods in order to calculate the area of a field that you want to sell. Is there anything else that you would like to know that is missing from the text?* My role was rather discreet and I intervened in the discussion with two hermeneutic tools. With the first, I oriented students' thinking to a hidden parameter of the source in order to move the discussion towards the goal of the activity (line 1). In order to uncover the essence and the source of Y's claim, the 'why' question followed (line 7). In addition, the hypothetical scenario in the journal writing after teaching served as an extra tool for accessing the students' views. In Sect. 12.5.3, I present the design of an anchoring activity and a discussion that shows the evolution of students' views.

12.5.3 Synthesis Part: Designing Discussions for R.Q.1a

If we see the whole as the development of adequate views about the impact of culture on the evolution of mathematics (R.Q.1a), the pieces were: (a) the mathematical content of the first source; (b) Cards of Philosophy with information about the predominance of Confucianism and the transmission of knowledge. I put the information into the words of imaginary people of the past who would discuss with students in first-person discourse; and (c) slides from the introductory presentation (e.g. the technological orientation of Chinese science and the problems of JZSS). For illustrative purposes, I gathered the 'pieces' in Fig. 12.3. The 'pasting' process followed the path: (a) ↔ *source* ↔ (b ↔ c). The first connection '(a) ↔ *source*' was an anchoring between the mathematics of the Analysis part and the source. More analytically, when we returned to problem 1.32 of JZSS after the exploration of Liu Hui's proof, the students found that, in the data and in two methods, π was 3. They realized for the first time that a mathematical textbook, JZSS, contained an error. When they read Liu Hui's comment on the incorrect ratio (third source), they already knew what he was talking about. Another connection between the excerpt and the historical proof in which Liu Hui said: "The ratio of the diameter to the perimeter [of the hexagon] is 1 to 3" and several lines below "if the ratio [i.e. 3] is

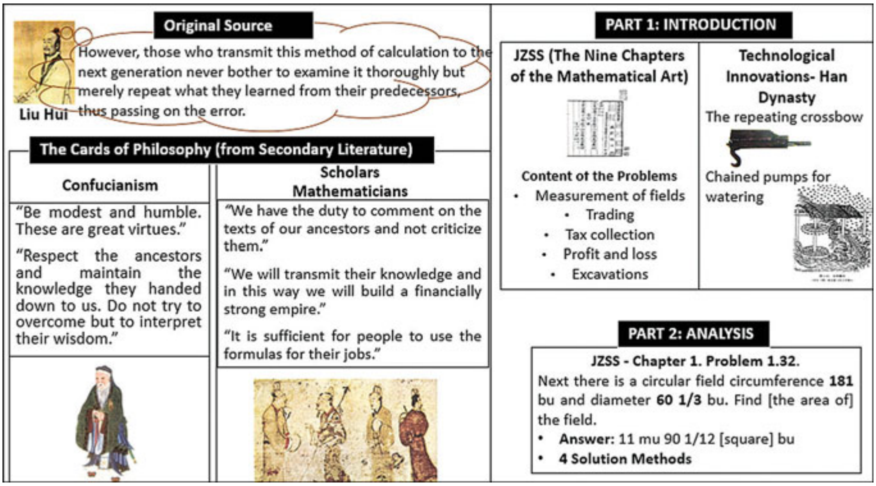

Fig. 12.3 Synthesis part: sources for appreciating the impact of culture on mathematics

used to compute the length of an arc, the result obtained is not the arc but the chord" (Lay-Yong and Tian-Se 1986, p. 336). During the Analysis part the excerpts were analyzed within a synchronic reading but the students could not explain Liu Hui's insistence on returning to the same issue; anchoring their knowledge with the meta-mathematical excerpt helped them understand why Liu Hui kept telling the aforesaid phrases. However, the rest of his words "merely repeat what they learned from their predecessors, thus passing on the error" did not make any sense. The first questions I raised were: *Who were responsible for transmitting the wrong ratio? What do we know about Liu Hui's ancestors?* From a hermeneutic approach, the questions are equivalent to the question: *What are the hidden assumptions in the source?* For students to be able to deal with the questions, the connection, *source* $\leftrightarrow$ (b $\leftrightarrow$ c), should follow. The reason for such a connection rested on the assumption that, by evoking (c), students would be able to interpret the scholars' views in (b). All of the above connections were expected to help students interpret the author's words, and to place their meaning in the context of the Han Dynasty.

12.5.3.1 An Example from Teaching

The students had already found the connection between the cards *Confucianism* and *Scholars Mathematicians* (Fig. 12.3). The classroom discussion revolved around the meanings embedded in the latter:

9. Te: To whom are the mathematicians referring? Which people?
10. P: To the other civil servants that gather taxes or the farmers and the excavators.

11. M: I do not get it; how can you calculate without understanding what you are doing?
12. P: We can't say if this right or wrong, but what mattered was to use mathematics for the welfare of the state.
13. Te: Do not forget that people of that time were influenced by the idea that the ancestors were holding the truth. So, what P said right now is important. It is not a matter of right or wrong. It is a matter of what counts as important or not, depending on the era you live in.
14. Y: I guess this worked and people did not ask for the reasons behind the math formulas.
15. Te: Should they ask for them?
16. Y: Liu Hui found that π was wrong. If they had searched… but how could they have searched? Did they know the mathematics he knew?
17. M: No, not the common people, the scholars!
18. Te: I wonder what made Liu Hui so different from other scholars that he searched for the reasoning behind the formulas of JZSS and, because of this, he corrected the wrong ratio. I mean, he was a scholar himself. Why did he start wondering about these things?

Compared to the previous dialogue, M is consistent with her view (line 11) that reasoning is significant, although the way she expressed her dissatisfaction with the scholars' views was like seeing the past through modern lenses. On the contrary, P justified the scholars by connecting their views with prior historical knowledge of China's technological orientation (lines 10 and 12), thus she made an effort to understand the past in its own terms. I tried to highlight P's contribution for a diachronic reading of the source (line 13) and this probably affected Y's remarks, who acknowledged the societal influence on peoples' thinking (line 14 and line 16). I observed a change in Y's view, compared to the previous dialogue, when he was asked to justify his claim (line 15). Y resorted to Liu Hui's contribution, the value of π, in order to stress the necessity of searching the reasoning behind mathematical concepts. We could say that, at this point, Y offered an anchoring argument, since he supported his meta-mathematical claims about people's reasoning by resorting to the mathematical content of the first source. This anchoring helped him to abandon the idea that common people do not need to know the reasoning behind formulas. Moreover, and based on his anchored argument, Y moved one step ahead and delved into the ancient era, by wondering how people could have searched for understanding (line 16). With his utterance, he added a new parameter (also connected with the Introduction part)—that of educational constraints—in order to express his sympathy for common people, who could not access knowledge. M's response (line 17) to Y's remark was significant for two reasons. First, her argumentation finally followed the other classmates in order to understand the ancient's reasoning. Secondly, she added new elements when she highlighted another aspect of the card and referred to the responsibility of scholars, and not common people, to test the knowledge, thus she implicitly accused scholars for the negative effects on mathematics. And, without realizing it, M went hand-in-hand with Liu Hui's intentions. In the end, I raised a question (line 18) that, from a hermeneutical approach, could be the following: *What is the reason that Liu Hui had different lines of reasoning from those of his predecessors?* It was a rhetorical question which aimed at introducing students to new historical events that positively influenced the development of mathematical thinking in China.

The discussion touched upon two meta-mathematical views. The first one was the necessity of knowing the reasoning behind concepts. Compared to the previous dialogue, all the participants embraced the view this time. This could be seen as the evolution of their subjective meanings in order to embrace those of the synchronous mathematics culture, resulting from the mathematical knowledge of the first two sources, and from the 'why'-oriented classroom's context. The second view was the main issue of the discussion, the cultural forces that prohibited the feeling of this necessity; however, such an aspect requires a diachronic reading. Due to my orienting questions and the social interaction between the students, the $(a) \leftrightarrow$ *source* anchoring (between mathematics and meta-mathematical ideas) became the springboard for the connection *source* $\leftrightarrow (b \leftrightarrow c)$, that is, between the original source and secondary literature. These conditions in turn enhanced a cultural interaction; the students' synchronous meanings widened for them to understand the past in its own terms and, as a result, they embraced the meanings of the author of the source.

12.5.4 Synthesis Part: Designing Discussions for R.Q.1b

Some of the meta-mathematical activities during the Synthesis part centered on R.Q.1b. Seen as a whole the development of the view that mathematical justifications are significant for acquiring understanding, I integrated the fourth source in which Liu Hui appraises the role of theoretical arguments and representations. I used as pieces (a) Cards of Philosophy that included images of the students' productions, and (b) real productions (constructions, worksheets, journal writing, etc.). These productions were the secondary artifacts carrying the students' mathematical and meta-mathematical signs and subjective meanings, rooted in activities with primary artifacts. The pasting process followed the path: *source* $\leftrightarrow a + b$, which means: reading of the source, resorting to the images and real productions in order to recall the mathematical content and reflect on the proving processes, returning to the source in order to understand its meaning. The reflections on the proving processes aimed at detecting the meanings they had constructed for the value of justifying mathematically, in line with the classroom's context (the intended meaning of the synchronous culture). Returning to the source aimed at a diachronic reading and involved the anchoring between the students' meta-mathematical meanings and those of the source. The background of such an activity from the Analysis part and the discussions from the Synthesis part are presented in Sects. 12.5.4.1 and 12.5.4.2.

12.5.4.1 The Design of an Activity and the Mathematical Background

The card of the activity was named *Liu Hui-Internal motive: To Explain and Justify Mathematical Concepts*. It contained the fourth original source and photos from students' activities (Fig. 12.4). My underlying message was: *What Liu Hui says here can be explained with our mathematical activities*. One such photo was from an activity of the Analysis part and concerned Liu Hui's excerpt: "If we multiply the radius by one side of the hexagon and then by 3, the product is the area of an inscribed dodecagon." During that part, students had inscribed a dodecagon based on a hexagon and reconfigured it to a rectangle; then, the task was to calculate its actual area. In order to find the dimensions of the rectangle, many students used measurements. Measurements would not lead to Liu Hui's generalized formula $Area\ of\ dodecagon = 3s_6 \times \frac{1}{2}d$, ($s_6$ = side of the hexagon and d = diameter). Theoretical arguments should enter the scene in order to relate the dimensions of the rectangle with the hexagon's sides and the radius. We reflected on both processes, in order to evaluate their efficacy and, in the Synthesis part, we returned in order to discuss in connection with the source.

12.5.4.2 Example from Teaching

The activity began when I asked students to recall the construction of the dodecagon and its reconfiguration to a rectangle (Fig. 12.4). However, the dialogue concerns the next step where the focus was placed on the proving methods.

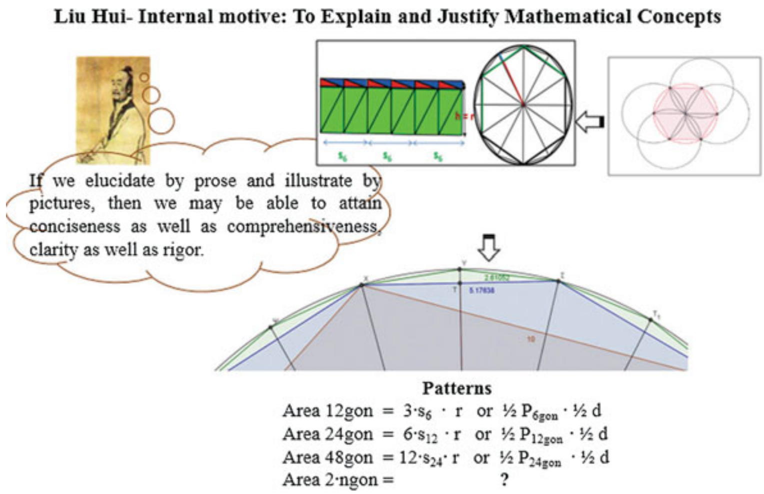

Fig. 12.4 Synthesis part: a card for appreciating the value of justifying mathematically

1. Te: Let's remember now how we had found the base of the rectangle. I remind you that, when we were taking this lesson, some of you measured the base while others tried to find relations with the circle and the hexagon.
2. M: No need to measure. The base is three hexagon sides. Every side is the radius.
3. Te: M said the hexagon's side equals the radius. How do we know that? Can someone explain M's thinking?
4. K: When we constructed the hexagon in the circle. Here… (*Points at the inscribed hexagon hanging on the wall, a students' construction in GeoGebra*). Every side was a different radius of a circle around the first one and it was 10 cm (*rotating hands imitating the compass movements*). [I asked for a better description of the construction process and K repeated with the help of other students.]

M reflected on the proving methods (line 2) with a deductive argument, a mathematical sign. The meaning of the sign was communicated by another student, K (line 4), who used body gestures (rotating hands and arms) and verbal descriptions. From a semiotic perspective, we could say that K has internalized cultural knowledge via the use of primary artifacts (the compass) and social interaction (manipulation of the artifact for the construction of the hexagon, the relation circle radius—hexagon side, and the description of the construction). In order to communicate with the rest of the class, K oriented the internalized knowledge outwards with various semiotic means, though informal in nature. Indeed, K's mental compass movements were linked to the concrete artifact use, and the verbal descriptions were lacking adequate syntax and terminology. Nevertheless, the communication of the meaning was substantial. And the benchmark for K to communicate M's meaning was the students' production hanging on the wall, the inscribed hexagon. At that point, it was not just a representation of the mathematical object of the hexagon; it became a *semiotic tool* the embedded signs of which were rooted in students' activities with primary artifacts and social interaction. In order to detect how students had internalized measurements and property relations as alternative proving processes, I asked them to compare the efficacy of both.

5. Te: So, we used our previous knowledge of the relation between the side of the hexagon and the radius in order to find the length of the rectangle's base. But think of something else know. Why not measure instead? Do you remember our discussions about this matter?
6. *No student responses*
7. Te: Would we have ended up with the formula for the area of the 12-gon if we had just measured the dimensions? This formula led us to the circle area formula in the end (*showing the consecutive double-sided polygons after the dodecagon and their formulas in the card*).
8. K: No. If we had measured, we would have just found a result.
9. Te: And something else. Would we be able to explain why the side equals 10 cm, and not 11 or 12?

10. M: I remember we said that again, when we wanted to prove that the triangles inside the dodecagon were isosceles (*meaning the triangles formed by the diagonals that pass from the center of the polygon*). K.'s team had measured, but we had found it in another way.
11. Te: Do you remember how?
12. M: The long sides were also radii. And the others had measured and found the same [result], but they couldn't explain why they found this result and not another.

My examples for the efficacy of the proving methods (line 7 and 9) were tied to the mathematics of the second source, and the specific numerical values. Based on the examples, students became able to recall the discussions on the proving processes. This was evident in K's comment, "If we had measured, we would have just found a result," an implicit reference to a conclusion we had arrived to during those discussions (line 8). Most importantly, M recalled another relevant example during which two groups were arguing on the efficiency of the two methods (line 10, 12). We could say that both students offered anchored arguments that are the discourse for mathematically-underpinned, meta-mathematical discussions which in turn revealed that students appreciated the value of proving-using relations. In order to provide a synthesis of the above ideas, I tried to anchor the students' reflections on the proving processes to Liu Hui's argument that theoretical arguments and figural representations promote understanding.

13. Te: In general, Liu Hui's phrase 'elucidate by prose' means to analyze my reasoning by using relations between figures or my previous knowledge, or describing a construction with adequate terminology, as K did before. In this way, the others understand what I am talking about and where I base my reasoning.
14. Te: Liu Hui also said "illustrate by pictures." How did we use shapes?
15. St: We cut shapes, we constructed on the worksheet, on the board, etc.
16. Te: Right. Do you think that we would have been able to understand, for example, the meaning of Liu Hui's excerpt in which he describes how to find the area of the dodecagon, if we had not used figures?
17. F: No way. Even now, when I read it, I cannot understand. Well… when I see the images again, I remember (*showing the figures in the card*).
18. Te: So, geometric figures help us clarify what the words are describing. And what tools did we use? Liu Hui does not explicitly refer to tools here, but we used them for the constructions.
19. Class: Rulers, compasses, gnomons, GeoGebra, our brain.
20. Te: So, do you agree with Liu Hui that analyzing with words and explaining with figures are important factors for making mathematical ideas understandable?
21. P: Yes! When I claim something, it is important to *do* it first and then confirm!

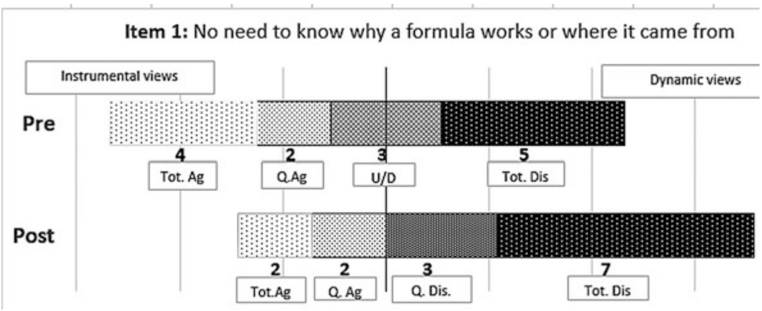

Fig. 12.5 Item 1: comparative results between Pre and Post

Besides showing the difficulties he encountered with the mathematics, F's utterance (line 17) is also a reflection on his meta-cognitive strategies, and an implicit agreement with the ancient author concerning the role of representations. In the end, another contribution was important. P's spontaneous meta-mathematical sign (line 21) could be seen as the external expression of the way she internalized the meaning of Liu Hui's entire excerpt (line 21). By saying "claim," "do" and "confirm" in the specific order, P informally captured the essence of argumentation. And, by emphasizing the simplistic, though active, verb "do" it was as if she wanted to stress the action of the learner, who, following Liu Hui's advice, he/she "elucidates by prose" and "illustrates by pictures." The entire dialogue showed that, to a certain extent, the students gave meaning to and embraced Liu Hui's pedagogical advice and appreciated that mathematical justifications are diachronic values, too.

12.6 Findings from the Questionnaires and Interpretations

I now present an example of the Likert-scale items with which I tried to detect students' views about the value of justifying connected with the R.Q.1b.[3]

Item 1: *It's enough to know a geometric formula and apply it to a problem. No need to know where it came from or why it works.*

In the Post (Fig. 12.5), I observed a positive shift towards the significance of knowing the reasoning behind a formula. The results in the Post are explained with the activities in the Analysis part, where the focus was placed on the proving

[3]The abbreviations for the Likert scaled used are defined as follows: Tot. Ag = Totally Agree; Q. Ag = Quite Agree; U/D = Undecided; Q. Dis = Quite Disagree; and Tot. Dis = Totally Disagree.

process of the circle area formula, and with the meta-mathematical discussions of the value of justifying in the Synthesis part.

Item 2 included factors that foster the development of mathematics. Especially items 2.1 and 2.3 are connected with R.Q.1a and R.Q.1b, respectively.

Item 2: *Mathematics has been developed because…*:

2.1. People insist on accuracy; for example, they want to correct wrong ideas.
2.2. People want to draw conclusions that apply generally and not only in specific problems.
2.3. People tend to prove what they say; that is, they justify their thinking based on a theory and on various representations such as symbols, shapes, etc.
2.4. People are curious to pose questions and then look for ways to answer.

Compared to the Pre-questionnaire, in the Post (Fig. 12.6), many students agreed with these dynamic views. Almost everyone agreed with the view of item 2.1, which also touches upon external factors (cultural reasons for the wrong ratio). The view expressed in item 2.3 also had a positive change. However, a remarkable

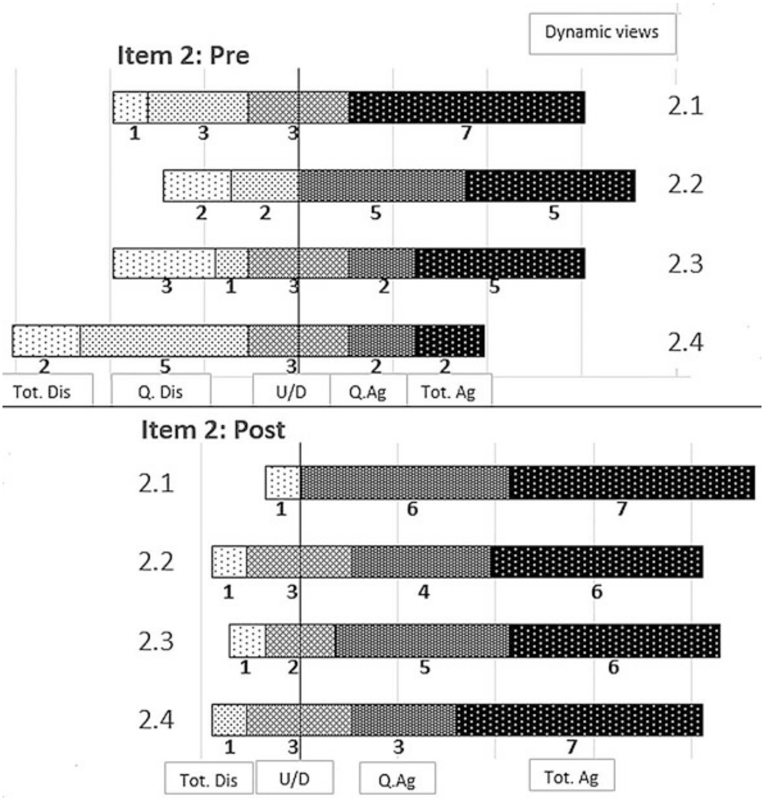

Fig. 12.6 Item 2: comparative results between Pre and Post

change was observed in the view of item 2.4. In the Pre, many students remarked that the idea of posing their own mathematical questions and pursuing their answer was strange. Thus, it came as no surprise that only four students aligned themselves with the idea. In the Post, all but one agreed. The connection with the intervention may rest on discussions where students passionately expressed their sympathy to Liu Hui for his curiosity to test the knowledge of the past and write his own proofs. This may also be attributed to the fact that they were asked to make hypotheses in order to predict Liu Hui's next steps during the proof exploration.

Item 3 was an open question with which I tried to detect students' views on the evolutionary nature of the discipline.

Item 3: *Do you believe that the mathematics in your textbook has always been the same?*

Due to the initial hypothesis, I was expecting that more students would give affirmative answers in the Pre (Fig. 12.7). Yet, half the students simply wrote, "I do not know," a sign that they never had such a dilemma. It is interesting, however, that four students saw mathematics as an evolving subject of human creation. One of the three students who said that mathematics has always been the same justified it by stressing the logical nature of mathematics. It is true, of course, that mathematics has a logical structure, and we may call the processes for justifying, stable ideas. However, it is unlikely that the young student has elaborated the stability of the mathematical ideas in terms of dynamic processes. In connection with his answer "mathematics has always been the same," he may be oriented by different views: either that mathematics is a logical discipline the stable ideas of which are closed sets of rules (an instrumental view) or it is infallible with stable eternal ideas, and, thus, it has always been the same (a Platonist view). In the Post (Fig. 12.7),

	ANSWER (STUDENTS)		EXPLANATION	CODING
PRE	YES	1	I believe it has always been the same	No justification
		1	Mathematics needs logic and this requires stable ideas	Platonist or instrumental view
		1		No explanation
	I DO NOT KNOW	7		
	NO	4	Through the years people's mind evolved and found new ways for solving a problem; It was gradually evolving; People new mathematics since the ancient times and probably it has changed	Evolving discipline
POST	NO	7	We invent new knowledge/ it evolves /is upgraded/ it has changed and we come up with new ideas/ things change/ people did not think of everything at once	Evolving discipline
		3	Over the centuries people correct errors; Ancient people believed that the ratio 3/1 is for the circle, but Liu Hui found that it was 3.14	Evolving discipline Intervention based argument
		2	Today we have different knowledge; People were thinking differently in the old times	Disconnect past from present
		2		No explanation

Fig. 12.7 Item 3: comparative results between Pre and Post

	ANSWER (STUDENTS)		EXPLANATION	CODING
PRE	YES	4	All these theories are related; I think, yes; I think they need it; I cannot explain	No Justification
	I DO NOT KNOW	3		
	NO	7	They are not related; Mathematics is for solving and for numbers; With philosophy we write essays; Mathematics is related to astronomy, geography and physics;	Disconnected areas
POST	YES	2	The Han dynasty adopted some of Confucius concepts and Yin-Yang's philosophy for ruling people and then mathematics. Today mathematics is influenced by technology	Influence: reference to the present or the to past
		7	The ideology of every era influences mathematics; these ideas are transmitted to mathematics; people' s ideas influence mathematics; In Liu Hui's era people were influenced by Dao and Ying Yang, but in our era I do not know which ideas we have	Diachronic influence
		3		No explanation

Fig. 12.8 Item 4: comparative results between Pre and Post

everyone answered that mathematics has changed. Half the students, without justifying it, stressed the evolutionary aspect in terms of 'change,' 'upgrade,' 'evolution,' 'invention.' Five students made either direct references to the intervention or, based on the intervention, gave general remarks as evidence, such as the impact of technology or the correction of mistakes, e.g. "over the centuries, people correct mistakes."

In Item 4, students were asked to position themselves in the connection between cultural factors and mathematics.

Item 4: *Do you think that mathematics is influenced by philosophy, namely people's ideas about the world?*

In the Pre (Fig. 12.8), the four students who saw a connection between mathematics and people's ideas about the world could not explain it. The rest of them could not find any relation and justified it by linking mathematics to other scientific domains or calculations. In the Post (Fig. 12.8), almost all students acknowledged the relation. Many saw a diachronic relation using expressions such as "depending on the era." They did not explicitly mention the intervention, but the words "ideology," "influence" and "transmitted" were connected with the terminology used in the intervention. Only three students gave concrete examples from the intervention.

12.7 Discussion

In this chapter, I discussed the integration of two excerpts from the history of ancient Chinese mathematics, for sixth-grade students to appreciate: (a) the impact of culture on mathematics, and (b) the value of justifying mathematically. Framed by cultural-semiotic theories of learning and a hermeneutic approach to history, I consider the development of these views as the merge of the students' subjective meanings with those embedded in the source, a process that requires a constant change between a synchronic and a diachronic reading. Regarding my objectives, I presented the design of two activities. The working material came from various sources included in the Cards of Philosophy, the meanings of which were related to the excerpts. Moreover, the organization of the material proved to be helpful in orienting students' thinking in significant parameters of the excerpts. Methodologically, the anchoring strategy facilitated the diachronic reading when the hidden assumptions of the excerpts were highlighted by mathematical concepts and processes of the related mathematical sources. However, for the realization of the cultural interaction between the author and the students, important conditions were provided by my close guidance and orienting questions, and the classroom's context, which facilitated the exchange of views.

Regarding the first objective, two points helped students to reflect on the hidden meanings of the mathematical sources. The first point was when students connected the excerpt with problem 1.32, and perceived Liu Hui's motive to comment on the specific problem and write his proof. The second point was between the excerpt and the proof where Liu Hui made recurrent references for what the relation 3:1 stands. The reason for his persistence in this issue was hidden during the synchronic reading, where the mathematical content had priority, but anchoring the mathematical experience to the author's words, students read the proof in a reflective way. However, the forces responsible for the error were hidden in the excerpt, so I oriented students to the material of the cards. In the discussion that followed, the above revelations fueled the emergence of the value of knowing the reasoning behind mathematical calculations. Compared to the dialogue before the intervention, the view was embraced by all the participants. A diachronic reading was performed during the dialogue and the students approached the author's view on the impact of culture on the development of the mathematics of his time. The factors that facilitated the reading were, on the one hand, the evocative power of the material and, on the other hand, the exchange of views between the students, who tried to understand the past in its own terms. Regarding the second objective, the connection with the mathematical discourse was more evident. The students became able to evoke their meta-mathematical views on the role and the efficacy of the various proving processes using various multimodal signs and arguments. From a semiotic perspective, the images of the card became semiotic tools due to their power to evoke the students' meanings, which in turn showed that they had appreciated the value of proving using relations, a value of the synchronous culture. For a diachronic reading, a link should follow between the students' reflections and Liu Hui's basic components of argumentation (theoretical arguments and figural representations). My approach was to anchor the students' contributions during the dialogue and from previous mathematical activities to these components. Although the students' argumentation was lacking analytical thinking, to a certain extent they reached an agreement with the author's views, a sign that they started appreciating the value of justifying mathematically, as a diachronic value.

The comparative results of the questionnaires showed that, in the Pre-questionnaire, many students held instrumental and Platonic views. In addition, many indicated that they were "undecided" or answered "I do not know." The latter may confirm Lester (2002), who stated that students may not be really aware of their views about mathematics. In the Post-questionnaire, these answers were almost eliminated, and a significant change towards dynamic views was observed. This was evident in items 1 (the value of knowing the reasoning behind a formula) and 2 (reasons for the development of mathematics), which are both connected with R.Q.1a and R.Q.1b. The change can be attributed to the activities concerning the proof and the culture of 'why,' and to those with the Cards of Philosophy. Regarding item 4, connected to R.Q.1, many students seem to have built a rather generalized view of the impact of culture on mathematics, although the excerpt used the Han Dynasty as a reference. Item 3 is connected to R.Q.1a. The wrong ratio was the only concrete example they knew, but it affected their views. While in the Pre, half the students said that they did not know whether mathematics changes or not, in the open-ended items of the Post, they justified with concrete examples from the original sources or with general arguments based on the intervention, and using terminology that revealed the evolutionary nature of mathematics.

12.7.1 Integrating Original Sources in Elementary School: Is It Feasible?

Siu (2006) listed objections to integrating the history of mathematics and I will try to discuss one of them with respect to integrating history in order to develop students' views. An objection is that students do not have enough general knowledge about culture to appreciate it. Jahnke (1994) stated that, when one is reading an original source, they must be able to feel the intellectual environment of the author, and this presupposes that the learner has a certain foundation. Due to this reason, I did not present Liu Hui's socio-historical background in the introductory activities alone (see paragraph 1 in Sect. 12.3.2 of this chapter), but I tried to build on students' historical knowledge (Jahnke et al. 2000), for them to appreciate that the Chinese culture of that time had specific similarities with their own cultural background, thus it was not something strange or bizarre. Of course, this cannot always be the case. I was lucky that Liu lived in an era with a historical counterpart that my students knew, yet other connections may be possible for other researchers. While the introductory part enabled somewhat straightforward information, the connections with the various historical and philosophical issues in the Synthesis part were a different case. One might say that elementary school students are not mature enough to 'feel' the intellectual context. I will discuss this issue from the perspectives of the original source, the teaching-learning process, and the effect on students, because I think that these three factors, and not the age of the students alone, should be taken into account before deciding whether such investigations could take place in elementary school. The appropriateness of the source has been discussed by Jankvist and Kjeldsen (2011), who claimed:

> By identifying concrete historical examples illustrating somewhat general features, students will be able to learn something about the historical development of mathematics in general, even though they may only be exposed to a single, but exemplary, case. (pp. 850–851)

I believe that the sources of the study (the excerpts) have general features related to concrete examples (the proof and problem 1.32). Liu Hui exposed his reasoning in his proof step by step, which not all the ancient authors were willing to do. I found that, with certain changes, students would be able to follow. In addition, the meta-mathematical meanings of the excerpts could be decoded with the help of secondary literature (Siu 1993). Thus, with respect to my two aims, the sources seemed appropriate. But were they appropriate for elementary school students?

And here enters the teaching-learning process. There are many parameters, but I will mention only those I discussed in this chapter. I believe that the reading of the source in a diachronic way fits the anchoring approach in elementary school, because the exemplary case substituted the students' limited background and decoded the general. Moreover, the material organized in the cards and adapted to the level of the students was helpful in enhancing discussions. Did this process have an impact on students? In other words, "… we need to know more whether this is really the case and, if so, to what degree" (Jahnke et al. 2000, p. 317). The dialogues and the questionnaire have shown a positive impact on students' views. But, if we observe the results, the positive change in many cases was from the "I do not know" or "Undecided" responses. Students are not always aware of their views (Lester 2002), and this may especially apply to elementary-school students. Based on the results, we may say that, for such students, a single historical case can foster them to gradually develop adequate views or at least to wonder about their views.

"To what degree?" Jahnke et al. (2000, p. 317) asked. I would add: … are elementary students capable of philosophical investigations in order to reflect and develop adequate views? To respond, I will provide some examples. The dialogue concerning the cultural forces that influenced Han's mathematics showed that many sixth-grade students were able to connect ideas from different sources in order to make hidden meanings apparent. Another example is item 2.4: *People are curious to pose questions and then look for ways to answer*, which had the greatest positive change. It was not directly connected with any objective, but instead with discussions about the change in Liu Hui's era. Many students passionately engaged in this discussion, comparing the Han period with the one of Liu Hui, and appraised his curiosity to test the knowledge and write his own proofs. Did all of the students discuss in this way? No. They were not all able to delve into such issues, but many of them, and despite their age, were capable of reasoning more deeply. Sometimes, such discussions influenced other students who usually remained silent. Nevertheless, reasoning does not come hand in hand with written language. Few students gave detailed descriptions of their reasoning to the open items.

I am fairly certain that Liu Hui became a teacher and shared his views with my students, but I use the word 'share' because this does not mean that his views were conceived by all students in the same way, since the meaning of cultural knowledge is reformed depending on the meanings we attach to the signs of culture. And here, a question is hidden: Is it possible that my students have developed new inadequate beliefs? For example, the connection of the wrong ratio and Confucianism must have played an important role for the change of views, but, on the other hand, one might say that a Platonic view may be hidden behind the phrase "over the centuries, people correct errors" (Fig. 12.7). It is possible that such a view may have emerged in connection with π, as something real and eternal? However, another interpretation could be that the student embraced the idea that mathematics in general has undergone changes, and he generalized by evoking the only concrete example he had in mind, in order to justify his negative answer to the question, *Has mathematics always been the same?* Inadequate beliefs may have arisen. Surely, the intervention could not have addressed all students' beliefs, but, once the issue was raised, it should be taken into account in the design of another project.

12.7.2 Conclusions

Is it, after all, feasible to integrate the history of mathematics in elementary school? I think it is. An appropriate source in connection with a design that promotes a diachronic reading and takes into account the limited background of elementary school students could make an intervention feasible, which, as the results showed, may influence the emergence of students' dynamic views. Obstacles do exist, such as the unwillingness of some students to participate in discussions or to express their views, especially in writing. These issues can be addressed with a more careful design and attractive activities. Moreover, new inadequate beliefs may arise that need to be analyzed in the design. But, on the other hand, elementary school students have not shaped persistent inadequate views, thus the sooner history is integrated in mathematics the more effectively teachers will shape these views. In addition, the students can show maturity and engage in meta-mathematical discussions and reflections, despite their young age. They are able, to some extent, to delve into relations and make conjectures and interpretations on historical texts. Therefore, with the inclusion of history "our students will no longer find themselves in front of an impenetrable alienating discourse, but rather will grow up as subjects of mathematics, as critical cultural subjects" (Radford 2014, p. 106).

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Chapter 13
Experimentation on the Effects of Mathematical Diversity

Using Ancient Cuneiform Mathematics on Conceptual and Nature of Sciences Aspects

Charlotte de Varent

Abstract We examine, with in-depth teaching recordings and interviews, how tenth grade (15–16-year-old) students react when confronted with an ancient cuneiform clay tablet. The question is whether mathematical diversity can produce new questions (to be further used by teachers) linked to area and measure concepts. We observed conceptual changes with regard to mathematics, but it was difficult for students to make them explicit. In terms of "nature of science" aspects, we were able to document a change in debate content, and we also formulated some precautions. We provide a methodological reflection. We are attentive to the consequences of historical constraints on making links between ancient and current mathematics.

