Spherically Symmetric Deformations of Micropolar Elastic Medium with Distributed Dislocations and Disclinations

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Abstract We discuss the problem of eigenstresses caused by distributed dislocations and disclinations in a hollow solid sphere of linearly elastic isotropic micropolar material. For any spherically symmetric distribution of dislocations and disclinations the exact solution of the boundary value problem is obtained. The derived solution is expressed in primary functions. The spherically symmetric eigenstresses problem is also resolved in the framework of the classical theory of elasticity that is without couple stresses.

Keywords The dislocations and disclinations densities · Couple stress
Spherically symmetric tensor fields · Eigenstresses · Exact solution

1 Introduction

A common feature of the structure of solids is a micro nonhomogeneity. To account the materials micro nonhomogeneity in the framework of continuum mechanics the model of the micropolar body can be applied, i.e. the model of a medium with couple stresses and the rotational interaction of material particles. The model of the micropolar medium, also called the Cosserat continuum, is often used for the description of grain polycrystalline bodies, polymers, composites, suspensions, liquid crystals, geophysical structures, biological tissues, nanostructured materials, see e.g. [1–9] and the extended bibliography therein.

Another important element of the microstructure of solids are defects of the crystal lattice such as dislocations and disclinations. In many cases, the number

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F. dell'Isola et al. (eds.), *Advances in Mechanics of Microstructured Media and Structures*, Advanced Structured Materials 87,
https://doi.org/10.1007/978-3-319-73694-5_19

of dislocations and disclinations in a bounded volume of the body is very large. So instead of considering a discrete set of defects it more efficient to analyze the continuous distribution of defects and use the theory of distributed dislocations and disclinations. The continuum dislocations theory in non-polar elastic bodies, i.e. in the simple materials was described, for example, in [10–19]. The theory of continuously distributed dislocations and disclinations in micropolar media is described in [16, 20, 21]. Up to our knowledge, nowadays in the literature there are practically no solutions presented for static boundary value problems of the micropolar elastic bodies with distributed dislocations and disclinations. This is because of the complexity of the system of governing differential equations, which in general case consists of six equilibrium equations for stresses and couple stresses and eighteen incompatibility equations regarding the metric and bending deformations. In the paper within the framework of the linear isotropic micropolar elasticity theory we find the exact solution of the eigenstresses problem in hollow solid sphere with spherically symmetric distribution of dislocations and disclinations. The solution is given in elementary functions. The solution is also compared with solution of the same problem obtained within the framework of the classic linear theory of elasticity of non-polar materials.

2 Input Relations

The system of static equations of a linear elastic isotropic micropolar body consists of the equilibrium equations for the stresses [4–6]

$$\mathrm{div}\mathbf{T} + \mathbf{f} = 0, \quad \mathrm{div}\mathbf{M} + \mathbf{T}_{\times} + \mathbf{h} = 0 \tag{1}$$

constitutive relations

$$\begin{aligned} \mathbf{T} &= \lambda \mathbf{E}\,\mathrm{tr}\,\boldsymbol{\varepsilon} + (\mu + \tau)\boldsymbol{\varepsilon} + (\mu - \tau)\boldsymbol{\varepsilon}^T \\ \mathbf{M} &= \nu \mathbf{E}\,\mathrm{tr}\boldsymbol{\kappa} + (\gamma + \eta)\boldsymbol{\kappa} + (\gamma - \eta)\boldsymbol{\kappa}^T \end{aligned} \tag{2}$$

and the geometric relations

$$\boldsymbol{\varepsilon} = \mathrm{grad}\mathbf{u} + \mathbf{E} \times \boldsymbol{\theta}, \quad \boldsymbol{\kappa} = \mathrm{grad}\boldsymbol{\theta} \tag{3}$$

