The Effect of Planarization on Width

David Eppstein[(✉)]

Department of Computer Science, University of California, Irvine, USA
`eppstein@uci.edu`

Abstract. We study the effects of planarization (the construction of a planar diagram D from a non-planar graph G by replacing each crossing by a new vertex) on graph width parameters. We show that for treewidth, pathwidth, branchwidth, clique-width, and tree-depth there exists a family of n-vertex graphs with bounded parameter value, all of whose planarizations have parameter value $\Omega(n)$. However, for bandwidth, cutwidth, and carving width, every graph with bounded parameter value has a planarization of linear size whose parameter value remains bounded. The same is true for the treewidth, pathwidth, and branchwidth of graphs of bounded degree.

1 Introduction

Planarization is a graph transformation, standard in graph drawing, in which a given graph G is drawn in the plane with simple crossings of pairs of edges, and then each crossing of two edges in the drawing is replaced by a new dummy vertex, subdividing the two edges [1–4]. This should be distinguished from a different problem, also called planarization, in which we try to find a large planar subgraph of a nonplanar graph [5–8]. A given graph G may have many different planarizations, with different properties. Although the size of the planarization (equivalently the crossing number of G) is of primary importance in graph drawing, it is natural to ask what other properties can be transferred from G to its planarizations.

One problem of this type arose in the work of Jansen and Wulms on the fixed-parameter tractability of graph optimization problems on graphs of bounded pathwidth [9]. One of their constructions involved the planarization of a non-planar graph of bounded pathwidth, and they observed that the planarization maintained the low pathwidth of their graph. Following this observation, Jansen asked on cstheory.stackexchange.com whether planarization preserves the property of having bounded pathwidth, and in particular whether $K_{3,n}$ (a graph of bounded pathwidth) has a bounded-pathwidth planarization.[1] This paper represents an extended response to this problem. We provide a negative answer

[1] See https://cstheory.stackexchange.com/q/35974/95.
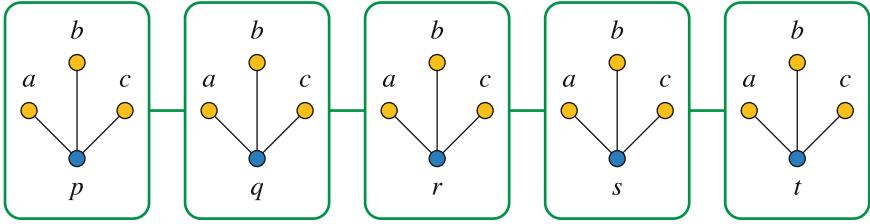
Fig. 1. A tree-decomposition and path-decomposition of $K_{3,5}$, with width three. Vertices a, b, and c (on one side of the bipartition) belong to all bags; vertices p, q, r, s, and t (on the other side) are each in only one bag.

to Jansen's question: planarizations of $K_{3,n}$ do not have bounded pathwidth. However, for bounded-degree graphs of bounded pathwidth, there always exists a planarization that maintains bounded pathwidth. More generally we study similar questions for many other standard graph width parameters.

Our work should be distinguished from a much earlier line of research on planarization and width, in which constraints on the width of planar graphs are transferred in the other direction, to information about the graph being planarized. In particular, Leighton [1] used the facts that planar graphs have width at most proportional to the square root of their size, and that (for certain width parameters) planarization cannot decrease width, to show that when the original graph has high width it must have crossing number quadratic in its width. In our work, in contrast, we are assuming that the original graph has low width and we derive properties of its planarization from that assumption.

1.1   Width Parameters in Graphs

There has been a significant amount of research on graph width parameters and their algorithmic implications; see Nešetřil and Ossona de Mendez [10] for a more detailed survey. We briefly describe the parameters that we use here.

