ICME-13 Monographs

Iliada Elia
Joanne Mulligan
Ann Anderson
Anna Baccaglini-Frank
Christiane Benz *Editors*

Contemporary Research and Perspectives on Early Childhood Mathematics Education

ICME 13
Hamburg 2016

Springer

ICME-13 Monographs

Series editor

Gabriele Kaiser, Faculty of Education, Didactics of Mathematics, Universität Hamburg, Hamburg, Germany

Each volume in the series presents state-of-the art research on a particular topic in mathematics education and reflects the international debate as broadly as possible, while also incorporating insights into lesser-known areas of the discussion. Each volume is based on the discussions and presentations during the ICME-13 Congress and includes the best papers from one of the ICME-13 Topical Study Groups or Discussion Groups.

More information about this series at http://www.springer.com/series/15585

Iliada Elia · Joanne Mulligan
Ann Anderson · Anna Baccaglini-Frank
Christiane Benz
Editors

Contemporary Research and Perspectives on Early Childhood Mathematics Education

Springer

Editors
Iliada Elia
Department of Education
University of Cyprus
Nicosia
Cyprus

Anna Baccaglini-Frank
Department of Mathematics
University of Pisa
Pisa
Italy

Joanne Mulligan
Faculty of Human Sciences
Macquarie University
North Ryde, Sydney, NSW
Australia

Christiane Benz
Department of Mathematics
and Computer Science
University of Education Karlsruhe
Karlsruhe
Germany

Ann Anderson
Department of Curriculum and Pedagogy
University of British Columbia
Vancouver, BC
Canada

ISSN 2520-8322 ISSN 2520-8330 (electronic)
ICME-13 Monographs
ISBN 978-3-319-73431-6 ISBN 978-3-319-73432-3 (eBook)
https://doi.org/10.1007/978-3-319-73432-3

Library of Congress Control Number: 2017962976

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Contents

Chapter 1
Research in Early Childhood Mathematics Education Today

Iliada Elia, Joanne Mulligan, Ann Anderson, Anna Baccaglini-Frank and Christiane Benz

Abstract This edited book brings together a collection of research-based work from different contexts across the globe to contribute to improving knowledge and understanding of major issues that early childhood mathematics education encounters today and to advancing research, development and practice in this field. The chapters of the book are based on the invited contributions in TSG 1: Early Childhood Mathematics Education at ICME-13. The chapters provide a wide scope for discussion of current themes, theoretical perspectives and methodological approaches to promoting teaching and learning of mathematics in the early years. This chapter includes an overview of the core focus of the chapters.

Keywords Early childhood mathematics · Pattern · Number development
Embodied mathematics · Technology · Early childhood educators

I. Elia (✉)
Department of Education, University of Cyprus, P.O. Box. 20537, 1678 Nicosia, Cyprus
e-mail: elia.iliada@ucy.ac.cy

J. Mulligan
Department of Educational Studies, Faculty of Human Sciences, Macquarie University, North Ryde, Sydney 2109, Australia
e-mail: joanne.mulligan@mq.edu.au

A. Anderson
Department of Curriculum and Pedagogy, University of British Columbia, 2125 Main Mall, Vancouver, BC V6T 1Z4, Canada
e-mail: ann.anderson@ubc.ca

A. Baccaglini-Frank
Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo, 5, 53127 Pisa, Italy
e-mail: anna.baccaglinifrank@unipi.it

C. Benz
Department of Mathematics and Computer Science, University of Education Karlsruhe, Bismarckstr. 10, 76133 Karlsruhe, Germany
e-mail: benz@ph-karlsruhe.de

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_1

1.1 Introduction

Early childhood mathematics education (ECME) involves creating learning environments and providing activities by professionals (practitioners and educators) to give young children (approximately ages birth 0 through 7 years) opportunities and experiences for their mathematical development in child-care centers, pre-kindergarten, kindergarten and primary schools. Children's development in mathematics can also be motivated outside educational settings, that is, in the children's home environment during their interactions with family members. Family-based activities are highly valued as the foundation on which ECME can build.

ECME is not new; it has a long history. It began in 1631 when Comenius in Poland, published his book 'School of Infancy' which initiated the creation of mathematics programs for young children relying on the use of concrete materials. Since then, many historical ideas of pioneers, including those of Pestalozzi, Fröbel, Owen, and Piaget, have contributed to the existing awareness of the importance of mathematics education for young children (Van den Heuvel-Panhuizen and Elia 2014; Saracho and Spodek 2009a, b).

Today and in the past few years ECME has gained a prominent position in the research community of mathematics education. The interest in, and recognition of the importance of, the teaching and learning of mathematics in early childhood, have grown through the research outcomes that provide evidence for the positive relation between children's early mathematical knowledge and their later mathematics learning and academic achievement (e.g., Duncan et al. 2007; National Research Council 2009). Further research highlights the significance of early childhood in brain development (Shonkoff and Philips 2000) and the strong emphasis many countries are putting on pre-school education.

This edited book aims to contribute to improving knowledge and understanding of key issues that ECME encounters today in different contexts and to propose ways of advancing research, development and practice in the teaching and learning of early childhood mathematics. This book brings together a collection of chapters on current research in ECME which are written by a broad range of scholars across the globe providing a wide scope in themes and approaches to promoting teaching and learning of early childhood mathematics. The chapters are mainly based on the invited contributions in TSG 1: Early Childhood Mathematics Education at ICME 13, which took place in July 2016. In addition, it is to be noted that many other presentations by researchers from different countries with different educational systems, cultures and research traditions took place in the sessions of TSG 1. Throughout the chapters (including the present one), this book incorporates references to the content of these contributions, which substantially enrich and advance the discussion about the present and the future of ECME from an international perspective.

To capture the diversity of topics and perspectives of the book, the chapters have been grouped into five key themes (although themes are diffused throughout the chapters): (1) Pattern and Structure, (2) Number sense, (3) Embodied action and context, (4) Technology and (5) Teacher professional issues and education. The core focus of the chapters within each theme is described below.

1.2 Theme 1 Pattern and Structure

The first theme of this volume focuses on current research on the development of mathematical structure. Three chapters present findings from recent studies of patterning and structural development in preschoolers and children in the first years of formal schooling. Mulligan and Mitchelmore (2018) provide an overview of the outcomes of the Australian Pattern and Structure Project comprising a suite of studies with four to eight-year olds. This introductory also paper provides segue to the two related and complementary papers that follow by Lüken (2018), and Lüken and Kampmann (2018) with German preschoolers and first graders respectively.

Mulligan and Mitchelmore focus attention on the current application of their studies over the past decade to the assessment of early mathematics and an aligned pedagogy based on pattern and structure. Building on their studies of patterning, counting, the numeration system and multiplicative thinking, the project focused on identifying, describing and measuring common structural characteristics in mathematical development, Awareness of Mathematical Pattern and Structure (AMPS). AMPS comprised two interdependent components: one cognitive—knowledge of structure, and one meta-cognitive—a tendency to seek and analyze patterns (Mulligan and Mitchelmore 2009).

A revised interview-based assessment, the Pattern and Structure Assessment (PASA) is described in three forms, validated in a recent study with 618 children in the first two years of formal schooling (Mulligan et al. 2015). The PASA assesses common core concepts including repeating and growing patterns, partitioning two-dimensional and three-dimensional shape and space, multiple counting and base ten structure, arrays and grids, distance and scale, and units of length, area, volume/capacity, and time and graphs. Qualitative analyses reliably classified responses into five ordered structural categories: pre-structural, emergent, partial, structural and advanced structural. Using Rasch modeling, an overall Awareness of Mathematical Pattern and Structure (AMPS) score was calculated and reported as a location on an AMPS scale. The AMPS scale makes it possible to compare children's level of AMPS across year (grade) levels, regardless of which PASA assessment form they are given.

The Pattern and Structure Mathematics Awareness Program (PASMAP) is described in two phases comprising 17 Pathways aligned with five structural groupings: sequences, structured counting, shape and alignment, equal spacing and partitioning. The interrelationships within and between these groupings demonstrate the highly integrated nature of AMPS learning and the importance of spatial structuring in developing pattern and structure. Implications for new research on spatial reasoning for young children are discussed.

The chapter by Lüken focuses explicitly on the relationship between the development of early repeating patterning competencies and later arithmetical achievement. Lüken's longitudinal explorative study tracks six children's patterning development from their first to third year of German kindergarten. Three task-based interviews are conducted over the two-year period. Results suggest that significant development takes place in children's repeating pattern competencies between the age of three and four years. Also, over the kindergarten years the children's strategies improved while they were describing, copying, repairing, extending, and translating repeating patterns. Steps in development were identified as the ability to identify the pattern, the ability to discern elements as identical and match them and the ability to alternate two elements. Children's strategies for completing different patterning activities were integrated into five increasingly sophisticated categories: no reference to pattern, use of pattern elements, comparison, focus on sequence, and a view of unit of repeat. In their first year of German kindergarten, no child was able to correctly copy a simple exposed pattern, let alone extend it or identify missing elements. What all children correctly did was reproduce the linear arrangement: aligning the cubes in one row. Lüken proposes that this gestalt view might be children's first distinction of the differences between patterns: two patterns are the same when they are both linear arrangements and they are different when one is linear and the other not. In the final assessment point when the children were aged 5 years they used strategies of comparison and focusing on the sequence. However, it was found that the view of unit of repeat still was not developed. This was considered critical to the children's mathematical thinking as Lüken highlights that mathematics is about analyzing the pattern's structure, in discerning the unit of repeat. The findings do not however preclude development of the concept of unit of repeat in the year prior-to-school or from ages four to five.

The chapter by Lüken and Kampmann presents a related study that found that patterning and structural abilities in the kindergarten can positively influence arithmetic skills in Grade 1. They describe an intervention study promoting patterning skills with 51 first graders that showed significant differences between pre- and post-test arithmetic achievement scores for the intervention group. This improvement was achieved after five months of explicit teaching of pattern and structure during regular mathematics lessons. The intervention lessons included recognizing, describing, explaining and creating patterns with an emphasis on structuring the base 10 system. The improvement was particularly beneficial for the lowest-achieving children. The differentiated analysis of achievement levels showed that half of the children with achievements below average succeeded in gaining age-adequate results by the end of the intervention. Consequently, they assert that fostering pattern and structure abilities might be the key to supporting lower-achieving children to develop their overall mathematical abilities. In contrast the high achievers' results were not affected either way. Lüken and Kampmann's interpretation was that "higher-achieving children, of their own accord, discover, seek out, and use pattern and structure in mathematics" (p. 64, this book).

1.3 Theme 2 Number Sense

The research presented in TSG 1 on the theme 'Number sense' examines different aspects of number development in different parts of the world. This theme involved three invited papers, oral presentations and posters. Before introducing each chapter of the theme, the oral presentations and posters will be referred to.

The oral presentation by Dorier and Coutat (2016) highlights the importance for pre-numerical learning called "enumeration" (see Briand 1999) from the background of French researchers' work within Brousseau's theory of didactical situations. Counting activities of children aged four and five years were investigated when reading and making lists for designating and representing some collections of objects with specific characteristics. Rinvold (2016) in his oral presentation presents a study where he investigated the use of numerical finger gestures and other bodily-based communication in order to facilitate the learning process of the first three number words.

Another study is reported on the early development of number sense by Adenegan (2016) in which he observed the link between writing skills and number development for children aged three to six years in Nigeria. In Schlicht's (2016) poster, a study of the number development of young children in Germany is presented. He reconstructs the children's mathematical knowledge to gain insight in the development of the concepts of sets and numbers and draws attention to the use of zero.

Later in number development children need to gain a conceptual understanding of numbers and their cardinality, including the understanding of place value. This aspect is highlighted in the poster by Young-Loveridge and Bicknell (2016). They present the impact of using multiplication and division contexts with five to seven-year olds on their number knowledge and operations and show that even five-year olds are able to solve multiplication and division problems when presented in familiar contexts.

The three chapters concerning number sense report studies that took place in contrasting cultures and contexts. The chapter by Bojorque et al. (2018) reports a study in Ecuador, Rathé et al.'s (2018) chapter examines the effect of picture book reading in a study conducted in Belgium, and Cheeseman et al.'s (2018) chapter focuses on the effect of a measurement curriculum for numerical development conducted in Australia.

The former two chapters of this theme address the Spontaneous Focusing On Numerosity (SFON). SFON indicates if someone spontaneously focuses and pays attention to the exact number of a set of items or incidents in daily life. Rathé et al. (2018) first present a review of the current research literature on the association between SFON in experimental tasks and everyday activities and then provides a short summary of two studies conducted in Belgium. They examined the association between children's SFON in experimental tasks and during everyday picture book reading. Bojorque et al. (2018) show that the SFON tendency and its relation to early numerical abilities can be observed in different cultures. They investigate if and how early numerical abilities contribute to Ecuadorian kindergartners' SFON tendency at the end of the kindergarten year and also whether the quality of early mathematics education in Ecuadorian kindergarten contributes to children's SFON development.

The chapter by Cheeseman et al. (2018) focuses on another aspect of number development. They report on a study with children of five and six years of age in Australia with a focus on the relation between measurement and number development. The children did not start with a typical number-focused curriculum but with a measurement-focused curriculum where numbers were included. Both quantitative and qualitative analyses show if and how a measurement-focused curriculum can contribute to number development.

1.4 Theme 3 Embodied Action and Context

The third theme of the book focuses on research on embodied action and context for early mathematics learning. Prior to introducing the three chapters in which this theme is present, we give a brief description of five additional presentations (oral communications and posters) in TSG 1 which are related to the theme.

Specifically, Karsli's (2016) study addresses the embodied nature of knowing and knowledge in a pre-kindergarten classroom using video-ethnographic research. The particular study shows that young children's bodies hold rich potential for different types of mathematics, including number, space, shape and measurement. Furthermore, the study suggests that early childhood teachers' attention to the embodied ways in which children engage with mathematics in different contexts in pre-school would substantially improve a teachable-moments approach focusing on children's spontaneous mathematics-related actions.

The study by Watanabe (2016) focuses on young children's spatial abilities. Particularly, five and six year-old children are examined when participating in activities when physically making a cube. It was found that children's ability to mentally visualise a three-dimensional shape from a two-dimensional net representation of a cube improved after using a polydron geometric toy to assemble and convert a two-dimensional plane figure into the three-dimensional shape of cube for a particular period of time.

Yagi (2016) reported a study of first grade children's mathematical processes in the context of whole group mathematics discourse in the classroom. The findings of the study suggest that children can communicate their mathematical thinking explicitly and their engagement in emergent forms of mathematical processes, such as reasoning and identifying mathematical structure, are manifested within whole group discourse.

The work by Henschen (2016) addressed children's engagement with mathematics in a different context, that is, in free play. In particular, the study focused on the mathematical ideas children encounter in blockplay activities and raised the methodological issue about how to identify and analyze the mathematical content in the observations of children's play through video-recorded data. Young children's experiences with mathematics in the context of play was also the focus of the study by Nakken et al. (2016). In this study children's play took place in a mathematics room which was designed for the needs of the particular research work. Preliminary findings indicate that guided and structured play in the room yielded deeper mathematical thinking and engagement with more specific mathematical concepts by children than free play.

The three chapters included in the third theme of the book are by Thom (2018), Elia (2018), and Anderson and Anderson (2018). The two former chapters refer to current research on spatial thinking in early childhood and give further insight into the crucial role of the body and other semiotic resources in young children's spatial-geometric reasoning.

In particular, the study reported in Thom's (2018) chapter investigates children's spatial reasonings and meaning-making in geometry from an embodied perspective, integrating gestures and diagrams. In particular, three children are engaged in an exploratory activity involving the interpretation of a photo of the circular basis of a cylinder. A narrative account of children's conversations, gestures, and drawings focuses on how children think and gain geometric awareness of challenging concepts at specific moments of the activity and also how these reasonings and conceptualizations change/evolve from one moment to another. For example, at first the children name the image as "circle", moving their fingers in the air to represent a closed circular path of a point. They then move on to extend their circle into a third dimension to become a cylinder, which is further explored in different orientations, in deconstructing it and in transforming it into a net. This evolution of children's thinking and awareness is found to be entirely (em)bodied as gestures, movements, drawings and words, indicating the integral role of the body in developing meanings for geometrical concepts and objects of different complexity.

Elia's (2018) chapter adopts a multimodal and semiotic theoretical perspective to address young children's thinking and learning in geometry with a focus on the role of gestures, through three case studies. These studies examine children's interactions with teachers and peers in whole classroom or individual activities involving the use of various semiotic resources and artefacts. The studies deal with different aspects of geometry and spatial thinking. The first study investigates how gestures contribute to the apprehension of geometrical figures while a kindergartner gives instructions for the construction of a composite two-dimensional geometrical representation with simple shapes using different artefacts. Irrespectively of the artefact used, the child uses gestures of iconic character as a tool to simulate (or represent) geometrical transformations, i.e., translation and rotation of two-dimensional shapes. The second study focuses on the interrelations of gestures with other semiotic resources in a whole group discussion in a kindergarten classroom about the different attributes of two-dimensional shapes. The gestures produced by the child under study materialize a part of his concept images of two-dimensional geometrical objects, without necessarily producing any words himself but by attending to a peer's corresponding words. The third study of the chapter analyzes the discourse of a 4-year old child with his teacher and peers about spatial concepts.

In the process of objectifying spatial concepts, such as “in” and “out”, the child’s gesture evolves by becoming more simplified and shortened. Overall the chapter gives insight into how gestures in association with other semiotic resources influence geometrical activities and contribute to different aspects of early geometry thinking and learning in various contexts.

Besides formal or informal learning in school contexts between children, teachers and peers, learning that occurs in the home between parents and children considerably affects children’s knowledge, understanding, skills and dispositions in mathematics. This effect possibly appears to a greater extent in early childhood than at any other school grade. The focus of the chapter by Anderson and Anderson (2018) is on mathematics learning in various contexts of home environments. Particularly, the authors study in a systematic and detailed manner the types of mathematics which young children engage in during their (verbal and non-verbal) interactions with family members at home in different contexts. The findings of the study provide evidence for the variety of the types of mathematics that emerge in children’s activities with parents at home, with the prevalence of geometric, spatial and measurement concepts compared to number-related concepts. Furthermore, the study shows the important role that materials and kinds of interactions play in specifying the types of mathematics shared within the activities. Also, according to the authors, the embodied ways in which pre-school children and parents might engage with geometry at home are also relevant and require further in-depth analysis.

1.5 Theme 4 Technology

Theme 4 of this book focuses on the use of technology in early childhood mathematics education. This theme was present in seven of the presentations given during TSG 1 sessions. Before introducing the content of the three chapters, based on invited papers, we briefly describe the other papers and posters presented during the TSG 1 sessions that touched on this theme.

In particular, Fletcher and Ginsburg (2016) presented a study involving software to facilitate discourse and promote understanding of symmetric transformations. This is an important and delicate topic that plays a vital role across branches and levels of mathematics, but that unfortunately is given minor attention in early childhood mathematics curricula in many nations. In their study, Fletcher and Ginsburg were interested in evaluating the effects of using such software on children’s discourse on symmetric transformations: their results suggest that the software can be indeed be utilized to strengthen symmetry-focused mathematics discourse in early childhood classrooms, thereby contributing substantially to students’ understanding of symmetry.

Also the work by Nivens and Geiken (2016) addresses learning mathematics through particular software. They describe a study on using a computer science-based board game, Robot Turtles, with the aims of identifying mathematical concepts children develop when playing the game, and explore how playing it also influences children's interest in computer-programming type games.

Two final contributions to TSG 1 address, respectively: the topic of augmented reality, in particular how it can be used to arouse children's curiosity and interest in learning numbers (Ong 2016); and a category system for analyzing pre-school teachers' use of ICT in their classroom practices, through the "instructive actions" of identifying, explaining, planning, supervising and analyzing (Sànchez García et al. 2016).

The three chapters in Theme 4 of this book are by Sinclair (2018), Baccaglini-Frank (2018) and Ginsburg et al. (2018); they refer to current research in the area of technology in early childhood mathematics education, and introduce interesting issues, both theoretical and practical, opening new research venues.

Sinclair's (2018) chapter is about multimodal, embodied and semiotic aspects of learning, and it discusses three innovative issues on the use of digital technology in early mathematics education that may transform the learning and teaching of mathematics. The first issue is that of temporalizing early childhood mathematics (time), which suggests that the use of dynamic geometry software and how a new multi-touch app for counting (TouchCounts) can promote the learning of sophisticated mathematical ideas through embodied actions highlighting dynamic aspects (which can only exist thanks to time) of mathematical objects. The second issue concerns the importance of children's contact with advanced mathematics (immersion) often beyond the curriculum, which is now possible thanks to less constrained digital environments. The third issue is the affordance of digital technology to support the articulation of signs in children's mathematical work, and especially the relations developed between digital technologies and paper-and-pencil environments. In doing this, Sinclair also considers the challenges that teachers face in integrating new technologies that differ significantly from existing paper-and-pencil modalities and physical manipulatives.

The second chapter of the section addresses the possibility of strengthening children's number sense through interaction with two iPad apps. The study presented by Baccaglini-Frank (2018) was part of an educational project that took place in Italian pre-schools, where an educator followed a previously tested protocol proposing the two apps to children between the ages of five and six, who interacted with the software through multiple-touch gestures on the screen. The chapter introduces the schemes developed by the children in response to the apps, and it includes a focus on the role the educator's interventions seemed to play in such development. In particular, analyses of the data collected suggest that the educator's interventions privileged and encouraged schemes involving counting, which limited the variety of schemes enacted and the aspects of number sense that could have been strengthened through the interactions with the apps.

The use of digital tools to support and possibly enhance adult-child joint activity is a topic addressed by Ginsburg et al. (2018) in their chapter, the third of this theme, which introduces key features of the digital resource he and his team designed and produced. As Ginsburg et al. indicate, interactive mathematics books, fiction and non-fiction, enveloped in a digital surround of supporting materials—

their “Friends”—can delight and educate young children as well as those (e.g., parents, teachers, siblings) who read with them, yet few such books now exist, and little is known about them. In the chapter the authors describe and illustrate the potential of interactive mathematics storybooks (IMS), which entail a special set of affordances that can promote young children’s mathematics learning, and the surrounding “Friends”, which can help the adult better understand both the mathematics and the child.

1.6 Theme 5 Early Childhood Educators’ Professional Issues and Education

Two chapters in this book, that is, by Cooke and Bruns (2018) and by Tsamir et al. (2018), focus on current research on the theme about early childhood teacher professional issues and education.

The chapter by Cooke and Bruns provides a comprehensive overview of the various contributions presented in the TSG 1 which address issues related to early childhood educators. From the perspective of the educators, the authors propose a categorization of the conditions for developing mathematical understandings into three levels: macro-, meso- and micro-level. At the macro-level, curricula provide a framework (aims, content to learn and activities) for mathematics teaching and learning in early childhood with varying views, expectations and enactments by educators which may impact early mathematics learning opportunities (Fosse and Lossius 2016; Lao 2016; Lembrér and Johansson 2016).

The central topic of the presentations linked to the meso-level is about early childhood educators’ competences, including mathematical knowledge, pedagogical knowledge, understandings, beliefs, and perceptions. These aspects constitute factors that influence teachers’ classroom decisions and actions (Bruns et al. 2016; Cooke 2016; Dunekacke et al. 2016; Eilerts et al. 2016; Goto 2016; Jenßen et al. 2016; Tsamir et al. 2018). Early childhood educators’ competencies can be supported and developed by professional learning (Feza and Bambiso 2016; Hassidov and Ilany 2016).

The micro-level refers to the mathematics educational programs and materials that may impact children’s engagement with mathematical activities, as well as to the required training for the educators’ to develop their capability to effectively select and implement such programs that address children’s mathematical needs (Fritz-Stratmann et al. 2016).

Previous research suggests that teachers’ practices and actions for the teaching of mathematics are related to their mathematical knowledge (e.g., Shulman 1986) and self-efficacy (e.g., Allinder 1994). However, knowledge and self-efficacy might vary in different mathematical domains and tasks. The chapter by Tsamir et al. (2018) reports on a study which focuses on early childhood teachers’ self-efficacy beliefs in a specific mathematical domain, that is, patterning, and the relationship between their knowledge for teaching in this domain and self-efficacy. Using the Cognitive Affective Mathematics Teachers Education (CAMTE) framework (e.g., Tirosh et al. 2014) as a research tool, Tsamir et al. (2018) investigate early childhood teachers' knowledge of evaluating solutions (including identifying repeating patterns, errors in repeating patterns and possible ways for continuing repeating patterns). Teachers are found to hold high self-efficacy beliefs about solving the patterning tasks. Their actual performance in identifying repeating patterns and mistakes in repeating patterns is high as well. However, teachers encounter difficulty in identifying appropriate continuations of repeating patterns that do not end with a complete unit of repeat. Also they are inclined to select continuations which end the pattern with a complete unit of repeat although continuations ending with a partial unit of repeat would be also correct. These findings indicate the need to provide teachers further experiences of the variation in patterning tasks, which could be met by appropriate professional development.

To conclude, we would like to thank the chapter authors for their insightful and creative perspectives on contemporary developments in early mathematics learning and teaching. We thank also the presenting authors of the contributions in TSG 1 at ICME-13 for their substantial and inspiring presentations and their enduring work in advancing the field. We hope that a broad audience of researchers, early childhood educators, pre-service teachers and doctoral students find the book challenging and useful for their current and future research work in early childhood mathematics education.

References

Adenegan, K. E. (2016, July). *An investigation into the writing and computational skills of school pupils of age 3–6: Implications for early childhood numeracy*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Allinder, R. M. (1994). The relationships between efficacy and instructional practices of special education teachers and consultants. *Teacher Education and Special Education, 17*(2), 86–95.

Anderson, A., & Anderson, J. (2018). Math-in-context: The types of math preschoolers 'do' at home. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Baccaglini-Frank, A. (2018). What schemes do preschoolers' develop when using multi-touch applications to foster number sense (and why)? In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Bojorque, G., Torbeyns, J., Hannula-Sormunen, M. M., Van Nijlen D., & Verschaffel, L. (2018). Ecuadorian kindergartners' spontaneous focusing on numerosity development: Contribution of numerical abilities and quality of mathematics education. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Briand, J. (1999). Contribution à la réorganisation des savoirs pré-numériques et numériques. Étude et réalisation d'une situation d'enseignement de l'énumération dans le domaine pré-numérique. *Recherches en Didactique des Mathématiques, 19*(1), 41–76.

Bruns, J., Eichen, L., & Blömeke, S. (2016, July). *Pre-school teachers' mathematics-related competencies.* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Cheeseman, J., Benz, C., & Pullen, Y. (2018). Number sense: The impact of a measurement-focused program on young children's number learning. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Cooke, A. (2016, July). *Pre-service teacher relationships with mathematics-creative? Anxious? Competent?* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Cooke, A., & Bruns, J. (2018). Early childhood educators' issues and perspectives in mathematics education. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Dorier, J. L., & Coutat, S. (2016, July). *Conceptualizing order in early school.* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Duncan, G. J., Dowsett, C. J., Clasessens, A., Magnuson, K., Huston, A. C., Klebanov, P., et al. (2007). School readiness and later achievement. *Developmental Psychology, 43,* 1428–1446.

Dunekacke, S., Eilerts, K., & Jenßen, L. (2016, July). *Relations of affective, cognitive and situations specific facets of preschool teachers' professional competence.* Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Elia, I. (2018). Observing the use of gestures in young children's geometric thinking. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Eilerts, K., Jenßen, L., Eid, M., Schmude, C., Koinzer, T., Blömeke, S., & Rasche, J. (2016, July). *Pro-komma: Effectiveness of preschool teacher education in the field of mathematics.* Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Feza, N., & Bambiso, N. (2016, July). *South Africa's educators' mathematics teaching journey: A case of 5–6 years old educator practices.* Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Fletcher, N., & Ginsburg, H. (2016, July). *Teaching symmetry in the early childhood classroom: Using software to facilitate discourse and promote understanding of symmetric transformations.* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Fosse, T., & Lossius, M. H. (2016, July). *Norwegian kindergarten teacher work with mathematics.* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Fritz-Stratmann, A., Ehlert, M., & Herzog, A. (2016, July). *From thought to reality—implementation of an in-school mathematics training in South Africa*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Ginsburg, H. P., Uscianowski, C., & Almeda, M. V. (2018). Interactive mathematics storybooks and their friends. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Goto, M. (2016, July). *Reform of the kindergarten teachers and child care workers in day-care center training curriculum in Japan.* Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Hassidov, D., & Ilany, B. S. (2016, July). *"Senso-Math" preschool program facilitators contribute to mathematics education in the preschool.* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Henschen, E. (2016, July). *Mathematical content of play activities in kindergarten, exemplified on blockplay activities*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Jenßen, L., Eilerts, K., Koinzer, T., Schmude, C., & Blömeke, S. (2016, July). *Development and validation of a test battery assessing preschool teachers' professional competence in the field of mathe.* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Karsli, E. (2016, July). *Young children's embodied mathematical practices in a pre-k classroom.* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Lao, K. L. (2016, July). Comparison of three early childhood curricula from the perspective of mathematics education. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Lembrér, D., & Johansson, M. C. (2016, July). Swedish preschool teachers' views of children's socialization. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Lüken, M. (2018). Repeating pattern competencies in three- to five-year old kindergartners: A closer look at strategies. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Lüken, M., & Kampmann, R. (2018). The influence of fostering children's patterning abilities on their arithmetic skills in Grade 1. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Mulligan, J. T., & Mitchelmore, M. C. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21,* 33–49.

Mulligan, J. T., & Mitchelmore, M. C. (2018). Promoting early mathematical structural development through an integrated assessment and pedagogical program. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Mulligan, J. T., Mitchelmore, M. C., & Stephanou, A. (2015). *Pattern and structure assessment (PASA): An assessment program for early mathematics (Years F-2) teacher guide*. Camberwell VIC: ACER Press.

Nakken, A., Grimeland, Y., Nergard, B., & Thiel, O. (2016, July). *Young children's play in a mathematics room*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

National Research Council. (2009). *Mathematics learning in early childhood: Paths toward excellence and equity*. Washington, DC: The National Academies Press.

Nivens, R. A., & Geiken, R. (2016, July). *Using a computer science-based board game to develop pre-schoolers' mathematics.* Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Ong, D. J. (2016, July). *Simple augmented reality (ar) for early childhood mathematics.* Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Rathé, S., Torbeyns, J., de Smedt, B., Hannula-Sormunen, M. M., & Verschaffel, L. (2018). Kindergartners' spontaneous focus on number during picture book reading. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Rinvold, A. R. (2016, July). *The difficulty of learning the three first numerosities*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Sànchez García, A. B., Monterrubio, M. C., Ramírez, E., & Martín-Domínguez, J. (2016, July). *Category analysis system for the educational practice with ICT resources in early childhood education*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Saracho, O., & Spodek, B. (2009a). Educating the young mathematician: A historical perspective through the nineteenth century. *Early Childhood Education Journal, 36,* 297–303.

Saracho, O., & Spodek, B. (2009b). Educating the young mathematician: The twentieth century and beyond. *Early Childhood Education Journal, 36,* 305–312.

Schlicht, S. (2016, July). *About the development of concepts of sets and numbers—a qualitative case study with 3- to 4 year-olds*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Shonkoff, J., & Philips, D. (Eds.). (2000). *From neurons to neighborhoods: The science of early childhood development*. Washington: National Academy Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Sinclair, N. (2018). Time, immersion and articulation: Digital technology for early childhood mathematics. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Thom, J. S. (2018). (Re)(con)figuring space: Three children's geometric reasonings. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Tirosh, D., Tsamir, P., Levenson, E., Tabach, M., & Barkai, R. (2014). *Early childhood teachers' self-efficacy and knowledge for defining and identifying triangles and circles.* Paper presented at the 20th MAVI (Mathematics Views) Conference, Sweden.

Tsamir, P., Tirosh, D., Levenson, E., & Barkai, R. (2018). Early childhood teachers' knowledge and self-efficacy for evaluating solutions to repeating pattern tasks. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Van den Heuvel-Panhuizen, M., & Elia, I. (2014). Early childhood mathematics education. In St. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 196–201). Dordrecht: Springer.

Watanabe, S. (2016, July). *Research on young children's spatial recognition ability*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Yagi, S. (2016, July). *An exploration of first grade students' engagement in mathematical processes during whole group discussions*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Young Loveridge, J., & Bicknell, B. (2016, July). *Tens awareness: A framework for early place value learning*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Part I
Pattern and Structure

Chapter 2
Promoting Early Mathematical Structural Development Through an Integrated Assessment and Pedagogical Program

Joanne Mulligan and Michael Mitchelmore

Abstract Early development of mathematical patterns and structures has been the focus of a suite of studies with four- to eight-year olds comprising the Australian Pattern and Structure Project over the past decade. Awareness of Mathematical Pattern and Structure (AMPS) has been identified and measured, and found to be indicative of general mathematical achievement. A revised interview-based assessment, the Pattern and Structure Assessment (PASA) is represented in three forms, validated in a recent study with children in the first two years of formal schooling. The Pattern and Structure Mathematics Awareness Program (PASMAP) is described as two phases of Learning Pathways according to five structural groupings: sequences, structured counting, shape and alignment, equal spacing and partitioning. These groupings were found to be critical to developing coherent mathematical concepts and relationships. Implications for research in early mathematical development are outlined.

Keywords Early childhood · Mathematics education · Patterns
Structural development · Intervention

2.1 Introduction

Research in early childhood mathematics education has witnessed increasing impetus over the past decade reflecting calls for a more integrated approach to investigating children's mathematical development (Mulligan and Vergnaud 2006). The influence of cross-disciplinary and interdisciplinary approaches, including

J. Mulligan (✉) · M. Mitchelmore
Macquarie University, North Ryde, Sydney 2019, NSW, Australia
e-mail: joanne.mulligan@mq.edu.au

M. Mitchelmore
e-mail: mike.mitchelmore@y7mail.com

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_2

various cognitive and neuro-cognitive theories has been recognized across many new studies (e.g., Butterworth 2015). The development of number concepts has been studied in investigations of processes such as subitizing (the rapid and accurate perception of small numerosities), comparison of numerical magnitudes, location on a number line, axis differentiation and symmetry (e.g., Dehaene 2011). Other approaches have identified children's Spontaneous Focusing On Number (SFON) (Hannula and Lehtinen 2005). New cross-disciplinary studies are providing strong evidence that spatial reasoning skills are malleable and are of critical importance in the early years, and that these skills impact on later mathematical development (Davis 2015; Verdine et al. 2013). Spatial ability has also been linked to development of patterning and early algebraic skills (Carraher et al. 2006; Clements and Sarama 2007; Papic et al. 2011; Warren and Cooper 2008). Related studies highlight the critical role of perceptual subitizing and the spatial structuring of groups in arrays (Starkey and McCandliss 2014). Such studies are informing pedagogical and curriculum initiatives centred on spatial approaches.

However, the aim to develop core numeracy knowledge and skills has remained central to studies focused on improving outcomes in later mathematical achievement (Aunio and Niemivirta 2010). Counting and arithmetic strategy development has dominated many early numeracy programs and assessment initiatives because it was expected that development of these strategies would improve later mathematical achievement. In some countries curriculum frameworks for pre-school mathematics education have emerged, emphasizing the importance of a range of mathematical domains including patterning and spatial skills, measurement, data exploration and mathematical reasoning (Clements and Sarama 2007; English 2012).

An emerging line of research, focused on *mathematical patterns and structures*, aims to provide a more coherent picture of the common underlying bases of mathematical development (Mulligan and Mitchelmore 2009). This research has identified how children can develop connected mathematical knowledge leading to emergent generalization—through the development of patterns and structural relationships. There are important synergies with current studies of children's spatial reasoning and geometry (Casey et al. 2008; Sinclair and Bruce 2015), and the role of communication (Thom and McGarvey 2015; van Oers 2013). Other cognitively-oriented studies have focused on quantitative relationships (Torbeyns et al. 2015) akin to aspects of the work on structural relationships. Another key influence has been the impact of embodiment theory (Elia 2018, this book; Radford 2009) on understanding children's development of semiotic systems. These approaches provide the context for a broader range of studies, several of which were presented at ICME-13, which may drive change in the field beyond traditional emphases on number and arithmetic (Lüken 2018; Lüken and Kampmann 2018, this book). While domain-specific studies are necessary, there seems to be greater consensus about the need for integrated studies focused on the big ideas, or underlying common processes. A common aim of the research of the ICME-13 TSG 1 on early childhood mathematics education (Elia et al. 2018, Chap. 16, this book) is to describe and explain the wide variation in early mathematical competence in order to develop more effective pedagogical approaches.

2.2 The Pattern and Structure Mathematics Project

A suite of related studies with four- to eight-year olds has focused on the assessment of a broad range of mathematical concepts and common structural features (Mulligan et al. 2013). These studies have taken into account the complexity of various components of mathematical competency by adopting a more integrated view to answering some complex research questions: What are common salient features of early mathematical development? Does the ability to recognize patterns and structures reflect innate mathematical ability or can it be developed?

Based on early studies on patterning, counting, the numeration system and multiplicative thinking the project focused on identifying and describing common structural characteristics, later coined as the term, *Awareness of Mathematical Pattern and Structure (AMPS).* AMPS comprised two interdependent components: one cognitive—knowledge of structure, and one meta-cognitive—a tendency to seek and analyze patterns (Mulligan and Mitchelmore 2009).

Through these studies students' responses to a wide variety of tasks, developed and administered as the Pattern and Structure Assessment (PASA) confirmed that first graders' responses could be reliably classified into five ordered structural categories defined in Table 2.1. Qualitative analyses of student profiles indicated that individual student's responses to different tasks may have varied from task to task, but students who responded at the highest structural levels on one task tended to respond highly on other tasks. This similar pattern was found for students who responded at the lowest pre-structural level (Mitchelmore and Mulligan 2017; Mulligan et al. 2013).

Table 2.1 The five response categories used in scoring PASA items

Response category	Characteristics of response
Advanced structural	An accurate, efficient and generalized use of the underlying structure
Structural	A correct but limited use of the underlying structure
Partial structural	Shows most of the relevant features of the pattern and structure but inaccurately organised
Emergent	Shows some relevant features of the pattern and structure but incorrectly organised
Pre-structural	Shows at most, limited and disconnected features of the pattern and structure

2.3 The Pattern and Structure Assessment—Early Mathematics (PASA)

A recent development is the validation of a new form of the *Pattern and Structure Assessment—Early Mathematics (PASA)* (Mulligan et al. 2015). PASA consists of three interview-based assessments designed for students in the first three years of formal schooling (students from five to eight years). Each PASA focuses on similar core concepts that underlie mathematical development ranging from 14 to 16 items from foundation to year (grade) 2. The concepts assessed included repeating and growing patterns, partitioning two-dimensional and three-dimensional shape and space, multiple counting and base ten structure, arrays and grids, distance and scale, and units of length, area, volume/capacity, and time and graphs.

2.4 Method

Two Sydney metropolitan schools were selected because they represented schools gaining scores within one standard deviation of the mean on state-based numeracy assessments in the third grade. Students were drawn from a diverse range of cultural and socio-economic contexts. Three forms of the PASA were administered by the research team to students in the foundation (F) year (kindergarten) and grade 1: PASA-F (n = 213), PASA-1 (n = 189) and PASA-2 (n = 216).

The PASA tasks included:

1. Partitioning lengths into thirds
2. Border patterns
3. Triangular arrays
4. Partitioning
5. Ten frames
6. Counting by threes: number track
7. Spatial pattern continuation
8. Square arrays
9. Structuring/using the hundred chart
10. Constructing analogue clock
11. Grid completion
12. Comparing triangles
13. Growing pattern continuation
14. Making a ruler
15. Constructing bar charts
16. Comparing capacities

The PASA interviews were conducted consistently following protocols from previous studies (see Mulligan et al. 2013). Each student also completed a PATMaths achievement test, comprising multiple-choice items across basic numeracy (Stephanou and Lindsey 2013). Six trained interviewers piloted protocols for conducting the interviews and coding of responses, with an inter-rater reliability of 0.82.

Consistent with the earlier forms of PASA, responses to each interview item were categorized by the interviewer as one of five broad levels of structural development: *pre-structural, emergent, partial, structural* and *advanced structural* using defined criteria (see Mulligan et al. 2013). Using Rasch modeling, an overall AMPS score was calculated and reported as a location on an AMPS scale, expressed as a scale score in a unit called amps (Mitchelmore and Mulligan 2017).

2.4.1 AMPS Levels on a Rasch Scale

Levels of AMPS are described along the Rasch measurement scale; response categories of each task are distributed differently along the scale depending on the level of AMPS required for a response to be assigned to a category. The location of a category of a task along the scale reflects the AMPS level required to respond in that category relative not only to categories of the same tasks but for categories of all PASA tasks. Four AMPS levels were identified and described on the scale, showing the expected development of AMPS from the lowest level 1 to the highest level 4, as follows:

1. Children struggle to recognize very simple patterns, e.g., they may copy block patterns only by matching the blocks one by one; draw individual squares in a grid pattern incorrectly, and count by ones up to three groups of two blocks.
2. Children recognize simple patterns, e.g., can identify unit of repeat and are aware of some relations to other patterns; recognize shapes embedded in a grid pattern.
3. Children are aware of fundamental structures such as underlying structure of unit of repeat, alignment in grids and arrays, use of equal spaces and scale and counting in multiples.
4. Children are aware of the generality of fundamental structures and can extend these; e.g., extend a growing pattern, systematically replicate or partition a pattern or area; explain the structure of numerals, draw and interpret graphs, and use multiplication and division.

Another form of validation was a comparison of AMPS measures with measures on another scale of mathematical achievement, the PATMaths scale, as all students in the same sample completed the PATMaths assessment (Stephanou and Lindsey 2013). Although the two assessments provide different information about the student's mathematical competence they were found to be highly correlated; Foundation (0.72), Year 1 (0.76) and Year 2 (0.84). The AMPS scale makes it possible to compare student's level of AMPS across year (grade) levels, regardless of which PASA assessment form they are given.

2.5 Structural Groupings

AMPS' levels can be described overall, or according to each of the following five individual structural groupings that represent the concepts embedded across the range of items.

Sequences: recognizing a (linear) series of objects or symbols arranged in a definite order or using repetitions, i.e., repeating and growing patterns and number sequences.

Structured counting and grouping: subitizing, counting in groups, such as counting by 2s or 5s or on a numeral track with the equal grouping structure recognized as multiplicative.

Shape and Alignment: recognising structural features of two- and three-dimensional shapes and graphical representations, constructing units of measure, such as co-linearity (horizontal and vertical co-ordination), similarity and congruence, and such properties as equal sides, opposite and adjacent sides, right angles, horizontal and vertical, parallel and perpendicular lines.

Equal Spacing: partitioning of lengths or other two- or three-dimensional spaces and objects into equal parts, and constructing units of measure. Equal spacing is fundamental to representing fractions, scales and intervals.

Partitioning: equal division of lengths and other two- or three-dimensional spaces, objects and quantities, into unequal or equal parts, including fractions and units of measure.

A separate scale is provided for each of these structural groupings so it is possible to view students' AMPS according to different conceptual structural groupings. This process provides valuable assessment data that can inform pedagogical planning and practice, and guide teachers to utilize the aligned program, PASMAP, or other curriculum support programs.

2.6 The Pattern and Structure Mathematical Awareness Program

Linked with the PASA, an innovative mathematics learning program comprising the first stage of the *Pattern and Structure Mathematical Awareness Program* (PASMAP) was developed and evaluated longitudinally with 316 students aged 4 to 6 years (Mulligan et al. 2013). Students were engaged in the experimental intervention PASMAP for the entire first year of formal schooling. The study found highly significant differences on the PASA between the program (intervention) students and the 'regular' group at the retention point ($p < 0.002$) and increased levels of structural development for intervention students. These levels were retained one year later although the program was not sustained during this subsequent year. The study validated the instrument (PASA) and constructed a Rasch scale indicating item fit. Students engaged in the PASMAP also showed consistent development of spatial reasoning, and structural relationships such as commutativity and equivalence, as well as the ability to form emergent generalizations. These competencies were assessed through individual students' responses to structured tasks, comprising the representing, visualizing and generalizing processes of the PASMAP pedagogical model, discussed later in the chapter.

Thus, further development and trials of the PASMAP learning experiences directed attention to role of spatial reasoning in developing AMPS. Several descriptions are provided of processes such as spatial structuring with a key finding that, coupled with visual memory, this was critical to forming and representing concepts. PASMAP learning experiences focused on such aspects as identifying objects or shapes in a foreground or background, alignment (co-linear or axis), unitizing the number line, space and shape, symmetry and transformations, and graphical representation of data.

Further development of the program (Mulligan and Mitchelmore 2016a, b) has resulted in two phases of PASMAP integrating most key concepts and processes across mathematics curricula, promoting a connected set of mathematical interrelationships. Each of the Learning Pathways focus on particular concepts or structures aligned with one or more of the five structural groupings described earlier. Table 2.2 lists the Learning Pathways developed in two phases of PASMAP.

Table 2.2 PASMAP pathways, phases 1 and 2, aligned with structural groupings

Pathway	Structural groupings
Phase 1	
RP: Repeating Patterns	Sequences, structured counting
SC: Structured Counting	Structured counting, equal spacing
GS: Grid Structure	Shape and alignment, structured counting
SS: Structuring Shapes	Shape and alignment, partitioning
PS: Partitioning and Sharing	Partitioning, equal spacing
BT: Base Ten Structure	Equal spacing, structured counting, shape and alignment
GP: Growing Patterns	Sequences, shape and alignment
SM: Structuring Measurement	Equal spacing, partitioning, shape and alignment
SD: Structuring Data	Equal spacing, shape and alignment
ST: Symmetry and Transformations	Shape and alignment, partitioning, equal spacing
Phase 2	
MP: Multiplication Patterns	Sequences, structured counting, equal spacing, shape and alignment, partitioning
FS: Fitting Shapes Together	Shape and alignment, equal spacing
PF: Partitioning and Fractions	Partitioning, equal spacing
PV: Place Value	Equal spacing, structured counting
MM: Metric Measurement	Shape and alignment, structured counting, equal spacing, partitioning
PD: Patterns in Data	Shape and alignment, equal spacing
AD: Angles and Direction	Shape and alignment

The design of the PASMAP intervention takes account of the assessment (PASA) that measures the student's level of AMPS. However the program can be utilized in conjunction with other assessments and intervention strategies. PASMAP was designed and trialed with young students aged four- to eight-years, of wide-ranging abilities including those with mathematics learning difficulties (MLD) and those assessed as gifted in mathematics. PASMAP is flexible in its implementation because the teacher can target specific mathematical structures with which the individual has most difficulty, or they can tailor the learning for students with advanced AMPS. The program is not intended to be lock–step in implementation, although there are some learning experiences that precede others in conceptual difficulty.

