On Vertex- and Empty-Ply Proximity Drawings

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Abstract. We initiate the study of the *vertex-ply* of straight-line drawings, as a relaxation of the recently introduced *ply* number. Consider the disks centered at each vertex with radius equal to half the length of the longest edge incident to the vertex. The vertex-ply of a drawing is determined by the vertex covered by the maximum number of disks. The main motivation for considering this relaxation is to relate the concept of ply to proximity drawings. In fact, if we interpret the set of disks as proximity regions, a drawing with vertex-ply number 1 can be seen as a weak proximity drawing, which we call *empty-ply* drawing. We show non-trivial relationships between the ply number and the vertex-ply number. Then, we focus on empty-ply drawings, proving some properties and studying what classes of graphs admit such drawings. Finally, we prove a lower bound on the ply and the vertex-ply of planar drawings.

1 Introduction

Constructing graph layouts that are readable and easily convey the information hidden in the represented data is one of the main goals of graph drawing research. Several aesthetic criteria have been defined to capture the user requirement for a better understanding of the data, e.g., resolution rules [13,18], low-density [14], proximity drawings [17]. The *ply number* [10] of a graph is another such criterion. We adopt the following notation: given a straight-line drawing Γ of a graph $G = (V, E)$, for each vertex $v \in V$ consider an open disk D_v (called the *ply-disk* of v) centered at v with radius r_v equal to half of the length of the longest edge incident to v. Over all points p on the plane, let k be the maximum number of ply-disks of Γ that include the point p in their interior. Then, the drawing Γ has *ply* k. The *ply number of* G is the minimum ply over all its drawings.

The ply number was originally proposed by Eppstein and Goodrich [12] in the context of interpreting road networks as subgraphs of disk-intersection graphs.

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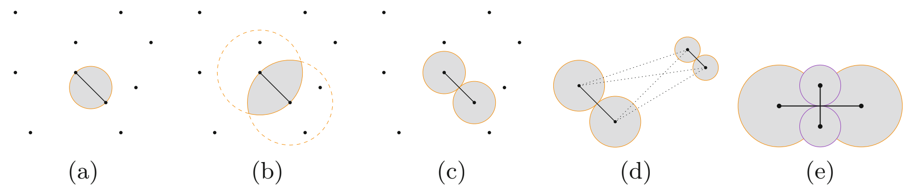


Fig. 1. (a) Gabriel, (b) Relative-neighborhood, and (c) Ply proximity regions. (d) A disconnected empty-ply graph. (e) A non-planar empty-ply drawing.

The concept of a ply number is also related to proximity drawings of graphs [17]. A *proximity drawing* of a graph G is a straight-line drawing of G in which for every two vertices u and v, there exists a region of the plane, called *proximity region* of u and v, that contains other vertices in its interior if and only if u and v are not connected by an edge in G. If G admits a proximity drawing, then it is a *proximity graph*. A proximity region specifies a set of points in the plane that are closer to u and v than to the other vertices, and different proximity regions lead to different definitions of proximity drawings. Regions can be *global*, e.g., Euclidean minimum spanning trees [19], or *local*, e.g., Gabriel graphs [15] (Fig. 1a), relative-neighborhood graphs [20] (Fig. 1b), and Delaunay triangulations [8,19]. Proximity drawings of graphs are also studied in the *weak* model [9], where the “if” part of the condition is neglected: i.e., if two vertices are not connected by an edge, then their proximity region may be empty.

In this work, we are interested in deepening the study of the relationship between the notions of ply number and of proximity drawings. In this direction, one can consider the local proximity region associated with a pair of vertices u and v as the one composed of the disks centered at u and at v, with radius equal to half of the length of the straight-line segment between u and v (Fig. 1c). Due to the possible absence of edges, this is a weak proximity model. However, a drawing Γ may have ply larger than 1 even if no proximity region contains a vertex different from the two which defined it, since the ply of Γ is only determined by the way in which different regions intersect each other.

To improve this relationship, we relax the definition of ply number and introduce the concept of *vertex-ply number*. Consider a straight-line drawing Γ of a graph G. Over all vertex-points p on the plane (i.e., points which realize a vertex of G), let k be the maximum number of ply-disks of Γ that include the point p in their interior. Then, the drawing Γ has *vertex-ply* k. The *vertex-ply number of* G is the minimum vertex-ply over all its drawings. In the special case in which Γ has vertex-ply 1, i.e., every disk D_v contains only v in its interior, we say that Γ is an *empty-ply* drawing. Note that an empty-ply drawing is in fact a weak proximity drawing with respect to the proximity region defined above, that is, a drawing is empty-ply if and only if all the proximity regions are empty.