Here $\mathbf{T}$ is the stress tensor, $\mathbf{M}$ is the couple stress tensor, $\boldsymbol{\varepsilon}$ is the non-symmetric metric strain tensor, $\boldsymbol{\kappa}$ is the bending strain tensor called also the wryness tensor, see [22, 23], $\boldsymbol{\theta}$ is the microrotation vector field, $\mathbf{u}$ is the displacement field of the elastic medium, $\mathbf{E}$ is the unit tensor. λ, μ, τ, ν, γ, η are the elastic modules, $\mathbf{f}$ is the volume density of mass forces, $\mathbf{h}$ is the volume density of mass moments. The div and grad operators are defined as in [24, 25]. The symbol $\mathbf{T}_{\times}$ denotes the vector invariant of a second-order tensor:

$$\mathbf{T}_\times = (T_{sk}\mathbf{r}^s \otimes \mathbf{r}^k)_\times = T_{sk}\mathbf{r}^s \times \mathbf{r}^k$$

where $\mathbf{r}^s$, $s = 1, 2, 3$, is a vector basis, see e.g. [25].

To introduce the dislocations density in the micropolar medium let us consider the problem of determination of the displacement field $\mathbf{u}(\mathbf{r})$ for a given strain tensor field $\boldsymbol{\varepsilon}(\mathbf{r})$ and microrotation vector field $\boldsymbol{\theta}(\mathbf{r})$ defined in multiply-connected domain v. Here $\mathbf{r}$ is the radius-vector of point in the 3D space. The fields $\boldsymbol{\varepsilon}$ and $\boldsymbol{\theta}$ are assumed to be differentiable and single-valued. According to (3)

$$\mathrm{grad}\mathbf{u} = \boldsymbol{\varepsilon} - \mathbf{E} \times \boldsymbol{\theta}, \tag{4}$$

in the case of the multiply-connected domain vector field $\mathbf{u}(\mathbf{r})$ can not be uniquely determined, in general. This results in the appearance of translational dislocations [10–12] in the body, each of which is characterized by the Burgers vector

$$\mathbf{b}_N = \oint_{\gamma_N} d\mathbf{r} \cdot (\boldsymbol{\varepsilon} - \mathbf{E} \times \boldsymbol{\theta}), \quad N = 1, 2, \ldots N_0 \tag{5}$$

Here γ_N is an arbitrary simple closed contour enclosing the axis of the Nth dislocation. The total Burgers vector of the discrete set of N_0 dislocations is defined according to (5) by the relation

$$\mathbf{B} = \sum_{N=1}^{N_0} \mathbf{b}_N = \sum_{N=1}^{N_0} \oint_{\gamma_N} d\mathbf{r} \cdot (\boldsymbol{\varepsilon} - \mathbf{E} \times \boldsymbol{\theta}) \tag{6}$$

Using the known properties of contour integrals the sum of integrals in (6) can be replaced by a single integral over the closed contour γ_0 surrounding the lines of all N_0 dislocations as follows

$$\mathbf{B} = \oint_{\gamma_0} d\mathbf{r} \cdot (\boldsymbol{\varepsilon} - \mathbf{E} \times \boldsymbol{\theta}) \tag{7}$$

Following [13, 14] we passed from a discrete set of dislocations to their continuous distribution, transforming the integral (7) by Stokes' formula

$$\mathbf{B} = \int_{\sigma_0} \mathbf{n} \cdot \mathrm{rot}(\boldsymbol{\varepsilon} - \mathbf{E} \times \boldsymbol{\theta}) d\sigma \tag{8}$$

Here σ_0 is the surface drawn over γ_0, $\mathbf{n}$ is the unit normal to σ_0. The relationship (8) allows to introduce the density of continuously distributed dislocations $\boldsymbol{\alpha}$ as a second-order tensor, whose flux across any surface yields the total Burgers vector of the dislocations crossing this surface

$$\mathrm{rot}(\boldsymbol{\varepsilon} - \mathbf{E} \times \boldsymbol{\theta}) = \boldsymbol{\alpha} \tag{9}$$

Let us assume that elastic body with continuously distributed dislocations occupies the multiply-connected domain and state the problem on rotation field $\boldsymbol{\theta}(\mathbf{r})$ determination in multiply-connected domain with single-valued and differentiable fields $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$. Now we do not use the requirement that the rotations are single-valued. By analogy with (4) the system of equations with respect to vector $\boldsymbol{\theta}$ takes the form

$$\mathrm{grad}\boldsymbol{\theta} = \boldsymbol{\kappa} \tag{10}$$

In the case of multiply-connected domain it has not uniquely defined solution, in general, that means existence in the body of rotational linear defects, i.e. disclinations [16–19]. The certain disclination is characterized by Frank's vector $\mathbf{q}_N$

$$\mathbf{q}_N = \oint_{\gamma_N} d\mathbf{r} \cdot \boldsymbol{\kappa}, \quad N = 1, 2, \dots N_0 \tag{11}$$

