Equisingularity and the Theory of Integral Closure

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Abstract This is an introduction to the study of the equisingularity of sets using the theory of the integral closure of ideals and modules as the main tool. It introduces the notion of the landscape of a singularity as the right setting for equisingularity problems.

Keywords Equisingularity · Multiplicity of ideals and modules · Integral closure of ideals · Integral closure of modules

Introduction

"Let me now take a new tack which promises a better wind. Instead of dealing with a pair of hypersurfaces, let us consider analytic families of hypersurfaces V_r, all having a singular point at the origin and depending on a set of parameters." O. Zariski, Presidential Address, Bulletin A.M.S. 77 No. 4 (1971), 481–491 [41].

Given a family of sets or maps, when are all the members the same? When are some of the members different? Equisingularity is the study of these questions. As Zariski noticed, it is easier to say when a member of family is different, than it is to say when two sets or two maps are the same. Often the change in a single invariant suffices to pick out the members which are out of step with the rest.

A basic question is what do we mean by "the same"? And how do we tell when a family of sets are the same using invariants of the members of the family? These questions are explored in these lectures.

As Zariski indicates earlier in his address, equisingularity had its roots in both differential topology and algebraic geometry, and both areas continue to contribute important ideas. The use of algebraic geometry naturally leads to the use of commutative algebra to count and to control.

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In answering the question of what "the same" means a topologist might ask that the members of the family be homeomorphic; a differential topologist would ask that some of the infinitesimal structure, such as limiting tangent planes and secant lines be preserved as well, while an algebraic geometer might ask that the singularities have the same multiplicity.

In these lectures we work in the complex analytic case using the Whitney conditions or Verdier's W, known to be equivalent in the complex analytic case [38], to say when the members of a family are the same. These conditions imply all three of the above possible answers. The theory of integral closure of ideals and modules provides an algebraic description of these conditions from which we may abstract the invariants which control them in families.

Here is an overview of my current approach to equisingularity questions. Given a set X, decide on the landscape that the set is part of. This means deciding on the allowable families that include the set, and the generic elements that appear in allowable families. Each set should have a unique generic element that it deforms to, and some elements of the topology of this generic element should be important invariants of our set. Describing the connection between the infinitesimal geometry of X and the topology of the generic element related to X is part of understanding the landscape. Based on the allowable deformations, determine the corresponding first order infinitesimal deformations of X. These make up a module $N(X)$. The Jacobian module of X, $JM(X)$ is the module generated by the partial derivatives of a set of defining equations for X. These can be viewed also as the infinitesimally trivial deformations of X. For the case of sets, the invariants we need for checking condition W come from the pair $(JM(X), N(X))$ and $N(X)$ by itself. A change at the infinitesimal level of the family is always tied to a change in the topology of the generic related elements.

Those who have studied maps using stabilizations [31] will recognize many elements of the overview in that context.

This paper is divided into three lectures with an afterword. They are designed to help you reach the point where the overview makes sense. In the afterword we will look at the overview again, using determinantal singularities as an example.

The first lecture introduces the Whitney conditions and Verdier's condition W, and shows how Verdier's condition W can be described using analytic inequalities. In the second lecture, the theory of the integral closure of ideals and modules is introduced, allowing us to recast the analytic inequalities of the first lecture in algebraic terms. This lecture contains a new and shorter proof of the integral closure formulation of Whitney equisingularity, Proposition 2.34. The third lecture introduces the main source of our invariants–the multiplicity of ideals and modules. In applications these multiplicities are infinitesimal objects, being intersection numbers connected with conormal spaces. The polar variety of a module is defined, and in the applications, these are local objects on our families. Through the Multiplicity Polar Theorem 3.22, they are connected to our infinitesimal invariants. The third lecture continues by applying all of these ideas to the study of determinantal singularities, which are a reasonable next step in complexity beyond complete intersections.

For complete intersections our families are obtained by varying the equations directly; for determinantal singularities we cannot vary the equations freely, but we can vary the entries of the matrix defining the singularity freely. This is the connection with complete intersections. However, since determinantal singularities are the inverse images of generic determinantal singularities, the polar varieties of the generic determinantal singularities contribute to the invariants we need to describe Whitney equisingularity in this context. (Cf. Theorem 3.28.)

Since these lectures are meant to be a tool for students to enter the subject, there are many exercises scattered through the lectures. I encourage you to try all of them. There are also some readings which fill in gaps in the proofs or provide deeper understanding. I encourage you to try these as well.

A first reading which gives an overview of how the material in these lectures developed can be found on the conference web site, along with the abstract for the course. It is a PDF of the talk I gave at Aussois in June '15 to celebrate the 70th birthday of Bernard Teissier. Teissier has made all of his papers available on his web site, (webusers.imj-prg.fr/ bernard.teissier/articles-Teissier.html) and many of the suggested readings can be found there.

It is a pleasure to thank the organizers of the conference for giving me the chance to speak about these beautiful ideas, and to share some of my thoughts about them.

1 Equisingularity Conditions

We start with some notation to describe a family of sets. In the diagram:

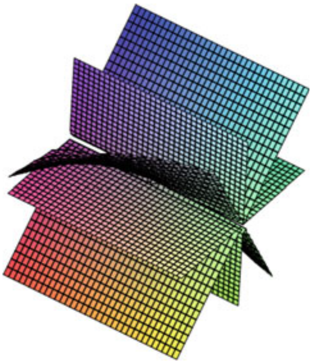
$$\begin{array}{ccccc} X^d(0) & \subset & \mathcal{X}^{d+k} & \subset & Y \times \mathbb{C}^N \\ \downarrow p_Y & & \downarrow \pi_Y & & \\ 0 \in & & Y = \mathbb{C}^k & & \end{array}$$

the parameter space is Y, $X(0)$ denotes the fiber of the family over $\{0\}$, $\mathcal{X}^{d+k}$ denotes the total space of the family which is contained in $Y \times \mathbb{C}^N$. We usually assume $Y \subset \mathcal{X}^{d+k}$, and $\mathcal{X} = F^{-1}(0)$, $X(y) = {f_y}^{-1}(0)$, where $f_y(z) = F(y, z)$.

Given a family of map germs as above, we say the family is holomorphically trivial if there exists a holomorphic family of origin preserving bi-holomorphic germs r_y such that $r_y(X(0)) = X(y)$. If the map-germs are only homeomorphisms we say the family is C^0 trivial.

Every subject needs a good example to start with. Here is ours:

Example 1.1 Let $\mathcal{X}$ be the family of four moving lines in the plane with equation $F(x, y, z) = xz(z + x)(z - (1 + y)x) = 0$. Here y is the parameter, the x and z axis are fixed, as is the line $z + x = 0$ while the line $z - (1 + y)x = 0$ moves with y. Here is a picture of the total space of the family:



This family is not holomorphically trivial as the next exercise shows, but it should be equisingular for any reasonable definition of equisingularity.

Problem 1.2 *Show that the family of 4 lines is not homomorphically trivial by following the hints and proving them: If r_y is a trivialization of the family of sets, $Dr_y(0)$ must carry the tangent lines of $X(0)$ to $X(y)$. If a linear map preserves the lines defined by $x = 0, z = 0, z = -x$ then the linear map must be a multiple of the identity. Hence r_y can't map $z = x$ to $z = (1 + y)x$, $y \neq 0$.*

Thus, we need a notion of equisingularity that is less restrictive than holomorphic equivalence.

The Whitney conditions imply C^0 triviality but also imply the family is well-behaved at the infinitesimal level.

If X is an analytic set, X_0 the set of smooth points on X, Y a smooth subset of X, then the pair (X_0, Y) satisfies **Whitney's condition A** at $y \in Y$ if for all sequences $\{x_i\}$ of points of X_0,

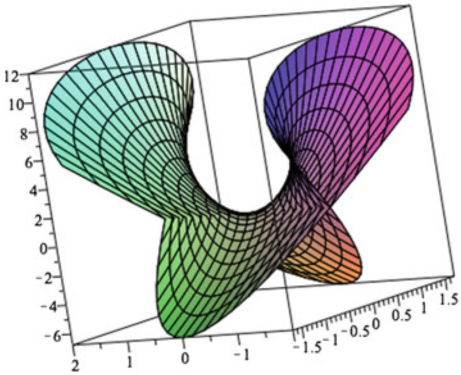
$$\begin{matrix} \{x_i\} \to y \\ \{TX_{x_i}\} \to T \end{matrix} \Rightarrow T \supset TY_y$$

The pair (X_0, Y) satisfies **Whitney's condition B** at $y \in Y$ if for all sequences $\{x_i\}$ of points of X_0,

$$\begin{matrix} \{x_i\} \to y \\ \{TX_{x_i}\} \to T \\ sec(x_i, \pi_Y(x_i)) \to L \end{matrix} \Rightarrow T \supset L$$

Problem 1.3 *Show that the family of 4 lines satisfies the Whitney conditions. (Hint: The family consists of submanifolds meeting pairwise transversely.)*

Example 1.4 This is a famous example used in many singularities talks. $\mathcal{X}$ is defined by $F(x, y, z) = z^3 + x^2 - y^2z^2 = 0$. The members of the family $X(y)$ consist of node singularities where the loop is pulled smaller and smaller as y tends to zero, becoming a cusp at $y = 0$. Here is a picture:



The singular locus is the y-axis. Whitney A holds because every limiting tangent plane contains the y-axis. But Whitney B fails. Notice that the parabola $z = y^2$ is in the surface, and letting $x_i = (0, t_i, {t_i}^2)$ and $y_i = (0, t_i, 0)$, t_i any sequence tending to 0, we see that the limiting secant line is the z-axis, while the limiting tangent plane along this curve is the xy-plane.

We see that the dimension of the limiting tangent planes at the origin is 1, while it is zero elsewhere on the y-axis. This kind of drastic change at the infinitesimal level is prevented by the Whitney conditions.

Reading You can read about the Whitney conditions in many places. Two references are the first chapter of [22], and Chap. III of [38]. The latter is more in the spirit of the way we are developing the subject, though harder. When you begin to study the polar varieties of a module in the third lecture, the lectures of Teissier [36] on the historical development of the polar variety of a space, and its connections to the Whitney conditions are very interesting. (Among other things, he explains why they are called "polar" varieties.)

Verdier's Condition W

The next condition, while equivalent to the Whitney conditions in the complex analytic case (proved by Teissier [38]) is easier to work with using algebra.

Condition W says that the distance between between the tangent space to X at a point x_i of X_0 and the tangent space to Y at y goes to zero as fast as the distance between x_i and Y. We first need to define what we mean by the distance between two linear spaces.

Suppose A, B are linear subspaces at the origin in $\mathbb{C}^N$, then define the distance from A to B as:

$$\operatorname{dist}(A, B) = \sup_{\substack{u \in B^\perp - \{0\} \\ v \in A - \{0\}}} \frac{\|(u, v)\|}{\|u\| \, \|v\|}.$$

In the applications B is the "big" space and A the "small" space. The inner product is the Hermitian inner product when we work over $\mathbb{C}$. The same formula also works over $\mathbb{R}$.

Example 1.5 For this example, we work with linear subspaces of $\mathbb{R}^3$. Let $A = x$-axis, B a plane with unit normal u_0, then the formula for the distance from A to B reduces to $\cos\theta$, where θ is the small angle between u_0 and the x-axis, in the plane they determine. So when the distance is 0, B contains the x-axis.

We recall Verdier's condition W.

Definition 1.6 Suppose $Y \subset \bar{X}$, where X, Y are strata in a stratification of an analytic space, and $\text{dist}(TY_0, TX_x) \leq C\text{dist}(x, Y)$ for all x close to Y. Then the pair (X, Y) satisfies **Verdier's condition** W at $0 \in Y$.

Problem 1.7 *Show that W fails for Teissier's example for X_0, Y where Y is the y-axis at the origin.*

As a first step to understanding the condition, we consider the case where X is a hypersurface in $\mathbb{C}^n$. We would like to re-write this condition in terms of F where F defines X. This will allow us to develop an algebraic formulation of the W condition.

Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ a family of hypersurfaces in $Y^k \times \mathbb{C}^{n+1}$.

Proposition 1.8 *Condition W holds for $(\mathcal{X}_0, Y)$ at $(0, 0)$ if and only if there exists U a neighborhood of $(0, 0)$ in $\mathcal{X}$ and $C > 0$ such that*

$$\|\frac{\partial F}{\partial y_l}(y, z)\| \leq C \sup_{i,j} \|z_i \frac{\partial F}{\partial z_j}(y, z)\|$$

for all $(y, z) \in U$ and for $1 \leq l \leq k$.

Proof In this set-up, Y is a k-plane, so we will set $A = Y$, and calculate the distance between Y and a tangent plane to $\mathcal{X}_0$ at (y, z) which is our B. At a smooth point of $\mathcal{X}^{k+n}$, we can use $\overline{DF(y, z)}/\|DF(y, z)\|$ for $u \in B^{\perp}$, and the standard basis for the vectors from A.

Then the distance formula says that condition W holds if and only if

$$\sup_{1\leq l\leq k} \frac{\|\frac{\partial F}{\partial y_l}(y, z)\|}{\|DF(y, z)\|} \leq C''\text{dist}((y, z), Y) = C' \sup_{1\leq i\leq n+1} \|z_i\|$$

This is equivalent to

$$\|\frac{\partial F}{\partial y_l}(y, z)\| \leq C \sup_{1\leq i\leq n+1} \|z_i\| \sup_{1\leq j\leq n+1} \|\frac{\partial F}{\partial z_j}(y, z)\|$$

From which the desired result follows. □

Denote the ideal generated by the partial derivatives of F with respect to the z variables by $J_z(F)$, and the ideal generated by z_j by m_Y. Then $z_i \frac{\partial F}{\partial z_j}$ are a set of generators for $m_Y J_z(F)$. The inequality above says that the partial derivatives of F with respect to y_l go to zero as fast as the ideal $m_Y J_z(F)$. We will examine the implications of this in the next section.

Reading After you read a little about the integral closure of ideals, reading pp. 589–605 [37] will give you a good background on the integral closure approach to Whitney equisingularity for hypersurfaces with isolated singularities.

2 The Theory of Integral Closure of Ideals and Modules

Many operations on ideals and submodules of a free module come from operations on rings. (For other examples of this, see [14, 15, 18].)

We illustrate this idea by reviewing the notions of the integral closure of a ring and the normalization of an analytic space.

Definition 2.1 Let A, B be commutative Noetherian rings with unit, $A \subset B$ a subring. Then $h \in B$ is integrally dependent on A if there exists a monic polynomial $f(T) = T^n + \sum_{i=0} f_i T^i$, $f_i \in A$ such that $f(h) = 0$. The integral closure of A in B consists of all elements of B integrally dependent on A.

Example 2.2 Let A be the ring of convergent power series in the germs t^2 and t^3, denoted $\mathbb{C}\{t^2, t^3\}$, $B = \mathbb{C}\{t\}$. Then if $f(T) = T^2 - t^2$ we have $f(t) = 0$, so t is integrally dependent on A. In fact, B is the integral closure of A in B.

Definition 2.3 Let A be the local ring of an analytic space X, x, B the ring of meromorphic functions on X at x; the space associated with the integral closure of A in B is the normalization of X.

Example 2.4 Let $A = \mathbb{C}\{t^2, t^3\}$, $B = \mathbb{C}\{t\}$. Then A is the local ring at the origin of the cusp $x^3 - y^2 = 0$, and since $t^3/t^2 = t$, the ring of meromorphic functions on X at the origin is $\mathbb{C}\{t\}$. So by the previous example the normalization of the cusp is a line.

In this context a ring A is normal if the integral closure of A in its quotient field is A. A space germ is normal if its local ring is normal. Normal spaces have nice properties–they are non-singular in codimension 1 and the Riemann removable singularities theorem is true for them. Given a space germ X, we always have a map π_{NX} from the normalization of X, denoted NX, to X which is finite and generically 1-1. NX and π_{NX} are unique up to holomorphic right equivalence. You can read proofs of these facts in [23] pp. 154–163, working backwards as necessary.

