Equisingularity

Lê Dũng Tráng

Abstract We give a quick survey of problems concerning Equisingularity.

Classifications 14B05 · 14B12 · 32S15

Introduction

A singularity is the germ of a complex analytic space (X, x). Equisingularity means the same singularity.

A naive view would be that two singularities are equisingular if they are analytically the same. It is known that two singularities (X, x) and (Y, y) are analytically the same if and only if the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic (see e.g. [8]).

In the case of complex singularities of hypersurfaces, it seems that one may use analytic isomorphism to define equisingularity, since for the most "simple" singularities analytic isomorphisms and ambient homeomorphisms between the singularities are equivalent.

For instance two complex cusps of plane curves are equally locally homeomorphic in the local ambient space or analytically isomorphic. One expresses this property by saying that the moduli of a complex cusp singularity is reduced to a point. More generally the moduli of a space or a singularity is the parameter space of a deformation of complex analytic spaces or singularities having the same "topological features", but being analytically different.

However, one knows that the moduli of a singularity is in general not reduced to a point. In the case of plane curves the moduli of the germs of plane curves with one Puiseux pair (3, 7) (see e.g. [25] Sect. 1 p. 284) is not reduced to a point. There is the famous example of Riemann of a family of cubic curves having different analytic structures. The cones on these cubic curves define a family of complex singularities having different analytic structures.

L. D. Tráng (✉)
Université Aix-Marseille, Marseille, France
e-mail: ledt@ictp.it

R. N. Araújo dos Santos et al. (eds.), *Singularities and Foliations. Geometry, Topology and Applications*, Springer Proceedings in Mathematics & Statistics 222, https://doi.org/10.1007/978-3-319-73639-6_10

Therefore, the analytic isomorphism of germs does not seem to be the adequate answer for equisingularity, since the analytic type can change continuously on a complex analytic space.

Several mathematicians like H. Whitney, R. Thom, O. Zariski have tried to give a good definition of equisingularity.

H. Whitney (see [30]) introduced a partition of a general complex analytic space, called a Whitney stratification (see below). The article [30] is the fruit of several discussions that H. Whitney had with R. Thom. Later R. Thom and J. Mather (in [13, 22]) proved that locally along the strata of a Whitney stratification the analytic space is a topological product.

In a paper published in 1937, O. Zariski used an argument very similar to equisingularity to prove that the fundamental group of the complement of a complex projective hypersurface of dimension $n \geq 2$ is isomorphic to the fundamental group of the complement of the complex curve intersection of the hypersurface with a general complex plane in the general plane section (see "A theorem on the Poincaré group of an algebraic hypersurface", Ann. of Math. **38** (1937), 131–141).

Unfortunately, at that time O. Zariski did not have a clear definition of what should be a general plane section and a fortiori of what should be equisingular sections. O. Zariski used to call jokingly that paper his last italian paper.

The term of equisingularity appears in the papers of O. Zariski (see [32, 34]). The viewpoint changed somehow. One considers a partition of the analytic space X such that, for two points x_1, x_2 of a stratum of the partition, the germs (X, x_1) and (X, x_2) are equisingular. For Zariski, he considers algebraic varieties and he wants that the partition is defined by algebraic data. In the case of a complex hypersurface the big stratum is the stratum of non-singular points and the stratum of codimension one is the one such that transversal sections by a plane of dimension 2 give a germ of plane curve with the same Puiseux pairs, if the germ is a branch, or a germ of plane curve with a given topology in the case of several analytic branches.

He could characterize the codimension one stratum with a new concept called the saturation (see [36]).

In the period from 1965 to 1968, O. Zariski introduced the notion of saturation of a ring. Then, he published an algebraic understanding of what he called equisingularity in several papers [31–36]. Surprisingly these papers attract little attention of the community of algebraic geometers. One of the reasons of this attitude might be because the notion of equisingularity was not clearly defined but in the case of plane curves for which equisingular germs of plane curves are germs of plane curves with isomorphic saturation rings. Unfortunately this definition does not work in dimension ≥ 2.

When the hypersurface singularity is isolated, in [14] (1968) J. Milnor has introduced a multiplicity that we call the Milnor number of the isolated singularity which is a topological invariant of the embedded topology of the hypersurface (see e.g. [26] Proposition p. 261). However, two isolated hypersurface singularities having the same Milnor number may not be topologically equisingular: two plane curves with one Puiseux pair (p_1, q_1) and (p_2, q_2) such that $(p_1 - 1)(q_1 - 1) = (p_2 - 1)(q_2 - 1)$ have the same Milnor number:

$$\mu = (p_1 - 1)(q_1 - 1) = (p_2 - 1)(q_2 - 1)$$

but, if $p_1 \neq p_2$, are not topologically equisingular.

In 1968 in a seminar at IHES, H. Hironaka made a conjecture that in a family of plane curves with Milnor number constant, the local topology of the plane curve does not change. In 1970 I found a proof of this conjecture (see [24] published in 1971). In 1971 together with C.P. Ramanujam I extended this result to complex hypersurfaces of dimension ≥ 3 (see [27] published in 1976). The restriction on the dimension came from the use of the h-cobordism Theorem.

This topological result showed that equisingularity can be understood either topologically, or algebraically, as Zariski tried to do for plane curves. The different ways to define equisingularity should at least imply topological equisingularity. Furthermore one should be able to "stratify" an algebraic variety with equisingular germs along each strata. The case of plane curves which should correspond to strata of codimension one in a hypersurface would be the typical first example.

Finally, the concept of equisingularity, although vague, can be formulated in the following way:

Let X be a complex analytic space. There is an analytic partition $X = \coprod_{i \in I} X_i$, such that:

- The definition of the analytic partition should be given by algebraic conditions on the local ring $\mathcal{O}_{X,x}$;
- All the germs (X, x) with $x \in X_i$ should be equisingular, e.g. topologically equisingular in the case of hypersurfaces;
- Following Zariski (see [31]) the multiplicity of (X, x) should be constant along X_i.

In these notes, in a quick way we shall present most of the aspects of Equisingularity theory that is known nowadays, hoping that it will motivate younger mathematicians to make research in this direction.