Treewidth. Treewidth has many equivalent definitions; the one we use is that the treewidth of a graph G is the minimum width of a tree-decomposition of G [10]. Here, a tree-decomposition is a tree T whose nodes, called *bags*, are labeled by sets of vertices of G. Each vertex of G must belong to the bags of a contiguous subtree of T, and for each edge of G there must exist a bag containing both endpoints of the edge. The width of the decomposition is one less than the maximum cardinality of the bags. Figure 1 shows such a decomposition for $K_{3,5}$.

Pathwidth. The pathwidth of a graph is the minimum width of a tree-decomposition whose tree is a path, as it is in Fig. 1 [10]. Equivalently the pathwidth equals the minimum vertex separation number of a linear arrangement of the vertices of G (an arrangement of the vertices into a linear sequence) [11]. Every linear arrangement of an n-vertex graph defines $n-1$ cuts, that is,

$n − 1$ partitions of the vertices into a prefix of the sequence and a disjoint suffix of the sequence. The vertex separation number of a linear arrangement is the maximum, over these cuts, of the number of vertices in the prefix that have a neighbor in the suffix. From a linear arrangement one can construct a tree-decomposition in the form of a path, where the first bag on the path for each vertex v contains v together with all vertices that are earlier than v in the arrangement but that have v or a later vertex as a neighbor.

Cutwidth. The cutwidth of a graph equals the minimum edge separation number of a linear arrangement of the vertices of G [12]. The edge separation number of a linear arrangement is the maximum, over the prefix–suffix cuts of the arrangement, of the number of edges that cross the cut.

Bandwidth. The bandwidth of a graph equals the minimum span of a linear arrangement of the vertices of G [12]. The span of a linear arrangement is the maximum, over the edges of G, of the number of steps in the linear arrangement between the endpoints of the edge.

Branchwidth. A branch-decomposition of a graph G is an undirected tree T, with leaves labeled by the edges of G, and with every interior vertex of T having degree three. Removing any edge e from T partitions T into two subtrees; these subtrees partition the leaves of T into two sets, and correspondingly partition the edges of G into two subgraphs. The width of the decomposition is the maximum, over all edges e of T, of the number of vertices that belong to both subgraphs. The branchwidth of G is the minimum width of any branch-decomposition [13].

Carving width. A carving decomposition of a graph G is an undirected tree T, with leaves labeled by the vertices of G, and with every interior vertex of T having degree three. Removing any edge e from T partitions T into two subtrees; these subtrees partition the leaves of T into two sets, and correspondingly partition the vertices of G into two induced subgraphs. The width of the decomposition is the maximum, over all edges e of T, of the number of edges of G that connect one of these subgraphs to the other. The carving width of G is the minimum width of any carving decomposition [13]. For instance, Fig. 6 depicts a carving decomposition of $K_{3,3}$ with width four, the minimum possible for this graph.

Tree-depth. The tree-depth of G is the minimum depth of a depth-first-search tree T of a supergraph of G (Fig. 2). Such a tree can be characterized more simply by the property that every edge of G connects an ancestor–descendant pair in T [10].

Clique-width. A clique-construction of a graph G is a process that constructs a vertex-colored copy of G from smaller vertex-colored graphs by steps that create a new colored vertex, take the disjoint union of two colored graphs, add all edges from vertices of one color to vertices of another, or assigning a new color to vertices of a given color. The width of a clique-construction is the number of distinct colors it uses, and the clique-width of a graph is the minimum width of a clique-construction [14].
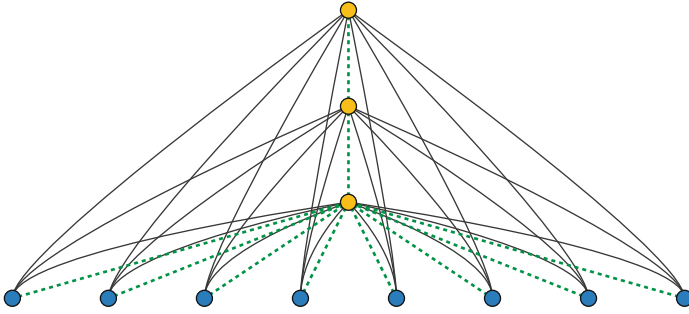
Fig. 2. $K_{3,8}$ has tree-depth three: The depth-three tree shown by the green dashed edges forms a depth-first search tree of a supergraph of $K_{3,8}$. (Color figure online)