2.7 Making Connections Between and Within Pathways and Structural Groupings

2.7.1 Pathways: Phase 1

PASMAP Pathways form an interconnected web of concepts and relationships that may develop in very different ways and routes for individual children. Table 2.2 provides an overview of the pathways aligned with one or more of the five structural groupings described earlier. The first pathway, *Repeating Patterns* exemplifies a *sequence* structure, for example, where a group of learning experiences is organised in a specific order, where each experience has a specific position in the sequence. Similar learning experiences can be grouped which are related to the concept of unit of repeat and the notion of multiple counting. Thus structured counting is interrelated with repeating patterns. This structure is linked in the PASMAP to multiplicative reasoning including measurement and data representation, as well as early algebraic relations and functional thinking.

Structured Counting also provides opportunity to develop patterns with an *equal spacing* structure (e.g., an empty number line) or when items are visualized and counted in the form of an array. Here we would expect to see the simultaneous development of the *shape and alignment* structure. The structure of arrays can be utilized to form grids (e.g., squares and rectangles), and this is where the development of co-linearity can occur. The alignment of the squares in horizontal rows and vertical columns is critical to using square units of measure.

Grid Structure is central to many mathematical concepts such as area measurement and because it is related to the development of spatial structuring it is fundamental to the PASMAP approach. Constructing and interpreting grids also relies on *equal spacing*. Grids are also connected with many spatial concepts such as congruence, co-linearity and juxtaposition.

The properties of, and relationships between two- and three-dimensional shapes enable exploration of the *shape and alignment* structure, developed in the *Structuring*

Shapes pathway. Investigating how shapes can be partitioned into smaller parts and then put together to form new shapes is explored further in *Partitioning and Sharing*. Situations that require that objects, shapes or spaces be partitioned into parts often require application of *structured counting*, *equal grouping* or *equal spacing* of unstructured collections or whole objects. The early development of multiplication and division as an inverse process is experienced through the reconstruction of shared equal parts as multiplication. Some students' experience of visualizing and constructing common fractions is unavoidably encountered.

The learning experiences of the pathway *Base Ten Structure* focuses on the numeration system, which relies on the early development of structured counting and equal grouping with explicit attention to grouping into tens. The notion of quinary-based structure i.e., grouping by fives supports the grouping by tens structure. There is a strong connection to the structure of the number line, an important example of *equal spacing*. Particular emphasis is placed on the notion of equal spacing of benchmarks 5, 10, 15 and 20. In connection, the experiences of partitioning and structured counting contribute to the student's estimation of equal parts or spaces. This pathway uses ten frames (2 by 5 grids), which can be extended to 20 and 50 frames to model the base ten system, provides a strong link to the *shape and alignment* structure. Construction of ten frames and the connection with counting and benchmarking on an empty number line extends *structured counting* to focus on counting by 5s and 10s. These structures are exploited to develop children's ability to visualize order and magnitude and to develop computational strategies for the four operations.

The *Growing Patterns* pathway focuses on the *sequences* structure developed in the initial pathway of *Repeating Patterns*. Early development of the notion of unit of repeat is considered pre-requisite to the development of growing patterns that are more complex than repeating patterns. The learner must move beyond the notion that extra units or 'chunks' can be added in repeating patterns or in sequences of equally-spaced numbers (e.g., counting by twos) to the idea of the whole pattern as multiplicative. *Growing Patterns* may lead to sequences of numbers, symbols, objects or shapes that are not equally spaced. The basic concept is that there must be a general rule for determining the next item in the sequence, whatever the items may be. Engagement in growing patterns that involve spatial concepts also contributes to children's awareness of the *shape and alignment* structure.

The pathway *Structuring Measurement* provides fundamental experiences of units of measurement: length, area, volume, capacity and mass. It is emphasized that to measure any quantity, a single, fixed unit must be used. Understanding how these units fit together in various contexts involves the *equal spacing* and *shape and alignment* structures. Although they are represented in the later parts of the learning pathways the experiences can be aligned with earlier pathways. The connection with *equal-sized units* or *equal spaces*, *grid structure*, and *partitioning* leads to an important generalization about relative size and structure of units, e.g., the smaller the unit the more (units) will be needed to measure the quantity, object or space. The measurement system that is developed through multiplicative and base ten concepts is interrelated across the learning pathways. Although the learning experiences are limited, the measurement of time is explored through the ideas of a timeline and a "time circle" as well as the structure of the analogue clock. These experiences also involve the *partitioning* structure, since it is often necessary to divide a unit of measurement (e.g., days) into more convenient sub-units (e.g., hours).

The pathway *Structuring Data* focuses on collecting simple categorical data, organising it in the form of a table and representing it pictorially as picture graphs. The basic concept is that each item in the data is best represented by a single symbol or *icon* (analogous to a unit of repeat and a unit of measurement). Data representation therefore involves the *equal spacing* structure. *Grid structure* is fundamental to constructing a picture graph, since the icons stand on a single base line and line up vertically and horizontally. This pathway thus involves the *shape and alignment* structure.

There is no direct connection intended between the *Structuring Measurement* and *Symmetry and Transformations* pathways. However *Symmetry and Transformations* are placed at this stage of the sequence of pathways because they involve many of the components of earlier pathways and the development of spatial structures. Simple transformations are initially represented as a way of moving a shape, and a shape that remains unchanged under a transformation is represented as having symmetry. The learning experiences in this pathway explore three two-dimensional transformations—reflection (flipping), translation (sliding) and rotation (turning)—and the corresponding symmetries. Exploring symmetry and transformations further develops children's awareness of the *shape and alignment* and the *equal spacing* structures. When children break a shape into parts to see if it is symmetrical, they are also developing their awareness of the *partitioning* structure.

2.7.2 Pathways: Phase 2

The second phase of the PASMAP develops seven pathways that develop further the learning experiences and five structural groupings of Phase 1. Multiplicative concepts including place value and fractions, units of measurement, graphical representation, angles and direction broaden the scope and depth of concepts developed in Phase 1. The two phases do not explore every mathematical concept in depth; other conceptual domains will be developed through the next group of learning Pathways (in preparation) such as concepts of chance and probability.

The *Multiplication Patterns* pathway develops all five structural groupings in an interrelated way. The emphasis is on *sequences* and *structured counting* in threes and fours, while construction and representation of the number line involves *equal spacing* and *partitioning*. Learning experiences also connect with spatial patterns related to multiplication. Children explore multiples of 2, 3, 4, 5 and 10 in the contexts of repeating patterns and grids (see Fig. 2.1). The number line is extended to at least 50 to encourage structuring and representing of counting sequences that repeat in the higher decades. Symmetry is embedded in some patterns and this connects with the structural grouping of *shape and alignment*.

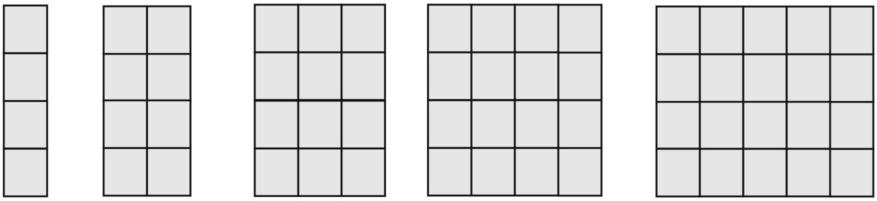


Fig. 2.1 Columns of four squares increasing in size

Figure 2.1 shows a sequence of grids, where children are encouraged to look for patterns in the number of squares, followed by construction of a table to represent the patterns (Table 2.3).

The table can be extended to larger grids with four rows which can encourage children to realize that the total number of squares is always four times the number of columns (Mulligan and Mitchelmore 2016b, p. 37).

In *Fitting Shapes Together* children investigate how simple two- and three-dimensional shapes fit together to make patterns and thereby deepen their awareness of the shape and alignment structure. For example, patterns are constructed by fitting two or more equilateral triangles together or forming tessellations, creating three-dimensional repeating patterns and making cuboids by folding nets. Some of the investigations also draw on the *equal spacing* structure implicit in grids.

The *Partitioning and Fractions* pathway highlights the importance of constructing unit fractions as a result of a partitioning process that creates a pattern. Mixed numbers are also introduced. Partitioning may be seen as the opposite of repeating patterns. This learning experience therefore also involves the *equal spacing* structure. A series of learning experiences investigate the patterns arising when a length or two-dimensional shape are partitioned into a number of equal parts, and how fractions are represented on a number line, including the idea of a mixed number.

The *Place Value* pathway explores patterns related to place value, where children develop their awareness of the multiplicative structure of the base-ten system and then extend their understanding to larger numbers and decimal currency. Estimation and computational strategies are developed through patterns and emergent generalizations. The *equal spacing* structure is connected with

Table 2.3 Pattern generated from the sequence of squares in Fig. 2.1

No. Columns	No. Squares	Total no. squares
0	4	0
1	4	4
2	4	8
3	4	12
4	4	16
5	4	20

exploration of the empty number line while an understanding of the place value of numerals is based on *structured counting* and multiplicative structure. For example experiences include strategies for addition and subtraction using an empty number line, constructing a hundreds chart and the structure of numerals beyond two digits.

The *Metric Measurement* pathway explores common physical quantities using a variety of units based on the metre, area using square metric units, cubic units, litres and millilitres, and time measurement. Understanding the relationship between length, area, volume and mass is encouraged. Learning experiences connect with *shape and alignment* and *structured counting* structures. For example, use of measuring scales involves the *equal spacing* structure, and the division of units into sub-units (e.g., metres into centimetres) involves the *partitioning* structure.

The learning experiences in the *Patterns in Data* pathway focus on the drawing and interpretation of column and line graphs, where categorical and continuous data are explored. The experiences focus on the relation between two quantities, one represented on a horizontal axis and one on a vertical axis. Drawing a graph freehand therefore involves the *shape and alignment* structure, and as quantities are often represented by a scale on an axis, the *equal spacing* structure is utilized.

The *Angles and Direction* pathway focuses on the relationships between directions, paths as movements in various directions, and perspective taking. For example, children investigate how to find and record the size of various corners which links turning to the concept of angle; and angles are linked with slides (translations) to create paths. Technological toys, such as Beebots are integrated into learning experiences. Perspective taking is developed through the idea that objects can look quite different from different viewpoints. The learning experiences draw heavily on the *shape and alignment* structure.

2.8 The Pattern and Structure Pedagogical Approach

Essentially, the PASMAP pedagogy is designed to move students towards generalization, albeit simple or emergent, with a view to representing and abstracting core structural elements and interconnecting them. The role of visual memory to visualize, construct and represent mathematical patterns and structures is emphasized. The pedagogical process is summarized as follows.

Modelling

Children copy, model or describe a pattern linked to a specified mathematical task, usually under teacher direction. The teacher ensures that children understand the essential features of the pattern or structural features that are the focus of the learning experience.

Representing

Children record usually by drawing, the pattern or a model of the pattern while visible without teacher direction. This experience helps children to isolate the essential features of the pattern or structure from the particular example in which it occurs. Digital technologies may also be utilized.

Visualizing

Children draw or symbolize the pattern or structure when it is not visible. Comparing their productions with the original pattern or structure highlights its essential features. The process of visualizing and comparing can be scaffolded and repeated until children have internalized the pattern or structure.

Generalizing

The teacher supports children (either individually or as a group) to make the pattern or structure explicit, find similar examples in other contexts, and express what is 'general' about the pattern.

Sustaining

Some learning experiences are provided that reinforce or extend children's development and application of the patterns or structures.

The following example describes a fundamental learning experience about *grid structure* that requires the child to grasp the structural features of a grid, by working through the phases of the PASMAP pedagogical process: initially the child models the simple grid structure with squares; copies the grid accurately, then represents it by drawing from memory. Structural features are highlighted: grids contain squares of same size (congruent); they are equally spaced, and their sides are always aligned (co-linear). Some children will discover that grids can be formed easily by forming a pattern of coordinating horizontal and vertical lines sequentially, and equally spaced. In connection with this basic structure, kindergarten children attempted to utilize the spatial features of the square to generate and represent the pattern of square numbers. In the following examples we see how two different children produce drawings of the pattern of squared numbers from memory using grid structure. In Fig. 2.2 the formation of a grid is limited to a 3×3 square drawn with nine individual squares and an attempt to draw a 4×4 and 5×5 square is limited by lack of spatial and multiplicative structure; and in contrast, in Fig. 2.3 we observe another child's representation of grid structure to represent the sequence of 1, 4, 9, 16, 25, 36 but the spatial structure of a square grid is limited to 5×5. The child's attempt to produce a 6×6 grid is represented as five additional vertical squares. Thus the child has not continued the growing pattern in two dimensions.

Fig. 2.2 Emergent grid structure

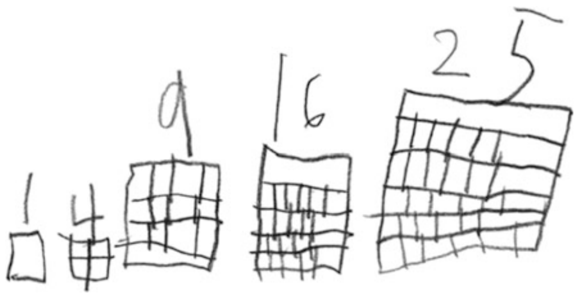
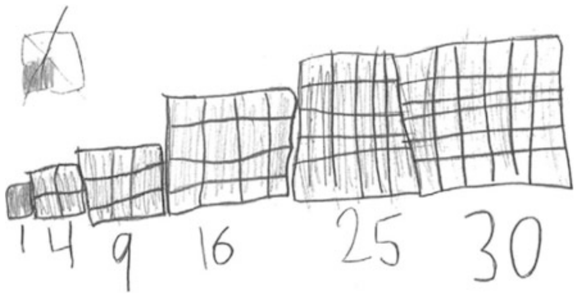


Fig. 2.3 Grid structure applied to a 5 × 5 square



2.9 Implications for Research and Practice

The Pattern and Structure project has provided an integrated assessment instrument and pedagogical program for early learners that support an alternative approach to developing mathematical concepts. However, it is still limited somewhat by the focus on domain-specific aspects of AMPS such as partitioning and two- and three-dimensional space. The evaluation studies of PASMAP have not yet fully articulated how the learner forms interrelationships between pattern, geometry and measurement, data and number through the development of the five structural grouping; further research is in progress on establishing how these connections are formed.

The role of AMPS has also been studied in a related project on data modelling and representation, that has described the development of structuring data—categorisation, scale and interval, and coordination of axes—as integral to meta-representational competence with young gifted children (Mulligan 2015).

The role of spatial reasoning in the development and use of AMPS is not yet fully understood; the relationship between pattern formation and spatial structuring needs more in-depth investigation. Further recent analyses, utilizing network analysis (Woolcott et al. 2015) has provided some visual links between the five structural groupings as network maps of connectivity. This form of analysis complements Rasch analysis because it highlights the connections, or lack thereof, that children make between structural groupings and specific concepts. This is one research domain that begs further development given the importance of spatial reasoning in related Science Technology Engineering and Mathematics (STEM) fields of learning.

Further investigation is ongoing through a new study (Mulligan et al. 2017), *Connecting Mathematics Learning through Spatial Reasoning* that investigates the interrelationships between structural groupings and spatial reasoning tasks. The project will develop an advanced form of the PASA, and a Spatial Reasoning Mathematics Program (SRMP) for Grades 3 through 5 (students aged approximately 7–11 years). The PASMAP will be expanded in both scope and depth to include a larger component on spatial reasoning, including transformations, spatial structuring of two- and three-dimensional shapes, axis differentiation, co-linearity, perspective taking and digitally-supported spatial representations.

In further studies it is imperative that the impact of PASMAP on mathematical achievement is assessed longitudinally with larger and diverse samples and compared with robust measures of mathematical development. Educators may need to adopt a broader, or more focused view of what is fundamental to assess and teach. This may require a reconsideration of what constitutes critical components of early mathematical development based on emerging transdisciplinary research evidence. The important question remains as to how educators can develop an understanding of, and consequently promote, connected mathematical structural development from an early age.

Acknowledgements The research reported in this paper was supported by Australian Research Council Discovery Grants No. DP0880394 and DP110103586 and the Australian Council for Educational Research (ACER). The authors express their particular thanks to Lyn English, Andrew Stephanou and John Lindsey.

References

Aunio, P., & Niemivirta, M. (2010). Predicting children's mathematical performance in grade one by early numeracy. *Learning and Individual Differences, 20,* 427–435.

Butterworth, B. (2015). Low numeracy: From brain to education. In X. Sun, B. Kaur, & J. Novotná (Eds.), *Proceedings of the 23rd ICMI Study Conference: Primary Mathematics Study on Whole Numbers* (pp. 21–33). Macao, China: University of Macau.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education, 37,* 87–115.

Casey, B. M., Andrews, N., Schincler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction, 26*(3), 269–309.

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *Journal for Research in Mathematics Education, 38,* 136–163.

Davis, B. (Ed.). (2015). *Spatial reasoning in the early years: Principles, assertions, and speculations*. New York: Routledge.

Dehaene, S. (2011). *The number sense: How the mind creates mathematics*. New York, NY: Oxford University Press.

Elia, I. (2018). Observing the use of gestures in young children's geometric thinking. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Elia, I., Mulligan, J., Anderson, A., Baccaglini-Frank, A., & Benz, C. (2018). Early childhood mathematics education: Conclusions and moving forward. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

English, L. D. (2012). Data modeling with first-grade students. *Educational Studies in Mathematics Education, 81,* 15–30.

Hannula, M. M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction, 15,* 237–256.

Lüken, M. (2018). Repeating pattern competencies in three- and four-year old Kindergarteners. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Lüken, M., & Kampmann, R. (2018). The influence of fostering children's pattern and structure abilities on their arithmetic skills in Grade 1. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Mitchelmore, M. C., & Mulligan, J. T. (2017). Developing a scale to measure awareness of mathematical pattern and structure (AMPS). In B. Kaur, W. K. Ho, T. L. Toh, B. H. & Choy (Eds.) *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 257–264). Singapore: PME.

Mulligan, J. T. (2015). Moving beyond basic numeracy: Data modeling in the early years of schooling. *ZDM—The International Journal on Mathematics Education, 47*(4), 653–663. (V. Geiger, M. Goos & H. Forgasz (Eds.), Berlin: Springer).

Mulligan, J. T., English, L. D., Mitchelmore, M. C., & Crevensten, N. (2013). Reconceptualizing early mathematics learning: The fundamental role of pattern and structure. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 47–66). New York: Springer.

Mulligan, J. T., & Mitchelmore, M. C. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21,* 33–49.

Mulligan, J. T., & Mitchelmore, M. C. (2016a). *Pattern and structure mathematics awareness program book 1*. Camberwell VIC: ACER Press.

Mulligan, J. T., & Mitchelmore, M. C. (2016b). *Pattern and structure mathematics awareness program book 2*. Camberwell VIC: ACER Press.

Mulligan, J. T., Mitchelmore, M. C., & Stephanou, A. (2015). *Pattern and structure assessment (PASA): An assessment program for early mathematics (Years F-2) teacher guide*. Camberwell VIC: ACER Press.

Mulligan, J. T., & Vergnaud, G. (2006). Research on children's early mathematical development: Towards integrated perspectives. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 117–146). London: Sense Publishers.

Mulligan, J. T., Woolcott, G., Mitchelmore, M. C., Davis, B. (2018). Connecting mathematics learning through spatial reasoning. *Mathematics Education Research Journal* (in press).

Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42,* 237–268.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70,* 111–126.

Sinclair, N., & Bruce, C. (2015). New opportunities in geometry education at the primary school. *ZDM—The International Journal on Mathematics Education, 47*(3), 319–329.

Starkey, G. S., & McCandliss, B. D. (2014). The emergence of "groupitizing" in children's numerical cognition. *Journal of Experimental Child Psychology, 126,* 120–137.

Stephanou, A., & Lindsey, J. (2013). *Progressive achievement tests in mathematics (PATMaths)* (4th ed.). Camberwell VIC: ACER Press.

Thom, J., & McGarvey, L. (2015). The act and artifact of drawing(s): Observing geometric thinking with, in, and through children's drawings. *ZDM—The International Journal on Mathematics Education, 47*(3), 465–481.

Torbeyns, J., Gilmore, C., & Verschaffel, L. (2015). The acquisition of preschool mathematical abilities: Theoretical, methodological and educational considerations: An introduction. *Mathematical Thinking and Learning, 17*(2–3), 99–115.

van Oers, B. (2013). Communicating about number: Fostering young children's mathematical orientation in the world. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 183–203). New York: Springer.

Verdine, B. N., Golinkoff, R., Hirsh-Pasek, K., Newcombe, N., Filipowocz, A. T., & Chang, A. (2013). Deconstructing building blocks: Preschoolers' spatial assembly performance relates to early mathematics skills. *Child Development, 85*(3), 1062–1076.

Warren, E., & Cooper, T. J. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8-year olds thinking. *Educational Studies in Mathematics, 67,* 171–185.

Woolcott, G., Chamberlain, D., & Mulligan, J. (2015). Using network analysis to connect structural relationships in early mathematics assessment. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 321–328). Hobart, Australia: PME.

Chapter 3
Repeating Pattern Competencies in Three- to Five-Year Old Kindergartners: A Closer Look at Strategies

Miriam M. Lüken

Abstract Activities with repeating patterns are common and important in early childhood education. Recent studies even show a (positive) relationship between early patterning competencies and arithmetical achievement at school. Nevertheless, there is insufficient research about the strategies children employ when completing patterning activities and about the development of these competencies in young children. This paper reports data from an explorative study that tracks six children's patterning development from their first to third year of German kindergarten. Three task- and material-based interviews are conducted over the course of two years. Results suggest that significant development takes place in children's repeating pattern competencies between the age of three and four years. Furthermore, strategy categories for describing children's construction of repeating patterns are developed.

Keywords Early childhood · Mathematics education · Repeating patterns
Strategies · German kindergarten

3.1 Repeating Patterns in Early Mathematical Learning

Pattern and structure are central components of mathematics knowledge, illustrated by two other chapters in this theme (Lüken and Kampmann 2018; Mulligan and Mitchelmore 2018). Both authors relate to a broad concept of pattern and structure: knowledge of structural relationships regarding many mathematical concepts and patterns in different facets as spatial, repeating and growing patterns.

M. M. Lüken (✉)
Bielefeld University, Universitätsstraße 25, 33615 Bielefeld, Germany
e-mail: miriam.lueken@uni-bielefeld.de

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_3

In this study, one type of pattern—repeating patterns is foregrounded. A repeating pattern is a sequence of elements (the unit of repeat) that is repeated indefinitely and therefore has a periodic structure. Repeating patterns can be represented with different shapes or areas: for example, they may cover a particular space or take the form of an object. In the study reported here repeating patterns are shown as linear arrangements (see Figs. 3.1 and 3.2) as this seems to be the most accessible kind of repeating pattern for very young children. Creating, copying and extending repeating patterns are very common activities found in early childhood learning settings (Economopoulos 1998; Miller et al. 2016; Papic et al. 2011). On the one hand, young children engage in these activities spontaneously (e.g., Fox 2005; Ginsburg et al. 1999). On the other hand early childhood educators use repeating pattern activities to initiate mathematical learning (see e.g., Fox 2006; Hoenisch and Niggemeyer 2004), as reflected in early childhood curricula internationally (e.g., National Council of Teachers of Mathematics [NCTM] 2013; Ministerium für Kultus, Jugend und Sport Baden-Württemberg [MKJS] 2014).

Lüken and Kampmann describe in Chap. 4 (this book) recent studies on patterning that provide strong evidence that patterning knowledge is central to mathematics achievement. These studies present empirical evidence that early patterning abilities have a positive influence on later mathematics learning. In an earlier study Mulligan and Mitchelmore (2009) found a nearly perfect correlation between Australian grade 1 students' (from 5 years 5 months to 6 years 7 months of age) general mathematical understanding and their pattern and structure competencies. Similarly, in an early study, Lüken (2012) showed a significant correlation on a medium level between patterning competencies of school starters (from 5 years 8 months to 7 years 2 months of age) and their early mathematical competencies, and a slightly lower correlation with their mathematical achievement at the end of grade 2. Lüken et al. (2014) also found a significant effect of 5-year old children's repeating pattern abilities on their mathematical competencies in kindergarten and the transition from kindergarten to school. This study showed that children who are able to reproduce, extend and explain a repeating pattern in the

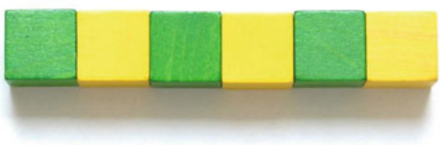


Fig. 3.1 AB pattern (green, yellow) (1st, 2nd and 3rd MP) (Color figure online)

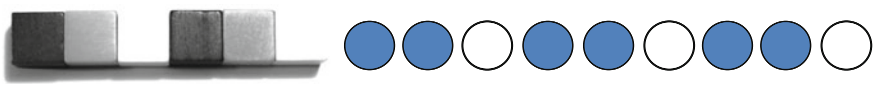


Fig. 3.2 ABC pattern (green, purple, orange) and AAB pattern (blue, blue, red) (2nd and 3rd MP) (Color figure online)

form ABCC one year prior to school are those who demonstrate elaborate number concept development in kindergarten and achieve at the highest level on a standardized mathematics classroom test at the end of Grade 1.

In a similar six-month intervention study, Kidd et al. (2013, 2014) assigned struggling first-grade students randomly (mean age of 6.63 years, $SD = 1.89$) to learning either patterns, numeracy, reading, or social studies. Children who received pattern instruction performed as well or better on several standardized mathematics assessments relative to children who received numeracy instruction, and systematically better than children who received reading or social studies instruction. Thus, multiple studies indicate that understanding patterns is important for mathematics achievement.

Experiences with repeating patterns provide early opportunities to identify and describe predictable sequences and as a conceptual stepping-stone in the development of pre-algebra (Threlfall 1999). The foundations can also be considered a precursor for functional thinking and algebra (e.g., NCTM 2000). Repeating patterns are also highlighted as important for measurement and as critical to the development of counting and multiplicative thinking (Mulligan and Mitchelmore 2009). Threlfall (1999) suggests that activities with repeating patterns develop general mathematical concepts in children such as ordering, comparing, sequencing, classification, abstracting and generalizing rules, and making predictions. However, Economopoulos (1998) points out that "To generalize and predict, children must move from looking at a pattern as a sequence of 'what comes next' to analyzing the structure of the pattern, that is, seeing that it is made of repeating units" (p. 230). Thus, children must learn to identify the pattern unit: the part of the pattern that repeats (Clements and Sarama 2009; Papic et al. 2011).

From my own observations as a teacher and as a researcher I have found that children do not have to be aware of the unit of repeat in order to correctly create, copy (duplicate), extend, or translate (abstract) a pattern. These activities can be correctly completed by object-matching strategies or by the help of a good memory and may not stand up to mathematical considerations (see also, Threlfall 1999). Therefore, a closer examination of children's views of repeating patterns is needed. This might be achieved by analyzing closely their strategies while they engage in patterning activities. Furthermore, despite the fact that the importance of patterning ability is widely acknowledged, the development of these abilities, especially with regard to children's strategies in the early years of childhood is seldom explored. However, early childhood educators need to be aware of young children's steps in the developmental journey towards mature pattern-making abilities as well as to provide appropriate contexts to support mathematical learning.

To address our limited knowledge of early development of patterning competencies, the aims of this study were to explore growth in repeating pattern competencies over the first three years of German kindergarten (ages three to five years) and to describe strategies children employ when doing repeating patterning activities. Before describing the current study, I summarize past research on the early development of patterning abilities with a particular focus on strategies and strategy development.

3.2 Research on the Early Development of Patterning Strategies Abilities

Several researchers observed patterning as part of children's everyday play (e.g., Ginsburg et al. 1999; Fox 2005, 2006; Gura 1992). In the context of such an observational approach, Garrick et al. (1999) reported a study on the development of early pattern-making skills. The research included a longitudinal study of 24 children, followed through from 3 years 6 months to 4 years 6 months of age. Data was collected in play contexts. The children were engaged in creating and copying patterns. The materials used were pegs, beads, and mosaic tiles, with which children were able to structure their patterns both spatially and in terms of color organization. As this paper's focus is repeating patterns in the form of linear arrangements, the findings of Garrick et al.'s (1999) research are described with regard to patterning by color.

Garrick et al. (1999) found the awareness of color and the purposive creation of similarities and differences by color as the very basic elements of pattern making. These elements were observed to appear relatively early in development and then led to the strategies of *chained*, *alternated*, *repeated*, *multiplied*, and *symmetrically placed*. *Chaining*, the simplest strategy and the first to be seen in children's work, refers to the successive placement of groups or series. The simple chaining shows no regularity concerning color or number. In patterns of advanced chaining, groups of equivalent size are chained, but without regularity concerning color placement. At this stage the child is following self-imposed rules for the selection of materials and thus "showing an early awareness of some key features of pattern" (Garrick et al. 1999, p. 11). Following closely in development is the *alternation* of (groups in) two colors (e.g. AB, AAB, AABB). The difficulty for the children here is to sustain an exact repetition of the size of the groups. Still, children's work is considered alternated when two colors are used successively regardless of exact quantity. At a more advanced level, children are able to create a *repeating* pattern with the unit of repeat consisting of three or more single elements (e.g. ABC, ABCDE). Another relatively late-appearing strategy in pattern making is creating repeating patterns where the elements within the unit of repeat are repeated (i.e., multiplied; e.g. AABBCCDDEE). After the basic elements of spatial and color organization are established, they also occur in combination. Children do not only create unidirectional, repeating patterns where the unit of repeat is geometrically translated, but they also place elements *symmetrically* around a center in a bidirectional way so that the unit of repeat is geometrically mirrored.

Papic et al. (2011) also described strategies young children employed when copying, creating, explaining, and extending a repeating pattern in a one-on-one interview situation. In the context of an intervention study, 53 participants (from 3 years 9 months to 5 years of age) were interviewed three times over the course of 18 months with a task- and material-based patterning assessment. The materials used were blocks, colored-pencils, and mosaic tiles, with which children were able to structure their patterns mainly in terms of color organization. The solution strategies fell in five main categories and are described here in increasing order of sophistication. The first strategy is *random arrangement*, where children choose and place elements randomly when copying or extending a repeating pattern. Matching one item at a time by *direct comparison* is a frequently observed strategy when copying a pattern. The most common strategy was *alternation*, where children focused on the sequence of individual colors (e.g. red, then blue, then green, then red, …) rather than on the unit of repeat (e.g. red-blue-green). The latter falls into the category *basic unit of repeat*. In this strategy, children were able to identify and use the unit of repeat to complete more complex tasks. After the intervention, where children's patterning skills were fostered, some children demonstrated and expressed simple generalizations about the unit of repeat (*advanced unit of repeat*).

In comparison, Garrick et al. (1999) describe a progression of repeating pattern's complexity levels in terms of what kind of repeating patterns children are able to create at different developmental stages. Papic et al. (2011) focus on the strategies children use in the activities with repeating patterns independent of the repeating patterns' complexity. These differences can be explained by the studies' different settings as observational versus interview investigating partially different patterning activities (self-initiated creating and copying versus task-based explaining, copying and extending). The studies demonstrate that different patterning activities and different pattern complexities may evoke different strategies in the children at different age levels. Detailed descriptions of children's competencies and strategies at different age levels are needed. In particular different patterning activities (i.e., copying, explaining, repairing, extending, translating, and creating a repeating pattern) and in-depth single case analyses are necessary to trace common developmental steps over a period of time.

3.3 Aims and Research Questions

This present study aims to further explore the development of repeating patterning competencies in three- to five-year old children. A purpose is to develop a system of strategy categories that integrates the existing research and develops it further. In particular this paper addresses the following questions:

- What competencies do three-, four- and five-year old German kindergartners display in explaining, copying, repairing, extending, and translating repeating patterns of different difficulty levels?
- What strategies do three-, four- and five-year old German kindergartners use in explaining, copying, repairing, extending, and translating repeating patterns of different difficulty levels? Can these strategies be integrated into broader strategy categories?
- Do common developmental steps emerge at this age (between the first to second and the second to third year of German kindergarten)?

3.4 Method

3.4.1 Participants

An explorative, longitudinal study was conducted in a kindergarten located in the suburb of a large German town. The six children (four girls, two boys) were purposively selected and attended the same kindergarten class. Four children spoke German as their first language and two children spoke German as second language. At the first interview the mean age of the children was 3 years 6 months (the youngest was 3 years 1 month, the oldest 3 years 11 months). The kindergarten did not use a specialized curriculum focused on patterning, and the children's educators reported that they had not taught anything patterning related during the first and second year of kindergarten. This was representative practice for German kindergartens.

3.4.2 Design

All children were interviewed individually three times over the course of two years, the first measuring point (MP) being the start of the children's first year of kindergarten (September, 2013). The second interview took place the following year in September 2014, the third in October 2015, the start of the children's third and final year of kindergarten. All interviews were video recorded.

3.4.3 Tasks and Materials

The one-on-one interviews were task- and material-based and were developed by the author for assessing young children's developing understanding in four areas: number, spatial pattern, repeating pattern, and growing pattern. This paper focuses on data about repeating patterns only.

At the first measuring point the children were shown an AB pattern out of green and yellow wooden cubes (see Fig. 3.1) and asked to *explain* the pattern ("Please, tell me about the pattern. What's the same? What's different?"), *copy* ("Build the same pattern as mine. Use the same colors."), *repair* ("A cube is missing. What color is the missing cube?"), and *extend* it ("What comes next?"). At the second measuring point these four activities were supplemented by *translating* the pattern ("Use these counters [different material and colors] to build the same pattern.") and conducted with AB, ABC and AAB patterns (see Figs. 3.1 and 3.2). Cubes in six different colors were available for the children to choose from.

In-depth single case analyses were conducted for each child and strategies were described. During several rounds of interpretation, strategy categories were developed. Answers and actions were coded by strategy and correctness; task solutions were then compared between the children and between the measuring points for each child. On this basis, hypotheses for the development of repeating patterning competencies are generated.

3.5 Results

For an easier understanding the strategy categories are described first, although they resulted as a last step from the comparison of the single-case analyses. In the second part of the results, children's strategies are presented with a focus on how these strategies change over the three measuring points, and how they are distributed across the five categories.

3.5.1 Description of Strategy Categories

During the analyses, children's strategies for different patterning activities and different patterns were summarized and integrated into five strategy categories. For the purpose of this paper the strategy categories are ordered hierarchically from a basic to an advanced understanding of repeating patterns.

3.5.2 Strategy Categories

In the first, most basic strategy category, *no reference to pattern*, children neither referred to the specific characteristics of the elements nor to the regularity of the pattern. They employed strategies that were based on guessing, personal preference, or random selection. For example, while copying or extending a pattern, children used different colors or shapes than those in the pattern. Still, most children recognized that the elements were arranged in a line. In the second strategy category, the *use of pattern elements*, children's strategies showed an understanding of singular aspects of the pattern. For example, they either used the same colors or the same shapes in the pattern, or they recreated the same length. Little regularity could be found in the children's patterns. The idea of regularity initially became visible in the third strategy category, *comparison*. Children compared the pattern's elements, and highlighted sameness within and between patterns on a basic, very concrete level (e.g., "The yellows are the same."; "Three purples and three blacks here. Three yellows and six oranges here"). A commonly observed strategy was extending a pattern step by step by looking at the beginning of the pattern, and comparing and matching the extension with the beginning. This procedure showed an emerging sense for some kind of regularity within a pattern.

The awareness of regularity grew in the fourth strategy category, *focus on sequence*, where children focused on the relations between successive elements of the pattern. A typical strategy for this category was cycling through the elements of the pattern over and over again, even chanting them rhythmically. The children were aware that the elements were ordered in some kind of regular way, without explicitly grasping the rule. They may have even formulated their first approaches as a general description for a singular relationship (e.g., for an ABC pattern: "It's always blue next to green"). Still, the elements of the pattern were seen as strung together. Children were not yet able to break the pattern down into the units of repeat. In the last, most advanced strategy category, *view of unit of repeat*, children finally grasped the structure of the repeating pattern. They knew that there was a smallest part that produced the sequence—they were able to identify this unit of repeat and use it during the activities.

Table 3.1 gives a condensed overview and description for each strategy category.

Table 3.1 Strategy categories with descriptions and examples of strategies

Strategy category	Description	Examples of strategies
No reference to pattern	Using different elements than given in the pattern AND no regularity at all in patterns	Randomly selecting new elements Selecting new elements for other reasons Guessing
Use of pattern's elements	Using singular aspects of the pattern (e.g. color, number, structure)	Using the same elements but in an arbitrary or different order Using the same structure with different elements Creating a 'difference series' Repeating the last element
Comparison	Comparing patterns or comparing singular elements of the pattern (e.g. color) or within the pattern	Finding sameness while comparing the pattern's elements (*explain*) Matching elements in a one-to-one correspondence (*copy*) Comparing and matching elements (*repair*) Looking at and comparing with the pattern's beginning (*extend*)
Focus on sequence	Focusing on the succession of elements; seeing a sequence of "what comes next"	Alternating or cycling through the elements without discerning the pattern's structure Focusing on the relation of successive items "Next to orange is purple, next to purple is green, next to green is orange, next to orange is purple, …" Rhythmical approach
View of unit of repeat	Identifying and using the unit of repeat	"It's always red-blue." Taking all elements of the unit of repeat at once out of the box

3.6 Children's Strategies and Distribution Across the Categories for the Three Measuring Points

This presentation of results focuses on comparing the children's task solutions and strategies. It is easy to mark a solution as right or wrong within our adult horizon of expectation. A strategy, however, is an interpretation that is based on children's explanations, actions and their observations; an unambiguous assignment of a strategy category is not always possible.

At the first measuring point (i.e., aged three years) four children compared the patterns' colors and sorted them together when *explaining* sameness in the AB pattern (e.g., "The yellows are the same. The greens are the same."). One child named the colors in order of appearance and one child did not refer to the pattern at all. No child was able to *copy* the AB pattern when it was hidden from view. Even if the pattern lay visibly in front of the children and the interviewer specifically asked them to use the same colors as in the example, five of six children did not refer to the pattern but either chose colors randomly (by grabbing a handful cubes without really looking) or by color preference (e.g., "I also took red, cause I like red, too"; Child puts green cube back in the box: "I don't like green") and aligned them in an arbitrary order (see Fig. 3.3 left-hand side). Afterwards, most children contrasted the length when their pattern was longer than six elements. One child succeeded to only use yellow and green cubes, using the same amount as in the given pattern, but put all the cubes of one color next to each other (AAABBB) as opposed to alternate the colors (ABABAB) (see Fig. 3.4 left-hand side). When asked to identify the missing cube (*repair*), all children guessed. Two children even named a color that was not part of the pattern. No child was able to *extend* the pattern. Five children randomly grabbed cubes in other colors than given in the pattern and elongated the AB pattern at one or even both ends without any distinguishable regularity (see Fig. 3.3 right-hand side). The same child mentioned above extended the pattern by repeating the same color as the last cube (yellow) over and over (see Fig. 3.4 right-hand side).

In randomly arranging and choosing new elements while copying, repairing and extending, the children did not refer to the repeating pattern. They showed no reference to the given pattern and to the pattern in general in terms of regularity. *No reference to pattern*, therefore, constitutes the first, most basic strategy category and is the most frequent category at the first measuring point (see Table 3.2). It corresponds to Garrick et al.'s (1999) "basic chaining" and partly to Papic et al.'s (2011) "random arrangement".

Still, for random arrangements an additional category is needed. Children's strategies at this age differed according to their reference or non-reference to the pattern. A simple reference to the pattern might be expressed by using the pattern's colors, even if they are arranged randomly or in another order. *Using the pattern's elements* seems to be an important first step and needs our recognition. Another strategy that falls into this category is the repetition of the last element in extending the pattern.

Fig. 3.3 Examples of the strategy category *no reference to pattern.* Left: Copying the AB pattern (green, yellow) with red, blue, purple, yellow, and green cubes. Right: Extending the AB pattern (green, yellow) with red, purple, and green cubes (Color figure online)

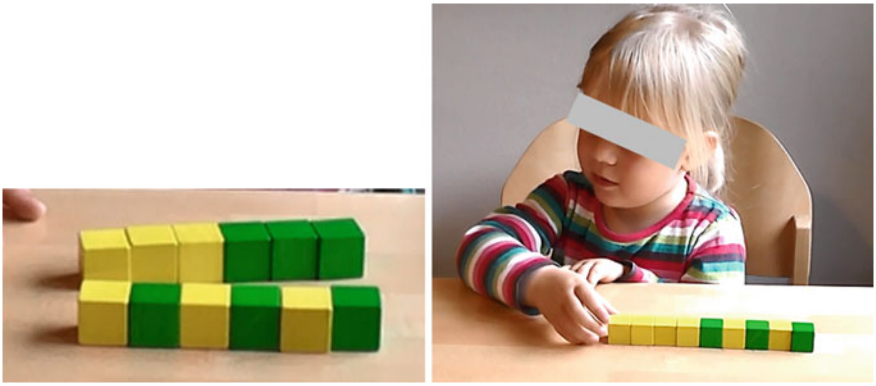


Fig. 3.4 Examples of the strategy category *use of pattern's elements.* Left: Copying the AB pattern (green, yellow) by grouping the yellow and green cubes. Right: Repeating the last element (Color figure online)

The third strategy category that was relevant at the first measuring point was *comparison* for the explaining activity. In comparing the pattern's colors the children recognized sameness and difference and matched identical colors.

At the second measuring point (i.e., aged four years) the most frequently used strategy in *explaining* the AB pattern was still comparing and sorting the colors, e.g., "yellow, yellow, yellow, green, green, green".

Child	"They are wrong, because this green and this green they don't match the yellows."
Interviewer	"Why don't they match?"
Child	"Because yellow has to match yellow and green has to match green."

Table 3.2 Strategy categories at first measuring point (MP) with AB pattern (frequently used strategies are highlighted in grey)

MP 1	No reference to pattern	Use of pattern's elements	Comparison	Focus on sequence	View of unit of repeat
Explain	AB	AB	AB		
Copy (Hidden)	AB	AB			
Repair	AB	AB			
Extend	AB	AB			

In contrast to the first point of measurement, all children were able to *copy* the AB pattern. Strategies differed according to the visibility of the pattern. If the pattern lay exposed in front of the children they mostly solved the task by matching the cubes in one-to-one correspondence. When hidden from view they all referred to the alternation of color in the pattern. One girl may even have referred to the unit of repeat. After putting down in turn, six yellow and green cubes, she stopped and exclaimed: "Still one step higher" She then put down one cube and a yellow and a green cube.

All children were able to correctly identify the missing colors in the AB pattern (*repair*). Two children still seemed to guess, but named colors that were in the pattern. Two other children compared and matched colors e.g., "Yellow…because there are only two yellows". One child referred to the sequence of the given pattern; "Because between green and green again a yellow is missing". Five children were able to correctly *extend* the AB-pattern; most of them alternated the two colors. One child pointed to the pattern's beginning: "I know it because here is green and there yellow". One child that did not extend the pattern created a difference series by using the four colors that were not part of the pattern (four purple cubes, four blues, two oranges and three reds). This task solution was interesting because at first sight the child did not refer to the pattern at all. She was neither focusing on amounts nor regularity; she did not use the pattern's elements. But with the child's explanation it became obvious that she was indeed referring to the pattern and was following a self-imposed rule in using all the colors. The following excerpt of transcript highlights this point.

Child "I take purple. Because there are already yellow and green."
Interviewer "Please, explain again."
Child "All the colors that we didn't have already belong there now, don't you think?"

The difficult task of *translating* the patterns into different materials and colors was achieved by four children for the AB pattern. They all focused on the alternation of color, e.g., "First red, then blue, now red is again, then blue is again, then red is again". It was not apparent, though, if the children really grasped the pattern's structure and translated it, or if they created an AB pattern without reference to the first pattern by alternating the two available colors. Another strategy seen in this task was matching a red counter with every cube so that both patterns had equal amounts of elements.

For the ABC and AAB patterns the children were unwilling to offer *explanations* at the second measuring point. The children that talked about the patterns named the pattern's colors and again compared and sorted these. All children were more or less able to *copy* the ABC and AAB pattern. Again strategies and correct solutions differed according to the visibility of the pattern. If the pattern lay exposed, they matched the cubes in a one-to-one correspondence with five of six children matching correctly. When hidden from view, half of the children remembered the sequence of the ABC pattern and alternated the colors correctly. The other half remembered only the ABC pattern's colors and used them in an arbitrary order. With the hidden AAB pattern, the singular strategy was the alternation of the pattern's elements. All children were able to identify the missing color in the ABC and AAB patterns (*repair*). Most children thereby referred to the sequence of the given pattern, e.g., "Because I did the sequence, first green, then blue, then red, then green, then blue, …". Another strategy for the ABC pattern was looking for the color next to the missing color, then locating that same color within the pattern and comparing it with the missing part.

Five children were able to correctly *extend* the ABC pattern; three were able to extend the AAB pattern. The most common strategy for the ABC pattern was that the children looked at the start or within the pattern and reproduced it from there step-by-step. Some emphasized the successive order of the colors (e.g. "After green is purple, after purple is orange, after orange is green, after green is purple, after purple, …"). For the AAB pattern, alternating two colors (i.e., alternating one red and two blues) was the singular strategy and the part where the same mistake happened for half of the children: they extended the AAB as an AB pattern by alternating not two but only one blue and one red counter. One child was able to *translate* the ABC pattern; another child correctly translated the AAB pattern. The strategies seemed to be the focus on the alternation of color (e.g., "Two purple, one orange, two purple, one orange, two purple, one orange"). The other children either did not refer to the pattern or refused this task completely.

The strategy category, *no reference to pattern*, was observed at the second measuring point only as a singular strategy in translating. A new strategy that falls into the category *use of pattern's elements* was creating a difference series (explanation see above). The most frequently observed categories were *comparison* and *focus on sequence* (see Table 3.3). The three strategies that made up the category *comparison* were: finding sameness while comparing the pattern's elements (explain), matching elements in a one-to-one correspondence (copy), and comparing elements within or at the beginning of the pattern (extend). The category

Table 3.3 Strategy categories at second measuring point with AB, ABC and AAB-patterns (frequently used strategies are highlighted in grey)

MP 2	No reference to pattern	Use of pattern's elements	Comparison	Focus on sequence	View of unit of repeat
Explain		AB ABC AAB	AB ABC		
Copy (Hidden)		ABC		AB ABC AAB	AB
Repair		AB	AB ABC	AB ABC AAB	
Extend		AAB	AB ABC	AB ABC AAB	
Translate	ABC AAB	AB		AB ABC AAB	

focus on sequence corresponds with Papic et al.'s (2011) "alternation" category and includes all strategies that focus on successive items. In the most advanced strategy category, *view of unit of repeat*, children are able to identify and use the unit of repeat. The children that were surveyed in this study did not show this competence at the age of four years.

