Disentanglements of Corank 2 Map-Germs: Two Examples

David Mond

Abstract We compute the homology of the multiple point spaces of stable perturbations of two germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ of corank 2, using a variety of techniques based on the image computing spectral sequence ICSS. We provide a reasonably detailed introduction to the ICSS, including some low-dimensional examples of its use. The paper is partly expository.

Keywords Disentanglement · Multiple-point spaces

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1 Introduction

In studying a singularity of mapping from n-space to $(n + 1)$-space, a rôle analogous to that of Milnor fibre is played by a stable perturbation of the singularity, and in particular by its image. The image of a map acquires non-trivial homology through the identification of points of the domain, and these identifications are encoded in the multiple point spaces of the map. For germs of corank 1, these multiple point spaces are well understood. For germs of corank > 1 the situation is radically different.

In this paper we study the multiple point spaces of stable perturbations of two map-germs of corank 2 from n-space to $(n + 1)$-space. In one case $n = 3$ and in the other $n = 5$. Previous work of Marar, Nuño-Ballesteros and Peñafort, in [16, 17] has explored the case where $n = 2$. Increasing the dimension introduces new difficulties. Confronting these will require a range of new techniques.

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Our work here is a preliminary exploration. Following the invitation of the editors to provide an accessible account, we have expanded the preliminary material on multiple-point spaces, disentanglements, and the image computing spectral sequence ICSS, our principle technical tool, and included, in Sect. 1.6, some examples of calculation using the ICSS.

The first of the two corank 2 map-germs we look at is the germ of lowest codimension in Sharland's list [24] of weighted homogeneous corank 2 map-germs $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$:

$$f_0 : (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0), \quad f_0(x, y, z) = (x, y^2 + xz + x^2 y, yz, z^2 + y^3). \tag{1.1}$$

This has $\mathcal{A}_e$-codimension 18.

The second is the lowest dimensional example of corank 2 map germ with $\mathcal{A}_e$-codimension 1,

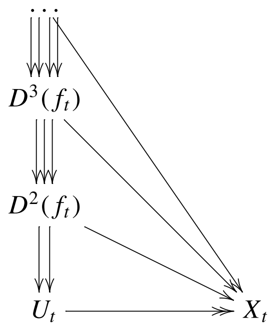
$$f_0 : (\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0), \quad f_0(x, y, a, b, c) = (x^2 + ax + by, xy, y^2 + cx + ay, a, b, c). \tag{1.2}$$

For each of these, we calculate a number of (topological) homology groups with rational coefficients, related to its disentanglement. By "disentanglement" we do not mean just the stable perturbation

$$f_t : U_t \twoheadrightarrow X_t \subset \mathbb{C}^{n+1}$$

of the germ f_0 (where U_t is a contractible neighbourhood of 0 in $\mathbb{C}^n$), as the term has been used by de Jong and van Straten in [4] and by Houston in [10] and subsequent papers. A richer picture is obtained by considering the "semi-simplicial resolution"

(1.3)



Here, for each integer $2 \le k \le n$, $D^k(f_t)$ is the closure, in U_t^k, of the set of k-tuples of pairwise distinct points ("strict" k-tuple points) sharing the same image, and the k distinct arrows $\pi_j^k : D^k(f_t) \to D^{k-1}(f_t)$, $1 \le j \le k$, are the restriction of the projections $U_t^k \to U_t^{k-1}$ obtained by forgetting the j'th factor in the product U_t^k.

For $k > n$, an $\mathcal{A}$-finite mono-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ will have no strict k-tuple points, since the dimension of $D^k(f_0)$ at a strict k-tuple point is $n-k+1$ (see Sect. 1.1 below). In this case $D^k(f_0)$ is defined by a slightly different procedure: we pick a stable unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}^d, 0)$ of f_0, define $D^k(F)$ as above, and then take $D^k(f_0)$ as the fibre over $0 \in \mathbb{C}^d$ of $D^k(F)$. We note that it is an easy consequence of the Mather–Gaffney criterion for $\mathcal{A}$-finiteness that if we apply this second procedure when $k \leq n$, we get the same space $D^k(f_0)$ as defined above.

The disentanglement, in this wider sense, contains complete information about the way that points of U_t are identified by f_t. The image X_t has the homotopy type of a wedge of n-spheres [25] whose number, the "image Milnor number" of f, $\mu_I(f)$, is the key geometric invariant of an $\mathcal{A}$-finite germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. Since the homology of X_t arises through the identifications induced by f_t, it is better described by the information attached to the diagram (1.3). This will become clearer in what follows.

Note that the π^k_j for fixed k and different j are left-right equivalent to one another thanks to the symmetric group actions on D^k and D^{k-1}, permuting the copies of U_t. In what follows we will consider only π^k_k, which we will refer to simply as π^k. We will denote the image of π^k in D^{k-1} by D^k_{k-1}, and, more generally, for $\ell < k$, we denote the image of $\pi^{\ell+1} \circ \cdots \circ \pi^k$ in D^ℓ by D^k_ℓ.

Remark 1.1 (1) Any finite map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is an embedding outside $D^2_1(f)$, which is the "non-embedding locus" of f. More generally each map $\pi^k : D^k(f) \to D^{k-1}(f)$ is an embedding outside $D^{k+1}_k(f)$, and each map π^{k+1} parameterises the non-embedding locus of its successor π^k. Thus the tower (1.3) shows a strong analogy with a free resolution of a module. If f is stable then $D^k(f)$, if not empty, is $n-k+1$-dimensional. It follows that the length of this resolution is at most n.
(2) For maps $M^n \to N^{n+1}$ with $n < 6$ there is no stable singularity of corank 2. Every $\mathcal{A}$-finite germ is stable outside 0, so if $n < 6$, any singularity outside 0 of an $\mathcal{A}$-finite germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, must be of corank 1. For stable germs of corank 1, all non-empty multiple point spaces are smooth [14]. It follows that for any $\mathcal{A}$-finite germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ with $n < 6$, $D^k(f_0)$ has (at most) isolated singularity. It also follows that a stable perturbation f_t has no singularities of corank > 1. Therefore all of the non-empty multiple point spaces $D^k(f_t)$ are smooth – indeed, are smoothings of the isolated singularities $D^k(f_0)$. For any map f, $D^\ell(\pi^k)$ can be identified with $D^{k+\ell-1}(f)$, by the obvious map

$$\Big((x_1, \ldots, x_{k-1}, x_k^{(1)}), (x_1, \ldots, x_{k-1}, x_k^{(2)}), \ldots, (x_1, \ldots, x_{k-1}, x_k^{(\ell)})\Big) \\ \longleftrightarrow \Big(x_1, \ldots, x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(\ell)}\Big) \tag{1.4}$$

– the left hand side here shows a point of $D^\ell(\pi^k)$, and the right hand side shows the corresponding point of $D^{k+\ell-1}(f)$. This observation is the basis of the "method of iteration" developed by Kleiman in [11]. From the smoothness of the $D^{k+j}(f_t)$ therefore follows smoothness of the multiple-point spaces of the projections π^k :

$D^k(f_t) \to D^{k-1}(f_t)$. The singularities of π^k are all of corank 1; this can be seen quite easily by writing f in linearly adapted coordinates, but see also [1]. By the characterisation of the stability of corank 1 map-germs by the smoothness of their multiple point spaces [14], it follows that provided $n < 6$, if f_t is stable then *all of the projections* π^k *are stable maps*.

1.1 Multiple Points

If $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is $\mathcal{A}$-finite then the set of strict k-tuple points is dense in $D^k(f_0)$, unless $D^k(f_0)$ consists only of the point $(0, \ldots, 0)$. The subset where f is an immersion at each of the x_i is still dense in $D^k(f_0)$. If $(x_1, \ldots, x_k)$ is such a k-tuple point, with $f_0(x_i) = y$ for $i = 1, \ldots, k$, then by the Mather–Gaffney criterion for $\mathcal{A}$-finiteness, the images of the germs $f_0 : (\mathbb{C}^n, x_i) \to (\mathbb{C}^{n+1}, y)$ meet in general position. It follows that their intersection has dimension $n + 1 - k$. This is therefore the dimension of $D^k(f_0)$, provided $k \leq n + 1$, and, similarly, of $D^k(f_t)$. If $k > n + 1$ then because f_t is stable, $D^k(f_t) = \emptyset$.

1.2 Alternating Homology

The developments in this section are due principally (but in some cases implicitly) to Goryunov in [6].
Notation For any topological space V, $C_k(V)$ is the free abelian group of singular k-chains in X, and $C_\bullet(V)$ is the singular chain complex. For a continuous map $\varphi : V \to W$, we denote by $\varphi_\#$ the map $C_j(V) \to C_j(W)$ induced by φ, and reserve the term φ_* for the corresponding map on homology.

Suppose $f : X \to Y$ is surjective. Recall the action of S_k on $D^k(f)$, permuting the copies of X. Define

$$C_j^{\mathrm{Alt}}(D^k(f)) = \{c \in C_j(D^k(f)) : \sigma_\#(c) = \mathrm{sign}(\sigma)c \ \text{ for all } \sigma \in S_k\}.$$

This gives a subcomplex, as $\partial(C_j^{\mathrm{Alt}}) \subset C_{j-1}^{\mathrm{Alt}}$, so we have *alternating homology*

$$H_j^{\mathrm{Alt}}(D^k(f)).$$

Now observe also that $\pi_\#^k : C_j^{\mathrm{Alt}}(D^k(f)) \subset C_j^{\mathrm{Alt}}(D^{k-1}(f))$. To see this, let $\sigma \in S_{k-1}$, and define $\tilde{\sigma} \in S_k$ by setting $\tilde{\sigma}(i) = \sigma(i)$ for $1 \leq i \leq k-1$ and $\tilde{\sigma}(k) = k$. Then $\mathrm{sign}(\tilde{\sigma}) = \mathrm{sign}(\sigma)$, and so if $c \in C_j^{\mathrm{Alt}}(D^k(f))$,

$$\sigma_\#(\pi_\#^k(c)) = \pi_\#^k(\tilde{\sigma}_\#(c)) = \pi_\#^k(\mathrm{sign}(\tilde{\sigma})c) = \mathrm{sign}(\sigma)\pi_\#^k(c).$$

In fact we have a double complex: on $C_j^{\mathrm{Alt}}(D^k(f))$, $\pi_\#^{k-1} \circ \pi_\#^k = 0$; for

$$\pi_\#^{k-1} \circ \pi_\#^k = \pi_\#^{k-1} \circ \pi_\#^k \circ (k, k-1)_\#,$$

and on alternating chains $(k, k-1)_\#$ is multiplication by -1. By the same argument, $f_\# \circ \pi_\#^2 = 0$. Thus, denoting X by $D^1(f)$, Y by $D^{-1}(f)$, and f by π^1, we have

Proposition 1.2 *$(C_j^{Alt}(D^\bullet(f)), \pi^\bullet)$ is a complex, and $(C_\bullet^{Alt}(D^\bullet(f)), \partial, (-1)^\bullet \pi_\#^\bullet)$ is a double complex.* □

The relevance to the homology of the image can be seen from two short calculations. In each, "c_j^k" *always denotes an alternating chain, when $k \geq 2$.*

Example 1 let $c_j^2 \in Z_j^{\mathrm{Alt}}(D^2(f))$.

$$\begin{array}{lccccl}
D^2(f) & 0 & \xleftarrow{\partial} & c_j^2 & & \\
 & & & \downarrow & & \\
X & 0 & \longleftarrow & \pi_\#^2(c_j^2) & \xleftarrow{\partial} & \exists\, c_{j+1}^1 \qquad \text{e.g. if } H_j(X) = 0 \\
 & & & \downarrow f_\# & & \downarrow \\
Y & & & f_\#\pi_\#^2(c_j^2) = 0 & \xleftarrow{\partial} & f_\#(c_{j+1}^1)
\end{array}$$

Because $f_\# \circ \pi_\#^2 = 0$ on alternating chains, $f_\#(c_{j+1}^1)$ is a cycle in Y. So from an alternating j-cycle c_j^2 in $D^2(f)$, we get a $j+1$ cycle on Y – provided $\pi_\#^2(c_j^2)$ is a boundary in X, i.e. provided $\pi_*^2[c_j^2] = 0$ in $H_j(X)$.