Keywords History of sciences · Units of measurement · Area
Mesopotamia · Cuneiform tablets · Interdisciplinary

13.1 Introduction

13.1.1 Framework and Purpose

The mathematical Sciences in the Ancient World project (SAW) has a main goal: to understand the ingenuity of the ancient mathematical thinking system, rather than project our own understanding and our own mathematical habits on ancient procedures. Seeking for a "transposition" of this current way of making history could be interesting on at least two levels: using a type of history that is connected to new research trends and using various mathematical solutions from ancient sources to

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question today's mathematical methods through diversity. Using the history of sciences in the classroom can take multiple forms, as described in Jankvist (2009). History can be seen as a tool or a goal in various ways; it can be oriented towards learning mathematics, working on the nature of science or on motivation and affectivity toward mathematics, seeing mathematics as a cultural activity, or providing new perspectives to teachers. We decided to work with a history of mathematics researcher accustomed to teach history "as a goal" and determine whether there would be effects on students from a mathematical perspective ("history as a tool" perspective). This was our research team's interpretation of the SAW context (our team was composed of history of sciences and science education researchers). In this regard, we placed ourselves in a type of "symmetrical" interdisciplinary approach; we tried to take both disciplines' constraints into account. These difficulties in such a balance had already been expressed, for instance by Fried (2008):

> The question at hand is whether the incorporation of history of mathematics in mathematics education is unproblematic in principle. […] the history of mathematics and mathematics education are disciplines, each with its own aims and its own conception of the subject. […] For mathematics education—at least as it is usually conceived (and this qualification is not trivial)—aims towards modern mathematics, but treats mathematics as it is conceived today as if it were mathematics tout court; […] In a way, this is the projection of the position of working mathematicians or working scientists who, to use Kuhn's famous terms, must, in a normal period, work within a paradigm, a set of concepts, procedures, and approaches fixed, as if eternal, in textbooks. […] Historians of mathematics are like anthropologists who study mathematical cultures very different from our own; at work, historians must consider mathematics as ever changing and having no eternal, fixed, reference. (pp. 2–3)

During one "history as a goal" session led by our historian, we decided to observe the possible conceptual effects of the mathematical differences born by an ancient cuneiform tablet. We also tried to describe some "nature of mathematics" and "nature of history" aspects by comparing two groups of students: a history of sciences group (HSG), which had followed our history sessions, and a control group, which had not.

13.1.2 Didactical Basis

The expectations observable a priori (Artigue 1992) from a conceptual point of view, on the one hand are based on a synthesis of didactic works on area and measure. On the other hand, an epistemological analysis of historical texts and an analysis of fifth-grade textbooks ensued from this analysis. Part of this work from our dissertation is presented in an article (de Varent 2015), though it is not possible to provide the full details here.

The following assumptions have been made thanks to this multi-dimensional analysis: the pupil's conception of area (Perrin-Glorian 1990) must go through a surface/area/number distinction (area can be expressed by different numbers depending on the chosen standard). We worked on how this previous work must be accompanied by a conceptualization of the units of measurement; this is linked to the underlying metrological system that creates implicit shortcuts, depending on how it was built. This distinction passes, among other things, by working on the unit of measurement concept in various "representation registers" (Duval 1993). For instance, fifth-grade textbooks present a "discovery activity" (using a grid to count tiles with 1 cm sides inside given squares or rectangles). We argue that linking this "discovery activity" (geometrical register) to a formula (arithmetical register) will be difficult because of unguided conceptual change in the unit of measurement status.

First, it is a square (with a given shape) with a given associated number, 1. It is noticeable that surface/area/number distinction does not seem to apply to the teaching of the unit of measurement. However, in this geometrical register, it still has the function of a standard (something used to measure by copying and shifting it as many times as necessary). In the formula, it is as if the standard has become part of a multiplication ($\text{cm} \times \text{cm} = \text{cm}^2$, which has its advantages, as Chevallard (2001) explains), when it is not completely erased of the action and added at the end (this is what we call the "literary role" of the unit). This is observable when students try to make up geometrical representations for "$\text{cm} \times \text{cm} = \text{cm}^2$." They can say, for example, "cm *by* cm creates a square," showing an "L" with their hands, or they can give an infinitesimal idea of adding "centimeter times a centimeter" to fill up an entire square. In any event, in the formula context, there is no more surface nor area nor number linked to the unit. It is no longer a measuring tool (standard function), either. We argue that the impossibility of linking these conceptions of units of measurement would create a difficulty in making sense of the formula and linking it to the "grid." This would, in the long run, prevent the student from using the grid anymore. Outhred and Mitchelmore (1992, 1996) provide insight into the "good conditions" needed for the grid to be linked to multiplication. Among others, they recall that it is not automatic for a student to link the number of tiles on a side with the number of centimeters of the side. In our perspective, we argue that this lack of connection can create a misconception about the unit of measurement: students then have to build an erroneous conception of the unit of measurement to make the geometrical and arithmetical registers compatible. Brissiaud et al. (2013), in their textbook, provide a half-erased grid that ensures that this connection must be made in order to solve the exercise.

We argue that the connection between the number of tiles on a side and the number of centimeters in the formula is the first step to ensuring conceptualization of unit of measurement that is compatible in both geometrical and arithmetical registers, which is an essential basis to ensure that students can keep a rich multi-level conception of area. If the concept of area can be replaced simply with multiplication, the risk is that we might believe that the students have evolved toward an algebraic conception of area and can begin working on integral and dimensional analysis when they cannot actually have a full understanding of these new notions, based on algebraic, geometrical, and measure-related registers. We argued that this difficulty in the conceptualization of the unit of measurement is implied by the underlying metrical system. Today, it is indeed possible to erase differences between the numerical value of the length and the number of tiles (and then to create misunderstood bridges between measuring units of lengths and areas), and between the result of the multiplication and the value of the area, because we constructed these possibilities when designing our metrical system. This provides an opening for quantities and units of measurement to be conceptually erased, leaving multiplication a central part and reducing the meaning of saying "in which unit is the result expressed" at the end of the computation. We make the hypothesis that these implicit links to our metrical system will ensure difficulties in at least three observable ways:

- the meaning students give to multiplication;
- the capacity to detail algorithmic steps and make explicit what is being operated on, with a possible blurring of the conceptualization of types of numbers (such as numerical values of length and area numbers, quantifying numbers and numbers to operate on with no special interest in their order of magnitude); and
- the conceptualization of the unit of measurement and underlying quantities in the formula (with the risk of mixing up the roles of measuring units of lengths and areas to build up erroneous geometrical representations of units of measurement and to forget about the "shifting" role-of-measuring-tool aspect of the unit of measurement).

The way that metrological systems, area-computation algorithms, concepts of area and units of measurement and quantities are balanced is different in every historical text. We decided to document whether these three previous items could be affected by a cuneiform tablet with a metrological system in which the systems for measuring length and area do not correspond. The main research question is: Is it possible for history of science sessions, presented in a way that is compatible with the historical discipline's constraints, to have an effect on the questions that students ask themselves in the context of the area of the square/rectangle (on the three observable points evoked above)? By this we mean, is it possible for history sessions to constitute a favorable "a-didactique *milieu*" (Brousseau 1986, p. 86) to be reinvested later by mathematics teachers to clarify some implicit aspects on a notion supposed to be stable. What are the conditions for the two types of mathematical systems (ancient and present-day) to affect one another (communicate) while at the same time preserving their diversity? This leads to a sub-question: What effects does such a meeting with mathematical thinking diversity have on nature of history and nature of mathematics perceptions?

13.2 Method and Implementation as a Teaching Situation

13.2.1 General Outline of Our Method

We document the use of a paleo-Babylonian cuneiform tablet in experimentation with tenth-grade students (15–16-year-olds). To be able to answer our research questions, implementation in the classroom needed to respect the following constraints:

- Material presented in a way that is acceptable for historical researchers. We chose to build our sessions with a specialist who was in charge of presenting them to students. This enabled us to observe the consequences of such a class without interfering with the variable consisting of the teacher's understanding of history. We will make some of our constraints explicit while detailing the teaching sequence. Using an unmodified clay tablet enabled us to preserve the cuneiform mathematical thinking system's specificity and diversity.
- Being a priori able to question student's representations about the area of the square/rectangle computation, which involves measuring units and the metrical system. We will make an extensive description of the possible effects this tablet might provoke a priori.

We do not exclude the hypothetical next phase, which would consist of implementing such a session by giving "ready to use" documents to teachers and studying their way of teaching it. We wanted to study this preliminary phase (in which a historian presented) to know more about its consequences before considering its further use. We have also not investigated the phase that would consist (for the mathematics teacher) of using the *milieu*'s (hypothetical) effects to clarify today's algorithm. Here we only try to document whether some effects exist, what they are, if they affect every child in the same way, and if it is possible to rely on some of these effects when building hypothetical next phases. We will mention these phases in our conclusion.

The tenth grade was chosen because it could provide students who would be able to understand some of the ancient text's features.[1] At the same time, tenth grade students would be confident enough with the area of the square algorithm. This can be related to level 3 in the sense of Airasian et al.'s approach (2013). We wanted to observe whether the tablet could make them reinvest in a notion that is supposed to be stable (opening a door for levels 4–6). The high school (Lycée Léonard de Vinci, Levallois-Perret) was of "average level." It had a 90% high school diploma success score in 2014, 3% lower than the national average. The two groups (control group and HSG) were taken from two tenth grade classes with approximately the same average grade in mathematics. The historian of medieval mathematics, M. Husson, led a history of sciences project in the tenth grade that allowed us to implement the cuneiform class for over 8 h, which was then followed by interviews.

To be able to document whether there was communication between the two mathematical thinking systems (ancient and current), we studied differences in the answers of the two groups. The "control group" had not taken the "history of sciences" teaching sequence, and only participated in our final interview. The HSG participated in the interviews after taking the history of cuneiform mathematics

[1]Discussion about the "right" level is still ongoing. It would be tempting to work with this text with sixth grade students, in both geometrical and metrological parts, when the concept of the area of the square is still "fresh." Although full understanding of mathematical creativity the numerical system provides is at the university level, when modulos are fully mastered, a balance must be struck between mathematical difficulty and full understanding of historical ingenuity.

sessions. Of course, this does not mean our control group never had any contact with the history of sciences, as we know this might have happened at some point in their mathematics education (from textbooks, mathematics teachers in their class introductions, their own conceptions of history, etc.). This parameter was not controllable. Thus, the study can only document a “state of the art” into our control group and whether or not the HSG bears the same features. If not, we can only make hypotheses about why they differ. The two groups were divided into sub-groups of three to four students during our interviews. As far as the HSG is concerned (32 pupils in total), these same sub-groups were already working together during the previous history sessions, and this was recorded. Students had to write down answers on papers that we collected after every session. Nine interviews (1 h each) were conducted and recorded in the HSG at the end of session 4. We were able to interview five sub-groups of four students from the control group (20 pupils in total). Semi-directed discussion formats were chosen to open a breach for spontaneous questions that we sought to document.

13.2.2 Implementation in the HSG: Preliminary Sessions

Special care was given to the type of history presented to the students in the HSG, who needed to know enough ancient mathematics to engage in the diverse mathematical thinking. Thus, the teaching sequence was time consuming, which we believe to be a real constraint on this type of history of sciences approach. It took four 2-h sessions to provide pupils with enough skills to be able to navigate the ancient mathematical system (although we will later indicate some shortening possibilities). The three preliminary sessions needed to understand the final clay tablet were also developed in terms of historical investigation and motivation.

***Teaching session* 1**: discovering sexagesimal place value notation (SPVN numbers) by deciphering an ancient multiplication table of 12 from paleo-Babylonian period (Nippur scribal schools) written in cuneiform. Students did not have trouble decoding numbers in cuneiform, and they understood it was a multiplication table fairly quickly. Arriving to “5 × 12,” they saw the same sign as for “1” as a result. For “6 × 12” they saw “1:12” and not the expected 72. A comparison with the way we write down hours on digital clocks then came naturally from the students. However, this does not make it explicit that 12 can mean 12, as well as “12 × 60,” or “12 × 60 × 60,” and so on (there is a loss of the order of magnitude, which is not the case in our digital clocks). Indeed, we might consider this notation as we would look at a digital watch. However, we will avoid translating 6:40 as 400, as we will follow Christine Proust’s interpretation of SPVN numbers. Indeed, she argues (Proust 2007, pp. 190–202; 2009, p. 11) in this numerical system, the order of magnitude is not taken into account (“floating numbers”). Of course, number position matters, and the 6 from 6:40 is sixty times superior to 40. However, this 6 could as well be 6 or 6 times 60 or 6 times 60^2, as long as 40 is sixty times inferior. In this way, the sexagesimal system can be used in all its ingenuity, providing easy computation (for example, it is possible to divide by multiplying with the inverse number in base 60 without having to care about the order of magnitude). The reader might ask the following: How to count things or measure, then? Another numerical system is used, which is “not floating.” The SPVN system is used only to compute.[2] This aspect needs to be made explicit by the speaker, and it can be observed in the way the result from “5 × 12” is not written to the left of the previous result (positions only matter relative to each other, as in “1:12,” where “1” is sixty times superior to twelve). This SPVN system is needed to understand the clay tablet (teaching session 4). Then, the students worked on deciphering a multiplication table of 18, table of squares and table of square roots. This part might be shortened for the sake of implementation in other contexts. The session ends with writing on real clay using chopsticks.

To summarize, this session provided skills with SPVN while preserving the diversity of this numerical system (a place value system with loss of magnitude order used to compute). It is also built in a motivational-investigational way (deciphering a numerical system written in cuneiform, making a hypothesis on broken parts of the tablets and working with material conditions, including clay, water and using chopsticks as reed pen *styli* to write on clay).

***Teaching session* 2**: discovering multiplication and division in SPVN. Students began by working on deciphering a new cuneiform table, which reveals itself to be an inversion table. While looking for the internal logic of the table, they understand it gives (in the right column) “the number by which you have to multiply the left column number to get 60” (which is 1). They are led to discover a new possibility with the SPVN system: multiplication by the inverse instead of division, which gives sense to its characteristics, particularly the loss of the order of magnitude. A second tablet is given in cuneiform, which gives a multiplication of 4:50 by itself (the result is 23:21:40).

Students had to guess that this was a way of writing down multiplication of a number by itself. An example of this was present in the top left corner of our cuneiform clay tablet in teaching session 4 (area of the square). This first tablet was also used to introduce multiplication in SPVN with beans and *pastas*. This part might be shortened for the sake of implementation in other contexts. Students worked on a multiplication abacus with an alternation of factors 6 (beans) and 10 (*pastas*) to go from one column to the next. It provided an opening for discussing the SPVN characteristics, such as place value notation (columns need to be respected for computation), with no general order of magnitude.

To summarize, this session deepens the understanding of the SPVN characteristics and its use in computation (with the possibility of replacing division by multiplication). It gives insights into the final tablet’s layout (top-left corner, multiplication), and the students were introduced to the final tablet’s multiplication

[2]The way this numerical system is used relative to other numerical systems in terms of preserving order of magnitude will be presented later.

of 20 by 20 (which is 6:40). It is also still built in a motivational-investigational way (deciphering tablets, making a hypothesis on the inversion table's meaning and working with hypothetical historical material conditions: abacus with columns alternating 6 and 10 groupings; we used *beans* and *pastas*: 6 *beans* equals one *pasta*, 10 *pastas* equals one *bean*).

***Teaching session* 3**: discovering metrological tables of lengths and surfaces by deciphering ancient tablets. Students began by finishing the abacus multiplication work from the previous session. Then, they were asked to translate "metrological tables" that consists of an increasing list of length (or area) measures, in the left column, with a corresponding mysterious SPVN number in the right. Interpretation of the use of these tables is not given at this point. This was the key issue in session 4. Length and surface measures are not expressed in SPVN. Indeed, SPVN does not allow the orders of magnitude to be given, so neither the measures nor the number of items are expressed in this system. Students were asked questions to help them understand the relations between the measuring units (factors between units) and the implications of these factors on the corresponding SPVN number for these measures. They were led to observe the cyclicity of the SPVN numbers in the right column, from 1 to 60 (several measures correspond to a SPVN number). This characteristic is important in session 4.

To summarize, this session provided an introduction to metrological tables whose meaning is the key question of session 4. It led students to observe them, to be introduced to the measuring units, to see how the tables went from small to large measures and to observe how there was a corresponding number on the right. They were led to differentiate numerical systems and recognize SPVN numbers as well as their cyclicity in the right column. They still worked directly with cuneiform sources, translating with the help of a small dictionary on measuring units. They added refined observation of lists to their history investigative skills. They observed how factors between length (or area) units might vary, and how systems of factors between area units might not be the square of length units.

13.3 A Priori Analysis and the Situation Given to Students

13.3.1 The Situation: Cuneiform Clay Tablet and the Students' Tasks

***Teaching session* 4**: discovering the area of the square on the cuneiform tablet UM 29-15-192[3] and the role of metrological tables. Students began by searching for SPVN number 10 and several corresponding length measures in the metrological table of lengths. They were asked to search for a relation between metrological tables of lengths and surfaces. At this point, students realized at least that

[3]Cuneiform transcription can be found in Proust (2007, p. 193) or CDLI (n.d.).

units of lengths and surfaces did not bear the same type of names as in cm → cm^2. Measuring units' names were translated in the student's document as they evoked tangible objects used to measure. Students were led by indications of the order of magnitude on the metrological tables and the corresponding SPVN numbers to observe that a measurement unit of lengths (*ninda*, order of magnitude of a house's side) would have to be linked to measurement unit of surface (*sar*, order of magnitude of a house's surface). One *ninda* corresponds to 1 SPVN number, and 1 *sar* does, too. Indeed, if they pursue the investigation and decide to use the corresponding SPVN number and squaring operation, a square of a 2 *ninda* side gives SPVN number 2; "2 × 2 = 4" can be carried out. They get a surface of 4 *sar,* which corresponds to SPVN number 4. However, this relation is not so simple when it comes to other measuring units. Indeed, 6 *šu-si* (meaning "fingers," order of magnitude of a tablet side) would give SPVN number 1 and not 6. At this point, students had to find a way to continue. Some students already understood the need to operate on SPVN numbers to be able to continue. Here, they computed "1 × 1 = 1." Using indications on the orders of magnitude, the surface measure corresponding to the order of magnitude of a tablet surface for SPVN number 1 is 3 gin_2 (grain),[4] which is the right answer. The speaker expressed the *sar* as the surface of a *ninda*-side square. She explained that the relation cannot be expressed this way between other measuring units because the factors between surface units do not correspond to the square of factors between length units. Here we introduced a copy of cuneiform tablet UM 29-15-192. Students were asked to translate it with the help of a small dictionary. Figure 13.1 shows the cuneiform tablet translation and a hand-written copy.[5]

Students deciphered cuneiform numerical writings on multiplication tables and a squaring tablet during the three previous sessions and were used to it. The dictionary was used only for words, such as "side" or "square." Written questions led them to look at the tablet's layout and make explicit the separation between a computation (top-left corner) and wording/result (bottom-right corner). Then they were asked to look for a link between the two parts of the tablet or express why they did not think it was possible. This question was intended to spur a reaction on the fact that the side is "2 fingers" and the computation is carried on the number "20." Students were led to use the documents at their disposal to interpret this tablet. The metrological tables were, of course, fresh in their minds, and their attention had also been focused on the relation between 6:40 and 1/3 of a grain in session 3, on the metrological table of surfaces. They were used to multiplication in SPVN following sessions 1 and 2 so that 20 times 20 could be naturally interpreted as a

[4]The measurement unit of surfaces gin_2 can be translated as "grain." It belongs to the metrological table of weights, and will be used in the sequel as terminus technicus. The beginning of metrological table of surfaces is exactly the beginning of metrological table of weights, until it reaches 1 ma–na. The gin_2 is also used in the metrological table of capacities but this metrological table is independent (Proust 2007, p. 311).

[5]The handwritten copy was made and adapted by Christine Proust for experimentation purposes. The original can be seen at CDLI (n.d.) and comes from Neugebauer and Sachs (1984, p. 251).

[20]
20
6:40

2 fingers the side of a square
Its surface how much?
Its surface is
1/3 grain.

Fig. 13.1 Tablet translation and hand-written copy

multiplication giving 6:40 as a result. This context led them to understand how to use metrological tables quite easily. In the area of the square computation, the two systems have to interact. How?

Computation (multiplication of 20 by itself) is not performed on "2" from "2 fingers" (2 *šu-si*),[6] but rather on 20. This symbolizes communication between the two numerical systems: SPVN numbers to compute and "non-floating numbers" to measure and count objects. The scribe uses a "dictionary" (metrological table of lengths, see left side of Fig. 13.2) to transform measures of length (2 fingers) into numbers that can be multiplied (here, number 20) in "floating numbers;" then a metrological table of surface (see right side of Fig. 13.2) is used to transform the result of multiplication (6:40) into area measures (1/3 of a grain). To select the "right" surface measure, he has to pay attention to the expected order of magnitude of the result (because of the SPVN number's cyclicity). If students do not think about using metrological tables, they are led to do so with the question, "How do you explain the use of metrological tables?" The ancient algorithm can be summarized this way:

Input: length measure.
Conversion in SPVN using metrological table of lengths.
Computation on this SPVN number (multiplied by itself).
Conversion of the SPVN result in area measure using the metrological table of lengths and expected order of magnitude.
Output: area measure.

The last two questions to the students were as follows: "Could you write this exact exercise with today's vocabulary?" and "Could you give your impressions of this exercise?" These questions were designed to provide records of the way history impacted students in terms of the current algorithm steps and if they were able to offer spontaneous remarks or questions on the fact that today, there are no steps

[6]*Šu-si* is a length unit which can be translated as "fingers."

Metrological table of lengths		Metrological table of surfaces	
1 finger	10	⅓ grain	6:40
2 fingers	20	½ grain	10
3 fingers	30	1 grain	20
4 fingers	40	2 grains	40
5 fingers	50	3 grains	1
6 fingers	1		

Fig. 13.2 Abstract from metrological tables of lengths and surfaces

corresponding to metrological tables' use (to make numerical values and multiplied numbers correspond; which is indirectly linked to our metrical system).

To summarize, this key session 4 was the meeting of the students with our *milieu*: cuneiform tablet UM 29-15-192. They were led to find an interpretation of the relation between the area of the square computation and metrological tables. They still worked directly with cuneiform sources, translating with the help of a small dictionary. They added refined observation of the tablet's layout to their investigative skills. They developed a historical interpretation using the sources at their disposal (area tablet, metrological tables, multiplication table of 20). They entered into the mathematical thinking system's special features: the use of SPVN numbers to compute (allowing them to get rid of division), the way to navigate between this numerical system and others (metrological tables), the use of the order of magnitude and special relations in a metrological system that differs from ours.

13.3.2 A Priori Expectations on Conceptual Aspects

In Table 13.1, we present items of the ancient text coming from mathematical diversity (first column). We show in the second column how this preserved diversity is expected to have an effect on the current mathematical conceptions of the student, first with regard to the general concepts (numbers, measurement, unit of measurement, etc.) and second, with the application via detailed questions linked to the area of the square. In the third column, we indicate the strategies developed to investigate these possible effects. Recall that we documented only the history of sciences sessions' effects and what meeting with ancient mathematics did to students *before* the intervention by the mathematics or history teacher. We sought to know more about the new ground on which teachers will institutionalize current mathematical notions, and whether we favored emergence of new spontaneous questions.

Table 13.1 A priori expectations

What the ancient text does to the reader	Expected effect on "current mathematics"	Investigation of this effect on the HSG
Observation of the existence of **two types of uses of numbers**: one numerical system to "measure" and one numerical system for computation (SPVN)	CONCEPT OF NUMBER Differentiation of the measuring step and computing step of the algorithm ↓ And: showing the importance of *measuring activity* in the algorithm	Questions on area algorithm steps (current & ancient ones) Task: "give an example of area computation with values you choose" (written record of steps)
Multiplication not computed on same values as measured: distinction between the value of a quantity (length, area) and numbers on which to compute Ex: 2 fingers corresponds to the SPVN number 20 with which computation occurs	CONCEPT OF NUMBER Difference between numerical value of length/area and numbers entering the computation + Possible spontaneous question: Why is this difference invisible today? ↓ Reflection on that which we compute (number of tiles, number of cm, and number of cm^2?) ↓ Also constitutes a step toward giving meaning back to geometrical representation and the action of *measuring*	Questions: "What is the '3'?" and "What do you multiply"? + Student's spontaneous hypothetical questions
Observation of the need to use an "intermediate system" (SPVN numbers) to go from length measures to area measures Observation of two **conversion** steps to go from length measures to area measures in ancient algorithm thanks to a tool (metrological tables) and a numerical system (SPVN) + Its absence (or invisibility) in our algorithm today	CONCEPT OF QUANTITIES IN AN ALGORITHM, MEASUREMENT Differentiation of length measuring system and area measuring system Looking at invisibility of conversion: a step towards giving back meaning to the measurement process + Why is it possible today to add the measuring unit quite simply at the end?	Why do we put "cm^2" as a unit of measurement here? We also expect fewer answers explaining "cm^2" by having chosen "cm" in the beginning than in the control group Or: formulation by students of spontaneous questions to themselves on this matter Questions on the area algorithm's detailed steps (modern & ancient ones): leading to the formulation by students of spontaneous explicit questions, such as the following:

(continued)

Table 13.1 (continued)

What the ancient text does to the reader	Expected effect on "current mathematics"	Investigation of this effect on the HSG
		– Why is there no such step today? – Why is it possible to use the same numbers to go from length measurement to multiplication? – Why can we use the "same" unit of measurement for length and area measures by just "adding" a square?
The use of **different unit of measurement names** for length and area metrological systems + Use of units of measurement names from **everyday life** (grains, fingers, etc.)	CONCEPTS OF UNIT OF MEASUREMENT AND MEASUREMENT Separation between measurement units' systems: a step toward giving meaning back to the measurement process + More than a "literary role" + Work on the idea cm^2 is: cm × cm	Why do we put "cm^2" as a unit of measurement here? What does "cm^2" mean to you?
Observation of the clarity with which it is possible to know at each step what the ancient algorithm is operating on: Input: length measure + Conversion + Computation (on numbers used to compute: SPVN numbers) + Conversion + Output: area measure	CONCEPT OF MULTIPLICATION IN AN ALGORITHM, MEASUREMENT Work on the distinction of steps in the algorithm and what the algorithm operates on at each step + Distinguishing actions: measuring lengths, making a multiplication and measuring area ↓ Giving meaning to the action of measurement (a first step toward reopening a discussion on the links between the algorithm and grid; the 5 can represent a number of tiles and can then be associated to a number of centimeters). + Work on the distinction of steps in the algorithm and what the algorithm operates on at each step: A step toward giving meaning back to multiplication	Question: "Is the "5" in "5 cm" the same as 5 in 5 × 5?" We expect more details about the current algorithm's steps than from the control group and we expected more spontaneous questions such as the following: Why is it possible today to use the same numbers to go from length measures to multiplication? What is this "5" exactly?

13.4 Results

In this section, we present some of the results from comparing answers between the control group's interviews and HSG's interviews.[7] Semi-directed discussions were chosen to provide an opening for spontaneous questions from students, which we sought to observe. The structure of the interview mainly consisted of two parts. First, we asked questions about the area of the square to both the HSG and control groups in order to (1) detail the mathematical basis we had made predictions on in the control group and (2) determine if we could record the effect of the history of mathematics sessions on current mathematical conceptions by comparing the HSG with the control group. Conclusions were based on indirect analysis of the answers regarding current mathematics questions (e.g. by observing spontaneous questions from HSG students on current area computation). Second, direct questions about mathematics, the history of sciences and history were asked to both groups to seek for nuances in their representation of these disciplines. We hypothesized that changes might emerge due to students being introduced to a type of history of sciences, in particular their immersion in another system of mathematical thinking, respecting its diversity. Students also had to work with history in an investigative way (developing an interpretation based on primary sources, translations from cuneiform and experimentation with tools, including clay, abacus, etc.).

13.4.1 Interviews on Current Mathematics

This part of the interview first aimed to verify that students' current mathematical state of knowledge was compliant with the hypothesis we drew from our preliminary analysis (control group study). Second, it aimed to provide a ground for the emergence of spontaneous questions as a result of this encounter with mathematical diversity in the form of the ancient cuneiform tablet, which would allow us to study the differences between the control and HSG groups and describe possible communications and effects between the two mathematical systems (ancient and current). Indeed, we detailed in our a priori analysis (see Sect. 13.3) how the ancient text might have had an impact on students' conceptions of numbers, measures, algorithm, etc. We sought for traces of these both in spontaneous questions and in the differences between the control group and the HSG in the context of the area of the square. We only present a brief overview of the results, though they are detailed thoroughly in the author's doctoral dissertation.

For this part of the experiment, there are two main conclusions. First, the hypotheses about (current) mathematics based on working with the ancient texts' influence were mainly verified, often more strongly than we had expected. The objects on which the computation algorithm operates are not clearly identified by

[7] Interview questions are presented in Appendix.

the students, and the concept of unit of measurement is not (or no more) related to the idea of shifting the same standard several times: the act of measuring. The unit of measurement seems to play what we had qualified as "literary role;" it was used to give contextual clarifications. The results reinforce our hypothesis that the loss (or lack of knowledge) of the geometrical grid taught in fifth grade plays a key role in these difficulties.

When students were asked to explain how to calculate the area of a square, the quasi-unanimous answer given by the two groups was very largely to give "the formula" (among them, six students in the HSG gave an erroneous one). In addition, 60.97% of students in the HSG did not attempt explanations, while 47.6% of the control group did not attempt an explanation. Only one pupil from each group proposed a "grid" representation at this stage; a student from the HSG also proposed an explanation related to the proportionality between the length and area. The other students related to external work ("it was taught this way/it has been proven") or responded by giving conditions for applying the formula (12.5% for the HSG, 20% for control group).

The hypothesis of this loss of the grid is largely reinforced by our last "pool" exercise. Students were asked to find the number of tiles (each of area 1 cm^2) needed to fill the bottom of a pool with 20 cm sides. From both groups, a strong majority of students chose explicitly to compute the area of the bottom (square) to find 400 cm^2 and then divide by the area of a tile (1 cm^2). Only a few performed a multiplication of 20 by 20, explicitly, to directly find the number of tiles.

Area conception seems to have lost a rich multi-level character which has been replaced only by multiplication, with almost no connection to quantities. An important confusion was expressed in one of the HSGs, as students considered the formula of the area of the square to operate on "one side." The formulas of the area of the triangle and, above all, of the circle (which has no sides) cannot apply the same type of operation, in their opinion. This is important because it exemplifies one of the risks of not knowing clearly which objects are involved in the calculation. It also shows to what extent the identification of the mathematical objects of an algorithm which here takes a "side" rather than a number of tiles as "entry," makes it mathematically difficult to gain a deep understanding of the operation. In detailing algorithmic steps, students focus only on the multiplication step. They can no longer connect the algorithm to tiles, nor "cm^2" geometrical units, nor "cm" geometrical length units (shifted on a side), but only to "the side: 5 cm" as a whole. When some of the students discussed measuring units during the interview, it was to remind them to "add" the "cm^2" at the end, but it was not physically conceptualized, and some students confused it with "cm." The majority of the students in the control group and the HSG expressed the fact that the unit of measure was used to "indicate that it is an area" (45% of the control group and 31.57% of the HSG). Furthermore, 10% of the students in the control group and 5.26% of the HSG students recalled that it was "because $cm \times cm = cm^2$." When asked what "cm^2" was, the students in the control group mostly answered "a surface" (37.5%), while 18.75% of the students specified "a tile." In the HSG, 53.12% of students directly indicated it was a tile or a square. However, the idea of a tile or surface does not mean that the students mobilized the idea of "shifting" or the conception of a "grid." Only in the HSG did there exist a reference to its use to measure (e.g. apartments), and thus indicates an indirect idea of shifting measurement unit. Students were also unable to link numbers (like 5) to the number of tiles or to the number of centimeters on a side.

We explain this non-persistence of the "grid" in part by the conceptual impossibility of making the "tile" geometrical representation of the unit of measurement in the grid coincide with its arithmetical multiplicative representation in the formula. Here, when asked about a "centimeter-square" length-measuring unit "cm" also pops into the picture and generates erroneous geometrical representations of relations between units of measurement (see the "L" representation in Sect. 13.1), but they are still never linked to an area algorithm. Thus, in the algorithm, the unit of measurement finds itself confined to a role that is not so much "mathematical" anymore. These results reinforce the interest in making explicit (at some point) the consequences of the metric system on the fluidity of navigation between quantities (governed by relations of proportion) and numbers (linked by operations). History has also had an important effect on the conception step of this experiment through the diversity of propositions it offers in connection with various metrological systems.

Questions arising from the meeting with history do not emerge spontaneously among pupils in this part of the interview, which is nevertheless suggestive. When asked to give the algorithmic steps (ancient and current) as they would for "a cooking recipe" or a computer, students from the HSG did not make spontaneous remarks about the absence of "conversion" in our system. The question "is the 5 in 5 cm the same as the 5 in 5×5?" shows a disparity between the HSG and the control group. In the control group, 62.5% numbers of pupils answered that they were "the same." This response was found in 28.57% of the HSG students. In the HSG, 50% responded that we compute on numbers that "represent something;" they express the idea of the remains of "side" (of the square) or "length" (of the side). In the ancient text, the number (SPVN number), whose value is not the same as the number associated with the measurements, is an independent mathematical construction that allows different elements to communicate: length measures and area measures. Here, the number on which to compute is not presented as "independent," as in the ancient system, by the students: it is precisely the interest of our metric system. However, interestingly, students also expressed the capacity of the number to "represent several other things." Consequently, this differentiates numbers (which according to their answers can represent measures and other mathematical objects) from measures, and it even underlines numbers as transitional objects. It is possible that here we have access to a communication trace between ancient and present-day mathematics, which is what we were seeking. It is conceivable that the teacher of mathematics can utilize this effect; although it should be noted it affected only 50% of the students in this study. This is one of the points we wished to describe: our ancient text as *a milieu* has different effects on different students, and therefore it cannot be taken for granted. Explicit spontaneous questions about the fact that it is possible today to use the "same" number values to measure length and to compute (whereas it is not possible in the ancient text) were not raised. The teacher would have to raise the question and explain the reasons for the possibility of expressing a calculation on values coming from measurement, due to the way in which the metric system is constructed, and thus make explicit the type of objects that enter a computing algorithm. He could then use the "grid" to re-explain where "numbers" come from at each step, linking them back to tiles (associated secondarily to the number of centimeters of the side).

Students could have raised explicit questions. In our system, measuring units of lengths and surfaces *seem* connected; why is this not the case in the ancient text? This question did not arise spontaneously. It would then be very interesting for the mathematics teacher to raise afterwards, and it would not transform history in a prejudicial way. One of our indirect questions seems to show an impregnation of the historical text: "Today, where do you shift from lengths to surfaces?" This question is not completely legitimate mathematically; although we used it to see if students would reply with explicit remarks, such as: "There is no conversion today," which could be used by the mathematics teacher to make the difference between length and area measuring systems more explicit (by expressing how we built a system that makes units correspond). Seventy-five percent of the students in the control group said that change occurred during multiplication, while only 64.7% of students in the HSG indicated this. A new option also emerged in HSG: "at the time of the result" for 14.7% of students. Formulations of the type "just after multiplication/between calculation and result" or "because of what one multiplies" were found, in minority, in both groups. The emergence of the new option could be explained by the encounter with the ancient text: after calculation, with SPVN numbers, we must look for the area measure in the metrological table. Thus, this could be related to the "time of the result" formulation. It should be noted that this effect, which again, impacted only 14.7% of students, did not lead to explicit questions, such as questions on the "absence of conversion" in our system, as we expected. Thus, the effect, probably linked to the historical text, remains unexploited and does not allow students to consciously question mathematics.