From the second to the third measuring point there was very little progression in strategy development as well as in the frequency of correct solutions (see Table 3.4). The children are now five years old and are in their last year of kindergarten; the year before they start primary school in Germany. The activity with the highest increase in strategy development was *explaining* the patterns. Children's explanations now focused on comparing, sorting and in parts enumerating the colors for all patterns (category *comparison*; e.g., "Yellow is three times and green is three times"; "Three red counters and six blue counters"). Some children explained the specific order of the colors or chanted the colors rhythmically (category *focus on sequence*; e.g. "Yellow, green, yellow, green, yellow, green"). For the activity *copying* a hidden pattern the only increase in strategy development was for the difficult ABC pattern. Children either used strategies from

Table 3.4 Strategy categories at third measuring point with AB, ABC and AAB patterns (frequently used strategies are highlighted in grey)

MP 3	No reference to pattern	Use of pattern's elements	Comparison	Focus on sequence	View of unit of repeat
Explain		ABC	AB ABC AAB	AB ABC AAB	
Copy (Hidden)			ABC	AB ABC AAB	AB
Repair			AB ABC	AB ABC AAB	
Extend		ABC AAB	AB ABC	AB ABC AAB	
Translate		AB ABC AAB	AB AAB	AB ABC AAB	

the categories *comparison* (e.g., child takes out two orange, two green and two purple cubes and then arranges them in order) or *focus on sequence* (e.g., "I remember orange, purple, green, orange, purple, green") while rebuilding an ABC pattern from memory.

One child used a strategy for extending the AB pattern that indicated an emerging view on the unit of repeat: She simultaneously took out one green and one yellow cube (this pattern's unit of repeat) out of the box and aligned them. The child repeatedly made a unit of repeat three times and then continued by taking singular cubes and successively aligning them. For *repairing* and *extending* the patterns, the strategies as well as the frequency of correct solutions, stayed mostly the same for all three patterns. Remarkably, the most frequently assigned strategy category for the AB and AAB patterns was a *focus on sequence* whereas the category for the more difficult ABC pattern was a lower level strategy category (*comparison*).

The most development took place when children *translated* the patterns. At the second measuring point most children *did not refer to the pattern* while translating. This strategy category was completely replaced by the *use of pattern elements* for the ABC and AAB patterns. In these cases children created other repeating patterns as a translation. For the AAB patterns the strategy category *comparison* was also frequently observed. Children counted the colors and created a pattern with the same numbers of the different colors, (e.g., “Three purples and three blacks here. Three yellows and six oranges here”). Alternatively they matched the colors one-by-one, (e.g., “Yellow to purple, orange to black”). The frequency of correct solutions regarding translating stayed nearly the same for the AB and ABC patterns (almost all children were able to translate the AB pattern; almost no child was able to translate the ABC pattern correctly) but this increased for the AAB pattern.

At the second measuring point only one out of six children was able to translate the AAB pattern, but at the third measuring point half of the children were able to do this. The common error was creating an AB pattern when translating the ABC and AAB pattern.

The strategy category, *no reference to pattern*, was not observed at all at the third measuring point; the category, *use of pattern's elements,* only as a seldomly-used strategy. One exception was the activity, *translate*, where most children used more lower level strategies than during the other activities. The most frequently observed categories were still *comparison* (especially for the ABC patterns), and *focus on sequence* (for the AB and AAB patterns). It became apparent that children at this age handled the AAB pattern similarly to the AB pattern. It seems that the big difference in pattern difficulty was not foremost the length of the unit of repeat but the number of different elements that made up the unit of repeat (two patterns of the study consisted of two different elements: A and B; the third pattern is made up from three different elements: A, B and C). The most advanced strategy category, *view of unit of repeat,* was still not observable in the work with five-year old children. However, an emerging view on the unit of repeat surfaced with individual children. For example when children chanted the pattern rhythmically or they took all the colors that make up the unit of repeat at once out of the box.

3.7 Discussion of Findings

In the present study, I examined the developing patterning competencies of German three to five-year old kindergartners with a special focus on the children's strategies when solving different patterning tasks. The findings are consistent with previous research on repeating pattern knowledge (Papic et al. 2011; Rittle-Johnson et al. 2013), indicating that kindergartners have a range of repeating patterning competencies that extend across activities with varying difficulty and that they develop over time. Further, children's strategies in solving different patterning tasks have been elaborated and strategy categories that combine similar strategies for different activities have been abstracted. The findings are discussed in turn. As the reported research is an explorative study with a very small sample, the conclusions are formulated as hypotheses that need to be tested in the future.

3.7.1 Developing Pattern Competencies of Kindergartners

The results suggest that children in their first year of German kindergarten, i.e., three-year olds may have very limited repeating pattern competencies. No child was able to correctly copy a simple exposed pattern, let alone extend it or identify missing elements. Most children did not refer to the pattern at all. When they took the pattern into account, they compared length, or they focused on sameness and difference related to color. A possible explanation might be that children this young are able to focus on only one aspect of the task. What all children correctly did is reproduce the linear arrangement: aligning the cubes in one row. This focus on the gestalt of the arrangement might be children's first view on repeating patterns; that is, two patterns are the same when they are both linear arrangements and they are different when one is linear and the other not.

After one year in kindergarten the children showed considerably increased abilities although there were no patterning activities explicitly taught by the kindergarten educators. The four-year olds were able to correctly solve more tasks, handle more complex patterns and, furthermore, use more elaborate strategies. The single case analyses show that this holds true for every child though not to the same extend for every child. The four-year olds' focus seemed to be finding sameness in the pattern's elements. In addition, they reliably learned to only use elements of the given pattern and to alternate them, though alternating three elements were still difficult for half of the children. This and the following findings are consistent with the research by Garrick et al. (1999).

Another challenge becomes apparent where patterns contain a double element (e.g., AAB). During the interview almost all children started to alternate only two elements (AB) at some point (mostly during extending or translating). A possible explanation might be the children's focus on comparing and grouping by color. This would consequently mean that patterns are the same for the children when they have the same number of *different* elements that alternate (i.e., two "kinds" of elements: blues and reds). At this early stage of their mathematical development they simply haven't learned that the exact number of each element, different or not, matters (i.e., two blues and one red). The most commonly used strategies, therefore, were *comparison* and *focusing on the sequence*. The *view of unit of repeat* was not developed yet.

In the third and last year of kindergarten the only increase in patterning competencies was observed with the activities of *explain* and *translate* and these were most pronounced with the ABC pattern. The development in describing the pattern might be explained by the natural development of language acquisition in early childhood. Still, the most common strategy was comparing the elements and grouping them instead of speaking about the relationships within the pattern.

As the children had no experience with repeating patterns other than what might have happened at home, during free-play, or what the children observed on television or digital media, another factor that might influence the developing patterning competencies (especially for the more difficult ABC pattern) could be the working memory. “For patterning, it seems likely that greater working-memory capacity facilitates children’s ability both to process and learn about pattern components, which may in turn increase awareness and understanding of relational similarities in repeating patterns” (Miller et al. 2016, p. 100). Although the biggest development in patterning competencies at this age became apparent for the task of translating a pattern, this activity explicitly showed that patterns were a sequence of colors for the children and that they did not grasp the pattern’s structure to its full extent. In contrast to Rittle-Johnson et al. (2015, p. 109) I argue that pattern abstraction (i.e., translating a pattern) can be completed successfully using a perceptual object-matching strategy or alternation strategy. It neither requires attending to the structure of the pattern as opposed to its surface features, nor does it require abstraction of the unit of repeat so that it can be translated to new materials or modalities. This becomes apparent if we explicitly look at the children’s strategies instead of their solutions.

The overall most commonly used strategies with the five-year olds were *comparison* and *focusing on the sequence*. One year before school entry the *view of unit of repeat* still was not developed. This was remarkable as the “mathematics” in patterning only unfolds in analyzing the pattern’s structure, in discerning the unit of repeat. In a large-scale study that is currently being conducted, it will be interesting to observe if German five-year olds in general have acquired the idea of unit of repeat. Nevertheless, the study by Papic et al. (2011) shows that the concept of unit of repeat can be developed before children start formal schooling.

3.7.2 Strategy Categories for Repeating Pattern Work

It was possible to observe and describe children’s patterning strategies for different activities and repeating patterns of different difficulty levels and integrate them into five strategy categories (see Table 3.1). These categories indicate that a strategy is not activity specific. The assignment of strategies to the categories furthermore indicates that diverse strategies are used for the same activities with patterns of different difficulty levels. For example, a child might be able to extend an AB pattern using an alternating strategy but has to rely on a comparison strategy (e.g. referring to the beginning of the pattern) while extending an ABC pattern. Furthermore, children used more simple strategies for more difficult activities like translating and explaining.

3.8 Conclusions and Implications

Over the kindergarten years, abilities as well as strategies while describing, copying, repairing, extending, and translating repeating patterns improved. In particular, it seemed that significant development took place in children’s repeating patterning competencies between the ages of three and four. The two big steps appeared to be first the ability to refer to the pattern and second the ability to discern elements as identical and match them and to alternate two elements reliably. With these findings, the beginning of a developmental journey starts to evolve.

To test the findings of this exploratory study and to evaluate the strategy categories a large-scale study is currently being conducted. The hope is to further measure the range of very young children's patterning competencies and to trace the development of the different strategies more detailed. To explore young children's view on the unit of repeat even further an additional task to identify the unit of repeat has been included in the interviews.

Acknowledgements The author thanks Alicia Neumann for providing her much valued view on the children's strategies.

References

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.

Economopoulos, K. (1998). What comes next? The mathematics of pattern in Kindergarten. *Teaching Children Mathematics, 5*(4), 230–233.

Fox, J. (2005). Child-initiated mathematical patterning in the pre-compulsory years. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Annual Conference of the International Group for the Psychology of Mathematics Education* (*PME*) (Vol. 2, pp. 313–320). Melbourne: PME.

Fox, J. (2006). Connecting algebraic development to mathematical patterning in early childhood. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities*, *cultures and learning spaces. Proceedings of the 29th Conference of the Mathematics Education Research Group of Australasia* (pp. 221–228). Adelaide: MERGA.

Garrick, R., Threlfall, J., & Orton, A. (1999). Pattern in the nursery. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 1–17). London: Cassell.

Ginsburg, H., Inoue, N., & Seo, K. H. (1999). Young children doing mathematics: Observations of everyday activities. In J. Copley (Ed.), *Mathematics in the early years* (pp. 88–99). Reston, VA: NCTM.

Gura, P. (1992). *Exploring learning: Young children and block play*. London: Paul Chapman.

Hoenisch, N., & Niggemeyer, E. (2004). *Mathe-Kings. Junge Kinder fassen Mathematik an*. Weimar, Berlin: Verlag das Netz.

Kidd, J. K., Carlson, A. G., Gadzichowski, K. M., Boyer, C. E., Gallington, D. A., & Pasnak, R. (2013). Effects of patterning instruction on the academic achievement of 1st-grade children. *Journal of Research in Childhood Education, 27*(2), 224–238.

Kidd, J. K., Pasnak, R., Gadzichowski, K. M., Gallington, D. A., McKnight, P., Boyer, C. E., et al. (2014). Instructing first-grade children on patterning improves reading and mathematics. *Early Education & Development, 25*(1), 134–151.

Lüken, M. M. (2012). Young children's structure sense. *Journal für Mathematik-Didaktik, 33*(2), 263–285.

Lüken, M., & Kampmann, R. (2018). The influence of fostering children's pattern and structure abilities on their arithmetic skills in Grade 1. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Lüken, M. M., Peter-Koop, A., & Kollhoff, S. (2014). Influence of early repeating patterning ability on school mathematics learning. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Mathematics education at the edge. Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 4, pp. 137–144). Vancouver, Canada: PME.

Miller, M. R., Rittle-Johnson, B., Loehr, A. M., & Fyfe, E. R. (2016). The influence of relational knowledge and executive function on preschoolers' repeating pattern knowledge. *Journal of Cognition and Development, 17*(1), 85–104.

Ministerium für Kultus, Jugend und Sport Baden-Württemberg (2014). *Orientierungsplan für Bildung und Erziehung in baden-württembergischen Kindergärten und weiteren Kindertageseinrichtungen*. Freiburg: Herder.

Mulligan, J. T., & Mitchelmore, M. C. (2018). Promoting early mathematical structural development through an integrated assessment and pedagogical program. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education.* (This book).

Mulligan, J. T., & Mitchelmore, M. C. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21*(2), 33–49.

National Council of Teachers of Mathematics. (2013). *Principles and Standards for School Mathematics*. http://standards.nctm.org/. Accessed May 29, 2017.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42*(3), 237–268.

Rittle-Johnson, B., Fyfe, E. R., McLean, L. E., & McEldoon, K. L. (2013). Emerging understanding of patterning in 4-year-olds. *Journal of Cognition and Development, 14*(3), 376–396.

Rittle-Johnson, B., Fyfe, E. R., Loehr, A. M., & Miller, M. R. (2015). Beyond numeracy in preschool: Adding patterns to the equation. *Early Childhood Research Quarterly, 31,* 101–112.

Threlfall, J. (1999). Repeating patterns in the early primary years. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 18–30). London: Cassell.

Chapter 4
The Influence of Fostering Children's Patterning Abilities on Their Arithmetic Skills in Grade 1

Miriam M. Lüken and Ralf Kampmann

Abstract Recognizing and using patterns and structures play a pivotal role in learning mathematics. Recent research shows that children's pattern and structure competencies are strongly linked to their general mathematical development. Furthermore, it is possible to foster pattern and structure competencies in young children. The study presented in this paper ties together these two main findings and asks if fostering pattern and structure competencies may even lead to advanced arithmetic achievement. Preliminary findings from an intervention study with 51 first grade students show an increase in the children's arithmetic skills after five months of explicitly teaching pattern and structure during regular mathematics lessons.

Keywords Early childhood · Mathematics education · Patterns
Intervention study

4.1 Patterns and Their Importance for Early Mathematics Learning

A mathematical pattern can be defined as any predictable regularity (Mulligan and Mitchelmore 2009). In the work with primary school children, three main types of mathematical patterns are used: spatial structure patterns, repeating patterns, and growing patterns. *Spatial patterns* are often used as (standard) number presentations to visualize numerical structures in a specific geometrical way. Examples of spatial structure patterns are spatial dot patterns and grids like the twenty frame. Particular characteristics of numbers can be illustrated in these ways and are used to develop

M. M. Lüken (✉)
Bielefeld University, Universitätsstraße 25, 33615 Bielefeld, Germany
e-mail: miriam.lueken@uni-bielefeld.de

R. Kampmann
Fröbelschule, Fröbelstraße 7, 33604 Bielefeld, Germany
e-mail: ralf.kampmann@uni-bielefeld.de

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_4

mental representations of numbers. The perception of spatial patterns and the ability to structure are the basis for subitizing too (Papic et al. 2011). *Repeating patterns* consist of a sequence of elements (the unit of repeat) that is repeated indefinitely (e.g., ABCABC…). In *growing patterns* a sequence of elements changes systematically (e.g., 1, 3, 5, 7…).

The value of patterning in the early years has been endorsed by many researchers. Threlfall (1999) suggests that patterning activities with repeating patterns develops general mathematical concepts in children such as ordering, comparing, sequencing, classification, abstracting and generalizing rules, and making predictions. Repeating pattern work is seen as a conceptual stepping stone which appears mostly in the area of algebra (or pre-algebra) and is highlighted as important for measurement as well as critical to the development of counting and multiplicative thinking (Mulligan and Mitchelmore 2009, 2013). The conceptual domain of *patterns and structures* is taken into account in national syllabi, often associated with algebra and hence connected with corresponding expectations of competence (e.g., National Council of Teachers of Mathematics 2013). National curricula often consider repeating and growing patterns as a precursor for functional thinking and algebra (NCTM 2013).

In Germany, the conceptual domain of *patterns and structures* was included in the national educational standards in 2004, requiring the German federal states to incorporate these into their state-specific syllabi (Sekretariat der Ständigen Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland 2005). Three German states have designated pattern and structures as an independent area. The other states refer to the cross-cutting aspects of patterns and structures.

4.2 Empirical Research on Pattern and Structure Competencies

Although the expectations of patterning abilities have mainly been derived from either observation, experience, or from theoretical considerations, recent studies provide evidence of the overall importance of pattern and structure competencies for early mathematical learning. Initially, we report two studies that measure the relation between children's pattern competencies and their mathematical understanding. Afterwards, we summarize selected studies on improving children's awareness of pattern and structure. In the preceding chapter by Lüken (2018) a study of three- to five-year olds' repeating patterning competencies is described.

Mulligan and Mitchelmore (2009) developed and refined a Pattern and Structure Assessment (PASA) (Mulligan et al. 2015) with tasks that required students to identify, visualize, represent, and replicate elements of pattern and structure with regard to number, measurement and space. In an initial study using PASA Australian first graders (n = 103), ranging from 5 years 5 months to 6 years

7 months of age were interviewed individually using the PASA. Responses were classified into four broad stages of development and students assigned to a stage of structural development accordingly. Analyses show a near perfect correlation between students' conceptual understanding of mathematics (measured by the same instrument) and their stage of structural development.

In another study Lüken (2012) conducted a longitudinal project with 74 children, ranging from 5 years 8 months to 7 years 2 months of age at the study's beginning. In the first and third assessment (i.e., the beginning of the first and the end of the second grade) the children's mathematical competencies were measured with standardized tests. During the second part of the assessment, task- and material-based interviews were administered to assess the children's ability to conceive, reproduce, copy from memory, use, extend, and create repeating and spatial structure patterns. Analyses show a significant correlation on a medium level between the pattern and structure abilities of the school starters and their early mathematical competencies, and a slightly lower correlation with their mathematical achievement at the end of grade 2. In summary, the two studies confirmed that young children who are competent in structuring as well as identifying, copying, creating and extending patterns are those who also demonstrated elaborate number concept development and who achieved best in a standardized mathematics classroom test at the end of grade 2.

Mulligan et al. (2006) conducted several studies on supporting children's understanding of pattern and structure in different settings, and measured the effect it had on children's patterning competencies as well as on their general mathematics understanding. Mulligan et al. (2006) reported a school-based intervention with 683 students from kindergarten to grade 6. A research team worked for a year with the associated 27 primary teachers to scaffold learning with small groups of students within regular classroom time. A numeracy initiative with an explicit pattern and structure approach (the Pattern and Structure Mathematics Awareness Program, PASMAP) was implemented across the school (Mulligan 2011). The number system, counting patterns, multiplication and division, partitioning, and fractions were the main focus. Improvements were found in the children's patterning skills and structure in their mathematical thinking and representations measured by PASA. Substantial improvements were also found in school-based and system-wide measures of numeracy achievement on the NSW Basic Skills Testing and the Schedule for Early Number Assessment (NSW Department of Education and Training 2002), although improvements were less pronounced in the upper primary years. As the instruments for the numeracy achievement were not standardized and there was no comparison group included in the study, the effect of the gains in mathematics learning could not be measured.

In another study an intervention program was implemented with ten kindergarten students, where a specially trained and experienced classroom teacher engaged the students in PASMAP tasks over 15 weekly teaching episodes (Mulligan et al. 2013, p. 51). Pre- and post-intervention testing was conducted with the PASA and two sub-tests of the Woodcock-Johnson mathematics test. Improvements were found on PASA scores but no significant gains were found on the Woodcock-Johnson test

(Woodcock et al. 2001). The sample size may have been too small to measure gains using the Woodcock-Johnson or the scale not sensitive enough to measure any effects. An alternative explanation is that the limited 15-week period did not allow sufficient time to show overall numeracy growth on a standardized test. However, the qualitative analyses of the levels of structure exhibited for each child showed improvements in the ability to recognize, represent, and continue simple repetitions, and develop basic spatial attributes of two-dimensional shapes.

Another study that informed the project reported in this paper is the intervention study by Papic et al. (2011) conducted with 53 pre-schoolers from 3 years 9 month to 5 years of age. The children's patterning knowledge was assessed with an *Early Mathematical Patterning Assessment* (EMPA) interview at the beginning of the pre-school year, six months later, and again at the end of their first year of formal schooling, complemented by a numeracy assessment (Schedule for Early Number Assessment [SENA], NSW Department of Education and Training 2002) at the third measuring point. Each child was placed on an instructional framework based on his/her responses in the first interview. The intervention was conducted by the pre-school teachers in the form of an enrichment program where individualized, problem-based patterning tasks were added to the existing pre-school program. Children worked in small groups or individually on spatial structure and repeating pattern tasks. One 30-min session was scheduled each fortnight for each child for a period of 18 weeks. The intervention did lead to gains in these children's understanding of simple repeating and spatial patterns well beyond those made by the comparison group. The improvement was maintained twelve months later. The intervention group also outperformed the comparison group on the SENA numeracy assessment. Numeracy development was not measured at the beginning of the intervention so it was not possible to ascertain any causal influence of the patterning program on the intervention children's advanced mathematical development.

In the *Reconceptualizing Early Mathematics Learning Project*, Mulligan et al. (2013) evaluated longitudinally the impact of the PASMAP with 316 kindergartners (first year of formal schooling in Australia) in an intervention study. As described in Chap. 2 (Mulligan and Mitchelmore 2018) two different mathematics programs were implemented: the PASMAP and the regular mathematics program. Students participating in the PASMAP program showed higher levels of Awareness of Mathematical Pattern and Structure (AMPS) than for the regular group. However, there were no significant differences found between groups on the standardized test of mathematics achievement, I Can Do Math (ICDM) (Doig and de Lemos 2000). It is questionable whether the multiple-choice ICDM paper and pencil format and limited scope and depth of the ICDM content were sensitive enough to measure any differences.

In another study focused on patterning a US-based intervention study adopting a psychological perspective, (Kidd et al. 2013, 2014) assigned 120 struggling first-grade students randomly to learn about patterns, numeracy, reading, or social studies. Teaching sessions were scheduled for 15 min three days per week over a period of six months. Patterning instruction included symmetrical patterns, growing patterns, and patterns involving a rotation. Children who received pattern instruction performed as well or better on several standardized mathematics assessments relative to children who received numeracy instruction, and systematically better than children who received reading or social studies instruction.

Summing up, two main results provide the basis of our research. Children's pattern and structure competencies are strongly linked to their general mathematical development, and it is possible to foster pattern and structure competencies. Furthermore, there are several indications from other studies that the development of patterning competencies may lead to advanced mathematics achievements, although it seems not always possible to measure reliably. In our research we aimed to follow the route of supporting children's general mathematics understanding through fostering their pattern and structure competencies.

In particular, we aimed to answer the following question:

- Does a focus on pattern and structure during regular mathematics lessons effect (positively) the arithmetic competencies of children in grade 1?

As a first approach to answer this question a small intervention study with 51 German first grade students was conducted by the second author. This pilot study was used to test the instruments, to develop, implement and evaluate the intervention lessons as well as to obtain some initial indications regarding possible intervention effects. Before the intervention the children were administered both a test on numerical-arithmetical skills, TEDI-MATH, (Kaufmann et al. 2009) and intelligence, the SON-R 2½-7 (Tellegen et al. 2007). Both are standardized, normalized individual tests. The TEDI-MATH consists of ten items. Four items assess numerical skills such as writing and reading numbers or comparing written and spoken numbers. Another six items test arithmetical skills, e.g., addition, subtraction, word problems and multiplication.

The intervention took place from January to May 2014. As a post-test the TEDI-MATH was used again in June 2014 (see Table 4.1 for an overview of the study's schedule). The children were aged from 6 years 1 month to 7 years 6 month of age (mean age: 6 years 8 month) at the study's beginning. One class of 25 children was used as an intervention group and another class of 26 children as comparison group. Each of the groups was taught by the same mathematics teacher using the same textbook Das Zahlenbuch 1 (Wittmann and Müller 2012). The

Table 4.1 Overview of the study's schedule

Measuring points		Instruments	Participants
November–December 2013	Pre-test	TEDI-MATH	class 1a (n = 25) and class 1b (n = 26)
October 2013–June 2014		SON-R 2½-7	class 1a (n = 12) and class 1b (n = 20)
January–May 2014	Intervention	13 pattern and structure lessons	class 1a (n = 25)
June 2014	Post-test	TEDI-MATH	class 1a (n = 25) and class 1b (n = 26)

thirteen 45-min intervention lessons were implemented within the regular mathematics lessons (mostly one intervention per week), so that the class had no extra mathematics lessons. The intervention group were taught by the second author ('teacher as researcher') and the regular teacher was present as an observer.

4.3 The Intervention Lessons

During the intervention spatial structure patterns and repeating patterns were regularly included as well as growing patterns being discussed at the end of the intervention. Tasks were used in which children could recognize, use, describe, and explain patterns and structures and also create their own. Discoveries could be made from unstructured quantities to represent the decadic system. Different representations and a variety of activities were deployed.

In the first lesson, beads in two colors were chained (see Fig. 4.1). Without any explicit request the children started creating repeating patterns and talked about regularity and order. With the help of the modeling with the children's bead-patterns the terms 'pattern', 'structure' and 'unit of repeat' were developed.

During the lessons the concept of repeating patterns was deepened. Repeating patterns were created by the children with different materials (different colored blocks, color-pencils, shapes, letters, numbers), on the basis of the unit of repeat. The children learned that pattern is not limited to color but can exist through a variety of material and that the unit of repeat is the crucial aspect to know about a repeating pattern. A strategy the children developed to check a block-pattern for correctness was to detach the first unit of repeat, and then move it alongside the pattern in order to compare unit by unit. It was discussed that repeating patterns could be extended to both sides or in both directions.

Another focus was to repair patterns. The children worked in pairs (see Fig. 4.2). One partner created a pattern and built in an error, or took away some elements while the other partner was not looking. The partner then had to repair the pattern by finding the missing elements or errors. Difficulty levels were differentiated by the length of the unit of repeat and the number of different elements (e.g., two, three

Fig. 4.1 Creating two-colored repeating patterns in lesson 1

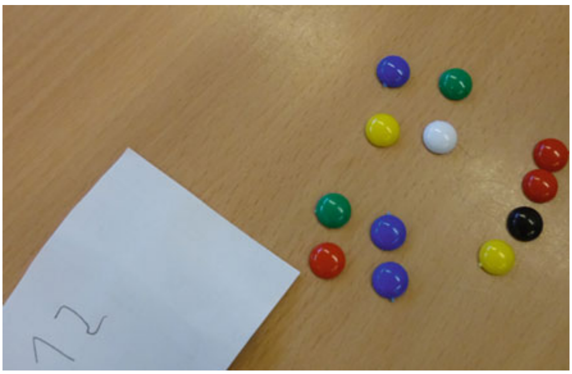
Fig. 4.2 Repairing a repeating pattern

or four colors). The activity of translating a repeating pattern within and between different modes of representation was emphasized too. For example, the children drew their bead-patterns from the first lesson onto paper. They then translated the drawn bead-patterns into different colored block-patterns. It was discussed that patterns that look different at first (because of different material or color) can have the same structure. It was also discovered that different unit of repeats can create the same pattern (e.g., ABA versus AAB).

During the work with spatial patterns (a total of five lessons) the children structured counters in order to quickly and easily grasp the number of counters (see Fig. 4.3). While working in pairs, one partner structured a given amount of counters and showed the pattern very quickly to his partner. The partner’s task was to grasp the pattern’s amount by subitizing. Afterwards the partners discussed if the pattern was correct and easy to subitize and to compare different views of the same pattern. Standard number representations like the twenty-field were looked at through a focus on patterns, and the inherent structures were made explicit. The children discovered patterns in the decadic system and learned to use them.

The content of the last two lessons were growing patterns. Children looked for patterns in geometrically-represented growing patterns and number patterns (see Fig. 4.4).

Fig. 4.3 Creating spatial patterns



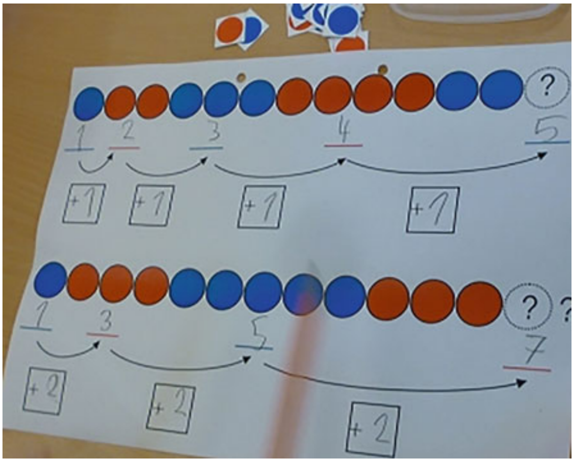


Fig. 4.4 Children's discoveries in growing patterns

Every lesson was framed through a collective meeting and following work in pairs. Lessons were completed by collecting and discussing the children's discoveries and by conducting a collective singing game that was also based on pattern.

4.4 Evaluation Methods and Analyses

For the data analyses descriptive statistics for each group were calculated and analyzed. A comparison of means in form of a t-test for independent samples was conducted with the t-values of TEDI-MATH and SON-R 2½-7. This enabled analysis of potential differences between the intervention and the comparison group with regard to arithmetical skills and intelligence. Calculation of effect size based on differences between means (Cohen's *d*) was used to evaluate the dimension of difference. Further, we investigated changes related to different competence levels within and between the classes and the two measuring points. In order to group children with similar arithmetic competencies together, we classified the TEDI-MATH results (t-values; min = 24, max = 76) into "arithmetic achievement levels". Kaufmann et al. (2009, p. 88) rated t-values from 46 to 55 as age adequate, smaller values as below average, and larger values as above-average achievements. We divided the range of t-values into units of tens (lowest from 20 to 25; see ibid, p. 100) and therefore we discriminated six levels: three levels with below-average, one with age-adequate, and two with above-average achievements (see Fig. 4.6).

4.5 Discussion of Results

The boxplots in Fig. 4.5 summarize the descriptive statistics of the TEDI-MATH results for both measuring points. It can be seen that the dispersion of t-values is wider for the intervention group. Both group's variances reduce slightly at the second measuring point. At the first measuring point the two groups' means are close (intervention group: mean = 44.8; comparison group: mean = 42.4), as are the medians (intervention group: median = 43; comparison group: median = 44). At the second measuring point the intervention group's mean and median increase (mean = 47.8; median = 48) whereas the comparison group's values decrease (mean = 41; median = 41).

Normality was tested via the Kolmogorv-Smirnov test. Both groups were found to have a normal distribution at both measuring points. The Levene-test revealed equality of variances only for the post-test results. For this reason the classic t-test was used for the post-test values. A t-test for independent groups with unequal variance was used for the pre-test data.

The t-test analyses of the SON-R 2½-7 results (pre-test) showed no statistically significant differences (t[30] = 0.106, $p = 0.916$). This means that the comparison group and the intervention group were comparable at the beginning of the intervention regarding the children's intelligence. The t-test analyses of the pre-test TEDI-MATH results also showed no statistically significant differences (t[40, 665] = 0.751, $p = 0.457$). This meant that the comparison group and the intervention group were also comparable at the beginning of the intervention on this measure. However, the groups showed statistically significant differences in the

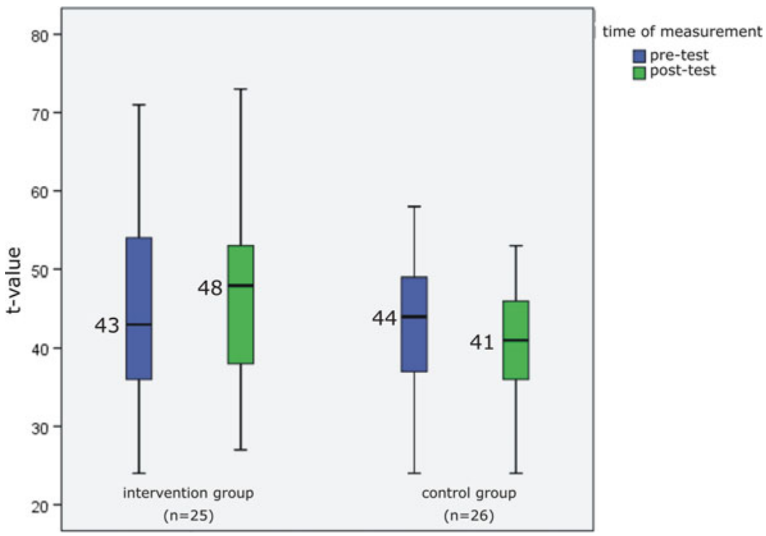


Fig. 4.5 Arithmetic achievements (TEDI-MATH) in pre- and post-test

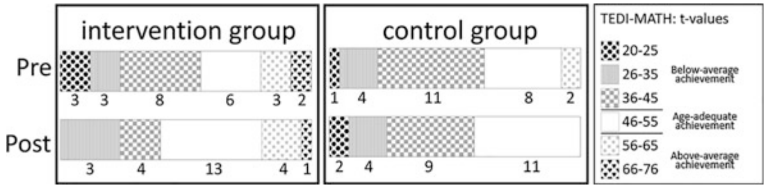


Fig. 4.6 TEDI-MATH results classified into arithmetic achievement levels (number of children shown in absolute frequencies)

post-test following the intervention (t[49] = 2.469, $p = 0.017$). The children from the intervention group achieved significantly better results in the TEDI-MATH. The calculation of effect size revealed ($d = 0.71$) a medium-to-high effect (we refer to Cohen 1988) for the interpretation of $d = 0.5$ as a medium effect and $d = 0.8$ as a high effect (see Rasch et al. 2014, p. 49).

The analysis of arithmetic achievement levels (see Fig. 4.6) indicated that 56% of the children in the intervention group showed below-average achievements in the pre-test. After the intervention only 28% were below average. The number of children with age-adequate achievement in the post-test more than doubled. In the comparison group, the number of children with below-average achievements decreased only slightly from 61 to 57%. The low-achieving children in particular, seemed to benefit from the intervention, which was considered remarkable.

4.6 Conclusions and Implications

In response to our research question this pilot study strongly suggests that a focus on pattern and structure during regular mathematics lessons does significantly effect the arithmetic competencies of first graders. This finding is consistent with previous research on supporting children's patterning competencies (Kidd et al. 2013, 2014; Mulligan et al. 2006; Papic et al. 2011). Analysis of the data from the two groups showed that the students who learned explicitly about pattern and structure achieved significantly better results in a numerical-arithmetical skills test than those students receiving traditional mathematics instruction. Focusing on patterns and structures in mathematics lessons and making patterning an explicit subject of discussion, not only helped children to gain an understanding of repeating and spatial patterns, but was found to have a strong positive impact on children's overall mathematical development.

This held true especially for lower-achieving children. The differentiated analysis of achievement levels showed that half of the children with achievements below average succeeded in gaining age-adequate results by the end of the intervention. The higher-achievers were not affected either way. Our interpretation is that higher-achieving children, of their own accord, discover, seek out, and use pattern and structure in mathematics. This may be the reason why they are high-achieving. Lower-achieving children on the other hand need support in noticing and using pattern and structure in doing mathematics (see Lüken 2012; Schipper 2002, p. 50; Wittmann and Müller 2007, p. 49). Consequently, fostering pattern and structure abilities might be the key to supporting lower-achieving children to develop their overall mathematical abilities.

It is important to note that the significant positive effect described in this study was achieved after only five months of intervention that took place during regular mathematics lessons; no additional class hours were acquired, and the lessons were implemented with only one teacher. Nevertheless, we need to interpret these results carefully. The sample was not randomly assigned to the two groups and the size of the sample was not large enough to permit generalization. Although the t-test was robust when preconditions were violated (e.g. no homogeneity of variances), it would have been advantageous to have a sample with $n > 30$ for each group (Rasch et al. 2014, p. 43). The medium-to-high effect size indicated, however, that even with the given sample the difference in the class's arithmetic achievements was unlikely to be an incidental finding. It is most likely to have been an outcome of the intervention. A second threat to reliability could have been that the intervention was conducted by the researcher, although the regular teacher acted an independent observer during the study.

To confirm the findings of this pilot study and to increase the reliability of the data, a new study with an increased sample size is planned. Lesson examples, based on the pilot work that can be used by the intervention groups' mathematics teachers are currently being developed. Furthermore, a measure of intelligence could be controlled for the whole sample.

Because we are convinced of the importance of pattern and structure awareness for early mathematics learning, we hope that in the long term teachers will prioritize the topic of patterns and structures in their mathematics lessons.

References

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences*. Hillsdale: Erlbaum.

Doig, B., & de Lemos, M. (2000). *I can do maths. Teacher's guide and test booklets levels A and B*. Camberwell, VIC: ACER Press.

Kaufmann, L., Nuerk, H.-C., Graf, M., Krinzinger, H., Delazer, M., & Willmes, K. (2009). *Test zur Erfassung numerisch-rechnerischer Fertigkeiten vom Kindergarten bis zur 3*. Klasse. Bern: Huber.

Kidd, J. K., Carlson, A. G., Gadzichowski, K. M., Boyer, C. E., Gallington, D. A., & Pasnak, R. (2013). Effects of patterning instruction on the academic achievement of 1st-grade children. *Journal of Research in Childhood Education, 27*(2), 224–238.

Kidd, J. K., Pasnak, R., Gadzichowski, K. M., Gallington, D. A., McKnight, P., Boyer, C. E., et al. (2014). Instructing first-grade children on patterning improves reading and mathematics. *Early Education & Development, 25*(1), 134–151.

Lüken, M. M. (2012). Young children's structure sense. *Journal für Mathematik-Didaktik, 33*(2), 263–285.

Lüken, M. M. (2018). Repeating pattern competencies in three- and four-year old Kindergarteners. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Mulligan, J. T. (2011). Towards understanding of the origins of children's difficulties in mathematics learning. *Australian Journal of Learning Difficulties, 16*(1), 19–39.

Mulligan, J. T., & Mitchelmore, M. C. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21*(2), 33–49.

Mulligan, J. T., & Mitchelmore, M. C. (2013). Early awareness of pattern and structure. In L. English & J. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 29–45). New York: Springer.

Mulligan, J. T., & Mitchelmore, M. C. (2018). Promoting early mathematical structural development through an integrated assessment and pedagogical program. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Mulligan, J. T., Mitchelmore, M. C., English, L. D., & Crevensten, N. (2013). Reconceptualizing early mathematics learning. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 47–66). New York: Springer.

Mulligan, J. T., Mitchelmore, M. C., & Stephanou, A. (2015). *Pattern and structure assessment* (*PASA*). *An assessment program for early mathematics* (*F–2*). Camberwell VIC: ACER Press.

Mulligan, J. T., Prescott, A., Papic, M., & Mitchelmore, M. C. (2006). Improving numeracy through a pattern and structure mathematics awareness program (PASMAP). In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces. Proceedings of the 29th Annual Conference of the Mathematics Education Research Group of Australasia (*Vol. 2, pp. 376–383). Canberra: MERGA.

National Council of Teachers of Mathematics (NCTM). (2013). *Mathematics in early childhood learning: A Position of the National Council of Teachers of Mathematics*. Reston, VA: NCTM.

NSW Department of Education and Training. (2002). *Count me in too: A professional development package*. Sydney: NSW Department of Education and Training.

Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42*(3), 237–268.

Rasch, B., Friese, M., Hofmann, W., & Naumann, E. (2014). *Quantitative Methoden 1. Einführung in die Statistik für Psychologen und Sozialwissenschaftler* (4th ed.). Berlin: Springer.

Schipper, W. (2002). Thesen und Empfehlungen zum schulischen und außerschulischen Umgang mit Rechenstörungen. *Journal für Mathematik-Didaktik, 23*(3/4), 243–261.

Sekretariat der Ständigen Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland. (2005). *Bildungsstandards im Fach Mathematik für den Primarbereich*. München: Luchterhand.

Tellegen, P. J., Laros, J. A., & Petermann, F. (2007). *SON-R 2 1/2-7: Non-verbaler Intelligenztest*. Göttingen: Hogrefe.

Threlfall, J. (1999). Repeating patterns in the early primary years. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 18–30). London: Cassell.

Wittmann, E. C., & Müller, G. N. (2007). Muster und Strukturen als fachliches Grundkonzept. In G. Walther, M. van den Heuvel-Panhuizen, D. Granzer, & O. Köller (Eds.), *Bildungsstandards für die Grundschule: Mathematik konkret* (pp. 42–65). Berlin: Cornelsen.

Wittmann, E. C., & Müller, G. N. (2012). *Das Zahlenbuch 1*. Klett: Schülerband. Stuttgart.

Woodcock, R. W., McGrew, K. S., & Mather, N. (2001). *Woodcock-Johnson III tests of achievement*. Itasca: Riverside Publishing.

Part II
Number Sense

Chapter 5
Ecuadorian Kindergartners' Spontaneous Focusing on Numerosity Development: Contribution of Numerical Abilities and Quality of Mathematics Education

Gina Bojorque, Joke Torbeyns, Minna Hannula-Sormunen, Daniël Van Nijlen and Lieven Verschaffel

Abstract Young children's spontaneous focusing on numerosity (SFON) predicts their later mathematical competencies. In this study we investigated the development of SFON in Ecuadorian kindergartners as well as the contribution of early numerical abilities and the quality of mathematics education to this development. The participants were 100 kindergartners drawn from 10 classrooms. Children received two SFON tasks, one at the beginning and one at the end of the school year, and an early numerical abilities achievement test at the beginning of the school year. The quality of mathematics education was assessed twice via the COEMET instrument. Results demonstrated limited SFON development during the kindergarten year, with large individual differences in and highly consistent SFON performances. Additionally, children's SFON development during the kindergarten year was predicted by their SFON tendency and early numerical abilities at the start

G. Bojorque (✉)
Faculty of Philosophy, University of Cuenca, Av. 12 de Abril s/n, Cuenca, Ecuador
e-mail: gina.bojorque@ucuenca.edu.ec

G. Bojorque · J. Torbeyns · L. Verschaffel
Center for Instructional Psychology and Technology, KU Leuven,
Dekenstraat 2, Post box 3773, 3000 Louvain, Belgium
e-mail: joke.torbeyns@kuleuven.be

L. Verschaffel
e-mail: lieven.verschaffel@kuleuven.be

M. Hannula-Sormunen
Turku Institute for Advanced Studies and Department of Teacher Education,
University of Turku, 20014 Turku, Finland
e-mail: minna.hannula-sormunen@utu.fi

D. Van Nijlen
Center for Educational Effectiveness and Evaluation, KU Leuven, Dekenstraat 2,
Post box 3773, 3000 Louvain, Belgium
e-mail: daniel.vannijlen@kuleuven.be

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_5

of the year. The quality of mathematics education did not contribute to children's SFON development. The scientific and practical implications of these results are discussed.

Keywords SFON · Early numerical abilities · Quality of math education Kindergarten

5.1 Introduction

Worldwide, scholars agree that early numerical competence is an important predictor of later mathematics achievement and of children's future professional career and life success. However, while most authors focus on children's early numerical *abilities* (i.e., their early numerical knowledge and skills; e.g., the ability to count, the ability to compare numerical magnitudes, the ability to decompose numbers, or the ability to approximate or estimate numerosities) (Andrews and Sayers 2015; De Smedt et al. 2009; Duncan et al. 2007; Geary 2011; Jordan et al. 2009), an increasing number of others is focusing on children's early numerical *dispositions* (e.g., the inclination to make sense of numerical situations, or the inclination to spontaneously focus on the numerical characteristics of daily-life situations) (Bojorque et al. 2016; Hannula and Lehtinen 2005; Mulligan et al. in press). This increasing interest into the dispositional side of numerical competence is in line with Perkins et al.'s (2000) definition of general thinking competence as consisting of abilities (e.g., the ability to consider hidden options, to search for evidence, to relate new information to old one) and dispositions (e.g., the inclination to be curious, to be open-minded, to pay attention to evidence). So, throughout this chapter, the following three different terms are used along the same lines: (a) *early numerical abilities*, to refer to children's early numerical knowledge and skills only; (b) *spontaneous focusing on numerosity* (*SFON*), to refer to children's early disposition to attend to numerosities only; and (c) *early numerical competencies*, to refer to both children's early numerical abilities and their SFON.

Given the growing number of studies providing empirical evidence on the importance of children's early numerical competencies for their further mathematical development (Aunio and Niemivirta 2010; De Smedt et al. 2009; Duncan et al. 2007; Geary 2011; Hannula-Sormunen et al. 2015; Jordan et al. 2009), it is surprising that empirical information on the development of early numerical competencies in Ecuadorian preschoolers and kindergartners is extremely scarce. However, studies with older Ecuadorian elementary and secondary school students indicate that they poorly perform in both national (Ministerio de Educación 2012) and international (UNESCO 2015) assessments in the domain of mathematics. Against this background, we aimed at investigating the development of Ecuadorian kindergartners' SFON during the kindergarten year, with special attention for the contribution of early numerical abilities and the quality of early mathematics education to this development. In the following, we first discuss prior research on

SFON and the relation between SFON and early numerical abilities. We next focus on the associations between the quality of mathematics education and children's mathematical development, and more specifically SFON. We end with our major research goals and questions.

5.1.1 Spontaneous Focusing on Numerosity

SFON refers to a process of spontaneously (i.e., in a voluntary way not prompted by others) focusing attention to the exact number of a set of items or incidents in daily life (e.g., noticing that there are two cats on the roof or that there are three cookies on the plate) when exact numerosity is utilized in action (Hannula and Lehtinen 2005; Hannula et al. 2007). According to Hannula and Lehtinen (2005), this attentional process is needed for eliciting exact number recognition and for using the recognized exact number in action because exact number recognition is not a totally automatic process that would occur every time a child is confronted with something to enumerate. SFON tendency is considered an indicator of the amount of a child's unguided or spontaneous practice in using exact enumeration in natural situations that are not explicitly numerical (Hannula and Lehtinen 2005) and differs from more general attention processes, enumeration skills or perceptual skills. Previous SFON studies revealed large inter-individual differences in young children's SFON tendency (Hannula and Lehtinen 2005; Hannula et al. 2005, 2007, 2010). According to Hannula and Lehtinen (2005), these individual differences are not due to children's lack of enumeration skills since SFON tasks involve only numbers within the children's enumeration capacity. These authors found that, although young children already possess some enumeration skills that enable them to count collections of up to three items, some children do not spontaneously focus on the aspect of number when confronted with novel, not explicitly numerical, tasks that involve such small collections. Furthermore, these authors showed that there is within-subject stability in children's SFON tendency across different task contexts and years of time. For instance, Hannula-Sormunen et al. (2015) reported stability in SFON tendency from the age of six to the age of 12 years.