Some relationships between proximity models are known, e.g., any Delaunay triangulation contains a Gabriel graph as a spanning subgraph, which in turn contains a relative-neighborhood graph, which in turn contains a minimum spanning tree [17]. It is hence natural to ask about the role of empty-ply drawings in these relationships. We first note that an empty-ply drawing may be non-planar (see Fig. 1e), which is not the case for Delaunay triangulations, and thus for any of the other type of proximity drawings. On the other hand, there exist empty-ply drawings that are not connected and that cannot be made connected by just adding edges while maintaining the empty-ply property (see Fig. 1d), which differs from the case for minimum spanning trees (and thus for all the other proximity drawings). These two observations imply that empty-ply drawings are not directly comparable with other types of disk-based proximity drawings.

The concept of empty-ply is related to *partial edge drawings* (PEDs) [4–6]. A PED is a straight-line drawing of a graph in which each edge is divided into three segments: a *middle part* that is not drawn and the two segments incident to the vertices, called *stubs*, that remain in the drawing and that are not allowed to cross. Our Theorem 2 in Sect. 3 shows that an empty-ply drawing also yields a PED whose stubs have nontrivial lengths.

Drawing graphs with low ply was first considered by Di Giacomo *et al.* [10]. They show that testing whether an internally triangulated biconnected planar graph has ply number 1 can be done in $O(n \log n)$ time and that the class of graphs with ply number 1 coincides with unit-disk contact graphs [3], which makes the recognition problem NP-hard. Angelini *et al.* [1] studied area requirements of drawings of trees with low ply. De Luca *et al.* [7] performed an experimental study demonstrating correlations between the ply of a drawing and aesthetic metrics such as stress and uniform edge-lengths. An interactive tool has been implemented by Heinsohn and Kaufmann [16].

We first demonstrate non-trivial relationships between the ply number and the vertex-ply number of graphs. In Sect. 2 we positively answer a question from [10] (Problem 4) regarding whether the ply number of an empty-ply drawing is constant. In Sect. 3 we study properties of empty-ply graphs. In Sect. 4 we provide several classes of graphs that admit empty-ply drawings and some classes that do not (we consider k-ary trees, complete (bipartite) graphs, and squares of graphs with ply number 1). Further, in Sect. 5 we answer another question posed in [10] (Problem 3), regarding the relationship between (vertex-) ply and crossings, by presenting graphs that admit drawings with constant ply and only 3 crossings but any corresponding planar drawing requires linear ply. We conclude in Sect. 6 with several open problems. For space reasons, some proofs have been sketched or omitted. Complete proofs can be found in the full version of the paper [2].

2 Relationships Between Ply and Vertex-Ply

We start with a natural question about the relationship between the ply number and the vertex-ply number of a graph.

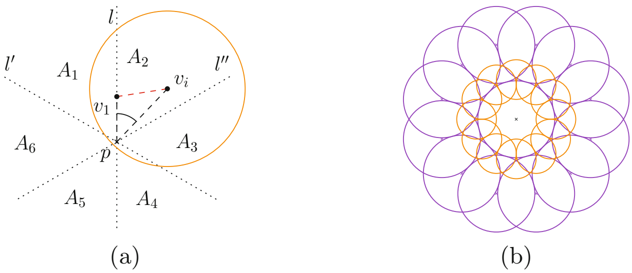


Fig. 2. (a) Illustration for the proof of Theorem 1. (b) An empty-ply drawing of a star of degree 24. For readability, edges are not drawn.

Theorem 1. *The ply of a drawing of a graph with vertex-ply h is at most $5h$.*

Proof. Let Γ be any drawing of a graph G with vertex-ply h. Let p be any point in the plane and let $v_1, \ldots, v_k$ be the vertices whose ply-disks contain p in their interior, appearing in this radial order around p; see Fig. 2a. Without loss of generality, assume that v_1 is the vertex closest to p. Let l be the line through p and v_1, and let l' and l'' be two lines through p creating angles $\frac{\pi}{3}$ and $-\frac{\pi}{3}$ with l. These lines determine a covering of the plane by six closed wedges $A_1, \ldots, A_6$ centered at p, each having $\frac{\pi}{3}$ as its internal angle.

Let A_1 and A_2 be the wedges delimited by the half-line starting at p and passing through v_1. For each vertex $v_i \in A_1 \cup A_2$ we have $\angle v_1 p v_i \leq \frac{\pi}{3}$. This implies that $|v_1 v_i| \leq |v_i p|$ and hence that v_1 belongs to the ply-disk D_{v_i}, since p belongs to D_{v_i}. Thus, if the union of the closed wedges A_1 and A_2 contains at least h vertices among $v_2, \ldots, v_k$, we obtain that v_1 belongs to at least $h+1$ ply-disks. This is not possible, since Γ has vertex-ply h.

We now prove that each wedge A_i with $3 \leq i \leq 6$ contains at most h vertices among $v_2, \ldots, v_k$. Namely if it contains at least $h+1$ vertices we can argue as above that the closest vertex to p among them belongs to the ply-disks of all the other h vertices. This completes the proof of the theorem that there exist at most $5h$ vertices whose ply-disks enclose p. □

Corollary 1. *The ply of an empty-ply drawing of a graph is at most* 5.