The second important conclusion of this descriptive experiment is that the text does not seem to act perfectly as an "a-didactic *milieu*," as spontaneous explicit questions cannot be raised by students. This shows the difficulty for students in having the ancient and current mathematical systems interact, although the students knew how to navigate in each system independently. This encourages precautions in all experiments using the history of science to improve current mathematical knowledge, as such interactions are not automatic. Furthermore, trying to force interactions almost always interferes with historical constraints in creating parallelisms between the two mathematical thinking systems (parallelisms that imply transforming ancient mathematics and its diversity for the sake of comparison). Even if a mathematics teacher utilized the ancient text's effects, he could not rely on uniform explicit influence of the text in his class. However, the divergence of results, sometimes important, between the control group and the HSG shows the emergence of historical influence in terms of current mathematical aspects. This does not mean pupils can use this influence to explicitly interrogate their own knowledge. To avoid speaking only to a part of the class, the teacher would also have to raise the hypothetical "spontaneous questions" we evoked. In this way and before establishing any mathematics-to-be-taught in its present-day form, he will help in making these changes explicit, which otherwise will remain tacit in students' discourse. This rejoins the deeper debate of investigative activities. However, these spontaneous questions can also be seen as a compromise that allows the use of the history sequence in mathematics classes without losing sight of history. Methodologically, we sought traces of conceptual impact in terms of both spontaneous questions and the differences between the control group and the HSG in the context of the area of the square. It might be interesting to search for a more direct impact on the concepts of number, algorithm, multiplication, etc. (see a priori analysis) without inquiring into the context of the area of the square, which could create a bias. Another methodological possibility would be to study the effects of history of science sessions led by teachers from each discipline (mathematics and history) in parallel, during preliminary analysis, with the help of a history education researcher. This would constitute a "double ingénierie" and would offer clear insights into where the parallelisms created to transpose mathematical knowledge today might affect history.

13.4.2 Interviews: "Nature of Sciences (Historical and Mathematical)"

This part of the interview aimed to study the "nature of science" effects of our sessions on representations of mathematics, history and history of sciences, based on the fact students had been introduced to historical mathematical diversity, in a motivational (or at least built to be motivating) context with historical investigation tasks. We only present a brief overview of the results, which are thoroughly detailed in the author's doctoral dissertation.

The question "What is important when presenting a historical text?" is methodologically interesting. The debate seems to continue from the previous question on the history of science, and some students answer based on what they would find important to say in a history of sciences class. The answers in the two groups are mostly related to "giving contextual clarification" on the source. Themes such as presenting "research steps" (6 occurrences) in history of sciences, or referring to the present (5 occurrences; e.g. How are these findings useful today? How can history be used to improve?), are absent in the HSG. One hypothesis is that conceptions about the possibilities of using the history of science have changed in the HSG because of the cuneiform sessions. The most important part of the results is that the HSG engaged in seven "discussions using diversity of arguments" that were not present in the control group:

1. Questions on the status of the source in relation to the subjectivity of the author (necessary step back):

S1: Well, for example, if we write it when we are at war, the adversary countries do not have the same version. So, if the text was written in one country… it's not… valid. […]

2. The difficulty and need (or not) to know the author depending on the type of source:

S2:	What I mean is, the author is not essential in every document, not in all types of documents. […] Well I mean for a novel, for an article and all, alright, we need it, I agree with S3.
Interviewer:	Why?
S2:	Because hum… to know, well for culture, to be able to compare… with others […] but hum, for instance a math problem, you will not ask yourself… who wrote it, I mean… whether it's your grandma' or some guy you ran into.
S3:	A math problem, it's different and… [inaudible debate] […]
S3:	The text will be famous and the problem won't be because the problem… It depends on the type of text. […]

3. The reliability of the scientific content of the text, with the risk of learning something incorrect [this could perhaps belong to one of the facets of what has been described as "dépaysement" (Barbin 1997; Guillemette 2015)]:

S5:	Well… if the source is not reliable, I mean… if it hasn't been demonstrated, we cannot give them [students] something… that might be wrong.
Interviewer:	How can it be wrong, for instance?
S5:	Hum… if we discovered afterwards that in fact this computation… did not work or… […]
S5:	It makes us take a… step back, right? [laughs]
Interviewer:	It makes you take a step back?
S6:	Well maybe it can also make us progress, because we already know it's not right, so we can already withdraw an… […] hypothesis, hum… […]
S5:	I mean if it's wrong and we know it, it's ok. However, if we think it's true and we learn it, hum…
Interviewer:	You're afraid to learn something wrong?
S5:	Exactly.

4. Nuances on the role of the historian in the presentation of a "true" interpretation of the text (this would need to be discussed with the help of a history teacher); also linked to a different form of *dépaysement*:

   S7: because hum… every historian must have had an hypothesis, they tried and all… and hum… they saw this [cuneiform sign] means "1" and tried on several tablets and so… this, hum… led them to believe it meant 1.
   S8: I think yes, she's right, many people must have, hum… described this, hum… tablet, some had right answers and some had wrong answers, and… I would not have liked to learn a wrong one.
   S7: It shouldn't… hum… it has to be a collective work, there must be several points of view.
   S8: Yes, that's it, it must be demonstrated and… […]
   S7: If there is a logical reasoning, hum… we also see… the greatest majority… even… everyone, gets the same result. This… they conclude on the same… hypothesis.
   S8: If we manage to convince everyone [it means it's true].

   This debate led to a debate on students' role and implication which could be linked with our "investigational" practice:

   S2: It's interesting but… in fact every student could also give his opinion, it could help historians.
   S8: Yes.
   S7: Because, hum… everyone can give his opinion and be right so, hum… maybe thanks to our hypothesis or our… commentaries it could lead them to think… something else which might lead them too, to find the right result (laughs).

5. The need to know the mathematical basis to understand the source.
6. The role of the context which is interpreted by the student as a motivating tool:

   S9: We should know… the context […] to know why… we found it today and what it was used for before.
   Interviewer: Alright. And why is it important?
   S9: Because if not… well to me we cannot understand the source. […]
   S10: Well, I would say you have to be, hum… well when we present something we have to be convincing, also, to make people want to listen. […]
   S9: A "full of life" type of class […]
   Interviewer: The context, to you, it gives interest? The context… of the source, for instance?

S10: I don't know.
S9: Well maybe not interest, but it helps understand […]
S10: Well yeah, I already had some teachers, when they explained something they used to put… the right tone. It made us want to listen. I think it's better.
Interviewer: Alright, when they presented the source?
S10: Yeah.

7. And the impact of status of the source (raw, to be deciphered or not) in motivating the student and/or taking a scientific approach:

S11: I would give it translated […] so we would get a first example, to know what they [scribes] did, and could try to put every letter in hum… the translation, for everyone to understand, and then try to translate by ourselves, with an example. […]
C: Alright. Do you agree?
S12: Yes, if we do not justify it's like preaching things. And so we cannot necessarily trust it.
C: Alright, you mean saying why it was translated this way? […] do they tell you this in history class?
All: No. Not much. […]
S13: I would give it raw. To me it's the only way… we can understand. We have to sort things out. […] to test, to sort things out, to understand by ourselves and then we can ask for the answer to check and then continue doing hum… computation, but to me, it's raw. Because with help it's too… easy. We get the answer and the hum… we just copy so… it's raw.

These discussions demonstrate a step back from the historical source, what is known, and what it takes to understand it. The various possibilities of presenting the source to the pupils are mentioned by one sub-group in a discussion on "pedagogy." This fact is not without interest, since it testifies to taking a step back in terms of the different presentation possibilities for the source as well as its foreseeable "political" use. It also testifies to taking a step back from possible scientific interpretation of the content of the text in relation to what is known today, based on the role of the historian in choosing an interpretation (although sublimated here, because he knew the "right" interpretation). All of these are interesting effects of the session. It could be guided by the history teacher after the session. Indeed, we studied the effect of the text on debates without teaching intervention. Doussot and Vézier (2014) recall differences in students' answers about the historical source, depending on the question addressed to them by teachers. The student discussions highlighted motivation to exchange on these themes. Variety in the use of arguments in the HSG may show a complexification of their position which can open an interesting breach to enrich their conceptions linked to history, which can be guided by the teacher.

In general, in this part of the experiment, the results reveal that the history of science teaching sequence may have had an effect on the wealth and number of debates concerning the nature of mathematics and history. We were able to document more varied arguments in the HSG, as well as more lengthy debates. It should be noted, however, that these positive results do not exclude the use of "condescending" formulations toward the past, even by some students in the HSG. Ancient mathematical work exposed to the class can be used as an argument to glorify the present and its simplicity, rather than to consider diversity. These "positivist" positions must, however, be nuanced, and they are not adopted all the time; the same pupil could adopt relativist position at other times. Indeed, many arguments proving the capacity to "put oneself in the shoes" of the ancient author, to consider the tools at his disposal and his possible objectives, are also recurring. It would be necessary to further document through experimentation, the possible increase in the number of relativist arguments between a control group and a history of science group. Here, the role of the history teacher, after the sequence, could be fundamental, and the debates described in the experimentation could be guided. In the same way, the spontaneous remarks that we have noted as interesting could serve as fertile soil in history class (the role of the historian, the availability or lack of availability of information on the author and his subjectivity, the diverse possibilities for presenting the source [untouched or translated], the room given to interpretation, etc.).

From a methodological point of view, in this part of the interview we used open questions, which could have induced rather general, even blurred, answers by the pupils and not provide access to all of their finesse. Furthermore, considering the hypothetical effects of our sessions shortly after they took place is questionable, so results should be only taken as a basis for further study and discussion. Results were built from comparison of both groups which prevented the risk for the HSG to be interviewed *before* attending HSG class, as they would have sought for our *didactical contract*.[8] Comparison of both groups made it possible to highlight differences in the types of arguments between the HSG and the control group. Our choice of semi-guided interviews gave free rein to discussion, which allowed, despite direct questions, spontaneous debates with varied arguments (as mentioned above) to emerge, allowing us to document the differences between the groups. This point should be investigated with a pre-session interview on history group in future studies. Finally, some direct questions indirectly led students to use history of science arguments (questions about the "nature of mathematics" and the "presentation of a historical source," for example), which provides hints to their conceptions. Thus, these open questions can give access to some aspects of the nature of history or nature of mathematics.

[8]See e.g. Mason and Johnston-Wilder (2004, Ch. 3).

13.5 Conclusion

The results of the "current mathematics" section show that the presence of pupils' difficulties, considered thanks to the ancient texts, are verified. It has also shown that the mathematics of the historical text effectively affected students' perception of current mathematics, as they acted on their conceptions of predicted mathematical objects. However, the effect is often difficult for the pupil to make explicit, and does not affect the whole class uniformly. The role of the mathematics teacher in the reuse of the history of science sequence is therefore necessary to make both mathematical systems communicate. However, it is a delicate role because of the disciplinary constraints (Décamp and Hosson 2015) imposed by history, which we mentioned above, and it also influenced the way we designed the experiment. We proposed methodological leads to investigate how to take into account this difficulty. The results of the "nature of science" section (historical and mathematical sciences) showed that attending the history of science sessions led to a higher number of arguments and spontaneous debates for both mathematics and history. It is necessary to qualify this: Students may appeal to condescending formulations toward history, sometimes in a non-uniform way. The same student can use this type of argument at one point in the interview and then demonstrate relativism by placing the objectives and mathematical tools in context. Moreover, a possible link between the number of "positivist" arguments and students' feeling of failure when interacting with ancient mathematics should be further documented. To frame these positions and debates, which could take place after the history of science sequence in history class, the history teacher could have a key role. It is possible for him to use this springboard to refine the students' relationship to historical sources, and to deepen their understanding of what it is possible or impossible to make ancient sources say. Given these complex interrelations, it seems all the more important to further balance history and mathematics teaching objectives in order to exploit historical sources in the classroom, while keeping a non-elitist perspective and without losing sight of historical and mathematical objectives.

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Appendix—Interviews' Content

I. *Questions: mathematical part*

Could you tell me how you calculate today the area of a square? (Individual reflection time)

1.

- If the student has given a formula: Do you know why this formula works? How would you explain it to someone?

For these first two questions, we do not assume that the results will be very different between the control and HSG groups. We expect some "grid" type of explanations and a majority of students with difficulties to answer, "I do not know" or students who simply give the formula. We can imagine, however, that the HSG may raise more spontaneous remarks such as "I wonder why the formula works" (see a priori analysis).

2.

- If the student has only given a formula, not applied to an example: Can you give me an example with a square the size of your choice? (Individual reflection time)

This question will allow us to analyze the space given to units of measurement in the application of the formula and their choice.

3. In your opinion, what are all the steps involved in calculating a square area, if you were to give them something like a kitchen recipe? (Individual reflection time)

This question has a key role; it will allow us to analyze a possible change in the ability of the students of the HSG to distinguish the mathematical steps and objects involved in each step of the algorithm after meeting with a different algorithm. I recall that we make the hypothesis of a difficulty distinguishing between steps (at least for the control group), with a tendency to summarize the area calculation in the single multiplication step.

4. In your formula, what is the "3" (which is multiplied)?

I recall that we make the hypothesis of a difficulty in explaining what is multiplied, especially for pupils unable to mobilize the idea of "grid," as well as a difficulty in distinguishing number and numerical value associated with a magnitude value. We expect a few answers like "a number of tiles" and a majority of answers like "length of the side," on which we could counter and ask for clarification, on why "it works."

This may be an opportunity to see a difference between the control and HSG groups. The latter could, for example, make spontaneous remarks such as "ah, but it is true that today it is both" (number multiplied and measure of length), while in Mesopotamia it is not the same (SPVN and length measurement numerical system).

5. In multiplication, what is multiplied?

Same remarks as in the previous question

6. Why do we put "cm^2" as a unit of measurement here?

I recall that we make the assumption of a general difficulty in explaining the choice of the unit of measurement, due to an automatic use of the area unit of measurement "corresponding" to the length unit of measurement.

This question may be the occasion to note the emergence of spontaneous questions in the HSG, due to the encountering with area units of measurement unrelated to lengths units of measurement, like: why do our units of measurement bear the "same name"?

7. What is "cm^2" to you?

We a priori expect answers of the type "a tile" and answers of the type "an area unit of measurement;" without distinction between the two groups (control and HSG).

8. Here, there are length measurements and there, surface measurements; when did we go from one to the other?

This question, although asked in a "partially mathematically legitimate" way, will allow us to draw students' attention to different type quantities at stake in the algorithm. Here we expect a difference between control and HSG groups. We believe that control group will respond "at the time of multiplication," giving a key role to the multiplication that seems to do a "magical" transformation. We believe that the HSG, which has found the existence of SPVN numbers and metrological tables, will be divided and hesitant. Perhaps spontaneous questions will emerge at this moment, like: "in the tablet we have made a correspondence between measure of length and numbers, then between numbers and area measure; I do not know how we do it today/I do not know where this step is today."

Some pupils in each group may use the idea of a "grid" to answer that length measurement does not really matter, and that it is actually a matter of knowing the number of tiles (the number of cm^2), with multiplication operating on the number of tiles per row and column. Then the length measurement simply gives the number of tiles on a line. It could be said that there is an implicit mapping of the length measurement from one side to the number of tiles on the side.

9. If I have a square (20 cm sided) pool, and if I want to tile it with tiles (squares) of 1 cm side, how many tiles do I need?

I recall (see previous analysis) that we make the hypothesis of a general difficulty in mobilizing the grid, that hence bears no meaning anymore, in the memorized formula. This question will show which method is preferred by the students: to count tiles or to calculate the area and then divide by the area of a tile. We assume a majority of use of this latter method, in both groups. We think the question may allow some of the students to remobilize the "grid."

10. Can you explain why the formula works?

We want to see if the students who "remobilized the grid" change their explanations.

II. *Questions: nature of sciences*

[The first six questions based on the cuneiform tablet (historical, mathematical) are omitted because they were asked only to the HSG.]

1. Do you like math? Yes/no, why?

This question shall be used to investigate the relationship of students with history of science sessions, based on their relationship to mathematics, and a possible explicit effect (or not) of the sessions on their tastes, by comparison with the control group.

2. In your opinion, what is important when presenting a historical text?

This very direct question will help us to know whether or not students are able to make explicit their relationship to the historical discipline; and to see possible changes between control group and HSG.

3. What is history of sciences, for you?

This very direct question will help us to know whether or not students are able to make explicit their relationship to the history of sciences; and to see possible changes between control group and HSG.

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Chapter 14
Making Domain-Specific Beliefs Explicit for Prospective Teachers

An Example of Using Original Sources

Susanne Spies and Ingo Witzke

Abstract The implicit effects of using history of mathematics in teachers' education on the individual beliefs of prospective mathematics teachers are widely discussed. However historical texts may also play an important role in making different mathematical worldviews and domain-specific beliefs explicit, as we discuss in this chapter. For this purpose, after sketching some connecting points between the history of mathematics on the one hand and individual beliefs of mathematics on the other and the short presentation of results of an empirical study on domain-specific beliefs of school calculus, we present an example from prospective teachers' education at the University of Siegen: Within a course on subject matter didactics of calculus a historical source is used to initiate discussions on students' beliefs.

Keywords Prospective mathematics teachers · History of mathematics
Mathematical worldviews · Domain-specific beliefs · Calculus

14.1 Introduction

It is widely discussed that teachers' individual beliefs (and related concepts such as mathematical worldviews, beliefs, belief systems, etc.) of what mathematics is, influences his or her beliefs about teaching and learning mathematics. This has an influence on the concrete classroom performance and the pupils' learning process (e.g. Philipp 2007; Wilson and Cooney 2002). Following this hypothesis, the

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research on mathematical worldviews takes place in teacher education with two different foci. Firstly, it is acknowledged that every teaching process at university leads students to individual experiences with mathematics, which influences students' belief systems. This rather unconscious development may eventually lead to an authentic and helpful belief system, which follows the hypothesis, 'mathematics is what mathematicians do.' Secondly, making different mathematical worldviews explicit could be helpful as well, for example, as explicit topics in seminars on the teaching and learning of mathematics. This process of making metacognitive aspects explicit to learners is in turn supposed to lead students to a more conscious and reflective approach on what mathematics is: "On the basis of rational and 'reflexive' considerations, mathematics is embedded in an individual idiosyncratic structure of beliefs" (Bauer 1990, p. 7; authors' translation).[1]

In this chapter we will make a case for using original sources from the history of mathematics within prospective teachers' education at university level, in the sense that it does not only work in the first implicit way, which was described above, but that it also can be fruitful to initiate explicit reflections on different—sometimes even problematic—belief orientations. Furthermore, we will discuss the capability of working with original sources especially within seminars on the teaching and learning of mathematics with a special focus on domain-specific beliefs on school calculus—which must be seen separately from courses on calculus itself.[2]

We will start by giving an outline on different aspects of the interplay between using history of mathematics and mathematical worldviews within teachers' education. In a second step, we present results of a questionnaire on calculus specific beliefs of high school students and point out some problems from the perspective of subject matter didactics. We will also mention parallel developments in the history of calculus, on an epistemological level. On this basis, we present an example from a seminar on the teaching and learning of calculus ('Didaktik der Analysis'), in which we used a part of Jean Bernoulli's (1667–1748) work on calculus to make the idea of an empirical orientation concerning main concepts of calculus explicit and to initiate students' reflections on possible consequences for their prospective calculus lessons in school.

[1]"Auf der Basis rationaler und 'reflexiver' Überlegungen wird die Mathematik in die eigene, individuelle Wertstruktur eingebettet" (Bauer 1990, p. 7).

[2]It is a special characteristic of the teacher education course at the University of Siegen that introductory content subject lectures are in general accompanied by appropriate Pedagogical Content Knowledge (PCK)-courses; 'Analysis courses' for example are accompanied by 'Didaktik der Analysis' (Didactics of Analysis).

14.2 History of Mathematics and Mathematical Worldviews

Many scholars are convinced that mathematical worldviews or beliefs are important to understand core processes of the learning of mathematics. Amazingly, the concept of beliefs remains a somewhat blurred construction with different thematic implementations (cf. Forgasz and Leder 2008). Some scholars have described the two dimensions of mathematics as a process versus mathematics as a product. As one interpretation of this duality we acknowledge the question of whether mathematics is a human activity and endeavor or an 'a priori' given corpus or a set of known rules and objects. Thereby, getting to know mathematics as a human endeavor is one of the prominent alleged effects of using history of mathematics within mathematics education. For example, for Jahnke et al. (2000, pp. 291ff.) the so called "replacement" is one of the "three general ideas" of using history of mathematics, with original sources in this special case:

> Integrating history in mathematics replaces the usual with something different: it allows mathematics to be seen as an intellectual activity, rather than as just a corpus of knowledge or a set of techniques. (p. 292)

Following this, it is not surprising that integrating history of mathematics—especially within teachers' education—is often linked with positive effects on the individual beliefs of prospective teachers (cf. Furinghetti 2007).

14.2.1 Changing Prospective Teachers' Beliefs Through History of Mathematics

Frequently changing or broadening teacher students' beliefs is formulated as an expected result when describing prospective teacher education programs, whereby the historical material is supposed to do its job in an implicit way [e.g. Beutelspacher et al. (2011), Fenaroli et al. (2014) and many others, as for example mentioned in Furinghetti (2007)]. This means, on the one hand, that only by the work with original sources for example, or by learning mathematics by its history, students are supposed to come to belief-changing experiences just by themselves, without focusing their attention explicitly on the transfer of certain beliefs. On the other hand, there are a few approaches relying on the possibility of historical material to initiate reflection on the mathematical objects, as a way to focus on students' mathematical worldviews explicitly [e.g. Jahnke et al. (2000), Jankvist (2015), or the approaches mentioned by Gulikers and Blom (2002)]. Additionally, there are a few empirical studies in which belief changes potentially caused by using history of mathematics in teachers' education were investigated. With respect to beliefs of mathematics in general see, for example, Charalambous et al. (2009), Liu (2007), Jankvist (2015, with focus on high school students) and others, such as those referred to by Bütüner (2015).

Somewhat remarkable is the fact that all of these studies, as well as the accompanying concrete examples, only focus on the nature of mathematics in general. Even if the emphasis often lies on courses concerning a special sub-discipline of mathematics (e.g. geometry, calculus, algebra) there is no attention to students' domain-specific beliefs.

There are only a few examples in literature in which history of mathematics is used in courses for teaching and learning mathematics, especially regarding subject matter didactics.[3] This is even more surprising considering that beliefs and mathematical worldviews—which are closely linked to the history of mathematics (see above)—are part of prospective teachers' curricula. Furthermore, "self-reflection" and "metacognition," as well as a "reflection of meaning and sense"—which are likely to be encouraged by working with historical sources—are metacognitive strategies alleged to profoundly make students' views on a piece of mathematics explicit (cf. Lengnink 2006). There are various reasons for the lack of a use of history in courses for teaching and learning mathematics: As we have already mentioned, there is a strong tendency to rely on the unconscious effects of historical material on students' beliefs while learning mathematics, rather than making different beliefs explicit. In addition, the common focus on beliefs concerns operations and motivations instead of the nature of mathematical objects. This focus may distract from the possibilities of historical material. Another reason why historical examples are rarely used in courses on subject matter didactics could be that there is not much secure knowledge on domain-specific beliefs of students in the literature thus far.

14.2.2 Classification of Beliefs

The categories which are used to classify beliefs differ quite a bit in each case, although many of them can be placed in the well-known catalogue of Grigutsch et al. (1998, p. 13; original in German, authors' translation), who formulate the toolbox aspect, the system aspect the process aspect and the utility aspect—described by Liljedahl et al. (2007) as follows:

> In the "toolbox aspect," mathematics is seen as a set of rules, formulae, skills and procedures, while mathematical activity means calculating as well as using rules, procedures and formulae. In the "system aspect," mathematics is characterized by logic, rigorous proofs, exact definitions and a precise mathematical language, and doing mathematics consists of accurate proofs as well as of the use of a precise and rigorous language. In the "process aspect," mathematics is considered as a constructive process where relations between different notions and sentences play an important role. Here the mathematical activity involves creative steps, such as generating rules and formulae, thereby inventing or re-inventing the mathematics. Besides these standard perspectives on mathematical beliefs, a further important component is the usefulness, or utility, of mathematics. (p. 279)

[3]An example connecting history and domain-specific beliefs of negative numbers is given by Hsieh (2000).

A closer look on these "classical" categories of Grigutsch et al. (1998) shows differences between the categories concerning the nature of objects of beliefs. While the toolbox aspect and the process aspect are connected to operations considered as central for mathematics in general, the system aspect and especially the utility aspect mainly take into account the motivation for doing mathematics in general. However, beliefs regarding the *nature of mathematical objects* are found only implicitly in the described system of categories. However, as these beliefs play a crucial role for the development of mathematical knowledge (cf. Burscheid and Struve 2010; Schoenfeld 1985; Witzke 2009) and open a field of possible connections to history of mathematics, we suggest extending the system of categories by two categories in the tradition of Schoenfeld (1985).

Schoenfeld introduced the term of a pure empiricist belief system, meaning that insights in the process of problem solving as well as the generation and verification of hypotheses depend solely on objects from physical aspects ('empirical' reality). In geometry, for example, insights for archetypal students in the sense of "pure empiricists" only come from drawings in the drawing plane while solving a problem. In such a "belief system mathematical proof is irrelevant to both the discovery and (personal, rather than formal) verification process" (Schoenfeld 1985, p. 161). Drawing from the Schoenfeldian belief system of "pure empiricism" Burscheid and Struve (2010) theoretically developed the notion of an "empirical belief system," where mathematics is ontologically bonded to reality but where deductive reasoning plays a major part (cf. the natural sciences) and a "formal(ist) belief system," which they ascribe in particular to modern professional mathematicians. In this model ideas and considerations are only generated and derived logically from existing knowledge in a formal way. For verification, hypotheses are exclusively deduced from known—meaning already proven—theorems and axioms, and verified logically: "The objects [of geometry] are taken as abstract entities, only visualized by drawings" (Burscheid and Struve 2010, p. 28; original in German, authors' translation). In such a setting at first, theorems are only expressive forms ('Aussageformen,' in Hilbert's expression), which may be interpreted semantically but do not have to exist in the empirical world.

As mentioned above, the concepts of the empirical and the formal belief system offer interesting links to the history of mathematics; they offer a special lens for looking on historical sources from an epistemological point of view. For example, the Leibnizian calculus could be reconstructed, by following the ideas of structuralism (Balzer et al. 1987), as an empirical theory, and as a prime example for mathematics developed in an empirical sense—meaning ontologically bonded to physically constructed curves on pieces of paper. This could be used again as a basis for further discussions about the teaching and learning of the subject matter (cf. Witzke 2009).

In the following sections, we first introduce a study on domain-specific beliefs of school calculus—including the broader notion of beliefs outlined above. Then, we present an example of using one original source within a course of subject matter didactics of calculus, used to discuss the empirical results of our study with our teacher students.

14.3 Domain-Specific Beliefs of School Calculus

14.3.1 Empirical Results of a Questionnaire Study

In order to describe and classify domain-specific beliefs of students at the very beginning of their mathematics studies regarding calculus in a qualitative setting we designed an open-ended questionnaire, in which the respondents were asked to associate and answer in their own words. The questionnaire consisted of three parts. The first part set the focus on one specific situation of a calculus course and the emotions the students connected to it:

> Recall and describe a special school lesson (e.g. the introduction of a new topic, classroom talk, a problem-solving situation…) of your calculus course. (Introduction of the first part)

The second part of the questionnaire made up the nucleus of the present investigation. Here we asked the students for associations on key terms regarding (school) calculus: derivative, function, integral, point of inflection, continuity, extreme value, tangent, limit, differentiability, graph, instantaneous velocity. Providing students with both lines and grid space, we gave them the possibility to express their associations in written and graphical terms and both possibilities were chosen by the respondents. In addition to the associations for single key terms of calculus, there were two more general questions that referred to the used objects and the legitimization of calculus in school with the possibility of free-text responses. These questions were:

22. What is computed in school calculus? What do the results stand for?
23. Why should calculus be taught at school?

The third part of the questionnaire was a standardized, closed-ended section, using prominent items of mathematics education for further triangulation purposes. The questionnaire concluded with personal data items. After piloting the questionnaire with a small number of respondents from different groups of interest (pupils, schoolteachers, university students and lecturers) it was given to a total of 83 first semester pre-service teachers at the University of Siegen and the University of Cologne in Germany. The students completed the questionnaires during their very first week at university, without any prior contact to university mathematics, and their responses to the second part of the questionnaire are the basis for the results we refer to in the present paper.[4] We used the categories of beliefs of mathematics that we previously discussed as a basis of the qualitative content analysis (e.g. Mayring 2002) conducted on the association section of the

[4]An extended description, including a link to the questionnaire as well as a detailed presentation of the results and a data-based analysis, is published in Witzke and Spies (2016). In the present paper we only provide a short outline of the instrument of analysis and describe the main results in a qualitative way, in order to focus on points of interest for our present topic.

questionnaire. After several iterations of the coding process we modified the theoretically based categories as given in the final coding framework (Fig. 14.1).

The analysis of the data by these categories shows several interesting results concerning particular items as well as the accumulation of typical combination of categories (cf. Witzke and Spies 2016). Summarizing the complexity of results from the first analysis of the data exhibited the following key findings:

- Ideas from elementary geometry generate dominant associations (and beliefs), even if the setting and the moment within the survey may suggest an explicit analytical calculus-context.
- The toolbox orientation, which is a mainly syntactical view on school calculus, still plays a remarkable role amongst students.
- The evaluated domain-specific beliefs of students do not reflect the paradigm of utility orientation in current efforts of subject matter didactics in Germany.
- If students associate semantic interpretations with key concepts of calculus these have an empirical or even geometrical character. That means the concept of function is associated with curves, the concept of derivative with slopes of tangents, the concept of integral with surface areas, the concept of extreme values with turning points, etc. Figure 14.2 displays an impressive example of how an empirical orientation is manifested by the students.

Even though these key findings have to be handled with care (since they need more empirical research), they give way for further interpretation, explanation and discussion. For example, the results of a follow up study of a small group of active teachers show remarkable differences. The teachers received a modified version of the association items, combined with questions concerning how they teach the core concepts of school calculus.[5] There was a much higher rate of utility orientation and a lower rate of symbolic or toolbox orientation than shown by the students. Furthermore, there were only a few answers coded as empirical orientation. [Only regarding the items to which a greater amount of students had no answer—differentiability, continuity—the teachers gave empirical oriented answers like “knickfrei” (without any kinks) or “ohne abzusetzen durch zeichnen” (draw in one uninterrupted line).] If they used drawings together with some kind of written definition, these were very often likely to be meant as visualizations of an abstract concept. These results show that teachers’ domain-specific beliefs are not directly transported into students’ beliefs. For example, pictures meant as visualizations by a logical-structural oriented teacher may be conceived of as mathematical objects by their students, which again may lead to an empirical orientation of students.

Nevertheless, it is worth making such discrepancies explicit and therefore the results are worth discussing with prospective teachers within a course on the teaching and learning of school calculus. Furthermore, the high rate of empirical orientated beliefs exhibited within the group of students makes for an interesting case to incorporate historical considerations.

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[5]For a more detailed analysis of the study involving teachers, see Spies and Witzke (2017).

Category	Definition in brief	Examples
Logical-structural orientation	Deduction and proofs, understanding/knowing about (inner-mathematical) connections of a concept and the underlying structure. **Integral**: Anti-operation to differentiation	13. Wendepunkt Extrempunkt von $f'(x) = \ldots$ **Point of inflection:** Turning point of $f'(x) = \ldots$ 12. Integral Die Gegenoperation zum Ableiten
Abstract-terminological orientation	Stressing formal rigor, using a precise mathematical language, mathematical objects are understood as abstract entities. **Function**: A function maps uniquely every x on to y	$f(x) = x^2$: „Parabel“ **f(x)=x²:** “Parabola” **f(x)=x²:** Eine Funktion ordnet jedem x genau 1 y zu
Toolbox orientation	Doing calculus means using rules, formulas and procedures in a schematic way. Association are about *how to* derivate, identify extreme values, and so on. **Point of inflection:** always setting equal to zero	20. $f(x) = x^2$ $f'(x) = 2x$ $f''(x) = 2$ **f(x)=x²:** $f'(x) = 2x$, $f''(x) = 2$ 13. Wendepunkt immer null setzen $f''(x) = 0$ $f'''(x) \neq 0$
Utility orientation	Extra-mathematical applications, mathematical modeling **Point of inflection:** Calculation for at what time or day a barrier lake loses water	13. Wendepunkt Berechnung, z.b. um wie viel Uhr oder Tag der Stausee wieder an Wasser verliert
Empirical orientation	Objects are related to the physical world. Basic concepts are derived from empirical perception. **Tangent:** “Touching in one point”	Integral d. Funktion (Flächeninhalt) Integral of a function (surface area) Tangente: „In einem Punkt berühren“
Symbolical orientation	Objects of calculus are identified with the symbols used in common.	11. Funktion $f(x)$ **Function:** $f(x)$

Fig. 14.1 Summary of the coding framework, including typical examples

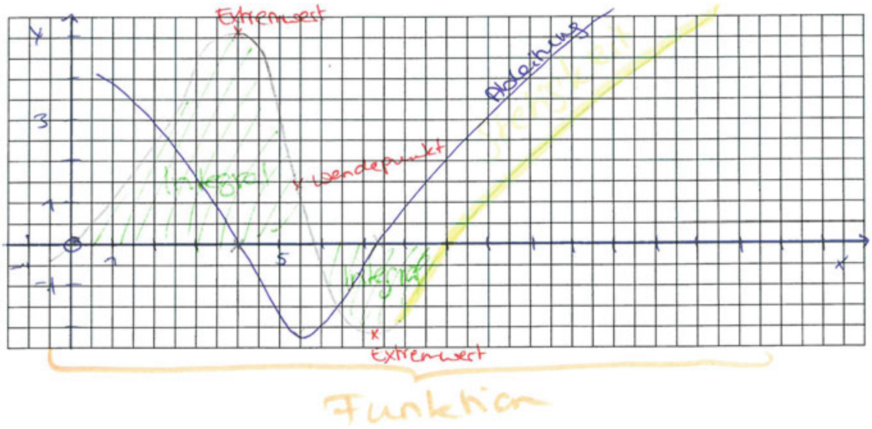

Fig. 14.2 Example from the association items of the questionnaire (answers to all items are given in one picture)

14.3.2 The Empirical Orientation in the Light of Examples from the History of Calculus

Searching for parallels and differences to such historical approaches that can be reconstructed as empirical theories as well, may help to interpret and judge the present results.

The first textbooks on calculus which were published at the end of the 17th century surprisingly worked without using a precise notion of limit. The central terms in Leibniz's *calculus differentialis* and *calculus integralis* (which is virtually equivalent to Newton's theory of fluxions) were respectively infinitesimal and infinitesimally small quantities. Looking at the first calculus textbook *Analyse des infiniment petits* by Marquis de l'Hospital in 1696,[6] today's reader may be surprised that instead of real-valued functions, curves are the objects of interest, as in many present-day school contexts. These were not defined by the help of numbers and variables, as it is common in modern analytical geometry, but were given by geometrical construction as empirical objects drawn on paper.[7]

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[6]Marquis de l'Hospital, in 1696, was a French nobleman who heavily relied on the knowledge of Jean Bernoulli and actually copied almost everything which could be found in the *Analyse* from Bernoulli's lectures of 1691–92. Leibniz—regardless of the priority controversy between himself, Bernoulli and l'Hospital—appreciated the book (Witzke 2009, p. 87) and was in contact with both.