The following exercise is easy assuming the facts in the last paragraph.

Problem 2.5 *Show that the normalization of an irreducible curve germ X, x is $\mathbb{C}, 0$.*

If you know a little bit about singularities of maps, the next exercise is also easy.

Problem 2.6 *Suppose* $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $n < p$ *and* f *is a finitely determined map-germ. Show* $(\mathbb{C}^n, 0)$, f *is a normalization of the image of* f.

Basic Results from the Theory of Integral Closure for Ideals

The operation of integral closure of rings creates, as we shall see, an operation on ideals, the operation of forming the integral closure of I, which is an ideal, denoted $\overline{I}$. Assume I is an ideal in $\mathcal{O}_{X,x}$, $f \in \mathcal{O}_{X,x}$. In discussing the properties of integral closure, sometime we work on a small neighborhood of X. In this case, I refers to the coherent sheaf I generates on U.

List of Basic Properties f is integrally dependent on I if one of the following equivalent conditions obtain:

(i) There exists a positive integer k and elements a_j in I^j, so that f satisfies the relation $f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + a_k = 0$ in $\mathcal{O}_{X,0}$.

(ii) There exists a neighborhood U of 0 in $\mathbb{C}^N$, a positive real number C, representatives of the space germ X, the function germ f, and generators $g_1, \ldots, g_m$ of I on U, which we identify with the corresponding germs, so that for all x in U we have: $\|f(x)\| \leq C \max\{\|g_1(x)\|, \ldots, \|g_m(x)\|\}$.

(iii) For all analytic path germs $\phi : (\mathbb{C}, 0) \to (X, 0)$ the pull–back $\phi^* f = f \circ \phi$ is contained in the ideal generated by $\phi^*(I)$ in the local ring of $\mathbb{C}$ at 0. If for all paths $\phi^* f$ is contained in $\phi^*(I) m_1$, then we say f is strictly dependent on I and write $f \in I^\dagger$.

Let NB denote the normalization of the blowup of X by I, $\bar{D}$ the pullback of the exceptional divisor of the blowup of X by I to NB by the normalization map. Then we have:

(iv) For any component C of the underlying set of $\bar{D}$, the order of vanishing of the pullback of f to NB along C is no smaller than the order of the divisor $\bar{D}$ along C. This implies that the pullback of f lies in the ideal sheaf generated by the pullback of I.

The set of all elements of $\mathcal{O}_{X,x}$ which are integrally dependent on I is the *integral closure of I* and is denoted $\overline{I}$.

Proposition 2.7 *If I is an ideal in* $\mathcal{O}_{X,x}$, *then so is* $\overline{I}$.

Proof We use property (iii). Let $\phi : (\mathbb{C}, 0) \to (X, 0)$ be any analytic curve, $g \in \mathcal{O}_{X,x}$, f_1, f_2 in $\overline{I}$. Then $(gf_1 + f_2) \circ \phi = (g \circ \phi)(f_1 \circ \phi) + (f_2 \circ \phi) \in \phi^*(I)$, since $\phi^*(I)$ is an ideal in $\mathcal{O}_1$. □

The proof of this for general rings is Corollary 1.3.1 of [35].

The first property is usually taken as the definition, and shows that integral dependence is an algebraic idea. This permits the extension of the concept to ideals in any ring. For the development of the idea of the integral closure of an ideal or module from the algebraic point of view see [35].

The second property is used to control equisingularity conditions. It already appeared in the discussion of Verdier's condition W in the hypersurface case earlier, and we will revisit it shortly.

The third property is convenient for computations, and often for proofs as the proof of the previous proposition shows. It is also helpful in understanding conditions involving limits. In the analytic setting, definitions that use sequences of points, such as the Whitney conditions, can be checked with curves, often leading to an interpretation of the condition in terms of the integral closure of an ideal or module. We will see an example of this in the study of limiting tangent hyperplanes in the next section.

The notion of strict dependence defined in the third property is used to describe properties like Whitney A, or Thom's A_f condition where integral dependence is insufficient–see the problem later on about Whitney A.

Given a curve $\phi(s)$, and a germ f, if $f \circ \phi$ is defined, it is equal to $cs^r \bmod m_1^{r+1}$ for $c \neq 0$ for some r. We call r the order of f on ϕ and write $f_\phi = r$, and J_ϕ for the order of an ideal J on ϕ.

Because the exceptional divisor of the blow-up of the Jacobian ideal tracks limiting infinitesimal information, the fourth property is perhaps the most important. Since NB is normal, each component of the exceptional divisor is generically a smooth submanifold of a manifold, so the ideal vanishing on the component is locally principal. This means we can talk about the order of vanishing on each component. The order of the divisor $\bar{D}$ is just the order of vanishing along the component of the pull-back of I to NB. Concretely, pick a local generator u of the ideal of the component, and write the elements of I in terms of u. The smallest power of u that appears is the order of I along C.

The fourth property also shows how a closure operation on rings gives a closure operation on ideals– start with a ring and an ideal, enlarge the ring by a closure operation, look at the ideal generated in the new ring, then intersect with the original ring to define the closure operation on the ideal.

Reading For detailed proofs of the equivalences between these properties see [28] pp. 18–27. You can download this paper from Teissier's list of publications–it is #15. Try this after reading the proofs of the equivalences contained here.

In the next example, we practice using the first property.

Example 2.8 Let $A = \mathcal{O}_2$, $I = (x^n, y^n)$. Suppose $f = x^i y^j$, $i + j \geq n$. Consider the monic polynomial $h(T) = T^n - (x^n)^i (y^n)^j$. Since $(x^n)^i (y^n)^j$ is in $(I^i)(I^j) \subset I^{i+j} \subset I^n$, and $h(f) = 0$, then $f \in \overline{I}$.

Now we do a computation using the third property.

Example 2.9 Let $A = \mathcal{O}_2$, $I = (x^a, y^b)$. Given $m = x^i y^j$ define the weight of m to be $bi + aj$, given $f(x, y)$, define the weight of f to be the minimum weight of all monomials appearing in a power expansion of f. We will show that $\overline{I}$ consists of all f such that weight of $f \geq ab$.

First, we'll show weight of $m \geq ab$ implies $m \in \overline{I}$. It suffices to check this for curves $\phi(t) = (t^r, t^s)$ as higher order terms don't affect the order of I or m on the curve. Since $\overline{I}$ is an ideal, this will show that $f \in \overline{I}$.

We have I_ϕ =min$\{ra, sb\}$; assume $ra \leq sb$.

It is convenient to think of the monomial $x^i y^j$ as the point (i, j) in the xy-plane. Consider the parallel lines $rx + sy = c$. Then if m is any monomial on this line, $m_\phi = c$, and $m_\phi > c$ if m lies above this line. If the weight of $m \geq ab$ then m lies above or on the line connecting $(a, 0)$ and $(0, b)$, so it will lie above or on any line passing through $(a, 0)$, which lies below or on $(0, b)$. This implies that $m_\phi \geq ra$ and shows $m \in \overline{I}$.

Suppose the power expansion of f contains a monomial m which lies below the line connecting $(a, 0)$ and $(0, b)$. Then the convex hull of the monomials appearing in f has a vertex m' which lies below the line connecting $(a, 0)$ and $(0, b)$. We can find a line passing through this vertex which lies below $(a, 0)$ and $(0, b)$. Then for the curve ψ defined by this line,

$$f_\psi = m'_\psi < I_\psi$$

which shows that $f \notin \overline{I}$.

This kind of reasoning is very useful in studying properties of ideals which are well connected to their Newton polygons. In this example, the Newton polygon of I is all the points of $\mathbb{R}^2$ above or on the line connecting $(a, 0)$ and $(0, b)$ in the first quadrant. For more examples and details see [39], which is #46 on Teissier's publication list or [34].

Next, we use property 2 to characterize Verdier's W in the hypersurface case.
Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ a family of hypersurfaces in $Y^k \times \mathbb{C}^{n+1}$.

Proposition 2.10 *Condition W holds for* $(\mathcal{X}_0, Y)$ *at* $(0, 0)$ *if and only if* $\frac{\partial F}{\partial y_l} \in \overline{m_Y J_z(F)}$ *for* $1 \leq l \leq k$.

Proof By the last proposition of the first section we know that W holds if and only if

$$\|\frac{\partial F}{\partial y_l}(y, z)\| \leq C \sup_{i,j} \|z_i \frac{\partial F}{\partial z_j}(y, z)\|$$

But, by property 2 this is equivalent to $\frac{\partial F}{\partial y_l} \in \overline{m_Y J_z(F)}$ for $1 \leq l \leq k$. □

If we have a curve ϕ on $\mathcal{X}^{k+n}$, $\phi(0) = 0$, and the image of ϕ in $\mathcal{X}_0^{k+n}$ except at 0, and $J(F)_\phi = r$ then we can calculate the limiting tangent hyperplane to $\mathcal{X}^{k+n}$ along ϕ as

$$\underset{s \to 0}{limit}(1/s^r)(DF(\phi(s)))$$

If $\frac{\partial F}{\partial y_l} \in \overline{J_z(F)}$ for $1 \leq l \leq k$, then the limiting plane is never vertical, but it does not necessarily contain Y.

Problem 2.11 *Show that if* $\frac{\partial F}{\partial y_l}$ *for* $1 \leq l \leq k$ *is strictly dependent on* $J_z(F)$ *then every limit of tangent planes along every curve* ϕ *not in* $V(J_z(F))$ *contains* Y.

Problem 2.12 *Show that if* $\frac{\partial F}{\partial y_l}$ *for* $1 \leq l \leq k$ *is strictly dependent on* $J_z(F)$ *then WA holds.*

We will prove a few of the implications showing the equivalence of the basic properties.

Proposition 2.13 *Property 1 implies property 3*

Proof Let f satisfy the relation $f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + a_k = 0$ in $\mathcal{O}_{X,0}$, and let $\phi : \mathbb{C}, 0 \to X, 0$. Choose $g \in I$ such that $g_\phi = I_\phi$. We may assume the image of ϕ does not lie in $V(I)$. Then

$$\frac{(f \circ \phi)^k}{(g \circ \phi)^k} + \frac{a_1 \circ \phi}{(g \circ \phi)} \frac{(f \circ \phi)^{k-1}}{(g \circ \phi)^{k-1}} + \cdots + \frac{a_{k-1} \circ \phi}{(g \circ \phi)^{k-1}} \frac{(f \circ \phi)}{(g \circ \phi)} + \frac{a_k \circ \phi}{(g \circ \phi)^k} = 0$$

and $\frac{a_i \circ \phi}{(g \circ \phi)^i}$ is holomorphic for all i. Since $\mathcal{O}_1$ is normal, it follows that $\frac{(f \circ \phi)}{(g \circ \phi)}$ is holomorphic, hence $f \circ \phi \in \phi^*(I)$.

Proposition 2.14 *Property 3 implies property 4*

Proof We will only prove this for the case where $V(I) = 0$.

Consider the components $\{C_i\}$ of $\bar{D}$. Since NB is normal and the C_i have codimension 1, we can pick out points c_i on each C_i and curves $\tilde{\phi}_i$, such that $\tilde{\phi}_i(0) = c_i$, and $\tilde{\phi}_i$ is transverse to C_i. We can choose c_i so that $\pi^*_{NB}(I)$ vanishes only on C_i in a neighborhood of c_i, and the same is true for $f \circ \pi_{NB}$. If u_i defines C_i at c_i, then we have $f \circ \pi_{NB} = h_i {u_i}^{f_i}, h_i$ a unit. The exponent f_i is the order of vanishing of f along C_i. Since $\tilde{\phi}_i$ is transverse to C_i at c_i, $u_i \circ \phi_i(t) = t$, so $f \circ \pi_{NB} \circ \phi_i(t) = h'_i(t) t^{f_i}$, h' a unit.

We can also find local generators of $\pi^*_{NB}(I)$ of form ${u_i}^{I_i}$ where I_i is the order of I along C_i. Now $\pi_{NB} \circ \tilde{\phi}_i$ is a map from $\mathbb{C}, 0 \to X, 0$, since $\pi_{NB}(C_i) = 0$, and hence $\pi_{NB}(c_i) = 0$. (This is the reason for restricting to this case.) Hence, if property 3 holds, $f_i \geq I_i$ for all i. If we work at any point of $\bar{D}$ since $\pi^*_{NB}(I)$ is principal, we can find $g \circ \pi_{NB}$ a local generator then $f \circ \pi_{NB}/g \circ \pi_{NB}$ is a meromorphic function which is well defined off a set of codimension 2. Since NB is normal, the function is analytic, so $f \circ \pi_{NB} \in \pi^*_{NB}(I)$. □

Proposition 2.15 *Property 4 implies property 2*

Proof Choose a compact neighborhood U of 0, and consider its inverse image in NB. The inverse image must be compact as well. So, since $f \circ \pi_{NB} \in \pi^*_{NB}(I)$, we can cover $\pi^{-1}_{NB}(U)$ with a finite number of sets and choose elements of I such that

$$\|f \circ \pi_{NB}(p')\| \leq C \max\{\|g_1 \circ \pi_{NB}(p')\|, \ldots, \|g_m \circ \pi_{NB}(p')\|\}$$

holds on $\pi_{NB}^{-1}(U)$. Then it is clear that

$$\|f(\pi_{NB}(p'))\| \leq C \max\{\|g_1(\pi_{NB}(p'))\|, \ldots, \|g_m(\pi_{NB}(p'))\|\}.$$

Since π_{NB} surjects on U, this finishes the proof. □

There is a nice corollary of the method of proof used in the previous proposition and of property 2 which we now describe. Given a subset S of an analytic set X, $f\ X, S \to Y, y$ where $S = f^{-1}(y)$ denotes the germ of an analytic map along S. Given an ideal I in $\mathcal{O}_{Y,y}$, $f^*(I)$ denotes the ideal sheaf along S obtained by pulling back I by f.

Proposition 2.16 *Suppose $f\ X, S \to Y, y$ where $S = f^{-1}(y)$, f proper and surjective. Suppose I an ideal of $\mathcal{O}_{Y,y}$, $h \in \mathcal{O}_{Y,y}$. Then $h \in \overline{I}$ if and only if $h \circ f \in \overline{f^*(I)}$ along S.*

Proof Since f is proper, S is compact, and as in the last proof we can cover S with a collection of neighborhoods such that on the union the germ of a function along S is in $\overline{f^*(I)}$ if an only if it satisfies an analytic inequality of the type described by property 2. Since f is surjective, the inequalities push down/pullback to Y, y. □

Problem 2.17 *Use the finite map $f(x, y) = (x^b, y^a)$ to give another proof that $\overline{(x^a, y^b)}$ consists of all g such that weight of $g \geq ab$.*

We have Proposition 2.10 to describe W for hypersurfaces, but what about sets of higher codimension? We will see that the theory of integral closure of modules provides the tools we need to describe the higher codimension case.

The Theory of Integral Closure for Modules: Motivation

Verdier's condition W is based on the distance between the tangent space TX_x to X at smooth points x and the tangent space T to Y. Recall this distance is defined as

$$\operatorname{dist}(T, TX_x) = \sup_{\substack{u \in TX_x^{\perp} - \{0\} \\ v \in T - \{0\}}} \frac{\|(u, v)\|}{\|u\| \, \|v\|}.$$

If $u \in TX_x^{\perp} - \{0\}$, then the set of points perpendicular to u consists of a hyperplane which contains TX_x. These hyperplanes are called *tangent hyperplanes*; denote a tangent hyperplane to X, x by H_x, and the collection of all tangent hyperplanes to X, x by $C(X)_x$. Then we can rephrase the distance formula as

$$\operatorname{dist}(T, TX_x) = \sup_{H_x \in C(X)_x} \operatorname{dist}(T, H_x)$$

If $X = F^{-1}(0)$ where $F\ \mathbb{C}^n \to \mathbb{C}^p$, then at a smooth point p of X, the projectivisation of the rowspace of the matrix of partial derivatives of F is $C(X)_p$. Since the tangent hyperplanes are what we need to control the distance between the tangent space of X, p and $TY, 0$, this suggests we should look at the module generated by the partial derivatives of F denoted $JM(X)$, just as we looked at $J(F)$ in the hypersurface case.