1.2 New Results

In this paper, we consider for each of the depth parameters listed above how the parameter can change from a graph to its planarization, when the planarization is chosen to minimize the parameter value. We show that for treewidth, pathwidth, branchwidth, tree-depth, and clique-width there exists a graph with bounded parameter value, all of whose planarizations have parameter value $\Omega(n)$. In each of these cases, the graph can be chosen as a complete bipartite graph $K_{3,n}$. (It was also known that the planarizations of $K_{3,n}$ have quadratic size [15].)
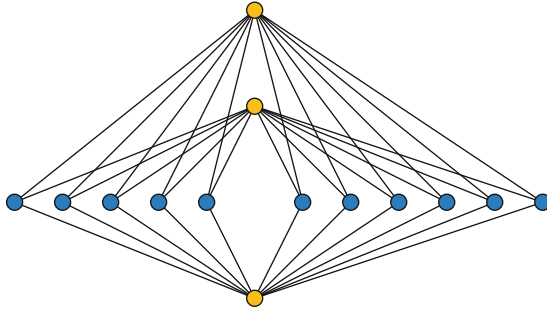
However, for bandwidth, cutwidth, and carving width, every graph with bounded parameter value has a planarization of linear size whose parameter value remains bounded. The same is true for the treewidth, pathwidth, branchwidth, and clique-width of graphs of bounded degree. (Graphs of bounded degree and bounded tree-depth have bounded size, so this final case is not interesting.)

2 Treewidth, Branchwidth, Pathwidth, Tree-Depth, and Clique-Width

In this section we show that all planarizations of $K_{3,n}$ have high width. We begin by computing the crossing number of $K_{3,n}$. This is a special case of Turán's brick factory problem of determining the crossing number of all complete bipartite graphs. For our results we need a variant of the crossing number, $\mathrm{cr}_{\mathrm{pair}}(G)$, defined as the minimum number of pairs of crossing edges (allowing edges to cross each other multiple times, but only counting a single crossing in each case) instead of the usual crossing number $\mathrm{cr}(G)$ defined as the number of points where edges cross [16]. We bound this number by an adaptation of an argument from Kleitman [17], who credits it to Zarankiewicz [15].

Lemma 1.

$$\mathrm{cr}_{\mathrm{pair}}(K_{3,n}) = \binom{\lfloor n/2 \rfloor}{2} + \binom{\lceil n/2 \rceil}{2} = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Fig. 3. A drawing of $K_{3,11}$ with 25 crossings, the minimum possible for this graph.

Proof. To show that a drawing with this many crossing pairs exists, place the n vertices on one side of the bipartition of $K_{3,n}$ along the x-axis, with $\lfloor n/2 \rfloor$ on one side of the origin and $\lceil n/2 \rceil$ on the other. Place the three vertices on the other side of the bipartition along the y-axis, with two points on one side of the origin and one on the other. Connect all of the pairs of points that have one point on each axis by a straight line segment, as shown in Fig. 3. A straightforward calculation shows that the number of crossings is as claimed.

In the other direction, we know as base cases that $\mathrm{cr_{pair}}(K_{3,2}) = 0$ and $\mathrm{cr_{pair}}(K_{3,3}) = 1$. For any larger n, let the vertices of the n-vertex side of the bipartition of $K_{3,n}$ be $v_1, v_2, \ldots v_n$. If every pair v_i, v_j form the endpoints of at least one pair of crossing edges, then the total number of crossings is at least $\binom{n}{2}$, larger than the stated bound; otherwise, order the vertices so that v_{n-1} and v_n do not form the endpoints of any pair of crossing edges.