The total Frank's vector of a discrete disclinations set can, in accordance with (11), be represented as

$$\mathbf{Q} = \sum_{N=1}^{N_0} \mathbf{q}_N = \sum_{N=1}^{N_0} \oint_{\gamma_N} d\mathbf{r} \cdot \boldsymbol{\kappa} \tag{12}$$

In a similar way we passed from a discrete set of disclinations to their continuous distribution and define the density of distributed disclinations $\boldsymbol{\beta}$ as a second-order tensor, whose flux across any surface yields the total Frank vector of all disclinations crossing this surface. This definition leads to the relation $\mathrm{rot}\boldsymbol{\kappa} = \boldsymbol{\beta}$.

Thus, in the presence of distributed dislocations and disclinations, the geometric relations (3) are transformed to the incompatibility equations with regard to the metric and bending deformations

$$\mathrm{rot}\boldsymbol{\varepsilon} - \boldsymbol{\kappa}^T + \mathbf{E}\mathrm{tr}\boldsymbol{\kappa} = \boldsymbol{\alpha} \tag{13}$$

$$\mathrm{rot}\boldsymbol{\kappa} = \boldsymbol{\beta} \tag{14}$$

The Eq. (13) is derived from the relationship (9) and expression of the tensor $\boldsymbol{\kappa}$ in (3). The incompatibility equation (13) and (14) are deduced earlier in [16] with another method. If $\boldsymbol{\alpha} \neq 0$ and $\boldsymbol{\beta} \neq 0$, the fields of displacements $\mathbf{u}$ and rotations $\boldsymbol{\theta}$ do not exist. If $\boldsymbol{\alpha} \neq 0$ but $\boldsymbol{\beta} = 0$, the displacements field does not exist and there exists a rotation field. In what follows we assume that the dislocations and disclinations densities are given tensor functions of coordinates, as mass loads $\mathbf{f}$ and $\mathbf{h}$. These functions cannot be taken arbitrarily, since they obey the equations of continuity [16].

$$\mathrm{div}\boldsymbol{\alpha} + \boldsymbol{\beta}_\times = 0, \quad \mathrm{div}\boldsymbol{\beta} = 0 \tag{15}$$

The Eq. (15) are easy to obtain as a necessary condition of solvability of incompatibility equations (13) and (14) by exclusion of the unknown functions $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$.

3 Spherically Symmetric State

Considering the problem for a hollow sphere we introduce spherical coordinates r, φ, θ by formula

$$x_1 = r\cos\varphi\cos\theta, \;\; x_2 = r\sin\varphi\cos\theta, \;\; x_3 = r\sin\theta,$$

here x_1, x_2, x_3 is the Cartesian coordinates, r ($r_1 \le r \le r_0$) is the radial coordinate, φ ($0 \le \varphi \le 2\pi$) is the longitude, θ ($-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$) is the latitude. Unit vectors tangent to the lines of spherical coordinates are denoted by $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_\theta$. The dislocations and disclinations density tensors take the form

$$\boldsymbol{\alpha} = \alpha_1(r)\mathbf{g} + \alpha_2(r)\mathbf{d} + \alpha_3(r)\mathbf{e}_r \otimes \mathbf{e}_r \tag{16}$$

$$\boldsymbol{\beta} = \beta_1(r)\mathbf{g} + \beta_2(r)\mathbf{d} + \beta_3(r)\mathbf{e}_r \otimes \mathbf{e}_r \tag{17}$$

$$\mathbf{g} = \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_\theta \otimes \mathbf{e}_\theta, \;\; \mathbf{d} = \mathbf{e}_\varphi \otimes \mathbf{e}_\theta - \mathbf{e}_\theta \otimes \mathbf{e}_\varphi \tag{18}$$