Example 2 let $c_j^3 \in Z_j^{\mathrm{Alt}}(D^3(f))$.

$$\begin{array}{lccccccl}
D^3(f) & 0 & \xleftarrow{\partial} & c_j^3 & & & & \\
 & & & \downarrow & & & & \\
D^2(f) & 0 & \xleftarrow{\partial} & \pi_\#^3(c_j^3) & \xleftarrow{\partial} & \exists\, c_{j+1}^2 & & \text{provided } \pi_*^3[c_j^3] = 0 \in H_j^{\mathrm{Alt}}(D^2(f)) \\
 & & & \downarrow & & \downarrow & & \\
X & & & 0 & \xleftarrow{\partial} & \pi_\#^2(c_{j+1}^2) & \xleftarrow{\partial} \ \exists\, c_{j+2}^1 & \text{provided } \pi_*^2[c_{j+1}^2] = 0 \in H_{j+1}(X) \\
 & & & & & \downarrow & \qquad\quad \downarrow & \\
Y & & & & & 0 & \xleftarrow{\partial} \ f_\#(c_{j+2}^1) &
\end{array}$$

Here, a j-dimensional homology class in $D^3(f)$ leads to a $j+2$-dimensional class in Y, provided certain homology classes vanish.

Note that in both cases, if c_j^k is the cycle we begin with, then

- if $c_j^k = \pi_\#^{k+1}(c_j^{k+1})$ for some $c_j^{k+1} \in C_j^{\mathrm{Alt}}(D^{k+1}(f))$ then $\pi_\#^k(c_j^k) = 0$, and
- if $c_j^k = \partial c_{j+1}^k$ for some $c_{j+1}^k \in C_{j+1}^{\mathrm{Alt}}(D^k(f))$, then we can take $c_{j+1}^{k-1} = \pi_\#^k(c_{j+1}^k)$ so the homology class we get in $H_{j+1}^{\mathrm{Alt}}(D^{k-2}(f))$ is zero.

So we are really interested in

$$\frac{\ker \pi_*^k : H_j^{\mathrm{Alt}}(D^k(f)) \to H_j^{\mathrm{Alt}}(D^{k-1}(f))}{\operatorname{im} \pi_*^{k+1} : H_j^{\mathrm{Alt}}(D^{k+1}(f)) \to H_j^{\mathrm{Alt}}(D^k(f))}.$$

1.3 The Image-Computing Spectral Sequence

Lurking behind the two calculations we have just gone through is the *Image-computing spectral sequence*, ICSS. This was introduced in [7] and further developed in [6]. It calculates the homology of the image X_t in terms of the alternating homology $H_*^{\mathrm{Alt}}(D^k(f_t))$ of the multiple point spaces $D^k(f_t)$. The version introduced in [7] worked with the subspace of $H_*(D^k(f);\mathbb{Q})$ on which S_k acts by its sign representation:

$$\mathrm{Alt}\, H_j(D^k(f);\mathbb{Q}) = \{[c] \in H_j(D^k(f);\mathbb{Q}) : \sigma_*([c]) = \mathrm{sign}(\sigma)[c] \text{ for all } \sigma \in S_k\}.$$

If we take the complex of alternating chains described in the last paragraph and replace integer coefficients by rational coefficients, then the two versions coincide:

$$\mathrm{Alt}\, H_j(D^k(f);\mathbb{Q}) = H_j^{\mathrm{Alt}}(D^k(f);\mathbb{Q}).$$

The ICSS has $E^1_{p,q} = H_q^{\mathrm{Alt}}(D^{p+1}(f_t))$ and converges to $H_{p+q}(X_t)$. The differential on the E^1 page, $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ is the simplicial differential $\pi_*^{p+1} : H_q^{\mathrm{Alt}}(D^{p+1}(f_t)) \to H_q^{\mathrm{Alt}}(D^p(f_t))$. In [7], a great deal hinges on the fact that for a stable perturbation f_t of an $\mathcal{A}$-finite germ f_0 of corank 1, the $D^k(f_t)$ are Milnor fibres of the isolated complete intersection singularities $D^k(f_0)$ (see [14]), and therefore their vanishing homology is confined to middle dimension. Since (over $\mathbb{Q}$) $H_i^{\mathrm{Alt}}(D^k(f_t)) \subset H_i(D^k(f_t))$, the vanishing alternating homology of $D^k(f_t)$ is also confined to middle dimension. From this it follows, in the case of a stable perturbation of a mono-germ, that the ICSS collapses at E^1: for all $r \geq 1$, $E^r_{p,q} = E^1_{p,q}$. The fact that the spectral sequence converges to $H_{p+q}(X_t)$ therefore means that, for map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, as $\mathbb{Q}$-vector space,

For a germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+c}, 0)$ with $c > 1$, the corresponding formula is

$$\begin{aligned} \tilde{H}_j(X_t) &\simeq H^{\mathrm{Alt}}_{n-(k-1)c}(D^k(f_t)) \quad \text{if } j = n-(k-1)(c-1) \text{ with } 2 \leq k \leq \frac{n}{c}+1, \\ &= 0 \qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

$$H_n(X_t) \simeq H_{n-1}^{\mathrm{Alt}}(D^2(f_t)) \oplus H_{n-2}^{\mathrm{Alt}}(D^3(f_t)) \oplus \cdots \oplus H_0^{\mathrm{Alt}}(D^{n+1}(f_t)). \tag{1.5}$$

The argument for collapse is as follows: for each space $D^{p+1}(f_t)$ there is at most one non-zero alternating homology group, $H_{n-p}^{\mathrm{Alt}}(D^{p+1}(f_t))$, and therefore either the source or the target of every differential at E^1 is equal to 0. Thus $E^2_{p,q} = E^1_{p,q}$. The higher differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ all vanish for exactly the same reason: for each one, either its source or its target is zero.

Notice that this is exactly what is needed to justify the assumptions we made in our two calculations in the previous paragraph. Whenever $D^k(f_t)$ has non-trivial alternating homology in dimension j, then $D^{k-1}(f)$ does not.

The situation for stable perturbations of multi-germs is slightly more complicated, as can be seen with the example of Reidemeister moves II and III in Sect. 1.6 below. Here $D^k(f_t)$ may have more than one connected component, and hence have vanishing alternating homology in dimension 0 as well as in middle dimension. As the calculations with Reidemeister moves II and III show, the differentials $\pi_*^k : H_0^{\mathrm{Alt}}(D^{p+1}(f_t)) \to H_0^{\mathrm{Alt}}(D^p(f_t))$ may not all be zero.

From (1.5) it follows that for a stable perturbation of a mono-germ

$$(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$$

$$\mu_I(f) = \sum_{k=2}^{n+1} \operatorname{rank} H_{n-k+1}^{\mathrm{Alt}}(D^k(f_t)). \tag{1.6}$$

In [10, Theorem 4.6], Kevin Houston showed the remarkable fact that that if f_t is a stable perturbation of an $\mathcal{A}$-finite mono-germ f_0 of *any* corank, then the *alternating* homology of $D^k(f_t)$ is once again confined to middle dimension, even though the ordinary homology of $D^k(f_0)$ may not be.[1] From Houston's theorem its follows that (1.5) and (1.6) hold for stable perturbations of mono-germs of any corank.

In both of our examples of corank 2 mono-germs, the multiplicity of f_0,

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,0}}{f_0^* \mathfrak{m}_{\mathbb{C}^{n+1},0}\, \mathcal{O}_{\mathbb{C}^n,0}},$$

is equal to 3, so f_t has no quadruple or higher multiple points, and (1.6) reduces to

$$\mu_I(f_0) = \operatorname{rank} H_{n-1}^{\mathrm{Alt}}(D^2(f_t)) + \operatorname{rank} H_{n-2}^{\mathrm{Alt}}(D^3(f_t)). \tag{1.7}$$

If $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is a germ with $\mu_I(f_0) = 1$, then (1.6) implies that the vanishing homology of the image comes from just one of the multiple point spaces. It is an interesting project to determine, for each such f_0, which one this is. It is possible to show that *the answer depends only on the isomorphism class of the local*

[1] In fact for the stable perturbation f_t of the germ $(\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$ described below, both $D^2(f_t)$ and $D^3(f_t)$ have non-trivial homology below middle dimension.

algebra of f_0. It is far from clear to me how to determine the answer from the local algebra. Nevertheless, examples support the following conjecture:

Conjecture 1.3 *If $(n, n+1)$ are in Mather's nice dimensions (i.e. $n < 15$) and if $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ has $\mu_I = 1$ then the vanishing homology in the image of a stable perturbation f_t comes from $D^k(f_t)$, where k is the dimension of the local algebra of f_0 (and is the highest integer for which $D^k(f_0) \neq \emptyset$).*

This is proved for germs of corank 1 in [2, Sect. 4].

1.4 Symmetric Group Actions on the Homology of the Multiple Point Spaces

From here on, and in the rest of the paper, we will consider only germs of maps from n-space to $n+1$-space, we will consider only homology with rational coefficients, and by $H_i(D^k(f_t))$ we will mean always $H_i(D^k(f_t); \mathbb{Q})$.

As we have seen, each multiple-point space $D^k(f_t)$ is acted upon by the symmetric group S_k, permuting the factors of U_t^k. The resulting representation of S_k on $H_*(D^k(f_t); \mathbb{Q})$ splits as a direct sum of isotypal components, whose ranks are the principle numerical invariants of the disentanglement. We have

$$H_i(D^2(f_t)) \simeq H_i^T(D^2(f_t)) \oplus H_i^{\text{Alt}}(D^2(f_t)),$$

where the two summands are the subspaces of $H_i(D^2(f_t))$ on which S_2 acts trivially, and by its sign representation, respectively, and

$$H_i(D^3(f_t)) = H_i^T(D^3(f_t)) \oplus H_i^{\text{Alt}}(D^3(f_t)) \oplus H_i^{\rho}(D^3(f_t)),$$

where now the summands correspond to the trivial, sign and irreducible degree 2 representation of S_3.