Then, in any drawing of $K_{3,n}$, the $K_{3,n-2}$ subgraph formed by deleting v_{n-1} and v_n has $\mathrm{cr_{pair}}(K_{3,n-2})$ crossings. Each of the $n - 2$ $K_{3,3}$ subgraphs induced by v_{n-1}, v_n, exactly one other v_i, and the three vertices on the other side of the bipartition supplies at least one additional crossing, because $\mathrm{cr_{pair}}(K_{3,3}) = 1$. None of these subgraphs share any crossings, because the crossings in the $K_{3,n-2}$ subgraph involve neither v_{n-1} nor v_n, while the crossings in the $K_{3,3}$ subgraph all involve exactly one of these two vertices and the one other vertex v_i included in the subgraph. Therefore, we have that

$$\mathrm{cr_{pair}}(K_{3,n}) \geq \mathrm{cr_{pair}}(K_{3,n-2}) + (n - 2)\,\mathrm{cr_{pair}}(K_{3,3}).$$

The result follows by induction on n.   □

This lemma shows that the *crossing graph* of a drawing, with a vertex in the crossing graph for each edge of $K_{3,n}$ and an edge in the crossing graph for each crossing of the drawing, has constant density, in the following sense. We define the *density* of a graph with m edges and n vertices to be $m/\binom{n}{2}$. This is a number in the range $[0, 1]$. For instance, the crossing graph of any planarization of $K_{3,n}$ has $3n$ vertices and (by Lemma 1) at least $(1 - o(1))n^2/4$ edges, so its density is at least

$$\left(1 - o(1)\right)\frac{n^2}{4} \bigg/ \binom{3n}{2} = \frac{1}{18} - o(1).$$

To prove that planarizations of $K_{3,n}$ have high treewidth, we need higher densities than this, which we will achieve using the following "densification lemma":

Lemma 2. *Let G be a disconnected graph with n vertices and m edges, such that the ith connected component of G has n_i vertices and m_i edges. Then there exists i such that $m_i/n_i \geq m/n$.*

Proof. We can represent m/n as a convex combination of the corresponding quantities in the subgraphs:

$$\frac{m}{n} = \sum_i \frac{n_i}{n} \cdot \frac{m_i}{n_i}.$$

The result follows from the fact that a convex combination of numbers cannot exceed the maximum of the numbers.   □

Given a tree-decomposition T of a drawing D of $K_{3,n}$, and any connected subtree S of T, we define a crossing graph C_S as follows. Let E_S be the subset of edges of $K_{3,n}$ with the property that, for each edge e in E_S, the only bags of T that contain crossings on e are the bags in S. (We do *not* require the two endpoints of e in $K_{3,n}$ to belong to these bags.) Then C_S is a graph having E_S as its vertex set, and having an edge for each pair of edges in E_S that cross in D. For instance, C_T is the crossing graph of the whole drawing, as defined earlier. We will use Lemma 2 to find subtrees S whose graphs C_S are more dense (relative to their numbers of vertices) than C_T. To do so, we use the separation properties of trees:

Lemma 3. *Let T be a tree-decomposition T of a drawing D of $K_{3,n}$, and suppose that T has width w. Let S be a subtree of T such that C_S has n_S vertices and m_S edges. Then there exists a bag $B \in S$ with the following properties:*

- *The removal of B disconnects S into subtrees S_i.*
- *For subtree S_i, the corresponding crossing graph C_{S_i} has at most $n_S/2$ vertices.*
- *The total number of edges in all of the crossing graphs C_{S_i} for all of the subtrees S_i is at least $S - 2(w + 1)(n - 2)$.*

Proof. We choose B arbitrarily, and then as long as it fails to meet the condition on the number of vertices in the graphs C_{S_i} we move B to the (unique) adjacent bag in which this condition is not met. After the move, the subtree containing the former location of B has at most $n_S/2$ vertices in C_{S_i}, because these vertices are disjoint from the ones in the large subtree before the move. Moving B also cannot increase the numbers of vertices in the crossing graphs of the subtrees formed from the partition of the large subtree, and it reduces the numbers of

bags in those subtrees. Therefore, this process must eventually terminate at a choice of B for which all crossing graphs have the stated number of vertices.