1 Basic Notions

1.1 Germs

Let X be complex analytic spaces and let x be a point of X. One calls germ of X at the point x the pair (X, x). Let (X, x) and (Y, y) be complex analytic germs. The germ at x of morphism from (X, x) into (Y, y) is the equivalence class of complex analytic morphisms defined on a neighborhood of x in X into Y such that the image of x is y and two such morphisms coincide on a neighborhood of x in X.

Germs of complex analytic spaces with germs of complex analytic morphisms form a category that we shall call *German*. The objects of this category are germs of complex analytic spaces and the arrows from (X, x) into (Y, y) are the germs of morphisms of (X, x) into (Y, y).

Similarly complex analytic algebras isomorphic to the local ring $\mathcal{O}_{X,x}$ of germs of complex analytic functions of some complex analytic space X at x form a category *Algan* in which the objects are complex analytic algebras and the arrows are $\mathbb{C}$-homomorphisms of these algebras.

We have a natural functor:

$$\Phi : \text{German}^{\circ} \rightarrow \text{ Algan}$$

where German° is the opposite category of German and such that $\Phi((X, x)) = \mathcal{O}_{X,x}$, where $\mathcal{O}_{X,x}$ is the local ring of germs at x of complex analytic functions on X.

It is known that (see [8] p. 13–02):

Theorem 1.1 *The functor Φ : German°→Algan is an equivalence of categories from the opposite of the category of complex analytic germs with the category of complex analytic local algebras.*

1.2 Analytic Equivalence

Of course one can classify singularities using analytic equivalence. Using Theorem 1.1 two singularities (X, x) and (Y, y) are analytically equivalent if the local analytic rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic. However such a classification is too fine and if $x \in X$ the analytic structure of $\mathcal{O}_{X,x}$ can vary continuously.

For instance, the complex surface of $\mathbb{C}^3$ given by $X(X-Y)(X+Y)(X+TY) = 0$ has a continuous analytic structure at the points $(0, 0, t)$.

1.3 Topological Equivalence

In the case of hypersurfaces, one has an notion of topological equivalence.

The germs of hypersurfaces (X, x) and (Y, y) of $\mathbb{C}^n$ are topologically equivalent if there exists a germ of homeomorphisms φ of $(\mathbb{C}^n, x)$ into $(\mathbb{C}^n, y)$ such that the image of (X, x) is (Y, y). In what follows, we shall say that two topologically equivalent hypersurface singularities are topologically equisingular.

This notion of topological equivalence does not extend to codimension ≥ 2 analytic spaces. For instance, two analytically irreducible germs of curves of $\mathbb{C}^n$ are topologically equivalent if $n \geq 3$.

1.4 Plane Curves

The case of germs of complex plane curves is the test case where all the criteria for a good equivalence are working.

As Zariski did, we shall call analytic plane branch an analytically irreducible germ of reduced complex plane curve. Let us suppose that an analytic branch $(C, 0)$ is defined by the equation $f = 0$ where f is an irreducible germ of complex analytic function of $(\mathbb{C}^2, 0)$ at the origin 0. Let us suppose that the coordinates X, Y of $(\mathbb{C}^2, 0)$ satisfy the Weierstrass type condition:

$$f(0, Y) \neq 0.$$

One can define the Puiseux exponents relatively to the coordinates X, Y (see [25]). Those Puiseux exponents define the knot type of the intersection $\{f = 0\} \cap \mathbb{S}_\varepsilon(0)$ of C and a sufficiently small sphere $\mathbb{S}_\varepsilon(0)$ centered at 0 with radius ε (e.g. see [25] Sect. 1).

Puiseux Theorem shows that one can parametrize the branch $(C, 0)$, i.e. there exist a convergent series $\Phi(X^{1/n})$ in $X^{1/n}$ such that $f(X, \Phi(X^{1/n})) \equiv 0$ and n equals the valuation of $f(0, Y)$:

$$\phi\left(X^{\frac{1}{x}}\right) = \Sigma a_k X^{\frac{k}{n}}$$

Let us now define the Puiseux exponents relatively to the coordinates (X, Y).

If $n = 1$, the Puiseux expansion is a formal series with coefficients in $\mathbb{C}$. In this case, there are no Puiseux exponent.

If $n > 1$, the set $E_1 = \{k/n \notin \mathbb{N},\ a_k \neq 0\}$ is not empty, since n is the smallest integer ℓ, such that $\Phi(X^{1/n}) \in \mathbb{C}[[X^{1/\ell}]]$.

Define the first Puiseux exponent relatively to the coordinates (X, Y):

$$\frac{k_1}{n} = \inf\{k/n \notin \mathbb{N},\ a_k \neq 0\}.$$

Then, either (k_1, n) are relatively prime and there is only one Puiseux exponent, or

$$\frac{k_1}{n} = \frac{m_1}{n_1}$$

and $n_1 < n$. The set $E_1 = \{k/n \notin (1/n_1)\mathbb{N},\ a_k \neq 0,\ k > k_1\}$ is not empty, otherwise $\Phi(X^{1/n})$ belongs to $\mathbb{C}[[X^{1/n_1}]]$.

Define the second Puiseux exponent by:

$$\frac{k_2}{n} := \inf\{\frac{k}{n} \notin \frac{1}{n_1}\mathbb{N},\ a_k \neq 0,\ k > k_1\}.$$

There is a unique way to write:

$$\frac{k_2}{n} = \frac{m_2}{n_1 n_2}$$

in such a way that (m_2, n_2) are relatively prime.

Then, either $n_1 n_2 = n$ and there are only two Puiseux exponents, or $n_1 n_2 < n$ and the set:

$$E_2 = \{\frac{k}{n} \notin \frac{1}{n_1 n_2}\mathbb{N},\ a_k \neq 0,\ k > k_2\}$$

is not empty.

By induction, one defines $m_h/n_1 \dots n_h$, where (m_h, n_h) are relatively prime. Either, $n_1 \dots n_h = n$ and there are h Puiseux exponents, or $n_1 \dots n_h < n$ and the set:

$$E_h = \{\frac{k}{n} \notin \frac{1}{n_1 \dots n_h}\mathbb{N},\ a_k \neq 0,\ k > k_h\}$$

is not empty, in which case $\inf E_h = k_{h+1}/n = m_{h+1}/n_1 \dots n_{h+1}$, where (m_{h+1}, n_{h+1}) are relatively prime and unique.

The process has to end, since n has a finite number of divisors.