[7]Only subsequently, these curves were described (following the ideas of Descartes) by coordinates to simplify the procedure of describing and analyzing the curves. After placing the curve in a coordinate system, mathematicians of the 17th and 18th centuries assigned geometrical lengths (length x of the abscissa or length y of the ordinate) to the points of the curve. These lengths, or what we want to call *quantity-functions* assigning lengths to points, were independent from each other; it seems as if the curve, as an empirical object, was defining the relations of the length belonging to certain points.

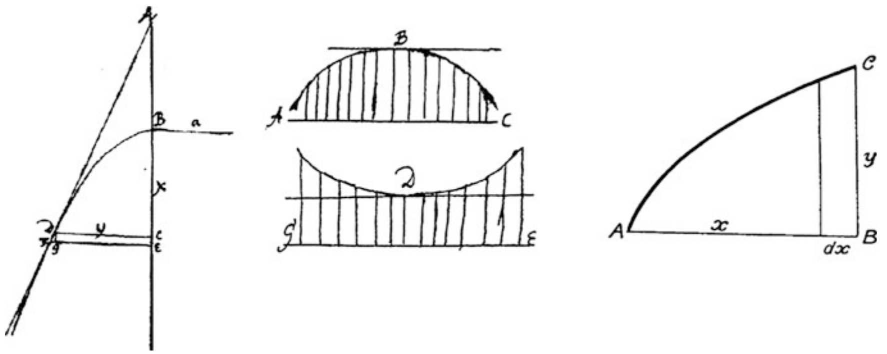

Fig. 14.3 Curves as objects of calculus, Bernoulli 1691/92 (Schafheitlin 1924, pp. 11 and 27; Kowalewski 1914, p. 12)

After short introductory remarks on the basic principles of the infinitesimal calculus and differentiation rules, Jean Bernoulli explained how to tackle mathematical problems with the new methods invented by Leibniz.

While the left-most curve in Fig. 14.3 is associated with the method of finding a tangent to a given point on a curve (cf. the example discussed with students later in this chapter), the middle one is the background for determining extreme values, and the right one shows the curve with which the author demonstrated his method to determine surface areas imbedded by a curve (integration). Therefore, he used in some sense heuristic arguments proposing that "every bent curve consists of infinitely many straight line segments" (Schafheitlin 1924, p. 11; original in German, authors' translation), which are formulated as postulates in the introduction.

The slopes of the curves, which Bernoulli and Leibniz observed—parabolas and root functions in today's terms—were of course well known at that time. Some historians conjecture that in principle they worked like natural scientists on an inductive path, extending the range of application of the calculus tentatively, e.g. from parabolas to arbitrary polynomials (cf. Boyer 1991). As already mentioned, the Leibnizian calculus works without a notion of limit; nevertheless, it relies respectively on the notion of differentials of quantities and infinitely small quantities. An expressive example here is the use of the so called characteristic triangle (cf. Fig. 14.4), which at the same time underlines parallels between the historical approach and today's discussion in subject matter didactics with outreach to modern textbooks used in schools.

Bernoulli and Leibniz assigned a right triangle to every point C of a curve, whose perpendicular sides are the differentials dx and dy and whose hypotenuse illustrates the bow length ds. The integral of ds—"meaning" the integral of $\sqrt{dx^2 + dy^2}$—can then be used to compute the bow length. This leads to the idea

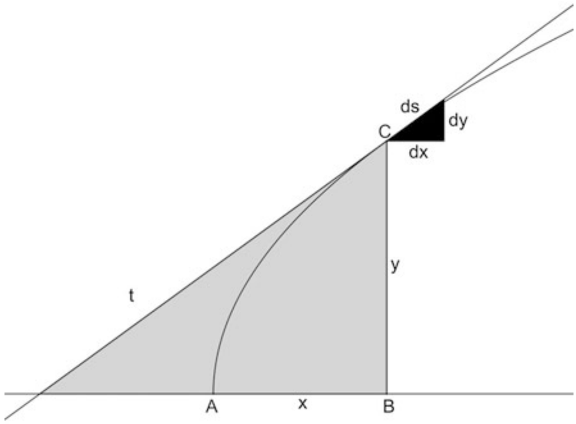

Fig. 14.4 Authors' illustration of Leibniz's characteristic triangle (cf. the left picture in Fig. 14.3)

that the characteristic triangle possesses a hypotenuse together with the curve; following this idea the tangent does not only contact the curve in a point *C* but 'shares' an infinitely small segment *ds* with the curve.

This characteristic triangle according to Leibniz, in Bernoulli's formulation, is geometrically similar to the tangent triangle, consisting of the tangent segment *t*, the sub tangent segment *m* and the ordinate *y* to a point *C* (cf. Fig. 14.4). In the Leibnizian calculus it holds that $ds{:}dx{:}dy = t{:}m{:}y$. From this, it easily follows that the quotient of $\frac{dy}{dx}$ gives the slope of the tangent as $dy{:}dx = y{:}m$ (for a detailed account of this simplified depiction compare with Witzke 2009).

This wonderfully ostensive deduction of the concept of derivative of course does not (aspire to) meet standards of precision of modern (university) mathematics; nevertheless, it works in an appealingly elegant way, and—especially interesting in the context of this paper—seems to correspond with modern didactical ideas of how to teach calculus at school. In some schoolbooks, we see that (especially in absence of a true concept of limit) a curve and tangent may have a line segment "in common" (see for example the popular German textbook Lambacher Schweizer; Brandt et al. 2014, p. 47). In special accompanying work assignments in the book students are confronted with the idea of magnifying a part of a curve with the help of the zoom function of the graphing calculator (GC), which eventually causes the same intuition. The idea of deriving graphically by the help of a "Funktionenemikroskop," or "optical microscope," was actually proposed by Kirsch (1979) and Tall (1980). The idea of zooming in, connected with the concept of local straightness in Tall's approach, forms an elaborated 'graphic calculus.'[8] Taking into account for example the second "postulate" made by Johann Bernoulli

[8]In Tall's understanding this way of teaching calculus can even lay foundations for an ε-δ-calculus or an integrated approach to non-standard-analysis giving proof to the thesis that we do not in general obstruct the development of precise student knowledge regarding calculus by "visual approaches." A detailed motivation and description can be found in Tall (2013, p. 298ff.).

in his lectures of differential calculus: "Every bent curve consists of infinitely many straight lines, which are infinitely small" (Schafheitlin 1924, p. 11), we see at first glance interesting parallels to modern 'visual' approaches in high school textbooks. But on the second glance there are remarkable differences; Bernoulli sees the need to justify his conclusion in a logical or at least heuristic way by referring to his own postulates stated at the beginning of the text or by referring to heuristic analogies taken from Euclidean geometry. He is clearly devoted to "more geometrico" although he was probably aware that some of his conclusions were set on what was seen as dubious logical foundations at the time (cf. Witzke 2009, pp. 130–136 and 178ff.). In contrast, today's widespread textbook argumentations in Germany are quite often based solely on "empirical evidence" taken from drawings. Without any further intervention by the teacher, this will, for many students, eventually lead to a naïve empirical orientation about calculus in the sense of Schoenfeld.

Another point worth mentioning regarding the comparison of empirical school calculus and historical examples is the modern criticism of a schematic-symbolic orientation of calculus in school curricula: when we look at Leibniz, the Bernoullis, and their students, we see that the algorithmic power of the "nova methodus" was a crucial point. The new differential and integral calculus offered standard routines for solving a variety of problems, like the determination of tangents, surfaces underneath curves, circles of curvature, etc., in a systematic way. The ancient Greeks could solve some of these problems with singular methods but it was Leibniz who first offered a systematic calculus for this. Looking at all the demands for a qualitative turn regarding modern calculus curricula, we should keep in mind that providing these algorithms is a major quality of differential and integral calculus. Nevertheless, the historical calculus always came together with intuition on curves to explain matters at least in a geometric manner. This in principle may lead to a more appreciative view on the combination of toolbox orientation and empirical orientation, which we have seen relatively often in our empirical investigation.

The detected resurgence of an empirical orientation in present-day school calculus gives one more reason for using original sources as a tool in teachers' education, as the following example makes a case for.

14.4 Reflecting on the Empirical Orientation with Bernoulli's *Lectiones de calculo differentialium*

At the University of Siegen, examples from the history of mathematics in general and original sources in particular have been used in a range of seminars and lectures for prospective teachers. According to the main topic or aim of the respective courses (e.g. university mathematics, school mathematics from an advanced point of view, courses on the teaching and learning of mathematics) different ways of introducing the historical examples are necessary (cf. Allmendinger et al. 2015).

The place in university curricula for learning about different domain-specific beliefs of students and teachers and their impact on the learning process of the chosen domain are seminars on subject matter didactics. Within such courses the phenomenon of different mathematical worldviews can be linked with questions regarding the teaching and learning of the chosen subject in current mathematics lessons, in the history of mathematics, as well as with results of empirical studies as described above. As experiments show, examples from the history of the subject matter are an appropriate starting point to this subject area. In addition to a range of general reasons for using original sources in teachers' education (cf. for example, the approaches compiled in Allmendinger et al. 2015), the issue of epistemic beliefs can be addressed via the history of mathematics for different reasons: On the one hand students get to know that during the historical development different belief systems were in place for solving the central problems of the discipline. Furthermore, the difficulties of understanding an old argumentation can sensitize for aims and assumptions which possibly are different from the current approaches. Making such differences explicit provides a fruitful starting point for discussions about the nature of the mathematical objects, the language used and how sophisticated the underlying theory is in each case. The following case study[9] proposes to concretize this.

14.4.1 Reading Bernoulli Within a Seminar of Subject Matter Didactics of School Calculus

Within a seminar of subject matter didactics of school calculus 'Didaktik der Analysis' at the University of Siegen, students got to know a well-known original source from the history of calculus: A translation of "exercise one" from Johan Bernoulli's *Lectiones de calculo differentialium* (1691/92; "How to find the tangent on a parabola") and as additional information his postulates and the rules about sums and products of differentials (Schafheitlin 1924, pp. 1–17). To avoid a totally anachronistic way of working with such a text, following the rules of a hermeneutic analysis of the source (e.g. Jahnke et al. 2000) at least some biographical information about the author and his oeuvre are necessary, as well as knowledge of the mathematical background. The latter was expected—as the participants were prospective high school teachers with a mathematical expertise regarding calculus; the former was given in a very condensed form. Due to strict time restrictions for the seminar lessons, students were provided with selected information as opposed to finding it on their own.[10]

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[9]The example presented is just part of reflective teaching reacting on the results of our empirical research on domain-specific beliefs presented earlier in the chapter. Since we have not completed further systematical evaluation of the course or analysis of the students' products, we can only describe our theory-based teaching design and link our concrete teaching experiences to theoretical considerations.

[10]It should be noted that it would have been more helpful for students to locate the relevant information on their own (see Allmendinger et al. 2015).

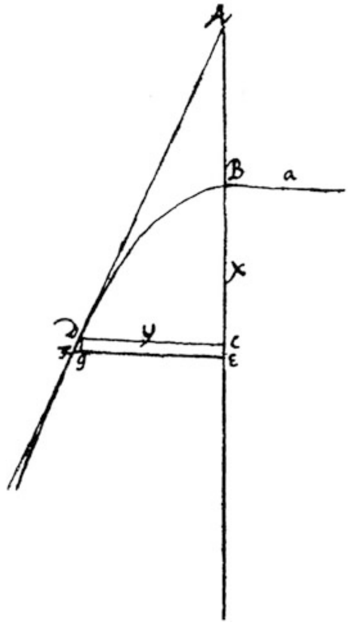

Fig. 14.5 Drawing of Bernoulli, (1691/92), Exercise one (Schafheitlin 1924, p. 11)

The following assignments for groups of three to four students guided the work on the source itself during the seminar lesson:

1. Read the excerpt from Johann Bernoulli's "Vorlesung über das Rechnen mit Differentialen" ("Lecture on the differential calculus") (1691/92) carefully (with paper and pencil):
   (a) Draw your own sketch while reading. Use the one of Bernoulli (Fig. 14.5) as an orientation.
   (b) How can the quantities dx and dy be interpreted within this construction? What is the so called 'sub tangent'? Which geometrical considerations are used?
2. Compare Bernoulli's text and current textbooks in school:
   (a) What is the starting point in each text? How is the problem formulated and motivated?
   (b) Which mathematical 'tools' (geometrical or algebraic considerations, a coordinate system, sketches, computer, etc.) are used?

Questions 1a and 1b should help students access the source and to understand the underlying mathematical content (see the explanation above). These questions already helped to focus on the geometrical character of Bernoulli's formulation of the problem and the solution. Even though all of the students knew how to find the slope of a tangent in a given point of a quadratic function, it seemed to be that this geometrical character of argumentation makes grasping Bernoulli's arguments

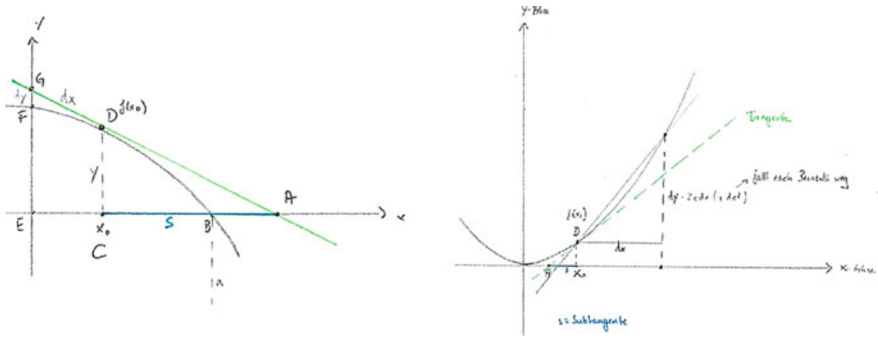

Fig. 14.6 Students' reconstruction of Bernoulli's solution by using today's coordinate system

difficult. Most students started to draw a coordinate system and reformulated the arguments by their notions of functions. This in fact is more complicated, especially because of the fact that Bernoulli uses the letter *x* for the ordinate and *y* for the abscissa (see Fig. 14.5)—which is the opposite of the common notion of today's calculus. Figure 14.6 shows two trials of reconstructing the situation using a coordinate system; in the first step students were confused by the change of the common orientations of the variables *x* and *y*.

Other difficulties arose when dealing with the infinitely small triangles. For example, students struggled with the pragmatic use of infinitely small parts of a sum when calculating [see the students comment in Fig. 14.6: "*dy* fällt nach Bernoulli weg" ("*dy* drops out according to Bernoulli")] or by trying to bring the notion of limits into account.

Another problem arose when interpreting the length "*a*"—a line used by Bernoulli to construct the curve of the parabola. Most students struggled with this line documented by comments like "Das nehme ich so hin" ("I take that for granted") or just ignored this line even in their drawings of the situation. Figure 14.7 gives an example where one student drew the parabola without the construction line, with today's picture of a full parabola in mind, including its entire graph with respect to its axis. The different backgrounds of the 17th century mathematician and today's teacher students who worked here became more obvious when they compared the former with today's textbooks. Even if they reconstructed Bernoulli's geometrical argumentation without using a coordinate system, they still talked of the graph of a function as modern textbooks do, and not of the curves as Bernoulli did.

In the previous lesson the students discussed different established approaches to the introduction of the derivative at school—here again, approaches from the history of calculus are considered as helpful (cf. Danckwerts and Vogel 2006, pp. 45ff.). In addition, students were made to recognize them within popular German textbooks. Question 2 led students to recapitulate these and to compare them with Bernoulli's approach.

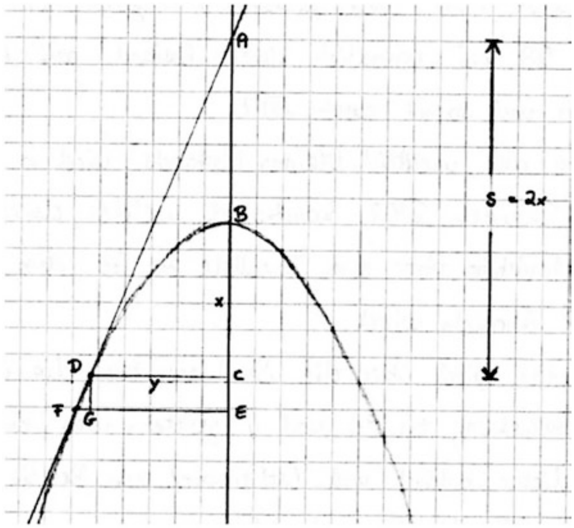

Fig. 14.7 Sketch of the parabola without using the construction-line

14.4.2 Reflecting on the Empirical Orientation Guided by Parallels and Differences

The aim of Bernoulli (finding a second point in the drawing plane to draw the tangent vs. determining the slope of the tangent in any point by getting the derivative function in modern approaches), as well as the discussed objects (curves in the drawing plane vs. functions or graphs of functions) and the mathematical tools (Bernoulli used theorems inspired by Euclidean geometry about congruence of (infinite small) triangles vs. algebraic rules and the limit value of the ratio of the differences used today), are very different from the modern differential calculus. Some of the students indeed needed quite some time to grasp especially the initial aim of Bernoulli. But on second glance there were also correspondences with the modern textbooks. In both cases, the parabola or the quadratic function is used as an introductory example. This somehow superficial observation can be the starting point of a discussion of the didactical quality of this example.

Another correspondence can open the discussion about different domain-specific beliefs of mathematics. That is, even if current textbooks often start interpreting the derivative as instantaneous rate of change, the argumentation is prominently supported by pictures of graphs of functions with their secants and tangents or supported by arguments like "zooming" in on the graph. The objects discussed and used as an argumentation basis in many cases are curves, which can be seen through the eyes of the observer: drawn on paper, generated by GCs, or taken from real contexts like bridges or trajectory parabolas. The consequent usage of GCs even supports this impression as the pictures of curves are potentially always present and are especially used to make qualitative statements or to motivate arguments. This may lead to an "empirical belief system," as introduced above, in which the function is identified with the sketch of the graph in the way Bernoulli did by arguing by means of the constructed curve in the drawing plane.

Here the next differences appear and lead the students to evaluate today's approaches. Whereas Bernoulli used the theorems of Euclidean geometry and his own axioms to justify and explain each step of his construction, current textbooks often refer only to what the students may see on the picture or within the screen of a computer algebra system (e.g. GC). This may be one reason for the (naïve) empirical orientation (Schoenfeld's pure empiricism) of a few students found in our study. The discussion may also achieve a possible difference between the intention of teachers and textbook authors and the impact on the belief system of their students. While the former may use pictures for didactical reasons to visualize abstract concepts, for the later the drawings may become the mathematical object itself.[11] This again gives rise to reflect on the ontological and epistemological status of the graph of a function and arguments based only on the empirical intuition.

In addition, the experience with the—in some ways not as different as expected—approaches in history and current school textbooks could be turned in a more constructive way by thought experiments like: Imagine if we would teach calculus in today's high school in the way Bernoulli did. What possible problems do we have to consider? In what ways would it be "another" calculus and would it be possible to link this to more formal or utility-orientated approaches of calculus? Taking our considerations into account it might be a sensible modern didactical perspective to develop a visual calculus leaning on (visual) concepts of infinitesimal calculus of the 17th/18th century. Teaching (variants) of Leibniz's calculus must not obstruct the further development of mathematical knowledge on a precise formal level. Therefore, it needs to be implemented properly—and not only for illustrative reasons—following a thoughtful conceptual design. This means to include deductive reasoning and the usage of infinitesimal arguments within visual approaches to calculus (cf. Jahnke 2006). These mathematical competencies must later be transferable to the formal level (Witzke 2014). Without these an empirical calculus will not be more than an "elementary algebra combined with the sketching of graphs" (Tietze et al. 2000; original in German, authors' translation) and will never reach the status of a "blended embodiment and symbolism" (Tall 2013) or, in our terminology, of a coherent empirical theory on calculus.

Of course, a profound reflection on such issues needs additional information (e.g. about different belief systems and the supposed impact on the learning and understanding) and quite some time of plenary discussion after working with the original source. But the special focus on domain-specific beliefs provides benefit on different levels. First, from a historical point of view it guides a deep and exact analysis of the given source. Furthermore, this is an example of how the effect of

[11]This hypothesis is supported by the results of a follow up investigation of domain-specific beliefs of high school teachers using a questionnaire similar to the one described above (cf. Spies and Witzke 2017).

alienation of a common notion by an original source leads to fruitful discussions about special issues on teaching and learning of mathematics within preservice teacher education.

14.5 Beyond the Special Topic: Original Sources in Courses on Mathematics Education in General

The extract of Bernoulli's *Lectiones* gave rise to reflect on a wide range of aspects with respect to domain-specific beliefs. Beyond the special example and the special topic in school calculus, there are many more positive accounts for experiences by using original sources within courses on mathematics education to foster discussions on belief systems and worldviews of mathematics at the University of Siegen. For example, the work on the transition problem from school to university mathematics heavily relies on the use of original sources for making belief systems regarding geometry explicit (Euclid vs. Projective Geometry and Non-Euclidean Geometries; Hilbert; Gödel; etc.) and discussing them with students (cf. Witzke 2015; Witzke et al. 2016). G. Stoffels' promising dissertation project focuses on the awareness for changing domain-specific beliefs of stochastics for pre-service teachers, on the work with original sources of R. von Mises and A. Kolmogoroff. To make students sensitive for the aesthetics of mathematics and to make the view of mathematics as a creative endeavor explicit, there are experiences with the reading of Al-Khwãrizmî's work on how to solve quadratic equations contrasted by today's formulas (for using and reflecting the source with middle school students see Allmendinger and Spies 2015).

By just looking only at these examples we get the insight that the implementation of original sources regarding beliefs in teacher education bears great potential, which brings additional value to sole usage in subject matter settings. A short exploration on the basis of the three "general ideas" for using original sources formulated by Jahnke et al. (2000) (replacement, reorientation and cultural understanding) may help to grasp possible effects more systematically.

By studying original sources, students are supposed to experience mathematics as an intellectual human endeavor instead of experiencing a fixed collection of a corpus of knowledge and techniques (replacement) (Jahnke et al. 2000, p. 292). Thinking of questions concerning mathematics education such experiences could lead to reflection on the nature of mathematics in general as well as on the role of the mathematician and the mathematical working processes. Furthermore, it makes students aware of the fact that even the great "masters of mathematics" struggled with the core ideas of their discipline (like infinitely small quantities for example) and were far away from today's rigor. By experiencing this, prospective teachers may learn to esteem such intuitive and abductive approaches instead of disregarding empirical graphical arguments in documents produced by pupils. In addition, historical sources as documents of authentic mathematical practices on an elementary level give the opportunity for reconstructing real problem-solving—or modeling processes—leaning on Freudenthal's idea of guided reinvention (Freudenthal 1991).

Additionally, reading sources from history helps to make the "familiar becomes unfamiliar" (reorientation) (Jahnke et al. 2000, p. 292). The positive effects of this "dépaysement" (cf. Barbin 1997) for learning mathematics have been widely discussed. With respect to topics of teaching and learning mathematics the impact may be different to some extent. The closer look on subject matter caused by the effect of "dépaysement" may draw the students' attention to the mathematical objects in detail, which is an essential condition for any serious discussion, at least in subject matter didactics. Regarding prospective classroom practice, this is supposed to be a helpful exercise for taking the pupils' perspective, for "learning to listen" (cf. Arcavi and Isoda 2007).

Furthermore, the detailed analysis necessary to reconstruct historical mathematical thinking processes underlying the sources on an epistemological level shows parallels to the analysis of pupils' products of problem-solving processes and their possible difficulties of understanding (e.g. epistemological gaps; cf. Bachelard 1991 or Sierpinska 1994). Depending on the selected historical text, it may also give examples of and lead to a systematic discussion of different modes of notation.

If students analyze original sources in a contextualized, not totally anachronistic way, they must pay attention to the sociocultural background of the text and his author. Once again, with this, mathematics may be experienced as a cultural endeavor (cultural understanding) and the influences of mathematics on society and vice versa can be discussed authentically. In addition, examples from old textbooks could give an authentic view into the history of teaching and learning mathematics.

Besides, there is at least one more fundamental reason for using original sources in courses of teaching and learning for prospective teachers: they deliver examples for bringing history into the mathematics classroom.

14.6 Closing Remarks

Although we cannot generalize the results from our relatively small data set, the case study provides evidence for the hypothesis that reading historical original sources does not only (implicitly or explicitly) lead to a deeper understanding of the mathematical content and an enrichment of mathematical worldviews, when used in content knowledge courses; when used and discussed in mathematics education courses they provide great potential for making domain-specific aspects and beliefs explicit, which lead to a more differentiated view on ways of how (in our example) calculus can be understood in a proper way. Our reflections give way to the consideration that especially regarding the empirical orientation, there is an interesting analogy between historical and present student beliefs. In both cases, it is the curve as a constructed and drawn "empirical" object on a piece of paper which is the object of interest—not a formal-abstract notion of function. The meaning of this gets again very clear as we look in the German mathematics standards (and textbooks designed according to them), in which it is required to teach the procedure of "graphical differentiation" meaning to draw a derivative curve from a given curve by virtually looking at characteristic points. This produces the impression that calculus is a branch of mathematics, which is about drawing objects in a certain manner—if we recall the reconstruction of Bernoulli's approach to finding a tangent to the parabola we see that it is about finding a second point *A* to virtually draw a tangent line to a curve constructed before on a piece of paper; and not to find the slope in a certain point as nowadays is intended. Making this explicit is a very important aspect, which might help to reflect with pre-service teachers why students in Germany quite often develop an empirical orientation by the time they graduate and how to deal with this phenomenon. This is especially valid for the problem of "naïve empiricism" which, following the terminology of Schoenfeld, may be attested when students see calculus as a matter about empirical objects (drawn curves). But in contrast to their historical predecessors, they do so without any sense of logical reasoning or structure, and in particular, regarding infinitesimal arguments.

More generally speaking, the use of original sources appears as an appropriate tool to discuss the prerequisites and consequences of the different orientations presented in this chapter (cf. Fig. 14.1). They give us the possibility to view the same content area in a totally different light—given the readiness to get involved with the historical approach without thinking of barriers.

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Chapter 15
Primary Historical Sources in the Classroom

Graph Theory and Spanning Trees

Jerry Lodder

Abstract I study student response to learning from a specific historical curricular module and compare this to advantages of learning from historical sources cited in education literature. The curricular module is "Networks and Spanning Trees," based on the original works of Arthur Cayley, Heinz Prüfer and Otakar Borůvka. Cayley identifies a compelling pattern in the enumeration of (labeled) trees, although his counting argument is incomplete. Prüfer provides an alternate proof of "Cayley's formula" by counting all railway networks connecting n towns that contain the least number of segments. Borůvka develops one of the first algorithms for finding a minimal spanning tree by considering how best to connect n towns to an electrical network.

Keywords Historical sources · Graph theory · Trees · Minimal spanning trees

15.1 Introduction

I examine the pedagogical benefits afforded by teaching from primary source documents by studying the details of one classroom module, "Networks and Spanning Trees" (Lodder 2013). This is only one of several curricular modules covering topics in discrete mathematics and computer science, available online (Barnett et al. 2013). "Networks and Spanning Trees" highlights the work of three scholars, Arthur Cayley (1821–1895), Heinz Prüfer (1896–1934) and Otakar Borůvka (1899–1995) on the use and enumeration of (labeled) trees as well as one of the first algorithms for finding a *minimal spanning tree*, all written before the subject of modern graph theory had been developed. These historical sources provide context to the subject matter, with the authors stating a compelling problem whose solution involves key concepts or constructions that have become abstract

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definitions or theorems in present-day textbooks. A verbal description of the problem, without specialized vocabulary, offers a more inviting and understandable entry into the subject. The sources provide motivation for study, once the original problem has been stated and its significance is understood. When arranged over time, the sources provide direction to the subject matter not apparent when reading the final axiomatized version in a textbook. Also, studying from primary sources leads to an interdisciplinary approach to learning, since the sources were almost always written before the modern division of scholarship into collegiate departments. Many concepts in the module "Networks and Spanning Trees" are germane to both discrete mathematics and computer science. In fact, algorithms for finding a minimal spanning tree have been a topic of research in computer science, while combinatorial arguments for counting trees are primarily taught in mathematics courses.

In terms of a theoretical framework, the classroom module "Networks and Spanning Trees" uses history-as-a-tool (Jankvist 2009) to learn the inner issues of graph theory and some of its applications. To elaborate, the primary goal of introducing historical sources in the classroom is to learn mathematics. The history of the subject is used as a tool to help in the understanding of mathematics. The inner issues refer to the lemmas, theorems, procedures or reasoning processes within the subject. In this module we read that Prüfer is motivated by finding all ways of connecting n towns to a railway network using the least number of railway segments as possible. Borůvka is motivated by a different problem, namely of all possible networks connecting n towns to an electrical grid, which network uses the least amount of electrical cable. Neither author uses any specialized vocabulary in the statement or solution to these problems, not even the word tree, which displays an advantage of learning from primary sources articulated by Jahnke and his colleagues (2000), namely an ease of understanding the motivational problem. Students and instructors may feel a bit of cognitive dissonance, or *dépaysement* (Barbin 1997) when reading a historical source and find no modern theorems. Primary source documents are not written like textbooks, and cognitive dissonance arises when the reader encounters the unexpected, particularly in what has become the formalized subject of mathematics. In the language of Sfard (2000, p. 161) Prüfer and Borůvka are using object-level rules to find solutions to their respective problems, while a meta-discursive discussion is needed to formulate these results in terms of modern theorems or algorithms in graph theory, a discussion worth pursuing in the classroom. Cayley uses the rules of algebra (associativity, commutativity, distributivity) when manipulating polynomials to represent trees, and these are the rules of the "objects" (object-level rules) in his treatment of counting trees. Prüfer counts trees with what today would be called Cartesian products of sets, which requires a different set of rules for the enumeration of their elements. To formulate the modern definition of a tree, however, we must go beyond counting arguments, and formalize the fundamental properties of the objects we wish to count. This requires reasoning beyond the object-level rules, and the beginning of a meta-discursive discussion. Throughout this chapter, I mention which definitions or theorems have evolved from observations in the primary source documents. In fact, Kjeldsen and Blomhøj (2012) suggest that reading original sources may be essential for raising students' awareness of the meta-discursive rules that govern the current mathematical paradigm.

15.2 Design Features of the Module

The module "Networks and Spanning Trees" was originally written to explore and explain the ideas behind the modern definition of a tree, appearing today in many textbooks on graph theory or discrete mathematics. There a tree is defined as a connected graph containing no cycles, which serves as the starting point for many lemmas and theorems about trees. While this definition has intuitive appeal after a study of the subject is complete, the definition remains opaque and in fact arbitrary for novices. The modern definition of a tree, stated by Veblen (1922), is an outgrowth of the study of the connectivity of a topological space and not an exploration of combinatorial problems that can be solved using the structure of a tree. In fact, explaining the mathematics behind any opaque definition or procedure could be the staring point of a historical curricular module. The pedagogical idea is to replace the memorization of technical definitions with the study of more engaging and compelling mathematical problems whose solutions involve the constructions appearing in modern definitions.

Once knowledge of the historical background of the topic is acquired, often from a few key primary sources, authorship of the module can begin. The modes of reasoning and standards of rigor from historical sources are often very different from those of today. Care should be taken to avoid an anachronistic or Whiggish (Fried 2001) view of history by evaluating sources in terms of the modern mathematical paradigm of the subject. For example, although Cayley uses the term "tree," he offers no mathematical definition of this term. Its use is intuitive and he arrives at a striking pattern for the enumeration of certain types of trees based on simple counting arguments that involve no specialized algorithms. Students are more often able to participate in the reasoning process when the cognitive demand (Schoenfeld and Floden 2014) is eased via the less formal description of a problem from historical sources. Careful study then reveals the need for more rigorous reasoning, often developed in later sources by scholars or mathematicians confronted with the same situation. Thus, we see Prüfer offers a rigorous proof of "Cayley's formula." Additionally we see how concepts and definitions evolve over time. Although Prüfer presents no formal definition of a tree, nor does he even use the word "tree," he seeks to count all railway networks between n-many towns that are connected and contain the least number of railway segments. This reflects the characterization of a tree as a connected, minimally-connected graph, which is logically equivalent to a connected graph containing no cycles (the textbook definition of a tree). By studying how modern textbook concepts evolved from solving problems of the past, students (and instructors) are able to resolve the cognitive dissonance encountered when first reading an original source.

A historical module should have a focal point, a main result with significance outside mathematical formalism. For "Networks and Spanning Trees," the ultimate goal is to understand Borůvka's algorithm for finding a minimal spanning tree, which is a tree of shortest total edge length that connects n-many towns (or points). The origin of this problem is to connect n towns to an electrical network using the least amount of cable, which Borůvka solved in 1926. Once the significance of this result is understood, we see how the concept of tree defines the domain of study for the minimal spanning tree algorithm, and we see how Cayley's and Prüfer's work enumerates the elements in the domain. Borůvka's work can then be understood as an algorithm for finding a tree of minimal total edge length over this domain. In this way we witness how the historical pieces fit together to form a coherent whole. Let us now outline the specific mathematical content of this module.

15.3 Cayley's "Theorem on Trees"

Although not motivated by a problem as broad in scope as those stated by Prüfer or Borüvka, Cayley (1857) does introduce the term tree, without definition, to describe the logical branching when iterating the fundamental process of (partial) differentiation. In a later publication Cayley (1889) counts trees in which each knot (his term for vertex or node) carries a particular label or letter. Two trees are counted as the same if and only if the same pairs of vertices are directly connected by an edge. Cayley associates to each tree a certain polynomial constructed from the vertex labels (letters). He then adds all polynomials for a fixed number of vertices and arrives at a striking pattern, which he claims (without proper justification) continues for all values of n, where n is the number of vertices. The object-level rules (Sfard 2000, p. 161) for working with polynomials are those of algebra, namely associativity and commutativity of addition and multiplication, and distributivity of multiplication across addition. To understand Cayley's use of polynomials, first consider trees with three fixed vertices, labeled α, β, γ in Fig. 15.1.

In tree I the vertices α and γ are not directly connected by an edge, while in tree II, α and β are not directly connected, and in tree III, β and γ are not directly connected. Should these three trees be counted as distinct? Cayley does so in his 1889 paper, and introduces a method of counting trees based on assigning polynomials to trees. Let us construct polynomials for the trees given in Fig. 15.1 by multiplying all pairs of vertices in the given tree that are directly connected by an

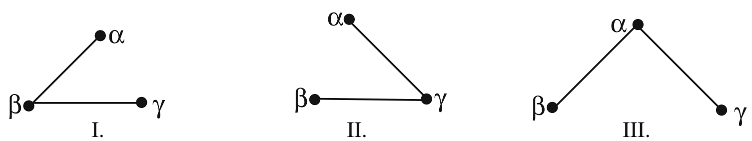


Fig. 15.1 Labeled trees on three vertices

edge, and then follow the rules of algebra. For tree I, α is directly connected to β and β is directly connected to γ. Thus, the Cayley polynomial is given by $(\alpha\beta)(\beta\gamma) = \beta(\alpha\beta\gamma)$, which uses the associativity and commutativity of multiplication. For tree II, α is directly connected to γ and γ is directly connected to β. In this case, the Cayley polynomial is $(\alpha\gamma)(\gamma\beta) = \gamma(\alpha\beta\gamma)$. For tree III, the Cayley polynomial is $(\beta\alpha)(\alpha\gamma) = \alpha(\alpha\beta\gamma)$. Adding all polynomials for labeled trees on three vertices, we have $(\alpha + \beta + \gamma)(\alpha\beta\gamma)$.

To follow Cayley's argument (Cayley 1889), the reader is asked to consider the number of possible trees on four vertices α, β, γ, δ. First, arrange the vertices in a fixed configuration, such as a diamond in Fig. 15.2.