Basic Results from the Theory of Integral Closure for Modules

Notation: $M \subset N \subset F^p$, F^p a free $\mathcal{O}_{X,x}$ module of rank p, M, N submodules of F. If M is generated by g generators $\{m_i\}$, then let $[M]$ be the matrix of generators whose columns are the $\{m_i\}$.

We will develop properties for modules similar to those for ideals; however a convenient entry way into the theory is:

Definition 2.18 If $h \in F^p$ then h is integrally dependent on M, if for all curves ϕ, $h \circ \phi \in \phi^*(M)$. The integral closure of M denoted $\overline{M}$ consists of all h integrally dependent on M.

A good very basic reference on properties of integral closure of modules is [9, pp. 301–307]. The development of these ideas in the setting of modules over commutative rings can be found in [35] starting with the chapter "Integral Closure of Modules".

Problem 2.19 *$\overline{M}$ is a module, $\overline{\overline{M}} = \overline{M}$*

Example 2.20 Let $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$, then $\overline{M} = m_2\mathcal{O}_2^2$.

It is clear that $\overline{M} \subset m_2\mathcal{O}_2^2$; we will show that $\begin{pmatrix} y \\ 0 \end{pmatrix} \in \overline{M}$.

Given a curve ϕ we can assume $y_\phi < x_\phi$ otherwise $\begin{pmatrix} y \circ \phi \\ 0 \end{pmatrix} \in \begin{pmatrix} x \circ \phi \\ 0 \end{pmatrix} \mathcal{O}_1$.

Then

$$\begin{pmatrix} y \\ 0 \end{pmatrix} \circ \phi = \begin{pmatrix} y \\ x \end{pmatrix} \circ \phi - x/y \circ \phi \begin{pmatrix} 0 \\ y \end{pmatrix} \circ \phi$$

where $x/y \circ \phi \in \mathcal{O}_1$.

Connection with the Theory of Integral Closure of Ideals I

Notation: Given an element $h \in F$ and a submodule M, then (h, M) denotes the submodule generated by h and the elements of M. Given a submodule N of F, $J_k(N)$ denotes the ideal generated by the set of k by k minors of a matrix whose columns are a set of generators of N. If M is an $\mathcal{O}_X$ module then the rank of M is k on a component V of X if $J_k(M) \neq (0)$ on V and k is the largest value for which this is true.

Theorem 2.21 (Jacobian principle) *Suppose the rank of (h, M) is k on each component of (X, x). Then $h \in \overline{M}$ if and only if $J_k(h, M) \subset \overline{J_k(M)}$*

Proof The complete proof appears in [9, p. 304]. The easy part is to show that $h \in \overline{M}$ implies $J_k(h, M) \subset \overline{J_k(M)}$.

We have

$$\phi^*(J_k(h, M)) = J_k(\phi^*(h, M)) = J_k(\phi^*(M) = \phi^*(J_k(M))$$

which implies the result.

The problem in the other direction is checking for curves which lie in the set of points where the rank is less than maximal, so that all the elements of $J_k(h, M)$ vanish, but h doesn't vanish. We approach this problem in two steps.

Assume first that the image of our curve ϕ does not lie entirely in $V(J_k(h, M))$.

Then, by hypothesis $\phi^*(J_k(h, M)) = \phi^*(J_k(M)) \neq 0$. So, there is a non-zero minor of the matrix of generators $[M]$, of M, $J(I, K)$ such that $J(I, K) \circ \phi$ is generator of $\phi^*(J_k(M))$. Here I is an index of the rows and K an index of the columns which comprise the $k \times k$ submatrix whose determinant is $J(I, K)$.

Consider $M_{I,K}$ the submodule of F^k defined using as matrix of generators the square submatrix of $[M]$ whose determinant is $J(I, K)$, and let h_I be the element obtained from h by using the entries indexed by I.

Applying Cramer's rule, we have that $h_I \circ \phi \in \phi^*(M_{I,K})$, where $h_I \circ \phi(t) = ([M_{I,K}] \circ \phi(t))\xi(t)$ for some column vector $\xi(t)$, given by composing the output of Cramer's rule with $\phi(t)$. Let $[M_K]$ be the submatrix of $[M]$ using the columns indexed by K. Consider $h_I \circ \phi(t) - ([M_K] \circ \phi(t))\xi(t)$. If this is zero, we have checked the condition for ϕ. If it is not zero, then $\phi^*(h, M)$ has rank greater than k which is a contradiction.

Now suppose the image of ϕ does lie entirely in $V(J_k(h, M))$, so $\phi^*(J_k(h, M)) = 0$.

Here the argument breaks into two parts again. We first assume X is smooth so that we can vary the curve freely, then we use the resolution of singularities to reduce to the smooth case.

Suppose $\phi^*(M) \neq \phi^*(h, M)$. Now, by the Artin–Rees theorem we know that there exists $\nu_0 > 0$, $\nu_0 \in \mathbb{Z}$ such that

$$m_1^l \mathcal{O}_1^p \cap \phi^*(h, M) = m_1^{l-\nu_0}(m_1^{\nu_0} \mathcal{O}_1^p \cap \phi^*(h, M)).$$

This implies, that in fact,

$$\phi^*(M) \neq \phi^*(h, M) \mod m_1^l \mathcal{O}_1^p$$

for any $l > \nu_0$. If not, then $h \circ \phi = g \mod \phi^*(M)$, with $g \in m_1^l \mathcal{O}_1^p$, and so

$$g \in m_1^l \mathcal{O}_1^p \cap \phi^*(h, M),$$

hence

$$g, h \circ \phi \in \phi^*(M) + m_1(m_1^{\nu_0} \mathcal{O}_1^p \cap \phi^*(h, M)).$$

Since $\phi^*(M) + m_1\phi^*(h, M) = \phi^*(h, M)$, Nakayma's lemma would imply the result.

Now choose $l > \nu_0$; since X is smooth, we can find a curve ϕ_1, by changing terms of the power series expansion ϕ of order $\geq l$, such that the image of ϕ_1 does not lie in $V(J_k(h, M))$.

This implies that

$$\phi_1^*(M) = \phi^*(M) \mod m_1^l \mathcal{O}_1^p$$

$$\phi_1^*(h, M) = \phi^*(h, M) \mod m_1^l \mathcal{O}_1^p$$

$$\phi_1^*(M) = \phi_1^*(h, M)$$

This gives a contradiction in this case.

If X is not smooth, then we can make a resolution, $\tilde{X}$, π, of singularities of X, lift ϕ to $\tilde{\phi}$ on $\tilde{X}$. Then $\phi^*(M) \neq \phi^*(h, M)$ if and only if $\tilde{\phi}^*\pi^*(M) \neq \tilde{\phi}^*\pi^*(h, M)$, then we can again vary $\tilde{\phi}^*$ as before. □

If $h \in \overline{M}$, this last proposition allows us to to do more than show $h \in M$ along curves.

Proposition 2.22 *Suppose $h \in \overline{M}$, then there exists an open cover $\{U_{I,K}\}$ of the complement of $V(J(M))$, such that on each $U_{I,K}$, $h = [M]\xi_{I,K}$, where the entries of $\xi_{I,K}$ are locally bounded on $U_{I,K}$.*

Proof The open cover $\{U_{I,K}\}$ is constructed by constructing an open cover $\{V_{I,K}\}$ of the fiber over the origin in $NB_{J(M)}(X)$ such that on each $V_{I,K}$, the pullback of $J(I, K)$ is a local generator of the pullback of $J(M)$. Then Cramer's rule applies, and the pullbacks of the $\xi_{I,K}$ are holomorphic, hence locally bounded on the images of the $V_{I,K}$ which are the $U_{I,K}$. □

As another application we can develop the analogue of property 2 for ideals.

Proposition 2.23 ([9], Proposition 1.11) *Suppose $h \in \mathcal{O}_{X,x}^p$, M a submodule of $\mathcal{O}_{X,x}^p$ of generic rank k on each component of X. Then $h \in \overline{M}$ if and only if for each choice of generators $\{s_i\}$ of M, there exists a constant $C > 0$ and a neighborhood U of x such that for all $\psi \in \Gamma(Hom(\mathbb{C}^p, \mathbb{C}))$,*

$$\|\psi(z) \cdot h(z)\| \leq C \sup_i \|\psi(z) \cdot s_i(z)\|$$

for all $z \in U$.

For each choice of ψ, the $\{\psi \cdot s_i(z)\}$ give a linear combination of the rows of $[M]$ at each point, while $\psi(z) \cdot h(z)$ is the analogous combination of the entries of h. So the inequality of the theorem relates the size of row vectors of $[M(x)]$ to corresponding combinations of the entries of h. The constant C and the neighborhood U depend on h and M but not on ψ.

Proof We will use the Jacobian principle to show that the inequality implies the integral closure inclusion, by using special ψ_i.

Let S_I be a $k \times (k-1)$ submatrix of $[M]$, going through all such submatrices as I varies, let h_I be a k-tuple gotten by dropping the same entries from h as rows from $[M]$ in forming S_I. Let $\psi_I(z)(h(z)) = \det[h_I(z), S_I(z)]$. Note that $\psi_I(z)s_i(z) = \det[s_i(z), S_I(z)]$, a generator of $J_k(M)$.

The inequality which we are assuming then shows that $J_k(h, M) \subset \overline{J_k(M)}$, which gives the result by the Jacobian principle.

A weaker version of the other direction is easy; if $h \in \overline{M}$, then for any curve ϕ, $(\psi(z) \cdot h(z)) \circ \phi \in \phi^*(\{\psi(z) \cdot s_i(z)\})$, hence $(\psi(z) \cdot h(z)) \in \overline{(\{\psi(z) \cdot s_i(z)\})}$. Then the result follows by property 2 for ideals. However, here the constant does depend on ψ.

Instead we argue like this. Let $\{s_i\}$ be a set of generators of M. Applying property 2 to the finite set of elements $\{g_i\}$ that make up the numerators of the entries of the $\xi_{I,K}$ in the last proposition, we have that there exists U and C such that if g_i is such a numerator, then

$$\|g_i(z)\| \leq C \sup \|J_{I,K}(z)\|.$$

We have that $J_{I,K}(z)h(z) = \sum g_i s_i$ for appropriate g_i. Then working first at $z \notin V(J(M))$

$$\|\psi(z) \cdot h(z)\| = \|\sum (g_i/J(I,K))(z)\psi(z) \cdot s_i(z)\| \leq CN \sup_i \|\psi(z) \cdot s_i(z)\|$$

where N is the number of terms in the sum. Since the inequality is between continuous functions and holds on an open dense subset of U it holds on U. □

Corollary 2.24 *Suppose $h \in \mathcal{O}^p_{X,x}$, M a submodule of $\mathcal{O}^p_{X,x}$ of generic rank k on each component of X. Then $h \in \overline{M}$ if and only if for each choice of generators $\{s_i\}$ of M, there exists a constant $C > 0$ and a neighborhood U of x such that for all $T \in \mathbb{C}^p$,*

$$\|T \cdot h(z)\| \leq C \sup_i \|T \cdot s_i(z)\|$$

for all $z \in U$.

Proof In one direction, take ψ to be constant; in the other we can replace T by ψ, using the fact that the constant C is independent of the choice of T. □

The corollary reflects the equivalence of $h \in \overline{M}$ and $\rho(h) \in \mathcal{M}$. (The notions of $\rho(h)$, $\mathcal{M}$ and the equivalence will be developed later.)

There is a useful variant of the last Proposition.

Proposition 2.25 ([17]) *For a section $h \in \mathcal{O}^p_X$ to be integrally dependent on M at 0, it is necessary that, for all maps $\phi : (\mathbb{C}, 0) \to (X, 0)$ and $\psi : (\mathbb{C}, 0) \to (Hom(\mathbb{C}^p, \mathbb{C}), \lambda)$ with $\lambda \neq 0$, the function $\psi(h \circ \phi)$ on $\mathbb{C}$ belong to the ideal $\psi(M \circ \phi)$.*

Conversely, it is sufficient that this condition obtain for every ϕ whose image meets any given dense Zariski open subset of X.

We will use these ideas to extend our criterion for condition W to equidimensional sets of any codimension, but first we develop the analogue of property 4 for modules.

Blowing Up Modules and Connection with Ideals II

We now develop the analogue of property 4 for modules. We will want a construction that works for pairs of submodules, not just a single submodule.

Given a submodule M of a free $\mathcal{O}_{X^d}$ module F of rank p, we can associate a subalgebra $\mathcal{R}(M)$ of the symmetric $\mathcal{O}_{X^d}$ algebra on p generators. This is known as the Rees algebra of M. If $(m_1, \ldots, m_p)$ is an element of M then $\sum m_i T_i$ is the corresponding element of $\mathcal{R}(M)$. Then Projan$(\mathcal{R}(M))$, the projective analytic spectrum of $\mathcal{R}(M)$ is the closure of the projectivised row spaces of M at points where the rank of a matrix of generators of M is maximal. Denote the projection to X^d by c, or by c_M where there is ambiguity.

Example 2.26 If M is the Jacobian module of X, then Projan$(\mathcal{R}(M))$ is $C(X)$, the projectivised conormal space of X.

If M is a submodule of N or h is a section of N, then h and M generate ideals on Projan $\mathcal{R}(N)$; denote them by $\rho(h)$ and $\mathcal{M}$. If we can express h in terms of a set of generators $\{n_i\}$ of N as $\sum g_i n_i$, then in the chart in which $T_1 \neq 0$, we can express a generator of $\rho(h)$ by $\sum g_i T_i / T_1$.

Example 2.27 If M is the Jacobian module of X and $N = F^p$ then $V(\mathcal{M})$ consists of pairs (x, L) where $x \in X$ and $L \in \mathbb{P}Hom(\mathbb{C}^p, \mathbb{C})$, and $L \circ DF(x) = 0$. If H is the hyperplane which is the kernel of L, then the image of $DF(x)$ lies in H.

Using Proposition 2.23 it is easy to show that h is integrally dependent on M at the origin, if and only the ideal sheaf induced from h is integrally dependent as an ideal sheaf on $\mathcal{M}$ along $0 \times \mathbb{P}^{p-1}$. In other words, if and only if $\rho(h)$ is integrally dependent on $\mathcal{M}$. The combination $\psi(t)$, $\phi(t)$ amounts to giving path on $X \times \mathbb{P}^{p-1}$. This is the second connection between integral closure of ideals and modules.

Looking at a pair (M, N) allows us to "strip out" one copy of N from M, as the following example shows.

Example 2.28 Let $M = I = (x^2, xy, z) = J(z^2 - x^2 y)$ and $N = J = (x, z)$. M is the Jacobian ideal of the Whitney umbrella, and N defines the singular locus of the umbrella. So, working on $\mathbb{C}^3$, Projan $\mathcal{R}(N) = B_J(\mathbb{C}^3)$, which has ring $R = \mathbb{C}[T_1, T_2]/(zT_1 - xT_2)$, and where the map from $\mathcal{R}(N)$ to R is given by $x \to T_1, z \to T_2$. Writing the generators of I in terms of the generators of J as $x^2 = x \cdot x, xy = y \cdot x, z = z$ the map from $\mathcal{R}(I)$ to R has image (xT_1, yT_1, T_2) and this induces the ideal sheaf $\mathcal{I}$ on Projan $\mathcal{R}(N)$. We see that this is supported only at the point $(0, [1, 0])$.