Note that the converse of Corollary 1 does not hold. If a graph G does not admit any empty-ply drawing, that does not imply that the ply number of G is larger than 5. A star graph with degree larger than 24 does not have an empty-ply drawing (see Theorem 3), but can be drawn with constant ply 2 [10].

3 Properties of Graphs with Empty-Ply Drawings

Let Γ be a straight-line drawing of a graph G. Let $\{D'_v, v \in V\}$ be the set of open disks where D'_v is centered at v, but with radius only $\frac{r_v}{2}$. We can think of these disks as obtained by shrinking the original ply-disks of Γ to half-length radius. Note that if Γ is an empty-ply drawing, then the disks in $\{D'_v, v \in V\}$ are pairwise disjoint. This observation implies the next result.

Lemma 1. *In an empty-ply drawing Γ of a graph $G = (V, E)$ the sum of the areas of all ply-disks $\{D_v, v \in V\}$ does not exceed 4 times the area of their union.*

Proof. Each disk D'_v has area four times smaller than D_v, but is drawn inside the union of all ply-disks.

In the rest of the paper we frequently use disk-packing arguments based on Lemma 1. Another consequence of the observation above is a relationship between empty-ply drawings and the most popular type of PED, called $\frac{1}{4}$-SHPED [5], in which the length of both stubs of an edge e is $\frac{1}{4}$ of e's length.

Theorem 2. *An empty-ply graph admits a $\frac{1}{4}$-SHPED.*

Proof. Let Γ be an empty-ply drawing of a graph $G = (V, E)$ with the set of disks $\{D'_v, v \in V\}$. Let Γ' be the drawing obtaining from Γ by keeping for each edge (u, v) only the two parts in the interior of disks D'_u and D'_v. By definition, both these parts cover at least $\frac{1}{4}$ of (u, v). Since no two such disks overlap, there is no crossing in Γ', and the statement follows. □

We now focus on the relationship between the radii of the ply-disks of adjacent vertices in an empty-ply drawing. For the following two lemmas we use that for each vertex v, and for each edge (v, w) incident to v, we have $r_v \leq |vw|$, as the drawing is empty-ply, and $r_v \geq \frac{|vw|}{2}$, by the definition of the ply-disk D_v.

Lemma 2. *In an empty-ply drawing, for any two edges (u, v) and (v, w) incident to the same vertex v, we have $\frac{1}{2} \leq \frac{|uv|}{|vw|} \leq 2$.*

Lemma 3. *In an empty-ply drawing, the radii of the ply-disks of two adjacent vertices u and v differ by at most a factor of 2, i.e., $\frac{1}{2} \leq \frac{r_u}{r_v} \leq 2$.*

We conclude the section by presenting a tight bound on the maximum degree of graphs that admit empty-ply drawings.

Theorem 3. *No vertex of an empty-ply graph has degree greater than 24.*

Proof. To obtain a contradiction, let Γ be an empty-ply drawing of a graph G with a vertex v of degree greater than 24. By Lemma 2, the lengths of all edges of v are in the interval $[m, 2m]$, where m is the length of the shortest edge. Note that there are at least 13 edge lengths either in the interval $[m, \sqrt{2}m]$ or in the interval $[\sqrt{2}m, 2m]$. In either case, there exist two neighbors u and w of v such that $|vu| \leq |vw| \leq \sqrt{2}|vu|$ and $\alpha = \angle uvw \leq \frac{2\pi}{13}$. Scaling Γ by a factor of $|vu|^{-1}$, we may assume w.l.o.g. that $|vu| = 1$ and that $|vw| = q \in [1, \sqrt{2}]$. By the law of cosines, $|uw|^2 = 1 + q^2 - 2q \cos \alpha$. As Γ is an empty-ply drawing, the vertex v does not belong to the open disk centered at w. Hence $|uw| \geq \frac{q}{2}$.

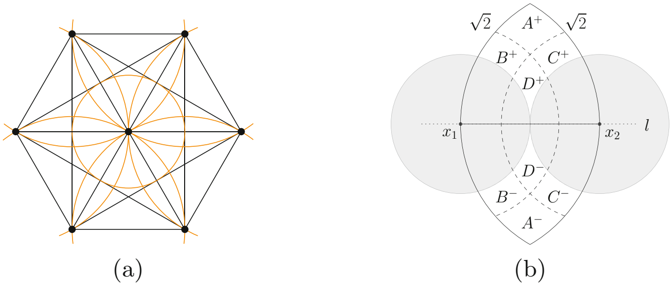


Fig. 3. (a) Empty-ply drawing K_7; note that there are edges drawn on top of each other. (b) Partition of the region where the vertices of K_8 can be placed.