Then, begin to connect the vertices via edges to form trees. Of course, one vertex, for example α, could be connected to each of the other vertices, β, γ, δ, forming the polynomial $(\alpha\beta)(\alpha\gamma)(\alpha\delta) = \alpha^2(\alpha\beta\gamma\delta)$. The list of all Cayley polynomials that can be constructed could be either a homework problem or an in-class activity. For reference, the complete list is:

$$\begin{array}{llll}
(\alpha\beta)(\alpha\beta\gamma\delta), & (\alpha\gamma)(\alpha\beta\gamma\delta), & (\alpha\delta)(\alpha\beta\gamma\delta), & \alpha^2(\alpha\beta\gamma\delta), \\
(\beta\alpha)(\alpha\beta\gamma\delta), & (\beta\gamma)(\alpha\beta\gamma\delta), & (\beta\delta)(\alpha\beta\gamma\delta), & \beta^2(\alpha\beta\gamma\delta), \\
(\gamma\alpha)(\alpha\beta\gamma\delta), & (\gamma\beta)(\alpha\beta\gamma\delta), & (\gamma\delta)(\alpha\beta\gamma\delta), & \gamma^2(\alpha\beta\gamma\delta), \\
(\delta\alpha)(\alpha\beta\gamma\delta), & (\delta\beta)(\alpha\beta\gamma\delta), & (\delta\gamma)(\alpha\beta\gamma\delta), & \delta^2(\alpha\beta\gamma\delta)
\end{array}$$

Although this list is a bit lengthy, adding all of the above terms, we have:

$$(\alpha + \beta + \gamma + \delta)^2(\alpha\beta\gamma\delta),$$

which hints at a simple pattern for counting labeled trees on n vertices. A discovery exercise for students could be to articulate what this pattern is. Note that the commutativity of polynomials, such as $\alpha\beta = \beta\alpha$, loses information about how the tree is constructed, a point the instructor may wish to explore with the class. Might there be a better symbolic device other than polynomials that encodes the construction of a tree? That question will be answered in the next section. Cayley (1889) discusses the number of trees on six vertices α, β, γ, δ, ε and ζ in detail, arriving at the expression $(\alpha + \beta + \gamma + \delta + \varepsilon + \zeta)^4(\alpha\beta\gamma\delta\varepsilon\zeta)$. After the six-vertex example, Cayley (1889) writes "It will be at once seen that the proof for this particular case is applicable for any value whatever of n," although the inverse correspondence between polynomials and trees is not mentioned. To illustrate the difficulties with the inverse correspondence, ask students to find all trees with polynomial $\alpha^2\beta^2(\alpha\beta\gamma\delta\varepsilon\zeta)$ in the six-vertex example. Nonetheless, a compelling

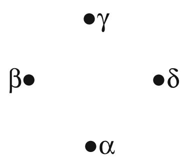


Fig. 15.2 Find the labeled trees

pattern in the number of labeled trees with n vertices has been identified, corresponding to the number of terms in an expansion of the form $(\alpha + \beta + \gamma + \cdots + \omega)^{n-2}$, where there are n-many letters in the list α, β, γ, …, ω. This suggests that there are n^{n-2} labeled trees on n vertices.

15.4 Prüfer's Enumeration of Trees

Heinz Prüfer begins his paper (Prüfer 1918) with the geometric problem of counting all railway networks connecting n-many towns so that: (1) the least number of railway segments is used; and (2) a person can travel from each town to any other town by some sequence of connected segments. The ideas expressed here, that the least number of railway segments is used, yet travel remains possible between any two towns, are recognized today as properties that characterize such a railway network as a tree. Since the town names (labels) are fixed, the modern concept of a labeled tree is an excellent model for this problem. Prüfer also states several properties about such a network that have become modern theorems in graph theory. For example, the statement that every network connecting n towns has exactly $n - 1$ many single segments has become the theorem that every tree on n vertices has $n - 1$ edges. Also, the statement that every network has an endpoint has become the theorem that every tree has a leaf (a vertex with only one edge connected to it). Prüfer assigns to each tree with n vertices a "symbol" consisting of $n - 2$ numbers (or characters) taken from the labels of the vertices. Moreover, he establishes that each tree corresponds to only one symbol, and each symbol corresponds to only one tree. Thus, the problem of counting trees is reduced to the problem of counting sequences of length $n - 2$ taken from a set of n numbers (or characters), where the characters may be repeated. Two symbols are considered the same if and only if all corresponding entries are the same. The resulting number of symbols is n^{n-2} for $n > 1$.

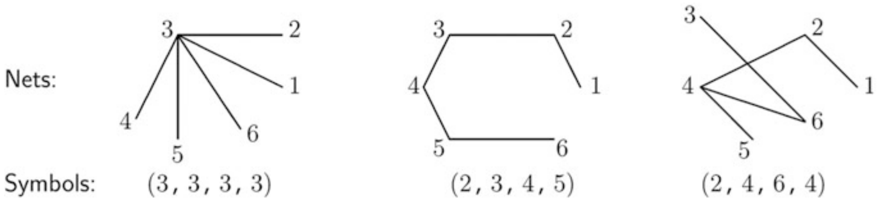
Prüfer (1918, 1976) writes:

> Consider a country with n towns. These towns must be connected by a railway network of $n - 1$ single segments (the smallest possible number) in such a way that one can travel from each town to every other town. There are n^{n-2} different railway networks of this kind.
>
> By a single segment is meant a stretch of railway that connects only two towns. The theorem can be proved by assigning to each railway network, in a unique way, a symbol $\{a_1, a_2, \ldots, a_{n-2}\}$, whose $n - 2$ elements can be selected independently from any of the numbers 1, 2, …, n. There are n^{n-2} such symbols, and this fact, together with the one-to-one correspondence between networks and symbols, will complete the proof. (p. 53)

Today, a Prüfer symbol would be written with parentheses as delimiters, i.e., $(a_1, a_2, \ldots, a_{n-2})$, and considered as an element of the Cartesian product $V^{n-2} = V \times V \times \cdots \times V$, where V is the set of vertex labels (or letters). The object-level rules (Sfard 2000, p. 161) of counting Prüfer symbols are the rules of enumerating the elements of Cartesian products, where elements are not subject to the commutativity of their components. To understand the construction of a Prüfer symbol, I quote from the original paper (Prüfer 1976):

> In the case $n = 2$, the empty symbol corresponds to the only possible network, consisting of just one single segment that connects both towns. If $n > 2$, we denote the towns by the numbers 1, 2, …, n and specify them in a fixed sequence. The towns at which only one segment terminates we call the endpoints.
>
> … In order to define the symbol belonging to a given net for $n > 2$, we proceed as follows. Let b_1 be the first town which is an endpoint of the net, and a_1 the town which is directly joined to b_1. Then a_1 is the first element of the symbol. We now strike out the town b_1 and the segment b_1a_1. There remains a net containing $n - 2$ segments that connects $n - 1$ towns in such a way that one can travel from each town to any other. (p. 53)

The above characterizes the new graph, after deleting vertex b_1 and edge b_1a_1, as a tree. The process may be iterated, or a recursive construction can be formulated to yield a Prüfer symbol. Prüfer offers several examples of how to construct symbols from nets (trees) in his paper, given below (Prüfer 1976, p. 53).



The ambiguity raised in the last section over which trees on six vertices have Cayley polynomial $\alpha^2\beta^2(\alpha\beta\gamma\delta\varepsilon\zeta)$ can now be solved. First, use vertex labels α, β, γ, δ, ε, ζ, ordered as $\alpha < \beta < \gamma < \delta < \varepsilon < \zeta$. Realizing that the Prüfer symbols for such trees are carried by the factor $\alpha^2\beta^2$, we see that there are six possible Prüfer symbols corresponding to $\alpha^2\beta^2$, namely:

$$(\alpha,\alpha,\beta,\beta),\ (\alpha,\beta,\alpha,\beta),\ (\beta,\alpha,\alpha,\beta),\ (\alpha,\beta,\beta,\alpha),\ (\beta,\alpha,\beta,\alpha),\ (\beta,\beta,\alpha,\alpha).$$

Then apply Prüfer's algorithm to produce trees from symbols, given in his original paper. Recall that Prüfer uses braces, {…}, to delimit his symbols. He writes (Prüfer 1976):

> Conversely, if we are given a particular symbol $\{a_1, a_2, \ldots, a_{n-2}\}$, other than the empty symbol, then we write down the numbers 1, 2, …, n, and find the first number that does not appear in the symbol. Let this be b_1. Then we connect the towns b_1 and a_1 by a segment. We now strike out the first element of the symbol and the number b_1.
>
> If $\{a_2, a_3, \ldots, a_{n-2}\}$ is also not the empty symbol, then we find b_2, the first of the $n - 1$ remaining numbers that does not appear in the symbol. Connect the towns b_2 and a_2. Then strike out the number b_2 and the element a_2 in the symbol.
>
> In this way we eventually obtain the empty symbol. When that happens, we join the last two towns not yet crossed out. (pp. 53–54)

Let us study how Prüfer's mostly verbal description of the tree corresponding to a symbol can be applied to $(\alpha, \alpha, \beta, \beta)$, using the modern notation for a symbol. Since Prüfer speaks of "the first town" or of "the first number," we see that the town names or vertex labels are ordered. For the vertex labels $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, simply use the alphabetical ordering. Now, set

$$(a_1, a_2, a_3, a_4) = (\alpha, \alpha, \beta, \beta)$$

Thus, $a_1 = \alpha$, $a_2 = \alpha$, $a_3 = \beta$, $a_4 = \beta$. The letters that appear in the symbol are just α, β. The first letter that does not appear in the symbol is γ. Set $b_1 = \gamma$. Since $b_1 = \gamma$ and $a_1 = \alpha$ are connected by a segment, the first edge in the construction of the tree has form as shown in Fig. 15.3.

The updated symbol is now $(a_2, a_3, a_4) = (\alpha, \beta, \beta)$. The updated list of vertex labels, after striking out γ, is: $\alpha, \beta, \delta, \varepsilon, \zeta$. The first symbol in this list that does not occur in (a_2, a_3, a_4) is δ. Thus, $b_2 = \delta$, which is connected to $a_2 = \alpha$ by a segment. Including the second edge in the tree, we have Fig. 15.4.

The updated symbol is now $(a_3, a_4) = (\beta, \beta)$. The updated list of vertex labels, after crossing out δ, is: $\alpha, \beta, \varepsilon, \zeta$. The first element in the above list that does not occur in (a_3, a_4) is α. Thus, $b_3 = \alpha$, and b_3 is connected to $a_3 = \beta$ by a segment. The tree now has form as shown in Fig. 15.5.

The updated symbol is now just $a_4 = \beta$. After crossing out α, the updated list of vertex labels is: $\beta, \varepsilon, \zeta$. The first element in the this list that does not contain a_4 is ε. Thus, $b_4 = \varepsilon$, and b_4 is connected to $a_4 = \beta$ by a segment. The tree now has form as shown in Fig. 15.6.

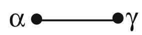


Fig. 15.3 The first segment of (α, α, β, β)

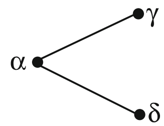


Fig. 15.4 The second step for constructing a labeled tree

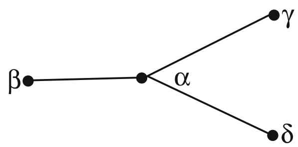


Fig. 15.5 The third step for constructing a labeled tree

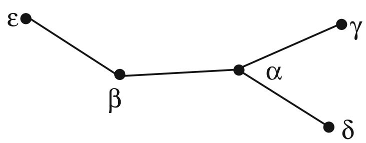


Fig. 15.6 The fourth step for constructing a labeled tree

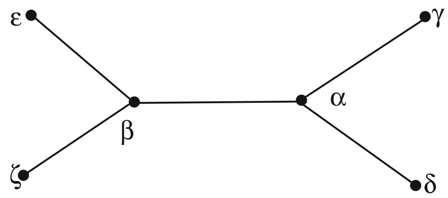


Fig. 15.7 A labeled tree for (α, α, β, β)

The updated symbol is now empty. The updated list of vertex labels, after crossing out ε, is just *β*, *ζ*. Prüfer writes: "In this way we eventually obtain the empty symbol. When that happens, we join the last two towns not yet crossed out" (1976, p. 54). Thus, the final step in the construction of the tree is to join *β* and *ζ* with a segment (Fig. 15.7).

Figure 15.7 is a tree corresponding to the Prüfer symbol (*α*, *α*, *β*, *β*). The exact positioning of the edges is not given by the algorithm to construct a tree and may depend on the location of the vertices, provided that information is given at the outset of the problem. An interesting exercise for students or instructors is to verify that the symbol for the above tree is actually (*α*, *α*, *β*, *β*). Also, trees corresponding to the symbols (*α*, *β*, *α*, *β*), (*β*, *α*, *α*, *β*), (*α*, *β*, *β*, *α*), (*β*, *α*, *β*, *α*), (*β*, *β*, *α*, *α*) could now be assigned as homework problems. Further exercises in the student module (Lodder 2013) develop a precise algorithmic formulation of the tree corresponding to a given symbol, which can be gleaned from Prüfer's original paper, excerpted above. The one-to-one correspondence between symbols and trees is further developed in the exercises of the student module, while a study of trees having the same Prüfer symbol leads to the idea of a graph isomorphism.

15.5 Borůvka's Solution to a Minimization Problem

Perhaps more important than counting trees are the applications that this structure have found in modern day mathematics and computer science. Well before graph theory was a subject in the present-day curriculum, Borůvka (1926a, b) published the solution to an applied problem of immediate benefit for constructing an electrical power network in the Southern Moravia Region, now part of the Czech Republic. He describes his own involvement in this project as (Graham and Hell 1985):

> My studies at polytechnical schools made me feel very close to engineering sciences and made me fully appreciate technical and other applications of mathematics. Soon after the end of World War I, at the beginnings of the 1920s, the Electrical Power Company of Western Moravia, Brno, was engaged in rural electrification of Southern Moravia. In the framework of my friendly relations with some of their employees, I was asked to solve, from a mathematical standpoint, the question of the most economical construction of an electric power network. I succeeded in finding a construction … which I published in 1926 …. (p. 50)

He phrased the problem as follows (Borůvka 1926b; Nešetřilm et al. 2001):

> There are n points in the plane (in space) whose mutual distances are all different. We wish to join them by a net such that: (1.) Any two points are joined either directly or by means of some other points; and (2.) The total length of the net would be the shortest possible. (p. 153)

How does this problem differ from that posed by Prüfer? Prüfer wishes to find a network that requires the least number of single segments, while Borůvka wishes to find a network of shortest possible total length. Both authors require that all towns in their respective applications be connected to the network (railway or electrical). Are these identical problems? No, since Prüfer never considers the length of a railway segment connecting two towns. Are these problems related? Yes, since a network of shortest total length is recognized today as a tree. Thus, of all possible n^{n-2} labeled trees on n points (towns), which tree or trees have the shortest possible total length? Borůvka offers a solution to this problem that is rather algorithmic in nature, and has become the basis for finding what today is called a *minimum spanning tree*. With the advent of the electronic programmable computer in the late 1940s and early 1950s, algorithms for finding minimal spanning trees became a topic of research in computer science, with both Kruskal (1956) and Prim (1957) publishing their own methods for finding such as tree. Some thirty years before this Borůvka (1926b) had published "A Contribution to the Solution of a Problem on the Economical Construction of Power Networks," which outlines how to find a network of shortest total edge length in a very visible and compelling example. He uses no modern terminology in his 1926 papers, not even the word "tree."

Borůvka proposes a simple algorithm to find such a net of minimum total length, based on the guiding principle "I shall join each of the given points with the point nearest to it" (Borůvka 1926b, p. 153; Nešetřilm et al. 2001). Of course, given points v_1, v_2, v_3, … in the plane, if the closest point to v_1 is v_2, then it is not necessarily the case that the closest point to v_2 is v_1. Also, if the only connections made are those resulting from connecting a vertex to its nearest neighbor, then a connected graph would not necessarily result, but would consist of several connected components. If this is the case, Borůvka uses the term "polygonal stroke" to refer to a connected component. He then devises an ingenious method to iterate this algorithm by connecting each polygonal stroke to its nearest polygonal stroke. The distance between two polygonal strokes G_0 and G_1 is given by $min\ d(v_i, v_j)$, where v_i ranges over the vertices of G_0, v_j ranges over the vertices of G_1, and $d(v_i, v_j)$ denotes the distance between v_i and v_j. Of course, after connecting a polygonal stroke to its nearest polygonal stroke, a connected graph may still not necessarily result, but the algorithm can be iterated until a connected graph does result, beginning with a finite number of vertices initially. Borůvka vividly illustrates his algorithm with the following example (Borůvka 1926b, p. 153). "Given the 40 towns (points) in Fig. 15.8 find the network of least total length that connects them."

First, "join each of the given points with the point nearest to it" (p. 153), resulting in "a sequence of polygonal strokes" (p. 153) in Fig. 15.9.

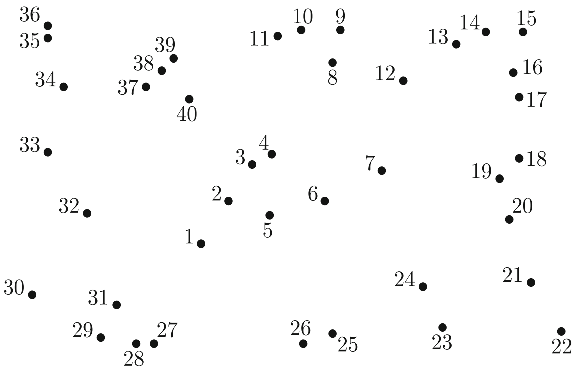


Fig. 15.8 Borůvka's 40 towns

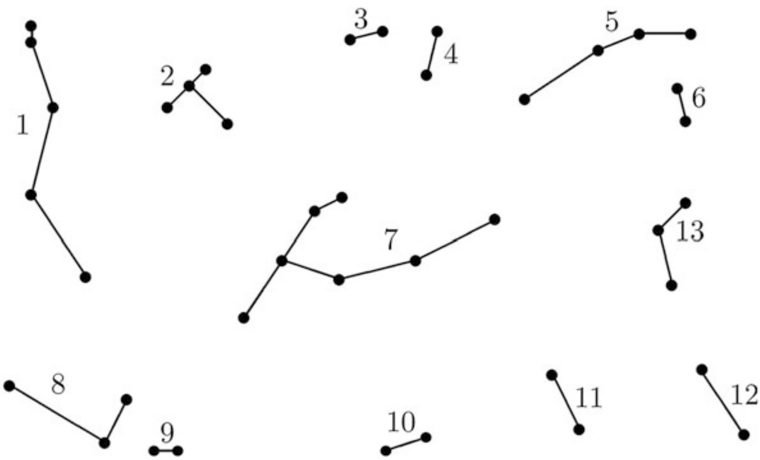


Fig. 15.9 The first step of Borůvka's algorithm

Then "join each of these strokes with the nearest stroke in the shortest possible way (resulting in Fig. 15.10)" (p. 153).

Borůvka concludes (Borůvka 1926b):

> I shall join each of these strokes in the shortest way with the nearest stroke. Thus stroke 1 with stoke 3, stroke 2 with stroke 3 (stroke 3 with stroke 1), stroke 4 with stroke 1. I shall finally obtain a single polygonal stroke (see Fig. 15.11) which solves the given problem. (p. 154)

It remains to be checked that Borůvka's algorithm produces a minimal spanning tree. Borůvka proves so in his first publication (Borůvka 1926a), which is a rather algebraic formulation of his algorithm using matrix notation. The visual appeal of

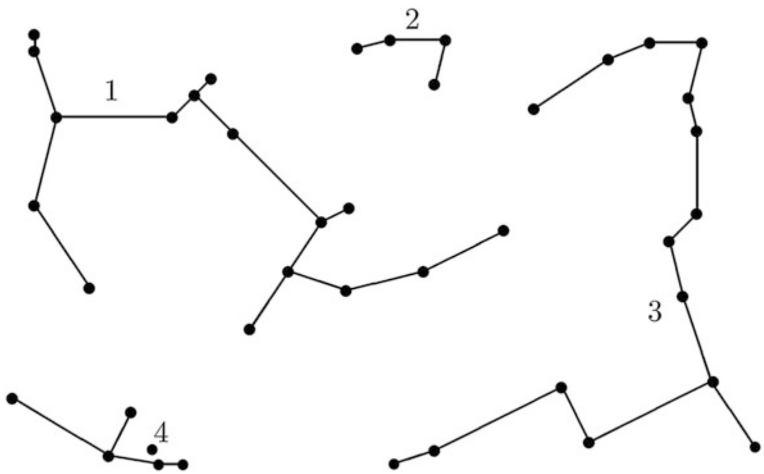


Fig. 15.10 The second step of Borůvka's algorithm

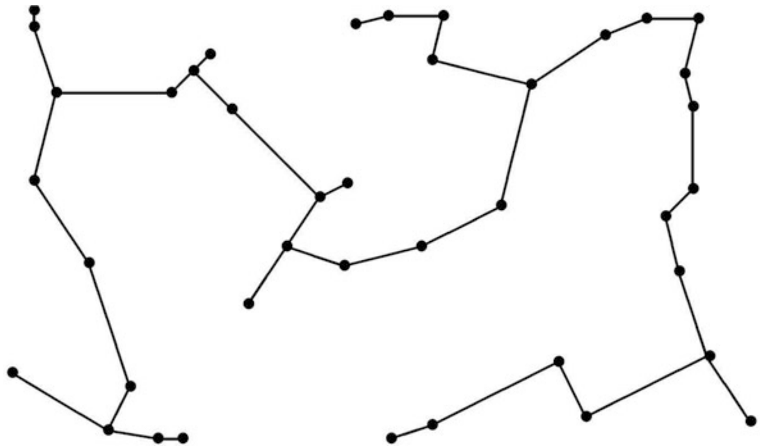


Fig. 15.11 The minimal spanning tree

the above example (Borůvka 1926b) remains a compelling illustration of this procedure. Under the assumption that all edge lengths are distinct, there is only one (a unique) solution to the minimal spanning tree problem for a given set of vertices. For further exercises on a precise formulation of Borůvka's algorithm, as well as an exploration of why a minimal spanning tree is produced, see the student module "Networks and Spanning Trees" (Lodder 2013).

15.6 Implementation of the Module

The module "Networks and Spanning Trees" has been tested at New Mexico State University in both a mathematics course on combinatorics and a computer science course on algorithm design during the years 2009–2011. Both of them were undergraduate courses. In the combinatorics course, student work on constructing trees with Cayley polynomial $\alpha^2\beta^2(\alpha\beta\gamma\delta\varepsilon\zeta)$ was graded with leniency, since the goal of this exercise is to raise doubt in the students' minds whether Cayley's formula for counting trees is correct. The complete solution to this problem must wait until the conclusion of the Prüfer section. While working through the Prüfer section, most students were able to construct the correct symbol from a given labeled tree. However, students needed guidance when working in the reverse direction, namely when constructing a labeled tree from a Prüfer symbol. During class, instructors may wish to work through the example of this given above. Also, Prüfer uses an induction argument on n, the number of vertices of a tree, to prove that there is a one-to-one correspondence between labeled trees and his symbols, which the instructor may wish to explore with the class.

For use in computer science courses, the need for an algorithm to find a minimal spanning tree can be vividly demonstrated from Borůvka's example of 40 towns. We know that there are $40^{38} \approx 7.55 \times 10^{60}$ labeled trees on 40 vertices, and finding the minimal spanning tree by checking the least value over this entire domain, even electronically, is virtually impossible. A more systematic method is necessary. The efficiency of Borůvka's algorithm can be explored by comparing the running time to find a minimal spanning tree with other algorithms, such as those proposed by Kruskal (1956) or Prim (1957). Today, Borůvka's algorithm is known as a "greedy" algorithm, since at each step, a vertex is connected to the vertex closest to it (in some iteration of the algorithm), and this is characterized as a "greedy" choice. In fact an entire subject, combinatorial optimization (Lawler 1976), has arisen to discuss these algorithms.

At the conclusion of these courses using historical curricular modules, students were asked to complete a questionnaire about their attitudes towards learning mathematics. Students were asked to offer, in free response, what are the benefits of learning from historical sources, and separately what are the drawbacks of learning from historical sources. Stated drawbacks include "the language may be difficult to read," and "math may not be state-of-the-art." Although anecdotal, stated benefits are encouraging, and include:

> "You get answers to questions like 'where did all of this come from?'"
>
> "It helps me understand the reason why things were put together like they are."
>
> "I like to see where everything comes from and how it works, especially when I am able to make sense of it."
>
> "You learn the concept."
>
> "It makes me care about learning."

15.7 Concluding Remarks

Pedagogical advantages of teaching from historical curricular modules include:

(1) The study of engaging problems rather than the memorization of technical definitions. Modern textbooks on graph theory begin with the formal definition of a graph and the definition of a tree, followed by lemmas and theorems about these structures and proofs of the results. By contrast, in a historical module we see from the outset what problems humankind confronted and the thought processes developed to find a solution. Cognitive dissonance or *dépaysement* (Barbin 1997) may occur when reading a verbal description of these problems without knowledge of the formal definitions of a graph or tree. A reader may not view the verbal description as mathematics.

(2) The description of problems without the use of specialized vocabulary. Both Prüfer and Borůvka offer verbal descriptions of problems they wish to solve without reference to any technical terms. This represents an advantage of learning from original sources articulated by Jahnke et al. (2000). No specialized use of vocabulary is required at the outset.

(3) An ease of the cognitive demand in understanding the reasoning process. Modern graph theory textbooks provide a proof of "Cayley's formula" for the number of labeled trees on n vertices, although such proofs are often difficult for students to follow, if students read these at all. The first attempts to justify a result are often intuitive and do not require the assimilation of a body of technical results, easing the cognitive demand (Schoenfeld and Floden 2014) for understanding. Thus, we see Cayley's attempt to count trees with an algebraic construct already familiar to him, and familiar to most students at this point, namely polynomials, subject only to the rules of algebra (associativity, commutativity, distributivity). Once it becomes clear how a monomial is constructed from a tree, then the object-level rules (Sfard 2000, p. 161) of the algebra of polynomials become the object-level rules of representing trees.

(4) An appreciation of rigor by observing how the subject has evolved over time. By studying the first attempts to verify a result, inconsistencies, omissions or unanticipated subtleties often arise. When students are afforded the opportunity to find or understand these gaps in reasoning, they appreciate the efforts of other scholars to offer a more rigorous justification of the same result. Thus, we see Prüfer provide a proof of "Cayley's formula" by using a one-to-one correspondence between labeled trees and Prüfer symbols. A Prüfer symbol for a labeled tree on n vertices can be interpreted as an element of the Cartesian product, $V^{n-2} = V \times V \times \cdots \times V$, where V is the set of vertex labels. Once the construction of a Prüfer symbol is understood, then the object-level rules of Cartesian products (as sets) become the object-level rules of counting trees. When moving from Cayley to Prüfer, we see the development of rigor by replacing one set of object-level rules (polynomials as objects) with a more subtle set of object-level rules (Cartesian products as objects). Of course,

elements of a Cartesian product are not subject to commutativity of their components.

(5) An understanding of the origin of modern mathematical definitions and procedures. Modern mathematics attempts to codify the key properties of a structure or procedure in the form of a definition or an axiom. After a study of trees and their applications is complete, we see why a tree might be defined as a "connected, minimally-connected graph," and why this might be reformulated as a "connected graph with no cycles," since a cycle always contains more edges than necessary in order for the graph to be "connected and minimally-connected." The formulation of a modern definition requires a knowledge of the various historical works (discursants) about the subject, and a meta-discursive (Sfard 2000) synthesis of these works into one definition. This offers an example of raising students' awareness of the meta-discursive rules that govern the current mathematical paradigm (Kjeldsen and Blomhøj 2012).

(6) Context, motivation and direction for the subject. As a summary statement, the three historical sources in this curricular module provide context, motivation and direction for a course on graphs and trees.

Instructors seeking more information on this teaching module should consult Lodder (2014).

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Part V
History and Epistemology of Mathematics: Interdisciplinary Teaching and Socio-Cultural Aspects

Chapter 16
The Pantograph: A Historical Drawing Device for Math Teaching

Silvia Schöneburg-Lehnert

Abstract Invented more than 400 years ago, the Pantograph—also called "Stork's beak"—is still known in modern times. Although it has lost its practical importance, it still invites users to play with geometry. This and the fact that the mathematical background to its working has strong links to modern curriculum suggest studying the Pantograph in class may be beneficial. Furthermore, the history of the Pantograph is described in historical sources and tells a lot about the history of mathematics. But can the history of the Pantograph be used in class to teach mathematics, Latin, history, or handicraft? We investigate this question in an interdisciplinary school project using the classical text by Christoph Scheiner of 1631 for studying the Pantograph with students of grades 8–11.

Keywords Intercept theorem · Similarities · Historical drawing devices
Pantograph · Christoph Scheiner · Latin

16.1 Introduction

"There is no Mathematics without its history" (Scriba 1983, p. 114; author's translation). Surely, it is not possible to argue against this statement, and in particular, mathematics teaching is more and more taking this into account. In the past years, several ideas and concepts from history of mathematics became increasingly important for the teaching of the subject. The success of this approach can be seen in a large amount of new material, publications and conferences on this subject, but also in the reconstruction of curricular frameworks (Fasanelli et al. 2000). This happened in spite of the doubt expressed in some debates on the benefit of this enterprise (cf. Glaubitz 2010, p. 1). Since history of mathematics, aligned with

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problems and personalities—seen in the context of “their” time—stimulates at all levels the “rigorous” investigations through all the different areas of mathematics, teaching of mathematics surely benefits from historical input.

For the understanding and consideration of mathematics within social context it is essential to conceive it as a self-developing science and as being part of the human culture. Questions about the decisive development of the aspects in mathematics emerging in this context are natural and may be included within the teaching of mathematics, though in different degrees.

We may refer here to Pythagoras or to the rule of Thales, but the mathematical and historical view should not be limited to these fundamental contents. It should rather be extended to many other topics, for which the mathematical and historical background is not always immediately obvious but still of the greatest interest.

One of these extensions, which can be integrated in the topic of “Similarities,” is the discussion of the Pantograph—also called “stork’s beak.” The word derives from the Greek. It is a compound word, consisting of the first part “πᾶν” (= each, all) and the second part “γράφειν” (= write, draw). So, its translation means “the all-writer.” It is a device capable of copying pictures at any scale. The device consists of four wooden rods, which are assembled to a parallelogram with mobile corners. A pencil, a stylus and a stationary mount are positioned collinearly on the device. The distance ratios of stylus and pencil to the mount are fixed and changing their ratio it is possible to choose how much the copy shall be smaller or larger than the original (Fig. 16.1). The mathematical background is easy to understand: the Pantograph is based on the intercept theorem. Hence, using the Pantograph one can apply dilations.

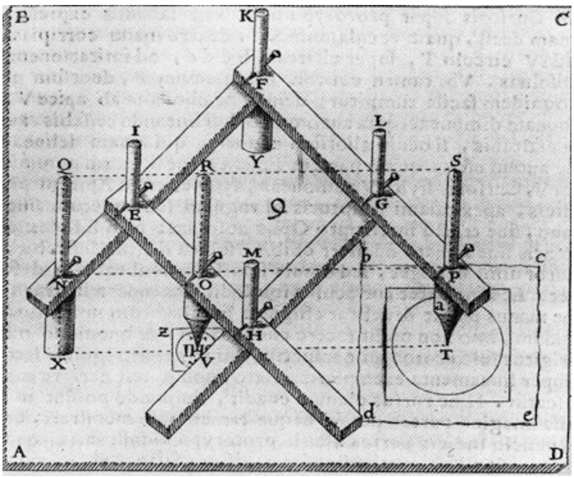


Fig. 16.1 The pantograph as shown in Christoph Scheiner’s *Pantographice seu Ars delineandi* (Scheiner 1631, p. 29)

16.2 Historical Remarks

Before we describe the actual school project let us start with some remarks about the history of the Pantograph. Following its history, one recognizes that the origin of this drawing device is not well documented and is difficult to reveal it down to the last detail. For instance, we find in a writing of the early 17th century the following statement:

> Diß Instrument / dessen gebrauch ich allhie in der kuertze zu beschreiben vorgenommen / ist zwar keine newe / und also auch nicht meine / sondern ein etwas alte Invention / welche aber / meines wissens / noch von niemanden in Truck befoerdert / oder zu Tag gegeben wurden. (Bramer 1617, p. 47)
>
> This device / the use of which I describe here shortly / indeed is not a new / hence not my / but a slightly old invention / which / up to my knowledge / nobody has printed yet / or communicated. (author's translation)

Hence, it seems that the Pantograph had already existed for a while, but there are no concrete references to the actual origin of this drawing device. Shortly after the year 1600, an increasing number of publications on this subject appeared, which shows that in the 17th century the Pantograph was broadly accepted and used as a copying device (Goebel et al. 2003, p. 9). Among the first known publications are the monographs of Benjamin Bramer (1588–1652), Daniel Schwenter (1585–1636) and Christoph Scheiner (1575–1650).

Benjamin Bramer was a German architect and mathematician. In 1617 Bramer published a short script: "Bericht eines Parallel Instruments" (Bramer 1617). On 12 pages Bramer describes how to build his drawing device and makes some detailed descriptions on the way it works. He describes how to use the device to copy figures at the same size and how to scale them up or down. He finishes his script with some short theoretical remarks on the mathematical background referring to the relevant parts of Euclid (Bramer 1617, p. 58).

Daniel Schwenter was a German orientalist. In the sixth book of the first treatise of his opus "Geometria e practicae novae et auctae" (Schwenter 1625), Schwenter describes the Pantograph. This description is very concise and hardly longer than a single page (Schwenter 1625, pp. 255–256). He finishes the treatment of the Pantograph in this overview with a reference to a more detailed description of the drawing device in a separate treatise.

> Weil aber diß ein sehr nuetzlich Instrument ißt / wil ich wils Gott davon ein sonderlich Tractetlein schreiben / und dem Leser viel schoener vorthel weisen zum abreissen / vergroessern / verjuengern / vergleichen. nicht allein mit dießem Instrument sonder auch mit einem andern und viel bequemern. (Schwenter 1625, p. 256)
>
> Since this is a very useful device / God willing I will write a separate tractate about this / and teach the reader much nicer how to copy / enlarge / reduce / compare. Not only with this device but also with a different and much more convenient one. (author's translation)

However, this “separate tractate” seems to be lost (as well as the much more convenient device). The type of Pantograph described in the work of Schwenter coincides with the device called Milanese Pantograph later on (Goebel et al. 2003, p. 12).

Starting in the 18th century, the Jesuit Christoph Scheiner is commonly considered to be the inventor of the Pantograph—even though the descriptions by Schwenter and Bramer appear superior in some technical aspects and Scheiner’s work was published more than ten years later.

16.2.1 The Jesuit Christoph Scheiner

The Jesuit, mathematician and scientist Christoph Scheiner was born on July 25, 1575 in Markt Wald close to Mindelheim, Germany. He entered the Jesuit order in 1595, studied philosophy in Ingolstadt and Dillingen, and earned the degree “Magister Artium” in 1605.

While studying in Dillingen he worked for three years (1602–1605) as a teacher for Latin at the Jesuit high school. At the same time, he temporarily held mathematics courses at the academy that was associated with the Jesuit high school. According to his description, he discovered the Pantograph during this period. We will come back to this, after the biographical sketch.

Fig. 16.2 Front page of *Pantographice seu Ars delineandi* (Scheiner 1631)

Scheiner studied theology in Ingolstadt earning a doctoral degree in 1609, the year of his ordination to the priesthood at Eichstätt. He took the three vows of poverty, chastity and obedience in 1617. From 1610 to 1616 he was professor of mathematics and Hebrew in Ingolstadt and held courses about sundials, practical geometry, astronomy and optics as well as a seminar on the telescope.