The next proposition and the ideas behind it, is very useful in the study of determinantal singularities. It is also a good example of stripping a copy of a module N from M.

Proposition 2.29 *Suppose $M \subset N \subset \mathcal{O}^p_{X,0}$ are $\mathcal{O}^p_X$ modules with matrix of generators $[M]$, $[N]$, and $[F]$ is a matrix such that $[M] = [N][F]$. Let $\mathcal{F}$ be the ideal sheaf induced on* $\operatorname{Projan}(\mathcal{R}(N))$ *by the module F with matrix of generators $[F]$. Then $\overline{M} = \overline{N}$ if and only if $V(\mathcal{F})$ is empty.*

Proof We are going to apply Proposition 2.25, so we must show that for all maps $\phi\ (\mathbb{C}, 0) \to (X, 0)$ and $\psi\ (\mathbb{C}, 0) \to (Hom(\mathbb{C}^p, \mathbb{C}), \lambda)$, that the order in t of $\psi(t)[M] \circ \phi(t)$ and $\psi(t)[N] \circ \phi(t)$ are the same. We have

$$\psi(t)[M] \circ \phi(t) = \psi(t)[N][F] \circ \phi(t).$$

Suppose the order of $\psi(t)[N] \circ \phi(t)$ in t is k. Then we can lift ϕ, ψ to a curve on $\operatorname{Projan}(\mathcal{R}(N))$ as follows. Define $\Phi : \mathbb{C}, 0 \to X \times \mathbb{P}^{g(N)-1}$, by $\Phi(t) = (\phi(t), [(1/t^k)(\psi(t)[N] \circ \phi(t)])$. We have $\Phi(0) = (0, \lim_{t\to 0}(1/t^k)(\psi(t)[N] \circ \phi(t))$, and the image of Φ for $t \neq 0$ clearly lies in $\operatorname{Projan}(\mathcal{R}(N))$.

Given an element $f \in \mathcal{F}$, the value of f along Φ is $(\phi(t), [(1/t^k)(\psi(t)[N]\tilde{f} \circ \phi(t)])$, where $\tilde{f}$ is the element of F which induces f. Then $V(\mathcal{F})$ is empty if and only if the order of $\mathcal{F}$ along all Φ is zero. Since $[M] = [N][F]$ this is equivalent to the order of M and N being the same on (ψ, ϕ). □

Notice that if $M \subset N$ and $\mathcal{F}$ are as above then the inclusion of M in N always induces a map from $\operatorname{Projan}(\mathcal{R}(N))\setminus V(\mathcal{F})$ to $\operatorname{Projan}(\mathcal{R}(M))$. The map is given by taking (x, p) to $(x, \mathcal{F}(p))$, where $\mathcal{F}(p)$ is evaluation of the set of generators of $\mathcal{F}$ which come from the columns of $[F]$. The next corollary includes this setting in our discussion of reduction.

Corollary 2.30 *Suppose M and N as above, then the following are equivalent:*

1. *M is reduction of N.*
2. *$V(\mathcal{F})$ is empty.*
3. *The induced map is a finite map from* $\operatorname{Projan}(\mathcal{R}(N))$ *to* $\operatorname{Projan}(\mathcal{R}(M))$.

Proof (1) and (2) are equivalent by the previous proposition. The material in Sect. 2 of [26] shows that the induced map is finite if and only if $V(\mathcal{F})$ is empty. □

Here is a typical way that (3) is used.

Proposition 2.31 *Suppose $N \subset F$, F a free $\mathcal{O}_{X,x}$ module, and suppose the fiber of* $\operatorname{Projan}\mathcal{R}(N)$ *over x has dimension k. Then N has a reduction M, where M is generated by $k + 1$ elements.*

Proof Let g be the number of generators of N, so we view $\operatorname{Projan}\mathcal{R}(N)$ as a subset of $X \times \mathbb{P}^{g-1}$. For a generic choice of plane P in $\mathbb{P}^{g-1}$ of codimension $k + 1$, the intersection of P and the fiber of $\operatorname{Projan}\mathcal{R}(N)$ over x is empty. We can choose coordinates on $\mathbb{P}^{g-1}$ so that the plane given by $T_1 = \cdots = T_{k+1} = 0$ is such a plane, T_i coordinates on $\mathbb{P}^{g-1}$. Choosing coordinates on $\mathbb{P}^{g-1}$ is equivalent to choosing generators on N. Let M be the submodule of N generated by the first $k + 1$ generators of

N after the new choice of generators. Then the projection onto the first $k+1$ coordinates of $\mathbb{P}^{g-1}$, when restricted to Projan $\mathcal{R}(N)$ gives a finite map to Projan $\mathcal{R}(M)$. Hence M is a reduction of N by (3). □

Corollary 2.32 *Suppose $N \subset F$, F a free $\mathcal{O}_{X,x}$ module, X^d equidimensional, N has generic rank e on each component of X, x, then N has a reduction with $d+e-1$ generators.*

Proof Since the generic rank of N is e, the generic fiber dimension of Projan $\mathcal{R}(N)$ is $e-1$, so the dimension of Projan $\mathcal{R}(N)$ is $d+e-1$. Then $d+e-2$ is the largest the dimension of the fiber of Projan $\mathcal{R}(N)$ over x can be, so N has a reduction with $(d+e-2)+1$ generators. □

Having defined the ideal sheaf $\mathcal{M}$, we blow up by it. The advantages of this we will see in the next section, as it gives a constructive/geometric way to calculate the multiplicity of a pair of modules. But for now, this gives the context for which property 4 in the ideal case holds. As an example of how the blow up comes up, if we are in the basic set-up, and $M = m_Y JM(\mathcal{X})$ then the blow up by $\mathcal{M}$ is the blowup of the conormal of $\mathcal{X}$ by the ideal defining the stratum Y. Teissier has shown [38] that condition W holds for the pair $(\mathcal{X}_0, Y)$ at the origin if and only if the exceptional divisor of this blow up is equidimensional over Y. We will see the proof of one direction of this in the next section as well.

To state our result some more notation is needed. Given M a submodule of $N \subset F^p$, $h \in N$, let $NB_{\mathcal{M}}$(Projan $\mathcal{R}(N)$), $\pi_{\mathcal{M}}$ be the normalized blow-up of Projan $\mathcal{R}(N)$ by $\mathcal{M}$ with projection $\pi_{\mathcal{M}}$ to Projan $\mathcal{R}(N)$.

Proposition 2.33 (Analogue of Property 4 for ideals) *In the above set-up $h \in \overline{M}$ if and only if $\pi^*_{\mathcal{M}}(\rho(h)) \in \pi^*_{\mathcal{M}}(\mathcal{M})$.*

Proof We give the proof for the case where N is free for simplicity. We apply Corollary 2.24, so $h \in \overline{M}$ if and only if for all ϕ $(\mathbb{C}, 0) \to (X, 0)$ and ψ $(\mathbb{C}, 0) \to (Hom(\mathbb{C}^p, \mathbb{C}), \lambda)$, we have the function $\psi(h \circ \phi)$ on $\mathbb{C}$ belongs to the ideal $\psi(M \circ \phi)$. Giving the pair (ϕ, ψ) is equivalent to giving a path on $X \times \mathbb{P}^{p-1}$, the order of $\rho(h)$ on the path is the order of $\psi(h \circ \phi)$. So 2.23 is equivalent to : $h \in \overline{M}$ if and only if the ideal sheaf induced by $\rho(h)$ is in the integral closure of the ideal sheaf $\mathcal{M}$. In turn, by property 4 for ideals, this implies the result. □

As an application we can extend our criterion for condition W to equidimensional sets of any codimension.
Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ an equidimensional family of equidimensional sets, $\mathcal{X}^{k+n} \subset Y^k \times \mathbb{C}^N$, $JM(X) \subset \mathcal{O}^p$.

Proposition 2.34 *Condition W holds for $(\mathcal{X}_0, Y)$ at $(0,0)$ if and only if $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \leq l \leq k$.*

Proof We re-work the form of Verdier's condition W to fit our current framework. If we work at a smooth point x of X, then a conormal vector u of X at x can always be written as $S \cdot DF(x)$, where $S \in \mathbb{C}^p$; S is not unique unless $DF(x)$ has rank p. Conversely, any such S gives a conormal vector. It is clear also that W holds if the distance inequality holds for the standard basis for the tangent space T of Y. Then

$$\operatorname{dist}(T, TX_x) = \sup_{\substack{u \in TX_x^{\perp} - \{0\} \\ v \in T - \{0\}}} \frac{\|(u, v)\|}{\|u\| \, \|v\|}.$$

becomes

$$\operatorname{dist}(T, TX_x) = \sup_{\substack{S \in \mathbb{C}^p - \{0\} \\ 1 \leq i \leq k,\, S \cdot DF(x) \neq 0}} \frac{\|S \cdot \frac{\partial f}{\partial y_i}\|}{\|S \cdot DF(x)\|}$$

because $\|u\| = \|S \cdot DF(x)\|$, and $\|v\| = 1$.

So Verdier's condition W becomes:

$$\sup_{\substack{S \in \mathbb{C}^p \\ 1 \leq i \leq k}} \|S \cdot \frac{\partial f}{\partial y_i}\| \leq C\|z\| \, \|S \cdot DF(x)\| \, .$$

Since the functions are analytic and the inequality holds on a Z-open set of X, we can assume it holds on a neighborhood of the origin.

Now consider the integral closure condition, $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \leq l \leq k$. Using Corollary 2.4, we have $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \leq l \leq k$ if and only if

$$\sup_{\substack{S \in \mathbb{C}^p \\ 1 \leq i \leq k}} \|S \cdot \frac{\partial f}{\partial y_i}\| \leq C \sup_{1 \leq i \leq n} \|z_i S \cdot DF(x)\| \, .$$

But this is easily seen to be equivalent to the previous inequality. □

This last result shows that Verdier's condition W is exactly the geometric meaning of the ideal sheaf induced by the $\frac{\partial f}{\partial y_i}$ being in the integral closure of the ideal sheaf induced by $m_Y JM(X)$ on $X \times \mathbb{P}^{p-1}$.

In the next section we will see how to describe and control equisingularity conditions using multiplicity of ideals and modules.

3 Multiplicities, Integral Closure and the Multiplicity-Polar Theorem

The multiplicity of an ideal or module or pair of modules is one of the most important invariants we can associate to an m-primary module. It is intimately connected with integral closure. It has both a length theoretic definition and intersection theoretic definition. We give the definition in terms of length first, for ideals, and submodules of a free module. Denote the length of a module M by $l(M)$.

Theorem/Definition 3.1 *(Buchsbaum–Rim [1]) Suppose $M \subset F$, M, F both A-modules, F free of rank p, A a Noetherian local ring of dimension d, F/M of finite length, $\mathcal{F} = A[T_1, \ldots, T_p]$, $\mathcal{R}(M) \subset \mathcal{F}$, then*

$\lambda(n) = l(\mathcal{F}_n/\mathcal{M}_n)$ is eventually a polynomial $P(M, F)$ of degree d+p-1.

Writing the leading coefficient of $P(M, F)$ as $e(M)/(d + p - 1)!$, then we define $e(M)$ as the multiplicity of M.

It is possible to compute simple ideal examples by hand as we show:

Example 3.2 Let $M = I = (x^2, xy, y^2) \subset \mathcal{O}_2$. Then $e(M) = 4$.

We have $p = 1$, $F = \mathcal{O}_2$, and we work with $\mathcal{F} = \mathcal{O}_2[T_1]$. (Notice that Projan $\mathcal{F} = \mathbb{C}^2$.)

Now $\mathcal{M}_n = I^n T^n = m_2^{2n} T^n$, so

$$l(\mathcal{F}_n/\mathcal{M}_n) = l(\mathcal{O}_2/m^{2n}) = (2n)(2n + 1)/2 = 4n^2/2! + (l.o.t.)$$

So $e(M) = 4$.

Problem 3.3 *Let $M = I = (x^2, y^2) \subset \mathcal{O}_2$. Show $e(M) = 4$. (Hint: Try to show that the terms that are missing in this problem due to the missing xy term, grow only linearly with n, so the leading term of the polynomial is the same.)*

It is possible to do the very simplest module examples by hand easily as well.

Problem 3.4 *Let $M = m_2\mathcal{O}_2^2$. Show $e(M) = 3$.*

The next problem is harder–try to use the same strategy as in Problem 3.3.

Problem 3.5 *Let* $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$. *Show $e(M) = 3$.*

If $\mathcal{O}_{X^d,x}$ is Cohen–Macaulay, and M has $d + p - 1$ generators where $M \subset F^p$, then there is a useful relation between M and its ideal of maximal minors and the multiplicity of both of them. The multiplicity of M is the colength of M, and is also the colength of the ideal of maximal minors, by some theorems of Buchsbaum and Rim [1], 2.4 p. 207, 4.3 and 4.5 p. 223. A proof of this theorem in the context of analytic geometry using the Multiplicity Polar theorem is given in [13]. Using this result, it is easy to do Problem 3.5.

Challenge Problem 3.6 *Buchsbaum and Rim showed $e(M) = l(F^p/M)$, if M has $d + p - 1$ generators, F a module over a Cohen–Macaulay ring. What is a generalization of this to $e(M, N)$? (If M and N are ideals there is something along these lines in [12] Theorem 2.3.)*

An important theorem both for computational and theoretical purposes was proved by Rees in the ideal case. A proof of a generalization to modules appears in [26].

Theorem 3.7 *Suppose $M \subset N$ are m primary submodules of F^p, and $\overline{M} = \overline{N}$. Then $e(M) = e(N)$. Suppose further that $\mathcal{O}_{X,x}$ is equidimensional, then $e(M) = e(N)$ implies $\overline{M} = \overline{N}$.*

Several generalizations of this result exist: Kleiman and Thorup [[26], (6.8)(b)] proved a similar result in which F^p is replaced by an arbitrary finitely generated module whose support is equidimensional; they also proved an additivity result in Theorem (6.7b)(i) of [26] for the three pairs of modules arising from three nested modules. Generalizations also exist where the multiplicity is not defined. Gaffney and Gassler did the case of ideals [16], and Gaffney for modules [10], while Ulrich and Valadoshti have an approach using the epsilon multiplicity.

For computational purposes, this is coupled with another result–given any $M \subset F^p$, M a module over a local ring of dimension d, there exists a submodule R of M with $d + p - 1$ generators such that $\overline{M} = \overline{R}$. Such an R is called a *reduction* of M.

So if $\mathcal{O}_{X^d,x}$ is Cohen–Macaulay, we can try to find a reduction R of M with the right number of generators $d + p - 1$, then calculate the length of F/R. (This length is also called the colength of R.) Here is a very simple example.

Problem 3.8 *Suppose I is any ideal in $m_2^n \mathcal{O}_2$ which contains x^n, y^n. Then $e(I) = n^2$.*

Now we want to give an intersection theoretic definition of the multiplicity. This definition applies to pairs of modules as well.

The next diagram shows the spaces that come into the definition.

$$\begin{array}{ccc} B_{\mathcal{M}}(\text{Projan}\,\mathcal{R}(N)) & \xrightarrow{\pi_N} & \text{Projan}\,\mathcal{R}(N) \\ \downarrow \pi_M & & \downarrow \pi_{XN} \\ \text{Projan}\,\mathcal{R}(M) & \xrightarrow{\pi_{XM}} & X \end{array}$$

On the blow up $B_{\mathcal{M}}(\text{Projan}\,\mathcal{R}(N))$ we have two tautological bundles. One is the pullback of the bundle on Projan $\mathcal{R}(N)$. The other comes from Projan $\mathcal{R}(M)$. Denote the corresponding Chern classes by c_M and c_N, and denote the exceptional divisor by $D_{M,N}$. Suppose the generic rank of N (and hence of M) is g.