An edge e in C_S (representing a crossing between two edges f and f' of $K_{3,n}$) will belong to one of the C_{S_i} unless B contains a crossing point on f or on f'. B may contain at most $w + 1$ crossings of D, and each may eliminate at most $2(n-2)$ edges of C_S (if it is a crossing of two edges in E_S and each has $n-2$ other crossings). Therefore, the total number of edges in all of the crossing graphs C_{S_i} for all of the subtrees S_i is as stated.   □

Theorem 1. *Every planarization of $K_{3,n}$ has treewidth $\Omega(n)$.*

Proof. Let D be an arbitrary planarization of $K_{3,n}$, and let T be a minimum-width tree-decomposition of D. Let $\epsilon > 0$ be a constant to be determined later. We will show that T either has width at least ϵn, or it has a subtree S whose crossing graph C_S has density strictly greater than one. Since no graph (without repeated edges) can have density so high, the only possibility is that T has width at least $\epsilon n = \Omega(n)$.

To find S, for drawings whose width is at most ϵn, begin with $S = T$. Then, repeatedly use Lemma 3 to partition the current choice of subtree S into smaller subtrees, and then use Lemma 2 to find one of these subtrees that is dense. Each such step reduces the number of vertices in C_S by at least a factor of two, while also reducing the number of edges by approximately the same reduction factor (approximately because of the $O(\epsilon n^2)$ edges of the crossing graph that are eliminated by the application of Lemma 3). Therefore, each step increases the density of C_S by a factor of $2 - O(\epsilon)$. When $S = T$, the density is at least $1/18 - o(1)$, so after at most five steps the density is $(32 - O(\epsilon))(1/18 - o(1))$.

To complete the argument, we need only choose ϵ to be small enough so that this expression, $(32 - O(\epsilon))(1/18 - o(1))$, has a value exceeding one.   □

Corollary 1. *For every planarization of $K_{3,n}$, and every parameter in {pathwidth, cutwidth, bandwidth, branchwidth, carving width, tree-depth, clique-width}, the value of the parameter on the planarization is $\Omega(n)$. Therefore, there exists a family of graphs for which each of these parameters is bounded but for each each planarization has linear parameter value.*

Proof. All of these parameters except clique-width are bounded from below by a linear function of the treewidth.

As with any complete bipartite graph, the clique-width of $K_{3,n-3}$ is two: it can be constructed from a disjoint union of single vertices of two colors, by adding edges between all bichromatic pairs of vertices (Fig. 4). The $\Omega(n)$ lower bound on clique-width follows from the facts that (as a planar graph) any planarization has no $K_{3,3}$ subgraph and that, for graphs with no $K_{t,t}$ subgraph, the treewidth is upper-bounded by a constant factor (depending on t) times the clique-width [18]. Equivalently, the clique-width of any planarization is lower-bounded by a constant times its treewidth, which by Theorem 1 is $\Omega(n)$.   □
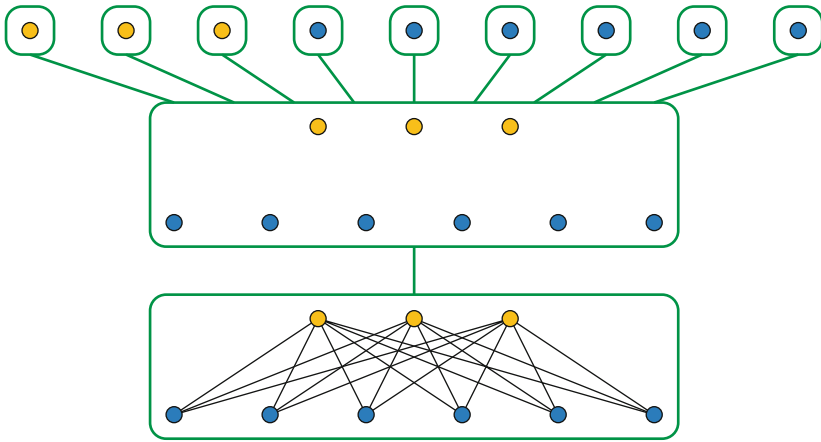
Fig. 4. Clique-width 2 construction of $K_{3,6}$ by a disjoint union of colored single vertices, followed by an operation that adds an edge between each bichromatic pair of vertices.