Children's SFON has been measured with different tasks. The most commonly used SFON tasks for children aged five (e.g., Hannula and Lehtinen 2005) are the Parrot Imitation task and the Mailbox Imitation task. Both tasks involve small quantities (i.e., up to three) and are introduced to the child as new pretend-play situations. The materials involved in the Parrot Imitation task are glass berries and a toy parrot. On each of four trials the experimenter introduces a given number of berries into the parrot's beak and then asks the child "to do exactly the same". In the Mailbox Imitation task, the experimenter posts some letters into a toy mailbox and then asks the child to do the same. The aim of these tasks is to obtain a reliable indicator of a child self-initiated focus on exact numerosity. Therefore, when presenting the task, the experimenter should not use any phrase that can suggest that the task is numerical or quantitative in nature (Hannula and Lehtinen 2005; Hannula et al. 2007).

Using different versions of these SFON tasks, Hannula and colleagues demonstrated that preschoolers' individual differences in SFON predicted both their concurrent early numerical abilities (Hannula et al. 2007) and their later school mathematics achievement (Hannula et al. 2010). Accordingly, findings of previous SFON studies reported a unique contribution of children's SFON to the development of their early numerical abilities, including subitizing-based enumeration, object counting, cardinality recognition, number sequence, and arithmetic competencies (Edens and Potter 2013; Hannula and Lehtinen 2005; Hannula et al. 2010, 2007; see Hannula-Sormunen 2015; Rathé et al. 2016). Furthermore, SFON tendency in kindergarten was demonstrated to be a significant, domain-specific predictor of arithmetical skills assessed at the end of grade 2 (Hannula et al. 2010). Kindergartners' SFON tendency also predicted their mathematical performance in grade 5 (Hannula-Sormunen et al. 2015), and it was even positively related to the development of numerical competencies up to the end of primary school (Hannula-Sormunen 2015). Regarding the mechanisms underlying the reported predictive relation between SFON and later mathematical performance, as summarized in Rathé et al. (2016), some authors have argued that children's cognitive factors such as their working memory, inhibition, language, and symbolic fluency play an important role in early mathematical development and thus also may be influencing the relationship between SFON and mathematical performance. Other authors explain this relation based on environmental factors such as children's spontaneous self-initiated practice in exact number recognition in daily situations. Finally, as reported in the next section, young children's SFON tendency can be enhanced through guided focusing activities (e.g., Hannula et al. 2005).

5.1.2 Quality of Early Mathematics Education

Studies that evaluate the quality of mathematics education are becoming increasingly important, as early numerical competencies predict later academic achievement (Kilday and Kinzie 2009). To the best of our knowledge, only one study previously addressed the influence of early mathematics education on the development of SFON. In a quasi-experimental study of Hannula et al. (2005), the personnel of a day care center was guided to create rich learning experiences with a view to intentionally direct three-year old children's attention towards variations in small numbers of objects or incidents in everyday situations and in structured games. An example of activities embedded in everyday situations is guiding the children to pay attention to (a small number of) slides of bread during lunchtime; an example of structured games is a board with removable animals that were changed in numerosity during the morning in the context of a singing game and then again secretly along the day. The authors found that children in the experimental group increased not only their SFON but also their counting skills compared to children in the control group. These findings suggest that it is important to give a central place to this feature of children's early numerical development in mathematics education at school, as SFON enhancement is possible and might help to prevent and overcome learning difficulties in mathematics (Hannula-Sormunen 2015).

More generally, the quality of mathematics education is shown to significantly influence students' school achievements (Hiebert and Grouws 2007), already at the kindergarten level (Fuson 2004). Children who attend high-quality pre-school programs make more substantial gains in their mathematical competencies than their peers who do not attend these programs (Sarama and Clements 2009a; Fuson 2004; Griffin 2004). Clements et al. (2013) found that low-SES children who participated in a high-quality, research-based mathematics intervention program from pre-school to grade 1, developed stronger early numerical competencies than their peers who did not participate in that program. It is important that these findings on the association between the quality of mathematics education and children's mathematical development are extended to other settings, and more specifically to children's acquisition of SFON.

5.1.3 The Ecuadorian Context

In Ecuador, the Ministry of Education is responsible for the organization of primary and secondary education. The Ecuadorian educational system comprises three levels, i.e., (1) Beginning level, involving pre-school, and intended for children up to four years; (2) Basic education, from grade 1 up to grade 10; with grade 1 referring to kindergarten; basic education is intended for students aged five to 14 years; and (3) High school, or the last three years of schooling, for students aged 15–17 years. Education is compulsory for all students in primary and secondary education (i.e., basic education and high school) but not for pre-school children (i.e., beginning level). Around 73% of Ecuadorian children attend public schools (39% attend public urban schools; 34% attend public rural schools), 21% attend private schools; the remaining 6% of the children attend municipal schools or schools financially assisted by both government and private sources (Ministerio de Educación 2013).

Kindergarten education is aimed for children aged five years. After one year of kindergarten, children are promoted to the first year of basic elementary school (for children aged six years). Kindergarten education is regulated by a mandatory national curriculum that prescribes the minimum requirements that students should master by the end of the school year. At this level, children attend school five days per week from 7:30 till 12:30.

5.2 The Present Study

All previous studies on the development of SFON have been conducted in developed countries, mainly in Finland, and thus it is not possible to generalize previous findings on children's SFON development to other, less developed countries such as Ecuador (United Nations 2016) that differs in its cultural and educational characteristics from Finland. Given the influence of SFON to young children's concurrent and later mathematical achievement, and the problematic poor performance of Ecuadorian children in the area of mathematics compared to their international peers (UNESCO 2015), our first goal was to examine Ecuadorian five to six-year olds' SFON development throughout the kindergarten year, focusing on both individual differences and stability in children's SFON development. Our second goal was analyzing the relationship between kindergartners' SFON development and their early numerical abilities. Finally, to complement current findings on the contribution of the quality of mathematics education to young children's SFON development, our third goal was to explore whether the quality of mathematics education Ecuadorian kindergartners receive at school is associated to the development of their SFON tendency.

Consistent with our three goals, we addressed three research questions:

(1) Does Ecuadorian kindergartners' SFON develop between the start and the end of the kindergarten year?
(2) Do Ecuadorian kindergartners' early numerical abilities at the start of the school year contribute to their SFON tendency at the end of the kindergarten year?
(3) Does the quality of early mathematics education in the Ecuadorian kindergarten contribute to Ecuadorian kindergartners' SFON tendency at the end of the kindergarten year?

5.3 Method

5.3.1 Participants

Participants were 100 kindergartners, with an average age of 5 years 3 months (SD = 3.7 months) at the start of the study. About 10 children were randomly selected from a convenient sample of 10 different schools of the three major school types in Ecuador (public urban, public rural, private). These schools were selected in view of their willingness to participate in the project. The inclusion of different school types was considered important in order to guarantee the representativeness of the sample. Table 5.1 shows the composition of the sample.

Table 5.1 Number of children and schools per school type

School type	Number of schools	Children			Mean age (SD)
		Boys	Girls	Total	
Public urban	3	14	15	29	5y 1 m (4.1)
Public rural	3	15	17	32	5y 3 m (3.2)
Private	4	23	16	39	5y 4 m (3.3)
Total	10	52	48	100	5y 3 m (3.7)

5.3.2 Measures and Procedure

Child measures *SFON Imitation tasks* (Hannula and Lehtinen 2005). Children's SFON tendency was measured using the Spanish version of two SFON Imitation tasks, namely, the Parrot Imitation task (Test 1) at the start of the school year and the Mailbox Imitation task (Test 2) at the end of the school year. We used these two SFON Imitation tasks given that they both were designed to capture young children' spontaneous attention for exact numerosity in non-mathematically focused situations and that they both are characterized by exactly the same task requirements and procedures, except from the concrete materials used and the overall context. These two SFON tasks were used in prior SFON studies with children aged four to six years (e.g., Hannula and Lehtinen 2005).

(1) The Parrot Imitation task consists of a toy parrot capable of swallowing different-colored small glass berries. The examiner starts the task by placing a case of eight red glass berries on the left, and a case of eight blue glass berries on the right, in front of the parrot, and by introducing the materials saying: "This is Elsi bird, she likes berries. Here are red berries and here are blue berries (pointing to the cases). Now, look carefully, what I do, and then you do exactly like I did". In the first trial the examiner puts two red berries and one blue berry into the parrot's beak, one at a time, and they drop into the parrot's stomach, making a bumping sound. Then the child is told: "Now you do exactly like I did". The number of berries in the second item is three green and two yellow; in the third item, two white and three brown; and in the fourth item, one transparent and two light-blue.
(2) The Mailbox Imitation task, consists of a mailbox to post different-colored envelopes. For the first trial, a pile of eight red envelopes is placed on the left, and a pile of eight blue envelopes is placed on the right, in front of the mailbox. The examiner starts with the task by saying: "This is a mailbox, and here are red envelopes and here are blue envelopes (pointing to the piles of envelopes). Now, please look carefully what I do, and then you do exactly like I did". The examiner puts two red envelopes and one blue envelop into the mailbox. Then s/he says to the child: "Now you do exactly like I did". For the second item, the examiner puts three green and two yellow envelopes, for the third item, two white and three brown envelopes, and for the last item, one orange and two light blue envelopes.

Each of the SFON tasks was administered in accordance with the procedure of Hannula and Lehtinen (2005). The examiner made sure that the child's attention was fully on the task while the trial was performed. She avoided the use of any phrases or other contextual hints that could have suggested that the task was somehow quantitative. The tasks included only very small numbers of items (i.e., 1–3). The child received a score of 1 if s/he responded by putting in the correct exact number of berries/envelopes and/or if s/he was observed doing any quantifying acts. By contrast, in each trial, the child received a score of 0 if s/he did not respond by putting in the correct exact number of berries/envelopes and did not present any quantifying act. Each child received a total score out of four. Both tests were administered individually and were checked for the quality of the task administration on the basis of the video recordings. Cohen's Kappa inter-rater reliability (on 10% of the data) of SFON scores was $K = 0.96$, $p < 0.001$ at Test 1; at Test 2, we obtained a perfect match.

Test of Early Number and Arithmetic (TENA) (Bojorque et al. 2015). Children's early numerical abilities at the start of the school year were measured using the TENA. The TENA is the only reliable and valid test available in Ecuador for assessing Ecuadorian kindergarten's early numerical abilities. This test was developed based on the Ecuadorian National Standards for kindergarten number and arithmetic. It consists of 54 items distributed among nine subscales (with six items per subscale), namely (a) quantifiers, (b) one-to-one correspondence, (c) order relations more than/less than, (d) counting, (e) quantity identification and association with numerals, (f) ordering, (g) reading and writing numerals, (h) addition, and (i) subtraction. The test is organized in two parts: an individual part with 29 items and a collective part with 25 items. Items are scored dichotomously: for each item, a score of 1 is assigned for a correct answer and a score of 0 for an incorrect answer (maximum score = 54). Cohen's Kappa (on 10% of the data) for the TENA scores revealed strong inter-rater reliability, $K = 0.92$, $p < 0.001$.

Classroom measures

Classroom Observation of Early Mathematics Environment and Teaching (COEMET; Sarama and Clements 2009b). The quality of mathematics education in children's classrooms was evaluated twice via the COEMET. We used this instrument for two reasons. First, the absence of valid observation instruments to assess the quality of early mathematics education in Ecuador. Second, the COEMET is the only evaluation instrument that focuses on the quality of *early mathematics* education without being linked to any specific curriculum (Kilday and Kinzie 2009). The COEMET was developed on the basis of research about the characteristics and teaching strategies of effective teachers in early childhood mathematics. The COEMET is a half-day administration instrument, specifically designed to assess the quality of mathematics education in early education settings by means of determining teaching strategies, mathematics content, clarity and correctness of mathematics teaching, and quality of student/teacher interactions. It has 28 items addressing the quality of the Classroom Culture (CC) (nine items) and the Specific Mathematical Activities (SMA) (19 items) on a five-point Likert scale (ranging from "strongly disagree" to "strongly agree"). Dimensions of the CC section are (a) environment and interactions and (b) teacher's personal attributes. An example of a CC item is: "The environment showed signs of mathematics: Materials for mathematics, including specific math manipulatives, were available and mathematics was enacted and/or discussed around them". With respect to the SMA, the COEMET distinguishes among seven dimensions, namely (a) mathematical focus, (b) organization, teaching approaches, interactions, (c) expectations, (d) eliciting children's solution methods, (e) supporting children's conceptual understanding, (f) extending children's mathematical thinking, and (g) assessment and instructional adjustment. An example of a SMA item is: "The teacher began by engaging and focusing children's mathematical thinking (i.e., directed children's attention to, or invited them to consider, a mathematical question, problem, or idea)". At each observation moment, two observers spent a half-day in each classroom from the beginning of the activities until lunch time, including the observation of a mathematics lesson. The observers took field notes and completed the COEMET scoring form after the observation on the basis of both their field notes and the videos of the lessons. The inter-rater reliability (on 10% of the data) of COEMET scores was $K = 0.88$, $p < 0.001$.

5.3.3 Data Analyses

The descriptive and inferential statistical analyses were conducted via IBM SPSS Statistics version 20.0. Due to the small number of schools included in this study, we used a non-parametric test, i.e., Spearman rank-order, to correlate the quality of early mathematics education between the two observations. Given that our SFON data do not follow a normal distribution, we calculated Wilcoxon signed-rank test between SFON scores at the start and the end of the school year to examine children's SFON development. Finally, to take into account the nested structure of our data (i.e., children nested within classrooms), we conducted multilevel analyses using the Mixed Models technique (Hayes 2006) as to analyze the contribution of children's early numerical abilities and the quality of early mathematics education to SFON development.

5.4 Results

We first present the descriptive statistics and analyses of the SFON, TENA, and COEMET scores. Then, we report the results concerning our three research questions.

5.4.1 Descriptive Statistics and Initial Analyses

The descriptive analysis of the data displayed in Table 5.2, first revealed that there were clear individual differences in children's SFON tendency both at the beginning and at the end of the school year. They also indicate a rather low SFON tendency of Ecuadorian children at both measurements. Moreover, only 37% of the kindergartners made progress in their SFON tendency throughout the kindergarten

Table 5.2 Means, standard deviations, and range of SFON, TENA, and COEMET scores

Measure	M	SD	Range
SFON (max. score = 4)			
Test 1	1.24	1.37	0–4
Test 2	1.66	1.61	0–4
TENA (max. score = 54)	25.42	9.30	8–49
COEMET			
Classroom culture (max. score = 45)	17.5	4.55	12–24
Specific math activities (max. score = 95)	41.85	5.56	34–51
Total COEMET (max. score = 140)	59.35	9.75	47.63–74.5

year; 44% of the children did not make any progress, whereas 19% of them decreased in SFON scores from Test 1 to Test 2. Furthermore, the correlation between children's SFON scores at the two measurement points was statistically significant (*Spearman's rho* = 0.40, $p = 0.01$), providing evidence for the consistency of the SFON construct. Second, regarding children's early numerical abilities, children's TENA scores were also low, again with large differences between individual children. Third, with respect to the quality of early mathematics education, it can be deduced from Table 5.2 that the quality of the mathematics education offered to the children tended to be low in the observed classrooms (i.e., only half of the maximum score per subscale as well as for the COEMET as a whole), with rather small differences between the participating classes. Typically, teachers' approach involved mainly whole-class and teacher-centered instruction supported by paper-and-pencil work sheets, with scarce individual teacher-child or child-child interactions, thought-provoking discussions or child-initiated activities. As mentioned above, teachers' classroom activities were observed twice throughout the kindergarten year. We found a highly significant positive correlation (*Spearman's rho* = 0.80, $p = 0.01$) between the COEMET scores on the two observation moments, supporting the stability of the COEMET construct.

5.4.2 Analyses Concerning Our Three Research Questions

To analyze whether Ecuadorian kindergartners' SFON develops between the start and the end of the kindergarten year (research question 1), we conducted a Wilcoxon signed-rank test on children's SFON scores at the start (Test 1) and the end (Test 2) of kindergarten. The results of this analysis indicated that SFON scores were significantly higher at the end of the school year (*Mdn* = 1.50) than at the beginning (*Mdn* = 1.00), $z = -2.415$, $p = 0.02$, meaning that there was development in Ecuadorian kindergartners' SFON tendency throughout the kindergarten year.

To examine whether children's early numerical abilities and the quality of early mathematics education contributed to children's SFON development throughout kindergarten (research questions 2 and 3), we conducted multilevel analyses. We evaluated the adequacy of three models for predicting SFON at Test 2, namely, (a) SFON Test 1 (Model 1); (b) SFON Test 1 and TENA (Model 2); (c) SFON Test 1, TENA, and COEMET (Model 3) (all children from the same class received the same COEMET score).

The outcome of the multilevel analyses presented in Table 5.3 indicates that children's SFON scores at Test 1 significantly and positively predicted their SFON scores at Test 2. Children's initial SFON scores accounted for 17% of the variance in their SFON score at the end of the school year. In addition, children's early numerical abilities accounted for a significant 20% of variance in SFON scores at Test 2 (Model 2), indicating that children's early numerical abilities at the beginning of the school year predict their SFON tendency at the end of the school year even when children's SFON score at the start of the school year is statistically controlled for. When adding the quality of mathematics education as the third predictor to the analyses (Model 3), the increase in the amount of explained variance in SFON scores at Test 2 was rather small (i.e., R^2 = 2%). The contribution of this predictor variable was not significant indicating that the quality of mathematics education children received did not predict their SFON tendency at the end of the kindergarten year. However, the increase in explained variance in the model cannot be used as the sole indicator of the importance of a variable. To compare the relative contribution of the different independent variables standardized betas (Everitt and Dunn 2001) were used. These indicated that children's early numerical abilities at the start of the school year are most predictive for their SFON tendency at the end of the school year, compared to children's SFON tendency at the start of the school year and the quality of mathematics education during the school year.

Table 5.3 Multilevel model of predictors of SFON scores at the end of the school year

Model	Predictor	Coeff	SE	Sig.	Stand. Beta	-2LL
1	Intercept	1.667	0.275			349.651
	SFON test 1	0.429	0.098*	***	0.366	
2	Intercept	−0.822	0.434			326.203
	SFON test 1	0.224	0.097**	*	0.191	
	TENA	0.087	0.017*	***	0.503	
3	Intercept	−2.161	1.157			324.783
	SFON test 1	0.223	0.097**	*	0.190	
	TENA	0.081	0.017*	***	0.468	
	COEMET	0.025	0.020		0.142	

Note $R^2 = 0.17$ (Model 1); $R^2 = 0.37$ (Model 2); $R^2 = 0.39$ (Model 3); *$p < 0.05$, **$p < 0.01$, ***$p < 0.001$

In sum, these results indicate that kindergartners with higher early numerical abilities and with higher SFON at the start of the school year develop higher SFON throughout the school year. The quality of the mathematics education received throughout kindergarten did not add to the prediction of children's SFON score at the end of the year. The latter result might be due to the generally low quality of early mathematics education as well as the small variance in observed quality in the participating classes (see above). However, given the small number of classrooms included in our study, we cannot make strong statements about the impact of the quality of the early mathematics education on the development of children's SFON tendency.

5.5 Discussion

5.5.1 Implications for Understanding Young Children's Early Numerical Competencies and Development

A first goal of our study was to examine Ecuadorian kindergartners' SFON development throughout the school year, focusing on both individual differences and stability in children's SFON development. First, our results demonstrate large inter-individual differences in SFON tendency among Ecuadorian kindergarteners, as well as consistency in their SFON tendency throughout the school year. These results are in line with previous findings in Finnish children (Hannula and Lehtinen 2005; Hannula et al. 2007, 2010). The similarities between our and Hannula and colleagues' findings suggest that the same structures and mechanisms underlie children's SFON development across different cultural and educational contexts. Second, we found that children's SFON scores were noticeably low at both the start and the end of the school year. Remember from prior findings that SFON contributes to children's mathematical performance in elementary school (Hannula-Sormunen et al. 2015; Rathé et al. 2016), thus it might be hypothesized that the low SFON tendency of Ecuadorian children may have a negative impact on their mathematics achievement during the elementary school years. As this is the first study on Ecuadorian kindergartners' SFON development, future studies are required to validate and refine our findings. Moreover, as we did not follow children's early numerical and later mathematical development during and after the kindergarten year, future studies need to longitudinally follow up kindergartners' SFON acquisition and its relation with their concurrent and later numerical and mathematical achievement at elementary school.

Our second goal was analyzing the relationship between kindergartners' SFON development and their early numerical abilities. Our results revealed a positive relation between children's early numerical abilities at the start of kindergarten and their SFON tendency at the end of kindergarten. Thus, the higher children's score on the early numerical abilities test at the start of kindergarten, the more children spontaneously focused on numerosity at the end of kindergarten. Importantly, this relationship was significant, even after controlling for SFON at the start of the school year. Moreover, the contribution of children's early numerical abilities to SFON development was stronger than the contribution of their initial SFON tendency. These results provide additional evidence for the relations between early numerical abilities and SFON (e.g., Hannula et al. 2010; Hannula-Sormunen 2015; Hannula-Sormunen et al. 2015; Rathé et al. 2016). As such, and as already stated for the first major research question, the highly similar results in Finnish and Ecuadorian kindergartners seem to indicate that SFON development relies on analogous developmental structures and processes in children coming from countries largely differing in cultural and educational characteristics. This study constitutes a first attempt to examine young children's SFON tendency in a developing country, i.e., Ecuador, however, further investigations are needed, to address the processes underlying Ecuadorian children's rather low SFON scores, and, to replicate and refine this study in other European and South-American samples, differing in general cultural and educational context, to allow more general conclusions.

5.5.2 Implications for Optimizing Early Mathematics Education

Our study did not only add to the theoretical understanding of SFON competencies and development in Ecuadorian children, but also offers new insights into the relation between SFON and the quality of mathematics education in current Ecuadorian kindergarten. Indeed, as outlined in our third research goal, we also aimed at examining the relationship between the quality of early mathematics education received in the kindergarten year and Ecuadorian children's SFON development throughout that school year. Surprisingly, our results revealed that the quality of early mathematics education that the Ecuadorian kindergartners received did not contribute to their SFON tendency at the end of the kindergarten year.

To the best of our knowledge, this is the first study that directly addresses the relation between children's SFON development for a one-year-time period and the quality of early mathematics education. As discussed above, Hannula et al. (2005) tried to stimulate (Finnish) children's SFON development via a focused intervention study and concluded that young children's SFON tendency can be enhanced through purposeful activities that guide their attention to the aspect of number. Our results are not in line with Hannula and colleagues' conclusions, taking into account the lack of contribution of the quality of mathematics education to SFON development throughout the kindergarten year. However, it should be noted that the teachers participating in our study were not trained to focus on enhancing children's SFON development, as was the case in the study of Hannula and colleagues. Moreover, as indicated by the rather low COEMET scores, children's early mathematics education did not only miss a focus on SFON enhancement but was also generally characterized as being of rather low quality, with teachers' approaches characterized as providing mainly whole-group teaching followed by individual work, with limited interactions and discussions between peers or between children and the teachers. These differences between this study and the previous studies of Hannula and colleagues might explain the observed differences. Additionally, the low number of schools included in this study, may also account for the lack of a relation between SFON and quality of mathematics education, thus urging the need for replication and extension in large-scale studies, not only in Ecuador but also other South-American and, more generally, other countries worldwide.

Although our results on the relation between children's SFON development and quality of mathematics education need to be confirmed and refined in future studies, they offer important building blocks for optimizing educational policy and practice in the domain of kindergarten mathematics in Ecuador. A first topic that requires considerable attention concerns the generally rather low quality of kindergarten mathematics education in Ecuador, as reflected in the low COEMET scores obtained by the participating classrooms when compared to previous studies conducted in the US, in which the authors reported COEMET scores of (about) 108 in experimental classrooms and scores of (about) 99 in control classrooms (Clements et al. 2011; Sarama et al. 2012). The low quality of early mathematics education in the participating classrooms might be due to the characteristics of current teacher training in Ecuador, with only marginal attention for both the core structures and processes involved in young children's mathematical development and the defining elements of powerful learning environments to effectively stimulate this development. Although the consistency in our classroom observations and the high inter-rater reliability in the COEMET instrument indicate a valid description of the educational practices in the participating classrooms, these observations need to be complemented with further observation studies. These may include more frequent classroom observations, teacher interviews, and fine-grained qualitative analyses of interactions during schooling and testing to provide a more detailed description and understanding of current educational practices in early mathematics education in Ecuador. The results of the presented study and of these future studies may allow us to make informed decisions in future educational reforms in Ecuador. Furthermore, the observations via the COEMET allow us to pinpoint both strengths and weaknesses in current educational practices; an overview of these strengths and weaknesses will enable focused reforms to address current weaknesses in both kindergarten classes and pre-service and in-service teacher training and, consequently, increase the quality of kindergarten mathematics education in Ecuador.

A second challenge for future studies on the role of early mathematics education on Ecuadorian children's SFON development refers to the contribution of the quality of mathematics education in the three major school types (i.e., public urban, public rural, private). There are some indications that the quality of mathematics education differs among these school types with children attending private schools having better educational opportunities than children attending public rural schools

(PREAL 2006). However, given the small number of schools per school type included in our study, it was not possible to reliably address the effect of school type on the quality of mathematics education in the different classrooms and children's SFON development during the kindergarten year. Therefore, further efforts that include a larger number of schools per school type and a larger sample of children are necessary to describe in more detail children's SFON tendency within schools as well as between school types in relation to the quality of early mathematics education. In these future studies, the complex interplay between the type of school children attend, the quality of the early mathematics education received and children's acquisition of SFON and early numerical abilities, requires careful consideration.

A third topic that needs further consideration relates to nonexistent contribution of the quality of Ecuadorian early mathematics education to Ecuadorian kindergartners' SFON development. A first hypothetical explanation for the absence of the assumed relation between children's SFON development and the quality of early mathematics education refers to the general low quality of mathematics instructional practices in the participating classrooms (see above). A second hypothetical explanation concerns the fact that the teachers in our study did not focus on enhancing their children's SFON development (cf. study of Hannula et al. 2005). Therefore, future intervention studies aiming at enhancing both general numerical abilities and SFON tendency in Ecuadorian kindergartners are needed. The implementation of the TRIAD/Building Blocks early childhood mathematics program (Clements and Sarama 2013) that has proven to be effective in North American countries provides a fruitful avenue for these future intervention studies. Moreover, it seems worthwhile to complement the TRIAD/Building Blocks program with guided activities that focus on directing children's attention to the aspect of number via structured games organized by the kindergarten teachers and also in everyday situations (Hannula et al. 2005). Our results and the results of these future intervention studies will offer important information for educational policy regarding the content of effective mathematics education in Ecuadorian kindergarten and for current educational practice in Ecuador with respect to the effective stimulation of young children's early numerical competencies.

Finally, in this study we used two instruments developed in Finland, namely the two SFON Imitation tasks, and one instrument developed in the US, namely the COEMET to assess the quality of early mathematics education in Ecuador. In this respect, one may question the fairness of analyzing Ecuadorian children's SFON tendency as well as Ecuadorian classroom practices with, respectively, a Finnish and US lens. Regarding the two SFON tasks being used in this study, namely the Parrot Imitation task and the Mailbox Imitation task, we argue that the contexts wherein these tasks are presented to the children, i.e., feeding a parrot and posting letters into a mailbox, respectively, are also closely familiar to Ecuadorian children and, thus, children easily became acquainted to them.

Moreover, meanwhile, these instruments have been successfully used in several different cultural settings (see Rathé et al. 2016). Regarding the use of the COEMET, we argue that this instrument was developed and based on vast international research literature about good early childhood mathematics teaching practices (Sarama and Clements 2009b) and can be used to measure the quality of early mathematics instruction in any classroom given that it "is not connected to any curriculum" (Clements and Sarama 2008, p. 461). We therefore reasoned that it might also be suitable for the Ecuadorian context. Moreover, the first authors' personal experience with early mathematics education in Ecuador allows to conclude that there is a good fit between the COEMET items and what is considered as good teaching practices in early mathematics education in Ecuador. Still, we are well aware of possible subtle influences of the cultural and educational context of the US on the development of the COEMET instrument. Consequently, it is important to conduct a more systematic evaluation of the suitability of the COEMET for the evaluation of the Ecuadorian early mathematics education teaching practice.

References

Andrews, P., & Sayers, J. (2015). Identifying opportunities for grade one children to acquire foundational number sense: Developing a framework for cross cultural classroom analyses. *Early Childhood Education Journal, 43,* 257–267.

Aunio, P., & Niemivirta, M. (2010). Predicting children's mathematical performance in grade one by early numeracy. *Learning and Individual Differences, 20*(5), 427–435.

Bojorque, G., Torbeyns, J., Hannula-Sormunen, M., Van Nijlen, D., & Verschaffel, L. (2016). Development of SFON in Ecuadorian kindergartners. *European Journal of Psychology of Education*, 1–14.

Bojorque, G., Torbeyns, J., Moscoso, J., Van Nijlen, D., & Verschaffel, L. (2015). Early number and arithmetic performance of Ecuadorian 4–5-year olds. *Educational Studies, 41,* 565–586.

Clements, D. H., & Sarama, J. (2008). Experimental evaluation of the effects of a research-based preschool mathematics curriculum. *American Educational Research Journal*, *45*(2), 443–494. https://doi.org/10.3102/0002831207312908.

Clements, D. H., & Sarama, J. (2013). *Building blocks-SRA, pre-kindergarten*. Columbus, OH: SRA/McGraw-Hill.

Clements, D. H., Sarama, J., Spitler, M. E., Lange, A. A., & Wolfe, C. B. (2011). Mathematics learned by young children in an intervention based on learning trajectories: A large-scale cluster randomized trial. *Journal for Research in Mathematics Education, 42,* 127–166.

Clements, D. H., Sarama, J., Wolfe, C. B., & Spitler, M. E. (2013). Longitudinal evaluation of a scale-up model for teaching mathematics with trajectories and technologies: Persistence of effects in the third year. *American Educational Research Journal, 50,* 812–850.

De Smedt, B., Verschaffel, L., & Ghesquiere, P. (2009). The predictive value of numerical magnitude comparison for individual differences in mathematics achievement. *Journal of Experimental Child Psychology, 103,* 469–479.

Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P., … Japel, C. (2007). School readiness and later achievement. *Developmental Psychology*, *43*, 1428–1446.

Edens, K. M., & Potter, E. F. (2013). An exploratory look at the relationships among math skills, motivational factors and activity choice. *Early Childhood Education Journal, 41,* 235–243.

Everitt, B. S., & Dunn, G. (2001). *Applied multivariate data analysis* (2nd ed.). Chichester: Wiley.

Fuson, K. (2004). Pre-K to grade 2 goals and standards: Achieving 21st century mastery for all. In D. H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in* *mathematics: Standards for early childhood mathematics education* (pp. 105–148). Mahwah, NJ: Lawrence Erlbaum Associates.

Geary, D. (2011). Cognitive predictors of achievement growth in mathematics: A 5-year longitudinal study. *Developmental Psychology, 47,* 1539–1552.

Griffin, S. (2004). Number worlds. A research-based mathematics program for young children. In D. H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics*: *Standards for early childhood mathematics education* (pp. 325–342). Mahwah, NJ: Lawrence Erlbaum Associates.

Hannula, M. M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction, 15,* 237–256.

Hannula, M. M., Lepola, J., & Lehtinen, E. (2010). Spontaneous focusing on numerosity as a domain-specific predictor of arithmetical skills. *Journal of Experimental Child Psychology, 107,* 394–406.

Hannula, M. M., Mattinen, A., & Lehtinen, E. (2005). Does social interaction influence 3 year-old children's tendency to focus on numerosity? A quasi-experimental study in day-care. In L. Verschaffel, E. De Corte, G. Kanselaar, & M. Valcke (Eds.), *Powerful learning environments for promoting deep conceptual and strategic learning (Studia Paedagogica)* (Vol. 41, pp. 63–80). Leuven: Leuven University Press.

Hannula, M. M., Räsänen, P., & Lehtinen, E. (2007). Development of counting skills: Role of spontaneous focusing on numerosity and subitizing-based enumeration. *Mathematical Thinking and Learning, 9,* 51–58.

Hannula-Sormunen, M. M. (2015). Spontaneous focusing on numerosity and its relation to counting and arithmetic. In R. Cohen Kadosh & A. Dowker (Eds.), *The Oxford handbook of numerical cognition* (pp. 275–290). Oxford, UK: Oxford University Press.

Hannula-Sormunen, M. M., Lehtinen, E., & Räsänen, P. (2015). Children's preschool subitizing, spontaneous focusing on numerosity and counting skills as predictors of mathematical performance 6–7 years later at school. *Mathematical Thinking and Learning, 17,* 155–177.

Hayes, A. F. (2006). A primer on multilevel modeling. *Human Communication Research*, *32*(4), 385–410. https://doi.org/10.1111/j.1468-2958.2006.00281.x.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on student's learning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Charlotte, NC: Information Age.

Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: Kindergarten number competence and later mathematics outcomes. *Developmental Psychology, 45,* 850–867.

Kilday, C. R., & Kinzie, M. B. (2009). An analysis of instruments that measure the quality of mathematics teaching in early childhood. *Early Childhood Education Journal, 36,* 365–372.

Ministerio de Educación. (2012). *Rendición de cuentas 2011 [Accountability 2011].* Ecuador: Quito.

Ministerio de Educación. (2013). *Archivo maestro de instituciones educativas AMIE* [Master file of educational institutions AMIE. Educational Records Report 2013–2014]. Reporte de registros educativos 2013–2014. Quito, Ecuador.

Mulligan, J., Verschaffel, L., Baccaglini-Frank, A., Coles, A., Gould, P., Obersteiner, A., Sinclair, N., & Roberts, N. (in press). Whole number thinking, learning and development. In M. G. Bartolini Bussi, & X. Sun (Eds.), *Building the foundation*: *Whole numbers in the primary grades*. The ICMI Study 23. New York: Springer.

Perkins, D. N., Tishman, S., Ritchhart, R., Donis, K., & Andrade, A. (2000). Intelligence in the wild: A dispositional view of intellectual traits. *Educational Psychology Review, 12,* 269–293.

PREAL. (2006). *Calidad con equidad*: *El desafío de la educación ecuatoriana. Informe del progreso educativo* [Quality with Equity: The challenge of Ecuadorian education. Educational progress report]. Ecuador. ISBN–9978–45–344–X.

Rathé, S., Torbeyns, J., Hannula-Sormunen, M. M., De Smedt, B., & Verschaffel, L. (2016). Spontaneous focusing on numerosity: A review of recent research. *Mediterranean Journal for Research in Mathematics Education, 15,* 1–25.

Sarama, J., & Clements, D. H. (2009a). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge.

Sarama, J., & Clements, D. H. (2009b). *Manual for Classroom Observation* (*COEMET*)*-Version 3*. Unpublished version.

Sarama, J., Lange, A. A., Clements, D. H., & Wolfe, C., B. (2012). The impacts of an early mathematics curriculum on oral language and literacy. *Early Childhood Research Quarterly, 27*, 489–502.

UNESCO. (2015). *Tercer estudio regional comparativo y explicativo. Logros de aprendizaje, Ecuador* [Third regional comparative and explanatory study. Learning achievements, Ecuador]. Santiago, Chile: Oficina Regional de Educación de la UNESCO para América Latina y el Caribe. http://unesdoc.unesco.org/images/0024/002435/243532S.pdf. Accessed: July 15, 2016.

United Nations. (2016). *World economic situation and prospects 2016*. New York.

Chapter 6
Kindergartners' Spontaneous Focus on Number During Picture Book Reading

Sanne Rathé, Joke Torbeyns, Bert De Smedt, Minna M. Hannula-Sormunen and Lieven Verschaffel

Abstract Children's Spontaneous Focusing On Numerosity (SFON) predicts later mathematics performance. This association is assumed to rely on children's self-initiated practice in number recognition during everyday activities, which would enhance their further mathematical development. Consequently, SFON in experimental tasks should be associated with SFON during everyday activities. The present contribution aims to enhance our understanding of this association by critically discussing the major results of two recently conducted studies on the association between SFON in experimental tasks and SFON during picture book reading. Study 1 revealed no association between children's SFON in an Imitation task and their number-related utterances during numerical picture book reading. Study 2, in which we contrasted two different SFON tasks and their association to picture book reading, revealed a positive association between children's SFON in the Picture task (but not in the Imitation task) and their number-related utterances during picture book reading. Theoretical, methodological, and educational implications are discussed.

S. Rathé (✉) · J. Torbeyns · L. Verschaffel
Center for Instructional Psychology and Technology, KU Leuven,
Dekenstraat 2, Postbox 3773, 3000 Leuven, Belgium
e-mail: sanne.rathe@kuleuven.be

J. Torbeyns
e-mail: joke.torbeyns@kuleuven.be

L. Verschaffel
e-mail: lieven.verschaffel@kuleuven.be

B. De Smedt
Parenting and Special Education Research Unit, KU Leuven,
Leopold Vanderkelenstraat 32, Postbox 3765, 3000 Leuven, Belgium
e-mail: bert.desmedt@kuleuven.be

M. M. Hannula-Sormunen
Turku Institute for Advanced Studies and Department of Teacher Education,
University of Turku, Room 222, 20014 Turku, Finland
e-mail: mimarha@utu.fi

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_6

Keywords Picture books · SFON · Kindergartners · Number

6.1 Theoretical and Empirical Background

Many children develop and use a rich diversity of early mathematical abilities from a young age (Torbeyns et al. 2015). During the past decade, a large number of studies have shown that these early mathematical abilities (e.g., counting or comparing numerical magnitudes) are highly predictive of later mathematics achievement in primary school (Aunio and Niemivirta 2010; Claessens et al. 2009; De Smedt et al. 2009; Nguyen et al. 2016; Vanbinst et al. 2015). Although a large body of research providing empirical evidence for the importance of these early mathematical *abilities* exists, recent research has revealed that children's mathematical *dispositions*, and in particular their tendency to spontaneously attend to and focus on numerosity (i.e., Spontaneous Focusing On Numerosity or SFON), is also important for explaining individual differences in early mathematical development (see Rathé et al. 2016a, for a review).

SFON is a recent construct developed by Hannula-Sormunen and colleagues, which refers to children's spontaneous (i.e., self-initiated, thus not prompted by others) focusing of attention on the aspect of exact number of a set of items or incidents and using this recognized numerosity in one's action (2010). SFON tendency indicates the amount of spontaneous practice in using exact number recognition across different task contexts and time (Hannula and Lehtinen 2005; Hannula et al. 2010). For example, some children spontaneously start to count the number of cars while playing with toy cars, while others do not explicitly focus on number, but instead pay attention to other, non-numerical aspects in the situation (e.g., colors or models of the cars).

So far, SFON has been measured primarily by using action-based Imitation tasks, in which children are required to imitate the experimenters' play behavior with toys (e.g., feeding berries into a toy parrot's beak; Hannula and Lehtinen 2005). More recently, SFON has also been measured with verbal description tasks, in which children are required to describe photos (Hannula et al. 2009) or the content of different cartoon pictures (Batchelor et al. 2015). Children are given a SFON score in a trial when they spontaneously focus on the numerical aspect of the task: For instance, feeding the correct number of berries in the Imitation task, or mentioning at least once an exact numerosity (e.g., I see *three* houses) while describing the pictures in the Picture task. In all SFON tasks, it is important that (1) only novel and not explicit mathematical tasks are used, (2) the experimenter does not provide any mathematical hints and has the full attention of the child on the task, (3) the task only includes a few trials as it aims to capture children's *spontaneous* attention on numerosity, and (4) the task includes numbers of items that are so small that all participating children should be able to recognize them (Hannula 2005).

Previous cross-sectional and longitudinal studies revealed that young children largely differ in their SFON tendency. Moreover, these individual differences in SFON were found to be associated with early mathematical abilities, such as counting and subitizing-based enumeration (Hannula and Lehtinen 2005; Hannula et al. 2007), and they were uniquely predictive of later mathematics achievement in the beginning (Hannula et al. 2010) and at the end of primary school (Hannula-Sormunen et al. 2015). The latter findings suggest that SFON might play a foundational role in children's early mathematical development.

In particular, SFON tendency is assumed to promote children's amount of self-initiated practice in exact number recognition during everyday activities, which in turn would enhance their further mathematical development (Hannula et al. 2010). In other words, the predictive association between children's SFON as measured via experimental tasks and their later mathematics achievement can be explained hypothetically by the amount of children's SFON during everyday activities and play. Although some research on this topic has been carried out, previous studies that investigated the assumed association between SFON in experimental tasks and SFON during everyday activities have provided inconsistent results (Batchelor 2014, Study 3; Edens and Potter 2013; Hannula et al. 2005).

Edens and Potter (2013), for instance, found no association between children's SFON as measured via various action-based Imitation tasks and their spontaneous activity choice (i.e., mathematics-related versus non mathematics-related activities) during free play in kindergarten. Their results suggest that higher SFON children do not per se choose more mathematics-related activities (e.g., block construction, jigzaw puzzles, and computer games) during free play in kindergarten in comparison to their peers with lower SFON scores. Children's SFON, however, is not limited to selecting mathematics-related activities, but can also occur during non-mathematics-related activities, such as picture book reading or making crafts. When testing the association between SFON in experimental tasks and SFON during everyday activities, it is important to consider children's concrete actions and/or numerical utterances when determining their SFON during everyday activities and play (Hannula 2005; Rathé et al. 2016b).

In this respect, the study of Batchelor (2014, Study 3) is highly relevant. In contrast to Edens and Potter (2013), Batchelor focused on children's and their parents' verbal expressions of SFON during everyday activities and play. Interestingly, results revealed a positive association between children's SFON as measured by the verbal Picture task and their verbal expressions of SFON as observed during a play session in which parents played different games (i.e., Hungry Hippos, Lego Duplo, and Picture Printing) with their children. Other researchers, who determined children's SFON on the basis of concrete actions (e.g., Imitation tasks), found evidence for an association between SFON in the posttest experimental tasks and SFON as observed by day-care professionals during everyday activities, but the same association was not found for SFON in the pretest experimental tasks (Hannula et al. 2005; Mattinen 2006).

Taken together, previous research findings on the association between SFON in experimental tasks and SFON during everyday situations have been inconsistent and contradictory. Moreover, recent findings of Batchelor (2014) show that it is important to consider children's verbal expressions of SFON as observed during everyday activities when investigating the underlying mechanism of SFON. Therefore, future research is needed to further explore these inconsistencies by correlating children's SFON as measured by different experimental SFON tasks with their SFON during other meaningful everyday activities, such as picture book reading, of which the potential benefits for children's early mathematical development have been shown (Elia et al. 2010; van den Heuvel-Panhuizen and van den Boogaard 2008).

Against this background, we set up two studies in which we aimed to investigate the association between children's SFON as measured by experimental tasks and their number-related utterances during everyday picture book reading. In Study 1 (Rathé et al. 2016b), we explored the association between children's SFON as measured by an action-based Imitation task and the frequency of their number-related utterances during numerical picture book reading. Based on the unexpected result obtained in this study, we conducted Study 2 (Rathé et al. 2017), in which we aimed to test whether the results of Study 1 might be explained by the way in which SFON was measured. More specifically, we contrasted two different experimental SFON tasks—an action-based Imitation task and a verbal Picture task—in their relation to everyday picture book reading. In the next sections, we summarize the method and the main findings of both studies and end with a discussion of some theoretical, methodological, and educational implications. For more details on the design and the results of the two studies, we refer the interested reader to the original research reports.

6.2 Study 1

The aim of Study 1 (Rathé et al. 2016b) was to address the assumed association between SFON in experimental tasks and SFON during everyday activities by investigating the association between children's SFON as measured by an action-based SFON Imitation task and their number-related utterances during numerical picture book reading. Based on Hannula and colleagues' hypothetical explanation (2010), we hypothesized that children with higher SFON scores in the experimental Imitation task would formulate more number-related utterances during numerical picture book reading.

Forty-eight kindergartners (28 boys, 20 girls, $M = 4$ years 6 months, $SD = 4$ months), drawn from five different schools in Flanders (Belgium), participated in the study. All children individually completed an Elsi bird SFON Imitation task including four trials (Hannula and Lehtinen 2005) and at least 10 days later, a numerical picture book reading activity, in which they individually were read aloud the numerical picture book *Farmer Boris* [Boer Boris] (van Lieshout and Hopman 2013). The picture book describes the story of Farmer Boris and his farm with the accompanying animals, fields, and machines, and is especially

Fig. 6.1 Materials in the Elsi bird Imitation task (Rathé et al. 2016b)

written for the purpose of learning the counting sequence from 1 up to 11. For an example page of the original numerical picture book, see Rathé et al. (2016b).

In the SFON Imitation task (see Fig. 6.1), the experimenter put small numbers (ranging from 1 to 3) of colored berries into a toy parrot's beak and then instructed the child to "do exactly the same". Children were scored as focusing on numerosity in a trial when they gave the correct number of berries and/or if they were observed doing verbal and non-verbal quantifying acts, such as counting or showing numbers with fingers. As there were only four trials, the maximum score on the Imitation task was 4.

During the numerical picture book reading activity, children were invited to spontaneously comment on the pictures in the book. The experimenter did not provide any numerical hints. We registered the frequency and the type of children's number-related utterances expressed before, during, and after reading the text on the front cover, page 1–12, and the back cover of the book. Children's number-related utterances were classified into seven types of utterances that were largely based on the framework of Elia et al. (2010): (N1) counting (e.g., *1*, *2*, *3*), (N2) determining the numerosity of a set of items (e.g., *There are six pigs*), (N3) recognizing a numerical symbol (e.g., *I see number 3*), (N4) comparing quantities (e.g., *There was 1 dog on the previous page, and now there is still 1 dog*), (N5) analyzing part-whole relationships (e.g., *There are 6 pigs, 5 pigs are walking around and 1 pig is playing in the mud*), (N6) using quantity concepts (e.g., *There are a lot of mice*), and (N7) using ordinal numbers (e.g., *The sixth pig plays in the mud*).

The results of Study 1 revealed large individual differences in kindergartners' SFON, and in the frequency and the type of their number-related utterances during numerical picture book reading. In the SFON Imitation task, about half of the children did not spontaneously focus on numerosity during the solution of the task (52%), while the others (48%) spontaneously focused on numerosity in at least one of the trials, and received SFON scores ranging from 1 to 4 (see Fig. 6.2).