Then the multiplicity of a pair of modules M, N is:

$$e(M, N) = \sum_{j=0}^{d+g-2} \int D_{M,N} \cdot c_M^{d+g-2-j} \cdot c_N^j.$$

Kleiman and Thorup show that this multiplicity is well defined at $x \in X$ as long as $\overline{M} = \overline{N}$ on a deleted neighborhood of x. This condition implies that $D_{M,N}$ lies in the fiber over x, hence is compact. Notice that when $N = F$ and M has finite colength in F then $e(M, N)$ is the Buchsbaum-Rim multiplicity $e(M, \mathcal{O}^p_{X,x})$.

Kleiman and Thorup also showed that $e(M, N)$ vanishes if and only if M and N have the same integral closure, provided the support of N is equidimensional. ([26], (6.3)(ii).)

Remark 3.9 We have seen that there is a map from Projan $\mathcal{R}(N)\backslash V(\mathcal{F}) \to$ Projan $\mathcal{R}(M)$. The diagram used in the definition of $e(M, N)$ can be used to make this more precise. Namely, the complement of $\pi_M D_{M,N}$ is the largest open subset V of Projan $\mathcal{R}(M)$ such that the map $\pi_M^{-1} V\backslash D_{M,N} \to V$ is finite. Plainly, π_N is an isomorphism over the complement U of $V(\mathcal{F})$, and $\pi_N^{-1} U$ contains $\pi_M^{-1} V$.

Let's re-calculate two examples using this definition.

Example 3.10 Let $M = I = (x^2, xy, y^2) \subset \mathcal{O}_2$. Then $e(M) = 4$.

Here $d = 2$, $p = g = 1$, Projan $\mathcal{R}(N) = \mathbb{C}^2$, Projan$(\mathcal{M}) = B_I(\mathbb{C}^2) = B_{\mathcal{M}}$ (Projan $\mathcal{R}(N)$), and Projan$(\mathcal{M}) \subset \mathbb{C}^2 \times \mathbb{P}^1$. So the only term we need to calculate is $\int D_{M,N} \cdot c_M$. We can calculate this term as follows: Intersect $B_I(\mathbb{C}^2)$ with $\mathbb{C}^2 \times H$, H a generic hyperplane in $\mathbb{P}^1$, which represents $c(M)$. Project this curve to $\mathbb{C}^2$, and calculate the order of I on the curve. Projecting the curve to $\mathbb{C}^2$ amounts to setting a generic combination of the generators to zero, and looking at the curve obtained, removing any components in $V(I)$. In this case a generic curve is $x^2 - ay^2 = 0$, $a \neq 0$. This consists of two branches ($x - y = 0$ and $x + y = 0$ if $a = 1$) and the colength of the ideal on each branch is 2 so the multiplicity is $2 + 2 = 4$.

Example 3.11 Let $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$. Show $e(M) = 3$.

Here $d = 2$, $p = g = 2$, $N = \mathcal{O}_2^2$, Projan $\mathcal{R}(N) = \mathbb{C}^2 \times \mathbb{P}^1$, Projan $\mathcal{R}(M) \subset \mathbb{C}^2 \times \mathbb{P}^2$, dimension of $B_{\mathcal{M}}$(Projan $\mathcal{R}(N)$) is 3. So we need to calculate $\int D_{M,N} \cdot c_M^2$, $\int D_{M,N} \cdot c_M \cdot c_N$ (Notice that $c_N^2 = 0$, since we are working on Projan $\mathcal{R}(N) = \mathbb{C}^2 \times \mathbb{P}^1$.) Now we have two choices: as before we intersect a representative of each class with the blow-up then push down to X, then see what the multiplicity of M is on each curve. Or, we can push down to Projan $\mathcal{R}(N)$ and evaluate $\mathcal{M}$ on each curve. (For details of how this approach works, the reader should consult [11] Theorem 3.1 and the two examples which follow.)

Taking the second route, projecting the intersection of the blow-up with a hyperplane from $\mathbb{C}^2 \times \mathbb{P}^1$ and a hyperplane from $\mathbb{C}^2 \times \mathbb{P}^2$, is a curve on $\mathbb{C}^2 \times \mathbb{P}^1$, defined by a linear relation $T_1 = aT_2$, and by setting one of the elements of $\mathcal{M}$ restricted to this set to zero. The restriction of $\mathcal{M}$ to the locus $T_1 = aT_2$ is the ideal generated by the entries of the linear combination of the first row and a times the second row from the original matrix. A generic curve is given by setting $x + ay = 0$, and the multiplicity of $\mathcal{M}$ on this curve is 1. So, $\int D_{M,N} \cdot c_M \cdot c_N = 1$.

Projecting the intersection of the blow-up with two hyperplanes from $\mathbb{C}^2 \times \mathbb{P}^2$, amounts to setting two generic elements of $\mathcal{M}$ to zero and removing any components of $V(\mathcal{M})$. Setting $xT_1 + yT_2$ and $yT_1 + xT_2 = 0$ gives two curves. One curve is $x = y$, $T_1 = 1 = -T_2$ and the other curve is $x = -y$, $T_1 = 1 = T_2$.

The restriction of $\mathcal{M}$ to the first curve is x so the multiplicity is 1; as it is on the second curve as well, for a total of 3.

Notice that in the last example $3 = e(M) \neq e(J(M)) = 4$. (J(M) is the ideal of maximal order non-vanishing minors, and is (x^2, xy, y^2) in this case.) But,

Problem 3.12 *Suppose $M \subset N \subset F$ are m primary $\mathcal{O}_{X,x}$ modules, X, x equidimensional. Show that $e(M) = e(N)$ if and only if $e(J(M)) = e(J(N))$.*

There are examples though, where there is a family of ICIS singularities where $e(JM(X_y))$ is independent of y, but $e(J(JM(X_y)))$ is not. In the example due to Henry and Merle, the embedding dimension of the singularity changes at $y = 0$–the singularity goes from being codimension 2 to being codimension 1, because one of the defining equations is no longer singular off the origin. Is this the only way for the connection between the two invariants to break?

Challenge Problem 3.13 *Give a geometric characterization of when $e(JM(X_y))$ is independent of y, but $e(J(JM(X_y)))$ is not.*

This problem is connected with the difference between using the conormal modification to study equisingularity conditions and using the Nash modification, which is why it is interesting. In the ICIS case a difference in the value of the multiplicity between the generic point y and the origin implies there is a jump in the dimension of the fiber of the exceptional divisor over the origin. So if the value of $e(JM(X_y))$ is independent of y, but $e(J(JM(X_y)))$ is not, then the set of limiting tangent planes has a jump in dimension at the origin, but the set of limiting tangent hyperplanes does not.

Reading In Sect. 3 of [11] these ideas are developed further. It also contains the example due to Henry and Merle mentioned above.

There is an important case where it is easy to calculate the multiplicity of the pair. Suppose we are given $\mathcal{O}_X$ modules $M \subset N \subset F$, where F is free, X has dimension 1, and $e(M, N)$ is defined. We want a procedure to calculate $e(M, N)$. The first step is to find a normalization $\tilde{X}, n$ of the curve. Then we can use the following proposition.

Proposition 3.14 *Suppose X is a curve singularity, then $e(M, N) = e(n^*(M), n^*(N))$.*

Proof This is a corollary of Theorem 5.1 of [25]. □

We'll illustrate the rest of the procedure with an example taken from [7]. The procedure is also described in [25].

The curves we consider are the X_l, defined by the minors of

$$F_l = \begin{bmatrix} z & x \\ y & z \\ x^l & y \end{bmatrix}.$$

We assume $l - 1$ is not divisible by 3. With this assumption we have a normalization given by $(\mathbb{C}, n_l)$ where $n_l(t) = (t^3, t^{2l+1}, t^{l+2})$. The assumption on l means that the exponents on the first and last terms in the formula for n are relatively prime. The form of n is a reflection of the fact that X_l is weighted homogeneous with weights $(3, 2l + 1, l + 2)$.

In this example the module N is $F_l^*(JM(\Sigma^2))$ where Σ^2 is the linear maps of rank < 2, and we view F_l as map from $\mathbb{C}^3 \to Hom(\mathbb{C}^2, \mathbb{C}^3)$. Then $M = JM(X_l)$.

The next step is to find a minimal set of generators for $n_l^*(N)$ and $n_l^*(M)$. Pulling back the generators of $JM(\Sigma^2)$ using $F_l \circ n_l$, we get:

$$n_l^*(N) = \begin{bmatrix} t^{l+2} & -t^3 & 0 & -t^{2l+1} & t^{l+2} & 0 \\ 0 & t^{2l+1} & -t^{l+2} & 0 & -t^{3l} & t^{2l+1} \\ t^{2l+1} & 0 & -t^3 & -t^{3l} & 0 & t^{l+2} \end{bmatrix}.$$

As this matrix has generic rank 2, $n_l^*(N)$ can be generated freely by 2 generators since we are working over $\mathcal{O}_1$, so a matrix of generators R_N of $n_l^*(N)$ with a minimal number of columns is

$$R_N = \begin{bmatrix} -t^3 & 0 \\ t^{2l+1} & -t^{l+2} \\ 0 & -t^3 \end{bmatrix}.$$

A calculation shows that $n_l^*(JM(X))$ is generated by the columns of:

$$R_{JM} = \begin{bmatrix} -t^3 & 2t^{l+2} \\ 2t^{2l+1} & -t^{3l} \\ t^{l+2} & t^{2l+1} \end{bmatrix}.$$

Note that

$$R_{JM} = R_N \begin{bmatrix} 1 & -2t^{l-1} \\ -t^{l-1} & -t^{2l-2} \end{bmatrix}.$$

Denote the submodule of $\mathcal{O}_1^2$ whose matrix of generators is the 2×2 matrix in the last line by K. Since $n_l^*(N)$ is freely generated, it is isomorphic to $\mathcal{O}_1^2$. The isomorphism carries the pair $(n_l^*(JM(X)), n_l^*(N))$ to $(K, \mathcal{O}_1^2)$. Then $e(n_l^*(JM(X)), n_l^*(N)) = e(K, \mathcal{O}_1^2)$. Since $\mathcal{O}_1$ is Cohen–Macaulay, the multiplicity of the second pair is the colength of the determinant of the matrix of generators of K, which is $2l - 2$.

Polar Varieties of a Module

Intuitively, the polar varieties of a module measure the "curvature" of Projan $\mathcal{R}(M)$, and we have encountered them in the examples of the previous paragraph. As we shall see, the projection of $B_{\mathcal{M}}(\text{Projan}\,\mathcal{R}(N)) \cdot c_M^2$ to $\mathbb{C}^2$, studied in Example 3.11 is the polar curve of M.

The *polar variety of codimension* l of M in X, denoted $\Gamma_l(M)$, is constructed by intersecting Projan $\mathcal{R}(M)$ with $X \times H_{g+l-1}$ where H_{g+l-1} is a general plane of codimension $g + l - 1$, then projecting to X.

So, in the setting of Example 3.11, $g = 2$, and $g + l - 1 = 2 + 1 - 1 = 2$, and the projection of $B_{\mathcal{M}}(\text{Projan}\,\mathcal{R}(N)) \cdot c_M^2$ to $\text{Projan}\,\mathcal{R}(M)$ is the intersection of $\mathbb{C}^2 \times H_2$ with $\text{Projan}\,\mathcal{R}(M)$. Thus the projection of $B_{\mathcal{M}}(\text{Projan}\,\mathcal{R}(N)) \cdot c_M^2$ to $\mathbb{C}^2$ is $\Gamma_1(M)$.

The polar varieties of M can be constructed by working only on X. The plane H_{g+l-1} consists of all hyperplanes containing a fixed plane H_K of dimension $g + l - 1$. By multiplying the matrix of generators of M by a basis of H_K we obtain a submodule of M denoted M_H.

Proposition 3.15 *In this set-up the polar variety of codimension l consists of the closure in X of the set of points where the rank of M_H is less than g, and the rank of M is g.*

Proof Since H_{g+l-1} is generic, the general point of $\text{Projan}\,\mathcal{R}(M) \cap X \times H_{g+l-1}$ lies over points where the rank of M is g. Choose coordinates so that a basis for H_K consists of the last $g + l - 1$ elements of the standard basis of $\mathbb{C}^j$, j the number of generators of M. We can find v such that $v[M_H] = 0$ but $v[M] \neq 0$ if and only if we are at a point where the rank of $M_H < g$. The existence of v is equivalent to being able to find a combination of the rows of $[M]$, such that the last $g + l - 1$ entries are 0. This row is a hyperplane which lies in H_{g+l-1}. □

Teissier [36, 38] defined the polar varieties of an analytic germ $(X^d, x) \subset \mathbb{C}^n$ of codimension l as follows: take a generic projection π of $X^d \to \mathbb{C}^{d-l+1}$, and take the closure of the critical points of the restriction of the projection to the smooth points of X. Using the last proposition, it is easy to see that these polar varieties are the polar varieties of the Jacobian module of X.

For, given $(X^d, x) \subset \mathbb{C}^n$, the generic rank g of the Jacobian module of X is $n - d$. The kernel of a generic projection to $\mathbb{C}^{d-l+1}$ has dimension $n - d + l - 1 = g + l - 1$. Let the fixed plane H_K in the previous proposition be the kernel of π. Then the rank of M_H is less than maximal at a smooth point of X if and only if the tangent space of X has larger than expected intersection with the kernel of π. Thus, a tangent hyperplane of X contains H_K at a smooth point of X if and only if x is a critical point for the restriction of the projection to X at x. Thus the two notions of polar variety coincide.

If M is an ideal and we are working on X, then M_H is a sheaf of ideals and the polar varieties are the closure of the set defined by this sheaf on the complement of $V(M)$.

Problem 3.16 *Given $M \subset N \subset \mathcal{O}^p_{X,x}$, M and N both $\mathcal{O}_X$ modules, M induces an ideal sheaf on* $\text{Projan}\,\mathcal{R}(N)$*, and we can define the polar varieties of this ideal sheaf. (To do this we must work on the fiber of* $\text{Projan}\,\mathcal{R}(N)$ *over x.) Show that the projection of the polar of dimension d defined in this way to X is $\Gamma^d(M)$.*

Thus, there are 4 different settings for studying the polar varieties. It is often useful in proofs to move between them.

There is a special case which will be important to us. The diagram below represents the smoothing of an isolated singularity.

$$\begin{array}{ccccc} X^d(0) \subset & \mathcal{X}^{d+1} \subset Y \times \mathbb{C}^N & \supset \mathcal{X}(y) \\ \downarrow & \downarrow p_Y & \downarrow \pi_Y \\ 0 \in & Y = \mathbb{C} & \supset y \neq 0 \end{array}$$

Let $M = JM_z(\mathcal{X})$, Then $\Gamma_d(\mathcal{X})$ by the previous proposition is defined by selecting $N-1$ generic generators of $JM_z(\mathcal{X})$, and looking to see where they have less than maximal rank. Assume coordinates chosen so that the first $N-1$ columns of $[JM(\mathcal{X})]$ are generic. Then the points where the polar intersects $\mathcal{X}(y)$ are the critical points of z_N restricted to $\mathcal{X}(y)$. The number of such points is the number of sheets of $\Gamma_d(\mathcal{X})$ over Y is the multiplicity of $\Gamma_d(\mathcal{X})$ over Y at the origin. If the smoothing is unique up to diffeomorphism, then the invariant is denoted $m_d(X)$. It is clear that the number of critical points of a generic linear form on a smoothing of X is important to the topology of $\mathcal{X}(y)$, so this number is an important invariant of X.

By construction, the existence of a polar variety of M at $x \in X$ is tied to the dimension of the fiber of Projan $\mathcal{R}(M)$ over x.