The tensor fields (16) and (17) have a spherical symmetry in the sense that their components in the basis $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_\theta$ on each spherical surface $r =$ conts are the same at all points of the spherical surface and the tensors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ invariant under rotations about the radial axis, i.e. about $\mathbf{e}_r$ vector. The last property means that for any function $\chi(r)$ there is the identity

$$\boldsymbol{\Omega} \cdot \boldsymbol{\alpha} \cdot \boldsymbol{\Omega}^T = \boldsymbol{\alpha}$$

$$\boldsymbol{\Omega} = \mathbf{g}\cos\chi(r) + \mathbf{d}\sin\chi(r) + \mathbf{e}_r \otimes \mathbf{e}_r$$

The first summand in (16) describes the distribution of screw dislocations the axes of which coincide with the parallels and meridians, the last summand describes the distribution of screw dislocations with radial axis. The meaning of the (16) corresponds to a distribution of edge dislocations.

The first summand in (17) describes the distribution of wedge disclinations the axes of which coincide with the parallels and meridians, whereas the last summand describes the distribution of wedge disclinations with radial axis. The meaning of the (17) corresponds to a distribution of twist disclinations.

In order to specify loadings, we assume spherically symmetric vector fields

$$\mathbf{f} = f(r)\mathbf{e}_r, \quad \mathbf{h} = h(r)\mathbf{e}_r \tag{19}$$

Using the constitutive relations (2) the equilibrium equations (1) can be easily converted into a system of two vector equations for tensor functions $\boldsymbol{\varepsilon}$ and κ. They should connect the two tensor incompatibility equations (13) and (14). Thus, we obtain the system of 24 scalar equations for 18 unknown scalar functions, i.e. for components of the tensors $\boldsymbol{\varepsilon}$ and κ. The spherically symmetric solution of this system will be obtained similar to (16)

$$\begin{aligned} \boldsymbol{\varepsilon} &= \varepsilon_1(r)\mathbf{g} + \varepsilon_2(r)\mathbf{d} + \varepsilon_3(r)\mathbf{e}_r \otimes \mathbf{e}_r \\ \boldsymbol{\kappa} &= \kappa_1(r)\mathbf{g} + \kappa_2(r)\mathbf{d} + \kappa_3(r)\mathbf{e}_r \otimes \mathbf{e}_r \end{aligned} \tag{20}$$

With (2) and (19) for isotropic material we get

$$\begin{aligned} \mathbf{T} &= t_1(r)\mathbf{g} + t_2(r)\mathbf{d} + t_3(r)\mathbf{e}_r \otimes \mathbf{e}_r \\ \mathbf{M} &= m_1(r)\mathbf{g} + m_2(r)\mathbf{d} + m_3(r)\mathbf{e}_r \otimes \mathbf{e}_r \end{aligned} \tag{21}$$

On the basis of (16) and (17), the tensor incompatibility equations (13) is transformed to the three scalar ordinary differential equations

$$\frac{1}{r}(r\varepsilon_2)^{/} = \alpha_1 - \kappa_1 - \kappa_3, \quad \frac{\varepsilon_3 - \varepsilon_1}{r} - \varepsilon_1^{/} = \alpha_2 - \kappa_2, \quad \frac{2\varepsilon_2}{r} = \alpha_3 - 2\kappa_1 \tag{22}$$

whereas the tensor incompatibility equations (14) is also results into the three equations

$$\frac{1}{r}(r\kappa_2)^{/} = \beta_1, \quad \frac{\kappa_3 - \kappa_1}{r} - \kappa_1^{/} = \beta_2, \quad \frac{2\kappa_2}{r} = \beta_3 \tag{23}$$

Two vector continuity equations (15) are reduced to two scalar differential equations

$$\alpha_3^{/} + \frac{2}{r}(\alpha_3 - \alpha_1) + 2\beta_2 = 0, \quad \beta_3^{/} + \frac{2}{r}(\beta_3 - \beta_1) = 0 \tag{24}$$

The continuity equations (24) does not include the density of edge dislocations function $\alpha_2(r)$. So this function can be arbitrary, including the Dirac delta-function.