From the above reasoning it follows that q should satisfy the quadratic inequality $\frac{q^2}{4} \le 1+q^2-2q\cos\alpha$, which yields that either $q \le \frac{4\cos\alpha-\sqrt{16\cos^2\alpha-12}}{3}$ or $q \ge \frac{4\cos\alpha+\sqrt{16\cos^2\alpha-12}}{3}$. This contradicts the fact that $q \in [1,\sqrt{2}]$, because: $4\cos\frac{2\pi}{13}-\sqrt{16\cos^2\frac{2\pi}{13}-12} \doteq 2.8 < 3$ and $4\cos\frac{2\pi}{13}+\sqrt{16\cos^2\frac{2\pi}{13}-12} \doteq 4.27 > 4.24 \doteq 3\sqrt{2}$. This concludes the proof of the theorem. □

Note that $K_{1,24}$ admits an empty-ply drawing with only two different lengths of edges (see Fig. 2b) and so the degree bound provided in Theorem 3 is tight.

4 Graph Classes with and Without Empty-Ply Drawings

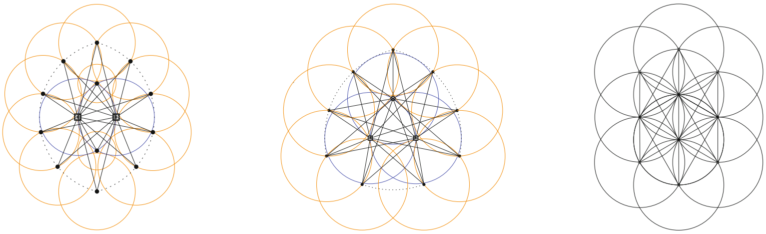
4.1 Complete Graphs

Theorem 4. *Graph K_n admits an empty-ply drawing if and only if $n \le 7$.*

Proof (sketch). For a contradiction, suppose that K_8 has an empty-ply drawing Γ. Let (x_1, x_2) be the longest edge of Γ, w.l.o.g. having length 2; assume that x_1 and x_2 lie on an horizontal line l. Since (x_1, x_2) is the longest edge, the remaining six vertices lie in the intersection of two disks centered at x_1 and x_2, respectively, with radius 2; also, by Lemma 2, they lie outside the two disks centered at x_1 and x_2 with radius 1; see Fig. 3b. This defines two closed regions in which these vertices lie: one above l and one below.

Using two circles centered in x_1 and x_2 with radius $\sqrt{2}$, we partition each of these two regions into four closed subregions, called A^+, B^+, C^+, D^+ and A^-, B^-, C^-, D^-, where the apex $^+$ or $^-$ indicates the region above or below l, respectively. Namely, any point in the interior of $A^+ \cup A^-$ (of $D^+ \cup D^-$) has distance larger (smaller) than $\sqrt{2}$ from both x_1 and x_2; while any point in the interior of $B^+ \cup B^-$ (of $C^+ \cup C^-$) has distance smaller (larger) than $\sqrt{2}$ from x_1 and distance larger (smaller) than $\sqrt{2}$ from x_2.

We show that any placement of the six remaining vertices in these regions leads to a contradiction. We denote by $|X^y|$, with $X \in \{A, B, C, D\}$ and $y \in \{+, -\}$,



(a) (b) (c)

Fig. 4. Empty-ply drawing of (a) $K_{2,12}$, (b) $K_{3,9}$, and (c) $K_{4,6}$. Note that the drawing of $K_{4,6}$ has edges drawn on top of each other.

the number of vertices in X^y. First note that each region can contain at most one vertex, except for D^+ and D^-, which may contain two vertices. In fact, if we place any vertex w in a region X^y, with $X \in \{A, B, C\}$ and $y \in \{+, -\}$, then the ply-disk D_w of w (defined, at least by the distance to x_1, x_2) covers the entire region X^y. Regions D^+ and D^-, on the other hand, have area with height 1 and width 0.5. Let $w \in D^+$ be the point at distance $\sqrt{2}$ from both x_1 and x_2 and D_w be its ply-disk. Then, set $D^+ \setminus D_w$ defines an area with diameter at most $\frac{1}{3}$ and it is not sufficient to place more than one vertex, since the ply disks would have at least a radius 0.5.

Combining the placement of the vertices in different regions, we can use similar arguments to prove that $|D^+ \cup D^-| \leq 3$ and $|A^+ \cup A^-| \leq 1$. Also, if $|A^+| = 1$ (resp. $|A^-| = 1$), then $|D^-| \leq 1$ (resp. $|D^+| \leq 1$). Thus, if $|A^+ \cup A^-| = 1$ then $|D^+ \cup D^-| \leq 2$. Also, if $|A^+| = 1$ (resp. $|A^-| = 1$) and $|B^-| = 1$ (resp. $|B^+| = 1$) then either $|B^+| = 0$ or $|C^+| = 0$ (resp. $|B^-| = 0$ or $|C^-| = 0$), i.e., $|B^+ \cup C^+| \leq 1$ (resp. $|B^- \cup C^-| \leq 1$). By symmetry, if $|A^+| = 1$ (resp. $|A^-| = 1$) and $|C^-| = 1$ (resp. $|C^+| = 1$) then either $|B^+| = 0$ or $|C^+| = 0$ (resp. $|B^-| = 0$ or $|C^-| = 0$), i.e., $|B^+ \cup C^+| \leq 1$ (resp. $|B^- \cup C^-| \leq 1$). Hence, if $|A^+ \cup A^-| = 1$, the other regions cannot contain 5 vertices.