During the numerical picture book reading activity, most children formulated at least one number-related utterance (87.5%). For the group of children who formulated at least one number-related utterance, we observed large individual differences in both the frequency and the type of their number-related utterances. The frequency of number-related utterances varied from 1 to 32 utterances (M = 8.96, SD = 8.21) and the type of number-related utterances varied from stating only one

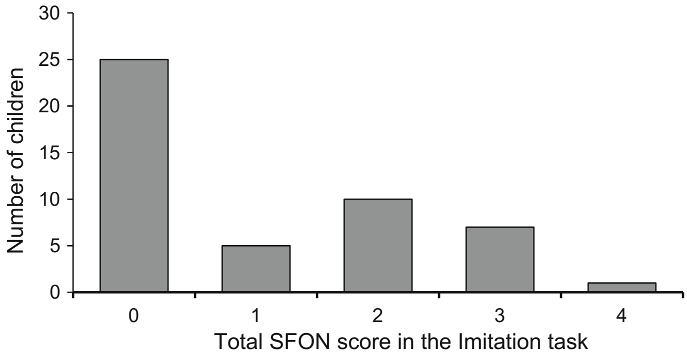


Fig. 6.2 Number of children per total SFON score in the Imitation task (N = 48)

type of utterance (e.g., N6; 9.5%) to formulating different types of number-related utterances (e.g., N2_N6; 26%). Regarding the observed frequencies of the different types of number-related utterances, the kindergartners most frequently mentioned numerosities (N2; 45.6%) and quantity concepts (N6; 32%).

A Spearman correlation analysis did not reveal the expected association between children's SFON in the action-based Imitation task and the frequency of their number-related utterances during numerical picture book reading ($r_s = -0.14$), also not after accounting for word count (i.e., the total number of words the children expressed during the picture book reading activity).

As discussed in Rathé et al. (2016b), there were different hypothetical explanations for this unexpected result. First, there was an important difference in response mode between the experimental SFON Imitation task and the numerical picture book reading activity. In the numerical picture book reading activity, children were required to *verbally* describe their mathematical thoughts and utterances (with attention for action-based numerical acts), while in the Imitation task children needed to use the information about numerosity in their *action*.

Second, the characteristics of the numerical picture book used in Study 1 might explain the absence of empirical support for an association between children's SFON and the frequency of their number-related utterances during numerical picture book reading. More specifically, the numerical picture book activity might not have captured children's *spontaneous*, but rather their *guided* focusing on numerosity (GFON), because the number words and number symbols on each page focused their attention toward numerosity. Finally, the absence of the expected association between children's SFON in the Imitation task and the frequency of their number-related utterances during numerical picture book reading might also be explained by the rather young age of the participating children (i.e., four to five-year olds) and their associated limited verbal skills.

6.3 Study 2

The aim of Study 2 (Rathé et al. 2017) was to investigate whether available inconsistent results on the association between SFON in experimental tasks and SFON in everyday activities (e.g., Batchelor 2014; Edens and Potter 2013; Hannula et al. 2005; Rathé et al. 2016b) could be explained by the way in which SFON is measured. Here we explicitly aimed to address and test the first hypothetical explanation as discussed in Study 1. We also took into account the second and third hypothetical explanation given in Study 1, but did not explicitly validate them.

With this aim in mind, we systematically contrasted children's SFON as measured by two different experimental tasks—an action-based Elsi bird Imitation task and a verbal Picture task—in relation to their number-related utterances during picture book reading, using a modified version of the picture book *Farmer Boris* without any number words and number symbols. Based on the reviewed literature above and the results of Study 1, we hypothesized that (1) children's SFON as measured by the verbal Picture task should be associated with the frequency of number-related utterances during everyday picture book reading, and (2) that children's SFON as measured by the action-based Imitation task should not be associated with the frequency of number-related utterances during everyday picture book reading.

In total, 65 kindergartners (31 boys, 34 girls, $M = 5$ years 5 months, $SD = 7$ months, range = 4 years 4 months to 6 years 4 months), coming from two different schools in Flanders (Belgium), participated in the study. All kindergartners were individually interviewed during two separate sessions, in which they first completed two experimental SFON tasks and a visuo-motor buffer task, and next were read aloud a modified version of the picture book *Farmer Boris*.

In the first session, all children individually completed an action-based Elsi bird SFON Imitation task including four trials (Hannula and Lehtinen 2005) and a verbal SFON Picture task including three trials (Batchelor et al. 2015). In the Imitation task, we used the same materials and procedure as in Study 1. In the Picture task, children had to verbally describe as precisely as possible the content of a set of three pictures that included, among other things, items that can be counted (see Fig. 6.3). To distract the children's attention from the numerical nature of the SFON tasks, they all completed a visuo-motor buffer task between both tasks.

Fig. 6.3 First and second trial used in the SFON Picture task (Batchelor et al. 2015)

In the second session, children were individually read aloud a modified version of the picture book *Farmer Boris* [Boer Boris] (van Lieshout and Hopman 2013). To identify children's *spontaneous* number-related utterances (and not their *guided* number-related utterances), we removed all the number symbols in the picture book pages. As an example, Fig. 6.4 shows page 9 of the modified picture book, which displays nine mice who are slinking secretly in the house of Farmer Boris. The picture book was read aloud according the same reading scenario as in Study 1.

In line with Study 2, we scored children's action-based responses and their verbal and non-verbal quantifying acts during the solution of the SFON Imitation task. The maximum score in the Imitation task was 4. In the SFON Picture task, children were scored as focusing on numerosity in a trial when they explicitly mentioned at least once an exact number (e.g., I see *three* chickens) and/or showed non-verbal quantifying acts (e.g., counting acts). The maximum score in the Picture task was 3. Children's number-related utterances expressed during the picture book reading activity (e.g., There are *six pigs* walking around in the mud) were scored using the same classification scheme as in Study 1.

In line with the results of Study 1, we observed large individual differences in children's SFON scores in the action-based Imitation task and in the verbal Picture task (see Fig. 6.5). Two thirds of the children (66%) spontaneously focused on numerosity in the Imitation task, whereas the other children (34%) did not show any evidence of SFON behavior during the solution of this task. During the administration of the Picture task, about half of the children (57%) spontaneously mentioned at least once an exact numerosity and/or showed quantifying acts while describing the pictures, receiving SFON scores ranging from 1 to 3; the other

Fig. 6.4 Page 9 from the modified picture book *Farmer Boris*. Accompanying text: "Farmer Boris has a farm. It also involves mice. Oh no, they do not belong there. They slink secretly in the house."

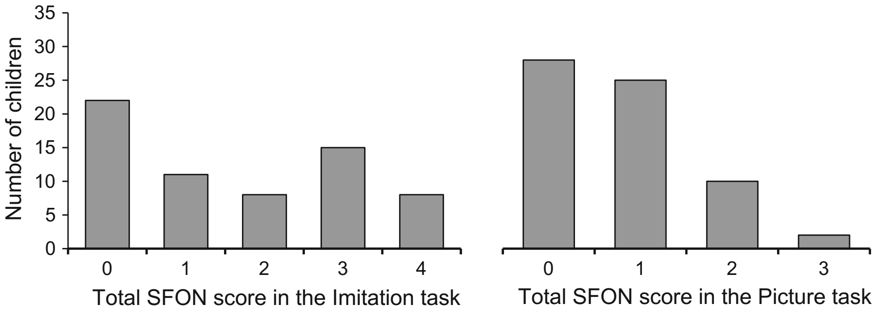


Fig. 6.5 Number of children per total SFON score in the Imitation task (N = 64) and in the Picture task (N = 65)

children (43%) did not spontaneously focus on exact numerosity and/or did not show any quantifying acts while describing the pictures and received a SFON score of 0.

As in Study 1, we observed large variety in the frequency and the type of children's number-related utterances during picture book reading. Whereas 28% of the participating children did not formulate a number-related utterance during the picture book reading activity, the other children (72%) formulated at least one number-related utterance. Interestingly, the mean frequency of number-related utterances in Study 2 (M = 5.08, SD = 6.56) was lower than in Study 1 and varied from 0 to 34. The combinations of types of number-related utterances differed from stating only one type of utterance (e.g., N6; 17%) to expressing different types of number-related utterances (e.g., N2_N6; 32%). With respect to the observed frequencies of the different types of number-related utterances, the children formulated altogether 329 number-related utterances during the picture book reading activity. As in Study 1, they most frequently focused on numerosities (N2; 62%) and quantity concepts (N6; 28.6%).

As we expected, Spearman correlation analyses revealed a non-significant association between children's SFON as measured via the action-based Imitation task and the frequency of their number-related utterances during picture book reading (r_s = 0.02), and a positive significant association between children's SFON as assessed via the verbal Picture task and the frequency of their number-related utterances during picture book reading, r_s = 0.47, also after accounting for word count. In line with recent results of Batchelor et al. (2015), we found no significant association between children's total SFON score in the Imitation and in the Picture task (r_s = 0.06).

6.4 Conclusion and Discussion

In recent years, young children's SFON tendency has been identified as a unique predictor of concurrent mathematical abilities and later mathematics achievement in primary school (Hannula-Sormunen et al. 2015; Hannula and Lehtinen 2005; Hannula et al. 2010). The main idea is that this SFON tendency promotes children's amount of self-initiated practice in number recognition during everyday activities and play, which consequently enhances their further mathematical development (Hannula et al. 2010). Thus far, results of studies that investigated the association between SFON in experimental tasks and SFON during everyday activities were inconclusive (Batchelor 2014, Study 3; Edens and Potter 2013; Hannula et al. 2005). These inconsistent results might be explained by the way in which SFON was measured during everyday activities and by the different types of experimental SFON tasks that were used.

To address these possible explanations, we conducted two closely related studies in which we examined the association between children's SFON in experimental tasks and their number-related utterances during picture book reading. More specifically, in Study 1 (Rathé et al. 2016b) we associated children's SFON as measured by an action-based Imitation task with their number-related utterances during numerical picture book reading. In Study 2 (Rathé et al. 2017), we explored the same association by contrasting two different experimental SFON tasks (i.e., an action-based Imitation task and a verbal Picture task) in relation to children's number-related utterances formulated during picture book reading, using a modified version of the picture book used in Study 1.

In line with previous research, both studies revealed large individual differences in kindergartners' SFON (e.g., Hannula et al. 2010; Hannula-Sormunen et al. 2015), and in the frequency and the type of their number-related utterances during picture book reading (Elia et al. 2010; van den Heuvel-Panhuizen and van den Boogaard 2008). In Study 1, we found no empirical evidence for the assumed association between children's SFON in the action-based Imitation task and their number-related utterances during numerical picture book reading. Yet in Study 2, we observed a significant positive association between children's SFON in the verbal Picture task (but not in the Imitation task) and the frequency of their number-related utterances during picture book reading, providing evidence for the first hypothetical explanation given in Study 1.

Although our studies yielded new insights in the association between SFON in experimental tasks and SFON during everyday activities, it should be noted that Study 2 revealed only partial empirical evidence for this assumed association. In particular, we did not investigate the association between children's SFON in the Imitation task and children's SFON during an action-based everyday activity, such as motor play with blocks, and we did not take into account (the development of) children's mathematical skills. Therefore, future research is required to investigate children's SFON as measured by different experimental tasks in relation to their

SFON as expressed during both verbal and action-based everyday activities, taking into account their (acquisition of) mathematical skills.

Related to our findings in Study 2 providing evidence for the first hypothetical explanation given in Study 1, SFON might be a multidimensional construct (including a verbal and an action-based aspect), which requires different experimental task contexts to measure these different aspects of SFON. However, as both verbal and action-based aspects of SFON have been shown to contribute to early mathematical development (Batchelor et al. 2015; Hannula-Sormunen et al. 2015), children's SFON as measured by the Imitation task and their SFON as measured by the Picture task, as well as their mutual relation, warrant further research.

In addition to these considerations, it could be argued that the positive association between children's SFON in the Picture task and their number-related utterances during picture book reading is not that surprising, given that the Picture task and the picture book reading activity are very similar. After all, both tasks assess children's number-related utterances when they are presented with pictures and are explicitly requested to comment on them. However, despite these similarities, there are some important differences between both tasks (see Table 6.1) that clearly indicate why the Picture task should not be considered an experimental task instead of an everyday activity, and vice versa (Rathé et al. 2017).

Finally, our studies may lead to some provisional implications for educational practice. More specifically, our findings suggest that, besides the acquisition of early mathematical abilities, children's mathematical dispositions, and in particular their SFON tendency, are also important for explaining individual differences in early mathematical development. Early childhood educators and parents could be informed about and helped in how to uncover and stimulate young children's tendency to attend to numerosities during primarily non-mathematically-focused everyday activities and play, including picture book reading. Interestingly, when

Table 6.1 Differences between the experimental Picture task and the everyday picture book reading activity (Rathé et al. 2017)

Experimental Picture task	Everyday picture book reading activity
• Unfamiliar experimental setting, in which the experimenter sits in front of the child, without seeing the pictures. The child is instructed to help the experimenter by describing the pictures, because the experimenter cannot see the pictures	• Familiar picture book reading setting, in which the experimenter sits next to the child on a chair or a pillow. The experimenter can see the pictures during the entire picture book reading activity. The child is invited to describe the pictures in the book, as they often do at home or in the classroom
• Neutral behavior of the experimenter, who merely asks the child to describe the pictures, but does not further interact with the child	• Less neutral behavior of the experimenter, who intervenes on each page by reading aloud the text, as a parent or teacher would do
• 3 random picture trials, coming from the same series, but including unrelated contents	• 11 picture trials, which all are part of a story line

comparing the results of both studies, we observed a lower frequency of children's number-related utterances during picture book reading in Study 2 than in Study 1, which suggests that the numerical characteristics (i.e., the number symbols in the text) of the picture book used in Study 1 might indeed have *guided* children's attention to number, instead of capturing their *spontaneous* attention to number. Moreover, this suggests that reading picture books with explicit numerical information included in the pictures and/or text, might be a promising tool to enhance children's SFON. Future observational and intervention studies, however, are needed to enhance our understanding on how to stimulate children's SFON during everyday activities and play. This might be an important first step in preventing later mathematical difficulties and stimulating positive attitudes toward mathematics in general.

References

Aunio, P., & Niemivirta, M. (2010). Predicting children's mathematical performance in grade one by early numeracy. *Learning and Individual Differences, 20,* 427–435.

Batchelor, S. (2014). *Dispositional factors affecting children's early numerical development.* Loughborough University, Leicestershire, United Kingdom. https://dspace.lboro.ac.uk/2134/. Accessed June 28, 2017.

Batchelor, S., Inglis, M., & Gilmore, C. (2015). Spontaneous focusing on numerosity and the arithmetic advantage. *Learning and Instruction, 40,* 79–88.

Claessens, A., Duncan, G., & Engel, M. (2009). Kindergarten skills and fifth-grade achievement: Evidence from the ECLS-K. *Economics of Education Review, 28,* 415–427.

De Smedt, B., Verschaffel, L., & Ghesquière, P. (2009). The predictive value of numerical magnitude comparison for individual differences in mathematics achievement. *Journal of Experimental Child Psychology, 103,* 469–479.

Edens, K. M., & Potter, E. F. (2013). An exploratory look at the relationships among math skills, motivational factors and activity choice. *Early Childhood Education Journal, 41,* 235–243.

Elia, I., van den Heuvel-Panhuizen, M., & Georgiou, A. (2010). The role of pictures in picture books on children's cognitive engagement with mathematics. *European Early Childhood Education Research Journal, 18,* 275–297.

Hannula-Sormunen, M. M., Lehtinen, E., & Räsänen, P. (2015). Preschool children's spontaneous focusing on numerosity, subitizing, and counting skills as predictors of their mathematical performance seven years later at school. *Mathematical Thinking and Learning, 17,* 155–177.

Hannula, M. M. (2005). *Spontaneous focusing on numerosity in the development of early mathematical skills* (Doctoral thesis, University of Turku, Turku, Finland). http://www.doria.fi/handle/10024/102190. Accessed June 28, 2017.

Hannula, M. M., Grabner, R., Lehtinen, E., Laine, T., Parkkola, R., & Ansari, D. (2009). Neural correlates of spontaneous focusing on numerosity (SFON). *NeuroImage, 47,* S39–S41.

Hannula, M. M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction, 15,* 237–256.

Hannula, M. M., Lepola, J., & Lehtinen, E. (2010). Spontaneous focusing on numerosity as a domain-specific predictor of arithmetical skills. *Journal of Experimental Child Psychology, 107,* 394–406.

Hannula, M. M., Mattinen, A., & Lehtinen, E. (2005). Does social interaction influence 3-year-old children's tendency to focus on numerosity? A quasi-experimental study in day care. In L. Verschaffel, E. De Corte, G. Kanselaar, & M. Valcke (Eds.), *Powerful environments for* *promoting deep conceptual and strategic learning* (pp. 63–80). Leuven, Belgium: Leuven University Press.

Hannula, M. M., Räsänen, P., & Lehtinen, E. (2007). Development of counting skills: Role of spontaneous focusing on numerosity and subitizing-based enumeration. *Mathematical Thinking and Learning, 9,* 51–57.

Mattinen, A. (2006). *Huomio lukumääriin: Tutkimus 3-vuotiaiden lasten matemaattisten taitojen tukemisesta päiväkodissa [Focus on numerosities: A study on supporting 3 year-old children's mathematical development in day care]*. Turku, Finland: Painosalama.

Nguyen, T., Watts, T. W., Duncan, G. J., Clements, D. H., Sarama, J. S., Wolfe, C., et al. (2016). Which preschool mathematics competencies are most predictive of fifth grade achievement? *Early Childhood Research Quarterly, 36,* 550–560.

Rathé, S., Torbeyns, J., De Smedt, B., Hannula-Sormunen, M. M., & Verschaffel, L. (2017). Verbal and action-based measures of kindergartners' SFON and their associations with number-related utterances during picture book reading. *British Journal of Educational Psychology.*

Rathé, S., Torbeyns, J., Hannula-Sormunen, M. M., De Smedt, B., & Verschaffel, L. (2016a). Spontaneous focusing on numerosity: A review of recent research. *Mediterranean Journal for Research in Mathematics Education, 15,* 1–25.

Rathé, S., Torbeyns, J., Hannula-Sormunen, M. M., & Verschaffel, L. (2016b). Kindergartners' spontaneous focusing on numerosity in relation to their number-related utterances during numerical picture book reading. *Mathematical Thinking and Learning, 18,* 125–141.

Torbeyns, J., Gilmore, C., & Verschaffel, L. (2015). The acquisition of preschool mathematical abilities: Theoretical, methodological and educational considerations. *Mathematical Thinking and Learning, 17,* 99–115.

van den Heuvel-Panhuizen, M., & van den Boogaard, S. (2008). Picture books as an impetus for kindergartners' mathematical thinking. *Mathematical Thinking and Learning, 10,* 341–373.

van Lieshout, T., & Hopman, P. (2013). *Boer Boris*. Haarlem, The Netherlands: Gottmer.

Vanbinst, K., Ghesquière, P., & De Smedt, B. (2015). Does numerical processing uniquely predict first graders' future development of single-digit arithmetic? *Learning and Individual Differences, 37,* 153–160.

Chapter 7
Number Sense: The Impact of a Measurement-Focused Program on Young Children's Number Learning

Jill Cheeseman, Christiane Benz and Yianna Pullen

Abstract Children begin to form mathematical concepts at an early age and many of these concepts are linked to measurement experiences. Often mathematics education in pre-school and at the beginning of school is focused on numbers. In order to acknowledge children's mathematical concepts and to build on them, a mathematics intervention program that focused on measurement replaced the usual mathematics curriculum for 40 children entering their first year of school in Australia. This chapter presents the results of the children's performance on a one-to-one task-based interview that tested their number knowledge at the beginning and end of the school year. In addition, two case studies and some classroom stories from the intervention are described. Findings indicate that a measurement-based curriculum can stimulate the development of children's number knowledge and their number sense.

Keywords Number sense · Measurement · Early mathematics education
Intervention · Design research

7.1 Introduction

Children acquire considerable mathematical knowledge before they enter school (Clarke et al. 2006). This knowledge is built through everyday playful experience and exploration in meaningful life contexts. Many examples could be offered to

J. Cheeseman (✉)
Monash University, Peninsula Campus, McMahons Rd, Frankston, VIC 3199, Australia
e-mail: Jill.Cheeseman@monash.edu

C. Benz
University of Education Karlsruhe, Bismarckstr. 10, 76133 Karlsruhe, Germany
e-mail: benz@ph-karlsruhe.de

Y. Pullen
Leibler Yavneh College, 2 Nagle Ave, Elsternwick, VIC 3185, Australia
e-mail: y.pullen@yavneh.vic.edu.au

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_7

illustrate ways in which young children's lives are rich in problem solving where children make decisions about number, position and size. Often, children's early mathematical experiences are in authentic measurement contexts (MacDonald and Rafferty 2015; Van den Heuvel-Panhuizen and Buys 2008). Mathematics education in pre-school and in school can be in stark contrast because many children experience a mathematics education which is heavily number focused (Benz 2012; Gasteiger 2010).

We contend that young learners in pre-school and in school deserve rich and meaningful mathematical experiences that acknowledge their mathematical strengths and connect their informal ideas to formal mathematics. Research shows that young children have intuitive and informal capabilities in both spatial and geometric concepts, and numeric and quantitative concepts (Bransford et al. 1999). In order to use these capabilities "children must learn to mathematize their informal experiences by abstracting, representing, and elaborating them mathematically" (Clements and Sarama 2011, p. 968). This applies especially for learning measurement because "measurement and geometry enable children to make connections with their daily environment" (Van den Heuvel-Panhuizen and Buys 2008, p. 10). We claim that children could learn number meaningfully through a measurement-centered curriculum where authentic experiences require children to solve problems. Therefore, we examined the possibilities of incorporating meaningful measurement problems into the mathematics program of children on entry to school. Knowing that many educators would be concerned about the effects of the program on children's number knowledge and skills, this chapter reports the assessment of children's number knowledge when they were engaged in a measurement-based program.

7.2 Theoretical and Empirical Background

The idea of using a measurement-focused curriculum is not new. Towards the end of the last century in Russia a curriculum was developed by Davydov et al. (1999) based on Davydov's earlier work (1975). These authors developed a mathematics program without numbers. The foundation of this program constituted describing and comparing physical attributes of objects (e.g., length, area etc.). It was intended that children should develop, through different activities, an understanding of equality and be able to describe comparisons with relational statements like $A < B$. After direct comparisons it was thought children should acquire knowledge of part-whole relationships when they transformed two unequal quantities into equal amounts. A comprehensive theoretical progression of children's thinking about measurement concepts was described by Davydov.

Dougherty and her colleagues adopted this theoretical approach and tested it in schools in Hawaii in their program *Measure Up* (Dougherty and Zilliox 2003). Both the Russian curriculum and the *Measure Up* (MU) project started with a theory of instruction from Vygotsky (1978) who:

> identified two ways of thinking about instruction leading to generalizations. One way is to teach particular cases and then build the generalizations from the cases. The other way is to start with a generalized approach and then apply the knowledge gained to specific cases. (Dougherty and Zilliox 2003, p. 22)

The MU project focused on the second of these two ways. The program used a general approach of instruction where an attribute of one concrete object, such as length, normally was not named with a concrete quantified result e.g., "5 sticks" but always in a relation to another length. The approach was seen as a basis for relational thinking, problem solving and early algebra. This was also due to the fact that letters were used instead of writing or representing the real objects. The project's mathematics sequence was primarily determined by Davydov's (1975) research and instructional approach. The MU project team worked on ways to deliver the theoretical approach in classrooms. The project identified at least six types of instruction: (1) giving information, (2) simultaneous recording, (3) simultaneous demonstration, (4) discussion and debriefing, (5) exploration guided, and (6) exploration unstructured. The order of instruction types from 1 to 6 represented a continuum from most teacher-active to most student-active. The program reported success: Dougherty and Zilliox (2003) and Sophian (2007) maintained that the relationship between measurement concepts and proportionality supported children's development of deep understandings of mathematical structures and properties of number.

However, the Russian approach and its embedding in the curriculum was not supported universally by researchers. Particularly due to its neglect of numbers, its use of letters, and its focus on early abstraction and generalization, the approach was criticized by authors such as Otte (1976), Freudenthal (1974) and Steinweg (2013).

7.2.1 Connections Between Number and Measurement

A contrasting view of the relationship between number and measurement was conceived by mathematics education researchers. For example, Steffe (2010) viewed discrete quantities and continuous quantities as connected: "We should not argue the operations that generate an awareness of numerosity [discrete quantity] are necessarily of a different genre than the operations that generate an awareness of length, distance, weight, area, volume" (p. 1). He named his four counting schemes as discrete quantitative measuring schemes and described four stages: *the perceptual counting scheme, the figurative counting scheme, the nested number sequence* and *the explicitly nested number sequence*. For Steffe, the development of awareness of continuous quantities was analogous to his four stages of awareness for discrete quantities.

Krajewski (2013) expanded her developmental model for the concept of number into a developmental model connecting the concept of number and magnitude. She highlighted the way in which discrete and continuous quantities have connections between magnitude and numbers. In her model of number children also need an understanding of size because number also represents the size of a quantity (see also Lorenz 2012). This thinking is reminiscent of Clements and Sarama (2009) who defined measurement as "the process of assigning a number to a magnitude of some attribute of an object, such as its length, relative to a unit. These attributes are continuous quantities" (p. 163). This definition emphasizes the links between number and measurement and was used in the present study.

7.2.2 Beginning Mathematics

The importance of measurement and geometry was highlighted by van den Heuvel-Panhuizen and Buys who said that measurement and geometry "lead to wonderment, and thus to the development of a mathematical disposition which is characterized by an exploring attitude, a certain perseverance in solving problems, and a sensitivity to the beauty of mathematical structures and solutions" (2008, p. 10).

Sophian (2007) questioned the common perspective that "children's thinking begins with the premise of counting, or some form of determining the numerical values of discrete quantities, [and] is the foundation for much of children's developing knowledge about mathematics" (p. 3). She described a contrasting position "that what is most fundamental for mathematical development is not counting or other mechanisms for apprehending numerosity, but rather basic ideas about relations between quantities" (p. 3). It is this comparison-of-quantity perspective that informed the present study.

In support of this approach another study of African American kindergarten children by Wang (2010) found that children from low-income families had higher mathematics achievement "if they had teachers who emphasized standardized measurement and comparison skills" (p. 301).

In the light of these findings and given the underachievement of Australian students from low-socioeconomic backgrounds in mathematics (Sullivan 2011) a teaching experiment was designed to change the emphasis from number to measurement in the first year of formal school.

7.3 Method

A year-long design research project was conducted with 40 children aged five to six years entering formal school in Victoria, Australia. A high proportion of these students came from Language Backgrounds Other Than English (LBOTE), had not been enrolled in pre-school, came from low socio-economic backgrounds and were members of newly arrived migrant families. In part, these factors were a stimulus for employing a different approach to mathematics learning as similar students in previous years had achieved limited success. The study aimed to answer the research question: What is the impact of a mathematics program based on measurement activities on young children's learning of number?

The intended curriculum (Van den Akker 2003) is documented in the *Australian Curriculum: Mathematics* (ACARA 2012) where the content strands Number and Algebra, Measurement and Geometry, and Probability and Statistics are described. The strands are intended to be integrated in practice, together with the proficiencies: understanding, fluency, problem-solving and reasoning. Curriculum outcomes for number and place value in the first year of school (Foundation) focus on counting numbers to 20, counting and subitizing small collections, and comparing collections. Measurement outcomes specify: "Use direct and indirect comparisons to decide which is longer, heavier or holds more, and explain reasoning" (ACARA 2012). This present study refers to this outcome of the *Australian Curriculum: Mathematics* but investigates a change of emphasis in the implemented curriculum from a number to measurement focus.

7.3.1 Implementing the Measurement Program

Detailed planning was undertaken by the teaching team, comprising two teachers who worked directly with the 40 children, led by the third author. The program was designed for a school that had a strong commitment to a Reggio Emilia philosophy of education (Rinaldi 2006) where children were offered *provocations* to learn. Mathematical activities were available to children every day in their classrooms and children could exercise choice over which tasks to complete. Every child was expected to find some mathematical task for exploration every day and learning challenges were offered children as mathematical open-ended problem-solving opportunities.

There was no intent to avoid the use of number. Rather, measurement was considered an authentic context in which children could be encouraged to use number. However, de-contextualised number was not taught at all—number was used to quantify attributes. A mathematics program was planned to include daily mathematics that focused on measurement tasks and problems. In addition to the mathematics planning, the teachers elaborated the mathematical potential of their integrated learning and teaching program. Children were expected to engage with mathematical thinking every day and teachers were expected to interact with children to challenge and extend their thinking.

The measurement-based program was characterized as being "student-active" (Dougherty and Zilliox 2003), where the planned experiences involved unstructured or guided exploration by the children. Daily mathematics focused on measurement tasks and problems. The classroom observations later in this chapter illustrate such problems. Teachers created detailed planning documents and observational records were kept of the children's actions and ideas. Teachers regularly met to discuss the mathematical learning of individuals and the group as a whole.

The study was conceived as design research because it was an interventionist, iterative, process and utility-oriented, practical in a real context and theory driven (Van den Akker et al. 2006). The mathematics intervention program was implemented with children throughout their first year of school. After it was implemented the merit of the program was evaluated by analysing the learning outcomes of the children. Three sources of data were collected to document the learning of the children: one-to-one task-based clinical interviews on number were conducted at the beginning and the end of the school year, case studies of seven selected individual children, and classroom observations where mathematics lessons were regularly observed, the general classroom context was described, and anecdotal notes were recorded. This chapter reports results of the analysis of the interview data on number for the cohort, two illustrative contrasting case studies, and two classroom stories.

7.3.2 Task-Based Interview for Number

A one-to-one task-based interview was used to assess the children's thinking about Number on entry to school and at the end of the first school year. The intention was to investigate the extent to which children learned number in a measurement context. The interview protocol was first developed in the Early Numeracy Research Project (ENRP) (Clarke et al. 2002) and subsequently used by educational sectors (Department of Education and Early Childhood Development (DEECD) 2006) and in other countries, for example Germany (Peter-Koop and Kollhoff 2015). The interview was constructed to match an ENRP Framework of Growth Points that was defined using the available research (Clarke et al. 2002). The interview schedule included 37 multi-part questions in the Number domain. Counting, place value, strategies for addition and subtraction, and strategies for multiplication and division were assessed. The interview was constructed using established criteria (Clarke et al. 2002). Based on the success of each child on the matched interview questions a Growth Point (GP) code was assigned to their responses. The tasks and the coding scheme is provided in detail in Clarke et al. (2002). In the present study the same interview was used on both occasions but due to the developing skills of the children additional tasks were given sometimes.

The third author interviewed the children in this study at the start of the school year using the established interview protocol. The second interview was conducted by an independently-trained interviewer. All student responses were independently coded numerically, representing the GPs (see theoretical framework according to the established coding protocols in Fig. 7.1, Clarke et al. 2002). The results of the counting and place value sections of the interview will be presented here and the original large-scale research project data are used here for purposes of comparison.

0. Not apparent.
Not yet able to state the sequence of number names to 20.

1. Rote counting
Rote counts the number sequence to at least 20, but is not yet able to reliably count a collection of that size.

2. Counting collections
Confidently counts a collection of around 20 objects.

3. Counting by 1s (forward/backward, including variable starting points; before/after)
Counts forwards and backwards from various starting points between 1 and 100; knows numbers before and after a given number.

4. Counting from 0 by 2s, 5s, and 10s
Counts from 0 by 2s, 5s, and 10s to a given target.

5. Counting from x (where x > 0) by 2s, 5s, and 10s
Given a non-zero starting point, can count by 2s, 5s, and 10s to a given target.

6. Extending and applying counting skills
Counts from a non-zero starting point by any single digit number, and can apply counting skills in practical tasks.

Fig. 7.1 ENRP growth points framework for counting (Clarke et al. 2002)

7.4 Findings

7.4.1 Interview Data

The mean GP codes of the children entering formal school for the first time (five to six-year olds) at the beginning and the end of their first year at school were calculated for each of the Number domains. These means were then compared to the three cohorts of reference school (control) data from the Early Numeracy Research Project (ENRP) (Clarke et al. 2002) because for three consecutive years data were collected from a representative range of schools matched to the research schools. Reference schools received no experimental treatment (intervention) and therefore could be considered suitable for comparison for children entering school and at the end of the first year of school in the state of Victoria, Australia. From this original large data set a sample was selected from the school where the third author taught. Her school had been a reference school for the original research project. The assessment instrument was identical to that used in the project.

ENRP Growth Points were defined to describe children's developing understanding of each mathematical domain. In the quantitative analysis only two Number domains will be used: counting and place value. The counting growth points (Fig. 7.1) described the development of children's counting by ones, as well as by twos, fives and tens. The growth points in counting identified children's production of number name sequences. However, the growth points were also concerned with children making the count-to-cardinal transition in word meaning described by Fuson (1982) where children are able to think about the number sequence to solve problems. The growth points in counting in Fig. 7.1 were devised to articulate the key steps taken by children in developing their understanding of the number sequence.

7.4.1.1 Analysis of the Interview Data

The mean GP results in counting and place value are presented in Table 7.1 where the cohorts of children from the original larger study for three years are labelled C1, C2, and C3, and the children in the intervention program are labelled ES. An original school data set is reported as OS. These data were collected in the same school over the three years of the ENRP study via a random sampling of the students. The purpose of constructing this sample is to create a comparative group of students from the same school with similar socio-economic backgrounds and Language Backgrounds Other Than English to those in the experimental group (ES). The children in the original school (OS) sample were taught the intended curriculum at the time of assessment that was largely number focused. First, these data concerning counting will be analysed, and later the GPs of place value will be presented and analysed.

Counting

An examination of the comparative results (Table 7.1) reveals that intervention program children (ES) came to school with counting knowledge that was not as sophisticated as most children in the original larger sample. On average they began school unable to recite the number names to 20 (mean GP = 0.33) and by the end of the year they had improved their rote counting skills (mean GP 0.93) but this was less than for the control groups. The mean for the state of Victoria was almost at Growth Point 2 where the child can reliably count a collection of around 20 objects.

Table 7.1 Comparison of mean growth point codes for counting and place value

	Counting		Place value	
	Mar	Nov	Mar	Nov
C1 (n = 438)	0.78	1.80	0.36	0.93
C2 (n = 504)	0.86	1.74	0.36	0.98
C3 (n = 523)	0.88	1.83	0.34	0.99
ES (n = 40)	0.33	0.93	0.34	0.95
OS (n = 51)	0.42	1.49	0.29	0.88

By the end of the first year of school on average the children in this study could recite the number names to 20 but not reliably count collections of 20 objects.

Comparisons Between the Original Group and the Intervention Group in Counting

The results of the experimental group (ES) compared to the original school data (OS) in counting (Table 7.2) shows that the two groups began the year with wide variations of achievement Growth Points. Almost three quarters of the ES children (73%) were unable to say the number name sequence to 20. Only 10% of this group could count a collection of over 20 objects. In comparison, Clarke et al. (2006) reported that 39% of Victorian children achieved this skill. The original control group had 36% at GP2 or above indicating that they were more aligned with the broader population on entry to school. Teachers' reasons for the poor counting knowledge on entry to school of the ES group were attributed to a combination of factors: a high proportion of students from Language Backgrounds Other Than English, few children had been to pre-school and many came from a low socio-economic background.

Table 7.2 shows the greatest differences between the two groups in the end-of-year (Nov) results. The ES children had learned to count in a measurement-focused curriculum but 14 (36%) remained on GP0 not yet able to state the number names to 20. A further 15 (38%) could verbally count but were unable to count a collection of objects and only 10 children could count reliably (26%). The two children who were able to count forwards and backwards from various starting points between 1 and 100, and knowing numbers before and after a given number (GP3) were exhibiting knowledge described in the intended curriculum as Year 1 outcomes. One student had achieved GP5 showing the ability to count from a non-zero starting point and to count by 2s, 5s, and 10s to a given target (a curriculum outcome for Year 2). In contrast in the control group (OS) the majority of children could reliably count collections at GP2 (68%) at the end of the year (see Fig. 7.2).

These data showed wide differences in counting growth points between the experimental and control group; although there was an improvement in both groups, at the end of the year only one quarter of the experimental group

Table 7.2 The numbers of children at each growth point in counting

Growth point	ES counting		OS counting	
	March n = 40 (%)	Nov n = 40 (%)	March n = 65 (%)	Nov n = 48 (%)
0	29 (73)	14 (36)	36 (55)	7 (15)
1	7 (17)	15 (38)	6 (9)	6 (13)
2	4 (10)	7 (18)	9 (14)	33 (68)
3	0	2 (5)	14 (22)	1 (2)
4	0	0	0	1 (2)
5	0	1 (3)	0	0

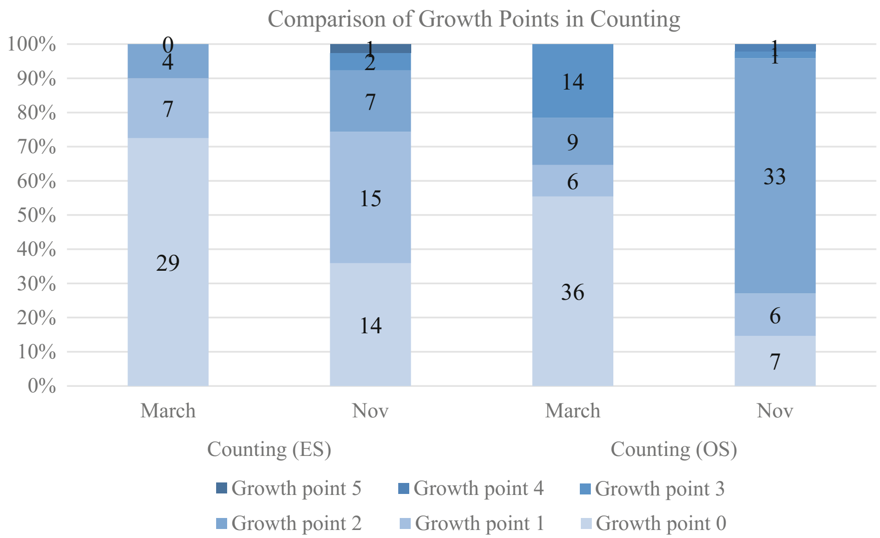


Fig. 7.2 Graph of the interview data in counting for experimental (ES) and control (OS) groups

(ES) compared to over two thirds of the control group (OS) were considered rational counters (Gelman and Gallistel 1978).

Place Value

An examination of the beginning and end of year data of each of the groups for place value revealed a very similar pattern of results (Table 7.1). By the end of their first year of school the children demonstrated that they had a sound understanding of single digit numbers. This relates to GP1 in the domain of place value in Fig. 7.3.

Comparisons Between the Original Group and Intervention Group in Place Value

Looking at matched group comparisons in place value (Table 7.3 and Fig. 7.4) shows patterns of results in the experimental (ES) and control (OS) groups of place value that are very similar.

In comparison with the counting data, 16% of the children in the measurement-based curriculum group (ES) has extended their knowledge of numbers and the number system beyond single digit numbers, reading writing and interpreting 2-digit (GP2) and 3-digit numbers (GP3) successfully. Place value knowledge indicates a developing awareness of the number system as a whole. The "top" (8%) of the experimental group had children who had mastered reading, writing and interpreting 3-digit numbers. This could possibly be attributed to the need to use larger numbers in the measurement context or the removal of a "ceiling effect" of the intended curriculum. Because the curriculum outcomes for number and place value in the first year of school focus on: counting numbers to 20, counting and subitizing small collections, and comparing collections, most mathematics programs limit the number range children meet to numbers less than 20. In

0. Not apparent

Not yet able to read, write, interpret and order single digit numbers.

1. Reading, writing, interpreting, and ordering single digit numbers

Can read, write, interpret and order single digit numbers.

2. Reading, writing, interpreting, and ordering two-digit numbers

Can read, write, interpret and order two-digit numbers.

3. Reading, writing, interpreting, and ordering three-digit numbers

Can read, write, interpret and order three-digit numbers.

4. Reading, writing, interpreting, and ordering numbers beyond 1000

Can read, write, interpret and order numbers beyond 1000.

5. Extending and applying place value knowledge

Can extend and apply knowledge of place value in solving problems.

Fig. 7.3 ENRP place value growth points framework (Clarke et al. 2002)

Table 7.3 The numbers of children at each growth point in place value

	ES place value		OS place value	
Growth point	March n = 40 (%)	Nov n = 40 (%)	March n = 51 (%)	Nov n = 48 (%)
0	29 (73)	13 (32)	38 (75)	14 (29)
1	10 (25)	21 (52)	13 (25)	29 (60)
2	1 (3)	3 (8)	0	5 (11)
3	0	3 (8)	0	0
4	0	0	0	0
5	0	0	0	0

the measurement-based program children chose to use units of measure that made sense to them and, in some instances, needed to use larger numbers in context to solve a measurement problem.

7.4.1.2 Discussion of Interview Data

While it seems at first sight that counting skills and place value skills would be closely linked, the results of this interview suggest otherwise. Patterns of results at the beginning and end of the year are quite similar for the experimental group (ES) and the control group (OS) for place value. However, patterns of results were very different for the counting section of the interview. The greatest difference in learning outcomes for number by the experimental group was in their poorer

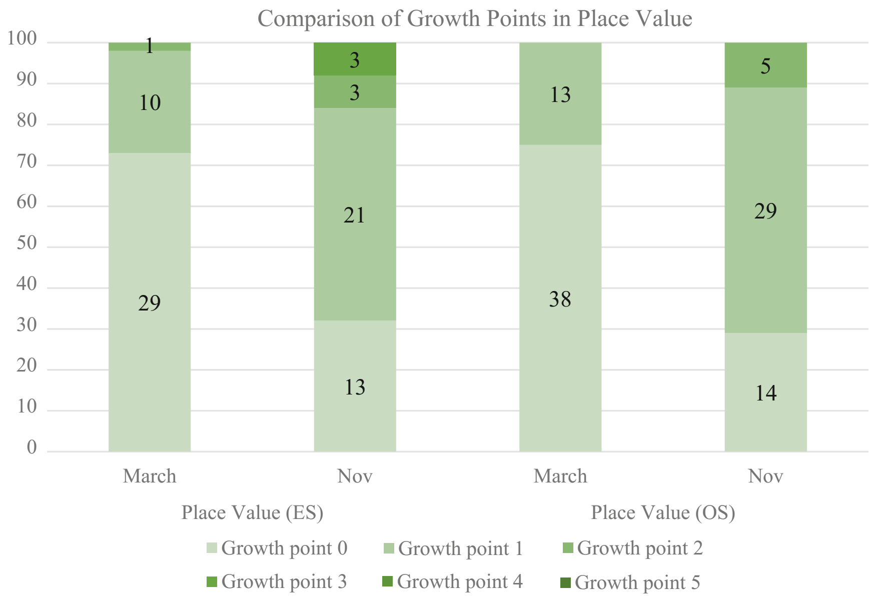


Fig. 7.4 Graph of the interview data in place value for experimental (ES) and control (OS) groups

knowledge of counting compared to the control group. This finding can perhaps be explained by the fact that the mathematics program experienced by children on entry to school is often based on counting. The *Australian Curriculum: Mathematics* (ACARA, p. 1) has a Number and Place Value section at Foundation level that lists the following outcomes:

- Establish understanding of the language and processes of counting by naming numbers in sequences, initially to and from 20, moving from any starting point (ACMNA001)
- Connect number names, numerals and quantities, including zero, initially up to 10 and then beyond (ACMNA002)
- Subitize small collections of objects (ACMNA003)
- Compare, order and make correspondences between collections, initially to 20, and explain reasoning (ACMNA289)
- Represent practical situations to model addition and sharing (ACMNA004)

As a consequence of these curriculum statements many teachers of children in their first year of formal schooling in Australia focus on the Number strand of the curriculum and in particular emphasize counting.

Even though there was an improvement in the outcomes of the children in the measurement-based program in specific aspects of counting, what is of concern is that children in this group have not demonstrated the same patterns of achievement in number name sequence as the control group (see Table 7.2). About half of the ES group children who were at GP0 in the beginning of their first year of schooling could still not state the sequence of number names to 20 at the end of the year. One reason could be that one of the fundamental principles of counting—the stable order principle (Gelman and Gallistel 1978), may have not been established for these children. We contend that knowing the number name sequence is a fluency skill that can be developed with practice. Therefore, based on the results presented here, we recommend that in addition to a program that contextualizes number children have regular opportunities to practice the number name sequence in meaningful ways. Furthermore, we recommend that teachers of young children continue to pay attention to each of the other principles of counting: one to one correspondence, the order irrelevance principle, and the cardinality principle (Gelman and Gallistel 1978). For children with a different language background comparing quantities, for example, is especially important as a way of generating number sense because this is a way of dealing with quantities without speaking and not necessarily needing the number sequence in the language spoken in school.

There is no doubt that counting is a key to young children's mathematical futures. It contributes to the development of "number sense" which is paramount for mathematical development. However, researchers have found that more than counting is fundamental for developing number sense (Benz 2014; Dornheim 2008; Mulligan and Mitchelmore 2009). Especially for the later development of calculation strategies, other competencies are essential such as grouping. Establishing learning contexts in the measurement-focused curriculum was an attempt to offer young children rich opportunities to learn mathematics. It did not abandon the use of numbers; in fact, it offered the children many opportunities to use numbers in different meaningful ways. The outcome may be seen in the development of the other domains of number like place value. The interview questions assessing place value focused on reading digits as numbers, modelling numbers with bundled and loose sticks, ordering numbers, writing numbers on a calculator, and mentally saying ten more or one hundred more than a number. In this domain children showed a similar performance to the original group who were taught the regular curriculum. When the children were provided with genuine contexts in which to use number, they built number skills and developed number sense.

It is also argued as essential that children's experiences should build on their intuitive mathematical knowledge (Howell and Kemp 2005; McIntosh et al. 1992). Therefore, teachers and parents are advised to "encourage young children to see themselves as mathematicians by stimulating their interest and ability in problem solving and investigation through relevant, challenging, sustained and supported activities" (AAMT and ECA 2006, p. 1).

7.4.2 Case Study Data

Having examined findings from the interview data, we present in the next section two case studies to illustrate the range of mathematical thinking elicited by the children at interview and to paint a picture of individual children's learning. Then two classroom observations are reported to give a sense of the richness of the measurement provocations teachers offered children as a context to learn mathematics.

7.4.2.1 Case Study Data: David

David (pseudonyms are used throughout) was 5 years old when he was interviewed in his first few weeks of formal school (March). His interview responses indicate that he entered school with numerical skills (Clarke et al. 2006). Tables 7.4 and 7.5 show a summary of the mathematical thinking revealed by the initial interview in the left hand column. The second column of each table shows the learning gains made by David by November. The content in the table is aligned to match the responses to the same interview questions where possible.

David had learned to count beyond 100 by the end of the year. He was showing signs of understanding the number system when he "reinvented" (Kamii 2003) the number after 110 to work out 111. David needed to recognise that the counting sequence beyond 100 follows the pattern of numbers less than 100 (Fuson 1988).