He spent several years in Innsbruck, Freiburg im Breisgau, Vienna and Rome and finally settled in Neisse, where he stayed at the Jesuit center founded by him. He died on July 18, 1650 (von Braunmühl 1891; Schönewald 2000).

As his oeuvre impressively shows, Christoph Scheiner was an outstanding mathematician, astronomer, physician and engineer of his times. He was always keen to use, understand, adopt and develop new instruments and methods in order to gain insight and to reproduce, verify and improve scientific progress. He made important contributions to the development of the astronomical telescope by inventing physical and optical devices like the "heliotropium teliscopium." He can be considered as cofounder of physiological optics (Daxecker 2014). In order to study conic sections, he invented his own ellipsograph, and in order to measure the orbit of a comet he developed within a single day a wooden sextant. But the reason why we discuss him here in great detail is of course his invention of the Pantograph.

According to his own words he was inspired for this invention by a painter. Following numerous considerations and unsuccessful attempts to recapture the painter's technique in copying images he finally found the solution to this problem in a dream. Using mathematical reasoning, he built the device in January 1603 and gave it the name "Parallelogrammos."

In the following years Scheiner continued to work on his Pantograph and made several improvements. It was not before 1631 that he published in Rome a description and assembling instruction titled *Pantographice seu Ars delineandi* (Fig. 16.2). In order to arouse readers' interest and curiosity, Scheiner tries to make the reader smile and lets his monograph start with the anecdote of the painter and the history of the invention. On the front page of the monograph the two possible applications of the Pantograph are shown. On the one hand the Pantograph can be used as a device for copying pictures on the plane. On the other hand, the Pantograph can be used as part of a perspective machine, with which one could draw any kind of object and project pictures on curvilinear surfaces.

In his monograph, Scheiner claims that this art of drawing was not widely known among his contemporaries in 1603. After inventing his drawing device, Scheiner started to teach how to use the Pantograph in order to copy pictures in the plane not only as a professor to his students at the college, but also privately (Scheiner 1631, p. 6). However, he did not reveal the full power of the new device and only few people were allowed to learn about the art of spatial drawing (Scheiner 1631, p. 6f). Nevertheless, he had to admit:

> …tametsi fieri posse non negem, ut aliqui fortassis extent alicubi, qui affine quid istis insinuent: at qui hac via, methodo, & arte progrediatur, esse puto neminem. Stereographicen certe istam totam novam esse primoq. partu in lucem prodire mihi persuadeo,[…]. (Scheiner 1631, p. 7)

> However, I shall not deny that it might happen, that eventually one might find somewhere others, who tell things close to this. But I believe that nobody exists who is proceeding this way, with this method, and with this art. I am convinced that the art of spatial drawing by itself is completely new and came to the world as a first birth, […]. (author's translation)

The monograph consists of two books describing both kinds of applications. The first book "Pantographices libellus primus" describes "the art of drawing in the plane": Given an arbitrary pre-image, it describes how to construct a similar image using the graphical tool Pantograph. The book is divided into two parts, the first being devoted to practical aspects only. In seven chapters, Scheiner reports about his invention, basic terms and definitions, materials and construction of the graphic Pantograph, tasks and assignments of different pieces and how to use the Pantograph and with which effect.

The second part deals with the theoretical background. Some aspects of the first part are taken up again and discussed from a theoretical point of view. In total fifteen "Propositiones" and several "Lemmata" are formulated and proved. In these proofs Scheiner's way of reasoning follows the commonly used approach to such problems. Starting from a general lemma, the problem is presented with a concrete example and a claim is stated. What follows is a mathematically well-thought-out proof.

In the second book, "Pantographices libellus secundus," Scheiner continues his studies. Reflecting directly on the function and meaning of the Pantograph, he focuses on several key aspects of his art of drawing. The book is divided into eight "Propositiones," which are discussed in detail. For instance, he discusses the art of spatial drawing. He concludes that in this case the pre-image cannot stay connected to the stylus. In the subsequent chapters he describes material, shape, and correct position of the stylus in spatial applications (Scheiner 1631, p. 89ff).

The construction manual and the description of how to use his tool in the art of planar as well as spatial drawing make Scheiner's monograph an essential resource about the Pantograph. A description as extensive as this had not existed before. Scheiner leads his readers to a much deeper understanding of the mathematics behind the Pantograph and shows a broader scope of possible applications than Bramer and Schwenter did. Compared to Scheiner's description the works of Schwenter and Bramer are only very brief introductions sketching only some main applications. Taking this into account, one understands the judgement that the Pantograph is Scheiner's invention, even if it has some technical disadvantages compared to the tools of Schwenter and Bramer.

The elementary mathematical features which characterize the Pantograph, its presence until nowadays—and not just as a widespread toy—as well as its historical embedding in a period that was very interesting and important from a scientific point of view, i.e. the Baroque, are good reasons to approach this topic with students. This suggests to go beyond the pure mathematical aspects of the device and to look at the mathematics of this period in its entire social context.

16.3 But Why Exactly the Scheiner Pantograph?

Even today the Pantograph is well known to several people. So, dealing with it in the mathematics teaching represents both for students and for teachers a natural link to the everyday life.

The unreflecting, playful use of the device offers a good starting point approaching its mathematical background. Thus, 9th grade students dealing with the topic of similarity transformations have the necessary prerequisites for handling the Pantograph.

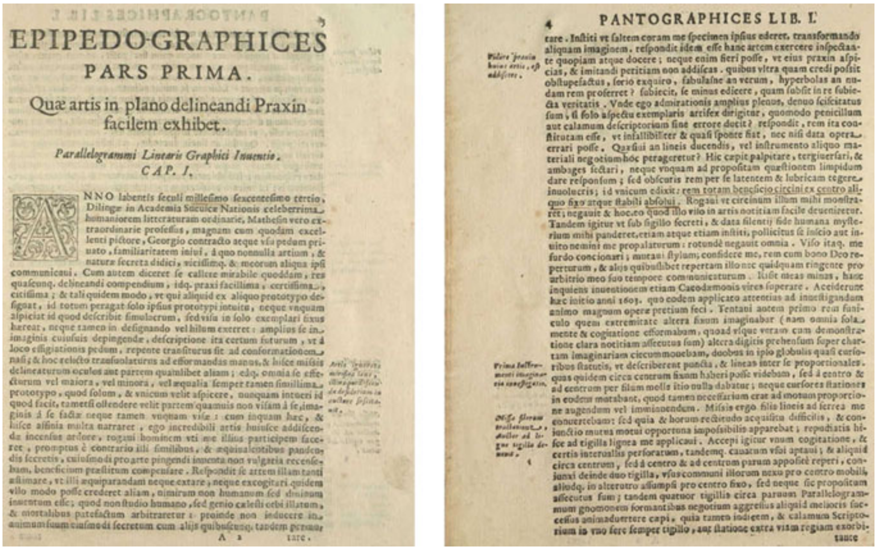
This argument is so obvious that we have to admit that we are not the only—not even the first—to suggest the use of the Pantograph in class. The Pantograph has even been included within the topics of some school books (e.g. Griesel et al. 2015, p. 247), though the respective exercises are usually detached from the historical context. However, embedding the Pantograph within a historical frame allows the teacher to transmit the idea that mathematics is a living part of the cultural development to the students.

The confrontation with the Baroque science, in our case, via the invention of a drawing device, is usually not considered within the teaching of history. Thus, working with the Pantograph can enrich both the teaching of mathematics and history. The style used by Scheiner in his description about the design and the functionality of the Pantograph and its mathematical background allows including his original textbook *Pantographice seu Ars delineandi* directly in class.

Nevertheless, it should be mentioned that there is a certain difficulty about the comprehension of the text, since it is written in Medieval Latin (cf. Fig. 16.3). However, this problem can be solved by connecting the teaching of Latin with the study of the Pantograph. Furthermore, the translation of selected excerpts allows gaining interesting and important information for both the mathematics and the history class. In collaboration with the teacher of Latin it is possible to create working material for the students, even for those who are still at an early stage of learning Latin (a concrete example of how this was done can be found in Sect. 16.5).

A particular advantage of dealing with the text on the Pantograph consists of the fact that the students have the possibility of facing the topic practically. Consequently, handling the toy-Pantograph, reproducing a Scheiner-Pantograph, as well as building a "modern" Pantograph can be very useful activities. The latter provides the possibility of cooperating with teachers of arts and handicrafts.

However, such a widespread topic is often not applicable due to the short time available in the mathematics class. For such cases, projects extending the standard teaching time would be suitable. Jankvist gave a categorization of approaches to integrate history of mathematics in the mathematics education (the "hows:" Jankvist 2009). From a methodological point of view, our project corresponds to what Jankvist called a "module approach."


EPIPEDOGRAPHICES

PARS PRIMA.

Quæ artis in plano delineandi Praxin facilem exhibet.

Parallelogrammi Linearis Graphici Inventio.

CAP. I.

PANTOGRAPHICES LIB. I.

Fig. 16.3 Excerpt of Christoph Scheiner's *Pantographice, seu Ars delineandi* (Scheiner 1631, pp. 3–4). The Latin source is not directly easily accessible to contemporary high school students

16.4 Is There Really a Benefit: From Using Historical Original Sources?

Obviously, this question is not at all new and a large number of authors contributed to an (often affirmative) answer (e.g. Arcavi and Bruckheimer 2000; Barbin 1991; Jahnke et al. 2000, 2006; Jankvist 2014; Pengelley 2011; Thomaidis and Tzanakis 2008). However, here we want to emphasize a special aspect of all this discussion that was of particular interest during the project. While most of the work on primary sources in the classroom deals with historical text sources, the Pantograph offers the chance to integrate a broader variety of historical sources in class.

> Quellen sind Objektivation und Materialisierung vergangenen menschlichen Handelns und Leidens. Sie sind in der Vergangenheit entstanden und liegen einer ihr nachfolgenden Gegenwart vor. (Pandel 2003, p. 11)

> Original sources are objectification and materialization of past human action and suffering. They have been created in the past and are available in a subsequent present. (author's translation)

Following this definition of original sources, the Pantograph as a historical drawing device is by itself an interesting historical source that is worthy to be rediscovered. Following this approach, van Randenborgh used the Pantograph—not only Scheiner's pantograph, but also Schwenter's and Bramer's pantograph—as one of two exemplary historic drawing devices in class (van Randenborgh 2015). In this study, van Randenborgh compared the use of the historical source with the use of a digital reproduction of it—using dynamic geometry environments. His results suggest that students perceive the digital reproduction as a simulation of the historical device (van Randeborgh 2015, p. 179). Another conclusion of this study is that the use of the Pantograph—or even its digital reproduction—in class provides special insights for the students:

> Historische Zeichengeräte beruhen auf der mathematischen Idee und genau diese soll im Unterricht von den Schülern entdeckt bzw. rekonstruiert werden. […] Durch dieses Aufdecken, Erklären und Herstellen von Zusammenhängen wird das Artefakt zum Instrument der Wissensvermittlung. (van Randenborgh 2015, p. 193f)
>
> Historical drawing devices rely on a mathematical idea and exactly this idea shall be rediscovered or reconstructed by the students […] This uncovering, explaining, establishing of relations turns the artifact into an instrument for the transfer of knowledge. (author's translation)

The monograph *Pantographice seu Ars delineandi* (Scheiner 1631) is another valuable source. On the one hand, it provides insight into the mathematical background, the construction and the use of the tool. On the other hand, it tells a story about Scheiner's motivation to construct such a tool, about the difficulties and problems and about the way people thought and talked about mathematics in the 17th century. This reason is enough to put not only the Pantograph itself—with its mathematical and mechanical ideas—but also Scheiner's original text and the critical reflection of his ideas into the focus of the school project.

Doing this it would be negligent to consider historical sources just as memory of knowledge and information. The additional educational potential of them originates in the particular way they tell their story—or can be brought to tell a story. This makes them monuments inviting us to study not only the source itself but also the broader context. Opening up minds for new kinds of questions the source provides more information than the author intended to put there.

Scheiner's textbook invites us to take the journey through the history of the Pantograph showing at the same time some disadvantages of the scientific style used in the 17th century. The students can compare the circumlocutory and (by modern standards) redundant mathematical language of Scheiner to the language used in modern textbooks and can make their own judgment on the value of our modern formal language of mathematics. In particular, this can be seen in Scheiner's proofs. This will be discussed further with the aid of a concrete example in Sect. 16.5.2.

16.5 The Project "The Pantograph of the Jesuit Christoph Scheiner"

Due to the long cooperation between the Cantor Gymnasium Halle and the Institute for Mathematics at the University of Halle-Wittenberg, and thanks to the open-minded spirit towards new forms of cooperation, the possibility arose for a project entitled "The Pantograph of the Jesuit Christoph Scheiner." The input to the project was given by a research project at the University of Halle-Wittenberg (Goebel et al. 2003; Richter and Schöneburg 2008) regarding Christoph Scheiner's book *Pantographice seu Ars delineandi*. The research project provided the working materials for the immediate school application within the frame of an interdisciplinary project.

The principal idea was to offer an extracurricular research project. The project was planned to (and did indeed) finish after one term. For the implementation of the project it was decided to hold a weekly meeting of a working group (an alternative to this would have been to have several "project days" during which the students would skip the regular classes in order to work exclusively on the research project). In this context, the students would have to develop a high level of self-activity for the arrangement and performance of the working group, thus making the project literally their own.

The definition of the thematic focal points for the working group was based on the suggestions and ideas of the students.

Christoph Scheiner and his time:

About Christoph Scheiner;
Universal Science in the Baroque;
the Pantograph and other drawing devices;
the Pantograph throughout the ages.

About the functioning and working of the Pantograph:

Scheiner's assembling instruction;
our reconstruction and our improved assembling instruction;
mathematical background;
testing and explanation of the Pantograph.

Presentation of results:

Wall newspapers;
crafting instructions;
theatre;
internet presentation.

The project was publicly announced at school and 18 students of grades 8 through 11 (ten students from grade 8, three from grade 10, and five from grade 11) volunteered to take part in the working group. Obviously, we had to cope with different levels of students' prior knowledge, in particular concerning Latin. This was not at all a disadvantage; on the contrary, it appeared to be an enriching feature of the project.

All students brought their own skills and, since they were used to working in groups, they were able to motivate each other via harmonizing their different skills and knowledge. With the aim of making the working group more interesting and

Fig. 16.4 The Latin text was elaborated in teamwork and discussed jointly

diversified, particular emphasis was put on the meaningful and alternating use of the linguistic, mathematical, historical and artistic aspects.

To be more precise, during the work on *Pantographice seu Ars delineandi* students started to study in groups excerpts of the Latin monograph and their results were discussed in class. The excerpts were chosen with a thematic focus on the tale of the invention of the Pantograph and the construction guidelines for its different parts. Along these lines, the teacher who was in charge of the history part of the project could easily lead the discussion to the historical research questions of the project, e.g. what was the Baroque period about and what are the characteristic elements of this epoch? Therefore, not only linguistic but also historical, mathematical and artistic aspects became crucial parts of the discussion while interpreting the excerpt (cf. Fig. 16.4).

The reproduction of the historical Pantograph—using the translation of the excerpt as a construction manual—was another part of the project. Here, the students needed not only artistic, but also mechanical skills (cf. Fig. 16.5). Originally, it was planned to construct only a paperboard Pantograph. After doing this and using the paperboard Pantograph for several drawings, the students came up with the wish to construct a "real" Scheiner Pantograph. Satisfying this wish turned out to be a valuable experience of constructing a "real tool."

A third part of the project was the development of the mathematical background. Obviously, this part demanded mainly mathematical skills. The students used their own Pantographs to search for mathematical reasons for the functionality and used their insights to explain what was going on. An important aspect of the project was to develop this explanation using mathematically correct language. This aspect was strengthened further by a subsequent discussion of one of Scheiner's propositions and its original proof (cf. Fig. 16.6).

Fig. 16.5 The main aim during the reconstruction of the Pantograph consisted of artistic and practical aspects

Fig. 16.6 Obviously, mathematics—with an emphasis on correct formulation—was the principal aspect of the project

16.5.1 How to Deal with Different Levels of Prior Knowledge?

As mentioned above, a consequence of allowing students of different grades to participate in the project was the variety of different levels of prior knowledge—not only in Latin but also in mathematics. Although sounding like a problem, this turned out to be a source for motivating teamwork, inspiring discussions and a fruitful learning environment. Nevertheless, this diversity was a challenge we had to face. Our strategy to do so is shown in Table 16.1.

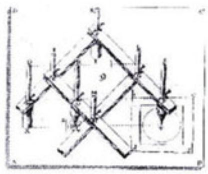
The following description of an exemplary phase of the project shall make more precise the above mentioned schematic description of our approach to face the varying level of students' prior knowledge. The phase of the project described below was used as an introduction to the project. It deals with the tale of the invention of the Pantograph.

Before handing out Scheiner's original description of this tale (cf. Fig. 16.3) students were prepared with working sheets, which were adjusted to the level of students' prior knowledge. Both versions—the working sheet for students of grade 8 (cf. Fig. 16.7) and the working sheet for students of grade 10/11 (cf. Fig. 16.8)—contain an edited version of the original text and selected vocabulary. While preparing the edited (Latin) text, the main goal was to stay as close to the medieval

Table 16.1 Working methods with respect to subjects and age groups

Age group characteristics relative to subject	Main working methods
Linguistic parts of project: Grade 8: Little prior knowledge—in particular, only basic knowledge of Latin; enriching the work in the project with enthusiasm, curiosity and creativity (both in terms of content and methodology) Grade 10/11: Solid prior knowledge, broader knowledge of Latin, higher level of autonomy	*Work with Latin source text* Working in parallel in groups of different sizes and in changing composition; Students are encouraged to help each other even beyond their own working groups
Mathematical and historical parts of project: Grade 8: High level of curiosity and creativity Grade 10/11: Analyzing different ideas, checking ideas for correctness, potential and possible consequences	*Work on the mathematics and the history of the Pantograph* Joint discussions interrupted by periods of teamwork (in groups of changing composition and size)

De vita et opere excellentis viri



Christoph Scheiner doctissimus vir erat. Multis rebus studebat. Scheiner excellentissimus physicus, astrologus et mathematicus erat. Dilingae linguam Latinam et mathematicam docuit. Eo tempore panthographicen invenit[1]. Pantographice est instrumentum delineandi[2]. Maiores vel minores vel aequales imagines[3] efficit[4]. Scheiner librum de inventione scripsit. In eo libro constructionem pantographices explicavit[5] et nonnullos modos[6] dixit.

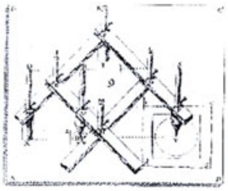
[1] invenire – to find, to invent
[2] delineare – to draw
[3] maiores imagines – larger pictures
[4] efficere – to cause, to produce
[5] explicare – to explain
[6] nonnulli modi – some rules

Exercises:

1.) Find the most important sentence regarding the functionality of the Pantograph!
2.) Describe Scheiner´s knowledge and skills!

Fig. 16.7 Introductory working sheet for grade 8 (exercises and vocabulary in English translated from German by the author)

De inventione mira



Anno millesimo sescentesimo tertio Dilingae magnam cum excellenti pictore Georgio amicitiam inivi. Dixit se scire mirabile res delineandi compendium[1] idque praxi facillima et tali modo:
„Qui aliquid ex aliquo prototypo designat[2], id totum peragit solo prototypi intuitu[3]."
Praeterea dixit se in prototypi descriptione ita certum futurum esse, ut a loco pedum repente[4] transiturus sit ad conformationem[5] nasi. Se ea omnia vel maiora vel minora vel aequalia semper tamen simillima prototypo effecturum esse. Artis addiscendae ardore incensus rogavi hominem, ut me illius participem[6] faceret. Respondit se in animum non inducere[7] suum secretum[8] cum aliis permutare. Id unicum dixit: rem totam beneficio circini[9] ex centro aliquo fixo[10] absolvi. Confisus sum[11] me rem cum bono deo reperturum esse.

[1] compendium – way, shortcut
[2] designare – design
[3] intuitus – sight, view
[4] repente – suddenly
[5] conformatio – conformation
[6] particeps – participant
[7] in animum inducere – to intend to
[8] secretum – secret
[9] beneficio circini – using a pair of compasses
[10] fixus – fixed
[11] confidere – to feel confident

Exercises:

1.) Find the most important sentence regarding the functionality of the Pantograph!
2.) Describe the problem Scheiner had with Georgius!

Fig. 16.8 Introductory working sheet for grades 10 and 11 (exercises and vocabulary in English translated from German by the author)

original as possible. So, while compiling the text in regard to its content, care was taken so that the students obtained a solid impression of Scheiner's use of Latin and his style in presenting the mathematical content. This impression was reinforced later on by comparing the edited text with Scheiner's original text.

16.5.2 How Does Scheiner Prove Statements?

As already mentioned, the second part of the first book of *Pantographice seu Ars delineandi* (Scheiner 1631) contains several mathematical statements and proofs. During the project, the students had to cope with several of them. To give an impression of this mathematical aspect of the project, we show the treatment of Lemma 3:

In modern mathematical language, the statement could be expressed as follows:

> A parallelogram in the Euclidian plane is uniquely determined by (the position and the length of) two of its sides.

However, Scheiner's formulation sounds—at least to students—very different as the following (rough) translation shows.

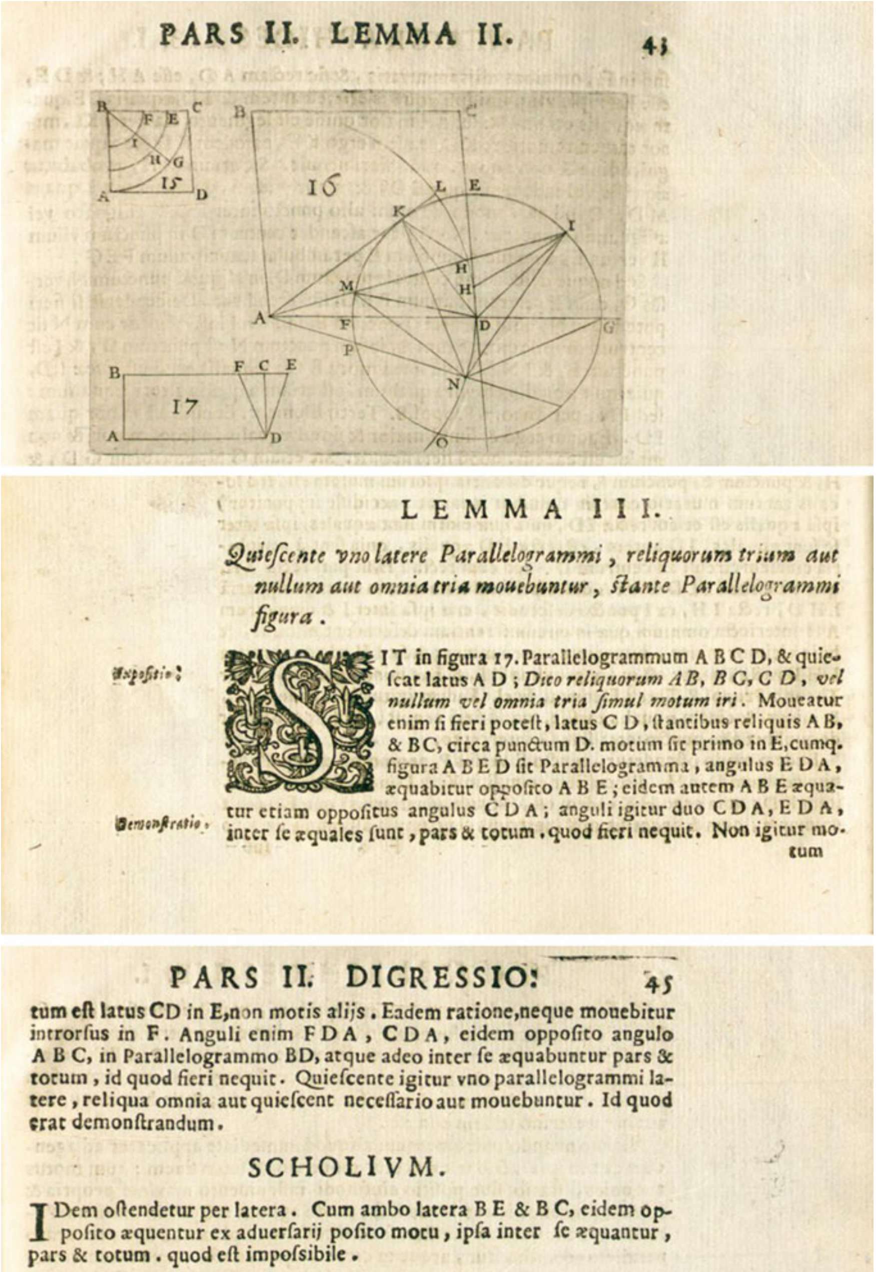
> If one side of a parallelogram stays fixed, then either none of the other three or all other three [sides] are moved to stay with a parallelogram. (author's translation; cf. Fig. 16.9)

Of course, Scheiner was thinking of his parallelogram as something representing his device consisting of four wooden rods. Therefore, it makes sense to speak of movements of sides. This monograph was meant to be a textbook for readers in the 17th century explaining the use of his device, not as a textbook explaining Euclidean Geometry.

A careful analysis of his "Expositio" shows that Scheiner seems to use either some hidden assumptions on a third angle to be fixed, or uses some unproven observations. In his proof he deduces that it is impossible to "move" just one side and concludes that all three sides have to be "moved." The case that just two sides are "moved" seems to be excluded per se. Although the original Latin text can be found in Fig. 16.9, we state here its translation:

> Expositio: Given Figure 17 of the parallelogram ABCD and the side AD that stays fixed. *I claim that either none of the other sides AB, BC, CD, or all three of them have to be moved*. If it happens that the side CD is moved around D in the beginning in E, while the other (sides) stay, then the figure ABED shall be a parallelogram, the angle EDA will be the same as the opposite angle ABE; this angle will be the same as the angle CDA; hence the two angles are equal to each other, a part and the total, which cannot happen.
>
> Neither has the side CD be moved to E, nor any other movement has happened. Similarly, it will not be moved inside to F. Since the angles FDA and CDA, opposite to the same angle ABC in the parallelogram BD, are equal, a part and a total, which cannot happen. Hence if one side of a parallelogram stays fixed, either none or all of the other three are moved. Qoud erat demonstrandum. (italics in the original; author's translation; cf. Fig. 16.9)

For several reasons, it was very fascinating to observe the students during this period of the project. First of all, this time they did not get an adapted version of the source but rather worked with the Latin source itself. Due to the mathematical content, the students were able to figure out the correct content. In particular, the use of Scheiner's sketch and the mathematical "symbols" ("A", "B" etc.) helped them to come up with the correct meaning of Scheiner's "Expositio." In contrast, translating the content in a modern language was much harder.



PARS II. LEMMA II. 43

LEMMA III.

Quieſcente vno latere Parallelogrammi, reliquorum trium aut nullum aut omnia tria mouebuntur, ſtante Parallelogrammi figura.

Expoſitio. SIT in figura 17. Parallelogrammum A B C D, & quieſcat latus A D; *Dico reliquorum A B, B C, C D, vel nullum vel omnia tria ſimul motum iri.* Moueatur enim ſi fieri poteſt, latus C D, ſtantibus reliquis A B, & B C, circa punctum D. motum ſit primo in E, cumq. figura A B E D ſit Parallelogramma, angulus E D A, æquabitur oppoſito A B E; eidem autem A B E æquatur etiam oppoſitus angulus C D A; anguli igitur duo C D A, E D A, inter ſe æquales ſunt, pars & totum, quod fieri nequit. Non igitur motum

Demonſtratio.

PARS II. DIGRESSIO. 45

tum eſt latus CD in E, non motis alijs. Eadem ratione, neque mouebitur introrſus in F. Anguli enim F D A, C D A, eidem oppoſito angulo A B C, in Parallelogrammo BD, atque adeo inter ſe æquabuntur pars & totum, id quod fieri nequit. Quieſcente igitur vno parallelogrammi latere, reliqua omnia aut quieſcent neceſſario aut mouebuntur. Id quod erat demonſtrandum.

SCHOLIVM.

IDem oſtendetur per latera. Cum ambo latera B E & B C, eidem oppoſito æquentur ex aduerſarij poſito motu, ipſa inter ſe æquantur, pars & totum. quod eſt impoſſibile.

Fig. 16.9 Lemma 3. Excerpt of the second part of the first book of *Pantographice seu Ars delineandi* (Scheiner 1631, pp. 43–45)

The students also realized that Scheiner's circumlocutory style carries some problems. Although he is using a lot of words it is not obvious why some of his conclusions are true. It was much easier for them to comprehend (8th grade) or write down (10th and 11th grade) a modern proof of the modern statement above.

16.6 What Was So Special in This Working Group?

Initially, there were concerns that through the encounter of students of different ages substantial social problems could arise, but those concerns were quickly canceled. The teamwork among the students was excellent and the different groups integrated each other smoothly. An advantage for the work was the fact that all participants brought their own experiences in the working group (from school and from other projects). The division of labor was made ad hoc and allowed each student to participate with her/his own creativity. The project was characterized by its serene working atmosphere. All students felt in the same way responsible for the organization, the contents and the design.

In order to make the work more interesting for the students, the activities were intentionally switched between working in groups and working with the whole class.

The students were so much interested in the project, that they continued working on the Pantograph topic at home, thus enriching the meetings with more content and new ideas. The chosen way shows good possibilities of following the skills and interests of the students and of implementing them productively and creatively.

The interaction of mathematics, Latin, history and handicrafts, united in studying the Pantograph, showed to be multifaceted in a way that cannot easily arise in everyday life or conventional school classes.

The students were guided through the project to an effective scientific and interdisciplinary work. It clearly appeared that many century-old original mathematics texts are suitable for being read, analyzed and elaborated by students. The students could gain experiences of many different kinds. The project "The Pantograph of the Jesuit Christoph Scheiner" was a very successful and rewarding venture.

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Chapter 17
Expanding Contexts for Teaching Upper Secondary School Geometry

Panagiota Kotarinou, Eleni Gana and Charoula Stathopoulou

Abstract This chapter describes how the theatrical performance based on the history of mathematics—'An Amazing Story: The Measurement of the Earth by Eratosthenes'—created the opportunity of a 'third,' expanding learning space, which allowed for new practices and tools to emerge. It also permitted students to approach mathematical concepts in an experiential way and (re)negotiate their own learning processes, their conceptions of mathematical Discourse, and the nature of mathematics. We analyze a one semester-long, interdisciplinary, didactical intervention for 10th grade students in a public school in Athens, where different funds of knowledge and Discourses expanded the boundaries of the official school Discourse. Our aim is to show how an experiential way of integrating the history of mathematics—a theatrical performance based on history—can create a 'third,' expanded learning space, where new tools and new Discourses are applied.

Keywords Eratosthenes' measurement · Theatrical play · Hybrid
Expanded space · Interdisciplinary

17.1 Introduction

The fact that students consider geometry as a difficult school subject and with different problems for them (Clements and Battista 1992) dictates, inter alia, the revision of teaching to enhance students' interest and active participation in class.

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Amongst the alternative approaches for teaching geometry which actively involve students' participation in the teaching and learning process, the use of applications from the history of geometry is suggested (Gulikers and Blom 2001).

Experiential learning theory defines learning as "the process whereby knowledge is created through the transformation of experience. Knowledge results from the combination of grasping and transforming experience" (Kolb 1984, p. 41, as cited in Kolb et al. 2000, p. 227). Experiential learning supports students' learning in an effortless, constant and permanent way through the 'real' experience and participation in activities that are related to themselves and that they find interesting.

Some of the experiential ways of integrating the history of mathematics include the enactment of theatrical plays (Fraser and Koop 1981; Prosperini 1999), the dramatization of different themes from the history of mathematics (Hitchcock 1999, 2000; Lawrence 2000; Ponza 2000), debates (Furinghetti 1997), historical debates based on a mathematical issue (Bartolini Bussi and Mariotti 1999; Bartolini Bussi and Mariotti 1998; Katz 1997), 'Mantle of the expert'—a 'Drama in Education' technique—(Pennington and Faux 1999) and oral story-telling from history of mathematics (Schiro and Lawson 2004; Selby 2009).

In this chapter, our aim is to discuss how one of the experiential ways of integrating the history of mathematics—a school theatrical performance with a theme from the history of mathematics—can create a 'third,' expanded learning space, where new tools and new Discourses are applied. Here we refer to Gee's (1996) conception of 'Discourses,' as the ways of knowing, doing, talking, interacting, valuing, reading, writing and representing oneself, produced and reproduced in several social and cultural communities in which they participate. The uppercase 'D' distinguishes this use of the term from a mere stretch of language, which Gee identifies as 'discourse' defined by a small 'd.' Any stretch of language (discourse) is always embedded in a particular way of knowing (Discourse) (Gee 1999). Gee's conception of the relationship between culture and Discourse refers not only to ethnic experiences and relations, but also to peer, social class and community relationships, among others. Therefore, if one situates this conception of Discourses in secondary school settings, it can be argued that content area classrooms represent communities that privilege different Discourses. In this perspective, our pedagogical endeavor challenged the boundaries of the diverse school subjects, encouraged students to approach mathematical notions experientially and through a multilingual corpus of literary texts offered them the potential to renegotiate their perceptions regarding mathematics nature and particularly Euclidean geometry.

17.2 'Third Space,' History of Mathematics and Proposals for Educational Practice

Borders or the boundary area between two fields, according to Bhabha (1994), a theorist of postcolonial studies, is often an overlap area or hybridization, i.e. a 'third space' that includes an unpredictable and changing combination of the features each of them carries. The theory of hybridity argues that people add meaning to their world through the integration and interaction of multiple available social and cognitive resources. This construction of a 'hybrid' space emphasizes the in-between space that gathers knowledge and Discourses by individuals and the various environments in which people participate today and which Discourses can be contradictory and competitive to each other across different contexts.

Moje and her colleagues (2004, p. 41) argued that the active integration of multiple funds of knowledge and Discourse is important to support youth in learning how to navigate the texts and literate practices that are necessary for 'survival' in secondary schools. In what comes to be called a 'third' or 'hybrid' space, the different knowledge, discourses and relationships one encounters in ways that collapse oppositional binaries, can actually work together to generate new knowledge, discourses and identities (Moje et al. 2004).

A number of studies examine the function of the third space to improve the teaching and learning of mathematics (e.g. Cribbs and Linder 2013; Flessner 2009; Razfar 2012; Thornton 2006). In each case the relevant research reveals the learning benefits when teachers undertake the responsibility to bridge the boundaries between the two 'worlds,' that of the lives of students outside the school and that in the classroom, describing the students' involvement in the learning process as substantial and lasting. Thornton (2006) examined how students' funds of knowledge play out in the mathematics classroom and casts forward to the creation of an environment that values and builds on the rich funds of knowledge brought by students as they enter high school mathematics.

In the educational context, the 'third space' could be also reconceptualized as the integration of the varied disciplinary Discourses in the school curriculum and the creation of a fruitful dialogue between their own Discursive practices in order to promote the acquisition of new knowledge (Wallace 2004). For example, Kotarinou et al. (2015) described how the reading of a literary work—*The Sand Reckoner*, concerning the work and life of Archimedes—created the opportunity of an expanding learning space, where tools and resources of different knowledge domains (funds of knowledge) came together to transform traditional classroom practices. Also, through a more recent project (Kotarinou et al. 2017), where multimodality was exploited in teaching geometry—literary, visual and performing texts and practices where used—it was noticed that such a framework would create a hybrid (third) space that enhances mathematics learning.