The final case where $|A^+ \cup A^-| = 0$ directly implies the claim for K_9. To prove this for K_8 we can see that if $|B^-| = 1$ and $|C^+| = 1$ (resp. $|B^+| = 1$ and $|C^-| = 1$), then $|D^+ \cup D^-| \leq 1$, which again leads to a contradiction. To conclude the proof, we present an empty-ply drawing for K_7 in Fig. 3a. We strongly believe that this drawing is unique.

4.2 Complete Bipartite Graphs

We now consider complete bipartite graphs. For proof-by-picture of the next theorem see Fig. 2b and Figs. 4a–c.

Theorem 5. *Graphs $K_{1,24}$, $K_{2,12}$, $K_{3,9}$, and $K_{4,6}$ admit empty-ply drawings.*

Note that Theorem 3 implies that $K_{1,25}$ does not admit any empty-ply drawing, and hence this is true for any complete bipartite graph $K_{n,m}$ with n or m

greater than 24. This leaves a wide open gap between the upper bounds on the values of n and m, and the lower bounds from Theorem 5.

For $K_{2,m}$, we give a negative result for $m \geq 15$ in the following theorem based on arguments similar to those in Theorem 4.

Theorem 6. *Graph $K_{2,m}$ with $m \geq 15$ does not admit any empty-ply drawing.*

4.3 Trees of Bounded Degree

A *d-ary tree T with k levels* is a rooted tree where all vertices at distance less than k from the root have at most d children and the remaining ones are leaves. If all the non-leaf vertices have exactly d children, we say that T is *complete*. Any tree with maximum degree Δ is a subtree of a $(\Delta - 1)$-ary tree.

Note that binary trees admit empty-ply drawings, as the drawings with ply 2 constructed by the algorithm in [10] are empty-ply drawings. Applying Corollary 1 to the class of complete 10-ary trees (which do not admit drawing with constant ply [1]) shows that they do not admit empty-ply drawing. But we can prove something stronger.

Theorem 7. *For sufficiently large k, the complete 4-ary tree T_k with k levels admits no empty-ply drawing.*

Proof. Assume without loss of generality that k is even and that T_k has an empty-ply drawing Γ where the ply-disk of the root v_0 has unit radius. We announce that for simplicity the following estimates are not stated in the tightest form. We will make use of the following consequences of Lemmas 2 and 3:

Claim (A). If a ply-disk of a vertex u in Γ has radius at least 2^i, then all the leaves of the subtree rooted at u have radii at least 2^{2i-k}.

Proof. Since $r_u \geq 2^i$, the distance between u and the root is greater than i by Lemma 3. Thus the path from u to its leaves has length at most $k - i$. □

Claim (B). If v is a leaf whose ply-disk has radius $r_v \in (2^{2i-k}, 2^{2i-k+2}]$, with $i \in \{0, k-1\}$, then its Euclidean distance from the root is $|v_0 v| \leq 2^{i+2}$.

Proof. Let $v_0, v_1, \ldots, v_k = v$ be the path from the root v_0 to v_k in T_k. Since $r_{v_0} = 1$, edge (v_0, v_1) has length at most 2. Also, by Lemma 2, the lengths of the edges can grow at most by a factor 2 along the path; hence, $|v_{j-1} v_j| \leq 2^j$ for $j \in \{1, \ldots, i\}$. If we traverse the path in the opposite direction from v_k, whose ply-disk has radius at most 2^{2i-k+2}, we get analogously that $|v_{k-j+1} v_{k-j}| \leq 2^{j+2i-k+2}$ for $j \in \{1, \ldots, k-i\}$.

The total distance is thus bounded by $|v_0 v_k| \leq |v_0 v_1| + |v_1 v_2| + \cdots + |v_{k-1} v_k| = \sum_{j=1}^{i} |v_{j-1} v_j| + \sum_{j=1}^{k-i} |v_{k-j+1} v_{k-j}| \leq \sum_{j=1}^{i} 2^j + \sum_{j=1}^{k-i} 2^{j+2i-k+2} = 2^{i+1} - 2 + 2^{i+1} - 2^{3+2i-k} \leq 2^{i+2}$, and the statement follows. □

We now distribute the 4^k leaves to k sets $L_0, \ldots, L_{k-1}$ (all logarithms binary):

(a) if $i \geq 3\log k$ then $L_i = \{v : r_v \in (2^{2i-k}, 2^{2i-k+2}]\}$
(b) if $i < 3\log k$ then $L_i = \{v : r_v \leq 2^{6\log k - k}$ and whose largest predecessor u has radius $r_u \in [2^i, 2^{i+1})\}$

In the first case, the radii of the leaves in L_i are sufficient to obtain a good bound on enclosing area of the disks in L_i. In the other case, the radius on the enclosing disk for L_i mostly depends on the presence of predecessors that are larger than the root disk.