With (21) the equilibrium equations (1) are equivalent to the following

$$t_3^{/} + \frac{2}{r}(t_3 - t_1) + f(r) = 0, \quad m_3^{/} + \frac{2}{r}(m_3 - m_1) + 2t_2 + h(r) = 0 \tag{25}$$

It's easy to check that the first equation of (22) is not independent since it follows from the third Eq. (22), the second Eq. (23) and the first Eq. (24). Similarly, the first equation of (23) is a consequence of the third relation (23) and the second relation (24). Thus, there are four independent incompatibility equations.

If we express in the equilibrium equation (25) the components of the stress tensor t_s and the couple stress tensor m_k with the help of constitutive relations (2) through the values ε_k and κ_s, we get two differential equations for functions $\varepsilon_k(r)$, $\kappa_s(r)$.

Adding to these another four incompatibility equations, we will have 6 ordinary differential equations with 6 unknown functions: $\varepsilon_1, \varepsilon_2, \varepsilon_3, \kappa_1, \kappa_2, \kappa_3$.

Thus, even in the total system of resolving Eqs. (1), (2), (13) and (14) the number of equations exceeds the number of unknown functions, in the case of a spherically symmetric deformation, the number of ordinary differential equations coincides with the number of unknown functions.

For the spherically symmetric state we rewrite the constitutive relations in the following form

$$t_1 = 2(\lambda + \mu)\varepsilon_1 + \lambda\varepsilon_3, \;\; t_2 = 2\tau\varepsilon_2, \;\; t_3 = 2\lambda\varepsilon_1 + (\lambda + 2\mu)\varepsilon_3 \tag{26}$$

$$m_1 = 2(\nu + \gamma)\kappa_1 + \nu\kappa_3, \;\; m_2 = 2\eta\kappa_2, \;\; m_3 = 2\nu\kappa_1 + (\nu + 2\gamma)\kappa_3 \tag{27}$$

Proceeding from (22) and (26) we obtain

$$\varepsilon_2 = \frac{r\alpha_3}{2} - r\kappa_1, \quad t_2 = \tau r\alpha_3 - 2\tau r\kappa_1$$

Therefore, the second equilibrium equations (25) will be sought in the following form

$$m_3^/ + \frac{2}{r}(m_3 - m_1) - 4\tau r\kappa_1 + 2\tau r\alpha_3 + h(r) = 0 \tag{28}$$

It follows from (23), (27) that

$$\kappa_3 = r\kappa_1^/ + \kappa_1 + r\beta_2,$$

$$m_1 = (3\nu + 2\gamma)\kappa_1 + \nu r\kappa_1^/ + \nu r\beta_2 \tag{29}$$

$$m_3 = (3\nu + 2\gamma)\kappa_1 + (\nu + 2\gamma)r\kappa_1^/ + (\nu + 2\gamma)r\beta_2 \tag{30}$$

$$m_3 - m_1 = 2\gamma r\kappa_1^/ + 2\gamma r\beta_2$$

$$m_3^/ = (\nu + 2\gamma)r\kappa_1^{//} + 4(\nu + \gamma)\kappa_1^/ + (\nu + 2\gamma)(r\beta_2)^/ \tag{31}$$

With (29)–(31) relation (28) becomes the equation for the function $\kappa_1(r)$

$$\begin{aligned} &r^2\kappa_1^{//} + 4r\kappa_1^/ - \frac{4\tau}{\nu + 2\gamma}r^2\kappa_1 = \\ &= -\frac{r}{\nu + 2\gamma}\left[(\nu + 2\gamma)(r\beta_2)^/ + 4\gamma\beta_2 + 2\tau r\alpha_3 + h\right] \end{aligned} \tag{32}$$

The right side of the Eq. (32) contains given functions $\beta_2(r)$, $\alpha_3(r)$, $h(r)$. After determining the unknown function $\kappa_1(r)$, the function $\kappa_3(r)$ can be determined from the second relation (23), and the function $\kappa_2(r)$ we can find directly.