The pairs $(m_1, n_1), \dots, (m_h, n_h)$ are called the Puiseux pairs of $(C, 0)$ relatively to the coordinates (X, Y) and the exponents:

$$\frac{m_1}{n_1}, \dots, \frac{m_h}{n_1 \dots n_h}$$

are called the Puiseux exponents of $(C, 0)$ relatively to the coordinates (X, Y).

One can prove:

Theorem 1.2 *Two plane branches* $(C_1, 0)$ *and* $(C_2, 0)$ *are topologically equivalent if and only if, there are coordinates for which their Puiseux exponents are equal.*

In [36] O. Zariski introduced the notion of saturation $\tilde{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X.x}$ when (X, x) is a complex analytic plane branch (see below in Sect. 3.2). The saturation $\tilde{\mathcal{O}}_{X,x}$ is a local ring which contains the local ring $\mathcal{O}_{X.x}$ and is contained in the normalization $\bar{\mathcal{O}}_{X,x}$:

$$\mathcal{O}_{X.x} \subset \tilde{\mathcal{O}}_{X.x} \subset \bar{\mathcal{O}}_{X,x}.$$

It is known that the normalization $\bar{\mathcal{O}}_{X,x}$ is the ring of germs of meromorphic functions whose restriction to (X, x) are bounded (e.g. see [18] Chapter VI).

Similarly F. Pham and B. Teissier have proved that the saturation $\tilde{\mathcal{O}}_{X,x}$ is the ring of germs of meromorphic functions on (X, x) which are Lipschitz functions (see [19] or [7]).

1.5 Hypersurfaces

In the case of reduced hypersurfaces (X, x) we proved in [26] (Proposition of the Introduction) that the local monodromy of the local Milnor fibration of (X, x) (see [14] Sect. 4 for the definition and existence) is a topological invariant of (X, x). In particular the Milnor numbers of two topological equisingular hypersurfaces $(X_1, 0)$ and $(X_2, 0)$ are the same.

It is remarkable that, in a smooth family of complex hypersurfaces containing the origin 0 and having at 0 the same Milnor number, the hypersurfaces are topologically equisingular (see [27]). However the dimension n of the hypersurfaces is $\neq 2$ because the proof uses the h-cobordism Theorem.

Conjecture: It is natural to conjecture that this result holds also in dimension 2.

1.6 Whitney Stratifications

As we have mentioned in the introduction, in 1965 H. Whitney introduced the notion of Whitney condition (see [30]).

Let X be a reduced complex analytic space. Let Σ_X the subset of singular points of X. It is known that Σ_X is an complex analytic subspace of X. We have the partition of X:

$$X = (X - \Sigma_X) \coprod \Sigma_X.$$

Defining by induction $X_1 = \Sigma_X$ and, for $i \geq 1$, $X_{i+1} = \Sigma_{X_i}$, we have:

$$X = (X - \Sigma_X) \coprod (X_1 - X_2) \coprod (X_2 - X_3) \coprod \dots$$

which has to be a finite partition since $X_i - X_{i+1}$ is a manifold and $\dim X_i > \dim X_{i+1}$ if $X_i \neq \emptyset$. It is called the partition by dimension of X.

The partition by the connected components of $X_i - X_{i+1}$ is called the *full partition* by dimension of X. If X is a complex analytic space, its full partition by dimension might not be finite but it is locally finite.

A complex analytic manifold Y contained in a complex analytic space X is *strict* if the closure $\bar{Y}$ of Y in X and the difference $\bar{Y} - Y$ are complex analytic subspace of X.

If Y is strict of dimension m, then Lemma 3.13 of [30] shows that the dimension of $\bar{Y}$ is m and the dimension of $\bar{Y} - Y$ is $< m$.

A *strict partition* of a complex analytic space X is a partition, which is locally finite, into strict manifolds. The elements of a strict partition are called the strata of the strict partition.

Lemma 18.2 of [30] states that the partition by dimension and the full partition by dimension of a complex analytic space X are strict partitions of X.

A strict partition $(X_i)_{i \in I}$ of a complex analytic space X satisfies *the frontier condition* if:

$$\forall i, j \in I, X_i \cap \bar{X}_j \neq \emptyset \Rightarrow X_i \subset \bar{X}_j \text{ and } \dim X_i < \dim X_j.$$

Definition 1.1 Let X be a complex analytic space. A stratification of X is a locally finite strict partition by connected strata which satisfies the frontier condition.

Now we can define the conditions of Whitney.

Definition 1.2 Let M and N be two complex analytic manifolds strict in the complex analytic space X. Assume that $N \subset \bar{M}$. Let $x \in N$. We may assume that locally at

x the are neighborhoods V of x in N and U of x in M such that $V \subset \bar{U} \subset \mathbb{C}^N$. One says that M satisfies the Whitney condition (a) at x along N if, for any sequence (x_n) of points of M converging x for which the sequence of tangent spaces $T_{x_n}(M)$ converge to T, we have $T_x(N) \subset T$.

Definition 1.3 Let M and N be two complex analytic manifolds strict in the complex analytic space X. Assume that $N \subset \bar{M}$. Let $x \in N$. We may assume that locally at x the are neighborhoods V of x in N and U of x in M such that $V \subset \bar{U} \subset \mathbb{C}^N$. One says that M satisfies the Whitney condition (b) at x along N if, for any sequence (x_n) of points of M and any sequence (y_n) of N converging x, for which the sequence of tangent spaces $T_{x_n}(M)$ converges to T and for which the sequence of lines $\overline{y_n x_n}$ converges to ℓ, we have $\ell \subset T$.

In [13] (Proposition 2.4) it is proven that If M satisfies the Whitney condition (b) at x along N, then it satisfies the condition of Whitney (a) at x along N.

We say that M satisfies Whitney condition (b) along N if it satisfies Whitney condition (b) at any point x of N along N.

Definition 1.4 A stratification $(S_i)_{i \in I}$ of the complex analytic space X is a Whitney stratification if, for any pair (S_i, S_j) of strata such that S_i is contained in the closure $\bar{S}_j$, the stratum S_j satisfies the condition of Whitney (b) at any point x of S_i along S_i.

In [30] (Theorem 19.2 p. 540, H. Whitney proved that any reduced complex analytic space has a Whitney stratification.