The educational value of integrating the history of mathematics in mathematics education has been intensively studied over the last four decades (e.g. Fasanelli and Fauvel 2006; Fauvel and van Maanen 2000; Katz and Tzanakis 2011), including a survey conducted on the recent developments in the field since 2000 by Clark and colleagues (2016). There are three different—though interrelated—types of contributions of the research on the role of history of mathematics in mathematics education: epistemological, cultural and didactical (Barbin and Tzanakis 2014). Tzanakis et al. (2000, p. 203) referred to five main areas in which mathematics teaching may be supported, enriched and improved through integrating the history of mathematics into the educational process: (a) the learning of mathematics; (b) the development of views on the nature of mathematics and mathematical activity; (c) the didactical background of teachers and their pedagogical repertoire; (d) the affective predisposition towards mathematics; and (e) the appreciation of mathematics as a cultural-human endeavor.

In our chapter, we claim that introducing the staging of a theatrical play based on the history of mathematics, exploiting of literary works and other genres in teaching geometry, has the potential to create a 'third space.' In this space, with new tools and new Discourse—a blend of standard and non-standard mathematics Discourse—a richer repertoire of students' participation possibilities is facilitated. In such a teaching environment—intertextual and cross-disciplinary—conditions and circumstances are created for the students to experience the learning process through a different educational management, as evidenced by the implementation of the project below.

17.3 The Study

Our aim in this chapter is to discuss how a group of adolescent students were engaged in the interdisciplinary project, "Hellenistic Alexandria: The Beacon of Knowledge," through a theatrical performance concerning Eratosthenes' measurement of the Earth. Students were encouraged to communicate mathematics through this theatrical performance (using both mind and body), through a variety of practices related to reading literature and various other activities. In this project, we explored the following main research questions:

1. How can a theatrical performance, based on a historical topic of mathematics, create a 'third,' expanding space in which students can renegotiate the dichotomy of the Discourses of Science and Humanities?
2. How might the students' experience of the expanded mathematical space influence their conceptualizations of mathematics and motivate their participation?

17.3.1 Data Collection

The use of ethnographic research techniques helped us to gather empirical evidence regarding the students' experiences, with the teacher-librarian and the mathematics teacher (the first author) being either participants or participant observers in all phases of the study. Both of them kept ethnographical notes, and the ethnographic material was supplemented by interviews with the students and video recordings of selected parts of the entire procedure. In our research we maintained a high standard of ethics. As is considered appropriate in the Greek context, we asked permission from students' parents to video record their children and to interview them; and we reassured students that their anonymity would be maintained at all times throughout the entire research process.

A questionnaire with open questions was administered to all students and a number of semi-structured interviews were conducted, which aimed to explore how students themselves perceived and processed their experience of participating in the project, mentally and physically. Our data collection was completed with a questionnaire given to students at the end of the project implementation, with open questions such as 'Which activities did you like more and why?', 'Which activities did you not like and why?,' and 'What positive thing do you think you gained from the activities?' A post-graduate student of the didactics of mathematics who acted as a non-participating observer in some of the activities interviewed the teachers and the students of the project.

17.3.2 The Project in Practice

The project was carried out for a whole school semester (February to May), in a State Lyceum, an inner-city school in Athens (Greece), with one class of 10th grade students. To meet the needs of the project, the students' mathematics teacher (first author), the Greek language teacher, and the teacher-librarian collaborated,[1] and in some cases co-taught. We chose the model of collaborative teaching practice which was based on the reiterative cycle of planning, researching, sharing resources, teaching collaboratively and finally assessing the outcomes of a lesson (Lawrence 2008). Working in this way, the teachers had the opportunity to renegotiate their practices and their professional development, entailing knowledge and practice. The main activity of the project, the theatrical performance of 'An Amazing Story: The Measurement of the Earth by Eratosthenes,' involved all the students of the class. This was the stimulus for the realization of all the activities of the project, aiming for students' acquaintance with the Hellenistic World through the History class, Ancient and Modern Greek Language, Mathematics, Religion and Computer Science classes.

For the whole project 21 teaching hours were required over a period of 10 weeks. The timeline of the activities in the project is shown in Table 17.1.

Throughout the project, different teaching interventions and activities were carried out every week according to the topics emerging from the theatrical play or the necessity of elucidating the historical era of the play. In this way, the teaching of different mathematical topics was presented as the elaboration of meanings constructed during the staging of the play. A combination of different tools and texts,

[1]Despina Koutli (Science teacher and teacher librarian at that time) and Anastasia Apostolopoulou Chrysanthaki (Greek language teacher).

Table 17.1 The timeline of the project

Week	Activities	Class
1st–5th	Reading and commenting on the dialogues of the play (one teaching period per week)	Geometry class
6th	a. Other measurements of the earth's meridian; a debate on the establishment of the meter	Science class
	b. Archimedes' "mechanical method"	Ancient Greek language class
	c. Resolving, as Archimedes did, mathematical problems with the use of physics	Geometry class
7th	a. Theatrical 'cold reading' of one of Renyi's 'Dialogues' between Archimedes and Ieron	Geometry class
	b. Simulation of Eratosthenes' Earth measurement	In computer lab
8th	a. Students' narrations about Archimedes	Geometry class
	b. Students' narrations about Septuagint	Religion class
	c. Students' narrations about Hellenistic Age and Alexandria	History class
	d. Students' narrations about Eratosthenes and Archimedes	English language
	e. Trigonometry in Hellenistic Age	In computer lab
	f. Alexandria through Cavafis' poems	Literature class
9th	a. Students' narrations about Archimedes, finding the area of an irregular shape, with the help of laws of physics	Geometry class
	b. Geometric constructions: drawing with compass and straightedge	Geometry class
	c. The unsolved geometric problems of Antiquity and Archimedes' method of angle trisection	Geometry class
10th	The theatrical performance	

coming from different contexts were used, challenging dichotomies like body-spirit, formal-informal learning, listening-acting, etc.

17.3.2.1 The Theatrical Play

At the beginning of the second semester, for five weeks (one teaching period per week) in the Geometry class students read aloud extracts from the work "*Les Cheveux de Bérénice*" by Denis Guedj (translated into Greek as *The Stars of Berenice*, Guedj 2005). These extracts consisted mainly of dialogues regarding the Earth's measurement carried out by Eratosthenes, which was based on measuring an arc of the meridian crossing the city of Alexandria in Egypt. This example, as Tzanakis (2016) mentions, is one of those that:

> …from a mathematical point of view, are elementary. However, the emphasis is on how elementary geometrical ideas and reasoning led historically to astronomically and physically non-trivial consequences with far-reaching cultural implications of the highest importance that can be posed didactically. (p. 89)

Fig. 17.1 A scene of the theatrical play

During the reading process, several issues such as the way time was measured in antiquity and the beliefs of that time about the shape of the Earth were further exploited. Notions that were included in the text such as the equator, the meridian, the latitude and longitude, as well as mathematical topics (the notion of proof, the measurement of an arc, the ratio of two arcs or two angles, the measurement of an angle with the help of the shadow of a set square, the Eratosthenes' model of Earth measurement and generally the mathematical model of a real problem) were elicited collaboratively.

Once the reading of the dialogues was completed, all three teachers helped after school with the staging of the play 'An Amazing Story: The Measurement of the Earth by Eratosthenes,' based on the aforementioned dialogues of Denis Guedj's book *Les Cheveux de Bérénice*, involving all the 10th grade students of the same class. We tried to involve all students in the theatrical play, without putting any pressure on anyone who did not wish to take part in the performance (Kontogianni 1998). After sharing the roles, the other students were assigned tasks such as the construction of scenery and costumes, music, soundtrack, lighting, designing, hairstyling and make up and stage managing. The 45-min play was performed for all school students on the last day of the school year and it was also presented to the parents (Fig. 17.1).

17.3.2.2 Activities During the Staging of the Play

After the reading of the play and along with the rehearsals, other activities were carried out. The students attended a debate between two 11th grade students (Fig. 17.2), concerning the choice by the French National Assembly during the French Revolution of a length measurement unit, which created the need of a new measurement of the Earth's meridian.

Prior to the debate, extracts from Denis Guedj's (2002) book *Le Mètre du Monde* (*The Meter of the World*) were read and the Physics teacher presented the pendulum and its principles. The references in the dialogues to the famous mathematician

Fig. 17.2 The debate

Fig. 17.3 Theatrical cold reading

Archimedes—contemporary and friend of Eratosthenes—gave us the opportunity of a theatrical 'cold reading' by students of one of the dialogues of the Hungarian mathematician Renyi's (1979) book, *Dialogues on Mathematics*. This imaginary dialogue is between Ieron, the tyrant, and Archimedes, regarding mathematical applications. After the reading of this dialogue by two students, we discussed pure and applied mathematics, focusing on the parabola and its properties, and parabolic reflectors (Fig. 17.3).

Students were assigned to make and present, through narration, thematic accounts of varied literary texts[2] relevant to Hellenistic Alexandria (Fig. 17.4). The topic of their narration (Fig. 17.5) referred to the foundation of Alexandria City, the Beacon and the Library, the Ptolemaic dynasty, the Septuagint and the mathematicians of the Museum: Archimedes, Euclid and Eratosthenes.

Each student was also assigned to write a short essay referring to the subject of his/her narration, finding all the information needed from a given bibliography. The essays (Fig. 17.6) were about the libraries and the books in antiquity, ancient and

[2]The literary books were the following: *The Bathtub of Archimedes* by Ortoli and Witkowski (1997), *The Parrot's Theorem* by D. Guedj (2000), *Euclid's Rod* by J.-P. Luminet (2003), *Pharos and Pharillon* by E. M. Forster (1991), *The Stars of Berenice* by D. Guedj (2005), *Anthology of Alexandria* by T. Psarakis (1992), and *The Lost Library of Alexandria* by L. Canfora (1981).

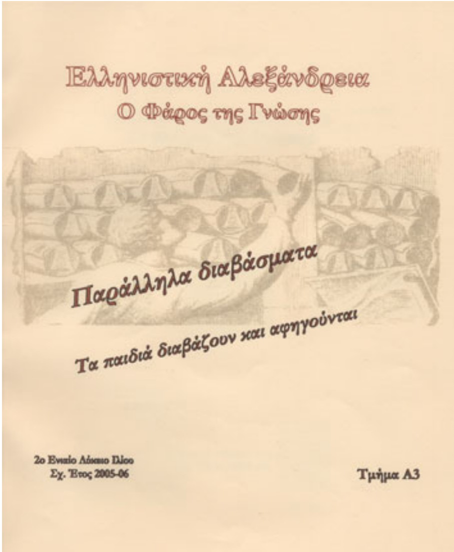


Fig. 17.4 The booklet with the literary texts for students' narration

Fig. 17.5 A student narrates in the school library

contemporary beliefs on the shape of the Earth, other measurements of the meridian, Eratosthenes' sieve, Archimedes' cattle problem and Archimedes' *The Sand Reckoner*. At the end of the school year, a school exhibition of the entire project's material was held (Fig. 17.7).

In the History class the unit, Hellenistic world—the Kingdom of Egypt and Hellenistic Culture—Alexandria, was presented. In Religion class the Translation of the Pentateuch in Greek language by the 72 elders representing the 12 tribes of Israel was discussed. In Greek language class students became acquainted with Alexandria through the life and work of the poet Konstantinos P. Cavafis, while in English language class the short text, "Stories about Eratosthenes," from Mary Brading's (1997) book, *Mathematics from History-The Greeks*, was read.

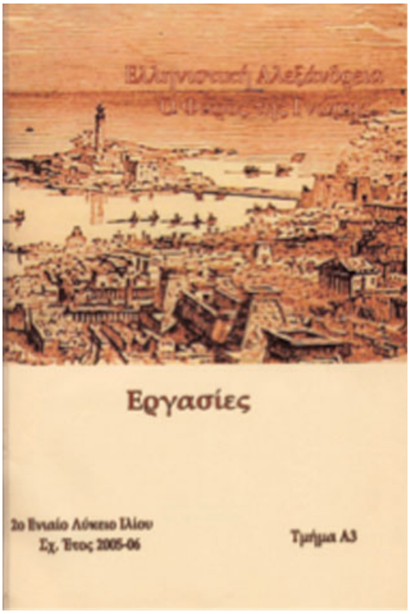


Fig. 17.6 The booklet with the students' written essays

Fig. 17.7 The exhibition in the end of the school year

The mathematical topics we chose to discuss were either mentioned in the theatrical play, or were related to Archimedes to whom several references are made in the play. Some of them also referred to Hellenistic mathematics. We ensured that the topics included were from the current class curriculum or known to students since high school, but presented or shown from another point of view.

In the following Sects. 17.3.2.3–17.3.2.6, some of the teaching issues related to mathematics are presented, followed by the practices and tools that were exploited.

17.3.2.3 Archimedes' "Mechanical Method": Finding the Area of a Parabolic Section Using the Law of the Lever

Due to Euclidean methodology, a 'deductivist style' of presentation of mathematics is adopted, leading to the concealment of the incentives, and the problems, questions and methods that led to the development of this field and the discovery and proof of the theorems (Lakatos 1976). As Lakatos (1976) characteristically noted, "the successive tentative formulations of the theorem in the course of the proof - procedure is doomed to oblivion while the end result is exalted into sacred infallibility" (p. 142).

In Archimedes' letter to Eratosthenes, a work referred to as *The Method* (in Greek: "Περὶ μηχανικῶν θεωρημάτων πρὸς Ἐρατοσθένη ἔφοδος")—the well-known Palimpsest, found by J. L. Heiberg in 1906—the heuristic method of Archimedes, with which he discovered many of his known theorems in geometry is described. This work has great importance due to the fact that it contains the only report of a mathematician of antiquity describing the method he utilized in discovering his theorems (Assis 2010, p. 38).

Using this "mechanical method," Archimedes, with the aid of the law of the lever, estimated the area of a parabolic segment, thus overcoming all the difficulties of the absence of limit methods. As Tzanakis (2016) remarked: "In teaching and learning either mathematics or physics, neither history should be ignored, nor the close interrelation of the two disciplines should be circumvented or bypassed" (p. 79).

To support this interactive interplay of mathematics and physics, and for the students to be acquainted with Archimedes' *Method*, mathematics and ancient Greek language were co-taught (Fig. 17.8) by teachers working together to translate and explain parts of the Archimedes' letter.

Students were asked to specify the content of mechanics taught in their physics lesson, while we referred to Archimedes' concept of mechanics. The first postulate[3] from his book *On the Equilibrium of Planes* (in Greek: Περὶ ἐπιπέδων ἰσορροπιῶν) was read and analyzed, as well as the law of lever: "Two magnitudes, whether commensurable (Proposition 6) or incommensurable (Proposition 7), balance at distances reciprocally proportional to the magnitudes" (Heath 2001, vol. II, p. 95). Students were then asked to establish the equilibrium condition of a solid body and to solve simple problems with levers.

Through the latter, Archimedes' method for the calculation of the parabolic section was presented. For example, for Archimedes to find this area, he balanced the unknown area of the parabolic section, with the known area of the triangle, so

[3]In postulate one, Archimedes states that "Equal weights at equal distances are in equilibrium" (Heath 2001, vol. II, p. 94).

Fig. 17.8 The two teachers, teaching together

the position of the fulcrum defined the relationship between their sizes.[4] However, students also realized that Archimedes used mechanics as a heuristic method, and then he confirmed his conclusions with a rigorous mathematical proof.

17.3.2.4 Resolving, as Archimedes Did, Mathematical Problems with the Use of Physical Principles

As an application in solving mathematical problems with the use of physical principles as Archimedes did, and more specifically, using the condition of static equilibrium of a solid body, a classroom activity asked students to find and prove Ceva's theorem. The activity consisted of three components.

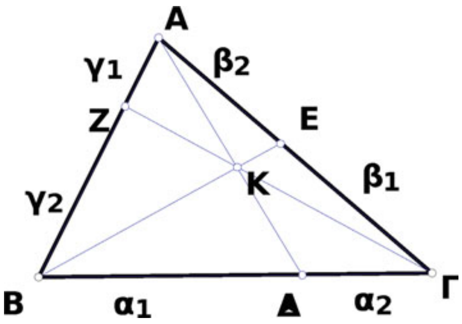
(a) Finding and proving Ceva's[5] theorem, using the properties of the center of gravity of a system of points.

Giovanni Ceva (1678), in his work *De lineis rectis se invicem secantibusstatica constructio* (construction of concurrent lines through statics), studied the application of propositions on the center of gravity of a system of masses in proving geometrical theorems (Loria 1972). As an alternative to the routine treatment of geometrical constructions with ruler and compass, he proposed the "replacement of lines with weights," placing at the points of intersection of the straight line segments weights that are inversely proportional to the lengths of these segments (cf. the law of the lever referred to above). The theorem bearing his name gives the condition for three line segments from the vertices of a triangle to the opposite sides

[4] If placed in a modern context, Archimedes assumes that the (algebraic) sum of the torques of the weights of each particle of the figure with respect to a given point, is equal to the torque of the weight of the entire shape with respect to that point, of course without referring to the mechanical concept of the torque of a force, which did not exist at that time.

[5] This activity was based on professor Stranzalos' (1999) lectures in a master's degree course in Didactics of Mathematics at the Mathematics Department of the National University of Athens. One can also find a proof of Ceva's theorem based on mechanical principles in the article of Hanna and Jahnke (2002).

Fig. 17.9 Ceva's theorem



to be concurrent. As an application of this theorem, the concurrency respectively of the bisectors, the medians and the heights of a triangle is derived.

Ceva's theorem states that a necessary and sufficient condition for the straight line segments *AΔ*, *BE*, *ΓZ* in a triangle *ABΓ* to be concurrent at *K*, is that the segments α_1, α_2, β_1, β_2, γ_1, γ_2 satisfy the relation $\alpha_1\beta_1\gamma_1 = \alpha_2\beta_2\gamma_2$ (see Fig. 17.9). In order to discover the condition of congruence, we asked students to assume that in the triangle *ABΓ*, the line segments *AΔ*, *BE*, *ΓZ* are rigid weightless rods passing through *K* and that weights β_A, β_B, β_Γ have been placed at *A*, *B*, *Γ* respectively so that the center of gravity of *ABΓ* is in *K*.

In the worksheet, the law of the lever was given[6] and it was also emphasized to the students that in any system of (point) masses:

1. The center of gravity is unique, defined as the point relative to which the (vectorial) sum of the torques of the points' weights vanishes; e.g. the center of gravity of β_B and β_Γ, is that point *Δ* on *BΓ* for which $\beta_B\alpha_1 = \beta_\Gamma\alpha_2$; and
2. The center of gravity of a system remains unchanged, if in the system we replace two weights by another one equal to their sum, applied to the center of gravity of the two masses.

The inverse of Ceva's theorem and the problem of the concurrency of a triangle's medians were given to students as homework.

(b) Finding the area of an irregular shape, with the help of laws of physics.

This activity was also given to students to try to make a change of context and solve a mathematical problem with the laws of physics.

Students were asked to measure, either directly or indirectly, the area of an irregular shape which is designed within a square (Fig. 17.10).

[6]Weights at the ends of a lever are in equilibrium at distances from the fulcrum inversely proportional to their weights; in other words, that the fulcrum is at the center of gravity of these two weights.

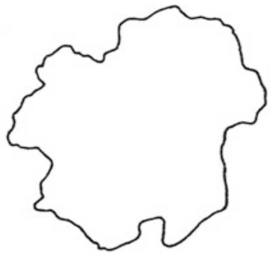


Fig. 17.10 The irregular shape designed within a square

(c) An activity concerning geometric solids' balance on a scale (a cylinder, a cone and a sphere).

Having encountered the solution of mathematical problems in a physics context, we wanted to see the reverse case, too; i.e. the solution of problems of physics with the help of laws which are traditionally considered to belong to the field of mathematics. The activity concerned a balance problem of solids mentioned in the above debate and whose volume was calculated by Archimedes: Let us assume that we have a cylinder (height R and base radius R), a cone (height R and R base radius) and a hemisphere (radius R), which are constructed from the same homogeneous solid material. What is the minimum number of objects of every shape, which is required to balance a scale with the minimum number of objects of another kind of shape? (Thomaidis et al. 1999, p. 343; see Fig. 17.11).

17.3.2.5 Archimedes' Method of Angle Trisection

Neusis (from the Greek word νεῦσις) is a geometric construction which consists of fitting a line element of given length in between two given lines, in such a way that

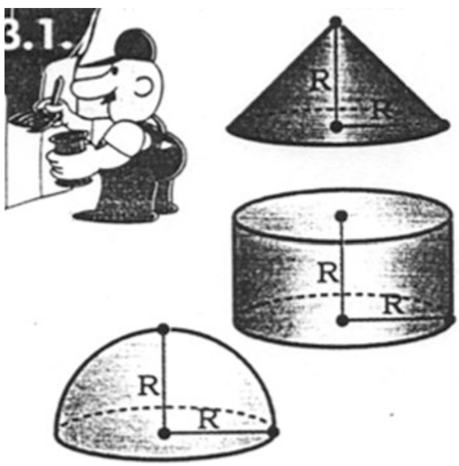


Fig. 17.11 Balancing a scale with geometric solids

the line element, or its extension, passes through a given point. Neusis was Archimedes' method of angle trisection, one of the unsolved geometric problems of Antiquity.

In order for the students to become acquainted with geometric constructions in the Geometry class, we drew compass and straightedge constructions such as: (i) the construction of the perpendicular bisector of a line segment, (ii) the construction of the bisector of an angle, and (iii) the construction of a line perpendicular to a straight line from a point not belonging to that line. Then, we stated the unsolved geometric problems of antiquity and presented different ways of solving the problems. We read Eratosthenes' letter to King Ptolemy concerning the Delian Problem and gave students as homework proposition 8 from the Book of *Lemmas* by Archimedes:

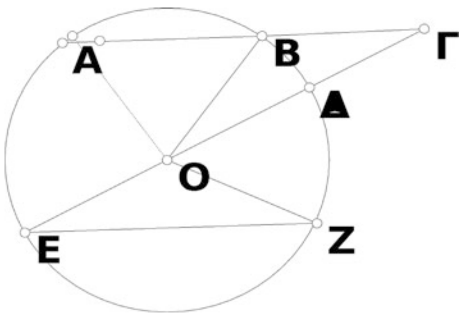
> If AB be any chord of a circle whose center is O, and if AB be produced to Γ so that $B\Gamma$ is equal to the radius; if further ΓO meets the circle in Δ and be produced to meet the circle the second time in E, the arc AE will be equal to three times the arc $B\Delta$. (Fig. 17.12)

The Neusis construction for the angle trisection (based on the previous Lemma) was presented, which called for fitting the line segment ΔE, in between the circle and the line AO, which also passes through the given point B (Thomaidis et al. 1999, p. 103; see Fig. 17.13).

17.3.2.6 Mathematics in the Computer Laboratory

(a) "The computer takes the place of Eratosthenes": A simulation in computer laboratory
While in the computer lab, students worked with a simulation of Eratosthenes' measurement of the Earth's circumference. There we had the opportunity to discuss on the non-mathematical assumptions such as the Earth is spherical and that the Sun is so far away that its light rays are practically parallel (Fig. 17.14).
(b) Trigonometry in Hellenistic Age
This session began with a brief historical overview on the genesis of trigonometry and its development in antiquity and included an activity with

Fig. 17.12 Proposition 8 from Archimedes' Book of Lemmas



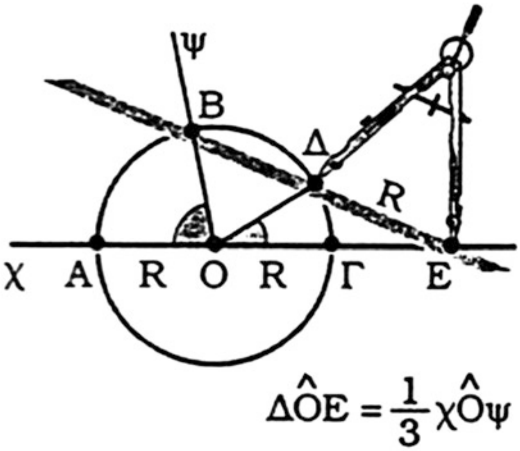


Fig. 17.13 Neusis construction

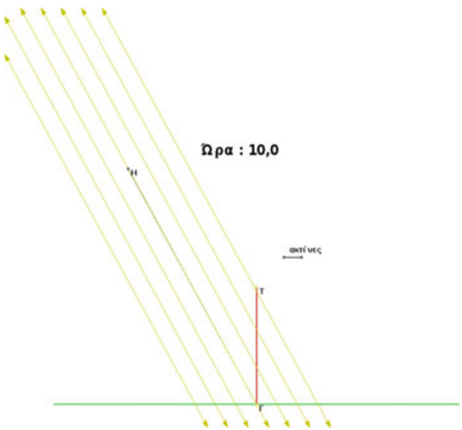


Fig. 17.14 The light rays are practically parallel

computers for the students to see that Ptolemy's table of chords is equivalent to the table of sines (Eves 1989, p. 115).

17.4 Discussion of Results

17.4.1 The Creation of the Third Space

Our first research question was if and how a theatrical performance about a historical topic of mathematics can construct an expanding learning space where different and apparently conflicting sources of knowledge and practices of the school curriculum enter in dialogue and synergy, allowing students to renegotiate technical imposed binaries like science versus humanities.

The theatrical stage itself was an expanding space, a space open to all possibilities, a space between truth and falsehood enabling students to experience both worlds, which are considered to be opposite. This space allowed the transition from the historical people and the historical context of this era to mathematics itself, with the protagonists of the play talking about and solving mathematical problems. In this space, the verb 'learn'—linked with school practice—and the verb 'play'—linked with outside classroom context—coexisted in a creative dialogue, creating conditions and prerequisites for a greater and a more effective participation of students.

The theatrical play in school mathematics teaching offered opportunities for incorporating unconventional and informal practices and also tools and practices of different activity systems of the school curriculum providing cross-disciplinary experiences. All of these generated a third space, where the different Discourses established a dialogue which expanded the boundaries of the official school mathematics Discourse and where students were actively involved.

For example, in the mathematics classroom the reading of a theatrical play or the reading of literary work introduced a classroom Discourse that was unfamiliar or even contradictory to the mathematics classroom Discourse, to which students have become accustomed. In the mathematics classroom students are used to reading texts written in the mathematical language and to solve problems, while in literature class students are used to reading poems, theatrical plays, and literary works.

The third space constructed in the ancient Greek language class with the reading and analyzing a text concerning mathematics, challenged and reshaped both disciplinary Discourses. Students did not only have to learn grammar and syntax and write the translation of the text; they had to transfer it to mathematical language and draw the geometrical shapes, as well as discuss the law of the lever, which they typically only meet in Science class.

17.4.2 Students' Experiences

We will now try to answer the second research question concerning students' experiences of the expanded mathematical space and how these influenced and changed their conceptualizations of mathematics and motivated their participation. The analysis of the interviews and the open questions of the questionnaire permitted us to stress a number of dimensions of processes of knowledge construction, the experiential character of learning, the role of the history of mathematics and changing stereotypic images of mathematics. Each of these is briefly analyzed in the following sections.

17.4.2.1 Challenging Students' Perceptions About Teaching and Learning Processes

From the questionnaire and the interviews conducted, it became clear that the pupils were motivated within these expanded contexts, and they were actively involved in the new teaching practices. The activities enabled students to experience situations cognitively, gaining new knowledge:

We learned about that era. (Maria)

We got historical information. (Eva)

We got new knowledge. (George)

Eratosthenes' way of thinking became understandable. (Katerina)

The activities also helped in establishing new ways of thinking:

It helped us think and discover things by ourselves. (Athina)

The project offered me knowledge and a new way of working. It also helped me to develop skills in other areas such as computing and handling of the computer; it motivated me to deal with the research, collection and evaluation of information and to look for ways to impart my knowledge. (Katerina)

Furthermore, the project enabled students to experience situations affectively:

'We cooperated', *'We came closer'*, *'We helped each other'*, *'We had new experiences'*,

'We had fun', *'It unfolded aspects of our character'*.

Overall, students' experience of the expanded mathematical space challenged their perceptions of mathematics teaching and learning:

The teachers chose the play as a different way for us to learn. From my point of view, what I learned, it was like a fairy tale. (Xenia)

The teacher wanted to teach math in an alternative way. The aim was to have fun and the learning came by itself. We loved it. (Sotiris)

The activities gave us the opportunity to get out of the standard mathematics of our textbooks. So, they helped us to think and to discover by ourselves the 'secrets' of mathematics. It's an alternative way of learning. (Athina)

The varied forms of activities developed during this project appear to be a positive experience for them:

I was motivated. (Maria)

Now I like math even more. (Valia)

This was true for students who already had a positive attitude towards mathematics:

I always liked math and these activities helped me to answer some questions I had and made me love it much more. (Marina)

And, also for others with negative experiences with mathematics:

I didn't like math at all. But my attitude changed. I want now to do relevant studies. This approach changed my beliefs about it…completely different from what I used to believe in the past. (Xenia)

17.4.2.2 Challenging Students' Perceptions About Geometry

These classroom experiences of the expanded mathematical space challenged students' stereotypic images of what constitutes mathematics knowledge:

I realized that math is simple and manageable. (Dimitra)

I understood that even difficult geometrical problems can be solved using and combining the knowledge we already know. (Heleni)

Moreover, the above-mentioned experiences helped students to understand the connection of mathematics to everyday life:

I saw maths related to practical applications. (Katerina)

Mathematics isn't simple theories, but helps us understand the world we live in. (Petros)

Now I see mathematics in a different way…more enjoyable! (Lydia)

Students' experience of the expanded mathematical space had an impact on the development of a stronger interest on the subject:

I got to know another aspect of math, related to the history. (Katerina)

The activities helped students to understand that mathematics is a historically-evolving human creation:

I liked it because I realized that there were some great people who thought such ideas that we are trying to understand by reading them. (Dimitra)

All of the students' responses offer considerable evidence of the effectiveness of the activities to challenge the students' dominant views of mathematics, to give students a humanistic image of mathematics and to modify their epistemological conceptions about mathematics.

17.4.3 The Experiential Character of Learning

The exploitation of theatrical practices added to the experimental dimension of teaching. To the question, 'Which activities did you like more and why?,' all 19 students chose the theatrical play, mainly due to its experiential dimension. Students experienced the way Eratosthenes managed to solve the problem, which led them to an understanding of the solution:

We experienced the Earth measurement. (Christina)

Every time the leading actor repeated his lines about the measurement of the Earth, during the rehearsals, we comprehended them better. (Lydia)

The play, because I was part of it. (Christos)

I liked the theatrical performance, because the representation of the era in which Eratosthenes, a great mathematician, measured the Earth was held in an experiential way. Yet, the process and the way of his thinking became understandable. (Katerina)

Acting or staging the play helped us realize the climate of that era and we better learned about Eratosthenes' measurement of the Earth from the theatrical play than from the narration of the dialogues by the teacher. (Marianna)

Ten students also chose the students' narration, with some of the students commenting that:

Because children narrate with an eloquent way and they communicate the new knowledge in their own way. (Katerina)

I became able to express myself verbally. (Dimitra)

I cultivate the ability of narrating. (Eva)

To the question, 'What activities did you not like at all?,' students chose the activities in which the majority of them remained passive listeners (eight replies were related to the 'cold reading' by their two classmates, seven to the debate between the two students, three to the reading of dialogues from the teacher). Students explained their choices, arguing that:

We didn't experience the persons as much as we experienced them when we played the roles. We did not have a direct role. (Christina)

17.4.4 The Role of History

According to Jankvist (2009), the arguments for using history are of two different kinds: those that refer to history-as-a-tool for assisting the actual learning and teaching mathematics and those that refer to history as-a-goal in itself, which focuses on the developmental and evolutionary aspects of mathematics as a discipline. In the whole project, we used both history as-a-goal (in the theatrical play) and history-as-a-tool (e.g. to highlight the interplay and mutual influence between mathematics and physics).

The reading of original historical texts gave us the opportunity to introduce methods that may not be taught today and helped to bridge mathematics with history, physics, astronomy, geography and philosophy. Eratosthenes' measurement revealed, as Tzanakis (2016) mentioned, the fruitful, far-reaching connections among elementary Euclidean geometry and modeling of physical situations.

The important role of the history of mathematics was highlighted by students:

I liked the fact that we got information about the Ptolemaic era and Alexandria and we learned the history of Earth measurements. (Eva)

I liked how Eratosthenes measured the circumference of the Earth so many years ago and was so close to the point! I liked it very much! (Mary)

Students believed that their teachers chose the play based on the history of mathematics for teaching purposes and they agreed with this choice:

I believe that the teachers chose this play for us to learn through the history of mathematics. We wouldn't have learnt with another theatrical play. (Petros)

Students who do not love mathematics claim that they liked not only the history of mathematics itself but also the teaching through its use. They believed that this is a teaching strategy suitable for students like themselves who have negative attitudes towards mathematics:

I don't like mathematics…I find it very difficult. But I liked the history of mathematics. With another play, we wouldn't have learnt anything…It would be good to teach mathematics in this way, to students who don't like it. (Panos)

17.5 Conclusion

Specialization in education, as a modern phenomenon, results in viewing present-day school mathematics as completely separate from other subjects of the curriculum. The way curricula are designed, on the one hand, and the way school administration and time-tabling of classes are organized, on the other, function as obstacles to the connections between subjects that should be taught (Fauvel and van Maanen 2000, p. 52). In our project, we used the history of mathematics as a resource to link different mathematical topics with topics of science and humanities, implementing interdisciplinary teaching and learning in a natural way. Students had the opportunity to explore the connections of the history of mathematics with the historical context and, at the same time, connections within mathematics and between mathematics and other disciplines, leading to them perceiving the subjects as meaningful.

Through this experience, we noticed changes (previously discussed), not only with respect to students but also with respect to teachers. The reflection of the experience of collaboration, of co-teaching, and furthermore, the use of several genres and modalities for mathematics teaching through history, contributed to teachers' professional development challenging their previous practices in mathematics classroom.

The staging of the theatrical play with a theme of the history of mathematics provided teachers with opportunities to incorporate into their teaching a variety of mediating tools and disciplinary discourses that generated an expanded learning space. This expanded space bridged different worlds and practices unifying usual dichotomies: the world of drama and the world of mathematics, the imaginary and the real, the spirit and the body, humanities and mathematics, the teacher who teaches and the student who learns. This space inspired greater participation of students in mathematical thinking and expression, as it was open to an emotional, physical and intellectual engagement of students. In these various forms of participation students could renegotiate both mathematics concepts and their own, personal perceptions of what constitutes mathematics knowledge. In such a context, the students face the challenge to see mathematics as a continuous spectrum which penetrates the various aspects of life, both now and in the future, fulfilling both individual and social needs.

As the post-graduate student (only an observer) wrote in her evaluation of the project:

> *Perhaps this performance didn't look like to a professional one but this is not what one would remember as the time goes by. The love of the three colleagues for their work and their willingness to challenge students' interest is the impressive feature of this experience. Working voluntarily and sacrificing much of their time, they managed to make the school a meeting place for discussions and reflections on the history of mathematics and the mathematics itself. Students responded to the call, perhaps not so much because of their diligence in the course of mathematics or of love for acting, as of the appreciation towards the interest of people who wanted to lead them to knowledge in this way.* (authors' translation form Greek)

Appendix

In this appendix, we provide the solutions of some of the given problems above, with an analysis of students' worksheets.