Now let us deduce an equation for the function $\varepsilon_1(r)$. Using (22) we get

$$\varepsilon_3 = (r\varepsilon_1)' + r\alpha_2 - \frac{1}{2}r^2\beta_3 \tag{33}$$

and on the basis of (26) we have

$$t_3 - t_1 = 2\mu(\varepsilon_3 - \varepsilon_1) \tag{34}$$

Substitute (33) into (34) we have

$$t_3 - t_1 = 2\mu\left[(r\varepsilon_1)' + r\alpha_2 - \frac{1}{2}r^2\beta_3 - \varepsilon_1\right] \tag{35}$$

Further, differentiating the third relation in (26) and using (33), we obtain

$$\begin{aligned} t_3' &= 2\lambda\varepsilon_1' + (\lambda + 2\mu)\varepsilon_3' = \\ &= 2\lambda\varepsilon_1' + (\lambda + 2\mu)\left[(r\varepsilon_1)'' + (r\alpha_2)' - \frac{1}{2}(r^2\beta_3)'\right] \end{aligned} \tag{36}$$

We transform (35) into

$$\frac{t_3 - t_1}{r} = 2\mu\left(\varepsilon_1' + \alpha_2 - \frac{1}{2}r\beta_3\right) \tag{37}$$

Substituting (36) and (37) to the first equilibrium equation (25), we obtain the equation for $\varepsilon_1(r)$

$$\begin{aligned} &(\lambda + 2\mu)r\varepsilon'' + 4(\lambda + 2\mu)\varepsilon_1' = \\ &= (\lambda + 2\mu)\left[\frac{1}{2}(r^2\beta_3)' - (r\alpha_2)'\right] + 4\mu\left(\frac{1}{2}r\beta_3 - \alpha_2\right) - f \end{aligned} \tag{38}$$

The homogeneous equation (38) is an equation of Euler's type and can be solved elementary. The inhomogeneous equations (38) can be solved by the above technique.

Once the functions ε_1 and κ_1 are found, the other unknowns can be determined directly with (22) and (23). And then with the use of (26) and (27) it is possible to find all stresses.

4 Solution of the Eigenstresses Problem in a Hollow Solid Sphere from Micropolar Material

Let us consider the differential equations (32) and (38)

$$r^2\kappa_1^{//} + 4r\kappa_1^{/} - br^2\kappa_1 = G(r), \quad b = \frac{4\tau}{\nu + 2\gamma} \tag{39}$$

$$G(r) = -\frac{r}{\nu + 2\gamma}\left[(\nu + 2\gamma)(r\beta_2)^{/} + 4\gamma\beta_2 + 2\tau r\alpha_3 + h\right]$$

$$(\lambda + 2\mu)r\varepsilon_1^{//} + 4(\lambda + 2\mu)\varepsilon_1^{/} = F(r) \tag{40}$$

$$F(r) = (\lambda + 2\mu)\left[\frac{1}{2}(r^2\beta_3)^{/} - (r\alpha_2)^{/}\right] + 4\mu\left(\frac{1}{2}r\beta_3 - \alpha_2\right) - f$$

where the right sides are expressed through the given functions describing the distribution of dislocations, disclinations and mass loads. As for the eigenstresses problem there are no external mass and surface loads we assume that $\mathbf{f} = \mathbf{h} = 0$, and the boundary conditions on the inner and outer spherical boundaries of the ball have the form

$$\begin{aligned} m_3(r)|_{r=r_1} &= 0, \quad m_3(r)|_{r=r_0} = 0 \\ t_3(r)|_{r=r_1} &= 0, \quad t_3(r)|_{r=r_0} = 0 \end{aligned} \tag{41}$$

In the case of $b > 0$ the general solution of the differential equations (39) has the form

$$\begin{aligned} \kappa_1(r) = {} & \frac{1}{r^3}\left(A_1 e^{-\sqrt{b}r}(br + \sqrt{b}) + A_2 e^{\sqrt{b}r}(br - \sqrt{b})\right) + \\ & + \frac{1}{2b^2r^3}\left[\mathrm{e}^{-\sqrt{b}r}(br + \sqrt{b})\int\left(\frac{1}{r} - \sqrt{b}\right)G(r)\mathrm{e}^{\sqrt{b}r}\mathrm{d}r - \right. \\ & \left. -\mathrm{e}^{\sqrt{b}r}(br - \sqrt{b})\int\left(\frac{1}{r} + \sqrt{b}\right)G(r)\mathrm{e}^{-\sqrt{b}r}\mathrm{d}r\right] \end{aligned} \tag{42}$$