Let $M_k(f_0)$ and $M_k(f_t)$ denote the set of *target* k-tuple points of f and f_0 respectively – points with at least k preimages, counting multiplicity. By e.g. [20], the germ $(M_k(f_0), 0)$ is defined by the $(k-1)$'st Fitting ideal of the $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module $f_{0*}(\mathcal{O}_{\mathbb{C}^n})_0$, that is, the ideal generated by the $(m-k+1) \times (m-k+1)$ minors of the $m \times m$ matrix of a presentation of $f_{0*}(\mathcal{O}_{\mathbb{C}^n})_0$.

Lemma 1.4 *Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ have multiplicity k and isolated instability, and suppose that $M_k(f_0)$ is non-singular. Let f_t be a stable perturbation of f_0. Then $H_i^T(D^k(f_t)) = 0$ for all $i > 0$.*

Proof Because the multiplicity of f_0 is k, f_t has no $(k+1)$-tuple points, and it follows that $M_k(f_t) \simeq D^k(f_t)/S_k$, and therefore $H_i(M_k(f_t)) \simeq H_i^T(D^k(f_t))$. Because $M_k(f_0)$ is smooth, $M_k(f_t)$ is contractible, and the result follows. □

Lemma 1.4, with $k = 3$, applies to both of the germs we consider. Smoothness of $M_3(f_t)$ can be seen in each case by considering a presentation of $f_{0*}(\mathcal{O}_{\mathbb{C}^n})_0$.

Suppose f has corank > 1. We have no closed formula for generators of the ideal defining $D^k(f)$ for $k \geq 3$, but for $D^2(f)$ there is a formula, introduced in [18], for germs of any corank. The ideal $(f \times f)^*(I_{\Delta_{n+1}})$ obtained by pulling back the ideal defining the diagonal in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ vanishes on $D^2(f)$, but also on the diagonal in $\Delta_n \subset \mathbb{C}^n \times \mathbb{C}^n$. To remove Δ_n and leave only the points in the closure of the set of strict double points, we proceed as follows. The ideal $(f \times f)^*(I_{\Delta_{n+1}})$, generated by $f_i(x^{(1)}) - f_i(x^{(2)})$, for $i = 1, \ldots, n+1$, is contained in I_{Δ_n}, which is generated by $x_j^{(1)} - x_j^{(2)}$, $j = 1, \ldots, n$. Thus for $i = 1, \ldots, n+1$ there are functions $\alpha_{ij}(x^{(1)}, x^{(2)})$ such that

$$f_i(x^{(1)}) - f_i(x^{(2)}) = \sum_{j=1}^{n} \alpha_{ij}(x^{(1)}, x^{(2)}) \left(x_j^{(1)} - x_j^{(2)}\right).$$

The $(n+1) \times n$ matrix $\alpha = (\alpha_{ij})$ restricts to the jacobian matrix of f on Δ_n. We take

$$I_2(f) = (f \times f)^*(I_{\Delta_{n+1}}) + \min_n(\alpha).$$

Lemma 1.5 *Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be $\mathcal{A}$-finite and not an immersion. Then $D^2(f_0)$, as defined by $I_2(f_0)$, is Cohen–Macaulay of dimension $n-1$, and normal.* □

The proof of Cohen–Macaulayness has been part of the folklore for some time, but has recently been written up carefully by Nuño-Ballesteros and Peñafort in [21]. When $n = 3$, $D^2(f_0)$ is therefore a normal surface singularity, and so by the Greuel–Steenbrink theorem, [9, Theorem 1], $H_1(D^2(f_t)) = 0$.

1.5 Calculating $\mu_I(f)$

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ have finite codimension and let

$$F : \left(\mathbb{C}^n \times \mathbb{C}^d, (0,0)\right) \to \left(\mathbb{C}^{n+1} \times \mathbb{C}^d, (0,0)\right), \quad F(x,u) = (f_u(x), u)$$

be a versal deformation. If G is a reduced equation for the image of F then for $u \in \mathbb{C}^d$, $g_u := G(\_, u)$ is a reduced equation for the image of f_u. By a theorem of Siersma [25], the image of g_u has the homotopy type of a wedge of n-spheres, whose number is equal to the number of critical points of g_u (counting multiplicity) which move off the zero level as u leaves 0. Note that the number of n-spheres is, by definition, the image Milnor number $\mu_I(f_0)$. We can therefore calculate $\mu_I(f_0)$ as follows: define the relative jacobian ideal J_G^{rel} by

$$J_G^{\rm rel} = \left(\frac{\partial G}{\partial y_1}, \ldots, \frac{\partial G}{\partial y_{n+1}}\right)$$

where $y_1, \ldots, y_{n+1}$ are coordinates on $(\mathbb{C}^{n+1}, 0)$. The relevant critical points of the functions g_t together make up the residual components of $V(J_G^{\rm rel})$ after removal of its components lying in $\{G = 0\}$. This residual set can be found as the zero-locus of the saturation $(J_G^{\rm rel} : G^\infty)$, defined as

$$\bigcup_{k\in\mathbb{N}}\{h \in \mathcal{O}_{\mathbb{C}^{n+1}\times\mathbb{C}^d,(0,0)} : hG^k \in J_G^{\rm rel}\}.$$

We denote the zero locus of $(J_G^{\rm rel} : G^\infty)$ by Σ. Thus the image Milnor number $\mu_I(f_0)$ is the degree of the projection $(\Sigma, 0) \to (\mathbb{C}^d, 0)$. This degree can be calculated as the intersection number $\left(\Sigma, \mathbb{C}^{n+1} \times \{0\}\right)_{(0,0)}$. If Σ is Cohen–Macaulay then

$$\mu_I(f_0) = \left(\Sigma, \mathbb{C}^{n+1} \times \{0\}\right)_{(0,0)} = \dim_{\mathbb{C}}\left(\frac{\mathcal{O}_{\mathbb{C}^{n+1}\times\mathbb{C}^d,(0,0)}}{(J_G^{\rm rel} : G^\infty) + (u_1, \ldots, u_d)}\right) \tag{1.8}$$

where $u_1, \ldots, u_d$ are coordinates on $(\mathbb{C}^d, 0)$.

In both of the examples considered here, this is the case, and it is a straightforward *Macaulay2* [8] calculation to follow this procedure (including to check the Cohen–Macaulayness of Σ) and find $\mu_I(f_0)$: it is 18 for the germ $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$, and 1 for the germ $(\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$.

If Σ is not Cohen–Macaulay, the intersection number can be calculated using Serre's *formule clef*, [23], which we use to calculate a related intersection number in Sect. 3.2 below.

Remark 1.6 The method outlined here gives no hint to any relation between $\mu_I(f_0)$ and the $\mathcal{A}_e$-codimension of f_0. It is conjectured that provided $(n, n+1)$ are nice dimensions, the standard "Milnor–Tjurina" relation holds, namely

$$\mathcal{A}_e\text{-codim } f_0 \le \mu_I(f_0) \tag{1.9}$$

with equality if f_0 is weighted homogeneous. In [19] another slightly more complicated method for calculating μ_I is explained, with a similar case-by-case justification – verification of the Cohen Macaulayness of a certain relative T^1 module, $T^{1\,\rm rel}_{\mathcal{K}_{h,e}} i$, and consequent conservation of multiplicity. The virtue of this second method is that the relation (1.9) is an immediate consequence, whenever Cohen–Macaulayness of the relative T^1 can be shown, since $T^1_{\mathcal{A}_e} f_0$ is a quotient of $T^1_{\mathcal{K}_{h,e}} i_0$.

1.6 Examples

Example I: the ICSS for a stable perturbation of $f(x, y) = (x, y^3, xy + y^5)$. Here we apply the calculations described in Sect. 1.2 to a stable perturbation of the germ of the title of this subsection, of type H_2. For any map-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of the form $f(x, y) = (x, f_2(x, y), f_3(x, y))$, $D^2(f_t)$ is defined in (x, y_1, y_2)-space by the equations (see [14])

$$\frac{\begin{vmatrix} 1 & f_i(x, y_1) \\ 1 & f_i(x, y_2) \end{vmatrix}}{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}} \quad i = 2, 3 \tag{1.10}$$

and $D^3(f)$ is defined in (x, y_1, y_2, y_3)-space by the equations

$$\frac{\begin{vmatrix} 1 & f_i(x, y_1) & y_1^2 \\ 1 & f_i(x, y_2) & y_2^2 \\ 1 & f_i(x, y_3) & y_3^2 \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}}, \quad \frac{\begin{vmatrix} 1 & y_1 & f_i(x, y_1) \\ 1 & y_2 & f_i(x, y_2) \\ 1 & y_3 & f_i(x, y_3) \end{vmatrix}}{\begin{vmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{vmatrix}} \quad i = 2, 3. \tag{1.11}$$

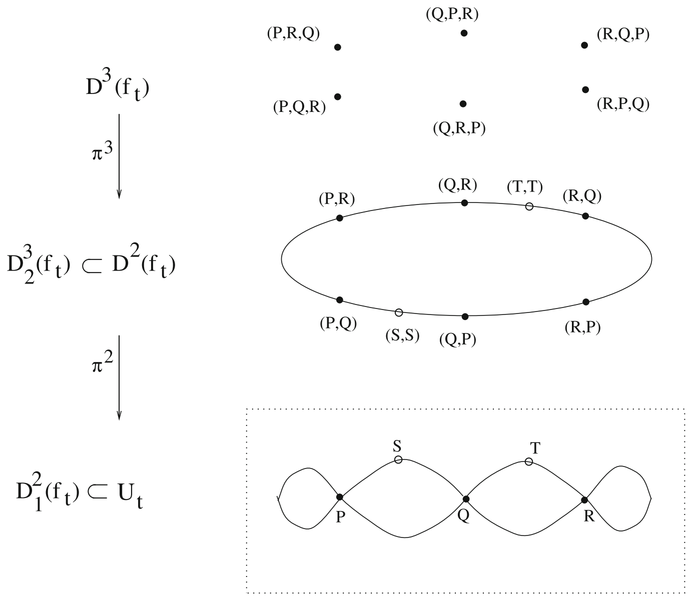
In this case these give

$$y_1^2 + y_1 y_2 + y_2^2, \quad x + y_1^4 + y_1^3 y_2 + y_1^2 y_2^2 + y_1 y_2^3 + y_2^4$$

for $D^2(f)$ and

$$P_2(y_1, y_2, y_3), \quad y_1 + y_2 + y_3, \quad x + P_4(y_1, y_2, y_3), \quad P_3(y_1, y_2, y_3)$$

for $D^3(f)$, where each P_j is a symmetric polynomial of degree j. Thus $D^2(f)$ is an A_1 curve singularity and $D^3(f)$ is a non-reduced point of multiplicity 6. If f_t is a stable perturbation then $D^2(f_t)$ is a Milnor fibre of the A_1 singularity, homotopy equivalent to a circle, and $D^3(f_t)$ consists of 6 points forming a single S_3-orbit. By judicious choice of parameter values u and v in the miniversal deformation $f_{u,v}(x, y) = (x, y^3 + uy, xy + y^5 + vy^2)$ (see [15]), one can arrange that the *real* picture of $D^2(f_t)$ and $D^3(f_t)$, and their projections $D_1^3(f_t)$, $D_1^2(f_t)$, are as shown in the following diagram.