Table 7.4 Summary of counting and place value responses by David at beginning and end of the school year

Counting skills in March	Counting skills in November
Counting	
– Counted a collection of 23 objects – Counted aloud forwards by ones from different starting points—1 to 32 and 3 to 62 with – 84 to 113 he stumbled at 99 unable to continue – Counted backwards by ones from 10 to 1 – Knew the number before and after 56 – Correctly counted to 100 by 10's	– Counted a collection of 24 objects accurately – Counted forwards and backwards proficiently with numbers to 100; counted beyond 100, and to 110 – Skip counted from 0 by 10s, 5s and 2s – Counted by 10s from 23 and 5s from 24; but not by 3s from 11
Place value	
– Read the numerals 3, 8, 36 – Read 18 as 81 – On the calculator read the tens numbers accurately – Unable to read or make a 3-digit number on the calculator – Successfully ordered all 1 digit, 2 digit, 3 digit and 4 digit cards. He used the hundreds digit to order the 4 digit cards however was unable to read any of the 4 digit numbers – Read the number 36 correctly but could not use the bundles of 10 sticks to model 36—he looked for 36 loose sticks	– Read, ordered and worked with numbers into the hundreds – Was able to work with some 4 digit numbers but not 6023 as he had difficulty with the 0 as a place holder – Understood two-digit numbers as tens and ones showing 3 bundles of 10 to make 30 and count on 6 to make 36 – Recognized number patterns the hundreds chart to find the blank square (58)

Table 7.5 Summary of addition and subtraction and multiplication and division responses by David at beginning and end of the school year

Mathematical thinking in March	Demonstrated learning gains in November
Strategies for addition and subtraction	
He added 4 teddies to the 9 teddies hiding under the ice-cream container lid by counting on. His first response was 14 which he corrected once he could see the 9, he started at 9 and counted on the 4 teddies again	He used his knowledge of known facts, and doubles, commutativity, and counting on He was able to count back to solve subtraction
	He could use derived strategies such as near doubles, bridge to ten, and add 10 take one to add 9 and knowledge of "fact families"
Strategies for multiplication and division	
He quickly put two teddies in each of 4 teddy cars and counted by ones to get the total as 8. When asked if he could count the teddies in a different way, he answered "There's 2 groups of 4, that's 8"	The teddy cars question he saw as double 4
The interview ended because David was unable to share the 12 teddies between the 4 mats	Sharing of 12 teddies on mats David said, "Three teddies sat on each mat because 4 groups of 3 make 12 and there are 12 teddies" The task shows one tin of three tennis balls and asks how many tennis balls in 4 tins. David began knowing that double 3 equals 6 then he added 3 and another 3

He had also generalized how to add 10 to any number and he knew how to count by 5s from variable starting points. He recognized counting patterns and could use them to find a missing number. It was clear that David had structured concepts of counting and that he developed an understanding of place value. He could bundle 10 ones into ten and using the ten as new unit. By the end of the year David was no longer having difficulties reading and writing teen numbers. He could interpret the value of a digit in the tens column, the hundreds column, and he was working towards reading, writing and interpreting 4-digit numbers.

The responses David made to the interview tasks were coded according to the prescribed protocols (Clarke et al. 2002) by a trained coder. The growth points for these two number domains are presented in Fig. 7.5.

David's strategies for early operations were limited to count on and doubling methods at the beginning of the year. However, by November he could manipulate numbers mentally using known facts, doubles and commutativity. He was using early multiplicative thinking as "groups of", and generated multiples by repeated addition.

The story of David is one of a child who entered formal schooling with knowledge of numbers and able to solve operations with numbers using counting. In Australia, 57% of children on entry to school could verbally count to 20 (Clarke et al. 2006). The same research showed that 39% of children could count a collection of at least 20 objects. Only 3% of children on entry to school could count forwards and backwards by 1s and say the number before and after a given number. Further only 0.3% could count from 0 by 2s, 5s and 10s. Based on these findings

Strategies for Addition and Subtraction

0. Not apparent

Not yet able to combine and count two collections of objects

1. Count all (two collections)

Counts all to find the total of two collections

2. Count on

Counts on from one number to find the total of two collections

3. Count back/count down to/count up from

Given a subtraction situation, counts back, counts down to or counts up from, without the need for physical modelling, such as using fingers.

4. Basic strategies (doubles, commutativity, adding 10, build to 10)

Given an addition or subtraction situation, all of the listed strategies (doubles, commutativity, adding 10, build to 10) are evident.

Strategies for Multiplication and Division

0. Not apparent

Not yet able to create and count the total of several small groups

1. Counting groups of items as ones

To find the total in a multiple group situation, refers to individual items only

2. Modelling multiplication and division (all objects perceived)

Models all objects to solve multiplicative and sharing situations.

Fig. 7.5 Growth point framework for strategies for addition and subtraction and for multiplication and division

we can say that David's knowledge of number would classify him as capable on entry to school.

It is interesting to note that David's number knowledge continued to develop markedly in a classroom where measurement was the focus of the mathematics curriculum and that while no number strategies were explicitly taught to him, he learned powerful strategic thinking. For example, he learned doubling using near doubles and bridging to ten, in measurement contexts. As we can see later in the classroom stories, numbers played a substantial role in the classroom activities and it was meaningful for him to deal with numbers in the learning activities. While we cannot be sure which activities in the end supported the extension of David's knowledge, we can say that when challenging and open measurement tasks were the focus of the program David could develop his number knowledge. This finding is reminiscent of the research of Young-Loveridge (1989) who monitored the mathematical thinking of five-year olds at entry to school in New Zealand. She

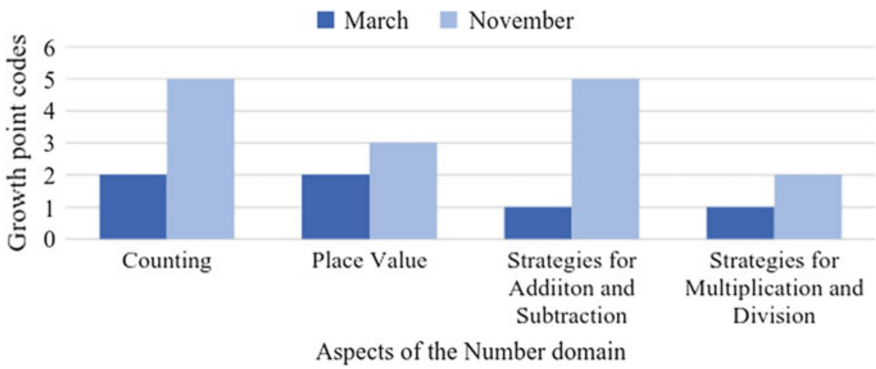


Fig. 7.6 Graph of interview data for David

followed the progress of the children as they were taught the standard mathematics curriculum. Young-Loveridge identified three categories of children's mathematical knowledge related to number and counting on entry to school: *expert*, *typical*, and *novice*. The children who were classified as expert on entry to school made mathematical learning gains that out-stripped the others. In fact, they developed number knowledge that was beyond the specified curriculum at that time. It seems that David would be categorized as mathematically expert when starting school and his number sense continued to grow and develop in a measurement-focused environment where he was not restricted to a limited number range (see Fig. 7.6).

7.4.2.2 Case Study Data: Maya

In order to get insight in Maya's mathematical competences data both of the interview and of the classroom will be analyzed and interpreted.

Interview Data: Maya

Maya's responses to the clinical interview make an interesting comparative story. English was Maya's second language and she spoke her first language at home and at pre-school. In addition she had refugee status. These two attributes meant that she was representative of over 70% of the children in the experimental group studied. Maya had been to pre-school with other children who spoke her first language. Starting school was a substantial cultural challenge for her and she had the mathematical "disadvantage" associated with students from a low socio-economic background (Sullivan 2011).

While Maya was just beginning to learn English, she had comprehension skills that indicated that she would be able to follow the task-based interview. She

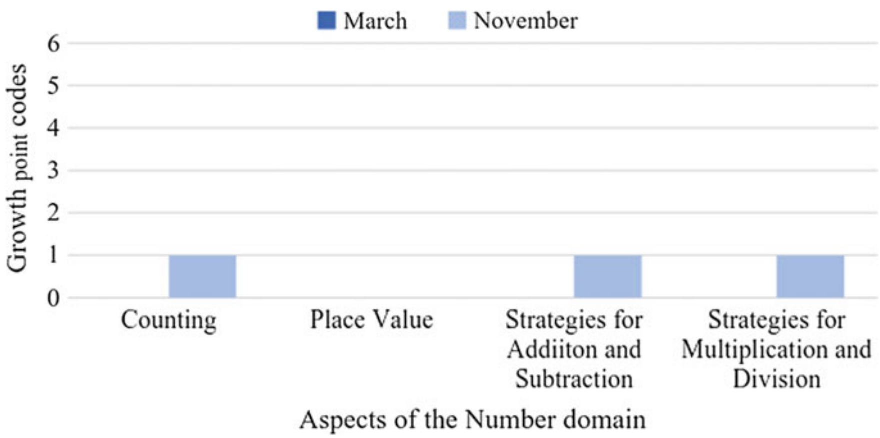


Fig. 7.7 Graph of interview data for Maya

displayed a keen interest in classroom tasks and was happy to work with her teachers and her peers and to share her ideas.

Before analysing the mathematical knowledge that Maya displayed during the interview, we will look at the GP codes indicated by her responses (Fig. 7.7). At first glance it seems that Maya knew nothing about number on entry to school because her responses were coded at zero in all four number domains. Further, the graph shows place value still at GP0 in November. So it appears that there was little development in her number knowledge as she could not reliably count a collection of 20 or read, write, and interpret single digit numbers at the end of one full year at school.

It was difficult early in the year for the interviewer to know whether Maya was struggling with mathematical thinking (was in fact a novice mathematically), was having difficulty finding the correct English words to express her mathematical thinking, or whether perhaps the language used in the interview was the challenge. Rather, it may have been a combination of all of these factors. Maya lacked experience with numbers especially number symbols because the numerals, other than 1 and 2, were unfamiliar to her. In fact, the only other symbols Maya knew were the letters needed to write her name. A closer examination of Maya's responses, as summarised in Table 7.6, revealed in more detail her mathematical thinking during the interview.

By comparing her end-of-year GP codes with the Early Numeracy Research Project students (Clarke et al. 2002) we can see how Maya stands alongside a representative sample of approximately 1500 students at the end of their first year of school.

- 5% were at Maya's counting growth point or below (p. 124);
- 7% were at Maya's place value growth point (p. 130);
- 52% of children could solve addition by counting all (as was Maya) or were unable to find a solution (p. 135);

Table 7.6 Summary of responses by Maya at beginning and end of the school year

Mathematical thinking in March	Demonstrated learning gains in November
Counting	
She estimated 6 teddies in her cup—actually there were 21	Maya's estimation of teddies in the cup was 12, the actual was 25
She could make and count a collection of 4 teddies She demonstrated one to one correspondence: she placed one teddy at a time, from the cup to form a line on the table, adding a number tag for each teddy. However, she did not have the number words after 5	She counted correctly to 15 looked at the teddies and said, "There are more." She had a sense of magnitude but could not count to find the quantity
She was able to verbally count to 5	She could count verbally by ones forward to 20 and backwards from 5 to 1
Place value	
She knew the symbols for 1 and 2 only	She could read the digits 1–5 only She could get 7 teddies when shown the numeral on a card
	She was able to order the cards 2, 5 and 9
Strategies for addition and subtraction	
She was not able to solve the tasks	She could solve 4 + 9 by counting all
Strategies for multiplication and division	
She followed instructions placing 2 teddies in each of the teddy cars however could not count them successfully, stumbling after 4	She counted by 2s four times to 8
	She shared the 12 teddies equally on 4 mats by putting 3 on each mat saying she "remembered it"

- 38% were solving multiplication and division by counting or unable to find a solution (p. 139).

What these figures show is that while Maya had poor counting and place value skills, in the bottom 10% of her age group, her strategies for early operations are comparable with her contemporaries who were also using counting to solve problems. Maya was able to improve her knowledge except in the domain of place value from GP0 to GP1. She still had problems reading numbers with the English number words but, in spite of that, she was able to make the quantity shown on the number cards. This could also reveal her struggle with the English language which may have led to some of the problems in mathematics revealed by the interview.

Maya in the Classroom

Classroom observation data of Maya adds to our picture of her mathematical thinking. Maya was seen measuring the length of a leaf with "fingers" (see Fig. 7.8). She explained:

Fig. 7.8 Maya

Maya "I keep my fingers like this (shows her fingers) and I measure the leaf".
Teacher "How big is your leaf Maya?"
Maya "It is four finger things—see?"

Interestingly, not long before the end of year interview, while working with one of the teachers in the classroom, Maya correctly ordered a set of single digit number cards, 1–9. When she was offered the 0, she placed it in line after the 9 saying, "It needs a number one next to it." Where she was clearly thinking of ten rather than zero. The teacher responded, "You have been doing a lot of measuring Maya, would thinking of your measuring tools help you find a place for this number without the one?" Maya immediately, with a smile on her face, put it before the 1 saying, "It's always in front, in front of the one on my measure". It is clear that the measurement context made the numerals meaningful for Maya. While we cannot say that she understands concepts of zero, she had a clear mental image of the sequence of single-digit numerals in a measurement context. However, in the interview without the context and with the expectation of the number names in English Maya struggled to display her knowledge. Maya had learned number concepts in her first year at school using measurement to make numbers sensible to her. She began with very little knowledge and experience and had gained much more of each.

An examination of the whole cohort interview results and case studies based on individual interview data, while interesting and revealing, tell only part of the complex story of the intervention project. The daily classroom anecdotes give a sense of the vigour and richness of the program.

7.4.3 Classroom Data

Two classroom anecdotes about length will illustrate some children's thinking in the program.

7.4.3.1 John's Height

In their first week at school children were invited to consider: How tall are you? How big are your feet? How much do you weigh? These provocations to explore were posted in the mathematics corner of the classroom. Materials were provided for the children: a height chart (in centimetres), scales, Unifix cubes, wooden sticks, assorted blocks, a measuring tape, and a 1 m ruler.

John (5 years) was observed sitting, pencil and paper in hand, looking very busy. School had not officially begun for the day but John had decided he wanted to draw himself against a height chart because he said that he already knew how tall he was. "Look I'm seventeen tall, see, not eighteen, because the line to my head is at 17." As John's drawing (Fig. 7.9) shows, on entry to school John understood that measuring height uses numbers and he knew the number sequence till 17 only with one number missing.

For him the 17 line matched his height. Despite the absence of a 15 in his number sequence, he drew and labelled spaced intervals to illustrate his

Fig. 7.9 John's drawing of measuring his height

understanding that he was 17 tall. His counting began at one and the origin of his measure is not drawn. The labels show that he understood that the count referred to the length of each interval. He carefully matched the line for 17 to the top of his head in the drawing, showing accuracy and an awareness of the end point of the measure. While it is not clear where the notion of 17 came from, John had in his mind that he was 17 tall and he could clearly show what this meant to him using an approximation of a number path.

7.4.3.2 Measuring the Autumn Leaves

The following conversation between Eli and his teacher shows how children's attention was naturally drawn to salient features of length measurement.

Eli “I don't know how much this leaf is. …It's 300”

Looked at the millimetres on the ruler he had placed along the length of the leaf

T “Where did you start your measuring?”
Eli “2.”
T “Why 2?”
Eli “I don't know, I think I should start at zero.”
T “Where is zero on your ruler?”

Eli pointed to the zero

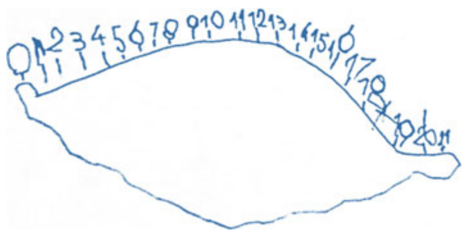
T “So, can you start measuring from zero? Does that help?”
Eli “Yes.”
T “Can you measure from the start of your leaf?'
Eli “Yes. I will draw a line so I know where the end is.”

Eli traced the outline of the leaf

T “What are you doing with your leaf?”
Eli “I'm putting numbers on something.”
T “What are the numbers for?”
Eli “Measuring something. Anything—like shelves, chairs and leaves.”
T “So why are you putting the numbers along the leaf?
Eli “To know how much the leaf is.”

This child knows that the purpose of numbers is to quantify a length although as yet he has no apparent concept of unit. His awareness of the origin of the measure was raised in conversation, and the number zero became meaningful for him in this context (Fig. 7.10).

Fig. 7.10 Eli's leaf



7.5 Final Remarks

In an attempt to find a way to make mathematics meaningful for children, a measurement-focused program replaced a counting-based program. Findings from the overall program will be reported elsewhere to show what measurement insights young children are able to achieve in an investigative play-based setting. Here quantitative results of pre- and post-intervention data have been analysed using a published assessment protocol to compare aspects of the knowledge of experimental to control groups of children.

Despite the improvement in their skills after beginning school, the results show inadequate learning by the children in the experimental group (ES) in counting. More than a third of the children could not rote count the number sequence to at least 20. More than half of the children in the experimental group were also not counting collections reliably by the end of their first year of school. These findings suggest that children entering school—at least those with poor English language skills and a low socio-economic background—seem to need explicit counting practice with the sequence of number names to 20 and with the one to one correspondence between number words and objects in addition to experiences with number in measurement contexts, where they were encouraged to count units aloud and compare continuous quantities.

The *Australian Curriculum: Mathematics* has a Number and Place Value section at Foundation level which lists five outcomes where counting as quantifying is emphasized. Mulligan and Mitchelmore (2009) emphasized that children need more than simply counting routines. The learning contexts of the measurement-focused program offered children many opportunities to use numbers in relevant and meaningful ways. The observation data showed that the children displayed a sense of 'the unit' with which they were measuring and counting. This experience may have contributed to place value understanding apparent in the interview data. Because the children needed to use larger numbers in context to solve a measurement problem, 16% of the children improved their skills in reading, writing, interpreting and ordering of two- and three-digit numbers.

The children displayed other skills for measurement like understanding of iteration of the unit to form a composite "whole", awareness that counting more units related to a sense of magnitude, and their ability to quantify to compare measures. We expected that in this domain the children would make great progress because many measurement activities took place. However, the genuine measurement context for number stimulated some children to go beyond the intended curriculum in measurement outcomes. The curriculum specifies only the use of direct and indirect comparisons. This study supports the findings of recent studies (Cheeseman et al. 2012; MacDonald 2012) which show that young children are capable of more sophisticated concepts of measurement than the curriculum specifies.

An implication of this study is that measurement contexts can be productively used with young children to stimulate number knowledge and reasoning. Like Sophian (2007), we question the perspective that the mathematical basis of children's thinking is counting or determining the numerosity of discrete quantities. This study supports a comparison-of-quantity perspective as an effective approach for young learners of mathematics. We are not advocating eliminating counting, we are advocating practising counting skills that can be used to determine numerosity in measurement contexts where they can be used to solve problems. This study, in particular the case studies and the classroom data, have shown that in principle when young children measure, they use numbers in meaningful ways and can acquire number competencies.

This project is similar to other effective intervention studies with young children reported by Clements and Sarama (2011, p. 970) that "challenge students to solve demanding mathematical problems … helping them to learn to think mathematically". The learning environment in this intervention study was not based on a strict step-by step curriculum. Proposing a learning challenge every day offered children mathematical open-ended problem solving opportunities. The number range was not restricted and children could choose to engage at their own level of thinking through the natural learning situations where novice mathematicians as well as expert mathematicians could grow in their thinking.

Regarding the special cultural and socio-economic background of the children the linguistic differences have to be considered. It is known that children from different cultural and socio-economic backgrounds proceed through the same developmental pathway in their early intuitive understanding of mathematics (Gelman 2000; Ginsburg 1982, 1997; Klein and Starkey 1988). By providing language rich, challenging, measurement provocations to young children who come to school with a socio-economic disadvantage and a linguistic disadvantage, we are confident that their counting skill will develop hand-in-hand with their number sense. The opportunities to learn through measurement problem solving also offered a different time frame for the children's development. The measurement provocations always offered an unrestricted number range for the children and therefore enabled children to set their own challenges and to learn in different number ranges.

We advocate that teachers consider enriching the early mathematical experiences that are offered to young children by maximising their use of measurement contexts while maintaining fluent counting skills. In this way authentic problem solving will lead to children posing problems of their own and developing ways of designing and testing mathematical experiments. Much work remains to be done in researching specific learning and teaching approaches and designing exemplary learning environments. However, this project has shown that in principle when young children measure, they use numbers and develop number sense.

References

Australian Association of Mathematics Teachers and Early Childhood Australia (AAMT & ECA). (2006). *Position paper on early childhood mathematics*. www.aamt.edu.au. Retrieved June 28, 2017.

Australian Curriculum Assessment and Reporting Authority (ACARA). (2012). *Australian curriculum: Mathematics*. http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10. Retrieved June 28, 2017.

Benz, C. (2012). Attitudes of kindergarten educators about math. *Journal für Mathematik-Didaktik, 33,* 203–232.

Benz, C. (2014). Identifying quantities—Children's constructions to compose collections from parts or decompose collections into parts. In U. Kortenkamp et al. (Eds.), *Early Mathematics Learning, Selected Papers of the POEM 2012 Conference* (pp. 189–203). New York: Springer.

Bransford, J., Brown, A., & Cocking, R. (Eds.). (1999). *How people learn*. Washington, DC: National Academy Press.

Cheeseman, J., McDonough, A., & Ferguson, S. (2012). The effects of creating rich learning environments for children to measure mass. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia)*. Singapore: MERGA.

Clarke, D. M., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., et al. (2002). *Early numeracy research project: Final report, February 2002*. Fitzroy, Victoria: Mathematics Teaching and Learning Centre, Australian Catholic University.

Clarke, B. A., Clarke, D. M., & Cheeseman, J. (2006). The mathematical knowledge and understanding young children bring to school. *Mathematics Education Research Journal, 18* (1), 78–102.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.

Clements, D., & Sarama, J. (2011). Early childhood mathematics intervention. *Science, 333*(6045), 968–970.

Davydov, V. (1975). Logical and psychological problems of elementary mathematics as an academic subject. In L. P. Steffe (Ed.), *Children's capacity for learning mathematics. Soviet studies in the psychology of learning and teaching mathematics* (Vol. VII, pp. 55–107). Chicago: University of Chicago.

Davydov, V. V., Gorbov, S., Mukulina, T., Savelyeva, M., & Tabachnikova, N. (1999). *Mathematics*. Moscow, Russia: Moscow Press.

Department of Education and Early Childhood Development (DEECD). (2006). *Mathematics Online Interview*. http://www.education.vic.gov.au/studentlearning/teachingresources/maths/interview/moi.htm. Retrieved March 4, 2012.

Dornheim, D. (2008). *Prädiktion von Rechenleistung und Rechenschwäche: Der Beitrag von Zahlen-Vorwissen und allgemein-kognitiven Fähigkeiten*. Berlin: Logos.

Dougherty, B. J., & Zilliox, J. (2003). Voyaging from theory to practice in learning: Measure up. In N. A. Pateman, B. J. Daugherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education (PME)* (Vol. 1, pp. 17–22). Honolulu: PME.

Freudenthal, H. (1974). Soviet research on teaching algebra at lower grade of the elementary school. *Educational Studies in Mathematics, 5*(4), 391–412.

Fuson, K. C. (1982). An analysis of the counting-on solution procedure in addition. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 67–81). Hillsdale, NJ: Lawrence Erlbaum.

Fuson, K. (1988). *Children's counting and concepts of number*. New York: Springer.

Gasteiger, H. (2010). *Elementare mathematische Bildung im Alltag der Kindertagesstätte: Grundlegung und Evaluation eines kompetenzorientierten Förderansatzes. Empirische Studien zur Didaktik der Mathematik:* Bd. 3. Münster: waxmann.

Gelman, R. (2000). The epigenesis of mathematical thinking. *Journal of Applied Developmental Psychology, 21,* 27–37.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, Massachusetts: Harvard University Press.

Ginsburg, H. P. (1982). The development of addition in the contexts of culture, social class, and race. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 191–210). Hillsdale, NJ: Lawrence Erlbaum.

Ginsburg, H. P. (1997). Mathematics learning disabilities: A view from developmental psychology. *Journal of Learning Disabilities, 30,* 20–33.

Howell, S., & Kemp, C. (2005). Defining early number sense: A participatory Australian study. *Educational Psychology, 25*(5), 555–571.

Kamii, C. (2003). *Young children continue to reinvent arithmetic—2nd grade: Implications of Piaget's theory*. New York: Teachers' College Press.

Klein, A., & Starkey, P. (1988). Universals in the development of early arithmetic cognition. *New Directions for Child and Adolescent Development, 1988*(41), 5–26.

Krajewski, K. (2013). Wie bekommen die Zahlen einen Sinn: ein entwicklungspsychologisches Modell der zunehmenden Verknüpfung von Zahlen und Größen. *Rechenstörungen bei Kindern: Neurowissenschaft, Psychologie, Pädagogik, 2,* 155–179.

Lorenz, J. H. (2012). *Kinder begreifen Mathematik – Frühe mathematische Bildung und Förderung*. Stuttgart: Kohlammer.

MacDonald, A. (2012). Young children's photographs of measurement in the home. *Early Years, 32*(1), 71-85.

MacDonald, A., & Rafferty, J. (2015). *Investigating mathematics, science and technology in early childhood*. South Melbourne, Victoria: Oxford University Press.

McIntosh, A., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12,* 2–8.

Mulligan, J. T., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21*(2), 33–49.

Otte, M. (1976). Die didaktischen Systeme von V.V. Davidov / D.B. Elkonin einerseits und L.V. Zankov andererseits, Skizze einer kritischen Auseinandersetzung. *Educational Studies in Mathematics, 6*(4), 475–497.

Peter-Koop, A., & Kollhoff, S. (2015). Transition to school: Prior to school mathematical skills and knowledge of low-achieving children at the end of Grade 1. In B. Perry, A. MacDonald, & A. Gervasoni (Eds.), *International perspectives: Mathematics and transition to school* (pp. 65–83). Singapore: Springer.

Rinaldi, C. (2006). *In dialogue with Reggio Emilia*. New York: Routledge.

Sophian, C. (2007). *The origins of mathematical knowledge in childhood*. New York: Lawrence Erlbaum Associates.

Steffe, L. P. (2010). *On children's construction of quantification*. http://www.uwyo.edu/wisdome/_files/documents/steffe.pdf. Retrieved June 28, 2017.

Steinweg, A. S. (2013). *Algebra in der Grundschule*. Heidelberg: Springer.

Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies*. http://research.acer.edu.au/aer/13. Retrieved June 28, 2017.

Van den Akker, J. (2003). Curriculum perspectives: An introduction. In J. van den Akker, W. Kuiper, & U. Hameyer (Eds.), *Curriculum landscapes and trends* (pp. 1–10). The Netherlands: Springer.

Van den Akker, J., Gravemeijer, K., McKenney, S., & Nieveen, N. (2006). Introducing educational design research. *Educational Design Research, 1*, 3–7.

Van den Heuvel-Panhuizen, M., & Buys, K. (2008). *Young children learn measurement and geometry*. Utrecht, The Netherlands: Freudenthal Institute.

Vygotsky, L. (1978). *Mind and society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wang, A. H. (2010). Optimizing early mathematics experiences for children from low-income families: A study on opportunity to learn mathematics. *Early Childhood Education Journal, 37*(4), 295–302.

Young-Loveridge, J. (1989). The development of children's number concepts: The first year of school. *New Zealand Journal of Educational Studies, 24*(1), 47–64.

Part III
Embodied Action and Context

Chapter 8
(Re)(con)figuring Space: Three Children's Geometric Reasonings

Jennifer S. Thom

Abstract Despite decades of research revealing the importance of and need for developing students' spatial reasoning skills, geometry receives the least attention in North American K-12 mathematics classrooms. This chapter focuses on three grade one children as they worked on a spatial-geometric task. The study as part of a larger research project inquired into the actual forms, activities and processes that constituted the children's reasonings and geometry during the three episodes. The findings contribute to current early years research by further explicating the body's role in the children's spatial-geometric reasonings, the impact of these on their conceptions, and how geometry emerged as an ongoing creative process of (re)(con)figuring space. Key implications are considered regarding young children's spatial-geometric reasoning in the mathematics classroom.

Keywords Primary children · Embodied · Diagrams · Gestures
Spatial reasoning · Geometric reasoning · Abstraction · Sensation
Figuring space

8.1 Rationale: Spatial Reasoning and School Geometry in the Early Years

Today in education and outside of the field, the focus on spatial reasoning continues to gain momentum. Outside of education, there is strong incentive, and pressure, for students to succeed in mathematics due to the increased demand for Science,

---

This study was supported by a grant from the Social Sciences and Humanities Research Council of Canada. I am most grateful to the research team, teachers, students, and assistants who took part in this work.

J. S. Thom (✉)
Department of Curriculum and Instruction, University of Victoria,
PO Box 1700 STN CSC, Victoria, BC V8W 2Y2, Canada
e-mail: jethom@uvic.ca

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_8

Technology, Engineering, and Mathematics (STEM) skills in the workplace. For more than two decades, research has shown the important role that spatial reasoning plays in the four subject areas (e.g., Newcombe 2010; Uttal et al. 2013). Spatial reasoning is not only a strong predictor in determining the likelihood of individuals' participation in STEM disciplines; it also serves as an indicator for their success in these areas (Kell et al. 2013; Shea et al. 2001; Wai et al. 2009). Even the most routine aspects of daily life require spatial reasoning skills. For example, travelling from one place to another, using a personal mobile device, drawing, reading, or even carrying out multi-step tasks involves thinking spatially.

With respect to education, The National Research Council (NRC) (2006, 2009) identified the need for "teaching spatial thinking to all students in all grades" (NRC 2006, p. 6). The NRC (2006) contended that while:

> spatial thinking itself is not a content-based discipline in the way that physics, biology, and economics are disciplines: it is not a stand-alone subject in its own right. Spatial thinking is a way of thinking that permeates those disciplines and… virtually all other subject matter disciplines…. Spatial thinking is the lever to enable students to achieve a deeper and more insightful understanding of subjects across the curriculum. (p. 7)

In the field of mathematics education, the National Council for Teachers of Mathematics (NCTM) (2006) clearly set out geometry, measurement, and spatial reasoning as key areas for mathematics teaching and learning in K-12 classrooms. The NCTM also advocated for an even greater emphasis of these areas in the early years. Moreover, as geometry naturally employs spatial reasoning, it seems obvious that students' spatial-geometric development be a focus in school mathematics. Clements and Sarama (2011) argued further, asserting that as geometry grounds all mathematics, geometry is what makes spatial reasoning possible in the first place. The authors contended that spatial reasoning and geometry should be the central focus in mathematics classrooms:

> Some mathematicians have claimed that, except for simple calculation, geometric concepts underlie all of the mathematical thought (e.g., Bronowski 1947). Smith (1964) argued that mathematics is a special kind of language through which we communicate ideas that are essentially spatial. (Clements and Sarama 2011, p. 134)

The argument connects well with Tahta's (1980) response to René Thom's claim that all (mathematics) magic, to the extent that it is successful, is geometry. The statement, Tahta pointed out, shifts attention away from the "what" of geometry and focuses on the activities that give rise to the creation and exchange of images. In relation to this, Tahta (1989) identified three overarching "powers" involved in working spatially. These were: (i) *imagining* (seeing what is said); (ii) *construing* (seeing what is drawn and saying what is seen), and (iii) *figuring* (drawing what is seen).

Still, despite these and other convergent perspectives that justify the importance of and need for developing students' spatial reasoning in school mathematics, geometry receives the least attention in North American K-12 classrooms compared to all other areas in mathematics such as number and operations, algebra, probability and statistics, and measurement (Clements and Sarama 2011; Lappan 1999;

Mammana and Villani 1998). Moreover, teaching and learning of geometry in the early years continues to center on children's development of vocabulary through identifying and sorting figures by properties. Little if any time is devoted to engaging children in manipulating, comparing, classifying, composing, decomposing or recomposing two- and three-dimensional figures (Sinclair et al. 2016).

8.2 Aims and Research Focus

This chapter features three grade one children. The intent of the research was to inquire into the actual forms, activities, and processes that constituted the children's spatial-geometric reasonings. In so doing, I aimed to further explicate the body's role in the children's spatial-geometric reasonings, the impact of these on their conceptions as well as how the children brought geometry into being. With new insight gained from the study, I discuss pedagogical and theoretical considerations for enabling and valuing young children's spatial-geometric reasoning. The specific research questions are presented and contextualized in the theoretical framework.

8.3 Literature

Within mathematics education, the focus on spatial reasoning, its relationship to school mathematics and in particular, geometry, continues to grow. For example, in the past four years, several publications devoted to spatial reasoning feature studies by researchers from around the world (e.g., see feature issues in *ZDM 46, 2014 and 47,* 2015; Davis and The Spatial Reasoning Group 2015; Sinclair et al. 2016). Much of the research addresses geometry and also investigates the usefulness of embodied theories to inform teaching and learning. How spatial reasoning and geometry relate to notions of embodiment can be attributed to the fact that just as geometry naturally employs spatial reasoning, the very phrase, "spatial reasoning," implicates the body as essential for such reasoning. In brief: "Spatial knowing, doing, and being as radically embodied is about how bodies engage with the world, regularly sensing and making sense of the situations in which we find ourselves (Thom et al. 2015)."

In the next three sections, I review recent literature that highlights the (em)bodied nature of students' spatial-geometric reasonings as well as the roles that diagrams, gestures, and the relationships between the two play in school mathematics.

8.3.1 Diagrams

Changes in theoretical perspectives, namely those that focus on notions of embodiment, challenge the assumption that diagrams (or drawings) as artifacts serve only as evidence of cognitive development (Bussi 2007). Drawing is a common activity in pre-school and primary classrooms. Often used to support geometric learning and spatial awareness, children's drawings as artifacts are also used to assess what they have learned. In this manner, children's drawings as representations reveal their internal cognitive schema—what they 'know' about geometry, such as their cognitive capabilities, spatial awareness, and conceptual understanding (Carlsen 2009; Davis and Hyun 2005; Goldin 2002; Kaput 1998; MacDonald 2013). Thom and McGarvey (2015) questioned these assumptions by raising the following issues:

> What if the act of drawing serves as a means by which children become aware of geometric concepts and relationships, rather than being viewed as a product of that awareness…. what might we learn about children's geometric thinking if we interpret drawings as a vehicle for thinking and not just an object of reasoning? Also, how do children's mark-making give rise to different ways of thinking geometrically? (p. 466).

In their study, the authors conceived children's mathematical drawings, thinking and meaning as both acts and artifacts that are always and inherently grounded in physiological, social, and cultural contexts. Drawing, then, is both a way of knowing as well as an embodied means of learning. As such, a child's drawing is "not merely a copy nor a perversion, or an expression of a reality; it *is* a multi-faceted reality itself" (Woodward 2012, p. 14). Children's drawings *as* spatial-geometric reasoning arise in the flow and creation of diagrams. Thus, drawing is "a matter of learning as much as it [is] a matter of thinking" (Cain 2010, p. 32).

Moreover, research conducted by Kaur (2015), Leung (2008a, b, 2012), and Ng and Sinclair (2015a, b) demonstrated other ways that children's drawings might be assessed that do not focus on identifying conceptual deficits. The findings of these studies enable an alternative view of children's spatial-geometric awareness and how their reasoning develops; namely, that the meanings students attribute to a given geometric concept are not necessarily indicative of sufficient or a lack of understanding, but a result of their shifts in attention while drawing.

8.3.2 Gestures

Recent studies on gestures emphasize the role of the body as it relates to spatial and geometric thinking. In particular, studies on spatial-geometric reasoning of young learners reveal the spontaneous and deliberate ways that students and teachers use their bodies as semiotic resources to communicate in the classroom.

Bussi and Baccaglini-Frank (2015) conducted a teaching experiment aimed at developing first grade students' conceptual understanding of oblongs and squares as rectangles. Here, the children programmed paths for a robot (i.e., "bee-bot"), enacted the bee-bot's paths, and represented the sequence of commands. The researchers observed that while the children's gestures occurred with and without other signs, their gestures independently communicated concepts and conceptual meanings about angles, sides, turns and directions. Also observed was how the teacher mirrored and utilized the children's "turning" gesture to develop the class' overall conception of rectangles.

In other studies, Elia and colleagues (Elia et al. 2014, 2016) applied McNeill's (1992) gesture classification—iconic, deictic, and metaphoric. The authors used the categories to identify the types of gestures generated by the kindergarten children and the geometric understandings demonstrated during story, sorting, and composite shape tasks. The findings of the research revealed the children's awareness and recognition of two-dimensional shapes, their articulation and comparison of shape attributes, as well as their activity in dimensional deconstruction, translations and rotations.

Ng and Sinclair's (2015a) research regarding the ways that second and third grade students worked in a dynamic geometry environment (DGE) also focused on the students' gestures but how they reasoned about the vertical, horizontal, rotational and oblique (as)symmetries of two-dimensional shapes composed of squares. Similar to the above studies, the researchers found that the students' gestures did not require other signs to communicate the movement of the squares. Additionally, the authors concluded that the children's signs and personal meanings demonstrated their internalization of the teacher's verbal signs and the visual signs from the DGE.

Research conducted by Thom, Roth, and colleagues exposes still other aspects of how the body supports, copes with, co-emerges with, and constitutes the development of geometric knowledge. For example, studies by the researchers evidenced how young children and their teacher produced and used communicative rhythms and patterns of sounds to attend to and articulate specific meanings of properties of three-dimensional objects (Bautista and Roth 2012; Thom et al. 2010). Differently, other studies revealed the role that kinetic movement played in the emergence and development of abstract geometrical understanding in primary classrooms (Bautista et al. 2012; Thom et al. 2015). Further still, Thom and Roth (e.g., 2009, 2011) explicated the different manners in which second grade children used touch and movement to quantify and spatialize attributes of three-dimensional objects; such as faces (e.g., flat hand), edges (e.g., straight finger) and vertices (e.g., fingertip) of different rectangular prisms.

8.3.3 Diagrams, Gestures, and the Body

While there is extensive research on gestures and diagrams separately, there is also increasing interest among educators to explore the interrelation between the two and the implications concerning the role of the body in school mathematics. Generally, current studies highlight the impact of diagrams and gestures on students' mathematical thinking and learning. Research specific to the early years includes how diagrams and gestures contribute to children's spatial and geometric reasoning. In this section, I examine additional literature in light of some of the discussed studies to address critical connections between diagrams, gestures, and geometry, ultimately revealing the primacy of the body.

De Freitas and Sinclair (2012) applied mathematician and philosopher, Châtelet's (1993/2000) notion that inventive reasoning requires and supports the co-emergence of diagrams and gestures by exploring how this occurs with adult learners. The researchers conceived the students' production of diagrams as creative and embodied acts that enabled new relationships for engaging in the mathematics and the environment. Consequently, similar claims can be made about the primary students in Ng and Sinclair's (2015a) study that examined the children's gestures, their diagrams, and the discussions that accompanied them. The students' explanations involved their invented use of arrows that illustrated the symmetric movement of the squares and hand gestures that communicated the same actions.

Also evident in the Bussi and Baccaglini-Frank (2015) investigation is the student teacher and student interaction where both the child's gesture and drawing (as well as oral words) illustrated the bee-bot's turn as a right angle. These instances not only demonstrate Châtelet's (1993/2000) contention that gestures and diagrams occasion each other, but the two together, engender new reasoning. In his elaboration on this latter point, Châtelet argued that gestures and diagrams should not be taken as simply representational but also as potential sources for further inquiry and discovery in mathematics.

Drawing on previous research by Duval (1995), Herbst (2004), O'Connor (1998), and Sfard (2001), Chen and Herbst (2013) compared the work of high school students who were given either a fully labeled or unlabeled geometric diagram. The researchers found that the students with the labeled diagram employed only pointing gestures whereas those given the unlabeled diagram extended their thinking by using gestures to lengthen the lines, create intersections, and angles. The students also generated new conjectures about objects and properties that then enabled new spatial-geometric knowledge.

Sinclair et al. (2012) observed similar findings in their study of grade one students. The researchers presented the class with a digital diagram of two lines that could be dragged on the screen. The students produced hand and arm gestures to conjecture where the point of intersection would occur by extending the lines off the screen. The students also used a finger and thumb gesture to mark the distance between the lines. Although the two studies above differ in the kinds of diagrams offered to the learners; that is, static and labeled, static and unlabeled, and dynamic diagrams, both investigations emphasize the same point: Critical to the development of students' geometrical reasoning, it is necessary that learners comprehend diagrams as generative and open rather than definitive and closed. Doing so affords students opportunities to engage with the diagrams, make reasoned conjectures (Herbst 2004), and create new knowledge.

Lastly, given the four different cases examined by Thom and McGarvey (2015), the authors questioned the pedagogical usefulness of separating diagrams from gestures or the two from the body writ large. The pre-school and primary students who explore concepts including number, vectors, and zero-dimensional (0D) to three-dimensional (3D) objects showed how their mathematical experiments in reasoning as well as their new conceptions continuously emerged and evolved *as* the very acts and artifacts of drawing. Thus, it was the children's simultaneous embodiment of mathematical knowledge and their bodying forth of spatial-geometric ideas and reasoning that were the marks on the paper as well as the gestures, sounds, movements and verbalizations. Drawing(s), gestures, and the body transcend one another, and in turn, co-generate new visual, aural, tactile, and kinetic activity with which geometric experience and conceptual insight can be explored, invented, and expanded.

8.4 Theoretical Framework: Embodied Cognition, Mathematics and Drawing

In this chapter, I explore how bodies of knowers (i.e., students April, Emma and Sophia and research assistant Mr. James) and bodies of knowledge (i.e., spatial and geometric) co-evolved immanently as emergent phenomena. This research draws on embodiment theories in cognitive science (e.g., Maturana and Varela 1991) and philosophy (e.g., Châtelet 1993/2000; Merleau Ponty 1945; Merleau-Ponty and Lefort 1968; Henry 2009) where cognition is assumed to depend "upon the experiences that come from having a body with various sensorimotor capacities" (Varela et al. 1991, p. 173). As geometry necessitates spatial reasoning and spatial reasoning engages perceptual-motor capacities, it is impossible to ignore the body and the role it plays in school mathematics, teaching and learning.

This point is not trivial; rather, it may appear insignificant only because our biological, social and cultural ways of knowing are so entangled in the historicity of everyday living that we take for granted the fact that, "the only mathematics we can know is the mathematics that our bodies and brains allow us to know" (Lakoff and Núñez 2000, p. 346). For example, consider the mathematical terms rotate, translate, and reflect. These seemingly abstract yet commonly used words are anything but disembodied symbols when viewed from a neuroscience perspective (Pulvermüller 2012). Inherently bodily, these terms like other language are, "'woven into action' at the level of the brain" (p. 423). In other words, the relationship between words and the objects to which they refer occurs as coordinated neuronal activity. The conceptualization of language emerges within concrete contexts and as such, later use of words even in the absence of their original contexts are shown to activate the motor and premotor areas in the cortex that first gave rise to the actions. In light of this, it is difficult if not impossible to ignore the significance of whole body sensations like turning, sliding, and flipping in making sense of and engendering dynamic meaning for the terms rotate, translate, and reflect.

Importantly, making sense through exploiting the senses is more than simply experiential; it is experience in*formed* by perception and action that is sensually and neurologically linked. The inseparability of perception and the body locates the body as the very site and means from which all knowing arises. Therefore, if perception is integral to cognition then perception is also vital for action-in-the-world (Merleau-Ponty and Lefort 1968). What exactly we come to *real*ize and that to which we choose to attend is contingent on how our biological bodies move through the world as well as the historically and culturally manners in which our experiences are specified (Maturana and Varela 1991; Thompson 2010; Vygotsky and Luria 1993). Action is neither a reflection nor a *re*presentation of knowing; rather, action is constitutive of knowing—full stop. Knowing, doing and being cannot be parsed out as separate phenomena but must be conceived radically as cognition itself. Further still, there can be no internal or external feature of knowing because knowing always and continuously reveals itself *as* action-in-the-world or, life (Henry 2009).

Related to these perspectives is Châtelet's (1993/2000) argument that gestures and diagrams are dynamic interrelated embodiments of mathematical understanding and inventiveness. In contrast to diagrams as static objects, gestures as distinct, and the two as *re*presentations of knowledge, diagrams are viewed as capturing "gestures in mid-flight… the moments when *being* is glimpsed smiling… the accomplices of poetic metaphor… [that] leap out in order to create spaces" (italics added, p. ix). Châtelet also contended that diagrams unlike metaphors, are never exhaustive or wear out; instead, they persist openly and generatively.

In this chapter, I examined how the 'drawing' bodies of young children continuously "(re)(con)figure space" (Châtelet 1993/2000). My inquiry focused on detailing the ways that children's bodies think and "explore [a] world in its own right instead of … simply… representing an outer world" (Cain 2010; Schneckloth 2008; Woodward 2012). More specifically, I asked, what are the manners in which young children bring geometry into being? And, how is it that the children's bodies as knowers and their bodies of knowledge occasion the exploration and expansion of their reasoning in spatial-geometric ways?

8.5 Method

8.5.1 Data

The three episodes in this chapter are part of a larger project that documented 19 mathematics sessions over 3 months. Here, a mainstream kindergarten and grade one class (N = 19; 11 boys, 8 girls) learned about geometry and spatial reasoning through partner, small-group and whole-class explorations. The activities were co-designed and co-taught by the classroom teacher, the Teacher of the Deaf/Hard of Hearing (TODHH), the two research assistants, and myself. A variety of contexts (e.g., stories, construction, outdoor spaces, and imaginary) and modes in which the children chose to work (i.e., drawing/recording, verbalizing, physical, or manipulative) were integrated into the lessons. All events during the study were recorded using two digital video cameras. Electronic pens and paper captured the children's drawing(s) and conversations. As well, class artifacts, drawings, field notes including physical, verbal, and gestural actions, transcripts of the video data, and photographs were collected.

8.5.2 Process

My discussion of the findings addresses key events as three grade one children worked with research assistant, Mr. James. Specifically, I examined the actual forms, activities and processes which comprised the children's geometry and reasoning during the session. Assuming that mathematical thinking is observable as socially and culturally structured bodily activity (Nemirovsky and Ferrara 2009), all forms of the data were in a similar way available for analysis. The analysis involved constantly (re)viewing and comparing the multi-layered data of moment-to-moment events as well as verifying conjectures against related theories on embodied cognition (Edwards 2009).

In the next section, I provide a narrative account of the conversations, movements, gestures, and drawing(s) by the children in the first part of one of the sessions. Included with this, is an analysis of critical events as they happened in each of the episodes. Specific attention was paid to the spatial-geometric conceptualizations of Emma (6 years), Sophia (7 years) and April (6 years) as they examined a photograph and worked to articulate what they saw and what the object might have been by imagining it in different dimensions and from multiple perspectives. What is clear from the analysis is the fully (em)bodied nature of the children's spatial-geometric awareness and reasonings as well the inventive ways that they brought geometry into being.