For the differential equations (39) in the case of $b < 0$ we obtain the general solution

$$\kappa_1(r) = \kappa_1(r) = \frac{1}{r^3}\left(A_1 e^{r\sqrt{-b}}(br + I\sqrt{b}) + A_2 e^{-r\sqrt{-b}}(br - I\sqrt{b})\right) + \\ + \frac{1}{2r^3 b}\left[e^{r\sqrt{-b}}(I\sqrt{b} + rb)\int \frac{G(r)r\left(I\sqrt{b} - rb\right)e^{-r\sqrt{-b}}}{r^2\sqrt{(-b)^3} + I\sqrt{b} - \sqrt{-b}}\mathrm{d}r - \\ - e^{-r\sqrt{-b}}(I\sqrt{b} - rb)\int \frac{G(r)r\left(I\sqrt{b} + rb\right)e^{r\sqrt{-b}}}{r^2\sqrt{(-b)^3} + I\sqrt{b} - \sqrt{-b}}\mathrm{d}r\right] \tag{43}$$

For the inhomogeneous differential equations (40) we obtain the general solution

$$\varepsilon_1(r) = B_1 + \frac{B_2}{r^3} + \frac{1}{\lambda + 2\,\mu}\int \frac{1}{r^4}\left(\int F(r) r^3 \mathrm{d}r\right)\mathrm{d}r \tag{44}$$

Constants A_1, A_2, B_1, B_2 have to be determined from boundary conditions (41). Thus, for (42), (43), and (44) with the boundary conditions (41) it is possible to find exact solutions of the eigenstresses problem.

5 Spherically Symmetric State with a Non-polar Elastic Medium with Distributed Dislocations and Disclinations

Let us also consider the problem of a hollow sphere equilibrium within the framework of the classic linear theory of elasticity, i.e. for a simple (non-polar) material. In this case, the Cauchy stress tensor **T** is symmetric and the system of equations of statics of an isotropic body in the absence of distributed defects has the form [24]

$$\mathrm{div}\mathbf{T} + \mathbf{f} = 0 \tag{45}$$

$$\mathbf{T} = \lambda\mathbf{E}\mathrm{tr}\mathbf{e} + 2\mu\mathbf{e} \tag{46}$$

$$\mathbf{e} = \frac{1}{2}\left[\mathrm{grad}\mathbf{u} + (\mathrm{grad}\mathbf{u})^T\right] \tag{47}$$

Derivation of incompatibility equations for a simple elastic material with dislocations and disclinations is similar to that outlined in Sect. 1. With (47) it is given by

$$\mathrm{grad}\mathbf{u} = \mathbf{e} - \mathbf{E}\times\boldsymbol{\varphi} \tag{48}$$

here $\boldsymbol{\varphi}$ is the linear rotation vector [24]. Considering the problem of determining the displacement field **u** for a given in multiply-connected domain unique fields of the symmetric strain tensor **e** and rotation vector $\boldsymbol{\varphi}$, and arguing as in Sect. 1, we come to the expression of a tensor density of dislocations

$$\mathrm{rot}\,\mathbf{e} - \mathrm{rot}(\mathbf{E} \times \boldsymbol{\varphi}) = \boldsymbol{\alpha} \tag{49}$$

Equation (49) can be transformed as follows

$$\mathrm{grad}\boldsymbol{\varphi} = (\mathrm{rot}\,\mathbf{e})^T + \frac{1}{2}\mathbf{E}\mathrm{tr}\boldsymbol{\alpha} - \boldsymbol{\alpha}^T \tag{50}$$

We consider (50) as a system of equations for determination of the rotation vector field in a multiply-connected domain with given fields $\mathbf{e}$ and $\boldsymbol{\alpha}$. Repeating the arguments of Sect. 1, we obtain an expression for the tensor of disclination density $\boldsymbol{\beta}$

$$\boldsymbol{\beta} = \mathrm{rot}(\mathrm{rot}\,\mathbf{e})^T - \mathrm{rot}\left(\boldsymbol{\alpha}^T - \frac{1}{2}\mathbf{E}\mathrm{tr}\boldsymbol{\alpha}\right) \tag{51}$$