Here S and T in U_t are the non-immersive points of f_t. At each, the germ of f_t is equivalent to the parametrisation of the Whitney umbrella, $(x, y) \mapsto (x, y^2, xy)$, since this is the only stable non-immersive germ in this dimension range. The non-strict double points (S, S) and (T, T) are the fixed points of the involution $(1, 2)$ on $D^2(f_t)$, which, in our picture, is induced by a reflection in the straight line joining them.

As a single faithful S_3-orbit, $D^3(f_t)$ carries an alternating cycle,

$$c_0 = (P, Q, R) - (P, R, Q) + (R, P, Q) - (R, Q, P) + (Q, R, P) - (Q, P, R).$$

The projection of this cycle to $D^2(f_t)$, $\pi^3_\#(c_0)$, is an alternating boundary, as in Example 2 of Sect. 1.2: for instance

$$\pi^3_\#(c_0) = (P, Q) - (P, R) + (R, P) - (R, Q) + (Q, R) - (Q, P) = \partial(c_1)$$

where c_1 is the alternating 1-chain

$$[(T, T)(Q, R)] - [(T, T)(R, Q)] + [(Q, P)(R, P)] - [(P, Q)(P, R)]$$

(here, for any two non-antipodal points $A, B \in D^2(f_t)$, $[A, B]$ denotes the singular 1-simplex parametrising the shorter arc from A to B). The projection of c_1 to U_t is a 1-cycle in U_t, and is the boundary of a 2-chain c_2 with support equal to the union of the first and third bounded regions of the complement of $D^2_1(f_t)$, counting from left to right. And by the argument above, $f_{t\#}(c_2)$ is a cycle in the image X_t, indeed one of the two generators of $H_2(X_t)$. Another generator comes from the alternating 1-cycle c_1' on $D^2(f_t)$ consisting of the anticlockwise arc $[(S, S)(T, T)]$ minus the clockwise arc $[(S, S)(T, T)]$. I encourage the reader to find a 2-chain c_2' on U_t such that $\partial c_2' = \pi^2_\#(c_1')$.

Example II: the Reidemeister moves. The Reidemeister moves of knot theory are versal deformations of the three $\mathcal{A}_e$-codimension 1 singularities of mappings from the line to the plane. It is instructive to look at their disentanglements (in the sense described above), and at the resulting ICSS. The codimension 1 germs are shown in the middle column of the table below, and the right hand column shows a 1-parameter versal deformation, which, fixing $t \neq 0$, gives a stable perturbation.

I	$f_0 : x \mapsto (x^2, x^3)$	$f_t : x \mapsto (x^2, x^3 - tx)$
II	$f_0 : \begin{cases} x \mapsto (x, \ \ x^2) \\ y \mapsto (y, -y^2) \end{cases}$	$f_t : \begin{cases} x \mapsto (x, x^2 - t) \\ y \mapsto (y, -y^2) \end{cases}$
III	$f_0 : \begin{cases} x \mapsto (x, \ \ x) \\ y \mapsto (y, \ \ 0) \\ z \mapsto (z, -z) \end{cases}$	$f_t : \begin{cases} x \mapsto (x, \ \ x) \\ y \mapsto (y, \ \ t) \\ z \mapsto (z, -z) \end{cases}$

(1.12)

For all three cases, the non-trivial modules in the E^1 page of the ICSS for f_t are contained in the single column

$$\begin{array}{c} 0 \\ \downarrow \\ H_0^{\mathrm{Alt}}(D^3(f_t)) \\ \downarrow \pi^3_* \\ H_0^{\mathrm{Alt}}(D^2(f_t)) \\ \downarrow \pi^2_* \\ H_0(U_t) \\ \downarrow \\ 0 \end{array} \tag{1.13}$$

Reidemeister I. Take $F : (x, t) \mapsto (t, f_t(x))$ as stable unfolding. Since in order that $F(t_1, x_1) = F(t_2, x_2)$, we must have $t_1 = t_2$, we can embed $D^2(F)$ in $\mathbb{C}^3$ with coordinates t, x_1, x_2. There, following the recipe preceding Lemma 1.5 above, we find that $D^2(F)$ is defined by the equations

$$\frac{x_1^2 - x_2^2}{x_1 - x_2} = \frac{x_1^3 - tx_1 - (x_2^3 - tx_2)}{x_1 - x_2} = 0. \tag{1.14}$$

Simplifying, this gives

$$x_1 + x_2 = 0 \quad x_1^2 + x_1x_2 + x_2^2 = t. \tag{1.15}$$

Thus $D^2(f_0)$ is a 0-dimensional A_1 singularity. Setting $t > 0$ for a good real picture, and denoting $\sqrt{t}$ by P and $-\sqrt{t}$ by Q, $D^2(f_t)$ is its Milnor fibre, the point-pair $\{(P, Q), (Q, P)\}$. Then for $t \neq 0$, $H_0^{\mathrm{Alt}}(D^2(f_t)) \simeq \mathbb{Q}$, generated by the class of $[(P, Q)] - [(Q, P)]$. Note that $H_0^{\mathrm{Alt}}(D^2(f_0)) = 0$, since when $t = 0$, $P = Q$. For both f_0 and f_t, $D^3 = \emptyset$.

In the E^1 page (1.13), $H_0^{\mathrm{Alt}}(D^3(f_t)) = 0$. We have $\pi_*^2 = 0$, for $\pi_*^2\big([(P, Q)] - [(Q, P)]\big) = [P] - [Q]$, and U_t is connected, so that $[P] = [Q]$. Hence the spectral sequence collapses at E^1, and for $t \neq 0$

$$H_0(X_t) = H_0(U_t) = \mathbb{Q}, \quad H_1(X_t) = H_0^{\mathrm{Alt}}(D^2(f_t)) = \mathbb{Q}.$$

Reidemeister II. Here both branches of the bi-germ f_0 are immersions, so all multiple points are strict. Denote by 0_x and 0_y the origins of the coordinate systems with coordinates x and y respectively. The domain of the stable perturbation f_t is a disjoint union $U_t = U_{x,t} \cup U_{y,t}$, where $U_{x,t}$ is a contractible neighbourhood of 0_x and $U_{y,t}$ is a contractible neighbourhood of 0_y. Thus $H_0(U_t) \simeq \mathbb{Q}^2$. There are no triple points, and $D^2(f_t)$ consists of

$$\{(x, y) \in (\mathbb{C}, 0_x) \times (\mathbb{C}, 0_y) : x = y, x^2 - t = -y^2\} \tag{1.16}$$

together with its image under the involution $(1, 2)$ sending (x, y) to (y, x). When $t = 0$ this is a pair of 0-dimensional A_1 singularities, interchanged by $(1, 2)$. To describe $D^2(f_t)$ for $t \neq 0$, denote the points in $(\mathbb{C}, 0_x)$ with x coordinates $\sqrt{t/2}$ and $-\sqrt{t/2}$ by P_x and Q_x respectively, and the points in $(\mathbb{C}, 0_y)$ with y coordinates $\sqrt{t/2}$ and $-\sqrt{t/2}$ by P_y and Q_y. Then for $t \neq 0$,

$$D^2(f_t) = \{(P_x, P_y), (P_y, P_x), (Q_x, Q_y), (Q_y, Q_x)\}, \tag{1.17}$$

with the involution $(1, 2)$ interchanging the first and second points, and the third and fourth. For $t = 0$, this collapses just to

$$D^2(f_0) = \{(0_x, 0_y), (0_y, 0_x)\}.$$

Thus for $t \neq 0$, $H_0^{\mathrm{Alt}}(D^2(f_t))$ is two-dimensional, with basis $[(P_x, P_y)] - [(P_y, P_x)]$, $[(Q_x, Q_y)] - [(Q_y, Q_x)]$, and for $t = 0$, $H_0^{\mathrm{Alt}}(D^2(f_0))$ has basis $[(0_x, 0_y)] - [(0_y, 0_x)]$. With respect to the basis of $H_0^{\mathrm{Alt}}(D^2(f_t))$ described above, and the basis $[P_x], [P_y]$ for $H_0(U_t)$, π_*^2 has matrix $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ when $t \neq 0$, and thus has 1-dimensional kernel and cokernel. The spectral sequence collapses at E^2, and

$$H_1(X_t) = E^2_{1,0} = \ker \pi^2_* \simeq \mathbb{Q}, \quad H_0(X_t) = E^2_{0,0} = \operatorname{Coker} \pi^2_* \simeq \mathbb{Q}.$$

Reidemeister III. We use the same conventions as for Reidemeister II. Let P_x and P_y denote the points in $U_{x,t}$ and $U_{y,t}$ with x and y coordinate t, and let Q_y and Q_z denote the points in $U_{y,t}$ and $U_{z,t}$ with y and z coordinate $-t$. Note that when $t = 0$, then $P_x = 0_x$, etc. Then

$$\begin{aligned} D^2(f_t) \cap \left(U_{x,t} \times U_{y,t}\right) &= \{(P_x, P_y)\} \\ D^2(f_t) \cap \left(U_{x,t} \times U_{z,t}\right) &= \{(0_x, 0_y)\} \\ D^2(f_t) \cap \left(U_{y,t} \times U_{z,t}\right) &= \{(Q_x, Q_z)\} \end{aligned} \tag{1.18}$$

and

$$D^3(f_0) \bigcap U_{x,t} \times U_{y,t} \times U_{z,t} = \{(0_x, 0_y, 0_z)\}.$$

Thus

$$\begin{gathered} H_0^{\mathrm{Alt}}(D^3(f_0)) \simeq \mathbb{Q} \quad H_0^{\mathrm{Alt}}(D^3(f_t)) = 0 \\ H_0^{\mathrm{Alt}}(D^2(f_0)) \simeq \mathbb{Q}^3 \; H_0^{\mathrm{Alt}}(D^2(f_t)) \simeq \mathbb{Q}^3 \\ H_0(U_0) \simeq \mathbb{Q}^3 \qquad H_0(U_t) \simeq \mathbb{Q}^3 \end{gathered} \tag{1.19}$$

with bases shown in the following table.