Some of the sets are empty by the definition, but it is irrelevant to our further deductions. By pigeonhole principle, either some L_i, $i \geq 3\log k$ satisfies $|L_i| \geq \frac{4^k}{2k}$ or some L_i, $i < 3\log k$ satisfies $|L_i| \geq \frac{4^k}{6\log k}$, since $\frac{k+1-3\log k}{2k} + \frac{3\log k}{6\log k} \leq 1$ when $k \geq \sqrt[3]{2}$.

The rough idea behind the distinction of these two cases is that in case a), when the diameters of leaves are sufficiently large, it suffices to consider twice smaller proportion than the uniform pigeonhole principle would use and show that the total area of ply-disks corresponding to leaves of L_i is still too large for an empty-ply drawing Γ. In case b) we use a slightly more elaborate argument considering also the area of the predecessors of the vertices in L_i.

Case (a). Assume that for some $i \geq 3\log k$ it holds that $|L_i| \geq \frac{4^k}{2k}$. The total area occupied by the disks in L_i is at least $\frac{4^k}{2k}\pi 4^{2i-k} = \frac{8^i\pi}{2k}$. By Claim (B), for every $v \in L_i$ it holds that $|v_0 v| \leq 2^{i+1}$, hence all ply-disks of L_i must be contained in a disk centered at the root of radius $2^{i+1} + 2^{2i-k+2} \leq 5 \cdot 2^i$, since for $i \in \{0, \ldots, k\} : i > 2i - k$. In particular this disk has area at most $25\pi 4^i$.

In order to apply Lemma 1, it suffices to choose k large enough such that $\frac{8^i\pi}{2k} > 4 \cdot 25\pi 4^i$ for all $i \geq 3\log k$, i.e., $k > \sqrt[5]{200} \doteq 2.9$.

Case (b). Assume that for some $i < 3\log k$ it holds $|L_i| \geq \frac{4^k}{6\log k}$. Any $v \in L_i$ has radius smaller than $2^{6\log k - k}$, as otherwise we would be in case a). To obtain the maximum distance between v and the root v_0 we argue that the first $3\log k$ disks along the path from v_0 to v may have radius at most 2^{i+1}. Analogously as in the proof of Claim (B), the j-th predecessor of v has radius at most $2^{6\log k - k + j}$. An upper bound of $|v_0 v| \leq 2^{i+2}(3\log k + 1)$ is obtained by summing up.

We now consider the subtree T' of T_k induced by the vertices of L_i and all their predecessors. Note that the drawing of the entire tree T' shall be contained within a disk of radius $2^{i+2}(3\log k + 1) + 2^{3\log k - k}$, i.e., in area at most $4^{i+3}\pi$. On the other hand, by Claim (A), each of the leaves has radius at least 2^{2i-k}. Thus, their total area is at least $\frac{4^k}{6\log k} 4^{2i-k}\pi = \frac{8^i\pi}{6\log k}$.

The number of parents of disks in L_i is at least $\frac{4^{k-1}}{6\log k}$, each of radius at least 2^{i-1}, hence they occupy area also bounded from below by $\frac{8^i\pi}{6\log k}$. Thus, all leaves in L_i and all their $k-i$ predecessors occupy space at least $\frac{8^i\pi}{6\log k}(k-i) \geq \frac{8^i\pi k}{12\log k}$. Again, to apply Lemma 1, it suffices to choose k large enough such that

$\frac{8^i \pi k}{12 \log k} > 4 \cdot 64\pi 4^i$ for all non-negative $i < 3 \log k$ (in particular for $i = 0$). A straightforward calculation verifies that the inequality holds e.g., for $k \geq 2^{16}$.

For $k = 2^{16}$ one of the two cases applies, which concludes the proof. □

Theorem 7 leaves open the question for 3-ary trees. We remark that the algorithm for binary trees [10] adopts a common drawing style: the orthogonal one with a shrinking factor of 1/2; see also [11]. We prove that this technique fails for 3-ary trees, for any shrinking factor in $(0, 1)$.

Theorem 8. *Rooted ternary trees do not admit empty-ply drawings constructed in orthogonal fashion with shrink factor q for any $q \in (0, 1)$, i.e., when the distance from a vertex to its children is q times the distance to its parent.*

4.4 Graph Squares

The *square* of a graph G is the graph obtained from G by adding an edge between each vertex and the neighbors of its neighbors.