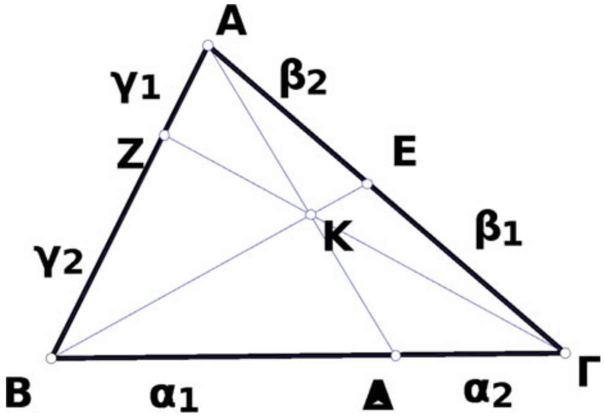
(i) ***Concurrent line segments in a triangle***

Let $A\Delta$, BE, ΓZ be concurrent at K. Imagine weights β_A, β_B and β_Γ suspended at A, B and Γ, respectively. The main idea for this solution is to choose the weights β_A, β_B and β_Γ so that the center of gravity of the triangle $AB\Gamma$ they define is on K. To this end we note that,

(1) Since the center of gravity of $AB\Gamma$ lies on ΓZ, the center of gravity of AB is at Z.
(2) For the same reason the center of gravity of $B\Gamma$ is at Δ.
(3) Similarly, the center of gravity of $A\Gamma$ lies on E.

By (1), (2), (3) and the definition of the center of gravity of two point masses, we have, respectively $\frac{\beta_A}{\beta_B} = \frac{\gamma_2}{\gamma_1}$., $\frac{\beta_B}{\beta_\Gamma} = \frac{\alpha_2}{\alpha_1}$, $\frac{\beta_\Gamma}{\beta_A} = \frac{\beta_2}{\beta_1}$.

Multiplying these three equations gives $\frac{\beta_A}{\beta_B} \cdot \frac{\beta_B}{\beta_\Gamma} \cdot \frac{\beta_\Gamma}{\beta_A} = \frac{\gamma_2}{\gamma_1} \cdot \frac{\alpha_2}{\alpha_1} \cdot \frac{\beta_2}{\beta_1} = 1 \Leftrightarrow \gamma_2\beta_2\alpha_2 = \gamma_1\beta_1\alpha_1$, that is, Ceva's theorem.



The **converse theorem** holds: In a triangle $AB\Gamma$ let Δ, E and Z be given on its sides $B\Gamma$, $A\Gamma$ and AB respectively, so that the following relation holds

$$\frac{\gamma_2}{\gamma_1} \cdot \frac{\alpha_2}{\alpha_1} \cdot \frac{\beta_2}{\beta_1} = 1$$

Then the three line segments $A\Delta$, BE and ΓZ are concurrent at a point K.

Indeed, if we choose the weights β_A, β_B and β_Γ so that the center of gravity of AB is at Z, and the center of gravity of $B\Gamma$ is at Δ, then we have that $\frac{\beta_A}{\beta_B} = \frac{\gamma_2}{\gamma_1}$ and $\frac{\beta_B}{\beta_\Gamma} = \frac{\alpha_2}{\alpha_1}$ and the center of gravity of $AB\Gamma$ should be on both ΓZ and $A\Delta$; that is, at their point of intersection K.

Then by the hypothesis, we have that $\frac{\beta_A}{\beta_\Gamma} = \frac{\gamma_2}{\gamma_1} \cdot \frac{\alpha_2}{\alpha_1} = \frac{\beta_1}{\beta_2}$, which shows that the center of gravity of β_A and β_Γ is at E. Consequently, K, the center of gravity of $AB\Gamma$ must be on BE as well, and therefore the three lines are concurrent at K.

The greatest difficulty faced by students during the proof of the theorem was the change of the context of the solution of the problem. The students have learnt to think of a mathematical problem only in a mathematical framework. Students were not able to prove neither the converse nor the theorem of the median, neither in the frame of mathematics (as an application of the Ceva's theorem) nor in the physics frame (i.e. to apply to the vertices of the triangle equal weights).

(ii) ***Finding the area of an irregular shape***

Here are two solutions: To calculate the area of the irregular shape indirectly we can initially weigh the entire square $AB\Gamma\Delta$ and measure its area. Then we have to cut the irregular shape of the frame and weigh it. The ratio of the weights is the ratio of the areas. We can also calculate the area by counting the number of squares contained in the irregular surface. To find the area of this irregular shape all our students chose to use millimeter-grid paper or created a grid of squares and counted the number of squares contained in the irregular surface. In some cases, students chose to split the irregular shape to other known shapes, as triangles or squares.

(iii) ***Using geometry to solve a problem in Physics***

To solve the above equilibrium problem students should work in a frame of geometry, after linking the mass of solids with their volumes through density. The solution is as follows: Since the objects are made from the same homogeneous material, it means that they have the same density and due to the formula $d = m/V$, which connects density (d), mass (m), and volume (V) of a solid, we conclude that the ratio of the volumes of the two objects is equal to the ratio of their masses.

In this equilibrium problem of solids, students were led to a geometric solution, i.e. the determination of the relation of the volumes, because the problem was given during the geometry class. Then, students calculated the volumes of the geometrical solids; they found the ratio of volumes and thus, they found the number of solids needed for the equilibrium. They were led automatically to this solution without proving it through the formula of the density of a solid body.

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Chapter 18
Learning New Mathematics from Old

Euclid's Art After Bath

Snezana Lawrence

Abstract This chapter presents a project for the learning of mathematics based on its relationship with art, conducted with secondary mathematics teachers in training. It aimed to use a reorientation process in order to reenergize students' interest in mathematics by giving them a problem that puts a mathematical concept under a new light, thus showing them different ways of teaching. The initial images were chosen by the author (Rafael's *The School of Athens*, and de'Barbari's *Luca Pacioli*, both containing mathematical diagrams referring to Euclid's *Elements*, book XIII) and their interpretations were investigated offering new insights related to the *Elements*. Students were then introduced to the project by putting mathematics in historical and cultural context through its relationship with art, and encouraged to seek new information in an area of mathematics they were already familiar with. The project's results relate to both the historical analysis of these images, and the use of such research to create opportunities to engage with the study of mathematics.

Keywords Euclid's *Elements* · Rafael · Pacioli · de'Barbari · Platonic solids
Adelard of Bath · Leonardo da Vinci

18.1 Euclid Goes from Athens to the Vatican

This section of the chapter will describe my research and how I came to formulate the task for my students, who were secondary mathematics teachers in training in the city of Bath, in the southwest of England. In other publications it is explained how, in my teacher education and training experience, a reorientation process has been used (Furinghetti 2007; Lawrence 2009, 2016) to reenergize students' interest in mathematics by giving them a problem that puts a mathematical concept under a

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new light in order to show them different ways of teaching. During the academic year 2015–16, I took my inspiration from art.

At the same time as this project was being planned, I organized a conference on *The Art of Learning Mathematics*, and was alerted by one of the speakers to the meaning of a geometric diagram that Euclid (possibly by him, but not directly identified) is showing, whilst seemingly proving a theorem in Rafael's (1483–1520) *The School of Athens*. This fresco was painted between 1509 and 1511, and can be seen in the Apostolic Palace in the Vatican. The detail (Fig. 18.1) from the fresco shows this diagram, but its exact shape is open to various interpretations.

18.1.1 Interpretations of the Diagram

The first interpretation of what *possibly* Euclid is showing, comes from Watson (2015). She suggests that the diagram refers to areas of certain shapes contained in a hexagonal star. The diagram of the original is given for reader's reference (Fig. 18.1).

Watson's interpretation (Fig. 18.2) claims that the image represents a right triangle contained within the hexagonal star. Then, Pythagoras' theorem, where areas of squares are being replaced by equilateral triangles, is applied so that *AEC* and *ABF* add to *BDC*. This interpretation also points to some conjecturing on the ratio of figures, *AEC* being 1/3 of *DGF* (Fig. 18.2). A reference is further made to the method of teaching to which the picture refers, namely the dialogue as that described in *Meno*, between Socrates and the slave boy, and modeling the universal teaching method via a dialogue (Lawrence 2013; Plato 2009; Watson and Mason 2009).

Fig. 18.1 The detail from Rafael's *The School of Athens* (1509–1511), showing a teacher (probably Euclid) demonstrating a theorem to a pupil. (Retrieved from https://commons.wikimedia.org/wiki/File:School_of_Athens_Raphael_detail_01.jpg (accessed 13/9/2017). This and other details from this famous painting are accessible online from many other sites; e.g. https://www.quora.com/What-are-the-important-figures-in-Raphaels-The-School-of-Athens or https://www.google.gr/search?q=Rafael%27s+The+School+of+Athens,+pictures&client=firefox-b&dcr=0&tbm=isch&tbo=u&source=univ&sa=X&ved=0ahUKEwjopIur45_WAhWHMhoKHcULCeYQsAQIJw&biw=1280&bih=878#imgrc=AU0wPOA0DYbdwM etc.)

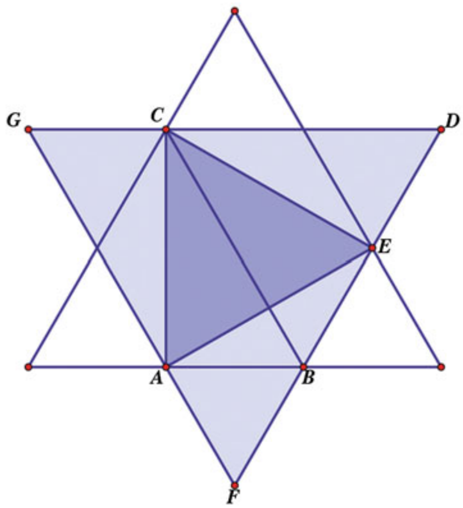

Fig. 18.2 A possible interpretation of the diagram from *The School of Athens*. (Produced by the author)

It occurred to me that a painting as old as this, with a diagram as clearly mathematical as this one is, must have had other interpretations. A brief search about the painting and mathematics contained within it found several papers written over the past quarter of a century or so. Although not the oldest from this group, we will now come to look at one of these interpretations, offered by Heilbron (2000).

Heilbron conjectures that the picture depicts Euclid teaching the properties of polygons, and in particular, of a hexagon. His diagram, reproduced here (Fig. 18.3), shows how a hexagonal star can be divided by a diagonal *PS*, on both sides of which, at equal distances, parallel lines are constructed. It can then be proved (Haas 2012; Heilbron 2000, pp. 229–230) that *AB* is equal to *CD*. Heilbron called this 'Rafael's theorem.' Although demonstrated figuratively by Euclid in Rafael's painting, the theorem does not appear in Euclid, and may well have been Rafael's original mathematical work. While we cannot say whether this was indeed original with Rafael as Heilbron suggests, it still gives us something important to consider: is it possible that art was at the time used to promote the study of mathematics and possibly some new findings that Rafael and his contemporaries worked on?

18.1.2 Pacioli's Euclid

Because the original diagram from Rafael's painting is linked in the already-mentioned literature to another painting from the same period, I looked at this to make a comparison. The second painting is of Luca Pacioli (ca. 1445–1517) attributed to Jacopo de'Barbari (1495). The painting portrays Luca Pacioli, celebrated mathematician, shown surrounded by mathematical instruments and objects,

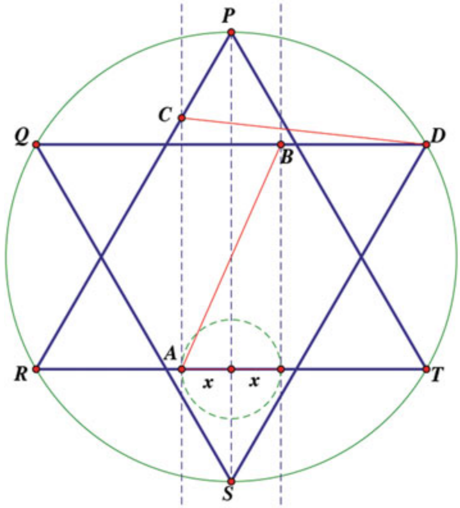

Fig. 18.3 Heilbron (2000) interpretation of the diagram from *The School of Athens*. (Produced by the author)

pointing to a drawing board and a book, and standing in front of (very possibly) Albrecht Dürer.

In de'Barbari's painting the diagram to investigate is also drawn on a small blackboard, being used possibly in some kind of teaching episode. In this painting, the diagram is perhaps clearer (Fig. 18.4). Its interpretation is however sometimes given as that referring to the theorem of Euclid XIII.8 (Baldasso 2010; Gamba 1999; Landrus 2001) or theorem XIII.12 (Mackinnon 1993). But let us distinguish the two theorems first.

The stick in Pacioli's right hand (Fig. 18.4) points in fact to an unfinished diagram, which could be either the theorem XIII.12 or XIII.8. However, Pacioli's left hand (e.g. index finger) points clearly to the text of the theorem XIII.8 from the book, with the diagram given in the margin (Fig. 18.5). On the blackboard, there are also, as Gamba (1999) points out, some numbers and a line divided in a ratio. One thing seems certain from Pacioli's posture, that there is a correlation between

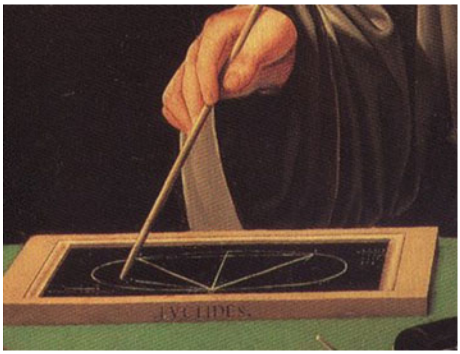

Fig. 18.4 Detail from de' Barbari's portrait of Luca Pacioli (1495), showing the diagram relating to Euclid's book XIII. (Retrieved from https://commons.wikimedia.org/wiki/File:Jacopo_de%27_Barbari_-_Portrait_of_Fra_Luca_Pacioli_and_an_Unknown_Young_Man_-_WGA1269.jpg (accessed 6/10/2017))

Fig. 18.5 Detail from de' Barbari's portrait of Luca Pacioli, Pacioli's index finger pointing to an edition of Euclid's Elements. (Retrieved from https://commons.wikimedia.org/wiki/File:Jacopo_de%27_Barbari_-_Portrait_of_Fra_Luca_Pacioli_and_an_Unknown_Young_Man_-_WGA1269.jpg (accessed 6/10/2017))

the diagram and the theorem to which the Pacioli points on the blackboard (Fig. 18.4) and the one which appears in an edition of Euclid's *Elements* (Fig. 18.5).

So, does it matter which theorem it is or whether the two pointers refer to the same one? Let us, for the moment, examine the theorems in question before attempting to make a conclusion about this.

18.1.3 Euclid's Theorems

Two theorems: XIII.8 and XIII.12 are different but they are in the same book and are indirectly linked.

Proposition 8, book XIII, refers to the golden ratio or section: *If in an equiangular pentagon straight lines subtend two angles [are] taken in order, then they cut one another in extreme and mean ratio, and their greater segments are equal to the side of the pentagon* (Heath 1908, p. 453).
This is nicely represented by the diagram in Fig. 18.6.

Proposition 12 of the same book states on the other hand that: *If an equilateral triangle be inscribed in a circle, the square on the side of the triangle is triple of the square on the radius of the circle* (Heath 1908, p. 466; Fig. 18.7).

What do these theorems, and indeed the diagrams, actually refer to?

We know that book XIII of Euclid's *Elements* refers to Platonic solids. In fact, there is a hypothesis, famously stated by Proclus, that one of the aims (perhaps the

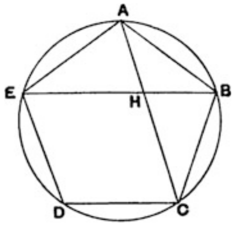

Fig. 18.6 Proposition XIII.8 as illustrated in Heath (1908). (Reproduced by the author)

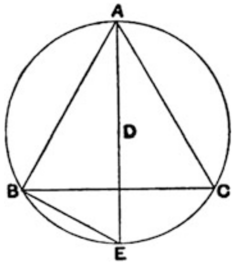

Fig. 18.7 Proposition XIII.12 as illustrated in Heath (1908). (Reproduced by the author)

most important one) of the *Elements* was the construction of the regular polyhedra (Proclus 1560; Sanders 1990).

If we look at the two theorems here stated, they refer to: XIII.8—the construction of a pentagon, the basis for the construction of dodecahedron, and XIII.12—the construction of tetrahedron. Let us then look a little more closely at the latter theorem, XIII.12. By closer inspection we can say that this theorem itself is closely related to what is previously mentioned as Rafael's theorem, easily resembling the first image (Fig. 18.1) as shown in the comparison (Fig. 18.8).

Why and how would the two be connected, apart from the obvious (they both refer to the book XIII, and the Platonic solids)? Further investigation will conjecture on this, but let us first examine the editions of the *Elements* to which both Rafael and Pacioli had access and to how these diagrams could be in any way relevant to our research.

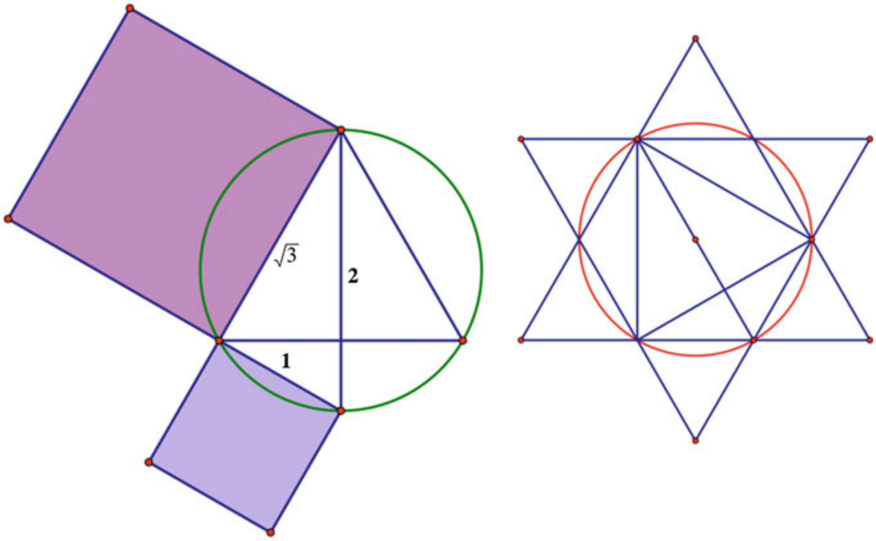

Fig. 18.8 Two diagrams compared—Rafael's theorem 18.3 with 18.6. (Produced by the author)

18.1.4 Edition of the Elements

It is possible, but not certain, that we can identify the edition of *Elements* to which Pacioli is pointing to with his left hand. As the date of the painting is most probably 1495, the only possible *Elements* to which Pacioli (and indeed de'Barbari) would have had access is the Venice edition from 1482 (Mackinnon 1993). This edition was the Latin translation by Johannes Campanus, Pope Urban IV's chaplain at the time. The book was illustrated and produced by Erhard Ratdolt, and, for my purposes of study, a copy of this can be found in Victoria and Albert Museum in central London. If one looks at the pages to which Pacioli is pointing in his portrait, they are quite identifiable as pages from this book—something that is analyzed at length by Baldasso (2010).

However, another interesting conjecture comes to mind—that of Taylor (1980)—in which he states that Pacioli may have written another book based on Euclid's *Elements* apart from the straightforward Latin edition published in 1509 (Swetz 2014). Taylor suggests that Pacioli would have written and possibly published this other book earlier, but that it did not survive to our time. If so, Taylor (1980) suggests that this edition would have also been based on Campanus' but written in Italian. Considering that another work of Pacioli's, his *De viribus quantitates*, written in the similar period (between 1496 and 1508), was only discovered in the University of Bologna library in 2007, it is conceivable that Taylor is right and that there may yet exist the books by Pacioli which he describes.

In any case, the two theorems also coincide with the apocryphal records about books XIV and XV of the *Elements*. These two books, apparently the first written by Hypsicles (ca. 190–120 BC) and possibly based on Apollonius' *Conics*, and the second by Isidore of Miletus (442–537), investigated Platonic solids inscribed in spheres.

If so, and if there are undiscovered books by Pacioli there is still much research to be done on the work relating to historical analysis of Pacioli and the diagrams we spoke about, as the two books by Hypsicles and Isidore of Miletus relate reportedly to:

(a) the ratio of the surfaces of dodecahedron and icosahedron inscribed in the same sphere being equal to their volumes; and
(b) the relationships of the edges and solid angles in Platonic solids (Boyer 1991).

18.2 Pedagogy Related to the History of Mathematics—How Does Euclid Make It to Bath?

In the previous section of this chapter I have recalled my search which began from Rafael's *The School of Athens* and ended up with one of the unsolved mysteries relating to the missing books by Pacioli and even Euclid. This research and the questions it raised inspired me to think further about the mathematics contained in the Rafael's and de'Barbari's paintings, bringing back the joy of learning new mathematics to my everyday working life. As I found such inspiration, I thought it would perfectly fit my original idea to find a source of inspiration for my students too, based on art.

This part of the chapter will deal with this particular aspect of the project: how I linked such personal inspiration and channeled and structured the work for student teachers. The aim of this was to try and recreate for them the investigation so that they too may have an aesthetically pleasing experience in the learning of new-but-old mathematics. This in turn, I hoped, would enable student teachers to seek similar sources of inspiration in the future, bringing them a recurrent 'reorientation' event every so often.

In the pursuit of this task, I looked for anchors on which to base the structure of the project for students. The building of teacher identity was the first aspect I wanted to pay attention to as this highly personal development is part of the official teacher training framework in the UK that is assessed in a formal way.

18.2.1 Teacher Identity

In building of the pedagogy for practice, my recent interests have focused on 'finding one's own voice' (Lawrence 2016). In the teacher education context, this question of identity is an interesting one to which I will now turn for some further thoughts and describe the experiences from the project.

The following question may be difficult to answer in some circumstances, but in our case it was quite easily dissected and deconstructed: "Who am I—personally, professionally, intellectually and so on?" One may ask oneself this question at regular intervals, and to which one may or may not find easy (or comfortable) answers (Brown 2011).

However, this project gave us ample opportunities to investigate precisely these types of issues and questions. They are more easily investigated when one has the Pacioli's portrait as a point of reference. Looking at it and seeing Pacioli as he was portrayed by de'Barbari, student teachers could easily ask themselves questions such as:

- Who am I? and consequently,
- What is it that defines me?
- What mathematics interests me, and what are my tools?
- Who is my most important colleague or collaborator?
- Who are my pupils (perhaps even my 'ideal' ones)?

All of these questions were easily posed and discussed with Pacioli's portrait as a starting point. My students came with engaging, and at times funny, answers—all of which were presented in self-portraits in response to these questions. These were presented visually and we organized an 'exhibition' of a kind where students portrayed themselves in a similar position in which Pacioli was depicted, with their preferred teaching and mathematical tools presented around them, and their most interesting recent discovery in front of them, with their 'ideal' student standing behind them. I found this exercise, apart from being an engaging one, to be the most (surprisingly) important one in the project, offering students opportunities to form opinions and position themselves in the new landscape of mathematics education they were just entering and exploring.

Linking Euclid's *Elements* and his depiction in Rafael's painting, and the tradition of teaching Euclidean mathematics, brought us to the question of its various editions and the loss and retrieval of the books and of knowledge in the Middle Ages. Here we came across one of the most celebrated moments of intellectual history of all time, the recovery of Euclid's *Elements* to the Western culture by Adelard of Bath (1080–1152). Adelard was a philosopher, traveler and translator, who brought to England the Latin translation of Euclid's most famous work. The fact that Adelard was born in Bath where the project took place, was an important factor in our project.

In terms of gender, the illustration below (Fig. 18.9) shows not only learned men, but also a female teacher showing geometrical constructions and/or, demonstrating geometrical theorems. This image is an illustration of Adelard's French translation, attributed to a Meliacin Master, dated approximately between 1309–1316. The female image brought about the question of gender balance into focus, not only apparent in ancient, but also the more modern or contemporary

Fig. 18.9 This image is an illustration of Adelard's French translation, attributed to a Meliacin Master, dated approximately between 1309 and 1316 (Meliacin Master 1309–1316)

mathematics scene. For example, in a study conducted recently in the UK, only 6% of all mathematics professors are women, and women roughly comprise about 30% of all university lecturers in the field (LMS 2013, p. 14).

This particular image further contains some other interesting information about the context and mathematical and teaching tools from the period. Let us pay attention to the circular space in front of the female teacher, in which geometric drawing tools can be seen, such as compasses, set squares and other objects. These tools are placed in a circular shape, most probably a sand tray, in which geometric diagrams were drawn for demonstration purposes in the teaching of mathematics (Fig. 18.9). My students did not question the gender of the teacher in the image and the general consensus between them was that there would have been more women in teaching roles in the history of the discipline than what the official records tell us.

Thus, we have the elements that can bring about a contemplation on the locality, individual's role in the teaching of mathematics and the circulation of mathematical knowledge and mathematics texts throughout the world.

18.2.2 Building the Pedagogy

Here I will describe how I went about building pedagogy to bring the students to both learning of new mathematics from the old, and to position themselves in relation to their chosen professional role of a mathematics teacher. I used the following sequence to introduce the students to the project.

Firstly, the students were given the original diagrams and images and the papers which analyzed these diagrams at the beginning of the project. Copies of Rafael's and de'Barbari's paintings and the enlarged details of the diagrams in question were given, from which students could work. The picture of Rafael was clearly related to Euclid, and from the same period there were other pictures related to doing mathematics and to Euclid, e.g. Pacioli's portrait. Once this was established, the images of Euclid's *Elements* were found that related to the city of Bath in England —the place where the project was taking place—via Adelard of Bath, the first translator of Euclid's *Elements* into Latin. These facts and images were presented and discussions were encouraged, along with further research on personalities introduced thus far (Euclid, Rafael, Pacioli).

The overall task they were presented with had two overarching aims:

1. To learn how to seek for cultural references in mathematics in order to gain understanding and insight into the different interpretations of what doing and learning of mathematics can be like (and how it is defined in different cultures and periods); and
2. To experience the process of learning some new mathematics via some old one, i.e. culturally referenced and historically bound mathematical analysis.

Secondly, I asked students to think, contemplate and not necessarily come up with immediate answers, on the questions such as:

1. What kind of mathematics teacher do I want to be?
2. How does mathematical knowledge travel through the world (and in this case, within Europe)?
3. Does my gender play any role in all of this, and if not for me, does it play one for others?
4. Formulate your own research question based on the context of these two paintings.

Students were then given five smaller tasks to complete in order to structure their work on this project, in which they were asked to:

- find everything they could about Rafael's and de'Barbari's paintings;
- analyze for themselves the images showing instruction on the blackboards depicted in these two paintings;
- compare interpretations of the diagrams;
- make mind-maps in groups to show all of the prerequisite knowledge needed to teach the theorems in question to pupils in secondary settings and
- construct an activity based on the theorems contained in the diagrams from two paintings, and connect it to the National Curriculum.

The original exploration and the connection between two images by Rafael and de'Barbari offered much more than was hoped for in the beginning. A link between the two images, and mathematics contained within them as we have shown, can be seen from the diagrams. It further transpired that these images, which put geometry in context of history and art so beautifully, also contain a wealth of further pathways to investigate. A few other aspects of pedagogy were considered in the project, and this is where we now turn.

18.2.3 Range of Pedagogical Tools: General and Mathematical

Finding one's 'voice' in the learning and teaching of mathematics is an important process as it is in any learning experience. This has been described in numerous examples, but I mention two as directly relevant: Fried (2008) and Lawrence (2016). This process of 'finding one's voice' related in our case to the dual aspect of being a learner and a teacher, and to finding an internal and external dialogue related to mathematical narratives in historical context. In our case, the narratives were manifold and related to:

- how something works;
- how this fits in with the greater structure of mathematical knowledge;
- how mathematical facts are interrelated; and
- how mathematical knowledge travels and is enriched by practitioners.

All of the above were easily contained within the framework of the *Elements*. Furthermore, the knowledge thus investigated, gained, networked and conveyed, could be questioned (Lawrence 2016). The questions we could pose while investigating our discussed diagrams, the way they came about and reading the relevant literature could be:

1. What further is there to learn from what we found out? and
2. How can that be translated into a mathematical language, a narrative and a mathematical pedagogy suited to a secondary classroom?

The answer to the first question remains open. I posed this in the conclusion of the first part of this chapter: there certainly is more to be discovered related to the relationship between tetrahedron and dodecahedron, and the apocryphal books XIV and XV of the *Elements*. Put in this light, this question could initiate further interest of future teachers (and their teacher) for years to come.

The second question is perhaps more difficult. It refers not only to the knowledge of mathematics, history of mathematics and pedagogy, but also to 'task design.' Task design is a complex skill and not easily taught to teachers in training. The necessary aspects of such an undertaking seem however to rely almost on the same principles that can be identified from the thread connecting the images of Euclid in Rafael's and de'Barbari's paintings: the variation of interpretation is needed (and necessary) and must be personal, to allow for sense making (Watson and Mason 2006).

My argument is that this type of research, starting from a historical artifact, in this case a painting, following some established and the seeking of new links, must be left to the teacher or educator to find her/himself in order to begin making sense of both mathematics and how to communicate it. When enough material is given, the teachers must be left free to rummage through it to make sense out of mathematics presented to themselves, to their peers, and to their pupils (Lawrence 2016). But how successful was this in our case? I will come to that in a moment, but let us now first pay attention to the artifacts and teaching techniques that the students learnt through this project.

18.2.4 Teaching Objects and Techniques

Through this investigation we have come across several artifacts and teaching techniques that are important for novice teachers to think about and investigate further.

First is certainly the role of demonstration in the teaching process. What tools of demonstration can one use and how have these developed over time? In our example there are at least two identifiable such objects and techniques of explanation: the blackboard, and the sand-table for drawing of geometrical objects and showing and proving their meaning. We may ask further—what are their modern equivalents?

There is also the *teaching dialogue* as implied by the Rafael's painting and the references we drew from it. The Socratic dialogue, the importance and the use of which never seem to fade in the learning process, has certain rules that a teacher in education would be prudent in knowing and exploring for themselves (Lawrence 2013; Watson and Mason 2009).

The exercise with the Pacioli's portrait was another important tool and artifact that was used in this project, as students made their own self-portraits. This exercise brought about a discussion on different ways not only of teaching but of learning mathematics. It gave the students ideas to repeat similar tasks in their secondary classrooms in order to give opportunities to their pupils to question and challenge their interest in mathematics and contemplate about their skills and abilities related to the subject.

18.3 Conclusion: What Does the Future Hold?

This project can certainly offer many further pathways for research. In terms of reinvigorating my own interest, I will certainly look further into the work of Piero della Francesca (1415–1492), one of the leading artists of the Renaissance with significant contributions to development of geometrical techniques, and who was connected to both Rafael and Pacioli. Leonardo da Vinci (1452–1519), one of the most celebrated artists and scientists of all time, was also closely working with Pacioli on his *Summa de Arithmetica, geometria, proportioni et proportionalita* (1494), having provided illustrations for it. Da Vinci's theorem (Fig. 18.10) is closely linked to the particular problem described at the beginning of this chapter. He was certainly at the time interested in the relationships between lengths, areas,

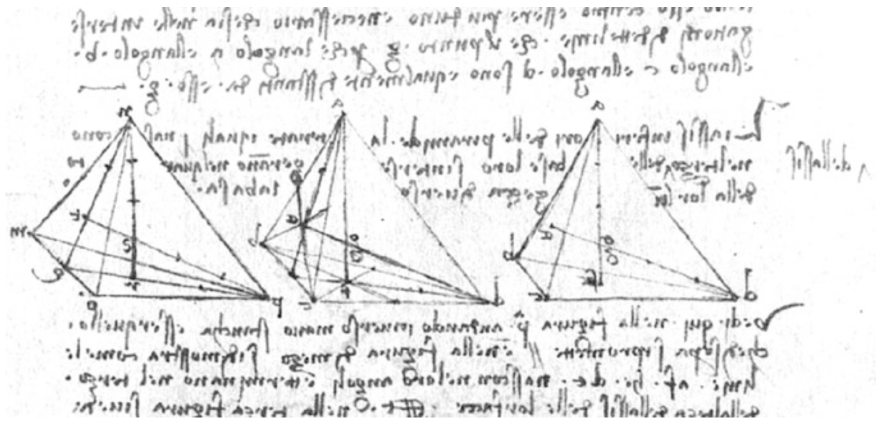

Fig. 18.10 Leonardo da Vinci, *Codex Arundel* (1478–1518), British Library manuscript Arundel 263, f215. (Reproduced with permission)

and volumes. In *Codex Arundel* (da Vinci 1478–1518; Duvernoy 2008), he calculates (Fig. 18.10) the center of gravity of a pyramid (Arundel 263, f215), further extending it to tetrahedron, as was the case with both instances of diagrams from which we began (Fig. 18.8).

To use this research further with teachers or learners, some discussions may be had about:

(a) the possibility of connecting two- and three-dimensional geometry seamlessly, showing the interconnectedness of mathematics;
(b) universality and beauty of mathematical concepts that transcend centuries, cultures and disciplines; and
(c) showing that mathematics is both an inspiration and a part of culture.

However, the most surprising results of this project could be put into two categories. The first relating to the teacher students, and the second to their teacher, i.e. this chapter's author.

(a) Teacher students' greatest benefit from this project was an opportunity to search for and find (or not) their own voice while making sense of mathematics they were trying to decipher. By making self-portraits based on de'Barbari's portrait of Pacioli, student teachers were able also to identify their own favorite mathematical tools, processes and facts, and imagine their favorite type of pupil.

Students engaged with this activity well and some used the auto-portraits throughout the year-long course. This activity—finding one's professional and literal image, articulating their own interest in mathematics, defining the tools with which to investigate, teach and explain mathematics most appropriately to their own interests, portray their own demeanor, illustrate and model the behavior of their own students (ideal or not) and collaborators—gave teachers the opportunities to question, examine and challenge themselves. It also gave them opportunities to do so with the images of mathematics and mathematicians from contemporary and historical cultures.

This aspect of the project was successful with all teacher students. Some discussions that followed this activity would be important in any learning environment and therefore they are described in the following paragraphs.

Firstly, what is familiar to learners locally is more likely to affect them positively as they begin making other connections with things they already know (Lawrence 2016). The local 'effect' had a positive influence as learners began to make other connections by drawing on their personal experience.

Secondly, the identification for both genders was a rich subject for many discussions—we had by a happy coincidence that both geography and gender were reflected in the image of Adelard of Bath's book (Fig. 18.8). We questioned the images of women as teachers of mathematics and discussed the possible interpretation—that such images were metaphors for sciences rather than images of women teaching in practice. Concluding remarks of my female students' were generally that such interpretations perpetuate the place of women in the history of teaching and learning of mathematics, rather than give accurate historical information.

In this complex project however, the engagement of some students was not positive. I was surprised to see that the thread relating to mathematical facts expounded in the first section of this chapter did not seem to be as interesting to students as I expected it to be (and as it was to me). One of the reasons for this was, I believe, due to the shortness of time given to teachers in training to investigate and contemplate mathematics. This stands in sharp contrast to the amount of time they are given to present mathematics in a utilitarian way (as prescribed by the schools and the National Curriculum).

So what was the main benefit of such a project? This brings us to the final point I wish to make in respect of the pedagogy relating to the history of mathematics as developed in this project.

(b) Through working in teacher education I have come across many approaches relating to pedagogy, and this project showed how one such approach can be developed starting from an image portraying teaching and learning of mathematics in art.

One aspect of pedagogy I have not come across in the related literature yet is about reigniting the interest in original research related to mathematics in the mathematics teachers' teacher.

This project showed me that this is, indeed, possible. Finding an initial impetus like our initial image, which gives opportunities for many interpretations, led me to new areas and images and a multitude of new potential research pathways. The further development of research in this project reminded me, a teacher educator, that one can learn many new things from old mathematics, and that doing so can indeed bring back the passion for learning in the teacher educator. This was probably the most satisfying and surprising result of this project. The hope is that the student teachers will be left with many unanswered questions relating to history of mathematics, showing them that the inspiration for their own engagement is always there, and that new mathematics can always be found in old artifacts and art related to mathematics.

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