Problem 3.17 *Suppose X^d, x equidimensional and M has the same generic rank g on each component of X at x. Show that $\Gamma_l(M, x)$ is non-empty if and only if the dimension of the fiber of* Projan $\mathcal{R}(M)$ *over x is greater than or equal to $l + g - 1$.*

There is a strong connection between polar varieties and integral closure thanks to an important result of Kleiman and Thorup [26, 27], which we next discuss. The following theorem ties the dimension of this fiber to integral closure conditions.
Set-up: X the germ of a reduced analytic space of pure dimension d, F a free $\mathcal{O}_X$-module, $M \subset N \subset F$ two nested submodules with $M \neq N$, M and N are generically equal and free of rank e. Set $r := d + e - 1$. Set $C :=$ Projan$(\mathcal{R}(M))$ where $\mathcal{R}(M) \subset \mathcal{S}ym\mathcal{F}$ is the subalgebra induced by M in the symmetric algebra on F. Let $c : C \to X$ denote the structure map. Let W be the closed set in X where N is not integral over M, and set $E := c^{-1}W$.

Theorem 3.18 (Kleiman-Thorup, [26, 27]) *If N is not integral over M, then E has dimension $r - 1$, the maximum possible.*

Proof Since this theorem is so important to us, we give a concise version, due to Kleiman [24], of the proof that appears in [27].

Given an element $h \in N$ that's not integral over M, let H be the module generated by h and M. Now we use the notation of the diagram used in the definition of $e(M, N)$. We have $D_{M,H}$ is nonempty by Remark 3.9, so of dimension $r - 1$ where $r :=$ dimProjan $\mathcal{R}(M)$. But π_H embeds $D_{M,H}$ in Projan $\mathcal{R}(M)$ because H/M is cyclic. Moreover, Remark 3.9 implies that N is integral over M locally off $\pi_M D_{M,N}$; so H is too; so Remark 3.9 implies that $\pi_M D_{M,N}$ contains $\pi_M D_{H,N}$. Plainly, $\pi_M D_{H,N}$ lies in E. Thus dim $E = r - 1$. □

A recent proof in a more general setting appears in [33].

We give an example the usefulness of this Theorem by giving a simple proof of one direction of a theorem of Teissier describing Whitney equisingularity.

Set-up: Suppose $Y^k, 0 \subset X^{d+k}, 0$, Y^k smooth, $\underline{y}$ coordinates on Y, $I(Y) = m_Y$. Set $M = m_Y JM(X)$, $N = M + \mathbb{C}\{\frac{\partial f}{\partial \underline{y}}\}$, then $\mathrm{Projan}(\mathcal{R}(M)) = B_{m_Y}(C(X))$, $M = N$ off Y.

Let E denote the exceptional divisor of $B_{m_Y}(C(X))$.

Theorem 3.19 (Teissier, [38]) *If the fibers of E, the exceptional divisor of B_{m_Y} $(C(X))$ over Y, have the same dimension, then the Whitney conditions hold along Y.*

Proof If the Whitney conditions fail along Y, they do so on a proper closed subset $S \subset Y$. Then S is the set where $\overline{M} \neq \overline{N}$ [9]. By the Kleiman-Thorup theorem there must be a component of E over S, so the fibers of E have larger dimension over points in S than over the generic point of Y. □

For the ICIS case we can use the machinery of multiplicities, together with the Kleiman-Thorup theorem to get criteria for a family of sets to be Whitney equisingular, in which the criteria depend only on the members of the family, not the total space. We describe how this developed.

The first theorem is a generalization of a result of Teissier, who used it in conjunction with hypersurfaces. This theorem is useful in showing that if invariants are independent of parameter then equisingularity conditions hold.

Theorem 3.20 (Principle of Specialization of Integral Dependence) *Assume that X is equidimensional, and that $y \mapsto e(y)$ is constant on Y^k. Let h be a section of a free $\mathcal{O}_X$ module E whose image in $E(y)$ is integrally dependent on the image of $M(y)$ for all y in a dense Zariski open subset of Y. Then h is integrally dependent on M.*

Proof Cf. Theorem 1.8 [17]. □

The proof of the PSID proceeds by showing that the constancy of the multiplicity means that M has a reduction M_R which is generated by $\dim(X(y)) + p - 1$ generators, which is the minimum possible if $e(M(y))$ is well defined for all y. To do this, first we find such an M_R whose restriction $M_R(0)$ to $X(0)$ is a reduction of M restricted to $X(0)$, so $e(M_R(0)) = e(M(0))$ by Theorem 3.7. Then the uppersemicontinuity of the multiplicity ([17], 1.1 p. 547), implies $e(M_R(0)) \geq e(M_R(y))$, while $M_R(y) \subset M(y)$ implies $e(M_R(y)) \geq e(M(y))$. This gives us the inequality:

$$e(M(0)) = e(M_R(0)) \geq e(M_R(y)) \geq e(M(y)) = e(M(0)).$$

Thus, by Theorem 3.7, $M_R(y)$ is a reduction of $M(y)$ for all y.

Now replace M by the submodule generated by M_R and g, where g may be h or any element of M not in M_R. A lemma ([17] 1.2, p. 548) shows that if the set of points where g fiberwise is not integrally dependent on M_R is a proper Zariski closed subset of Y, then the set of points where g is not integrally dependent on M_R is also a proper Zariski closed subset W of Y. This implies that M_R is a reduction of (M, h) off a Zariski closed set of Y as this is true fiberwise.

Now, the dimension of the fiber of Projan$(\mathcal{R}(M_R))$ over our base point $x_0 \in X$ is at most $\dim(X(y)) + p - 2$, which is one less than the number of generators. Now the inverse image of W in Projan$(\mathcal{R}(M_R))$ must have dimension at most $\dim(X(y)) + p - 2 + k - 1$. Then since

$$\begin{aligned}\dim(X(y)) + p - 2 + k - 1 &\leq (\dim(X(y)) + k) + (p - 1) - 2 \\ &= (\dim(X) + p - 1) - 2,\end{aligned}$$

the Kleiman-Thorup theorem then shows that $\bar{M}_R = \bar{M}$, which gives the result.

In order to show that the equisingularity condition implied that the invariants were independent of y more ideas are necessary. These are discussed in the proof of the next theorem.

Theorem 3.21 *Suppose $(X^{d+k}, 0) \subset (\mathbb{C}^{n+k}, 0)$ is a complete intersection, $X = F^{-1}(0)$, $F : \mathbb{C}^{n+k} \to \mathbb{C}^p$, Y a smooth subset of X, coordinates chosen so that $\mathbb{C}^k \times 0 = Y$. Then the following are equivalent:*

(i) the pair $(X - Y, Y)$ satisfies W at 0;
(ii) The sets $X(y)$ are complete intersections with isolated singularities and $e(m_y JM(X_y))$ is independent of y for all $y \in Y$ near 0.

Proof For the proof that (ii) implies (i), the condition on $e(m_y JM(X_y))$ implies that the singularities do not split, so that $X - Y$ is smooth. Since the integral closure condition is a generic condition, the PSID applies.

For the proof that (i) implies (ii) the proof is more complicated. An expansion formula shows that $e(m_y JM(X_y))$ is a sum of multiplicities. Each multiplicity that appears is the sum of two Milnor numbers of plane sections of the ICIS $X(y)$. Since Whitney equisingularity of X implies the Whitney equisingularity of the plane sections of X, and the Milnor numbers of the sections are topological invariants, the multiplicities, and hence their sum is invariant as well. □

With this result you can see that the Whitney conditions imply in the ICIS case, that the fiber of $B_{m_Y}(C(X))$, the blow-up of the conormal modification along Y, is equidimensional over Y. For the Whitney conditions imply that the multiplicity of $m_Y JM(X(y))$ along Y is constant. Then the technique of proof used in the Principle of Specialization implies that we can pick $d + p - 1$ elements of $m_Y JM(X(0))$ which generate a reduction N first of $m_Y JM(X(0))$, then of $m_Y JM(X)$. This implies that there exists a finite map from $B_{m_Y}(C(X))$ to Projan$(\mathcal{R}(N))$. Now since Projan$(\mathcal{R}(N)) \subset X \times \mathbb{P}^{d+p-2}$, the fiber dimension of $B_{m_Y}(C(X))$ over $0 \in X$ is less than or equal to $d + p - 2 = n - 2$ which is the minimum possible.

For an ICIS X, we use the multiplicity of $m_Y JM(X)$ to control the Whitney equisingularity type. What do we use when $e(m_Y JM(X))$ is not defined? Since $e(m_Y JM(X))$ is defined only when $JM(X)$ has finite codimension in $\mathcal{O}_X^p$, it is only defined for ICIS.

Looking at the ideas relating $e(m_Y JM(X))$ to the Whitney conditions, though the connections are beautiful, the proofs that Whitney implies the constancy of the multiplicities seem unnecessarily round about. The Whitney conditions themselves are described by the behavior of the exceptional divisor of $B_{m_Y}(C(X))$. Is there a direct link between $e(m_Y JM(X))$ and the exceptional divisor, so that it would not be necessary to go through topology to show that Whitney implies the constancy of $e(m_Y JM(X))$?

To answer the first question, start with thinking about the pair of modules $(JM(X), \mathcal{O}^p_X)$. The module $JM(X)$ can be viewed as the module of infinitesimal, first order trivial deformations of X. (Trivial with respect to biholomorphic equivalences of $\mathbb{C}^n$.) The module $\mathcal{O}^p_X$ is then the module of all infinitesimal, first order deformations of X since we can deform the equations of X freely, and get a family of ICIS. It is known that if X has an isolated singularity, then again the codimension of $JM(X)$ inside the module $N(X)$ of all infinitesimal, first order deformations of X is finite. This suggests using $e(JM(X), N(X))$.

However, two problems surface. We want specialization of N from the total space of a family to the fibers. This is necessary if the results are to depend only on the fibers of the family and not on the total space. This will be true, provided any first order linear infinitesimal deformation of a space lifts to a deformation of the family. However this is clearly false, if the base space of the versal deformation space has components. If the base space of the verbal deformation space is smooth for example, then the specialization property is true.

Another problem enters because $N(X)$ may have curvature. Here we are making an analogy between N and $JM(X)$. Moonen [29] has shown that the multiplicities of the polar varieties of X, x are related to the curvature of X at x. (In the real case see also [4]) This curvature then is related to the limiting tangent hyperplanes of X at x. Since the polar varieties of N are related to limiting hyperplanes defined by row vectors of a matrix of generators of N, it is reasonable to call the phenomena picked up by polar multiplicities of N as the curvature of N. How this curvature enters into the invariants we want will be a main theme of the next section.

In the next section we give also an example which shows the multiplicity of the pair may be zero, but the curvature contribution of N gives a non-zero invariant.

Since the Whitney conditions are controlled by the dimension of the fiber of the exceptional divisor of $B_{m_Y}(C(\mathcal{X}))$, and the dimension of the fibers are detected by the presence of the polar varieties of the relative Jacobian module, it is reasonable to look for a connection between invariants associated with integral closure and those associated with polar varieties.

An approach for linking the behavior of the multiplicity of an ideal in a family to the degree of the exceptional divisor is given by Teissier in [38, p. 345]. We include an excerpt from this reference where this idea is mentioned.

Soient $f : (X,0) \to (\mathbb{D},0)$ un morphisme d'espaces réduits, I un idéal de $\mathcal{O}_X$ définissant un sous-espace $Y \subset X$ tel que $f|Y : Y \to \mathbb{D}$ soit fini, $p : X' \to X$ l'éclatement de Y, $D_{vert.}$ la réunion des composantes du diviseur exceptionnel D (non nécessairement réduites) dont l'image ensembliste par p est 0, $\deg D_{vert.} = \deg(\mathcal{O}_{D_{vert.}}(1))$. Pour tout représentant suffisamment petit du germe de f en 0, on a l'égalité

$$\deg D_{vert.} = e(I \cdot \mathcal{O}_{X(0)}) - e(I \cdot \mathcal{O}_{X(s)}) \quad (\text{pour } s \neq 0) \quad .$$

En particulier, on a "$e(I.\mathcal{O}_{X(s)})$ est indépendant de $s \in \mathbb{D}$" si et seulement si $\dim p^{-1}(0) = \dim X - 2$.

Here is how we can understand Teissier's formula. The fiber of the exceptional divisor over $0 \in X^{d+1}$ is a projective variety so it has a degree. When we intersect this variety with a linear space of complementary dimension, on the one hand, the number of points we get is the degree of the variety, on the other, because intersecting $B_I(X)$ with this linear space defines the polar curve of I, it is the number of points in the polar curve over a generic t value. Call this number $m_d(I, X)$. Now one way to define the polar curve is to pick d generic elements of I, chosen so that they define a reduction of $I(0)$ and are a reduction of I on the total space over $\mathbb{D} - 0$, and see where they are zero. Call this ideal J. By construction the points where they are zero outside of $V(I)$, will be a Z-open and dense set of the polar curve, and at points of $V(I)$, $\overline{I(y)} = \overline{J(y)}$ and so $e(I(y)) = e(J(y))$ at such points. Since J is generated by d elements, a lemma shows that $e(J(y))$ is independent of y. So

$$degD_{vert} = m_d(I, X) = e(J \cdot \mathcal{O}_{X(0)}) - e(J \cdot \mathcal{O}_{X(y)}) = e(I \cdot \mathcal{O}_{X(0)}) - e(I \cdot \mathcal{O}_{X(y)}).$$

If we extend this approach to pairs of modules we find that the polar variety of N enters as well as the polar variety for M.

Set-up: $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y, with equidimensional fibers, Y smooth, $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y.

Let $\Delta(e(M, N)) = e(M(0), N(0), \mathcal{O}_{X(0)}, 0) - e(M(y), N(y), \mathcal{O}_{X(y)}, (y, x))$ be the change in the multiplicity of the pair (M, N) as the parameter changes from y to 0.

Theorem 3.22 (Multiplicity Polar Theorem [6, 11])*:*

$$\Delta(e(M, N)) = mult_y\Gamma_d(M) - mult_y\Gamma_d(N)$$

Many applications of this theorem can be found in: [7, 11–13, 19].

To show its power we give a simple proof of Theorem 3.21 which links $e(JM(X_y))$ and the Whitney conditions. The proof that that (ii) implies (i) avoids the use of topology.

Proof (*of* 3.21): (i) implies (ii) The Whitney conditions imply that the fiber of $D \subset B_{m_Y}(C(X))$, the exceptional divisor is equidimensional over Y. Because the dimension of the fiber is small, there is no polar variety of codimension d for $m_Y JM(X)$. Since $\mathcal{O}_X^p$ has no polar varieties, the Multiplicity Polar Theorem implies that $e(mJM(X_y))$ is independent of y.

(ii) implies (i) The independence of $e(mJM(X_y))$ from y implies that there is no polar variety of codimension d for $m_Y JM(X)$, and hence the fiber of D over Y^k is equidimensional. At this point we apply the theorem of Kleiman-Thorup (3.18). We know that $JM_Y(X)$, the submodule generated by the partial derivatives taken with respect to coordinates on Y, is in the integral closure of $m_Y JM(X)$ at points in a Z-open subset of Y. Since the dimension of the set of points of Projan$(m_Y JM(X))$ over the set of points where the integral closure condition does not hold is at most $(k-1)+(d+g-2)<(d+k)+(g-1)-1$, it follows that $JM_Y(X)$ is in the integral closure of $m_Y JM(X)$ at all points of Y. □

In the next part, we examine an important class of singularities for which the module N of first order deformations does specialize as we desire.

Determinantal Singularities

We begin with F, a $(n+k,n)$ matrix, with entries in $\mathcal{O}_q$; we view F as a map from $\mathbb{C}^q \to Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$. Let Σ^r denote elements of $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ of rank less than r. Let I_r be the ideal in $\mathcal{O}_{n^2+nk}$ generated by the minors of size r of elements of $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$. It is easy to check that the codimension of Σ^r in $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ is $(n-r+1)(n+k-r+1)$. The elements of $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ of rank r, $0 \leq r \leq n$ give a stratification of $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ which we call the rank stratification.