Theorem 9. *Let G^2 be the square of a graph G. If G admits a drawing with ply 1, then G^2 admits an empty-ply drawing. Also, if G is a subgraph of a triangular tiling, then G^2 admits an empty-ply drawing with ply at most 4.*

Proof. Let Γ be a straight-line drawing of G with ply 1. As proved in [10], all the edges of G have the same length, say 1, in Γ, and every two non-adjacent vertices are at distance at least 1 from each other. Hence, adding the edges of $G^2 \setminus G$ to Γ produces a drawing Γ^2 of G^2 in which each edge has length at most 2. This implies that every ply-disk has radius at most 1 in Γ^2, and thus Γ^2 is an empty-ply drawing. Note that Γ^2 may contain edge overlaps.

For the second part of the statement, recall that if G is a subgraph of a triangular tiling, then it admits a drawing Γ in which all edges have the same length and all the angles are multiples of $\frac{\pi}{3}$. Hence, Γ has ply 1. Also the drawing Γ^2 obtained by adding the additional edges of $G^2 \setminus G$ to Γ is an empty-ply drawing. In this case, however, we can also prove that the ply of Γ^2 is at most 4; recall that an upper bound of 5 to the ply of Γ^2 is already implied by Corollary 1.

W.l.o.g. let the triangular tiling be of unit edge length. Consider the open disk of unit radius, which is centered at an arbitrary point p on the plane. If p is not a vertex of the triangular tiling, at most four vertices of the triangular tiling may fall in this disk. In the case where p is a vertex of the triangular tiling, no other vertex of the tiling falls in the disk, but only on its boundary. Thus, any point p can be internal to at most four ply-disks of the tiling vertices. □

5 Ply and Vertex-Ply of Planar Drawings

In the original paper on the ply number it was observed that considering only plane graph drawings may prevent finding low ply non-plane drawings [10]. In particular, for the class of *nested-triangles* graphs the "most natural" planar

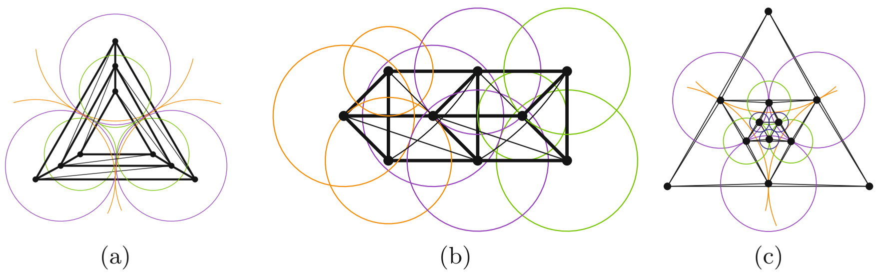


Fig. 5. Nested triangles graph: (a) "The most natural" drawing. (b) A non-planar drawing with ply 5. (c) A planar drawing with ply 4. The disks of three vertices at the same level do not properly overlap, and disks at levels i and $i+3$ do not overlap.

drawing has ply $\Omega(n)$ (see Fig. 5a), while there exist non-planar drawings (with edge overlaps) with ply 5 (see Fig. 5b). Note that however a "less natural" planar drawing with ply 4 can always be constructed; see Fig. 5c.

We strengthen this observation by providing a planar 3-tree G admitting a non-planar drawing (with only 3 crossings) with ply 5, such that *any* planar drawing of G has ply $\Omega(n)$; the same linear lower bound holds even for vertex-ply when the outer face is fixed. Recall that a *planar* 3-*tree* can be constructed, starting from a 3-cycle, by repeatedly adding a vertex inside a triangular face and connecting it to all three vertices of this face.

Our result also gives a negative answer to an open question posed in [10] on whether there exists a relationship between the number of crossings and the ply number of a drawing. Our example shows that one can reduce the ply number from $\Omega(n)$ to $O(1)$, by introducing only $O(1)$ crossings.

Theorem 10. *There exists an n-vertex planar* 3*-tree* G *such that any planar drawing of* G *with a fixed outer face has vertex-ply* $\Theta(n)$*, and hence ply* $\Theta(n)$*, while* G *admits a drawing with ply* 5 *and vertex-ply* 4 *with three edge crossings.*

Proof. Graph G has three vertices v_1, v_2, and v_3 on the outer face, and a vertex u that is connected to all of v_1, v_2, and v_3. Refer to Fig. 6a. In addition, it contains three paths $x_1, \dots, x_m$, $y_1, \dots, y_m$, and $z_1, \dots, z_m$, each on $m = \frac{n-4}{3}$ vertices. The edge set further contains edges (u, x_1), (u, y_1), (u, z_1) and also (x_i, v_1), (x_i, v_2), (y_i, v_2), (y_i, v_3), (z_i, v_1), (z_i, v_3) for each $i \in \{1, \dots, m\}$.