The remarkable result of Mather and Thom is that for any Whitney stratification of a complex analytic space X the topology of X along any strata is a local product. Namely let (S_i) be a Whitney stratification of X, for any point $x \in S_i$, there is a neighborhood U_x of x in X, such that U_x is homeomorphic to the product $(U_x \cap S_i) \times (N_x \cap U)$ where N_x is a slice of X transverse at x to S_i in a local smooth ambient space.

Since the strata of a Whitney stratification are pathwise connected, the topology of the germ of N_x at x does not depend on the point x in a stratum.

In fact, the notion of stratification as well as Whitney conditions can be extended to subanalytic spaces or even to definable spaces. We shall not consider this extension in these notes.

1.7 The Concept of Equisingularity

Equisingularity is up to now a rather vague concept. We shall try to fix some properties which should be satisfied by a proper definition of equisingularity.

As we said in the introduction, roughly speaking two germs of complex analytic spaces should be equisingular if their singularity are somehow the "same". We already mentioned that considering complex analytic equivalence is too strong, because in a family the analytic structure might change continuously.

We can give some basic features which should characterize Equisingularity:

1. It is an equivalence relation in the class of complex analytic germs;
2. Two equisingular hypersurfaces should be topologically equisingular;
3. If X is a complex analytic space, the disjoint subspaces:

$$\mathcal{S}_x = \{y \in X \mid (X, y) \text{ is equisingular to } (X, x)\}$$

   define a strict partition of X.
4. Two equisingular spaces (X, x) and (Y, y) should have the same multiplicity.
5. Equisingularity should be characterized algebraically.

In the paper of Zariski, "A theorem on the Poincaré group of an algebraic hypersurface", quoted in the introduction above, one of the main arguments of the proof is that two general hyperplane sections of a projective hypersurface and their embedding in their hypersurface are homeomorphic or equivalently the germs of their cones at the origin are topologically equisingular.

2 Whitney Equisingularity

A possible definition of Equisingularity is Whitney Equisingularity. Let X be a reduced complex analytic space. Let x and y be points of X.

Definition 2.1 The singularities (X, x) and (X, y) are Whitney equisingular if there is a Whitney stratification $(S_i)_{i\in I}$ of X such that x and y belong to the same stratum S_i.

We shall see that Whitney equisingularity satisfies the features mentioned above.

2.1 Topological Properties

Let X be a reduced complex analytic space. Let $\mathcal{S} = (S_i)_{i\in I}$ be a Whitney stratification of X. There is a local topological triviality of X along the strata of the stratification $\mathcal{S}$ in the following sense:

As we have said above, for any $x \in X$, let $S_{i(x)}$ be the stratum of the stratification $\mathcal{S}$ of X which contains x, then there exist an open neighborhood V of x in $S_{i(x)}$ and a *slice* $\mathcal{N}_x$, i.e. in a local embedding $(X, x) \subset (\mathbb{C}^N, x)$ the intersection of X with a linear subspace of $\mathbb{C}^N$ transverse to $S_{i(x)}$ at x in a neighborhood of x in X, such that a neighborhood of x in X is homeomorphic to the product $V \times (\mathcal{N}_x \cap V)$.

This result was announced by R. Thom in [22] and one can find a sketch of proof by J. Mather in [13].

As a consequence, using the tubular neighbourhoods of J. Mather (see p. 480 of [13]), we can prove that, for any point $x \in S_i$, the slices $\mathcal{N}_x$ are diffeomorphic.

In particular, if X is a hypersurface, if $x, y \in X$ are points of X, since they belong to the same Whitney stratum of some Whitney stratifications the germs (X, x) and

(X, y) are homeomorphic germs of hypersurface, so they are topologically equisingular.

2.2 Equimultiplicity

Let X be a reduced complex analytic space. Let $\mathcal{S} = (S_i)_{i \in I}$ be a Whitney stratification of X. In his paper [9] Corollary 6.2, H. Hironaka proves that for any points $x \in S_i$, the multiplicity of X is the same. Then, along its Whitney strata, a reduced analytic space is equimultiple.

2.3 Polar Varieties

Let X be an equidimensional reduced complex analytic space of dimension d and let x be a point of X. Consider the integers k, $2 \leq k \leq d + 1$.

We may embed $(X, x) \subset (\mathbb{C}^N, x)$. In [28] (2.2.2) we show that the set of germs of projection:

$$p : (X, x) \to (\mathbb{C}^k, 0)$$

induced by surjective affine maps $(\mathbb{C}^N, x) \to (\mathbb{C}^k, 0)$ contains an Zariski dense subset Ω_k such that, for any $p \in \Omega_k$, the critical locus $C(p)$ of the restriction of p to the non-singular part $X \setminus \Sigma_X$ is a reduced complex analytic space and the multiplicity $m_k(X, x)$ of germ of the closure $\overline{C(p)}$ at the point x does not depend on $p \in \Omega_k$.

For $p \in \Omega_k$ the germ $(\overline{C(p)}, x)$ is called a polar variety $P^{k-1}(X, x)$ of (X, x) of dimension $k-1$. Beware that $P^{k-1}(X, x)$ can be empty in which case its multiplicity at x is 0.

Therefore, one can associate a d-uple $M(X, x) = (m_2(X, x), \ldots, m_{d+1}(X, x))$ to the germ (X, x). Notice that some $m_k(X, x)$ can be 0 and $m_{d+1}(X, x)$ is the multiplicity of X at x, because $P^d(X, x) = (X, x)$.

We have the following algebraic characterisation of Whitney stratification due to B. Teissier (see [21] Chapitre 5 Théorème 1.2) which gives somehow an algebraic characterisation of Whitney equisingularity:

Theorem 2.1 *Let X be a reduced equidimensional complex analytic space. Let $\mathcal{S} = (S_i)_{i \in I}$ be a stratification of X (see Definition 1.1). Suppose that, for any pair (S_i, S_j), such that $S_i \subset \overline{S}_j$, the* $\dim(S_j)$*-uple $M(\overline{S}_j, x)$ is constant for $x \in S_i$. Then, the stratification $\mathcal{S}$ is a Whitney stratification of X.*

2.4 Vanishing Euler Characteristics

Let X be a d-equidimensional reduced complex analytic space and x be a point of X. We may assume that $(X, x) \subset (\mathbb{C}^N, x)$. We have seen that, for any $p \in \Omega_k$, where Ω_k is a Zariski dense open subset of the space of projections of (X, x) onto $(\mathbb{C}^k, 0)$ induced by affine maps $(\mathbb{C}^N, x) \to (\mathbb{C}^k, 0)$, we can define Ω_k, such that the general local fiber $p^{-1}(u) \cap \mathbb{B}_\varepsilon$, where $\mathbb{B}_\varepsilon$ is the ball centered at x of radius ε in $\mathbb{C}^N$, $0 < \|u\| \ll \varepsilon$ and $u \in \mathbb{C}^k$ is a general point, of p at x has a homotopy type which does not depend on p (see 3.1.2 in [23]).