3   Cutwidth and Bounded-Degree Pathwidth

Cutwidth behaves particularly well under planarization:[2]

Theorem 2. *Let G be a graph with n vertices and m edges, of cutwidth w. Then G has a planarization with $O(n + wm)$ vertices, of cutwidth at most w.*

Proof. Consider a linear arrangement of G with edge separation number w, and use the positions in this arrangement as x-coordinates for the vertices. Assign the vertices y-coordinates that place them into convex and general position, draw the edges of G as straight line segments between the resulting points, and planarize the drawing by replacing each crossing by a vertex. Here, by "general position" we mean that no two points have the same x-coordinate, no five points form a pentagon in which two crossing points and a vertex have the same x-coordinate, no six points form a hexagon with three coincident diagonals, and no eight points form an octagon in which the crossing points of two pairs of diagonals have the same x-coordinate. This will all be true of a rotation by a sufficiently small but nonzero angle of any convex placement. In the resulting drawing, there can be no intersections of vertices or edges other than incidences and simple crossings, and no two vertices or crossing points can have the same x-coordinate. An example is shown in Fig. 5.

We use the ordering by x-coordinates of the planarization as a linear arrangement of the planarization. The edge intersection number is the maximum number of edges in the drawing that can be cut by any vertical line, unchanged between G and its planarization.

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[2] After the appearance of the preprint version of this paper [19], we learned that this result has been obtained independently by van Geffen et al. [20].
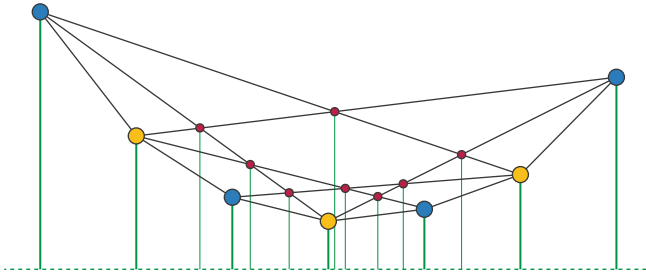
Fig. 5. Planarizing a graph of low cutwidth (here $K_{3,4}$, drawn with edge separation number six) by lifting its linear arrangement to a convex curve.

Because of the convex position of the vertices of G, each edge (u, v) of G can only be crossed by other edges that cross exactly one of the two vertical lines through u and v; there are $O(w)$ such edges, so the number of crossings per edge is $O(w)$ and the total number of crossings is $O(wm)$.   □

The lower bound of Corollary 1 does not contradict Theorem 2 because $K_{3,n}$ does not have bounded cutwidth. Its cutwidth is at least $3\lceil n/2 \rceil$, obtained in any linear arrangement at the cut between the first $\lceil n/2 \rceil$ vertices on the n-vertex side of the bipartition (together with any vertices from the other side that are mixed among them) and the remaining vertices of the graph. For instance, the drawing of $K_{3,4}$ in Fig. 5 achieves the optimal cutwidth of six for this graph. An example showing that the planarization size bound is tight is given by a disjoint union of $O(n/w)$ bounded-degree expander graphs, each having $O(w)$ vertices and crossing number $\Theta(w^2)$.