Consider any planar drawing Γ of G. Suppose, w.l.o.g., that (v_1, v_2) is of unit length and that it is the longest edge in Γ among the three edges incident to the outer face, that is, $|v_2v_3|, |v_1v_3| \leq 1$. Since vertex u lies inside the triangle $v_1v_2v_3$, we have $|uv_1|, |uv_2| < 1$. Hence, it is possible to cover the whole region of the plane delimited by triangle uv_1v_2 with a set of 28 disks, each having radius $\frac{1}{8}$, as illustrated in Fig. 6b. Thus, at least one disk D out of these 28 contains in its interior at least $\frac{m}{28} = \frac{n-4}{84}$ vertices out of $x_1, \dots, x_m$.

Consider any vertex $x_i \in D$. Since x_i is connected to both v_1 and v_2, the longest of its incident edges has length at least $\frac{1}{2}$, and hence the radius of the ply-disk of x_i is at least $\frac{1}{4}$. Hence the ply-disk of x_i entirely contains the disk D

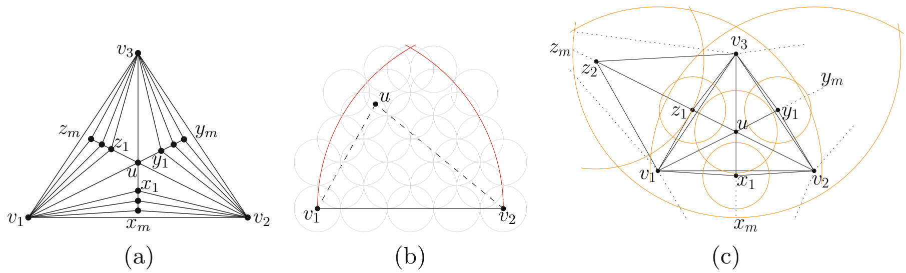


Fig. 6. (a) The planar 3-tree G in the proof of Theorem 10. (b) A set of 28 disks of radius $\frac{1}{8}$ covering the whole region delimited by triangle uv_1v_2 when $|v_1v_2| = 1 > |uv_1|, |uv_2|$. (c) A non-planar drawing of G with ply 5 and vertex-ply 4.

in its interior, and thus it contains all the vertices inside it. Since this is true for all the $\frac{n-4}{84}$ vertices inside D, the first part of the statement follows.

A non-planar drawing of G with ply 5 and vertex-ply 4 is depicted in Fig. 6c. Here vertices v_1, v_2 and v_3 form an equilateral triangle with barycenter u. Vertices $x_1, \ldots, x_m$ are arranged along the axis of the segment v_1v_2 at distances growing exponentially by a factor of 2, analogously for vertices $y_1, \ldots, y_m$ and $z_1, \ldots, z_m$. The disk D_u overlaps with D_{x_1}, D_{y_1}, and D_{z_1}, without enclosing these vertices. The drawing of the subset of vertices $\{u, x_1, y_1, z_1\}$ is empty-ply and of ply 2. After considering the remaining vertices, the disks of v_1, v_2, v_3 may contain all of them in their interior. Thus we obtain ply 5 and vertex-ply 4.

6 Conclusions and Future Work

We defined and studied the vertex-ply of a straight-line drawing, paying particular attention to the special case of empty-ply drawings, whose vertex-ply is 1. We conclude with several natural open problems.

1. We know that binary trees admit empty-ply drawings [10] and that 4-ary trees do not (Theorem 7). What about 3-ary trees? Note that Theorem 8 rules out a large class of possible drawings (orthogonal and shrinking).
2. Another way of generalizing binary trees is to maintain the degree restriction, leading to the question: do (planar) max-degree-3 graphs admit empty-ply drawings?
3. In Theorem 9 we proved that the square G^2 of a graph G with ply 1 admits an empty-ply drawing, which has ply at most 5 by Corollary 1. On the other hand, if G is a subgraph of a triangular tiling, then the empty-ply drawing of G^2 has ply at most 4. Does the square of every graph with ply 1 admit an (empty-ply) drawing with ply 4? Note that there are ply 1 graphs that are not subgraphs of a triangular tiling.
4. Looking at empty-ply drawings from the proximity perspective, it is natural to consider the generalization in which ply-disks do not need to be empty, but can contain at most k vertices. We call a drawing with this property

a *k-empty-ply* drawing, in compliance with the definition of k-Gabriel and k-relative-neighborhood drawings [17]. With the argument of Theorem 10 there exist n-vertex graphs whose any planar drawing is $\Omega(n)$-empty-ply.

5. In Theorem 4 we proved a tight bound of 7 on the size of complete graphs admitting empty-ply drawings. For complete bipartite graphs $K_{n,m}$, we have a tight bound of $m = 24$, for $n = 1$, and an almost tight bound of $12 \leq m \leq 14$, for $n = 2$, with larger gaps between the bounds for larger values of n.

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