Module	Basis
$H_0^{\mathrm{Alt}}(D^3(f_0))$	$[(0_x, 0_y, 0_z)] - [(0_x, 0_z, 0_y)] + [(0_z, 0_x, 0_y)] - [(0_z, 0_y, 0_x)]$ $+[(0_y, 0_z, 0_x)] - [(0_y, 0_x, 0_z)]$
$H_0^{\mathrm{Alt}}(D^2(f_t))$	$[(P_x, P_y)] - [(P_y, P_x)],\ -[(0_x, 0_z)] + [(0_z, 0_x)],$ $[(Q_y, Q_z)] - [(Q_z, Q_y)]$
$H_0(U_t)$	$[0_x] = [P_x],\ [P_y] = [Q_y],\ [Q_z] = [0_z]$

With respect to these bases, the differentials π^k_* have the following matrices (with the first only for $t = 0$):

$\pi^3_* = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\pi^2_* = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

In the spectral sequence for f_0, the image of π^3_* kills the kernel of π^2_*. When $t \neq 0$, D^3 vanishes, along with its homology, while $H_0^{\mathrm{Alt}}(D^2(f_t))$ remains unchanged.

The spectral sequence collapses at E^2, and

$$H_1(X_t) = E^2_{1,0} = \ker \pi^2_* \simeq \mathbb{Q}, \quad H_0(X_t) = E^2_{0,0} = \operatorname{Coker} \pi^2_* \simeq \mathbb{Q}.$$

2 New Examples: Disentanglements of Two Germs of Corank 2

2.1 Summary of Results

2.1.1 A Germ of Corank 2 from 3-Space to 4-Space

Let

$$f_0(x, y, z) = (x, y^2 + xz + x^2 y, yz, z^2 + y^3).$$

The rows of the following table show relations between the ranks of the isotypal subspaces of the homology groups of $D^2(f_t)$ and $D^3(f_t)$ and of the homology groups of their projections to U_t, $D^2_1(f_t)$ and $D^3_1(f_t)$, and VD_∞, the number of Whitney umbrellas on $D^2_1(f_t)$, which plays a crucial role in our calculation. The left hand column shows where in the paper the calculation is made. Blank spaces indicate zeros.

datum	$H_2^T(D^2)$	$H_2^{\mathrm{Alt}}(D^2)$	$H_2(D_1^2)$	$H_1^T(D^3)$	$H_1^{\mathrm{Alt}}(D^3)$	$H_1^\rho(D^3)$	$H_1(D_1^3)$	VD_∞	
(1.7)		1			1				= 18
$\delta(D_1^3)$ in §4.2							1		= 8
$\delta(M_3)$ in §4.2				1					= 0
(3.4)				1	1	1	−2	−1	= −1
(3.6)								1	= 10
(3.7)				−1	1			−1	= −1
(3.9)			1						= 27
(3.10) − (3.13)	1	1	−1		1	$\frac{1}{2}$			= 0

(2.1)

The rank of the matrix of coefficients is 8, so we are able to compute all of the invariants. The following table shows their values.

$H_2^T(D^2)$	$H_2^{\mathrm{Alt}}(D^2)$	$H_2(D_1^2)$	$H_1^T(D^3)$	$H_1^{\mathrm{Alt}}(D^3)$	$H_1^\rho(D^3)$	$H_1(D_1^3)$	VD_∞
1	9	27	0	9	16	8	10

(2.2)

2.1.2 A Germ of Corank 2 from 5-Space to 6-Space

Let

$$f_0(x, y, a, b, c) = (x^2 + ax + by, xy, y^2 + cx + ay, a, b, c).$$

We are able to show

(a) $H_3^{\mathrm{Alt}}(D^3(f_t)) \simeq \mathbb{Q}$ and $H_4^{\mathrm{Alt}}(D^2(f_t)) = 0$, so the vanishing homology of the image comes from the triple points.
(b) $H_1(D^3(f_t)) = 0$, $H_2(D^3(f_t)) = H_2^{\rho}(D^3(f_t)) \simeq \mathbb{Q}^2$, and $H_3(D^3(f_t)) = H_3^{\mathrm{Alt}}(D^3(f_t)) \simeq \mathbb{Q}$.
(c) $H_1(D^2(f_t)) = 0$, $H_2(D^2(f_t)) = H_2^{T}(D^2(f_t)) \simeq \mathbb{Q}$.
(d) $\dim_{\mathbb{Q}} H_4(D^2(f_t)) = \dim_{\mathbb{Q}} H_3(D^2(f_t)) \leq 1$. Both groups are S_2-invariant, by Houston's theorem [10, Theorem 4.6]

Statements (a) and (b) are shown in Sect. 4.3, and (c) and (d) are shown in Sect. 4.4.

This is the first example I know of a stable perturbation of a map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ for which the vanishing homology of the multiple point spaces is not confined to middle dimension, though of course many such examples are to be expected when f_0 has corank > 1.

3 Calculations for the Germ $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$

3.1 Triple Points

Since no closed formula for a set of generators for the ideal defining $D^3(f)$ in $(\mathbb{C}^3)^3$ is known, we do not have direct access to any of the invariants of $D^3(f_t)$. However, we are able to build up a complete picture of the representation of S_3 on its homology, and in particular to calculate the dimension of $H_1^{\mathrm{Alt}}(D^3(f_t))$, by working our way up from its image under projection to U_t, $D_1^3(f_t)$.

Lemma 3.1 *$D_1^3(f_t)$ is a smoothing of $D_1^3(f)$*

Proof We have to show both that $D_1^3(f_t)$ is smooth, and that it is the fibre of a flat deformation of $D_1^3(f)$. The first statement is a consequence of the classification of stable map-germs. Up to $\mathcal{A}$-equivalence, the only stable germs of maps $\mathbb{C}^3 \to \mathbb{C}^4$ are

(a) a trivial unfolding of the parameterisation of the Whitney umbrella:

$$p_1(u, v, w) = (u, v, w^2, vw);$$

(b) a bi-germ whose two branches are a germ of type (a) and an immersion, meeting in general position in $\mathbb{C}^4$;
(c) a multi-germ consisting of k immersions meeting in general position, for $k = 1, 2, 3, 4$ (we denote these by (c1), ..., (c4)).

Since f_t is stable, every one if its germs is one of these types, and one can easily check that for each of them, except for (c4), the triple point locus D_1^3, where non-empty, is smooth. In the mapping f_t there are no points of type (c4), so $D_1^3 f_t)$ is smooth.

For the second statement, let $F : (\mathbb{C}^3 \times S, (0,0)) \to (\mathbb{C}^4 \times S, (0,0))$ be a stable unfolding of f over a smooth base S. Then $D^3(F)$ has dimension $1 + \dim S$. By the principle of iteration, $D_1^3(F) = M_2(\pi^2 : D^2(F) \to \mathbb{C}^3 \times S)$ (where M_2 means the set of double points in the target). Now $D^2(F)$ is Cohen–Macaulay, and π^2 is finite and generically 1-1, so $M_2(\pi^2)$ is also Cohen Macaulay [20]. Flatness of the projection $D_1^3(F) \to S$ now follows from the fact that the dimension of its fibre, $D_1^3(f_t)$, is equal to $\dim D_1^3(F) - \dim S$. □

It follows from the lemma that rank $H_1(D_1^3(f_t)) = \mu(D_1^3(f), 0)$. We find μ by using Milnor's formula $\mu = 2\delta - r + 1$ [13], where δ is the δ-invariant of a curve-germ and r the number of its branches. We find $D_1^3(f)$ as the zero locus of the ideal $f^*(\mathrm{Fitt}_2)$, where $\mathrm{Fitt}_2 := \mathrm{Fitt}_2(f_* \mathcal{O}_{\mathbb{C}^3,0})$ is the second Fitting ideal of $\mathcal{O}_{\mathbb{C}^3,0}$ considered as $\mathcal{O}_{\mathbb{C}^4,0}$-module via f^*. *Macaulay2* [8] gives the following presentation of $f_*(\mathcal{O}_{\mathbb{C}^3})$:

$$\begin{pmatrix} -X^2U^2 - 2XUV + V^2 - UW & X^4 + U^2 + X^3V & X^3U + 2X^2V + XW \\ X^4 + U^2 + X^3V & -X^6 - 2X^2U - XV - W & -X^5 - XU + V \\ X^3U + 2X^2V + XW & -X^5 - XU + V & -X^4 - U \end{pmatrix} \tag{3.1}$$

from which we see that

$$\mathrm{Fitt}_2 = (X^4 + U, V, X^2U + W) \tag{3.2}$$

and

$$f^*\mathrm{Fitt}_2 = (x^4 + x^2y + y^2 + xz, yz, x^3z + z^2). \tag{3.3}$$

Primary decomposition of the ideal (3.3) shows that the curve $D_1^3(f)$ has three smooth components:

$$C_1 = V(y, x^3 + z), \quad C_2 = V(z, y - \xi x^2) \quad C_3 = V(z, y - \xi^2 x^2)$$

where $\xi = e^{2i\pi/3}$, with parameterisations

$$\gamma_1(t) = (t, 0, -t^3), \quad \gamma_2(u) = (u, \xi u^2, 0), \quad \gamma_3(v) = (v, \xi^2 v^2, 0).$$

Denoting by $\mathcal{O}_{\tilde{\Sigma}} := \mathbb{C}\{t\} \oplus \mathbb{C}\{u\} \oplus \mathbb{C}\{v\}$ the ring of the normalisation of Σ, we find that

$$n^*(\mathcal{O}_{\Sigma,0}) = (t^3) \oplus (u^3) \oplus (v^3) + \mathrm{Sp}\{(1,1,1), (t,u,v), (t^2,u^2,v^2), (0, \xi u^2, \xi^2 v^2)\} \subset \mathcal{O}_{\tilde{\Sigma}}.$$

Hence

$$\delta(D_1^3(f)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\tilde{\Sigma}}, 0}{n^* \mathcal{O}_{\Sigma,0}} = 5,$$

and

$$\text{rank } H_1(D_1^3(f_t)) = \mu(D_1^3(f)) = 2\delta - 3 + 1 = 8.$$

The projection $D^3(f_t) \to D^3_1(f_t)$ is a double cover, with points (a, b, c) and (a, c, b) sharing the same image, and no points of higher multiplicity, as f has no quadruple points. The cover is simply branched at triple points of the form (a, b, b); there are no triple points of the form (a, a, a), since if there were, then f_t would have multiplicity ≥ 3 at a, and as we see in the list of stable germs in the proof of Lemma 3.1, none has multiplicity > 2. Thus

$$\chi(D^3(f_t)) = 2\chi(D^3_1(f_t)) - \#\text{branch points} = -14 - \#\text{branch points}. \qquad (3.4)$$

To complete the calculation of the Euler characteristic of $D^3(f_t)$, we have to compute the number of branch points. This seems not to be straightforward. Though the branch points are points of intersection of $D^3_1(f_t)$ and the non-immersive locus $R(f_t)$, both of these are curves so their intersection in U_t is not a proper intersection. Both curves lie in the surface $D^2_1(f_t)$, where the intersection is proper, but $D^3_1(f_t)$ is the singular locus of $D^2_1(f_t)$ and so again calculation of the intersection number is difficult. Instead we use the fact that the branch points are Whitney umbrella points of $D^2_1(f_t)$, which we explain in the next section, and count them using a theorem of Theo de Jong in [3].