Considering the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ tensor fields as given quantities whereas the symmetric strain tensor $\mathbf{e}$ as the unknown function, from (51) we get the incompatibility equation of the classic theory of elasticity

$$\mathrm{rot}(\mathrm{rot}\,\mathbf{e})^T = \mathrm{rot}\left(\boldsymbol{\alpha}^T - \frac{1}{2}\mathbf{E}\mathrm{tr}\boldsymbol{\alpha}\right) + \boldsymbol{\beta} \tag{52}$$

As it is known [24], the tensor $\mathrm{rot}(\mathrm{rot}\mathbf{P})^T$ is symmetric, if $\mathbf{P} = \mathbf{P}^T$. Therefore, a necessary condition for the solvability of equations (52) is the symmetry requirement of the right side. This leads to the relation $\mathrm{div}\boldsymbol{\alpha} + \boldsymbol{\beta}_\times = 0$. The second solvability condition can be obtained by applying divergence operator to Eq. (52) and has the form $\mathrm{div}\boldsymbol{\beta} = 0$. Thus, the conditions of solvability of the incompatibility equations (52) coincide with the continuity equations (15) of the micropolar theory of elasticity.

Because of the symmetry of tensors $\mathbf{e}$ and $\mathbf{T}$ the spherically symmetric solution of equilibrium equations (45) and the incompatibility equations (52) for simple linearly elastic isotropic material should be sought in the form

$$\mathbf{e} = e_1(r)\mathbf{g} + e_3(r)\mathbf{e}_r \otimes \mathbf{e}_r, \quad \mathbf{T} = t_1(r)\mathbf{g} + t_3(r)\mathbf{e}_r \otimes \mathbf{e}_r \tag{53}$$

Considering (16), (17) and continuity equations (24) the right part of the incompatibility equations (52) takes the form

$$\mathrm{rot}\left(\boldsymbol{\alpha}^T - \frac{1}{2}\mathbf{E}\mathrm{tr}\boldsymbol{\alpha}\right) + \boldsymbol{\beta} = \left[\beta_1 - \frac{1}{r}(r\alpha_2)'\right]\mathbf{g} + \left(\beta_3 - \frac{2\alpha_2}{r}\right)\mathbf{e}_r \otimes \mathbf{e}_r \tag{54}$$

Components of the dislocation densities α_1, α_3 and the component β_2 of disclination density are not included in the expression (54). This means that the components of distributed defects do not affect the stress state of solid sphere made of linear elastic nonpolar material, whereas in the micropolar material, these defects manifest themselves, i.e. creating their own stresses. Note that the dislocation densities α_1

and α_3 manifest themselves also in the framework of nonlinear elasticity theory of simple materials [26]. In other words, these dislocations in non-polar material cause nonlinear effects. Using (53) and taking into account the continuity equations (15) the tensor incompatibility equations (54) is reduced to one scalar equation

$$\frac{de_1}{dr} + \frac{e_1 - e_3}{r} = \frac{r}{2}\beta_3 - \alpha_2 \tag{55}$$

Vector equilibrium equations (46) with $\mathbf{f} = 0$ on the basis of (53) results in one scalar equation. By using constitutive relations (46) this equation is converted to a differential equation for functions $e_1(r)$ and $e_3(r)$. The latter by means of (55) is the equation of the second order with respect to function $e_1(r)$. This equation not differs from Eq. (36) corresponding to the micropolar material.

6 Conclusion

Using the concept of spherical symmetric tensor field, we reduced a complex system of differential equilibrium equations of the micropolar elastic medium with distributed dislocations and disclinations to two ordinary differential equations. We demonstrate that it is possible to find exact solution of the problem of eigenstresses in a hollow solid sphere made of micropolar material for any spherically symmetric distribution of dislocations and disclinations. This problem is also solved within the framework of the classical theory of elasticity which does not take into account the couple stresses. We established that in this case some components of the dislocations and disclinations density tensors do not affect the stress state of the solid sphere, i.e. the effect of these defects may not be identified in the framework of the classical theory of elasticity.

Acknowledgements This work was supported by the Russian Foundation for Basic Research (grant 18-01-00203).

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