We call the general local fiber $p^{-1}(u) \cap \mathbb{B}_\varepsilon$ of p at x a *local vanishing fiber of* (X, x) *of dimension* $d - k$. When $k = d + 1$ the local vanishing finer is empty, so we put $\chi_{d+1}(X, x) = 0$ for the Euler characteristic of the empty fiber.

Definition 2.2 We call the Euler characteristic of the local vanishing fiber of $p : (X, x) \to (\mathbb{C}^k, 0)$ the vanishing Euler characteristic $\chi_k(X, x)$. The vanishing Euler characteristics of (X, x) is the dim X-uple:

$$\mathbb{K}(X, x) = (\chi_2(X, x), \ldots, \chi_{\dim X+1}(X, x))$$

In [23] (Théorème (5.3.1)) we have the following characterization of Whitney stratification:

Theorem 2.2 *Let X be an equidimensional reduced complex analytic space and let* $\mathcal{S} = (S_I)_{i \in I}$ *be a stratification of X. Suppose that for any pair* (S_i, S_j) *of strata of* $\mathcal{S}$*, such that* $S_i \subset \overline{S}_j$ *we have that the vanishing Euler characteristics* $\mathbb{K}(\overline{S}_j, x)$ *is constant for* $x \in S_i$*, then the stratification* $\mathcal{S}$ *is a Whitney stratification.*

As it is noticed in [23] (5.3) this theorem can be understood as a converse of Thom-Mather first isotopy theorem.

In fact, Theorem 2.2 is a consequence of Teissier's Theorem 2.1 stated above by using Théorème 4.1.1 of [23].

2.5 Summary

All this results show that Whitney equisingularity satisfies the requirements of 1.7.

The Theorem 2.2 is given to show that a Whitney stratification can be characterized by topological data and leads naturally to the question:

Can a Whitney stratification on a real analytic space (or a subanalytic space) be characterized by a real version of Theorem 2.2?

3 Saturation

In this section we essentially follow O. Zariski in [36].

3.1 Definition

Let $\mathcal{O}$ be a ring with identity. Let K be its total ring of fractions and Let $L \subset K$ be s a subfield of K. We assume:

1. The ring has no divisor of zero $\neq 0$;
2. In view of the preceding hypothesis, the total ring of fractions K being noetherian, the ring K is the direct sum of finite number of fields:

$$K = K_1 \oplus \cdots \oplus K_r;$$

3. The field L contains the unit of K, or equivalently no element $\neq 0$ is a zero divisor of K;
4. Let ε_i be the unit of K_i. Then K_i is a finite separable extension of $L\varepsilon_i$;
5. If $R = \mathcal{O} \cap L$ then, the ring $\mathcal{O}$ is integral over R.

Let us fix an algebraic closure Ω of L. Consider L-homomorphisms of K into Ω. Let ψ such a homomorphism. Then, for some i, $\psi(\varepsilon_i) = 1$ and, for $j \neq i$, $\psi(\varepsilon_j) = 0$. Then, for $j \neq i$, $\psi(K_j) = 0$ while ψ induces an isomorphism of K_i onto its image and $\psi(\alpha\varepsilon_i) = \alpha$ for any $\alpha \in L$. According to the hypothesis 4 above the number of L-homomorphisms of K into Ω is finite.

For any given i,$1 \leq i \leq r$, the compositum K_i^* of the fields $\psi(K_i)$ as ψ varies, i.e. the smallest field of Ω which contains the $\psi(K_i)$'s, is a finite Galois extension of L. Similarly, the compositum K^* of the fields $\psi(K)$ is a finite Galois extension of L.

Following O. Zariski, we shall say for two elements ξ and η of K, ξ dominates η if for any pair of homomorphisms ψ_1 and ψ_2, either $\psi_1(\eta) \neq \psi_2(\eta)$ and the quotient:

$$\frac{\psi_1(\xi) - \psi_2(\xi)}{\psi_1(\eta) - \psi_2(\eta)}$$

is integral over R, while $\psi_1(\eta) = \psi_2(\eta)$ implies $\psi_1(\xi) = \psi_2(\xi)$.

Note that if. for some i, ψ_1 and ψ_1' are L-homomorphisms of K into Ω such that $\psi_1(\varepsilon_i) = \psi_1'(\varepsilon_i) = 1$, then there is a L-monomorphism ϕ_0 of $\psi_i(K)$ into Ω such that $\psi_1' = \phi_0\psi_1$. The monomorphism ϕ_0 can be extended to a L-automorphism of the compositum K^* of the fields $\psi(K)$. Thus, for any element η of K, the set of elements $\psi_1'(\eta) - \psi_2(\eta)$ is the set of ϕ-images of the elements $\psi_1(\eta) - \psi_2(\eta)$ (ψ_1' and ψ_1 being fixed as above).

It yields that, if one fixes for each $i = 1, \ldots, r$ (where r is the number of fields in the hypothesis 2 above) a L-homomorphism $\psi_1^{(i)}$ of K into Ω such that $\psi_1^{(i)}(\varepsilon_i) = 1$, then, in order to verify that ξ dominates η, it is sufficient to verify the conditions of domination only for the pairs (ψ_1, ψ_2) where ψ_1 ranges over the set $\{\psi_1^{(1)}, \ldots, \psi_1^{(r)}\}$ and ψ_2 is any L-homomorphism of K into Ω.