Corollary 2. *Let G be a graph with bounded pathwidth and bounded maximum degree. Then G has a planarization with linear size and bounded pathwidth.*

Proof. If a graph has pathwidth w and maximum degree d, it has cutwidth at most dw [12], and so does its planarization (Theorem 2). Because the planarization has cutwidth at most dw, it also has pathwidth at most dw, because the vertex separation number of any linear arrangement is at most equal to the edge separation number (with equality when the separation number is achieved by a matching).   □

4   Bandwidth

The same construction used for planarizing graphs with low cutwidth also works for graphs of low bandwidth.

Theorem 3. *Let G be a graph with n vertices and m edges, of bandwidth w. Then G has a planarization with $O(n+w^2m)$ vertices, whose bandwidth is $O(w^4)$.*

Proof. We lift a linear arrangement of G with low span to a convex curve in the plane, as in the proof of Theorem 2. Within the span of any edge e of G, there are $O(w^2)$ other edges and $O(w^4)$ crossings of those edges, so the span of e in the planarization is $O(w^4)$. This bound applies also to the span of any segment of e created by crossings with other vertices. Each edge may be crossed by $O(w^2)$ other edges, so the total number of dummy vertices added is $O(w^2 m)$.   □

It may be possible to reduce the bandwidth of the planarization by introducing artificial crossings to break up edges with long spans, but we have not pursued this approach as we do not believe it will lead to better graph drawings.

5   Carving Width and Bounded-Degree Treewidth

If a graph has low carving width, we can use its carving decomposition (a tree with the vertices at its leaves, internal degree three, and with few edges spanning the cut determined by each tree edge) to guide a drawing of the algorithm that leads to a planarization with low carving width.

It is helpful, for our construction, to relate carving width to cutwidth.

Lemma 4. *If a graph G has cutwidth w and maximum degree d, then G has carving width at most $\max(w, d)$.*

Proof. We form a carving decomposition of G in the form of a caterpillar: a path with each path vertex having a single leaf connected to it (except for the ends of the path which have two connected leaves). The ordering of the leaves is given by a linear arrangement minimizing the edge separation number. Then the cuts of the carving decomposition that are determined by edges of the path are exactly the ones determining the edge separation number, w. The remaining cuts, determined by leaf edges of the tree, are crossed by the neighboring edges of each vertex, of which there are at most d. An example of this construction can be seen in Fig. 5: the dashed horizontal green line represents the path from which the carving decomposition is formed, the heavy vertical green lines correspond to the leaf edges of the carving decomposition of $K_{3,4}$, and the thin vertical green edges correspond to the leaf edges of the carving decomposition of a planarization of $K_{3,4}$.   □

Theorem 4. *If an n-vertex graph G has carving width w, then G has a planarization with $O(w^2 n)$ additional vertices that still has carving width at most w.*

Proof. Let T be the tree of a carving decomposition of G with width w. Draw T without crossings in the plane, with straight-line edges, and thicken the vertices of T to disks and the edges of T to rectangles without introducing any additional self-intersections of the drawing. Place each vertex of G in the disk of the corresponding leaf vertex of T. Route each edge of G as a curve through the rectangles and disks connecting its endpoints, so that within each rectangle it forms a monotone curve (with respect to the orientation of the rectangle) crossing at
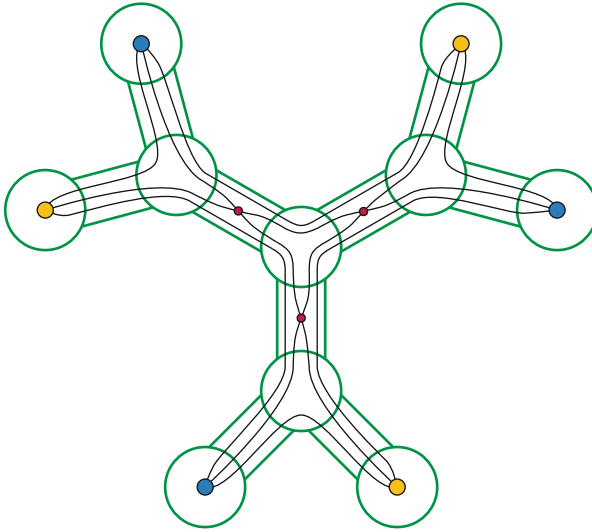
Fig. 6. Using a carving decomposition of $K_{3,3}$ to guide a planarization.