Assume F is transverse to the rank stratification of $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ on $\mathbb{C}^q - 0$. Let $\Sigma^r(F) := V(F^*(I_r))$, then $F^*(I_r)$ is generated by the minors of size r of F. $\Sigma^r(F)$ is determinantal i.e. codim $\Sigma^r(F)$ =codim Σ^r. If $q < (n-r+2)(n+k-r+2)$ then $\Sigma^r(F)$ has a smoothing, because when we deform F so that it is transverse to the rank stratification there will be no points where the rank$< r-1$.

We fix the class of deformations and fix a unique smoothing by only considering deformations of $\Sigma^r(F)$ which come from deformations of the entries of F. As we shall see, the geometric meaning of the invariants we develop is tied to the topology of the smoothing.

We may freely vary the entries of F and deformations of the entries of F induce deformations of the generators of $F^*(I_r)$; first order deformations define the module $N(X_F)$. Generators of $N(X_F)$ are tuples of minors of F of size $r-1$. If F and r are understood we simply write $N(X)$.

Properties of $N(X)$

The operation of forming $N(X)$ has some nice properties.

- N is universal. If the entries of F are coordinates on $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_F) = F^* N_U$.
- N_U is stable; $N_U = JM(\Sigma^r)$. Coupled with universality this implies $N(X_F) = F^* JM(\Sigma^r)$, which explains why the generators of $N(X_F)$ are tuples of minors of F of size $r-1$. (We say that the first order linear infinitesimal deformations are

stable if they are trivial. Here the first order linear trivial infinitesimal deformations are deformations are $JM(\Sigma^r)$.)

- Stability implies the polar varieties of Σ^r are the polar varieties of N_U.
- Universality implies $\Gamma_i(N(X_F)) = F^*\Gamma_i(N_U)$.
- Together they imply if $\tilde{F}$ defines a smoothing $\tilde{X}$ of X^d_F, then

$$mult_{\mathbb{C}}\Gamma_d(N(\tilde{X}_{\tilde{F}})) = F(\mathbb{C}^q) \cdot \Gamma_d(\Sigma).$$

In general, the intersection number $F(\mathbb{C}^q) \cdot \Gamma_d(\Sigma)$ is defined as follows. Work on $\mathbb{C}^q \times Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ and consider the intersection of the graph of F with $\mathbb{C}^q \times \Gamma_d(\Sigma)$, where, since $\Gamma_d(\Sigma)$ is the polar variety of codimension d in $\Gamma_d(\Sigma)$, the graph of F and $\mathbb{C}^q \times \Gamma_d(\Sigma)$ have complementary dimension in $\mathbb{C}^q \times Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$. If F is one to one, then the intersection number is that of the image of F with $\Gamma_d(\Sigma)$ in $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$.

If $r = n$ which is the case that I_r is the ideal of maximal minors, $F(\mathbb{C}^q) \cdot \Gamma_d(\Sigma)$ is computed in terms of the entries of F in [7]. We give a brief introduction to the formula in this paper in order to continue the study of curve singularities begun at the end of the section on multiplicities. This will also show why singularities defined by maximal minors are easier to study.

To study the polar varieties of $F^*JM(\Sigma^n)$, we need to understand Projan($\mathcal{R}(F^*JM(\Sigma^n))$). At a smooth point M of X_F, consider pairs (l_1, l_2) where $l_1 \in ker M^t$, $l_2 \in ker M$. Here $l_2 \in \mathbb{P}^{n-1}$ is unique, while the set of $l_1 \in \mathbb{P}^{n+k-1}$ has dimension k. Take the closure of this set in $X_F \times \mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1}$. This is the M-transform of X, denoted X_M. In [7], this is shown to be isomorphic to Projan($\mathcal{R}(F^*JM(\Sigma^n))$). The isomorphism is defined by

$$\Phi(x, (T_1, \dots, T_{n+k}), (S_1, \dots, S_n)) = (x, T \cdot S),$$

where $T \cdot S$ is an element of $\mathbb{P}Hom(n, n+k)$.

If $\tilde{F}$ is defines a smoothing $\mathcal{X}$ of X^d, then we want to calculate the degree over the base $\mathbb{C}$ of the polar curve of $\tilde{F}^*JM(\Sigma^n)$, denoted $m_d(F^*JM(\Sigma^n))$. Ideally, we would want to find the equations of the polar variety of Σ^n of complementary dimension to q, pull them back to $\mathcal{X}$ and take degree. This seems difficult. Instead, we will define "mixed polars" for which we can find equations, and which will define Cohen–Macaulay germs. To define these we look again at the construction of the polar varieties of Σ^n and their pull backs–the polars of $\tilde{F}^*JM(\Sigma^n)$.

First, denote the fiber over the origin in $\mathcal{X}$ of Projan $\mathcal{R}(\tilde{F}^*JM(\Sigma^n))$ by E. The generic rank of $\tilde{F}^*JM(\Sigma^n)$ is the same as the generic rank of $JM_z(\mathcal{X})$ which is $k+1$, the codimension of the generic fiber of $\mathcal{X}$. Then the polar curve is gotten by intersecting Projan $\mathcal{R}(\tilde{F}^*JM(\Sigma^n))$ with $d+k$ hyperplanes and projecting to $\mathcal{X}$. The degree of the polar curve over $\mathbb{C}$ is just $E \cdot h^{d+k}$ in $\mathbb{P}Hom(n, n+k)$, where h is the hyperplane class of $\mathbb{P}Hom(n, n+k)$. Now we use the isomorphism between Projan $\mathcal{R}(\tilde{F}^*JM(\Sigma^n))$ and $\mathcal{X}_M$. Denote the hyperplane classes on $\mathcal{X} \times \mathbb{P}^{n-1}$ and

$\mathcal{X} \times \mathbb{P}^{n+k-1}$ by h_2 and h_1 respectively. As classes, the pullback of h to $\mathcal{X} \times \mathbb{P}^{n+k-1} \times \mathbb{P}^{n-1}$ by the Veronese V is $h_1 + h_2$. So,

$$m_d(F^*JM(\Sigma^n)) = \sum_{i=0}^{d+k} \binom{d+k}{i} h_1^i h_2^{d+k-i} \cdot E.$$

The simple description we have of $C(\Sigma^n)$ which permits the decomposition of the last formula seems to be unique to $r = n$. This decomposition is the key to being able to write $m_d(F^*JM(\Sigma^n))$ as the alternating sum of colengths of ideals defined using the entries of F.

Define $\Gamma_{i,j}(\tilde{F}^*JM(\Sigma^n))$ to be $\pi_{\mathcal{X}}(\mathcal{X}_M \cap h_1^i h_2^j)$. We call these the mixed polars of type (i, j) of $\tilde{F}^*JM(\Sigma^n)$. Denote the degree of this mixed polar over $\mathbb{C}$ by $h_1^i h_2^j$.Then

$$m_d(F^*JM(\Sigma^n)) = \sum_{i=0}^{d+k} \binom{d+k}{i} h_1^i h_2^{d+k-i}.$$

It is shown in [7] that the mixed polars are related to certain determinantal varieties, and that the $h_1^i h_2^j$ are the alternating sum of degrees of these determinantal varieties. These degrees are just the lengths of the rings gotten by modding out the local ring of the associated determinantal variety by the coordinate on $\mathbb{C}$. In turn, these are just the lengths of the pullbacks by F of the rings defining the corresponding varieties on Σ^n. So, these numbers depend only on the component functions of F.

Now we consider again the determinantal space curves X_l defined by $F_{X_l}^{-1}(\Sigma^2)$,

$$F_{X_l} = \begin{bmatrix} z & x \\ y & z \\ x^l & y \end{bmatrix}.$$

We have $n = 2, k = 1, d = 1$, so

$$m_1(F_{X_l}^*JM(\Sigma^n)) = h_1^2 + 2h_1h_2 + h_2^2.$$

The h_2^2 term is zero, because we are working on $X_l \times \mathbb{P}^2 \times \mathbb{P}^1$, and the square of the hyperplane class on $\mathbb{P}^1$ is zero.

To calculate h_1^2, note that if we choose $(1, 0, 0)$ as the point of intersection of our two hyperplanes on $\mathbb{P}^2$, the ideal of $\Gamma_{2,0}$ for this choice on $Hom(2, 3)$, is $(a_{1,1}, a_{1,2}, a_{2,1}a_{3,2} - a_{2,2}a_{3,1})$, for these are the points of Σ^2 for which $(1, 0, 0)$ is in the kernel of M^t, $M \in Hom(2, 3)$. Pulling this ideal back by $F_{X_l}^*$ gives (x, z, y^2), which has colength 2, so $h_1^2 = 2$.

To compute h_1h_2, choose $(0, 1)$ as the point on $\mathbb{P}^1$ defined by the hyperplane, and let $(0, 0, 1)$ be the hyperplane on $\mathbb{P}^2$. So, we are looking for M such that $(0, 1)$ is in the kernel of M and some line defined by $(a, b, 0)$ is in the kernel of M^t. The ideal that defines this set is $(a_{2,1}, a_{2,2}, a_{2,3})$. This is already determinantal, so our procedure simplifies in this case. We get h_1h_2 is the colength of (x, y, z) which is 1, so $m_1(F^*_{X_l} JM(\Sigma^n)) = 2 + 2(1) + 0 = 4$ for all l.

Putting together our previous work, we see that if $l = 1$, then $e(JM(X_1), F_1^*(JM(\Sigma^2))) = 0$, but $e(JM(X_1), F_1^*(JM(\Sigma^2))) + m_1(F_1^*(JM(\Sigma^2))) = 3$. In fact, for isolated space curve singularities, the invariant $e(JM(X_F), F^*(JM(\Sigma^k))) + m_1(F_1^*(JM(\Sigma^k)))$ is never zero, since the polar of codimension 1 of Σ^r is non-empty for $r > 1$. (If $r = 1$, then F defines an ICIS, and $e(JM(X_F), F^*(JM(\Sigma^1))) = e(JM(X_F)) \neq 0$.)

It is important to understand when an invariant is zero. The next proposition gives a geometric criterion for when $e(JM(X_F), F^*(JM(Y)) = 0$, and also relates this invariant to the map F.

Proposition 3.23 *Suppose $F\ \mathbb{C}^q, 0 \to \mathbb{C}^n, 0$, $(Y, 0) \subset \mathbb{C}^n$, Y reduced and X_F defined with reduced structure also. Then $e(JM(X_F), F^*(JM(Y))) = 0$ if and only if no limiting tangent hyperplane to Y along the image of F contains the image of $DF(0)$.*

Proof Let $G = 0$ define Y with reduced structure. By hypothesis, $G \circ F$ defines X with reduced structure also. This implies that $JM(X_F) \subset F^*(JM(Y))$, by the Chain rule. The condition that $e(JM(X_F), F^*(JM(Y))) = 0$ is equivalent to $\overline{JM(X_F)} = \overline{F^*(JM(Y))}$. By Proposition 2.29 this is exactly the condition that the ideal sheaf induced by $JM(X_F)$ on Projan $\mathcal{R}(F^*(JM(Y)))$ is irrelevant ie. does not vanish on the fiber of Projan $\mathcal{R}(F^*(JM(Y)))$ over $0 \in X_F$. Since Projan$(JM(Y))$ is $C(Y)$, the fiber of Projan $\mathcal{R}(F^*(JM(Y)))$ over 0 is just limiting tangent hyperplanes to Y along the image of F.

The set Projan $\mathcal{R}(F^*(JM(Y)))$ is a subset of $X_F \times \mathbb{P}^{n-1}$. By the Chain Rule we know $DG(F(x)) \circ DF(x) = D(G \circ F)(x)$. Now, $DF(x)$ induces an ideal sheaf on $X \times \mathbb{P}^{n-1}$, because F has n component functions. If we restrict this sheaf to Projan $\mathcal{R}(F^*(JM(Y)))$, we get the ideal sheaf induced by $JM(X)$ on Projan $\mathcal{R}(F^*(JM(Y)))$, because this ideal sheaf arises from writing the generators of $JM(X)$ in terms of the generators of $F^*(JM(Y))$, and this is exactly what the Chain Rule does for us. Denote this sheaf by $\mathcal{F}$.

The condition that this ideal sheaf vanish at a point $(x, H) \in$ Projan $\mathcal{R}(F^*(JM(Y)))$ is just that the linear form defining H when applied to each of the generators of $\mathcal{F}$ give zero. For, the value of the i-th generator, $\sum_1^n \frac{\partial F_j}{\partial z_i} T_j$ on $(x, (a_1, \ldots, a_n))$ is $\sum_1^n \frac{\partial F_j}{\partial z_i}(x) a_j$. Since the fiber over 0 is the limiting tangent hyperplanes to Y along F at the origin the result follows. □

We explore the case of three lines in $\mathbb{C}^3$ $(l = 1)$ further. It is simpler to do this if we use the map

$$F = \begin{bmatrix} z & 0 \\ y & y \\ 0 & x \end{bmatrix}.$$

For this F, X_F is the coordinate axes.

Example 3.24 The fiber of Projan $\mathcal{R}(F^*JM(\Sigma^2))$ over 0 consists of three copies of $\mathbb{P}^1$, namely,

$$\begin{bmatrix} 0 & 0 \\ 0 & a \\ 0 & b \end{bmatrix}, \begin{bmatrix} a & 0 \\ b & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a & -a \\ 0 & 0 \\ b & -b \end{bmatrix}, (a, b) \in \mathbb{P}^1.$$

Further, the image of $DF(0)$ is not contained in any element of the fiber of Projan $\mathcal{R}(F^*JM(\Sigma^2))$ over 0.

To see why these assertions are true, note that the fiber of Projan $\mathcal{R}(F^*JM(\Sigma^2))$ is constant over the z axis for $z \neq 0$. This is because F and Σ^2 are homogeneous.

We have a general result which describes the fiber of $C(\Sigma^r)$ which we can apply here, which we now describe.

We know that the fiber to the normal bundle to the smooth manifold $\Sigma^{r+1} - \Sigma^r$ at $M \in \Sigma^{r+1} - \Sigma^r$, is $Hom(K(M), C(M))$ where $K(M)$ denotes the kernel of M and $C(M)$ denotes the cokernel, which we think of as the vectors in $\mathbb{C}^{n+k}$ which annihilate the image of M.

So up to some identifications, the fiber of $C(\Sigma^{r+1})$ at M is inside $\mathbb{P}Hom(K(M), C(M))$. Let $\Sigma_j(M)$ denote the elements of $Hom(K(M), C(M))$ of kernel rank j.

Let X_j denote the projective variety determined by $\overline{\Sigma}_j$. If $M \in Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$, then we denote $\mathbb{P}(\overline{\Sigma}_r(M))$ by $X_r(M)$.

Theorem 3.25 (Conormal fiber Theorem) *Suppose M is in Σ_s, $s > r$. Then the fiber of the conormal of $C(\overline{\Sigma}_r)$ at M is $X_{s-r}(M)$.*

Proof See the Conormal Fiber Theorem at the end of Sect. 2 of [7]. □

In the case of singularities defined by maximal minors if we know the M-modification of X_F we can compute these fibers. For example, at points on the z axis of X_F, $z \neq 0$, we see that the fiber is $(0, a, b) \times (0, 1)$, because $(0, 1)$ is the kernel of $F(0, 0, 1)$, and $(0, a, b)$ is the kernel of $F^t(0, 0, 1)$. Then a point of the fiber maps to

$$\begin{bmatrix} 0 \cdot 0 & 0 \cdot 0 \\ 0 \cdot a & 1 \cdot a \\ 0 \cdot b & 1 \cdot b \end{bmatrix}$$

The condition that the image of $DF(0)$ is contained in a limiting tangent hyperplane implies that

$$\frac{\partial F}{\partial x} \cdot \begin{bmatrix} 0 & 0 \\ 0 & a \\ 0 & b \end{bmatrix} = 0, \ \frac{\partial F}{\partial y} \cdot \begin{bmatrix} 0 & 0 \\ 0 & a \\ 0 & b \end{bmatrix} = 0.$$

Expanding we get:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 & 0 \cdot 0 \\ 0 \cdot 0 & 0 \cdot a \\ 0 & 1 \cdot b \end{bmatrix} = 0, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 \cdot 0 & 0 \cdot 0 \\ 1 \cdot 0 & 1 \cdot a \\ 0 & 0 \cdot b \end{bmatrix} = 0.$$

This implies that $a = b = 0$; thus no element of the fiber which is a limit of tangent hyperplanes to Σ^2 along the image of the z axis in Σ^2 can contain the image of $DF(0)$.