In particular if $\mathcal{O}$ is a domain of integrity K is a field and $r = 1$. So, we may assume that $K \subset \Omega$. Then, the compositum K^* in Ω is the smallest Galois extension of L containing K. One can take $\psi_1^{(1)}$ to be the injection map of K into Ω. Then, the definition of domination is the following:

The element ξ dominates η if, for any element σ of the Galois group of the compositum K^* over L, the following holds: if $\sigma.\eta \neq \eta$, then the quotient $(\sigma.\xi - \xi)/(\sigma.\eta - \eta)$ is integral over $R = \mathcal{O} \cap L$, while $\sigma.\eta - \eta = 0$ implies $\sigma.\xi - \xi = 0$.

Now, we can define:

Definition 3.1 Let $\overline{\mathcal{O}}$ be the integral closure of $\mathcal{O}$ in K. The ring $\mathcal{O}$ is said to be saturated with respect to the field L if it contains every element of $\overline{\mathcal{O}}$ which dominates an element of $\mathcal{O}$.

Since the integral closure $\overline{\mathcal{O}}$ is saturated with respect to the field L, the set of saturated rings with respect to L which contain $\mathcal{O}$ and are contained in $\overline{\mathcal{O}}$ is not empty.

The intersection of two rings saturated with respect to the field L which contain $\mathcal{O}$ and are contained in $\overline{\mathcal{O}}$ is also saturated with respect to L. It implies:

Proposition 3.1 *The set of saturated rings with respect to the filed L which contain $\mathcal{O}$ and are contained in $\overline{\mathcal{O}}$ has a smallest element for the order induced by inclusion.*

Definition 3.2 The smallest element of the set of saturated rings with respect to the field L which contain $\mathcal{O}$ and are contained in $\overline{\mathcal{O}}$ is called the saturation of $\mathcal{O}$ with respect to L and is denoted by $\tilde{\mathcal{O}}_L$.

3.2 Dimension 1

We shall be interested in complex analytic local rings, i.e. local rings isomorphic to quotients of a ring of convergent series by an ideal. It is known that a complex analytic local ring is noetherian (see e.g. [10]).

A complex analytic local ring $\mathcal{O}$ is isomorphic to the ring $\mathcal{O}_{X,x}$ of germs of complex analytic functions on complex analytic space X at a point x.

If $\mathcal{O}$ is reduced the normal closure of $\overline{\mathcal{O}}$ of $\mathcal{O}$ in its total ring of fractions is isomorphic to the germ of meromorphic functions which are bounded in a neighborhood of x in X.

In the case of a complex analytic local ring of dimension 1 a result of F. Pham and B. Teissier proves that the saturation $\tilde{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$ with respect to the quotient field of the ring of convergent series in a parameter u of $\mathcal{O}_{X,x}$ is the ring of meromorphic functions which are Lipschitz in a neighborhood of x in X (see [19]).

Then, they prove that two germs of plane branches are topologically equisingular if the saturations of their local rings with respect to the quotient field of the ring of convergent series in a general parameter are isomorphic.

In fact in [7] A. Fernandes proved a geometrical result:

Theorem 3.1 *Let $(X, 0)$ and $(X', 0)$ be germs of complex analytic curves in $\mathbb{C}^2$ with branches X_i, $i \in I$ and X'_j, $j \in J$:*

$$X = \cup_{i \in I} X_i \text{ and } X' = \cup_{j \in J} X'_j.$$

Then the following conditions are equivalent:

1. *There exists a germ of the subanalytic bi-Lipschitz map $F : (X, 0) \to (X', 0)$;*
2. *There exists a bijection $\sigma : I \to J$ such that $\beta(X_i) = \beta(X'_{\sigma(i)})$ for all $i \in I$, where $\beta(\Gamma)$ is the Puiseux exponents of the branch Γ at 0, and such that $(X_i, X_j)_0 = (X'_{\sigma(i)}, X'_{\sigma(j)})_0$, for all $i, j \in I$, where $(\bullet, \bullet)_0$ denotes the intersection multiplicity at the point 0;*
3. *$(X, 0)$ is topologically equivalent to $(X', 0)$;*
4. *There exist an integer d, a germ of the curve $(C, 0) \subset (\mathbb{C}^d, 0)$, and two linear projections $p, p' : \mathbb{C}^d \to \mathbb{C}^2$, both general for C at 0 and such that $p(C) = X$ and $p'(C) = X'$.*

In summary two germs of plane curves at 0 have isomorphic saturations with respect to the quotient field of the ring of series in a transversal parameter, i.e. a parameter whose valuation in the normalization of the local rings is equal to the multiplicity of the local rings, if and only if there is a bijection between the branches such that corresponding branches have the same topology and pairwise intersection numbers at 0 of branches and their corresponding branches are equal, i.e. if and only if the two germs of curves at 0 are topologically equisingular.

3.3 Zariski Equisingularity

In [35] O. Zariski introduced the notion of equisingularity in codimension one for an algebraic variety. It is easy to adapt his definition to define equisingularity in codimension one for a germ of complex analytic set.

Definition 3.3 Let (X, x) be a germ of a reduced equidimensional complex analytic space. Let Y be a codimension one complex analytic subspace of X which is smooth at x. We suppose that (X, x) is embedded in $\mathbb{C}^N$. We say that X is equisingular in codimension one if the intersections of X with smooth spaces S_v transverse to Y in $\mathbb{C}^N$ define germs of curves $(S_v \cap X, S_v \cap Y)$ which are equisingular for any point $S_v \cap Y$ is a neighborhood of x.

Here equisingularity is concerning curves and is taken in the sense of [34]. According to what is said above, equisingularity means that the saturations of the local rings $\mathcal{O}_{S_v \cap X, S_v \cap Y}$ with respect to the quotient field of the ring of convergent series of a general parameter of the local rings.

In the case the singular locus has not codimension one, e.g. in the case of a normal germ of complex analytic space, one cannot use equisaturation to define equisingularity since the saturation of a ring is contained in its normalization and, when the singular locus has codimension 2, the local ring might be normal.

This is why O. Zariski imagines to define equisingularity by induction (see [37] Definition 3 p. 589):

Let (X, x) be a germ of d-equidimensional reduced complex analytic space. Let (Y, x) the germ of a smooth subspace of (X, x) which is contained in the singular locus. Let $p : (X, x) \to (\mathbb{C}^d, 0)$ a general projection of (X, x) onto $(\mathbb{C}^d, 0)$. Then, p is finite and one can define the discriminant of p. Let $\Delta(p)$ be the discriminant of p. The reduced germ $(|\Delta(p)|, 0)$ contains $(p(Y), 0)$ which is smooth, since p is a general projection. Then, (X, x) is equisingular along (Y, x) at the point x if $(|\Delta(p)|, 0)$ is equisingular along $(p(Y), 0)$ at the point 0.