most once each other edge routed within the same rectangle, and so that, at each end of each rectangle, the curves are sorted by the ordering of their destination leaves in the planar embedding of T. With this sorted ordering, there need not be any crossings within the disks representing internal vertices of T, nor in the rectangles representing leaf edges of T (Fig. 6). The $n - 3$ remaining edges of T each contain at most $\binom{w}{2}$ crossings. So the total number of crossings is at most $(n - 3)\binom{w}{2} = O(w^2 n)$.

This drawing cannot yet be recognized as a carving decomposition of a planarization of G, because some of its vertices (the dummy vertices introduced at crossings) are now placed along the edges of T rather than at leaves. However, by topologically sweeping the arrangement of monotone curves [21] within each rectangle corresponding to an edge of T, we can arrange the crossing points within that rectangle into a linear sequence, such that the portion of the drawing within that rectangle has edge separation number at most w for that sequence. Applying Lemma 4 (replacing the edge of T by a carving decomposition in the form of a caterpillar, with a leaf of the decomposition for each vertex added in the planarization to replace a crossing of G, and with the ordering of these leaves given by a topological sweep of the arrangement) produces a carving decomposition of the planarization with width w, as required. □

We note that this planarization technique resembles the "simple planarization" method of Di Battista et al. [2] for clustered graphs. In this respect, we may view the carving decomposition of G as a clustering to be respected by the planarization.

In an appendix to the preprint version of this paper [19], we prove the following strengthening of Theorem 4:

Theorem 5. *If an n-vertex graph G has carving width w, then G has a planarization with $O(w^{3/2}n)$ additional vertices that still has carving width $O(w)$.*

An example showing that Theorem 5 is tight is given by a cluster graph consisting of $O(n/\sqrt{w})$ disjoint cliques of size $O(\sqrt{w})$, each requiring $\Theta(w^2)$ crossings in any drawing.

Corollary 3. *Let G be a graph with bounded treewidth or branchwidth and bounded maximum degree. Then G has a planarization with linear size and bounded treewidth and branchwidth.*

Proof. Treewidth and branchwidth are always within a constant factor of each other [13] so we may concentrate on the results for branchwidth, and the corresponding results for treewidth will follow automatically.

A carving decomposition may be converted into a branch decomposition by replacing each leaf of the carving decomposition (representing a vertex of the given graph) with a subtree (representing edges adjacent to the given vertex), in such a way that each edge is represented at exactly one of its endpoints. This increases the width of the decomposition by at most a factor equal to the degree. In the other direction, a branch decomposition may be converted into a carving decomposition by replacing each leaf of the branch decomposition (representing an edge of the given graph) by a subtree of zero, one, or two leaves (representing endpoints of the edge) in such a way that each vertex is represented at exactly one of its incident edges. This increases the width of the decomposition by at most a factor of two. So, the carving width is at most the degree times the branchwidth, and at least half the branchwidth [22].

Therefore, if G has bounded branchwidth and bounded maximum degree, it has bounded carving width, and Theorem 4 tells us that it has a planarization of linear size that also has bounded carving width. The same planarization also must have bounded branchwidth. □

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