Problem 3.26 *Prove the rest of the assertions of the last example.*

We can use this simple example to get some idea of the possible ways our invariants can change in a family. Given a family of singularities $\{X_t^d\}$, with parameter t, let $e(JM(X_t), F_t^* JM(\Sigma^r), t)$ denote the sum of $e(JM(X_t), F_t^* JM(\Sigma^r), x)$ over all $x \in X_t$; let

$$e_\Gamma(M, F_t^* JM(\Sigma^r), x) = e(JM(X_t), F_t^* JM(\Sigma^r), x) + m_d(F_t^* JM(\Sigma^r), x),$$

and define $e_\Gamma(M, F_t^* JM(\Sigma^r), t)$ in a way similar to $e(JM(X_t), F_t^* JM(\Sigma^r), t)$.

Example 3.27 Let $F_t = \begin{bmatrix} z & 0 \\ y - t & y + t \\ 0 & x \end{bmatrix}$. Let $X_t = X_{F_t}$, then X_t for $t \neq 0$ consists of three lines which intersect in two plane curve singularities–both ordinary nodes. Further $e(JM(X_t), F_t^* JM(\Sigma^2), t)$ is 0 for $t = 0$ and 4 for $t \neq 0$, hence is not upper semicontinuous. The invariant $e_\Gamma(M, F_t^* JM(\Sigma^2), t) = 4$, for all t.

The example shows that $e_\Gamma(M, F_t^* JM(\Sigma^2), t) = 4$ being independent of t does not prevent the singularity from splitting. If we assume the parameter space is embedded in X as $\mathbb{C} \times 0$, and ask that $e_\Gamma(M, F_t^* JM(\Sigma^2), (t, 0))$ is independent of t, then splitting cannot occur because $e_\Gamma(M, F_t^* JM(\Sigma^2), t)$ is upper semicontinuous, and $e_\Gamma(M, F_t^* JM(\Sigma^2), x)$ is always non-zero in the curve case if x is singular.

Equisingularity of Determinantal Varieties

In this section we bring together many elements of these lectures to prove a theorem on the Whitney equisingularity of families of determinantal singularities.

The key invariant is the generalization of the invariant $m_d(X^d)$ in the ICIS case. As in the definition of $m_d(F^* JM(\Sigma^n))$ we pick a smoothing $\tilde{F}$ of F. We can extend the sheaf $JM(X_F)$ over $X_{\tilde{F}}$ by considering the sheaf of modules generated by the partial derivatives of $\tilde{F}$ with respect to the variables of $\mathbb{C}^q$, the ambient space of X_F. Denote this by $JM_z(X_{\tilde{F}})$. Now assume $X = X_F = F^{-1}(\Sigma^r)$; for simplicity, assume X has a smoothing. Applying the MPT to this set-up (3.22), we know that

$$m_d(X) = e(JM(X_F), F^*(JM(\Sigma^r))) + F(\mathbb{C}^q) \cdot \Gamma_d(\Sigma^r) := e_\Gamma(JM(X_F), F^*(JM(\Sigma^r))).$$

In an analogous way we can define $m_d(mJM(X))$, and again we have as a corollary of the MPT,

$$\begin{aligned} m_d(mJM(X)) &= e(mJM(X_F), F^*(JM(\Sigma^r))) + F(\mathbb{C}^q) \cdot \Gamma_d(\Sigma^r) : \\ &= e_\Gamma(mJM(X_F), F^*(JM(\Sigma^r))). \end{aligned}$$

(In picking the smoothing it is necessary to ensure that $\tilde{F}(t,0) \notin \Sigma^r$ for $t \neq 0$.) We use the notation e_Γ for the multiplicity of a pair corrected by the curvature of the larger module.

If we have a family of sets X_F defined by $F : \mathbb{C}^t \times \mathbb{C}^q \to Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$, $Y = \mathbb{C}^t \times 0 \subset X_F$, we show that $m_d(mJM(X))$ controls the Whitney conditions for the open stratum of X_F along Y. The precise statement follows.

Theorem 3.28 *Suppose* $(X^{d+t}, 0) \subset (\mathbb{C}^{q+t}, 0)$, $X = F^{-1}(\Sigma^r)$, $F : \mathbb{C}^{q+t} \to Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$, *$Y$ a smooth subset of X, coordinates chosen so that $\mathbb{C}^t \times 0 = Y$, F induced from a deformation of the presentation matrix of $X(0)$, X equidimensional with equidimensional fibers, of expected dimension, $X(y)$ has only isolated singularities for all y.*

(A) Suppose the singular set of X is Y. Suppose $e_\Gamma(m_y JM(X_y), F_y^* JM(\Sigma^r))$ is independent of y. Then the union of the singular points of $X(y)$ is Y, and the pair of strata $(X - Y, Y)$ satisfies condition W.

(B) Suppose the singular set of X is Y and the pair $(X - Y, Y)$ satisfies condition W. Then $e_\Gamma(m_y JM(F_y), F_y^* JM(\Sigma^r))$ is independent of y.

Proof First, we prove (A). We can embed the family in a restricted versal unfolding with smooth base $\tilde{Y}^l$. Consider the polar variety of $m_Y JM_z(F)$ of dimension l, and the degree of its projection to $\tilde{Y}^l$ along points of Y. The hypothesis on $e_\Gamma(m_y JM(X_y), F_y^* JM(\Sigma^r))$ implies by the multiplicity polar theorem that this degree is constant over Y. In turn this implies that the polar variety over Y does not split, hence the polar of the original deformation is empty. This implies that the fiber of the exceptional divisor of B_{m_Y} Projan$(JM_z(F))$ cannot be maximal, since there is no polar variety. By the theorem of Kleiman-Thorup on the dimension of this fiber, it then follows that $JM_Y(F) \subset \overline{m_Y JM_z(F)}$ which implies W.

This also implies that $\overline{JM(F)} \subset \overline{JM_z(F)}$. Hence the union of the singular points of F_y which is the cosupport of $\overline{JM_z(F)}$ is equal to the cosupport of $\overline{JM(F)}$ which is Y. Then the inclusion $JM_Y(F) \subset \overline{m_Y JM_z(F)}$ implies W for $(X - Y, Y)$. (Cf. [9].)

Now we prove (B). W implies $JM_Y(F) \subset \overline{m_Y JM_z(F)}$ which implies that $\overline{m_Y JM(F)} = \overline{m_Y JM_z(F)}$. We know by [38] that condition W implies that the fiber dimension of the exceptional divisor of $B_{m_Y}(C(X))$ over each point of Y is as small as possible. The integral closure condition $\overline{m_Y JM(F)} = \overline{m_Y JM_z(F)}$ implies that the same is true for $B_{m_Y}(\text{Projan}\,\mathcal{R}(JM_z(F)))$. This implies that the polar of $m_Y JM_z(F)$ is empty, hence by the multiplicity polar formula the invariant $e_\Gamma(mJM(F_y), F_y^* JM(\Sigma^r))$ is independent of y. □

We also have a geometric description of our invariant based on the smoothing and the existence of a unique Milnor fiber.

Theorem 3.29 $e(JM(X_y), F_y^* JM(\Sigma^r)) + F(y)(\mathbb{C}^q) \cdot \Gamma_d(\Sigma^r) = (-1)^d \chi(X_{s,y}) + (-1)^{d-1} \chi((X \cap H)_{s,y})$, *$X_{s,y}$ a smoothing of $X(y)$.*

Proof (Cf. [11, p. 130], [32].) □

Example 3.30 Consider the family of curves X_l, defined by the minors of

$$F_{X_l} = \begin{bmatrix} z & x \\ y & z \\ x^l & y \end{bmatrix}.$$

Then by our previous work we have $e_\Gamma(JM(X_l), F_l^* JM(\Sigma^2)) = 2l + 2$, for $l = 1$ or $l - 1$ not divisible by 3, $l > 1$. Since $-\chi(X_{s,l}) + \chi((X \cap H)_{s,l}) = \mu(X_l) + m(X_l) - 1$, we have $\mu(X_l) = 2l$ recovering a result of Watanabe et al. [30].

Challenge Problems and Further Directions in Determinantal Singularities

- In the maximal minor case, the work of [7] gives a formula for the Euler characteristic of a smoothing of a nondeterminantal singularity. Can we say something about the Betti-numbers of a smoothing when there is more than 1? (Frühbis–Krüger and Zach have some results for three-folds. Cf. [5, 40].)
- What is the connection between the results of [7] on the Euler characteristic of a smoothing and Damon–Pike [3] in the (2,3) case?
- What is the relation in the curve case between between the results of [7] and those of Greuel and Buchweitz [2] and Rosenlicht differentials?
- For what determinantal singularities is the invariant $m_d(X^d) = 0$? Hopefully, we can classify them. In May 2015, work was done giving the dimensions in which they can appear, and a transversality condition that must be satisfied. In September of 2016 as part of a project with Ruas and Pedersen, normal forms for the space curve-maximal minor case were found.
- What additional invariants are needed to ensure the singular locus of a family does not split? In the ICIS case the independence from parameter of $m_d(X_y)$ ensures the singular locus is the parameter axis. Because some determinantal singularities have $m_d(X) = 0$, this is not true for families of determinantal singularities, even in the maximal minor (2, 3) case.
- Is there a way to connect the terms that appear in the calculation of the multiplicity of the polar of $F^* JM(\Sigma^n)$ with the geometry of X_F in the $(n, n + k)$ case?
- What is a formula in terms of the entries of the presentation matrix for $F(\mathbb{C}^q) \cdot \Gamma_d(\Sigma^r)$, $1 < r < n$?
- What can we say about EIDS (Essentially Isolated Determinantal Singularities)? These include determinantal singularities which are isolated, but cannot be smoothed, because the dimension of the domain is too large, as well as determinantal singularities which are non-isolated, but which are well behaved away from the origin.) Some work on these has been done in [7, 20] and other papers mentioned in their bibliographies.
- Can we calculate the multiplicities of the polar varieties of $\Sigma^r \subset Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ at the origin of $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$? This is known for the cases $r = n, 2$ ([8]). This will give a lower bound on the size of the contribution of $F(\mathbb{C}^q) \cdot \Gamma_d(\Sigma^r)$ to $e_\Gamma(JM(X), F^* JM(\Sigma^r))$. Since the Σ^r are homogeneous, their ideals define projective varieties, and these multiplicities will be the degrees of the polar classes of the projective varieties.

- There are other invariants associated with X such as the index of differential forms and the Milnor number(?) of functions with isolated singularities. Compute these in terms of infinitesimal invariants similar to those of these lectures.(Cf. [19] for a framework for doing this.)

4 Afterword: Examples of the Point of View of the Introduction

We will talk about two examples of our point of view.

Hypersurfaces with isolated singularities are our first example. Suppose $X^n, 0$ has an isolated singularity at the origin, $X = f^{-1}(0)$.

Choose the landscape This is done by looking at the possible deformations of X. We see we can deform f freely, and still, for small deformations, get a hypersurface with at most isolated singularities. So, the landscape will be all hypersurfaces in $\mathbb{C}^{n+1}$ with at most isolated singularities. The generic element that X deforms to is its Milnor fiber.

Describe the connection between X and its generic element To do this, deform X to its Milnor fiber, using $F(y, z) = f(z) - y$. Then the ideal $J_z(F)$, when restricted to the graph, vanishes only at $(0, 0)$, so its polar curve is given by the vanishing of the first n partial derivatives, in generic coordinates. Applying the MPT, we get $e(J(f), \mathcal{O}_{X,0}) = mult_{\mathbb{C}}\Gamma_n(J_z(F))$.

In turn $mult_{\mathbb{C}}\Gamma_n(J_z(F))$ is the colength of the ideal $(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})$ in $\mathcal{O}_{n+1}$. This is $\mu(X) + \mu(X \cap H)$, H a generic hyperplane.

Determine the first order infinitesimal deformations Since $f \to f + tg$ where g is arbitrary, is a first order defomation, and the corresponding infinitesimal first order deformation is $f \to \frac{\partial f + tg}{\partial t} = g$, the first order infinitesimal deformations are just $\mathcal{O}_{X,0}$.

Our invariant for controlling Whitney equisingularity is $e(mJ(f), \mathcal{O}_{X,0})$.

If we have a family of hypersurfaces $\mathcal{X}$, then if $\mu(X) + \mu(X \cap H)$ changes, then so must $e(J(f), \mathcal{O}_{X,0})$, and the exceptional divisor of $B_{J_z(F)}(\mathcal{X})$ must pick up a vertical component and vice-versa. The change in the topology of the landscape is reflected in a dramatic change in the fibers of the exceptional divisor, which is the infinitesimal information.

For determinantal singularities the story is similar.

If we look at all possible deformations, then we have examples where the same singularity can be deformed in two different ways, even giving Whitney equisingular families in which the generic fiber has non-homeomorphic smoothings [7]. So, we restrict our deformations by using the same size presentation matrix. The entries of the matrix can be deformed freely.

Then, the landscape will be the determinantal singularities corresponding to a matrix of fixed size. The generic element associated to X will be smooth, given some dimension restriction; otherwise we can say what the stabilizations of the singularity are, and can begin to study those [20].

In the case of smoothable singularities, by use of the multiplicity polar theorem and some topology, we get Theorem 3.29 which gives the connection between the topology of smoothing and the algebraic invariants of the singularity, which are connected to its infinitesimal geometry. This is generalized in [20] to the EIDS case.

The first order infinitesimal deformations of X can be explicitly computed; deform an entry of the presentation matrix by t, calculate the minors of the order used to define X; taking derivative with respect to t then gives a map from the defining equations for X into tuples in $\mathcal{O}^g_{X,0}$, where g is the number of defining equations. These give the generators of $N(X)$. It is clear from this formulation that N is universal and specializes well in families. We can calculate $JM(\Sigma)$ explicitly–the partial with respect to the (i, j) entry of the matrix is just the corresponding generator of N. So Σ is stable. The geometric representation of $C(\Sigma^n)$ in terms of kernels of M and M^t gives the formula for computing $mult_{\mathbb{C}}\Gamma_d(N(\tilde{X}_{\tilde{M}}))$ using the presentation matrix, but leaves the formula in terms of the entries still to be determined in general.

Once again, a change at the infinitesimal level of the family is always tied to a change in topology of the generic related elements. Here, the infinitesimal level of a family $\mathcal{X}^{t+d} \subset \mathbb{C}^{t+q}$ is the relative conormal modification $C_Y(\mathcal{X})$ of $\mathcal{X}$, which is the limits of tangent hyperplanes in $\mathbb{C}^q$ to the fibers of $\mathcal{X}$ over $\mathbb{C}^t$. Assume the singular locus of the family is $\mathbb{C}^t \times 0$. By a change at the infinitesimal level, we mean that the dimension of the fiber of $C_Y(\mathcal{X})$ over the origin in $\mathcal{X}(0)$ jumps in dimension from the generic value of $q - d - 1$ to at least $q - 1$. This is equivalent to the polar variety of dimension t of the module $JM_z(\mathcal{X})$ at (0, 0) being non-empty. In turn by the MPT, this implies that $m_d(\mathcal{X}(0)) > m_d(\mathcal{X}(y))$, y a generic value of $\mathbb{C}^t$. By Theorem 3.29, this implies that the topology of the smoothings of $\mathcal{X}(0)$ and $\mathcal{X}(y)$ are different.

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