Then, Zariski equisingularity can be defined by induction on the dimension of the ambient space.

In the case of a hypersurface, if the germ $(Y, 0)$ has codimension one in (X, x), then $(p(Y), x) \subset (|\Delta(p)|, 0)$, and we know that (X, x) is topologically equisingular at x along (Y, x) if and only if it is equisaturated along Y at x, which means that the Milnor number of the plane curve, intersection of a plane transversal P to Y at $P \cap Y$, plus its multiplicty minus 1 is constant along Y in a neighborhood of x in Y which implies $(p(Y), x) = (|\Delta(p)|, 0)$.

In the case of a subspace (Y, x) of higher codimension little is known. Recently W. Neumann and A. Pichon have studied hypersurfaces of dimension 3 and have related Zariski equisingularity with Lipschitz equisingularity which we shall define in the following section.

4 Lipschitz Viewpoint

Although F. Pham and B. Teissier were the first to relate Lipschitz meromorphic function and Saturation of local rings (see [19]), T. Mostowski introduced Lipschitz equisingularity where instead of homeomorphisms, he considers Lipschitz homeomorphisms (see [15–17]).

In particular, T. Mostowski proved that any complex analytic space have a Lipschitz stratification (see [15]).

Little has been done about Lipschitz equisingularity. L. Birbrair and T. Mostowski have introduced the notion of normal embedding in [3]. For instance, suppose that the reduced complex analytic space $X \subset U \subset \mathbb{C}^N$, where U is an open set of $\mathbb{C}^N$, then X is endowed by two metrics: the outer and the inner metrics. The outer metric is the metric induced by the embedding $X \subset U$. The inner metric is the the metric defined by $d(x, y) = \inf l(\gamma)$ where γ is a piecewise C^1 continuous path and $l(\gamma)$ is the length of γ. It may happen that these two metrics are different. When, they are the same one says that the embedding $X \subset U$ is normal.

In fact, in [7] A. Fernandes proved that topological equisingularity and Lipschitz equisingularity are the same for germs of plane curves.

For the case of surfaces, there are several papers on the Lipschitz structure of a germ of surface which begins with [1] until [4].

In higher dimensions little is known about the Lipschitz structure.

A lot is to be done with Lipschitz viewpoint.

Let us cite the recent result of [2] where it is proved that a germ of d-equidimensional reduced complex analytic space (X, x) which is bi-Lipschitz homeomorphic by a subanalytic map to $(\mathbb{C}^d, 0)$ is non-singular. The results on the Lipschitz structure of germs of complex surfaces should encourage new results on germs of reduced complex analytic spaces of higher dimension.

5 Open Problems

In this section we shall list some open problems on equisingularity.

5.1 Zariski Multiplicity Conjecture

Among basic problems about equisingularity, there is a basic problem by O. Zariski (see [31] and [37] p. 483):

Conjecture 1 Let (X, x) and (X', x') be topologically equisingular hypersurfaces. Their multiplicity $e(X, x)$ and $e(X', x')$ are equal.

In fact, we can weaken this conjecture:

Conjecture 2 Let (X_t, x_t) be a complex analytic family of topologically equisingular hypersurfaces. The multiplicty $e(X_t, x_t)$ is constant.

Both of these conjectures are true for complex analytic plane curves.

In a natural way, one may ask the same conjecture in the case of Lipschitz singularities.

5.2 Do the Diverse Definitions of Equisingularity Satisfy the Conditions of 1.7?

Above in 1.7 we give some hints which should be satisfy by an notion of equisingularity on a given complex analytic space.

For instance:

Question 3 Does Lipschitz equisingularity have an algebraic definition?

We saw above that this conjecture has a positive answer for a hypersurface along a codimension one stratum of the singular locus where one has topological equisingularity.

Question 4 Does Lipschitz equisingularity imply equimultiplicity?

This is nearly proved by G. Comte in the case the constants of the bi-Lipschitz homeomorphism satisfy some inequality in relation with the multiplicities of the singularities (see [6]). His proof might imply Question 4 for an analytic family of reduced complex analytic spaces.

5.3 What Are the Relations Between the Diverse Equisingularites

It was asked by Zariski (see [37] p. 487):

"Does topological equisingularity implies differential equisingularity (i.e. Whitney equisingularity)?"

It was proved by J. Briançon and J.P. Speder that the answer is negative in [5].

However, one should investigate other relations between the diverse notions of equisingularity.

For instance, a result of R. Thom and J. Mather (see [22] and [13]) shows that Whitney equisingularity implies topological equisingularity on a reduced complex analytic space. By definition Lipschitz equisingularity implies topological equisingularity.

Recent results of W. Neumann and A. Pichon assert that Lipschitz equisingularity is equivalent to Zariski equisingularity in dimension ≤ 3. Of course, it remains to understand the general case.

5.4 Is There Any Other Type of Equisingularity?

Then, it remains to find if there are other types of equisingularity.

Since it could be required that equisingularity is defined by algebraic data, we should study algebraic invariants for some equisingularity. For instance, Teissier proved that Whitney equisingularity is defined by the constancy of the multiplicities of some Polar varieties (see [21]). One should investigate the meaning of the constancy of Lê numbers or Lê cycles introduced by D. Massey in [12] (see also [11]).

5.5 Real Case

As we have mentioned above, is there a result similar to Theorem 2.2 in the real case?

Is there a characterization of Whitney stratification in the real case?