3.2 Double Points

Lemma 3.2 *$(a, b, b) \in D^3(f_t)$ if and only if $(a, b) \in D^2(f_t)$ is a Whitney umbrella point of the projection $\pi^2 : D^2(f_t) \to U_t$.*

Proof This is, once again, the principle of iteration. The map

$$(a, b, c) \mapsto ((a, b), (a, c))$$

identifies $D^3(f_t)$ with $D^2(\pi^2 : D^2(f_t) \to U_t)$. A point of the form (a, b, b) becomes a fixed point of the involution $((a, b), (a, c)) \mapsto ((a, c), (a, b))$, and thus a non-immersive point of π^2. By Remark 1.1, this must be a Whitney umbrella point. □

From Lemma 3.2 we see that to find the dimension of $H^{\text{Alt}}_1(D^3(f_t))$ we must count the number of Whitney umbrellas on $D^2_1(f_t)$. Let $W(D^2_1(f_t))$ denote the set of all such points. They appear when f_t has a bi-germ of type (b) in the list in the proof of Lemma 3.1: the Whitney umbrella appears on $D^2_1(f_t)$ at the source point of the immersive member of the bi-germ. If $R(f_t)$ is the set of non-immersive points of f_t, then $W(D^2_1(f_t)) = D^3_1(f_t) \cap R(f_t)$, so one might hope to calculate the number of points in $W(D^2_1(f_t))$ as an intersection number. But as remarked above, the intersection is improper: both $D^3_1(f_t)$ and $R(f_t)$ are curves. We are forced to look further afield, and use a theorem of Theo de Jong [3]. The *virtual number of D_∞ points* on a germ of singular surface $(S, x_0) \subset \mathbb{C}^3$, with 1-dimensional singular locus Σ, and with reduced equation h, is defined as follows. Let $\theta(h)$ be the restriction to Σ of the germs of vector fields on $(\mathbb{C}^3, x_0)$ tangent to all level sets of h.

Then $\theta(h) \subset \theta_\Sigma$. Let $\tilde{\Sigma}$ be the normalisation of Σ. Since vector fields lift uniquely to the normalisation we can consider the quotient $\theta_{\tilde{\Sigma},\tilde{x}_0}/\theta(h)$. De Jong defines

$$\mathrm{VD}_\infty(S) = \dim_{\mathbb{C}}\left(\frac{\theta_{\tilde{\Sigma},\tilde{x}_0}}{\theta(h)}\right) - 3\delta(\Sigma) \tag{3.5}$$

and shows ([3, Theorem 2.5]) that $VD_\infty(S)$ is conserved in a flat deformation of S which induces a flat deformation of Σ.

Let us apply this to the case where S is the surface $D^2_1(f)$ for a finitely determined map-germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$. In this case $\Sigma = D^3_1(f)$. A deformation of f over a smooth base S induces a flat deformation of $D^2_1(f)$, since this is a hypersurface. We have already seen that $D^3_1(f)$ deforms flat over S. Thus we may apply de Jong's theorem. The special points on $D^3_1(f_t)$, where $D^2_1(f_t)$ is not a normal crossing of two sheets, are of two types: Whitney umbrella points and triple points. We have already seen how Whitney umbrella points arise; triple points correspond to quadruple points of f_t, in which four pieces of $\mathbb{C}^3$ are mapped immersively and in general position. We denote the number of these by Q.

Corollary 3.3 $VD_\infty(D^2_1(f)) = |Fix(1, 2)| - 8Q$

Proof Each Whitney umbrella point contributes 1 to $\mathrm{VD}_\infty(D^2_1)$. Each quadruple point gives rise to four triple points on $D^2_1(f)$. Each triple point contributes -2 to $\mathrm{VD}_\infty(D^2_1))$ [3, Example 2.3.3]. So

$$\mathrm{V}D_\infty(D^2_1(f)) = \#\text{Whitney umbrellas} - 2\#\text{triple points} = |\mathrm{Fix}(1, 2)| - 8Q.$$

□

Now we return to the map germ f of Sharland that is the focus of our interest.

To compute $VD_\infty(D^2_1)$ we need to find lifts to the normalisation $\tilde{\Sigma}$ of $D^3_1(f)$ of the vector fields annihilating the equation h of $D^2_1(f)$. A *Macaulay* calculation finds that modulo the defining ideal of $D^3_1(f)$, these vector fields are generated by

$$\chi_1 = (x^3y^2 + 2xy^3 - 3xz^2)\frac{\partial}{\partial x} + (2x^2y^3 + 4y^4)\frac{\partial}{\partial y} - 9z^3\frac{\partial}{\partial z}$$

$$\chi_2 = (y^4 + x^2z^2)\frac{\partial}{\partial x} - (2x^3y^3 - 2xy^4)\frac{\partial}{\partial y} + 3xz^3\frac{\partial}{\partial z}$$

These lift to

$$\tilde{\chi}_1 = \left(-3t^7\frac{\partial}{\partial t}, (2+\xi^2)u^7\frac{\partial}{\partial u}, (2+\xi)v^7\frac{\partial}{\partial v}\right), \quad \tilde{\chi}_2 = \left(t^8\frac{\partial}{\partial t}, \xi u^8\frac{\partial}{\partial u}, \xi^2v^8\frac{\partial}{\partial v}\right)$$

in $\theta_{\tilde{\Sigma}} = \mathbb{C}\{t\}\partial_t \oplus \mathbb{C}\{u\}\partial_u \oplus \mathbb{C}\{v\}\partial_v$. The $\mathcal{O}_{\mathbb{C}^3}$-submodule of $\theta_{\tilde{\Sigma}}$ they generate is equal to

$$(t^{10})\partial_t \oplus (u^{10})\partial_u \oplus (v^{10})\partial_v + \mathrm{Sp}_{\mathbb{C}}\{\tilde{\chi}_1,\ x\tilde{\chi}_1,\ x^2\tilde{\chi}_1,\ \tilde{\chi}_2,\ x\tilde{\chi}_2\}.$$

Hence $\dim_{\mathbb{C}}(\theta_{\tilde{\Sigma}}/\theta(h)) = 25$ so that

$$VD_\infty(D_1^2) = 25 - 3 \times 5 = 10. \tag{3.6}$$

We have proved

Lemma 3.4 *The involution* (2, 3) *has* 10 *fixed points on* $D^3(f_t)$. □

Corollary 3.5 $dim_{\mathbb{Q}} H_1(D^3(f_t); \mathbb{Q}) = 25$.

Proof By the lemma and (3.4), $\chi(D^3(f_t)) = -24$. □

Proposition 3.6 $dim_{\mathbb{Q}} H_1^{Alt}(D^3(f_t)) = 9$, $dim_{\mathbb{Q}} H_1^{\rho}(D^3(f_t)) = 16$, *and* $dim_{\mathbb{Q}} H_2^{Alt}(D^2(f_t)) = 9$.

Proof We use the Lefschetz fixed point theorem:

$$10 = \#\text{fixed points of } (2,3) = \sum_{k\geq 0}(-1)^k \left(\text{trace}(2,3)_* : H_k(D^3(f_t)) \to H_k(D^3(f_t))\right)$$

$$= 1 - \text{trace}(2,3)_* : H_1(D^3(f_t) \to H_1(D^3(f_t)) = 1 + \dim_{\mathbb{Q}} H_1^{\text{Alt}}(D^3(f_t)) - \dim_{\mathbb{Q}} H_1^T(D^3(f_t)). \tag{3.7}$$

The last equation here follows from the fact that the trace of (2, 3) on the irreducible sign representation of S_3, on the trivial representation and on the irreducible 2-dimensional representation is −1, 1 and 0 respectively. It is straightforward to check that each fixed point of (2, 3) is non-degenerate and therefore has Leftschetz number 1. Since $H_1^T(D^3(f_t)) = 0$, we obtain the first equality in the statement of the corollary. The second equality now follows by Corollary 3.5 and the third by the fact that $18 = \mu_I(f) = \dim_{\mathbb{Q}} H_2^{\text{Alt}}(D^2(f_t)) + \dim_{\mathbb{Q}} H_1^{\text{Alt}}(D^3(f_t))$. □

Now we compute $H_2(D^2(f_t))$. Although we have a formula for the ideal defining $D^2(f)$, we have no method of deriving from it a formula for the rank of the homology of $D^2(f_t)$. So once again we proceed indirectly, by calculating the homology of the image of its projection to U_t, $D_1^2(f_t)$.

Lemma 3.7 $D_1^2(f_t)$ *has the homotopy type of a wedge of 27 2-spheres.*

Proof We use the technique explained in Sect. 1.5, based on Siersma's theorem [25] that the rank of the vanishing homology of $D_1^2(f)$ is equal to the number of critical points of a reduced defining equation of $D_1^2(f)$ which move off the zero level as t moves off 0. The unfolding

$$F(t_1, t_2, t_3, x, y, z) = (t_1, t_2, t_3, x, y^2 + xz + x^2y, yz + t_1y + t_2z, z^2 + y^3 + t_3y)$$

is stable, by Mather's algorithm for the construction of stable germs as unfoldings of germs of rank 0, and $D_1^2(f_t)$ is the fibre of $D_1^2(F)$ over $t \in \mathbb{C}^3$. Let G be an equation for $D^2_1(F)$, let g_t be its restriction to $\{t\} \times \mathbb{C}^3$, and let J^{rel}_G be the relative jacobian ideal $(\partial G/\partial X, \partial G/\partial Y, \partial G/\partial Z)$. As in Sect. 1.5, we compute the number of critical points of a reduced defining equation of $D^2_1(f)$ which move off the zero level as t moves off 0, as the intersection multiplicity

$$\left(V(J^{\mathrm{rel}}_G : G^\infty) \cdot (\{0\} \times \mathbb{C}^3)\right)_{(0,0)}. \tag{3.8}$$

In fact calculation shows that in this case $(J^{\mathrm{rel}}_G : G^\infty)$ is equal to the transporter $(J^{\mathrm{rel}}_G : G)$. However, unlike the situation discussed in Sect. 1.5, $V(J^{\mathrm{rel}}_G : G^\infty)$ is not Cohen–Macaulay; it has projective dimension 5 as $\mathcal{O}_{\mathbb{C}^6,0}$-module, while its codimension is 3. To compute the intersection multiplicity, we have to use Serre's *formule clef*, from [23]. Denote $(J^{\mathrm{rel}}_G : G^\infty)$ by Q; then

$$(V(Q), \{0\} \times \mathbb{C}^3)_0 = \sum_j (-1)^j \dim_{\mathbb{C}} \mathrm{Tor}^{\mathcal{O}}_j \left(\frac{\mathcal{O}}{Q}, \frac{\mathcal{O}}{(t_1, t_2, t_3)}\right)$$

where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^6,0}$. Since t_1, t_2, t_3 is a regular sequence there are at most three non-vanishing Tor modules, for $j = 0, 1, 2$. A *Macaulay* calculation shows that they have dimension 29, 3, 1 respectively, so that

$$\dim_{\mathbb{Q}} H_2(D^2_1(f_t)) = (V(Q), \{0\} \times \mathbb{C}^3)_{(0,0)} = 29 - 3 + 1 = 27. \tag{3.9}$$

□

It is striking that in this case $V(Q)$ is not Cohen–Macaulay. In all of the examples I know, where one uses the procedure of Sect. 1.5 to calculate μ_I, and G is the defining equation of the image of the stable unfolding F, the corresponding space $V(J^{\mathrm{rel}}_G : G^\infty)$ is Cohen Macaulay.