8.6 Results: Findings and Discussion of the Children's Work

Mr. James asks the children to look at the photograph (see Fig. 8.1).

In unison, Emma, Sophia and April say, "circle!" April traces around the outline with her finger while Emma and Sophia move their fingers in circles in the air. They then each draw a circle on separate pieces of paper (see Fig. 8.2)[1]:

It is important to notice that none of the children simply verbally identify the image as a circle. Rather, the tracing and drawing motions first with their fingers in

[1]Throughout the larger research project from which these episodes are taken, there was no particular mode that the children were expected to demonstrate their thinking. However, making mathematical ideas and thinking available to others and for the class' further exploration was certainly modeled, discussed, and encouraged in all lessons. As such, a variety of materials were on hand for students to use if they wished.

Fig. 8.1 Digital photograph presented to Emma, Sophia and April

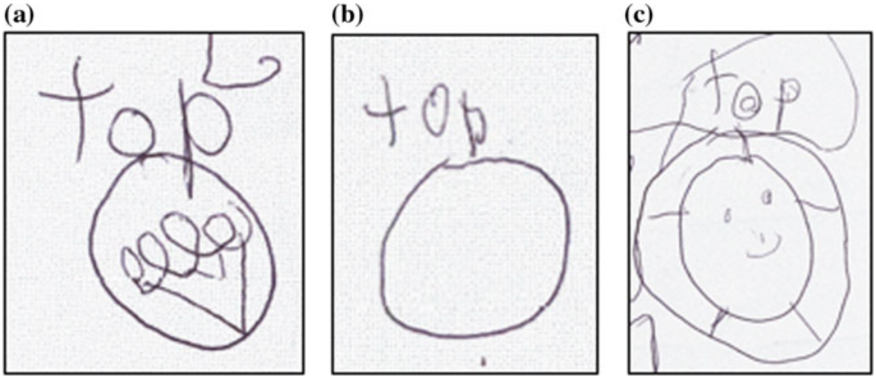


Fig. 8.2 Circle drawings by Emma (**a**), Sophia (**b**), and April (**c**)

the air and then with pen on paper that follow their verbalizations, bring forth the idea of a circle and its further meaning. Here, the children's articulation of the circle as a continuous curved line that is joined end to end emerges from and as whole body events (Châtelet 1993/2000).

Sophia:	*[Drawing a circle]* This is a circle. It's round, no corners.
Mr. James:	It's round, no corners. How many faces?
Sophia:	*[Looks at Mr. James and holds up one finger]* One. *[Pauses. Looks away and then back again at Mr. James]* Two *[holds up two fingers]*.
Mr. James:	Where's the other face?
Sophia:	On the… *[pauses]* if you have the block (see Fig. 8.3) then there are two *[holds hands flat, apart, and parallel to each other and then shakes her hands twice up and down]*.
Mr. James:	Oh…! So on the opposite side *[gestures with his hand]*.
Olivia:	Yes *[smiles at Mr. James]*.

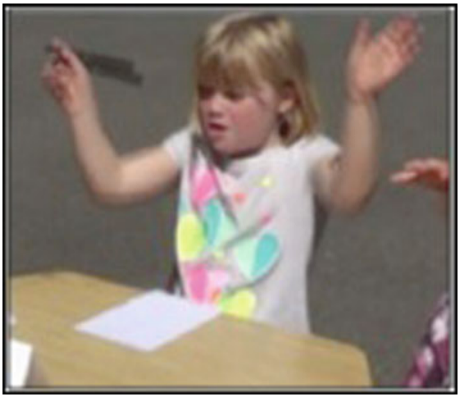


Fig. 8.3 Sophia locates two circular faces at opposite ends of the block

In this episode, Sophia further identifies "it"—a circle as "round, no corners". And in the next moments when she pauses, looks away, and then back again at Mr. James, what she initially expressed as "it" with one circular face, transforms into two circular faces. With fingers together and flat upright hands, Sophia holds them parallel to and facing each other, shaking them in an up and down motion twice. While the gestures and motion locate two circular faces, they also express a new orientation and dimensional space in which the circles exist; earlier, the circle was in front of her or viewed from above. Now "it" is two circular faces, one at each end of a 3D object.

Mr. James then asks the group:

Mr. James: If this is the front *[pointing to the photographed image]*, what would the back look like?
Emma: Same.
Mr. James: Same?
Emma: Because this is what the side would look like…

While Emma says this, she holds up her left hand and points in the air with her index finger. She looks at her hand and pauses. She then closes her eyes, makes a fist, and holds it at a slight angle in a vertical position (see Fig. 8.4). With eyes still closed, Emma pauses again, unclenches her fist and makes a curved shape with her thumb and index finger. She holds her fingers in this new position and moves her hand diagonally downward through the air (see Fig. 8.5a). Emma opens her eyes and looks at the group as she completes the motion and comes to a stop (see Fig. 8.5b–c). Taking the pen in her hand, she draws the side view of the block on the piece of paper (see Fig. 8.6).

Emma's three different hand shapes and the diagonal downward movement demonstrate the work she accomplishes to make new meaning of the prediction. The first gesture of pointing a finger in the air compared with the second gesture of a clenched fist clearly shows a shift from simply indicating or referencing an object to physically creating an object that closely resembles a cylinder. Further,

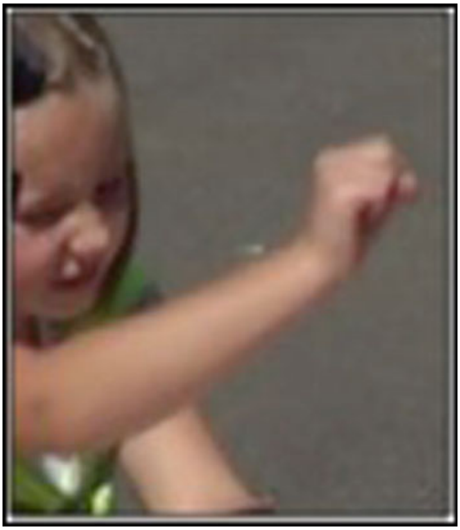


Fig. 8.4 Emma's second hand gesture

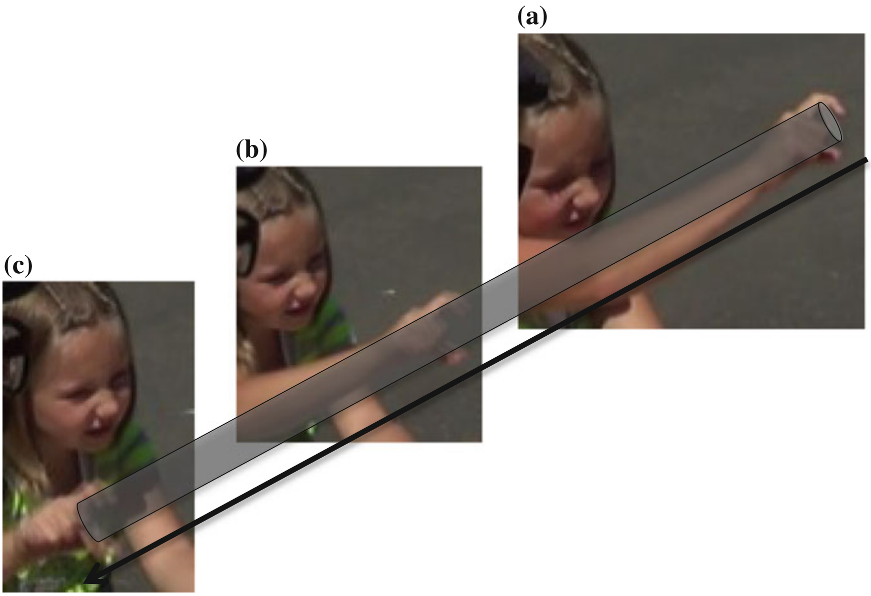


Fig. 8.5 **a–c** Emma's third hand gesture and accompanying motion

when Emma transforms the closed fist into what appears to be a circle/ellipse moving along a straight path, not only does she explicate why the front and back faces of the object will be the same shape, but also, how a curved surface is formed

Fig. 8.6 Emma's 'side' drawing

between the beginning and end of the path of the circle/ellipse. In these ways, the resulting cylinder gives rise to new and important temporal, spatial, and geometric properties that extend beyond generating and justifying the conjecture.

Similarly, just as the children's verbalizations, gestures, and movements emerge moment to moment as spatial-geometric concept(ion)s and reasonings, so too do their drawings (Thom and McGarvey 2015). For example, of her own accord, Emma changes focus from the front and back faces of the object to the side of the object. Then, as she draws the side of the block, new meanings of the cylinder arise.[2] Here, orienting the object horizontally and viewing it from the side enables the surface of the object between the two circles to become a flat oblong (Duval 2014).

While Emma prints "saiDe" (sic) next to the image, Sophia shows Mr. James her drawing (see Fig. 8.7). Differently from her previous 3D object, she attends to the space between the two circular faces. Sophia maintains the horizontal position, but this time, draws an oblong to be the side of the object.

April also focuses on the side of the block and announces that she is drawing a semi-circle.

Mr. James: Why?

April: Because it has a semi-circle on the side…. This is where this one starts and this is where it ends *[traces "this" as a semi-circle with the pen]*. And this is where this one starts and this is where this one ends *[traces "this" as a second semi-circle with the pen]* (see Fig. 8.8)

April's reenactment of her drawing evidences how her spatial-geometric reasoning changed from conceptualizing the 2D "top" of the cylinder as a circle—seen from above—to drawing the two semicircles that wrap around the top and form the side of the block.

Mr. James then asks what the entire block might be if he picked it up and gave it to them. Without any talk, the three children draw objects that look very similar (see Fig. 8.9a–c).

[2]This shift in attention carried through to the next drawing(s) and conversations with Sophia and April where the students, also on their own, chose to focus on the side of the object.

Fig. 8.7 Sophia's 'side' drawing

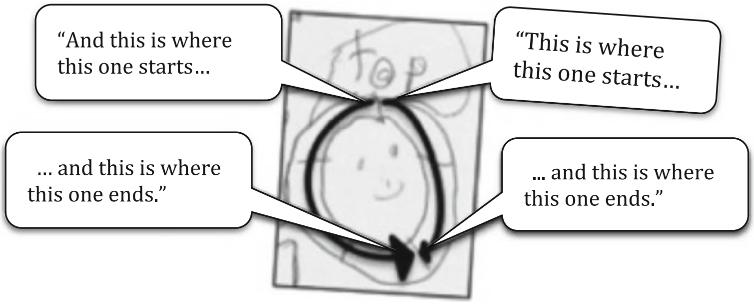


Fig. 8.8 April's "semi-circle" sides drawing, reenactment and explanation

Although alike in appearance, key differences are observed by examining how each child attends to and accomplishes the task. Additional distinctions can be made regarding Emma's mark making of her 'side' drawing (see Fig. 8.6) and the current one which appears to be one and the same. With the first cylinder, she quickly draws the circles/ellipses only then to slow down and carefully make two parallel lines from one circle/ellipse to the other. The change in speed with which Emma draws suggests her particular attention to this part of the three-dimensional object (see Fig. 8.11a). Both the focus and care that Emma demonstrates could well be an extension of the parallel lines she previously made visible and tangible as she moved her curved thumb and finger in the diagonal downward motion (see Fig. 8.10).

Emma later draws the circles/ellipses and parallel lines of the second cylinder with the same speed and fluidity. And in the moments of drawing the circles, she also speaks in a singsong voice, lilting, "two…" as she forms the first circle and in the same tone says, "circles…" as she makes the second circle. This is followed by a sound that Emma spontaneously invents while she draws the two parallel lines. Here, she voices, "doo—oo—oo—ip" each time she draws one of the lines in a left

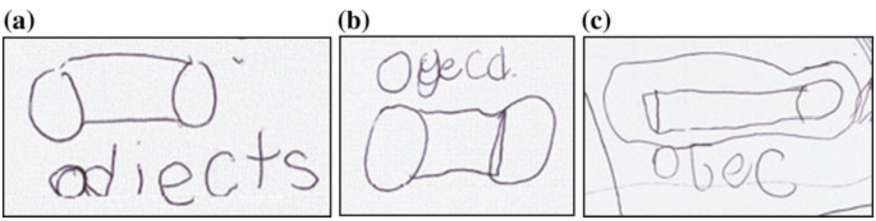


Fig. 8.9 Emma (**a**), Sophia (**b**), and April's (**c**) drawn objects

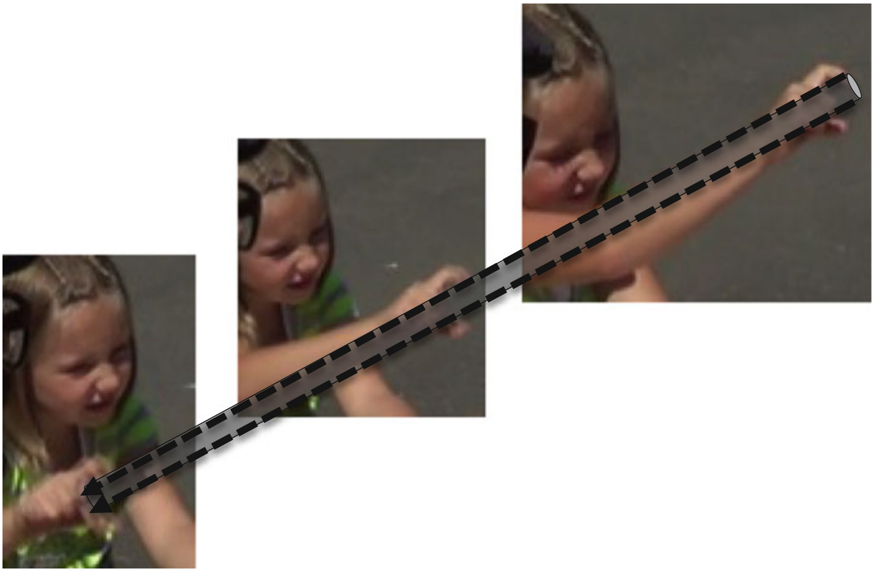


Fig. 8.10 The parallel lines Emma makes visible and tangible

to right direction. Emma's intentional and creative work not only elucidates the composition and paired attributes of the two-dimensional cylinder as visual, verbal, tactile, and dynamic forms, but also sonorously as rhythm and sound (see Fig. 8.11b).

Sophia's third drawing contrasts with her previous one in which she created an oblong (see Fig. 8.7). This time, she draws an oblong with two circles. Working in a left to right manner, she makes one circle first, then the oblong, and the other circle (see Fig. 8.12).

It is also interesting to compare how April drew the previous object (see Fig. 8.8) and how she draws it now. No longer drawing from the perspective of looking down at a cylinder standing on its circular base, April draws from a side view of the object as it appears horizontally (see Fig. 8.13a). She draws a curved line on the right side of the paper (see Fig. 8.13b) and slowly lengthens the two

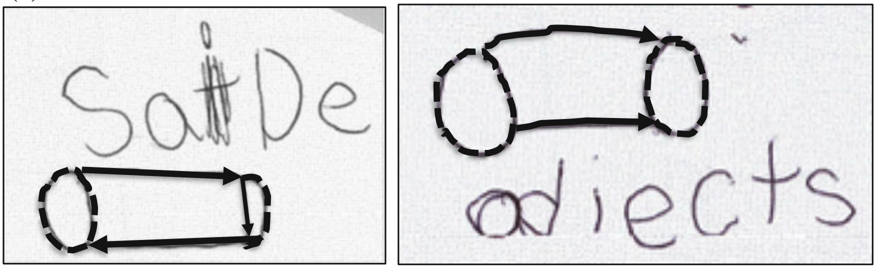


Fig. 8.11 **a**–**b** Emma's second and third conjectures of what the side of the object and the whole object might look like

Fig. 8.12 Sophia's third conjecture of what the whole object looks like

ends leftward, keeping the lines parallel to each other (see Fig. 8.13c). She then joins the ends of the line by quickly making two short vertical strokes that serve to mark the two edges that separate the three surfaces of the cylinder. Here, April's drawing shows how the two-dimensional cylinder originates from its circular base and eventually becomes what she describes next as "one of those stretchy l-o-n-g circles." Mr. James points to April's drawing (see Fig. 8.9c) and asks what its name might be.

April: *[Shrugs her shoulders]* I forget. *[with hands and arms aligned with center of the body, she moves them in an outward motion]* (see Fig. 8.14) … but it's one of those stretchy l-o-n-g circles

April proceeds to wave her arms outwards and inwards several times in the air (see Fig. 8.15). She then sits still. Following this, she curls her fingers into the palms of her hands, positioning them out in front, beside each other, and thumbs facing inward. She then moves her hands three times in a continuous, horizontal, center-out motion (see Fig. 8.16).

The way in which April works in response to Mr. James' question reveals new spatial, geometric, and temporal meanings as well as the evolution of these as she creates each of the cylinders. In the first part of the excerpt (see Fig. 8.14), the concept of "stretchy" reappears. However, in contrast to the cylinder she draws that forms lengthwise and two-dimensionally on paper from its circular base, the new object that comes into being is not a flat cylinder but a three-dimensional one

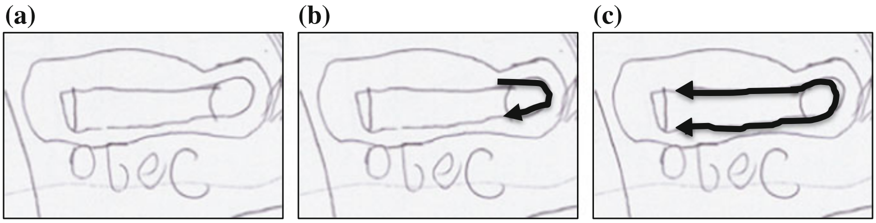


Fig. 8.13 **a–c** April's conjecture of the whole object, the first part of her drawing, and the second part of her drawing

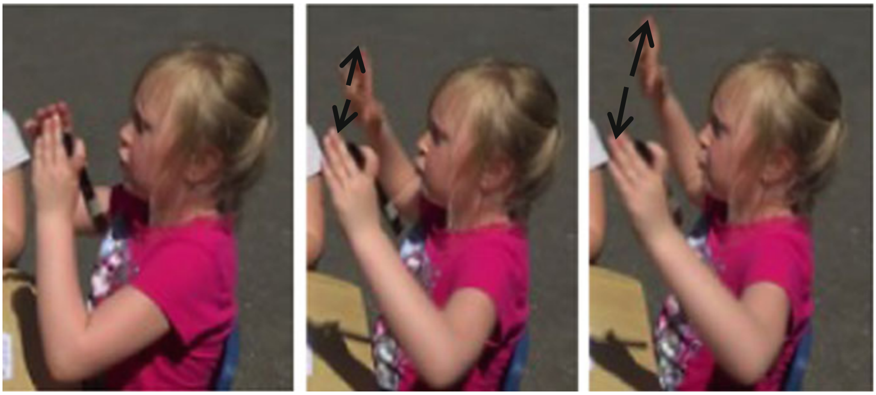


Fig. 8.14 April's first gesture and movement that creates, "one of those stretchy l-o-n-g circles"

Fig. 8.15 April's second gesture and movement that "erases" the first 3D object

positioned between two hands that face each other. As the hands move apart and April slowly and deliberately says, "long" as "l—o—n—g", a cylinder emerges and continues to grow in length as it extends outward from the center.

And, when April waves her hands and arms in this 3D space and then sits still, her actions appear to 'erase' or 'clear' the 3D space in which she works, signaling a shift in thinking (see Fig. 8.15). What is observed next is the new meanings she

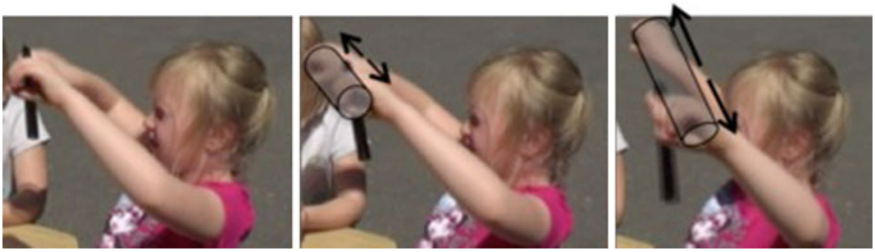


Fig. 8.16 April's third gesture and movement that creates a second yet different "one of those stretchy l-o-n-g circles"

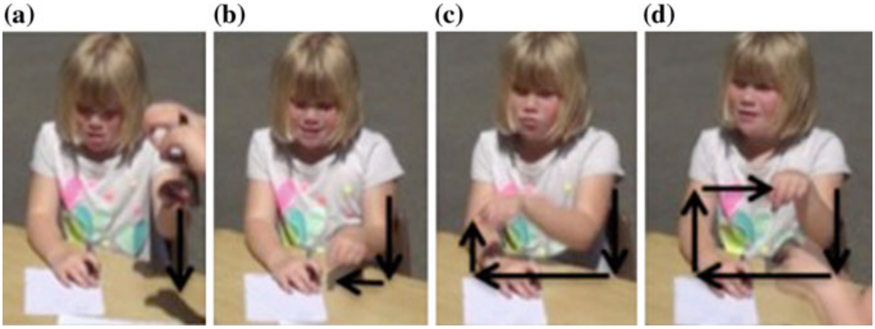


Fig. 8.17 a–d Sophia draws a rectangle in the air with her finger

makes as she creates the next cylinder. This cylinder, like the previous one, is positioned horizontally and extends in length like "one of those stretchy l-o-n-g circles". What differs from this and previous cylinders, however, is the way April's curled fingers create a circle from which she can then physically articulate the shape as a closed curve/disc. The new shape with the subsequent motion of her hands effectively adds another dimension to the circle by repeatedly stretching it from the center-out in both directions to form a cylinder (see Fig. 8.16).

During this part of the episode, Sophia spends the entire time looking at her drawing(s) (see Figs. 8.2b, 8.7 and 8.9b). Slowly and deliberately, she addresses the group:

Sophia: It has *[looking at her drawing(s)]* a rectangle *[draws in the air with her finger]* (see Fig. 8.17a–d) and then two circles—at the bottom (see Fig. 8.18a) and top (see Fig. 8.18b) *[moves flat hand in circular motion, each circle diagonally opposite from each other]*

What Sophia draws is unlike any of the group's previous work. During the moments when she re-views the drawings, engages in talking, and invents three new gestures-in-motion, she transforms "it" yet again into a completely different object. The time that Sophia spends on studying the drawings along with the

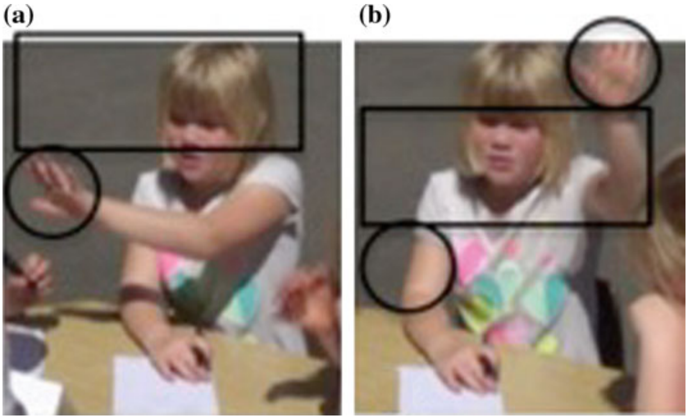


Fig. 8.18 **a–b** Sophia creates "[t]wo circles—at the bottom and the top"

unhurried manner in which she moves her gaze from the paper to the space in front of her, not only suggests her intention to conceptualize the cylinder in a new way but also the effort required to achieve this task. Sophia works slowly and methodically on what appears to be a virtual vertical plane. Eventually, she invents a net that if folded and rolled together, forms a three-dimensional cylinder. Her conception of the cylinder's net begins as an oblong that later integrates two circles. She creates the oblong by using her finger to draw it in the air. Facing Sophia, the first circle appears along the side of the oblong's bottom left corner while the second circle is placed diagonally opposite—along the outer side of the top right corner. In each of the two locations, she holds her left hand out, open, and flat, moving it in the same circular motion. The action not only produces two equivalent circles, it also allows for spatializing at least two important geometric properties: first, the space covered by Sophia's *flat* hand becomes the interior of each figure and second, the boundary that results from the points that her hand's outer edge bumps against as she *rotates* it *around*, effectively becomes the circumference of the circles.

8.7 Conclusion

This study contributes to current research on spatiality by further elucidating the critical role that the body plays in young children's spatial-geometric reasoning. The activities, processes, and products that constituted Emma, Sophia and April's classroom geometry revealed up close, how their awareness and thinking arose fully (em)bodied "in the flesh" as gestures, movements, rhythms, patterns, drawing(s), sounds, imagery, and verbalizations (Henry 2000). While the children certainly developed verbal and visual meanings for the concepts that emerged and the objects they created, it is impossible not to notice the tactile, temporal, and audible features that expressed a characteristic feel, a distinct motion, or a particular sound that was integral to their geometries. It is here at a phenomenological level that the children's bodies as knowers and of knowledge were observed as constantly co-emerging and co-evolving—not only within the contexts of a photograph, drawing(s), and conversations, but importantly and seemingly, out of thin air!

The fluid manner in which the children's spatial-geometric concept(ion)s and reasonings developed continuously occasioned: perspective-taking, orienting, visualizing, locating, comparing, decomposing, recomposing, symmetrizing, rotating, multiple dimensions, static and dynamic objects as well as curved and flat surfaces. As with Châtelet (1993/2000) and Thom and McGarvey (2015), it is clear that Emma's, Sophia's, and April's work in (re)(con)figuring space cannot be conceived as simply the outcome of their reasoning, but more precisely, *as* their geometric awareness and the extending of their conceptual thinking, ideas, and meanings.

The children's work, clearly effortful, while at the same time, unpredictable yet recursive, must also be viewed as inherently creative in both the forms of reasoning and the geometries they (em)bodied forth (Châtelet 1993/2000; Merleau-Ponty and Lefort 1968). Recognizing these critical aspects as well as those observed and discussed in the study offer untapped areas to explore how spatial-geometric reasoning might live out in rich ways in mathematics classrooms.

8.8 Implications

While spatial reasoning is integral to all areas of mathematics, it is consequential to school geometry. In the final section of the chapter, I revisit particular issues identified in the reviewed literature, reflect on these in relation to the findings of the study, and suggest further pedagogical and theoretical considerations for enabling classroom contexts that provoke young children's spatial-geometric reasoning.

8.8.1 Theoretical Perspectives

In the 13th International Congress of Mathematics Education (ICME) survey team report, *Recent Research on Geometry Education*, the authors postulated that future research "may well require the use of theoretical frameworks that are capable of integrating discursive and embodied components" (Sinclair et al. 2016, p. 734). While the position taken by the team addressed the research community, another conversation arises about the need for recognition and enaction of such theoretical perspectives in the very places where early years geometry teaching and learning happen. As described in the first section of the chapter, despite decades of research including studies that expose the (em)bodiment of geometry, geometry continues to receive little time in the classroom and is taught and learned in manners that not only reduce the potential for developing children's spatial-geometric thinking, but also limit opportunity for their deep and connected conceptual development across all areas of mathematics. Moreover, as classroom mathematics remain confined in cognitive perspectives that emphasize abstract mental operations, the enaction of embodied theories would arguably make it impossible to ignore the (em)bodied nature of knowing as well as the ways in which children come to know what they know spatially and geometrically in the classroom.

Examining Emma, Sophia and April's work from an (em)bodied perspective, the children's spatial-geometric reasonings are anything but disembodied or abstract representations. Rather, what these children demonstrate is not only the emergence and growth of significant spatial and geometric concepts but also the observable aspects that constitute their thinking, moment to moment. Emma, Sophia, and April's ongoing (em)bodiment of mathematics; that is, very physical, rhythmic, patterned, audible, verbal, gestural, and kinetic spatial-geometric thinking happens within and as language woven into action (Pulvermüller 2012). In this way, the children's thinking persists irreducibly as action-in-the-world (Merleau-Ponty and Lefort 1968) where complex and sophisticated transformations of the circle arise as they are recursively *real*ized. Such reasonings are not only available to the student who expresses these, but importantly, to all of those for whom they are made available. Attention to these aspects of thinking presents opportunity for them to be used, by teacher and students, in ways that expand and occasion new geometric experiences in the classroom.

8.8.2 Interacting and Transacting with Diagrams

Chen and Herbst's (2013) study highlights the value in extending high school students' reasoning beyond making distinctions and generalizations about diagrams to interacting with diagrams through "reasoned conjectures". This approach to spatial-geometric reasoning complements other approaches such as "restorative" activities where students complete a partial image of an assumed figure through various transformations (e.g., Perrin-Glorian et al. 2013). However, tasks that require students to make reasoned conjectures, promote opportunities for the invention of objects as well as new ways of thinking about them.

Similar benefits were also gained by grade one students Emma, April, and Sophia as they generated conjectures and inquiries about the photograph, their drawing(s), and the objects they created. The children's transactions[3] played a critical role in moving their thinking beyond the identification of the image in the photograph as circle and occasioning their further spatial-geometric reasonings and

[3]In contrast to the concept of interaction, transaction implies the irreducibility of the children's agency as well as the material and spatial-geometrical structures that emerge.

conceptualizations. It is also important to note that Mr. James did not begin the session by asking the children to look at the photograph and answer, "what is this?" Rather, the conversation that developed from the prompt, "what do you see?" deliberately opened a conceptual space for a variety of spatial-geometric responses, depending on the manners and contexts in which these occurred. The questions, for instance, that followed the first prompt in the study can be viewed as provoking the children to recursively generate conjectures and reasonings about other surfaces that might exist if the image in the photograph was a 3D object. It is curious that none of the children commented on the presence of the rectangular shadow in the photograph (see Fig. 8.1). As a result, it is unclear whether the shadow contributed to the transformations of the circle into only cylinders. However, later on in the session, Emma, April and Sophia posed conjectures that eventuated more than one 3D object from two other photographed images which did not include shadows that the children distinguished as triangle and square.

8.8.3 The Potential of Mathematical Diagrams and Objects

It is natural to assume that digital technologies, especially those that involve dynamic geometry environments (DGE), maximize opportunities for students to develop their spatial and geometric thinking (e.g., Baccaglini-Frank 2016; Fletcher and Ginsburg 2016; Sinclair 2018). However, as discussed in the previous section, spatial-geometric reasoning that involves diagrams also depends on *how* students and teachers transact with diagrams, regardless of whether these are digital, physical, or virtual. Therefore, it is not so much the particular form of a diagram that is critical as it is its *potential* to occasion ideas, meanings, and ways of reasoning that are dynamic or "elastic" (Châtelet 1993/2000). Châtelet asserted that attention should not be on diagrams (or, objects) themselves, "but the register they provide for asking how we discover geometric space and how such space in turn becomes used as space for thinking." (K.J. Knoespel, as quoted in Châtelet 1993/2000, p. xxi).

The work of Emma, April, and Sophia exemplifies this point well. Here, the photograph as a visual representation appeared at the start of the lesson to be a singular static image. However, as soon as the children explored the possibilities of the object as being more than a circle, all sorts of concept(ion)s and spatial-geometric ways of making meaning of the figure emerged. The geometric contexts in which the children and Mr. James found themselves can be seen as specifically shaped or (re)(con)figured by the ideas and meanings they brought forth; in other words, a direct result of the children's work to recursively articulate what it was that they saw moment to moment and why they thought this might be so. For example, as soon as the children identified the image as "circle", they moved on to transform it into a single point in space where each of their fingers moved around in a circular path, returning the point to its original position. This geometric space gave rise to connected yet diverse contexts in which semi-circles and circles extended into a third dimension, growing out in different directions to become cylinders. The children and Mr. James then used the cylinders to further inquire into the effects of different orientations, imagine the cylinder in two- and three-dimensions differently, deconstruct the cylinder, and flatten it into a net.

In these ways, the photograph as well as the children's drawing(s) and the objects they created, continuously opened spaces for them to develop their spatial-geometric reasonings about the circle(s), constantly shift their attention and necessitate different concept(ion)s to make new meaning. Remarkably, the photograph, drawing(s), and objects ceased to be still, fixed, or spent. Rather than focusing on a particular image or object, the children and Mr. James explored the potential of each where:

> Potential is what, in motion, allows the knotting together of an 'already' and a 'not yet"; it gives some reserve to the act, it is what ensures that act does not exhaust motion and, in giving some scope to the grasping of the motion… as an indefinitely suspended actualization". (Châtelet 1993/2000, p. 19).

8.8.4 Sensation and Abstraction: Spatial-Geometric Sense and Reasoning

In the first chapter of the book, *Spatial Reasoning in the Early Years*, Whitely et al. (2015) elaborated on Tahta's (1980) three powers (i.e., imagining, construing, and figuring). The authors presented a list of dynamic processes that characterize spatial reasoning but do not require concurrent work with language. Examples of these processes included: navigating, locating, orienting, and balancing. In the final chapter, Davis et al. (2015) reimagined the list as circular in structure to express spatial reasoning as inherently emergent, non-sequential and complex. Reflecting on the authors' revised representation and characteristic processes of spatial reasoning, not only am I reminded of Tahta's (1980) assertion: "To think mathematically is to work in some such way with images" (p. 6), but also the point he related to it; that is, imagery is not only visual constructs but includes images that may well only be *felt*.

In connection to this, Artist Kandinsky (1926/1979) identified the complementary nature between events which can be seen and those which can only be felt in his colour theory. The former he described as abstraction that involves phenomena that can be externalized, made visible, and logical. Take for instance, the *green colour* of the trees as an abstraction. The colour can be represented and recognized in an external way such as a swatch of fabric, a paint stroke on a canvas, or even as it appears here in written text as "green". In contrast, the latter—those events that can only be felt, involve sensation or affect. Originating from the Latin word *impressionem*, meaning, "pressing into," sensation entails those invisible aspects of phenomena that give rise to how the colour green makes me feel. Sensation, unlike abstraction, is conceived as a dynamic force that only occurs in life (Zordan 2013).

Further, just as language is woven in action and originates from contexts that are sensually experienced and neurologically connected (Pulvermüller 2012), abstraction too then can be understood as live(d) experience that first emerged from sensation (Thompson 1995). Complementary and interrelated rather than disparate, abstraction and sensation are inherently coincidental (Henry 2009).

Moreover, philosopher Michel Henry (2009) extended Kandinsky's conception of sensation as affect, locating it within what he referred to as the Internal. Henry explained why affect as felt phenomena cannot be seen, represented, or externalized:

> To prove itself is to show itself, but in its own way and no longer that of the world…. There is no inner world. The Internal is not the fold turned inward of a first Outside. In the Internal, there is no putting at a distance and no putting into a world—there is nothing external, because there is no exteriority in it. (p. 7)

It is in this way that the affect happens in life and why it is impossible for us to abstract it or hold it at a distance because we too "are always and already in life" (Henry 1999, p. 364). And despite the fact that the affect cannot be abstracted, its presence, integrity, and complexity are neither compromised nor negated. Rather, Henry argued, it is only the linguistic manners with which we attempt to convey affectivity that prove inadequate.

Consequently, as I watched Emma's hand gestures and movements project a circle through the air, I could not help but wonder what spatial-geometric instances, were felt but not seen or heard (Merleau-Ponty and Lefort 1968). What kinds of sensations and sense had the children and Mr. James experienced while Emma closed her eyes, formed a fist and then a "c" shape as she moved it in a diagonal and downward motion? Or when April drew the side of block as semi-circles on the piece paper? What other spatial-geometric sense in*formed* the cylinder that Sophia created as she drew a rectangle in the air and moved her hand in a clockwise motion to make two circles? To be certain, the fact that such images cannot be pulled from their contexts or captured with words or even a gesture does not mean that they could not exist or might not have contributed to the children's spatial-geometric reasoning in vital ways (Luria 1968; Sfard 1994); rather, it only means that the presence of sensation or affect did not exist as some*thing*. Here begins a new conversation—one that provokes inquiry into young children's (re)(con)figuring of space as both abstraction *and* sensation as well as the ways in which these give rise to their geometric reasoning *and* sense.

References

Baccaglini-Frank, A. E. (2016, July). *Educational multi-touch applications, number sense, and the homogenizing role of the educator*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Bautista, A., & Roth, W. M. (2012). Conceptualizing sound as a form of incarnate mathematical consciousness. *Educational Studies in Mathematics, 79*(1), 41–59.

Bautista, A., Roth, W. M., & Thom, J. S. (2012). Knowing, insight learning, and the integrity of kinetic movement. *Interchange, 42*(4), 363–388.
Bronowski, J. (1947). Atomic bombardment and its effects in relation to fire engineering. *The British Chemical Digest, 2*(2), 84–88.
Bussi, M. G. B. (2007). Semiotic mediation: Fragments from a classroom experiment on the coordination of spatial perspectives. *ZDM Mathematics Education, 39*(1–2), 63–71.
Bussi, M. G. B., & Baccaglini-Frank, A. (2015). Geometry in early years: Sowing the seeds towards a mathematical definition of squares and rectangles. *ZDM Mathematics Education, 47*(3), 391–405.
Cain, P. (2010). *Drawing: the enactive evolution of the practitioner*. Chicago, ILL: University of Chicago Press.
Carlsen, M. (2009). Reasoning with paper and pencil: The role of inscriptions in student learning of geometric series. *Mathematics Education Research Journal, 21*(1), 54–84.
Châtelet, G. (1993/2000). *Les enjeux du mobile*. Paris: Seuil. (Engl. transl., by R. Shore & M. Zagha: *Figuring space: Philosophy, mathematics and physics*). Dordrecht: Kluwer Academy Press.
Chen, C. L., & Herbst, P. (2013). The interplay among gestures, discourse, and diagrams in students' geometrical reasoning. *Educational Studies in Mathematics, 83*(2), 285–307.
Clements, D. H., & Sarama, J. (2011). Early childhood teacher education: the case of geometry. *Journal of Mathematics Teacher Education, 14*(2), 133–148.
Davis, B. & The Spatial Reasoning Group. (2015). *Spatial reasoning in the early years: Principles, assertions, and speculations*. New York, NY: Routledge.
Davis, G., & Hyun, E. (2005). A study of kindergarten children's spatial representation in a mapping project. *Mathematics Education Research Journal, 17*(1), 73–100.
Davis, B., Okamoto, Y., & Whitely, W. (2015). Spatializing school mathematics. In B. Davis (Ed.), *Spatial reasoning in the early years: Principles, assertions and speculations* (pp. 139–150). New York, NY: Routledge.
De Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics, 80*(1–2), 133–152.
Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 142–157). Berlin: Springer.
Duval, R. (2014). Commentary: Linking epistemology and semio-cognitive modeling in visualization. *ZDM Mathematics Education, 46*(1), 159–170.
Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics, 70*(2), 127–141.
Elia, I., Evangelou, K., & Gagatsis, A. (2016, July). *Gestures and their interrelations with other semiotic resources in the learning of geometrical concepts in kindergarten*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.
Elia, I., Evangelou, K., Hadjittoouli, K., & van den Heuvel-Panhuizen, M. (2014). A kindergartner's use of gestures when solving a geometrical problem in different spaces of constructed representation. *Revista Latinoamericana de Investigación en Matemática Educativa, 17*(4), 199–220.
Fletcher, N. & Ginsburg, H. (2016, July). *Teaching symmetry in the early childhood classroom: using software to promote understanding of symmetric transformations*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.
Goldin, G. A. (2002). Representation in mathematical learning and problem solving. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 197–218). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.
Henry, M. (1999). Material phenomenology and language (or, pathos and language). *Continental Philosophy Review*, *32*(3), 343–365.
Henry, M. (2000). *Incarnation: Une philosophie de la chair [Incarnation: A philosophy of the flesh]*. Paris: Seuil.
Henry, M. (2009). *Seeing the invisible: On Kandinsky*. (S. Davidson, Trans.). London: Continuum.

Herbst, P. (2004). Interaction with diagrams and the making of reasoned conjectures in geometry. *ZDM Mathematics Education, 36*(5), 129–139.

Kandinsky, W. (1926/1979). *Point and line to plane: Contribution to the analysis of the pictorial elements* (H. Dearstyne & H. Rebay, Trans.). New York, NY: Dover.

Kaput, J. J. (1998). Representations, inscriptions, descriptions and learning: A kaleidoscope of windows. *Journal of Mathematical Behavior, 17*(2), 265–281.

Kaur, H. (2015). Two aspects of young children's thinking about different types of dynamic triangles: prototypicality and inclusion. *ZDM Mathematics Edcuation, 47*(3), 407–420.

Kell, H. J., Lubinski, D., & Benbow, C. P. (2013). Who rises to the top? Early indicators. *Psychological Science, 24*(5), 648–659.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.

Lappan, G. (1999). Geometry: The forgotten strand. *NCTM News Bulletin, 36*(5), 3.

Leung, A. (2008a). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning, 13*(2), 135–157.

Leung, I. K. C. (2008b). Teaching and learning of inclusive and transitive properties among quadrilaterals by deductive reasoning with the aid of SmartBoard. *ZDM Mathematics Education, 40*(6), 1007–1021.

Leung, A. (2012). Variation and mathematics pedagogy. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics education: Expanding horizons. Proceedings of the 35th Conference of the Mathematics Education Research Group of Australasia (MERGA)* (Vol. 2, pp. 435–442). Singapore: MERGA Inc.

Luria, A. R. (1968). *The mind of a mnemonist: A little book about a vast memory*. Harvard University Press.

MacDonald, A. (2013). Using children's representations to investigate meaning-making in mathematics. *Australasian Journal of Early Childhood, 38*(2), 65–73.

Mammana, C., & Villani, V. (1998). Introduction. In V. Villani & C. Mammana (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 1–8). The Netherlands: Kluwer Academic Publishers.

Maturana, H. R., & Varela, F. J. (1991). *The tree of knowledge: The biological roots of human understanding* (Rev. ed.). Boston, MA: Random House. (Original work published 1987).

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago, ILL: University of Chicago Press.

Merleau-Ponty, M. (1945). *Phénoménologie de la perception [Phenomenology of Perception]*. Paris: Gallimard.

Merleau-Ponty, M., & Lefort, C. (1968). *The visible and invisible* (A. Lingis, Trans.). Evanston, IL: Northwestern University Press.

National Council of Teachers of Mathematics (NCTM). (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: NCTM.

National Research Council. (2006). *Learning to think spatially: GIS as a support system in the K-12 curriculum*. Washington, DC: National Academies Press.

National Research Council. (2009). *Mathematics learning in early childhood: Paths toward excellence and equity*. Washington, DC: National Academies Press.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70*(2), 159–174.

Newcombe, N. S. (2010). Picture this: Increasing math and science learning by improving spatial thinking. *American Educator, 34*(2), 29–43.

Ng, O., & Sinclair, N. (2015a). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM Mathematics Education, 47*(3), 421–434.

Ng, O., & Sinclair, N. (2015b). "Area without numbers": Using touchscreen dynamic geometry to reason about shape. *Canadian Journal of Science, Mathematics and Technology Education, 15*(1), 84–101.

O'Connor, M. C. (1998). Language socialization in the mathematics classroom: Discourse practices and mathematical thinking. In M. Lampert & M. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 17–55). Cambridge, UK: Cambridge University Press.

Perrin-Glorian, M.-J., Mathé, A.-C., & Leclercq, R. (2013). Comment peut-on penser la continuité de l'enseignement de la géométrie de 6 a 15 ans? *Repères-IREM, 90,* 5–41.

Pulvermüller, F. (2012). Meaning and the brain: The neurosemantics of referential, interactive, and combinatorial knowledge. *Journal of Neurolinguistics, 25*(5), 423–459.

Schneckloth, S. (2008). Marking time, figuring space: Gesture and the embodied moment. *Journal of Visual Culture, 7*(3), 277–292.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics, 14*(1), 44–55.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1), 13–57.

Shea, D. L., Lubinski, D., & Benbow, C. P. (2001). Importance of assessing spatial ability in intellectually talented young adolescents: A 20-year longitudinal study. *Journal of Educational Psychology, 93*(3), 604–614.

Sinclair, N. (2018). Time, immersion and articulation: digital technology for early childhood mathematics. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank and C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Sinclair, N., Bussi, M. G. B., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., et al. (2016). Recent research on geometry education: an ICME-13 survey team report. *ZDM Mathematics Education, 48*(5), 691–719.

Sinclair, N., de Freitas, E., & Ferrara, F. (2012). Virtual encounters: The murky and furtive world of mathematical inventiveness. *ZDM Mathematics Education, 45*(2), 239–252.

Smith, I. (1964). *Spatial ability: Its educational and social significance*. San Diego, CA: Knapp.

Tahta, D. (1980). About geometry. *For the Learning of Mathematics, 1*(1), 2–9.

Tahta, D. (1989). Is there a geometric imperative. *Mathematics Teaching, 129,* 20–29.

Thom, J. S., D'Amour, L., Preciado, P., & Davis, B. (2015). Spatial knowing, doing, and being. In B. Davis (Ed.), *Spatial reasoning in the early years: Principles, assertions and speculations* (pp. 63–82). New York, NY: Routledge.

Thom, J. S., & McGarvey, L. M. (2015). The act and artifact of drawing (s): Observing geometric thinking with, in, and through children's drawings. *ZDM Mathematics Education, 47*(3), 465–481.

Thom, J. S., & Roth, W. M. (2009, April). The absence and emergence of collectivity in the mathematics classroom. A paper presented at the *American Educational Research Association Conference*, San Diego, CA.

Thom, J. S., & Roth, W. M. (2011). Radical embodiment and semiotics: toward a theory of mathematics in the flesh. *Educational Studies in Mathematics, 77*(2–3), 267–284.

Thom, J. S., Roth, W. M., & Bautista, A. (2010). In the flesh: Living growing conceptual domains in a geometry lesson. *Complicity: An International Journal of Complexity and Education*, *7*(1), 7–24.

Thompson, E. (1995). *Colour vision: A study in cognitive science and the philosophy of perception*. New York, NY: Routledge.

Thompson, E. (2010). *Mind in life: Biology, phenomenology, and the sciences of mind*. Cambridge: Harvard University Press.

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., et al. (2013). The malleability of spatial skills: a meta-analysis of training studies. *Psychological Bulletin, 139*(2), 352–402.

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Vygotsky, L. S., & Luria, A. (1993). *Studies on the history of behavior, ape, primitive, and child*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology, 101*(4), 817–835.

Whitely, W., Sinclair, N., & Davis, B. (2015). What is spatial reasoning? In B. Davis (Ed.), *Spatial reasoning in the early years: Principles, assertions and speculations* (pp. 3–14). New York, NY: Routledge.

Woodward, M. (2012). *A monstrous rhinoceros (as from life): Toward (and beyond) the epistemological nature of the enacted pictorial image*. Plymouth: Transtechnology Research.

Zordan, D. (2013). Seeing the invisible, feeling the visible. *CrossCurrents, 63*(1), 77–91.