References

1. Birbrair, L.: Local bi-Lipschitz classification of two-dimensional semialgebraic sets (Portuguese). Rev. Semin. Iberoam. Mat. Singul. Tordesillas **2**, 29–34 (1998)
2. Birbrair, L., Fernandes, A., Lê, D.T., Sampaio, J.E.: Lipschitz regular complex algebraic sets are smooth. Proc. Am. Math. Soc. **144**, 983–987 (2016)
3. Birbrair, L., Mostowski, T.: Normal embeddings of semialgebraic sets. Mich. Math. J. **47**, 125–132 (2000)
4. Birbrair, L., Neumann, W., Pichon, A.: The thick-thin decomposition and the bilipschitz classification of normal surface singularities. Acta Math. **212**, 199–256 (2014)
5. Briançon, J., Speder, J.P.: La trivialité topologique n'implique pas les conditions de Whitney. C. R. Acad. Sci. Paris Ser. A - B **280**(Aiii), A365–A367 (1975)
6. Comte, G.: Multiplicity of complex analytic sets and bi-Lipschitz maps. Real analytic and algebraic singularities (Nagoya/Sapporo/Hachioji, 1996). Pitman Research Notes in Mathematics Series, pp. 182–188. Longman, Harlow (1998)
7. Fernandes, A.: Topological equivalence of complex curves and bi-Lipschitz homeomorphisms. Mich. Math. J. **51**, 593–606 (2003)
8. Grothendieck, A.: Techniques de construction en Géométrie analytique VI: Etude locale des morphismes. Germes d'espaces analytiques, Platitude , Morphismes simples, Sém. Henri Cartan, 13 ème année, Exposé XIII, Secrétariat Math., 11 Rue Pierre Curie, 75005 (1962)
9. Hironaka, H.: Normal cones in analytic Whitney stratifications. Inst. Hautes Études Sci. Publ. Math. **36**, 127–138 (1969)
10. Houzel, C.:Géométrie analytique locale, I, in Séminaire H. Cartan 1960–1961, Familles d'espaces complexes, Exposé 18, Secrétariat Mathématique, IHP, 11 rue Pierre et Marie Curie, Paris (1962)
11. Massey, D.B.: Lê Cycles and Hypersurface Singularities. Lecture Notes in Mathematics, Xii+131 pp. Springer, Berlin (1995)
12. Massey, D.B.: The Lê varieties I. Invent. Math. **99**, 357–376 (1990); The Lê varieties II. Invent. Math. **104**, 113–148 (1991)
13. Mather, J.: Notes on topological stability. Bull. Am. Math. Soc. **49**, 475–506 (2012)
14. Milnor, J.: Singular points of complex hypersurfaces. Annals of Mathematics Studies **61** Princeton University Press, Princeton, New Jersey; University of Tokyo Press, Tokyo (1968) iii+122 pp
15. Mostowski, T.: Tangent cones and Lipschitz stratifications, Singularities (Warsaw, 1985), Banach Center Publ. vol. 20, pp. 303–322. PWN, Warsaw (1988)
16. Mostowski, T.: Lipschitz equisingularity. Diss. Math. (Rozprawy Mat.) **243**, 46 (1985)
17. Mostowski, T.: A criterion for Lipschitz equisingularity. Bull. Pol. Acad. Sci. Math. **37**(1–6), 109–116 (1989)
18. Narasimhan, R.: Introduction to the Theory of Analytic Spaces. SLN, vol. 25. Springer, Verlag (1966)
19. Pham F.: (work with B. Teissier), Fractions lipschitziennes et saturation de Zariski des algèbres analytiques complexes, Actes du Congrès International des Mathèmaticiens (Nice, 1970), Tome 2, pp. 649–654. Gauthier-Villars, Paris, (1971)
20. Teissier, B.: Cycles èvanescents, sections planes et conditions de Whitney, Singularitès á Cargèse (Rencontre Singularitès Géom. Anal., Inst. tudes Sci., Cargèse, 1972). Asterisque, Nos. 7 et 8, Société Mathématique de France, Paris, pp. 285–362. (1973)
21. Teissier, B.: Variétés polaires II. Multiplicités polaires, sections planes, et conditions de Whitney, Algebraic geometry (La Rábida, 1981), 314–491, Lecture Notes in Math. **961**, Springer, Berlin, 1982
22. Thom, R.: Ensembles et morphismes stratifiés. Bull. Am. Math. Soc. **75**, 240–284 (1969)
23. Lê, D.T., Teissier, B.: Cycles évanescents, sections planes et conditions de Whitney II, Singularities, Part 2 (Arcata, Calif., 1981). Proceedings of Symposia in Pure Mathematics, pp. 65–103. American Mathematical Society, Providence, RI (1983)

24. Lê, D.T.: Sur un critre d'équisingularité. C. R. Acad. Sci. Paris Ser. A - B **272**, A138–A140 (1971)
25. Lê, D.T.: Sur les nœuds algébriques. Compos. Math. **25**, 281–321 (1972)
26. Lê, D.T.: Calcul du nombre des cycles évanouissants d'une hypersurface complexe. Ann. Inst. Fourier **23**, 261–270 (1973)
27. Lê, D.T., Ramanujam, C.P.: The invariance of Milnor's number implies the invariance of the topological type. Am. J. Math. **98**, 67–78 (1976)
28. Lê, D.T., Teissier, B.: Variétés polaires locales et classes de Chern des variétés singulières. Ann. Math. **114**, 457–491 (1981)
29. Lê, D.T., Teissier, B.: Limites d'espaces tangents en géométrie analytique. Comment. Math. Helv. **63**, 540–578 (1988)
30. Whitney, H.: Tangents to an analytic variety. Ann. Math. **81**, 496–549 (1965)
31. Zariski, O.: Contributions to the problem of equisingularity, Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969), pp. 261–343. Edizioni Cremonese, Rome (1970)
32. Zariski, O.: Equisingular points on algebraic varieties (1965) in Seminari 1962/63 Anal. Alg. Geom. e Topol., Vol. 1, Ist. Naz. Alta Mat. pp. 164–177, Ediz. Cremonese, Rome
33. Zariski, O.: Equisingular points on algebraic varieties (1966). Some Recent Advances in the Basic Sciences In: Proceedings of the Annual Science Conference on Belfer Graduate School Science, Yeshiva University, New York, vol. 1, pp. 163–175. (1962–1964)
34. Zariski, O.: Studies in equisingularity. I. Equivalent singularities of plane algebroid curves. Am. J. Math. **87**, 507–536 (1965)
35. Zariski, O.: Studies in equisingularity. II. Equisingularity in codimension 1 (and characteristic zero). Am. J. Math. **87**, 972–1006 (1965)
36. Zariski, O.: Studies in equisingularity. III. Saturation of local rings and equisingularity. Am. J. Math. **90**, 961–1023 (1968)
37. Zariski, O.: Some open questions in the theory of singularities. Bull. Am. Math. Soc. **77**, 481–491 (1971)