To relate the homology of $D^2_1(f_t)$ to the homology of $D^2(f_t)$, we use the image computing spectral sequence: $D^2_1(f_t)$ is the image of the projection $\pi^2 : D^2(f_t) \to U_t$. Taking account of the facts that f_t has no quadruple points, so that π^2 has no triple points, and that $H_1(D^2(f_t)) = 0$, the E^1 term is reduced to

$$\begin{array}{ccc} 0 = H^{\mathrm{Alt}}_0(D^2(\pi^2)) & H^{\mathrm{Alt}}_1(D^2(\pi^2)) & 0 \\ \downarrow & \downarrow d_1 & \\ H_0(D^2(f_t)) & 0 & H_2(D^2(f_t)) \end{array} \tag{3.10}$$

and the spectral sequence collapses here. So

$$27 = \dim_{\mathbb{C}} H_2(D^2_1(f_t)) = \dim_{\mathbb{C}} H_2(D^2(f_t)) + \dim_{\mathbb{C}} H^{\mathrm{Alt}}_1(D^2(\pi^2)). \tag{3.11}$$

Recall from Remark 1.1 the isomorphism $i : D^3(f_t) \to D^2(\pi^2 : D^2(f_t) \to U_t)$, given by $(a, b, c) \mapsto ((a, b), (a, c))$. The involution on $D^2(\pi^2)$ lifts to the transposition (2, 3) on $D^3(f_t)$. Thus under the induced isomorphism of first homology, $H_1^{\text{Alt}}(D^2(\pi^2))$ corresponds to the -1 eigenspace of $(2,3)_*$ on $H_1(D^3(f_t))$. On each copy of the 2-dimensional irreducible representation ρ, and on each copy of the sign representation, (2, 3) has 1-dimensional -1 eigenspace. Thus, using Proposition 3.6 for the second equality,

$$\dim_{\mathbb{Q}} H_1^{\text{Alt}}(D^2(\pi^2)) = \dim_{\mathbb{Q}} H_1^{\text{Alt}}(D^3(f_t)) + \frac{1}{2}\dim_{\mathbb{Q}} H_1^{\rho}(D^3(f_t)) = 17, \quad (3.12)$$

and, by (3.11),

$$\dim_{\mathbb{Q}} H_2(D^2(f_t)) = 10. \quad (3.13)$$

4 Calculations for the Germ $(\mathbb{C}^5, 0) \to (\mathbb{C}^6, 0)$

The germ

$$f_0(x, y, a, b, c) = (x^2 + ax + by, xy, y^2 + cx + ay, a, b, c)$$

has $\mu_I = \mathcal{A}_e$-codimension $= 1$, and versal unfolding

$$F(x, y, a, b, c, t) = (x^2 + ax + by, xy, y^2 + cx + (a+t)y, a, b, c, t).$$

Let $U_t \xrightarrow{f_t} X_t$ be a stable perturbation of f_0, with contractible domain $U_t \subset \mathbb{C}^5$. By (1.7),

$$1 = \operatorname{rank} H_5(X_t) = \operatorname{rank} H_4^{\text{Alt}}(D^2(f_t)) + \operatorname{rank} H_3^{\text{Alt}}(D^3(f_t)).$$

As in the previous section, we approach $D^3(f_t)$ via its projection to U_t, $D_1^3(f_t)$. As before, $D_1^3(f_t)$ is defined by the pull-back of the second Fitting ideal of $\mathcal{O}_{\mathbb{C}^5,0}$ considered as $\mathcal{O}_{\mathbb{C}^6,0}$-module. The $\mathcal{O}_{\mathbb{C}^6\times\mathbb{C},(0,0)}$-module $F_*\left(\mathcal{O}_{\mathbb{C}^5\times\mathbb{C},(0,0)}\right)$ has presentation

$$\begin{pmatrix} Y^2 - XZ - abZ - bcY + atY & aY + cX + tY & aY + bZ \\ aY + cX + tY & -Z - ac & Y - bc \\ aY + bZ & Y - bc & -X - ab - bt \end{pmatrix}. \quad (4.1)$$

so

$$\text{Fitt}_2 = (Z + ac, Y - bc, X + (a+t)b)$$

and

$$F^*\mathrm{Fitt}_2 = (y^2 + y(a+t) + xc + ac, xy - bc, x^2 + xa + yb + ab + bt)$$
$$= \min_2 \begin{pmatrix} -y & -c \\ x+a & -y-a-t \\ b & x \end{pmatrix}$$

The corresponding ideal for $t = 0$ defines the 3-fold singularity $D^3_1(f_0)$. A *Macaulay2* [8] calculation shows that the T^1 of $D^3_1(f_0))$ has dimension 1. Therefore $D^3_1(f_0)$ is isomorphic to the unique non-ICIS codimension 2 Cohen Macaulay 3-fold singularity with $\tau = 1$, which one can find in the table on p. 22 of [5]. This table also lists the Betti numbers of a smoothing, from which we obtain

$$h_0(D^3_1(f_t)) = 1, \quad h_1(D^3_1(f_t)) = 0, \quad h_2(D^3_1(f_t)) = 1, \quad h_3(D^3_1(f_t)) = 0. \quad (4.2)$$

In particular, $\chi(D^3_1(f_t)) = 2$.

Now $D^3(f_t)$ and $D^3_1(f_t)$ are smoothings of $D^3(f_0)$ and $D^3_1(f_0)$. Let $\pi = \pi^2 \circ \pi^3$ be the projection from $D^3(f_t)$ to U_t, $\pi(P, Q, R) = P$. Then $D^3(f_t)$ is a branched double cover of $D^3_1(f_t)$: for a generic point $P \in D^3_1(f_t)$, which shares its f_t-image with Q and R, $\pi^{-1}(P) = \{(P, Q, R), (P, R, Q)\}$. Because there are no quadruple points, the branching is of two types:

- over a point P where f_t has a stable singularity of type $\Sigma^{1,1,0}$, $\pi^{-1}(P) = \{(P, P, P)\}$. The set of all such points P is denoted $\Sigma^{1,1} f_t$. It lies in the closure of the set of branch points of the second kind:
- if $f_t(P) = f_t(Q)$ with f_t an immersion at P and of type $\Sigma^{1,0}$ at Q, then $\pi^{-1}(P) = \{(P, Q, Q)\}$, so (P, Q, Q) is a branch point.

We denote the set of all such points P by $D^3_{1,0}(f_t)$. Note that (Q, P, Q) and (Q, Q, P) are not branch points.
Thus

$$\chi(D^3(f_t)) = 2\chi(D^3_1(f_t)) - \chi(D^3_{1,0}(f_t)) = 4 - \chi(D^3_{1,0}(f_t)). \quad (4.3)$$

4.1 Equations for $\Sigma^{1,1} f$

The ramification ideal R_f, generated by the 5×5 minors of the jacobian matrix J of f_0 defines the non-immersive locus Σf of f_0. Then $\Sigma^{1,1}(f)$ is defined by the ideal of maximal minors of the matrix obtained by concatenating J with the jacobian matrix of a set of generators of R_f (see e.g. [12]). By removing from this ideal an $\mathfrak{m}$-primary component we obtain the ideal

$$S := (3y + a, 3x + a, ac - 3bc, ab - 3bc, a^2 - 9bc),$$

easily recognised as defining a curve isomorphic to the germ of the union of the three coordinate axes in $(\mathbb{C}^3, 0)$. This has $\delta = 2$ and therefore $\mu = 2\delta - r + 1 = 2$. It is not quite evident that this is deformed flat in a deformation of f_0, but nevertheless this is the case. The corresponding locus for the 1-parameter versal deformation F of f_0 has an $\mathfrak{m}$-primary component, whose removal leaves a 2-dimensional Cohen–Macaulay component which restricts to $\Sigma^{1,1} f_0$.

4.2 Equations for $D^3_{1,0}(f_0)$

By the description above, $D^3_{1,0}(f_0)$ is the "shadow component" of $f_0^{-1}(f(\Sigma f_0))$, that is, the closure of $f_0^{-1}(f_0(\Sigma f_0)) \smallsetminus \Sigma f_0$. To find equations for it, we first look for equations for the support of $f_{0*}(\mathcal{O}_S/\mathcal{R}_{f_0})$. Let I_0 be the radical of the zero'th Fitting ideal of $f_{0*}(\mathcal{O}_S/\mathcal{R}_{f_0})$, let $I_1 = f_0^*(I_0)$, and let I_2 be the saturation $I_1 : \mathcal{R}_{f_0}^\infty$, in this case equal to $I_1 : \mathcal{R}_{f_0}^2$. After some effort one finds that I_2 is the ideal of maximal minors of the 2×4 matrix

$$\begin{pmatrix} a & b & x & y \\ -3y + a & x + a & -y - a & 3y - a + 4c \end{pmatrix}.$$

This is isomorphic to the cone over the rational normal curve of degree 4 (Pinkham's example). In the versal deformation F, the same construction leads to the ideal of maximal minors of the 2×4 matrix

$$\begin{pmatrix} a & b & x & y + t \\ -3y + a + t & x + a & -y - a - t & 3y - a + 4c - t \end{pmatrix}$$

One checks that this defines a smoothing of $D^3_{1,0}(f_0)$, over the Artin component of the base space (since it is given by the minors of a 2×4 matrix). So the only non-zero reduced Betti number is $\beta_2 = 1$ (see e.g. [22] p. 173). In particular

$$\chi(D^3_{1,0}(f_t)) = 2. \tag{4.4}$$

4.3 Homology of $D^3(f_t)$

By (1.7) and Lemma 1.4,

$$H_i(D^3(f_t)) = H_i^{\mathrm{Alt}} \oplus H_i^{\rho}. \tag{4.5}$$

Denote by h_i^{Alt} and h_i^{ρ} the ranks of these summands.

Because there are no quadruple points, $D^3_1(f_t)$ is the quotient of $D^3(f_t)$ by the $\mathbb{Z}_2$-action generated by the transposition $(2,3)(P,Q,R)=(P,R,Q)$. So $H_i(D^3_1(f_t))$ is the part of $H_i(D^3(f_t))$ invariant under $(2,3)_*$. Since $H_i^T(D^3(f_t))=0$ for $i>0$, the $(2,3)_*$ -invariant part of $H_i(D^3(f_t))$ is the $(2,3)_*$-invariant part of $H_i^\rho(D^3(f_t))$, and thus isomorphic to the sum of copies of the subspace of ρ invariant under (2, 3). The (2, 3)-invariant subspace of ρ is 1-dimensional. Thus,

$$h_i(D^3_1(f_t))=\frac{1}{2}h_i^\rho(D^3(f_t)) \tag{4.6}$$

for $i>1$. Hence, by (4.2),

$$h_1^\rho(D^3(f_t))=0,\ \ h_2^\rho(D^3(f_t))=2,\ \ h_3^\rho(D^3(f_t))=0. \tag{4.7}$$

On the other hand, as $D^3(f_t)$ is a branched cover of degree 2 of $D^3_1(f_t)$, branched along $D^3_{1,0}(f_t)$, it follows that

$$\chi(D^3(f_t))=2\chi(D^3_1(f_t))-\chi(D^3_{1,0}(f_t))=2.$$

Putting this together with (4.6), we have

$$2=\chi(D^3(f_t))=1-\left(h_1^\rho+h_1^{\text{Alt}}\right)+\left(h_2^\rho+h_2^{\text{Alt}}\right)-\left(h_3^\rho+h_3^{\text{Alt}}\right)=1-h_1^{\text{Alt}}+h_2^{\text{Alt}}+2-h_3^{\text{Alt}}.$$

so

$$-1=-h_1^{\text{Alt}}+h_2^{\text{Alt}}-h_3^{\text{Alt}}. \tag{4.8}$$

By [10, Theorem 4.6], the alternating homology of the multiple point spaces of a stable perturbation of a finitely determined map-germ is concentrated in middle dimension. As $D^3(f_t)$ is a 3-fold, this means $h_i^{\text{Alt}}(D^3(f_t))=0$ for $i\neq 3$ and so from (4.8), $h_3^{\text{Alt}}(D^3_1(f_t))=1$. Since here $\mu_I=1$, we conclude from (1.7) that $h_4^{\text{Alt}}(D^2(f_t))=0$. Also, from (4.7) and (4.5), we conclude that $H_1(D^3(f_t))=0$, $\dim_{\mathbb{Q}}H_2(D^3(f_t))=2$ and $\dim_{\mathbb{Q}}H_3(D^3(f_t))=1$.

Remark 4.1 An application of the extended version of the Lefschetz Fixed Point Theorem gives the same conclusion: the fixed set of the involution (2, 3) on $D^3(f_t)$ is homeomorphic to the branch locus $D^3_{1,0}(f_t))$, and so by the extended version of the Lefschetz Fixed Point Theorem,

$$2=\chi(\text{Fix}(2,3))=\sum_i(-1)^i\left(\text{Tr}(2,3)_*:H_i(D^3(f_t))\to H_i(D^3(f_t))\right).$$

Because $\chi_\rho(2,3)=0$ and $H_i^T(D^3(f_t))=0$ for $i>0$, this alternating sum is equal to $1+h_1^{\text{Alt}}-h_2^{\text{Alt}}+h_3^{\text{Alt}}$. This gives us the same information as (4.8).

4.4 Double Points

The double locus of F is defined by $F^*(\mathrm{Fitt}_2(F_*(\mathcal{O}_S)))$, which is a principle ideal generated by the composite with F of the determinant of the lower right 2×2 submatrix of (4.1). We will call this composite G. As in Lemma 3.7, $D^2_1(f_t)$ has the homotopy type of a wedge of 4-spheres, whose number is the intersection number of $V(J^{\mathrm{rel}}_G : G^\infty)$ with $\mathbb{C}^5 \times \{0\}$. A *Macaulay2* [8] calculation gives

$$(J^{\mathrm{rel}}_G : G^\infty) = (J^{\mathrm{rel}}_G : G) = (3c - t, 3b + t, 2a + t, 6y + t, 6x - t). \tag{4.9}$$

The zero-locus of this ideal is a smooth curve of degree 1 over the t-axis, so $D^2_1(f_t)$ is homotopy-equivalent to a single 4-sphere, by Siersma's theorem [25].

The map of pairs $(D^2(f_t), D^3_2(f_t)) \xrightarrow{\pi^2_1} (D^2_1(f_t), D^3_1(f_t))$ induces a morphism between the long exact sequences of reduced homology of the pairs. Because $\pi^2_1 : D^2 \to D^2_1$ is an isomorphism outside D^3_2, the morphisms $H_i(D^2, D^3_2) \to H_i(D^2_1, D^3_1)$ are isomorphisms for all i. From the segment

$$\begin{array}{ccccccccc}
\cdots \to & H_2(D^2, D^3_2) & \to & H_1(D^3_2) & \to & H_1(D^2) & \to & H_1(D^2, D^3_2) & \to 0 \\
& \| & & \downarrow & & \downarrow & & \| & \\
\cdots \to & H_2(D^2_1, D^3_1) & \to & H_1(D^3_1) & \to & H_1(D^2_1) & \to & H_1(D^2_1, D^3_1) & \to 0
\end{array}$$

and the fact that $H_1(D^2_1) = 0 = H_1(D^3_2)$ (the latter equality because $D^3_2 \simeq D^3$, as there are no quadruple points) we see that $H_1(D^2(f_t)) = 0$. Because $H_2(D^2_1, D^3_1)$ is sandwiched between 0's in the lower sequence, continuing to the left we have

$$\begin{array}{ccccccccccccc}
 & & \mathbb{Q} & & & & & & \mathbb{Q}^2 & & & & \\
 & & \| & & & & & & \| & & & & \\
H_4(D^2, D^3_2) & \to & H_3(D^3_2) & \to & H_3(D^2) & \to & H_3(D^2, D^3_2) & \to & H_2(D^3_2) & \to & H_2(D^2) & \to & 0 \\
\| & & \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
H_4(D^2_1, D^3_1) & \to & H_3(D^3_1) & \to & H_3(D^2_1) & \to & H_3(D^2_1, D^3_1) & \to & H_2(D^3_1) & \to & H_2(D^2_1) & \to & 0 \\
\| & & \| & & \| & & \| & & \| & & \| & & \\
\mathbb{Q} & & 0 & & 0 & & \mathbb{Q} & & \mathbb{Q} & & 0 & &
\end{array}$$

We deduce successively

- $H_3(D^2, D^3_2) \simeq \mathbb{Q}$ and $H_3(D^2, D^3_2) \to H_2(D^3_2)$ is injective, and therefore
- $H_2(D^2) \simeq \mathbb{Q}$.
- $H_3(D^3_2) \to H_3(D^2)$ is surjective

The left end of the upper sequence is thus

$$0 \longrightarrow H_4(D^2) \longrightarrow H_4(D^2, D^3_2) \longrightarrow H_3(D^3_2) \longrightarrow H_3(D^2) \longrightarrow 0$$

with the two inner modules both isomorphic to $\mathbb{Q}$.

4.5 Homology of $M_2(f_t)$

By comparing the homology of $D^2(f_t)$ and $M_2(f_t)$ (which we will shortly determine), we might hope to gain some information about the homology of $D^2(f_t)$. All of the homology groups of $M_2(f_t)$ vanish. This can be seen with the help of the morphism f_{t*} from the long exact sequence of the pair $(U_t, D^2_1(f_t))$ to the long exact sequence of the pair $(X_t, M_2(f_t))$. Because f_t is an isomorphism outside $D^2_1(f_t)$, the morphisms of relative homology groups

$$f_{t*} : H_i(U_t, D^2_1(f_t)) \to H_i(X_t, M_2(f_t))$$

are all isomorphisms. From the top row of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 = H_5(U_t) & \longrightarrow & H_5(U_t, D^2_1) & \longrightarrow & H_4(D^2_1) & \longrightarrow & H_4(U_t) = 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_5(X_t) & \longrightarrow & H_5(X_t, M_2) & \longrightarrow & H_4(M_2) & \longrightarrow & H_4(X_t) = 0
\end{array}$$

we see that $H_5(U_t, D^2_1) \simeq \mathbb{Q}$. Hence $H_5(X_t, M_2) \simeq \mathbb{Q}$ also, and then from the bottom row it follows that $H_4(M_2) = 0$. A similar argument shows that $H_i(M_2) = 0$ for $0 < i < 4$.

In fact Houston shows in [10] by a rather more sophisticated argument that for a stable perturbation f_t of an $\mathcal{A}$-finite germ f_t, all of the $M_k(f_t)$ have the homotopy type of wedges of spheres in middle dimension. In this case the number of spheres in the wedge homotopy-equivalent to $M_2(f_t)$ is 0.

4.6 Relation Between D^2 and M_2

There is a surjective map $f_t^{(2)} : D^2(f_t) \to M_2(f_t)$, $f_t^{(2)}(P, Q) = f_t(P)$. The multiple point spaces of $f_t^{(2)}$ are related to those of f_t, but are not identical. Consider the following maps:

$$\alpha : D^2(f) \to D^2(f_t^{(2)}), \quad (P, Q) \mapsto \big((P, Q), (Q, P)\big)$$
$$\beta : D^2(f_t^{(2)}) \to D^2(f_t), \quad \big((P, Q), (R, S)\big) \mapsto (P, R).$$

Denote by $(1, 2)$ the usual involution on D^2. The diagrams

$$\begin{array}{ccc} D^2(f_t) & \xrightarrow{(1,2)} & D^2(f_t) \\ \alpha\downarrow & & \downarrow\alpha \\ D^2(f_t^{(2)}) & \xrightarrow{(1,2)} & D^2(f_t^{(2)}) \end{array} \qquad \begin{array}{ccc} D^2(f) & \xrightarrow{(1,2)} & D^2(f) \\ \beta\uparrow & & \uparrow\beta \\ D^2(f_t^{(2)}) & \xrightarrow{(1,2)} & D^2(f_t^{(2)}) \end{array}$$

both commute, and $\beta \circ \alpha$ is the identity on $D^2(f)$. It follows that α and β induce morphisms

$$H_i^{\mathrm{Alt}}(D^2(f_t)) \underset{\beta_*}{\overset{\alpha_*}{\rightleftarrows}} H_i^{\mathrm{Alt}}(D^2(f_t^{(2)}))$$

and $\beta_* \circ \alpha_*$ is the identity.

However α is not surjective and β is not injective. Suppose that $(P, Q, R) \in D^3(f_t)$ with P, Q, R pairwise distinct. Then

$$\big((P, Q), (Q, P)\big), \big((P, Q), (Q, R)\big), \big((P, R), (Q, P)\big), \big((P, R), (Q, R)\big)$$

all lie in $D^2(f_t^{(2)})$ and all are mapped by β to (P, Q). And, of these, only $\big((P, Q), (Q, P)\big)$ is in the image of α.

We draw no further conclusion from this, but ask whether further consideration of the multiple point spaces of the map $f_t^{(2)}$ and indeed of $f_t^{(k)}$ for higher k may provide useful information.

Nevertheless, from the vanishing of $H_1(M_2(f_t))$, and the image-computing spectral sequence, we obtain a second argument that $H_1(D^2(f_t)) = 0$.

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