Chapter 9
Observing the Use of Gestures in Young Children's Geometric Thinking

Iliada Elia

Abstract This chapter addresses children's geometry thinking and learning in early childhood with a focus on the use of gestures. The chapter begins with the theoretical frameworks which underlie this work and some background information about geometry learning and gestures. The next parts of the chapter aim to give insight into the role of gestures in young children's geometric thinking in different contexts. Specifically, three case studies are discussed which investigated different aspects of geometry understanding: two-dimensional shapes, composition and transformations of two-dimensional shapes and spatial concepts. Finally, a number of concluding remarks are discussed about the multiple uses and contributions of gestures in association with other semiotic resources in the evolution and communication of early understanding of shapes and space.

Keywords Early geometry learning · Gestures · Semiotic resources
Shape composition · Spatial concepts · Geometrical figures

9.1 Introduction

Geometry is a core component of contemporary early childhood curricula and educational programs (e.g., Sarama and Clements 2009). In an attempt to endorse the importance of geometry in the early years of education and to propose new directions, research in geometry learning and teaching at this age span has currently received increasing attention (Sinclair and Bruce 2015). Nevertheless, there is still much to be done to gain insight into young children's development in this domain (Dindyal 2015), particularly in relation to the body and gestures. Recent research has shown that young children's bodies hold rich potentials in different mathematics strands, including number, geometry and measurement, and also that a teachable moments' approach would benefit if early childhood teachers attend to

I. Elia (✉)
Department of Education, University of Cyprus, P.O. Box. 20537, 1678 Nicosia, Cyprus
e-mail: elia.iliada@ucy.ac.cy

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_9

children's mathematics-related embodied actions (Karsli 2016). Furthermore, it is known that gestures are used extensively by children between the ages three and five (McNeill 2005), while written symbols do not yet have a primary role in mathematical cognition. This chapter addresses young children's geometrical thinking with a focus on gestures, which is an important source of developing abstract thinking (Radford 2009) and an additional window to the mind of the developing child (Goldin-Meadow 2000).

9.2 Multimodal Approach, Gestures and Objectification

"A sign, or representamen, is something which stands to somebody for something in some respect or capacity" (Peirce 1931/1958, 2. 228). Peirce's definition for sign is a broad definition which refers to a variety of phenomena involved in semiotic processes, including gestures, glances, drawings, and extra-linguistic modes of expression. Furthermore, multimodality is a core characteristic of human cognition (Gallese and Lakoff 2005). This means that in reasoning and communicating, people use multiple modes of thinking and expression, including sight, hearing, touch, movement and so on. Taking such a broad semiotic and multimodal perspective in studying mathematical activity, Arzarello (2006) introduced an enlarged notion of a semiotic system, the semiotic bundle. The semiotic bundle includes all the signs that are produced simultaneously, by a student or a group of students who interact in order to solve a problem and/or discuss a mathematical question. In turn, the mechanism that teachers use to harmonize with the semiotic resources produced by the students and then to guide the development of knowledge according to these resources is called a semiotic game (Arzarello 2006).

Within the wide conception of sign and the broad notion of semiotic bundle, gestures are viewed as important semiotic resources in relation with the signs that are traditionally discussed in mathematics education (such as spoken or written language and mathematical symbols). According to Sabena (2008) gestures refer to "all those movements of hands and arms that subjects (students and teachers) perform during their mathematical activities and which are not a significative part of any other action" (p. 21). In his work, McNeill (2005) classified gestures using a number of dimensions. Of these, iconic, deictic and metaphoric gestures are the ones of most interest in mathematics. Deictic gestures refer to pointing movements towards existing or virtual objects and actions in space. Iconic gestures are related to the semantic content of speech, that is, iconic gestures visually represent the content of concrete entities and actions. Metaphoric gestures refer to the representation of an image of an abstract object or idea.

In the course of mathematical activity gestures appear to provide specific ways of carrying out the semiotic process. Firstly, gestures, as a semiotic resource inherently connected to bodily movements have a dynamic nature, which can endorse the dynamic features of mathematical meanings. The condensed or blended character of gestures (and of the semiotic bundle) is regarded as another important contribution of gestures to the mathematical activity (Sabena 2008). It describes the potential of gestures to convey simultaneously different aspects of the mathematical objects in the activity through a hand movement, e.g., the orientation and the shape of a geometrical figure. Often, learning is dependent on the context of the semiotic resources that are used in a mathematical activity, e.g., confined to a specific drawing on the classroom board. Gestures contribute to the shifting from grounding the reference to a contextual dimension of the mathematical activity which is taking place to the embodiment of a certain character of generality (i.e., a third contribution of gestures) (Sabena 2007). A part of this transition is the phenomenon of semiotic convergence, which includes students' progressive coordination of the linguistic, gestural and inscribed (e.g., written symbols, graphs) resources that occur in a mathematical activity. This multimodal semiotic convergence is considered an important step in the process of knowledge evolution (Sabena 2007).

Furthermore, gestures usually focus on certain aspects that play key roles in an unfolding process of knowledge objectification, through which students gradually encounter and develop mathematical meanings and reasoning (Radford 2002). Etymologically objectification refers to actions which aim to make something a conscious object (Radford 2003). It is related to the processes by which learners encounter and give meaning to historical and cultural forms of mathematical thinking and acting. According to the theoretical perspective of objectification, signs (and artefacts) are a central component of these processes as they convey human cognition and historical forms of human production, which determine the ways one learns and understands the world. The semiotic resources that learners use in the context of objectification, including gestures, body position of students and teacher, formulae, written language, speech, objects and so on, are named as semiotic means of objectification.

In the process of objectification and evolution of meanings in semiotic activities and multimodal discourses, the notions of catchment (McNeill 2005; Radford and Sabena 2015) and semiotic contraction (Radford 2008; Radford and Sabena 2015) are of great importance when students use gestures. Catchment refers to the recurrence of a gesture form in at least two (not essentially successive) gestures and this phenomenon is interpreted as an indication of consistency of visuospatial imagery in learners' thinking (Radford and Sabena 2015). Semiotic contraction is related to the process of simplification that learners' gestures and words undergo in their semiotic activity, indicating increasing consciousness of mathematical meanings and more abstract thinking. In particular, this simplification in gesture could involve the loss of movement and shortening of duration. In speech, simplification refers to disappearance of terms and details.

9.3 Geometry Learning and Gestures

Geometrical knowledge involves dealing with theoretical objects, properties and relationships, which are accessible mainly through two semiotic systems, the graphical and the linguistic (Lavergne and Maschietto 2015). In geometrical activities, both geometrical figures and verbal statements have an essential role. According to Duval (2014) language production and visualization need to be coordinated with each other in doing geometry.

Therefore, both visual abilities and verbal skills need to be addressed and developed in the teaching and learning of geometry (Hoffer 1981), starting already from the early years of education (Dindyal 2015). In play activities, either free (Henschen 2016) or guided, involving two- or three-dimensional shapes (like block play), young children often talk together. In general, children use language to share their experiences and reflections with others and thus develop their abilities to describe visual images (e.g., geometrical figures or configurations), spatial concepts, relations and reasoning (Van den Heuvel-Panhuizen and Buys 2008). Developing children's geometry language entails developing their knowledge and understanding of geometrical terms for shapes and also of naming and describing actions and transformations that are performed with shapes and other objects, such as rotating, moving, and identifying their position (i.e., descriptions of configuration productions) (Duval 2014).

By using language related to geometrical activities children learn better how to use geometry language and this can support spatial visualization and reasoning. Language, as a fundamental source of knowledge, includes verbal as well as non-verbal means of communication (Sabena 2017). Gesture is thus an indispensable component of children's communication and thinking in geometry (Elia et al. 2014). It serves as a dynamic representational tool of abstract mathematical ideas through which children may develop a deeper level of consciousness of their meaning. Furthermore, gestures along with speech may have a significant role in the complex transition from perceptual experiences with materials in space to semiotic representations and inscriptions (e.g., on worksheets, the blackboard, a computer screen) often used when geometry is taught in school (Sabena 2017).

However, only a few studies have discussed the role of gestures in the development of space and shape concepts in young children. Ehrlich et al. (2006) explored the strategies 5-year-old children used to solve tasks on spatial transformation. The findings of the study showed that children produced gestures whose meaning was not necessarily involved in the accompanying speech. Children whose gestures represented spatial information, which was not found in their speech, were more likely to succeed. These findings indicate that gestures have the potential to improve early spatial abilities. In particular, hand movements may support children to mentally simulate transformations in space (Newcombe and Frick 2010).

In a teaching experiment Sabena (2017) investigated how spatial competencies are developed using robot-based activities in the kindergarten. Observing and analysing gestures revealed that different conceptualizations of space, including for example a static and a dynamic perspective, may co-exist in a child's thinking and also that relevant gestures often were accompanying speech when new spatial terms were used for the first time by children.

Thom (2018) investigated the forms, (em)bodied acts and processes that constitute children's reasoning while working on a spatial-geometric task. A photograph of a cylinder evoked different mathematical ideas and ways of reasoning in three grade one children, which were materialized as gestures, movements, drawings, imagery and verbalizations. The findings of the particular study provided evidence for the critical role of the body in how children's geometry reasoning and conceptions developed, including transitions between multiple dimensions, visualisation, decomposition, re-composition, perspective taking, dynamic objects, rotation, symmetry, curved and flat surfaces.

The next sections of this chapter aim to give further insight into how gestures contribute to young children's thinking, learning and communication in geometry in different contexts. The findings reported are mainly derived from three qualitative case studies in which children (and teachers) use gestures in different types of shape and spatial activities and settings with different tools and semiotic resources. According to Radford (2009) "[t]o better weigh the role of gestures and bodily actions in mathematics cognition, more detailed investigations are required." (p. 124). In these case studies, the activities were video-recorded and the data from the obtained videos were analysed using the microgenetic method (Siegler 1995). This analysis focused on the children's and/or teachers utterances and gestures during the activities. It is my contention that the microgenetic approach can shed some light on the processes the children go through while thinking of, and communicating mathematical meanings related to geometrical figures and spatial concepts during their interaction with others (i.e., peers or adults).

9.4 Solving a Shape Configuration Problem in Different Spaces of Constructed Representation: The Role of Gestures

The semiotic registers of geometrical figures and linguistic statements as well as the cognitive processes of visualization and reasoning are essential components in geometrical cognition (Duval 1998). Visualization includes the recognition of figural parts in a configuration of shapes as well as figural treatment. The first process is based on the perceptual apprehension of figures, while the second is a major component of the operative apprehension of geometrical figures. Perceptual apprehension refers to the recognition of a shape in a plane or in depth, the recognition of shapes in a perceived figure and the naming of shapes. Operative apprehension refers to the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way entails the partitioning of the whole figure into various shapes and the combination of them in another figure (reconfiguration), the optic way is when one makes the figure larger or narrower, or slant, while the place way refers to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations (Duval 1995). Operative apprehension of shapes is a complex process as it involves spatial visualization abilities and the creation of mental images. However, a recent study by Watanabe (2016) on young children's spatial abilities indicated that children as young as five and six years of age could use operative apprehension on the three-dimensional shape of cube when participating in activities of physically making a cube. Particularly, children's ability to mentally modify a two-dimensional net representation of a cube into the three-dimensional shape improved after using a polydron geometric toy to assemble and convert a plane figure into the three-dimensional shape for a particular period of time.

To investigate the interrelations between geometrical figure apprehension processes in two-dimensional shapes and gesturing we set up a study in which we examined a kindergartner, that is, a 5-year-old girl, while interacting with her kindergarten teacher in a geometrical activity with the use of two different artefacts respectively (Elia et al. 2014). A shape configuration problem was used which involves perceptual and operative apprehension of geometric shapes. The activity that was used also involved the solver's discursive processes pertaining to the shape configuration problem, that is, explanations about which shapes to use and about their proper position and orientation in the composite figure. Thus, the geometrical activity used in this study incorporates two registers of representation, shape configuration and linguistic statements, as well as conversion processes from one register to the other. Furthermore, the activity takes place within a 'microspace' (Brousseau 1983). Microspace refers to a space of interactions tied to the manipulation of small objects (Brousseau 1983). The geometrical representation that was included in this study was constructed in two different types of micro-space: objects made of paper (representing two-dimensional shapes) and a computer screen, illustrating a digital mathematical applet, the Patch Tool (http://illuminations.nctm.org/ActivityDetail.aspx?ID=27). We named these different types of micro-space as Spaces of Constructed Representations (SCR). SCRs are likely to differ in the processes they evoke as "materialized" through gestures and words, while children construct a composite two-dimensional geometrical representation with shapes.

The goal of the activity in each SCR was the child to give appropriate instructions to the researcher so that the researcher could make a composite figure with a given outline, by selecting appropriate shapes and by putting them in the proper position and orientation. During the activity, the researcher encouraged the child to express her thinking by probing and asking questions to the child without providing any guidance for the solution of the puzzle.

At the first SCR, the computer, the user (in this case the researcher following the instructions of the child) was able to move a shape using the mouse, rotate a shape using the spin tool and remove a shape using an eraser. These operations could be implemented, also, in the second SCR (paper) by the researcher, using his hands. As it is illustrated in Fig. 9.1, the same outline of composite figure and the same shapes (triangle, rhombus, square, trapezoid and hexagon) were used in both SCRs.

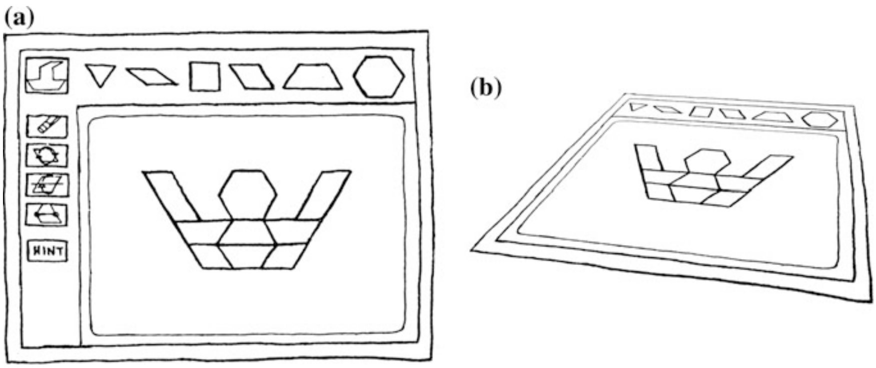


Fig. 9.1 **a** First space of constructed representation (computer screen), **b** second space of constructed representation (paper), from Elia et al. (2014)

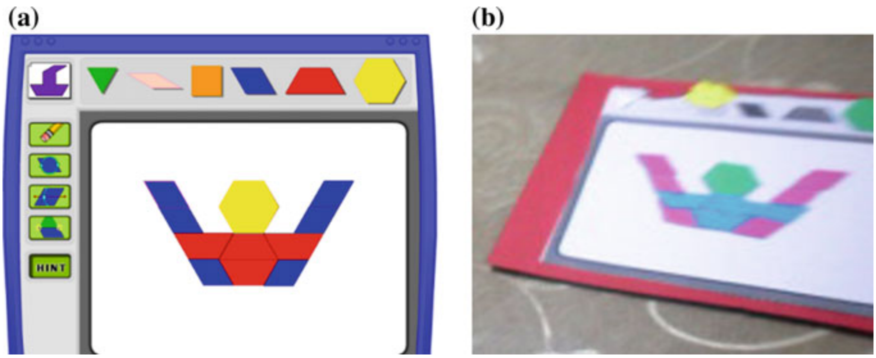


Fig. 9.2 The shape composition made by child to fill the larger figure with the use of **a** computer (lines 1–30) and **b** paper (lines 31–54), from Elia et al. (2014)

Figure 9.2 includes the child's constructed representations using the two SCRs respectively. The figure is followed by extracts of the child's (C) and researcher's (R) talk and gestures in the first activity and the second activity (Elia et al. 2014, pp. 210, 212–213).

1	R:	I would like you to show me the first shape that you want to use
2	C:	I want to start with this shape (*shows with her pointing finger the rhombus*)
3	R:	And where do you want to place it?
4	C:	I want to place it here (*shows with her pointing finger down, on the left side of the figure*)
5	R:	Ok, I am starting the game. This one is a rhombus (*puts the shape in the indicated place*). Is it right how I put it?

(continued)

(continued)

6	C:	No
7	R:	What do you want to do?
8	C:	You have to turn it (*makes a rotation with her pointing fingers, using both of them as moving points*)
9	R:	Helen, you are amazing. I will turn it (*makes a rotation using her pointing fingers*). I am taking this tool and I am starting to turn it. Is it right here?
10	C:	No, you have to turn the shape once more (*makes a rotation with her pointing fingers, using both of them as moving points*)
11	R:	Once more. Is it right?
12	C:	You have to turn it again (*makes a rotation with her pointing fingers, using both of them as moving points*)
13	R:	Ok, I will turn it again. Is it right?
14	C:	Yes
15	R:	Wonderful. Is the place absolutely right?
16	C:	No
17	R:	What do I have to do?
18	C:	You have to turn it on the left (*puts her palms opposite to one another in a vertical direction and she moves them on the left*)
19	R:	On the left (*shows with her pointing finger on the left*). Nice. Is it right?
20	C:	Yes
21	R:	Nice. We have placed the shape on the right location. Let's place the second shape. Tell me, show me
22	C:	(*She shows the trapezoid with her pointing finger*)
23	R:	Do you know its name?
24	C:	Yes I know it…
25	R:	It's tr…
26	C:	A rectangle
27	R:	No, it's a trapezoid
28	C:	The trapezoid
29	R:	Excellent. So, I am choosing a trapezoid. And, where do you want to place it?
30	C:	Here (*shows with her pointing finger the location on the bottom of the figure*)

In the second activity:

31	R:	Show me the next shape that you want to continue with
32	C:	With this (*shows with her pointing finger the rhombus*)
33	R:	With the rhombus, and where do you want to put it?
34	C:	Here (*shows with her pointing finger close to the correct place, down on the left side of the outline*)
35	R:	Nice, I will put it here. Is it right here?
36	C:	No. You must turn it little bit here (*moves her hands from right to left*)
37	R:	Here? Is this right?

(continued)

(continued)

38	C:	Yes
39	R:	Wonderful. Let's continue. Choose a shape
40	C:	(*She shows with her pointing finger the trapezoid*)
41	R:	Do you remember its name? (…) Trapezoid
42	C:	Trapezoid
43	R:	Where do you want to place it?
44	C:	Here (*shows with her pointing finger the correct place, on the bottom of the figure*)
45	R:	I have put it. Is it right?
46	C:	No
47	R:	What do I have to do?
48	C:	You have to turn it (*she makes a rotation with her pointing fingers using both of them as moving points*)

At the end of the second activity:

49	R:	Choose a shape
50	C:	(*She shows the trapezoid with her pointing finger*)
51	R:	You chose the trapezoid again. I think you like this shape more than the other shapes. Where do you want to place it?
52	C:	Here (*shows with her pointing finger a correct place on the right side of the outline*)
53	R:	Is it right here?
54	C:	Yes but move it little bit here *(she moves her right hand from the left to the right)*

The SCR with which the child interacted in the geometrical activity was found to differentiate her gestural production. Although in both SCRs the child produced the same two types of gestures, iconic and deictic, on the second SCR, the child produced fewer iconic gestures (n = 10) than on the first SCR (n = 26). This difference could be explained by the fact that the visual features of the first SCR, e.g., the image of the spin tool, and its clear dynamic character, e.g., the slow and step-by-step rotational function of the tool on shapes, stimulated the use of iconic gestures depicting spatial transformations. It is likely that the mathematical applet helped the child to generate relevant spatial images through her body and express them by gesturing. This inference is further supported by the child's perseverance in using the same types of iconic gestures also in the second SCR, the paper material. The recurrence of these gestures in two distinct moments and occasions (with an interval of two weeks) provides evidence for the phenomenon of "catchment" (McNeill 2005; Radford and Sabena 2015), indicating a strong consistency of the child's visuospatial imagery about the meaning of a shape's rotation and translation.

The child used different types of gestures when activating the perceptual or the operative apprehension of geometrical figures in the shape configuration problem. In both SCRs the child produced deictic gestures when she activated operations related to recognizing a shape (Figs. 9.3a and 9.5a) or specifying the placement of a

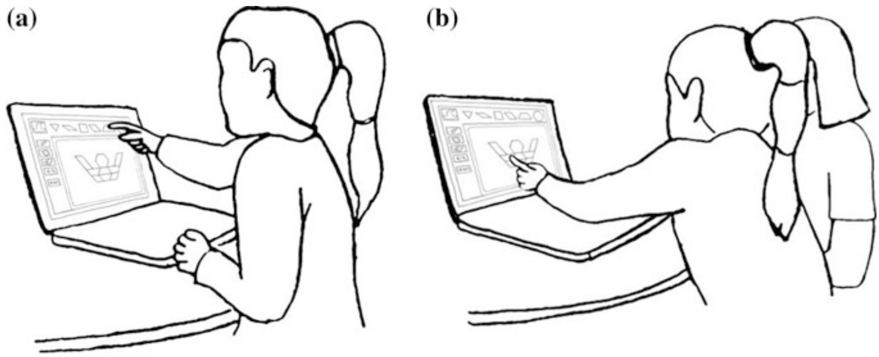


Fig. 9.3 **a** Deictic gesture for the shape of trapezoid on computer (line 22), **b** deictic gesture for the place of rhombus on composite figure on computer (line 4), from Elia et al. (2014)

shape (Figs. 9.3b and 9.5b). In the first case, through deictic gestures (e.g., lines 40, 50) the child recognized all the shapes composing the figure although she was unable to name most of the shapes (e.g., lines 41–42). Identifying shapes in a perceived composite figure is an important component of perceptual apprehension. This indicates that the deictic gestures made "visible" the child's perceptual apprehension competences, and also were an indispensable component of her visual thinking (Fig. 9.5).

In the second case, the child used deictic gestures to point to the placement of a recognized shape in the configuration without giving a precise verbal description about its location or spatial relations with other shapes in the outline. She either did not use any verbal utterances or used the word "here" (e.g., lines 43–44, 51–52) along with the deictic gestures. This indicates that the deictic gestures were a major component of the child's spatial communication and thinking and conveyed information that was not found in the child's speech (as in the first case, in the recognition of shapes).

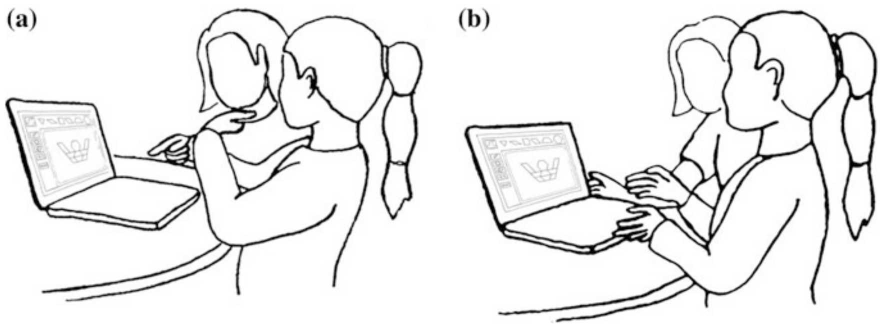


Fig. 9.4 **a** Iconic gesture for the rotation of shape on computer (line 8), **b** iconic gesture for the shape translation on computer (line 18), from Elia et al. (2014)

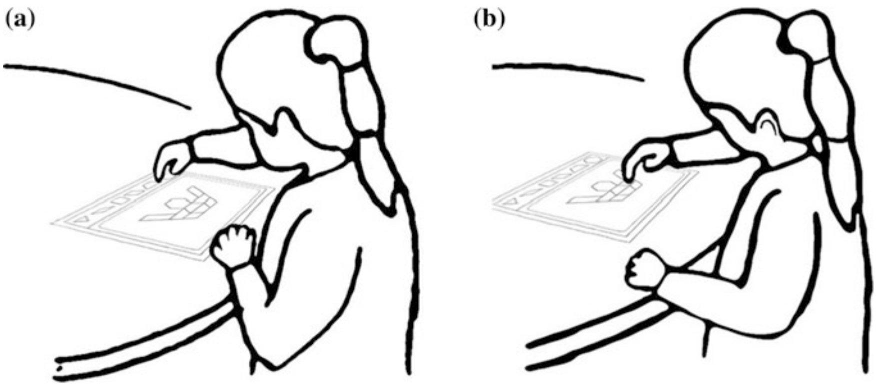


Fig. 9.5 **a** Deictic gesture for the recognition of trapezoid on paper (line 40), **b** deictic gesture for the recognition of the place of shape on paper (line 52), from Elia et al. (2014)

Additionally, in both SCRs, the child applied the place way of modifying a figure, which is an important component of the operative apprehension of geometrical figures. Specifically, when she selected a shape and a location for putting it in the configuration, she produced the rotation by using her pointing fingers as moving points in the air (line 8, Fig. 9.4a and line 48, Fig. 9.6a) and sometimes the translation of the particular shape by putting her hands opposite to each other in a vertical orientation and moving them on the left or on the right (line 18, Fig. 9.4b and line 36, Fig. 9.6b) so that the teacher moved the shape accordingly to fit on a specific place. Thus, these shape modifications were generated and conveyed by the child through words and also, by iconic gestures which were used as a representational tool of these geometrical transformations. It is noteworthy that when the child used an incorrect verbal expression (e.g., line 18) for horizontal translation, gestures had a significant role to generate and convey her thinking. In particular, at the beginning of the activity in each SCR the child referred to two shape transformations (rotation and horizontal translation) using the same verbal expression, that is, "turn", but two distinct iconic gestures (e.g., lines 8 and 18). A possible explanation could be that the spatial concepts which are involved in these two transformations, left and right, are used both as directions of moving horizontally a shape (horizontal translation) but also as directions of turning a shape (rotation).

Furthermore, at the beginning of the activities the wrong term "turn" for a shape's horizontal translation was accompanied with a gesture using both her hands to represent this transformation (Figs. 9.4b and 9.6b). By the end of the first and the second activity the child discriminated the two different geometrical transformations, rotation and horizontal translation, both by gesturing and by using two distinct words, "turn" (e.g., line 48) and "move" (line 54). This progress was accompanied by another change: the use of only one hand to generate the transformation of horizontal translation at the last minutes of the activity with paper (Fig. 9.7) indicating the shortening and simplification of the gesture.

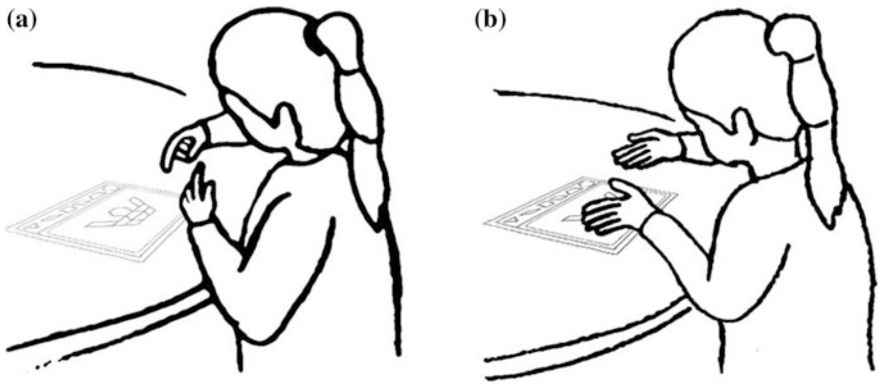


Fig. 9.6 **a** Iconic gesture for the rotation on paper (line 48), **b** iconic gesture for the translation of the shape on paper (line 36), from Elia et al. (2014)

Fig. 9.7 Shortened and simplified gesture for the translation on paper (line 54), from Elia et al. (2014)

This simplification of the child's activity could be regarded as a semiotic contraction (Radford 2008), signifying a progressively higher stage in the child's objectification of the transformation of horizontal translation.

In sum, deictic gestures' production was linked to the recognition of geometrical shapes and their position in the composite two-dimensional representation and had a major role in the perceptual apprehension of shapes. Iconic gestures, which were used to represent shape transformations of different complexity, had an important role in the operative apprehension of shapes. Both types of gestures together with oral speech and the SCRs were an integral part of the child's reasoning for the solution of the problem. This included the combination of geometrical shapes to compose the figure, the recognition of geometrical shapes in different and not necessarily prototypical positions (e.g., horizontal base) and the implicit inference that a shape remains the same in different positions and orientations.

9.5 Gestures and Their Interrelations with Other Semiotic Resources in Gaining Awareness of Two-Dimensional Shapes

Plane geometry, and particularly, identifying, describing and classifying two dimensional shapes have been the focus of a major corpus of research on geometry learning in mathematics education literature in the past few years. The Van Hiele (1985) model has strongly influenced mathematics education research in this domain at all age levels (Sinclair and Bruce 2015). According to this model, children move from conceiving a shape as a whole based on its appearance using visual reasoning, to identifying attributes and many common shapes, to recognizing relationships between attributes and figures in the same class (the concept of rectangle includes the concept of squares) (Levenson et al. 2011).

Research on the role of gestures in the understanding of plane geometrical shapes or its development is rare. A few studies discussed the use of gestures when describing shapes. For example, a study by Graham and Argyle (1975) investigated the accuracy of the information conveyed when gestural communication was allowed in the communication of irregular two-dimensional shapes. It was found that adding gestures to speech enhanced the accuracy with which shapes were communicated.

To explore how children's gestures are related to oral language, and other semiotic systems in geometry reasoning and in generating, visualizing and conveying geometrical meanings about two-dimensional shapes, we conducted longitudinal observations in a kindergarten classroom in a natural setting mainly during whole group discussions. Through whole group discussion in the classroom children can communicate their mathematical thinking explicitly and thus their engagement in emergent forms of mathematical processes, such as reasoning and identifying mathematical structure, are manifested (Yagi 2016). The focus of the observations in the particular classroom was on the use of semiotic resources by a group of children and teacher in their interactions in geometrical activities guided by the teacher. A child that belongs to this group and is studied here is Louis. An episode is provided and analyzed below, which is a part of a geometry lesson aimed at the recognition and sorting of shapes with different criteria. By the time this lesson took place the children had been taught various types of shapes and lines such as triangle, rectangle, square, straight line and curved line. The lesson started with a theatre game, in which the teacher (T) as an alien presented to the children the planets of the shapes that she met in space. At first she met the circular planet, then the triangular, the rectangular and finally the planet of the squares. Then she asked children to sort a collection of shapes made of paper using their own criteria. After making groups according to the type of shape, Louis (L) made an observation that initiated the following conversation:

55	L:	All of them have angles… all of them except the circles
56	T:	Why circle doesn't have any angle?
57	L:	Because it's only circular (*he makes circles with his hands on the air*)
58	T	Hmm… Does it matter? What does it need to make an angle?
59	L:	You have to turn
60	T:	So what kind of lines I need?
61	Andis:	Two. One straight line and one like this
62	L:	(*he makes a right angle using his pointing fingers*)

Specifically, at this episode Louis, without any prompt by the teacher, moved into a higher level of shape categorization than the other children who identified and classified shapes based on their type, as he discriminated circles from all the other categories of shapes. Louis observed that all shapes had angles except the circle, and this motivated the teacher to begin a semiotic game in which she asked children to explain why the circle does not have any angle. Louis's verbal and gestural answer (line 57, Fig. 9.8a) revealed his focus on the round form of the circle. His gesture corresponded to his verbal representation and had a metaphoric character about the shape of circle, as it represented a general case of circle and not a specific model of circle in the class that the child was seeing. In the context of the semiotic game, the focus of the discussion turned onto the concept of angle, which brought to light the child's ideas about it through his words and gestures. Louis' initial idea about angle was based on the dynamic notion of turning (line 59). Then, another child, Andis, entered the semiotic game by saying that it is necessary to have two lines, "one straight line and one like this" to respond to the teacher's question about the kind of lines that are needed. Influenced by his peer's answer, Louis produced a

Fig. 9.8 Louis' metaphoric gestures about **a** "circular" (line 57), **b** angle (line 62)

gesture representing a different notion of angle than before, a "static" right angle made of two line segments, one of which was horizontal ("straight") (line 62, Fig. 9.8b). It is to be noted that Louis' gesture did not have a communicative intention. It seemed to be a part of an internal reflection as it was observed by the absence of words and his body orientation. However, this incident provided evidence for the phenomenon of coordination between different semiotic systems enacted by different children, i.e., Louis and Andis. This is an "inter-personal synchronization" (Sabena et al. 2005, p. 136), which may have a role in objectifying geometrical notions.

Throughout this episode, Louis used an analytic way of thinking by focusing on some critical attributes of shapes, i.e., the roundness of a circle and the complex concept of angle, in his attempt to identify the differences between the circle and the other shapes and to explain what an angle is through his gestures and words. His thinking was decontextualized from the theatre game, detached from the material used and more generalized, as his words and gestures did not make any reference to the particular context but exemplified imaginary and general geometrical objects (e.g., angle). This explains the prevalence of metaphoricity in the child's gestures.

Looking into the interrelations between gestures and oral speech indicated that gestures enabled the child to materialize implicit aspects of his images of geometric shapes and relevant attributes and reinforce or complement the child's words. In most cases, such as when referring to the roundness of circle, the child's gestures were synchronized with his use of oral language and there was a semiotic convergence between these two semiotic resources. Nevertheless, this was not the exact case when the child made a reference to the concept of angle. While in his words he used the dynamic notion of turn to explain how to make an angle, he did not use any gesture. Then, within the semiotic bundle between Louis, the teacher and another child, in an internal reflection moment, Louis used a gesture by making a right angle on the air using his pointing fingers to represent the static notion of angle, which was simultaneously described verbally by the other child and not by himself. This inconsistency in the child's verbal and gestural productions could be a result of the complexity of the concept of angle for which the child was probably at an early stage of developing.

Analyzing this episode indicates that without considering the child's gestural production and its interrelations with other semiotic resources, it would be impossible to access the images for two-dimensional shapes he used in making sense and communicating meanings during the classroom geometry activity. For example, in the case of the static view of angle, the production of gesture revealed much more information than his single verbal description for the idea of turn. It provided insight into the child's prototypical image of a right angle with one horizontal ray pointing to the left for the concept of angle.

Overall, the analysis of this episode showed that body and gestures are integral components of early learning and thinking in plane geometry. Together with artefacts (which initiated the child's thinking) and oral language, gestures played a key role in the kindergartner's process of gaining geometric awareness for two-dimensional shapes, their attributes and attribute-based differences between shapes.

9.6 Gestures in the Learning of Spatial Concepts in a Natural Classroom Setting

Spatial reasoning is regarded as a critical cognitive ability. We need it to discover and better understand the world around us. Also Clements (2004) suggests that "[g] eometry and spatial reasoning form the foundation of much learning of mathematics and other subjects." (p. 267). For example, spatial reasoning could be advantageous in solving mathematical problems, through the use of diagrams and drawings (Casey et al. 2008). Moreover, a great number of studies have shown that at all levels of schooling spatial abilities are of great importance for learning processes in STEM disciplines (e.g., Wai et al. 2009) and that children should be given opportunities to further develop their spatial skills from a young age (Newcombe 2010; Verdine et al. 2014). In the NCTM (2000) *Standards for Geometry in K-2 grades*, a spatial skill that receives considerable attention is specifying locations, which includes interpreting relative positions in space. This was the focus of the mathematics lesson we observed in a kindergarten classroom to gain further insight into the body and gesture's role in young children's spatial reasoning.

There is growing evidence about the link between spatial skills and gestures (e.g., Kita and Davies 2009). The visuospatial type of gesture makes it suitable for capturing visuospatial information. A considerable body of research stresses that gesture is well suited to conveying spatial information (McNeill 1992; Kita and Özyürek 2003), cognitive processing of spatial content and maintaining spatial images in working memory (Alibali 2005). Gesture along with speech is a good vehicle for representing spatial relationships (Wagner et al. 2004). Krauss (1998) found that gestures are used more often in defining spatial words than non-spatial words. Gestures are frequently produced when describing how to navigate through space in a town (e.g., Emmorey et al. 2000), giving directions (e.g., Allen 2003), or describing motion in space (e.g., Kita and Özyürek 2003).

To better understand the role of gestures in the learning of spatial concepts at a young age, we carried out a study in which we analyzed a kindergartner's gestures as a semiotic resource and their dynamics with speech and other semiotic resources while interacting with others (teacher and peers). This case study was conducted in a natural mathematics classroom setting addressing spatial concepts (Elia and Evangelou 2014).

A kindergarten class of children from four years and a half to five years of age was observed during a mathematics lesson. The data analysis concentrated on a 5-year-old boy, who was identified to interact continuously with the teacher and produce many gestures during the lesson (George, pseudonym). The focus of the lesson was on the spatial concepts 'in' and 'out', 'up' and 'down', 'on' and 'under'. Emphasis was given on recognizing the meaning of these concepts and using them when talking about a location or a movement. The teaching approach that was used was the theatre game, in which the teacher and the children explored the concepts by using a box and two cartoon puppets, the Halloween Pumpkin and Golson.

In a lesson's episode which is provided below (from Elia and Evangelou 2014, pp. 52–53, Copyright © EECERA, reprinted by permission of Taylor & Francis Ltd, http://www.tandfonline.com on behalf of EECERA) the teacher (T) gave some everyday examples verbally about the use of the words 'in' and 'out' with reference to virtual objects, the car and the house. He used the particular inverse 'in'–'out' gesture (line 63, Fig. 9.9), thus implicitly stressing the opposite relation between those terms. Then he asked children to give their own examples and George (G) answered by giving another example referring to a virtual object (cupboard). In doing this, he used words and a similar gesture as the one his teacher produced, moving his finger on the right when referring to the word 'in' and moving his hand to the opposite direction when referring to the word 'out' (line 64, Fig. 9.10). Then, George used the same gesture for a different example, referring this time to a real object which existed in the room (fridge) (line 66). The recurrence of a gesture for two different examples for "in" and "out" can be interpreted as a "catchment" in the child's evolution of the meaning of these spatial concepts.

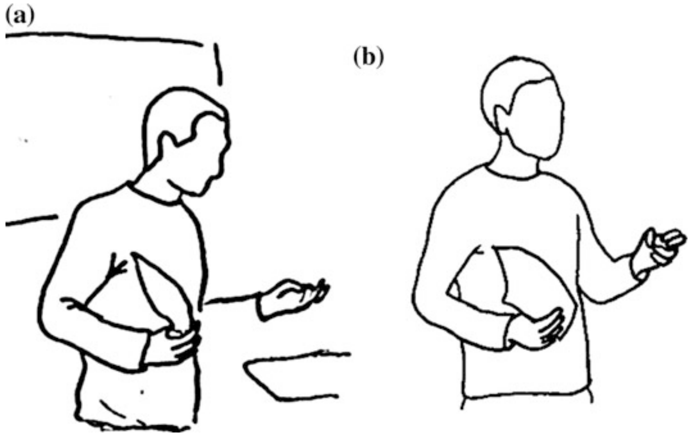


Fig. 9.9 Teacher moves his left hand **a** on the left saying "in the car" and **b** on the right, close to his body saying "out of the car" (line 63), from Elia and Evangelou (2014). Copyright © EECERA, reprinted by permission of Taylor & Francis Ltd, http://www.tandfonline.com on behalf of EECERA

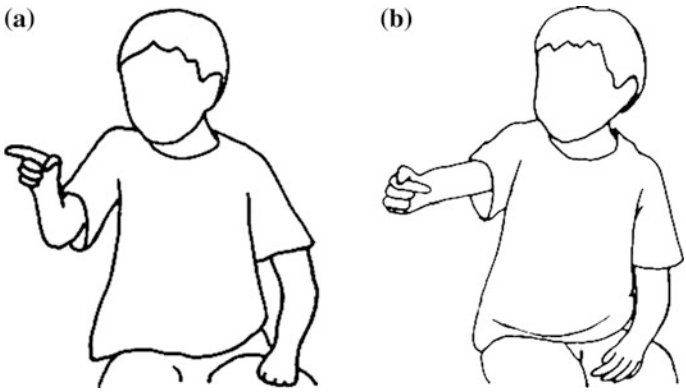


Fig. 9.10 Child's deictic gestures for the verbal expression **a** "in the cupboard" and **b** "out of the cupboard" respectively (line 64), from Elia and Evangelou (2014). Copyright © EECERA, reprinted by permission of Taylor & Francis Ltd, http://www.tandfonline.com on behalf of EECERA

63	T:	What else can we say with 'in' and 'out'? Maybe in the car (*Teacher moves his left hand to the left*), out of the car (*Teacher moves his left hand to the right, close to his body*); in the house (*Teacher moves his left hand to the left*), out of the house (*Teacher moves his left hand to the right, close to his body*)
64	G:	In the cupboard (*Moves his right hand on the left while showing with his pointing finger to the left*) and out of the cupboard (*Moves his hand to the opposite direction*)
65	T:	In the cupboard, out of the cupboard (*Teacher imitates George's gesture*)
66	G:	In the fridge (*Shows with his pointing finger the fridge in the room*), out of the fridge (*Points with his pointing finger to the opposite direction*)
67	T:	Bravo. In the fridge, out of the fridge (*Teacher shows with his pointing finger the fridge*). Bravo George. Very good

In this episode, the child's gestures were deictic (as most of the gestures produced by the child in the lesson) and were used flexibly in coordination with words based either on an existential mode of signification (fridge) or on an imaginative mode of signification (cupboard) (Sabena et al. 2005).

Considering the child-teacher interaction, in this episode the child did not only use gestures that were similar to the teacher's gestures in comparable situations, but he somehow 'extended' the teacher's gesturing by producing similar gestures in different situations for the spatial concepts 'in' and 'out' expressed through his verbal utterances. This indicates that the child gave meaning to the opposite relation between the inverse 'in'–'out' gesture that his teacher produced, by observing and mimicking the teacher's gesture flexibly in different situations from the ones the teacher gave for the spatial concepts 'in' and 'out'. Thus, besides pointing to an existing or a virtual object, at the same time the child's gesture referred to the opposite meaning of the two spatial terms with respect to the corresponding object as a reference point. This is indicative of the blended character of gestures with respect to the meanings they exemplify.

The coordination between speech and gestures had a significant role in the process of objectification of the concepts 'in' and 'out'. However, the opposite spatial relation between the concepts 'in' and 'out' was never expressed verbally by the teacher or the child during the lesson, but began to be objectified thanks to the pair of inverse deictic gestures he produced. This provides evidence to the importance of speech-gesture dynamics (in which gestures supplement and enrich speech) in the child's thinking processes. Furthermore, it is noteworthy that towards the end of the lesson, when the children were singing a theme about spatial relations including the terms 'in' and 'out', George, without moving his hand, pointed his finger in front of him, when using the verbal expression 'in' and then he pointed to the opposite direction using the opposite spatial term 'out' (Fig. 9.11). In this song, for the terms 'in' and 'out' he produced a simplified and shortened pair of inverse deictic gestures without making any reference to an object, existing or virtual, in his speech. This is an indication of a semiotic contraction in the child's activity, by which he could imagine and generalize the meaning of the spatial concepts.

Overall, it can be claimed that the objectifying deictic gestures (Radford 2002), which were initiated by the child-teacher's interaction, played a role in the evolution of the objectification about the spatial relation between 'in' and 'out', and specifically in the child's process towards the generalization and abstraction of this spatial aspect.

In another part of the lesson the teacher asked questions with respect to the position of specific objects in the classroom. When such questions were answered

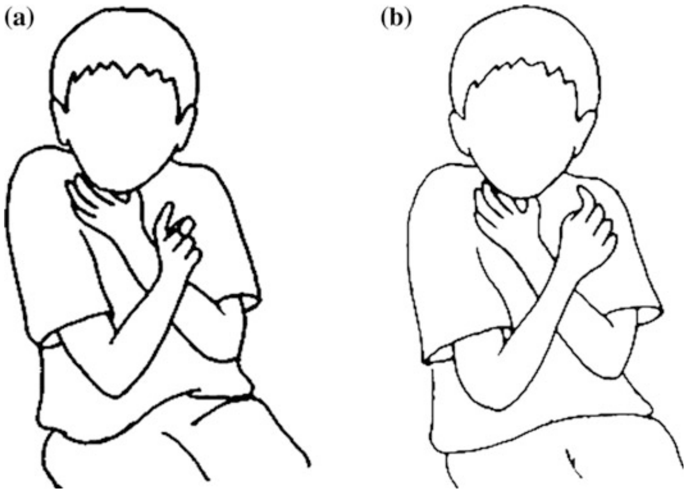


Fig. 9.11 Child's deictic gestures for the words **a** "in" and **b** "out" while singing, from Elia and Evangelou (2014). Copyright © EECERA, reprinted by permission of Taylor & Francis Ltd, http://www.tandfonline.com on behalf of EECERA

loudly by all the children to describe verbally the required location of objects, sometimes the child under study was showing with a deictic gesture the particular location without producing any utterance. For example, in one episode when all the other children said "on the table" to explain the position of Halloween Pumpkin, George pointed to the table without any verbalization. In the previous episode, a coordination was identified between two different semiotic systems, speech and gestures, activated by the same child in the process of objectification of spatial relations. In line with the study described previously, the latter episode provided evidence for the phenomenon of "inter-personal synchronization" (Sabena et al. 2005, p.136) between different semiotic systems enacted by the child studied here and the other children, which may contribute to the objectification of spatial relations.

9.7 Concluding Remarks

The work described above investigated geometry learning in the early years with a focus on children's gestures in three different aspects of the development of understanding of shapes and space: two-dimensional shapes, composition and transformations of two-dimensional shapes and spatial concepts. Despite the different geometrical content, including plane or spatial geometry, the different activities, settings, didactical tools and artefacts that were involved in this work, all three studies provided evidence for the essential role of gestures in geometrical thinking and communication. This work also revealed how young children used gestures in these varying conditions, indicating the multiple uses of gestures and their interconnections with other semiotic resources in early geometry learning.

It is noteworthy that in spite of the aforementioned variations in the research approaches and in the uses of gestures in children's semiotic activity, some common patterns are identified in all three studies or in at least two of the studies, indicating a certain degree of generality and reliability of our findings. For example, multimodal semiotic convergence was a commonly found characteristic in children's geometric thinking and communication in all the studies. In a few occasions though, this semiotic convergence appeared to be interpersonal rather than personal. For example, a child generated a static conceptualization in oral speech about the notion of angle while another child, without producing any utterances, produced a concurrent embodied form of conceptualization for the same notion, again static, (interpersonal semiotic synchronization). This was, however, different from the latter child's previously generated verbal form of dynamic conceptualization for angle. In a different occasion, for the transformation of horizontal translation, a child's speech (use of the word "turn") was not in line with her produced gesture or spatial transformation on the SCR which embodied the appropriate meaning. These occasions of lack of (personal) semiotic convergence are likely due to the complexity of shape and space aspects involved (e.g., operative apprehension of geometrical figures, deconstruction of geometrical objects) and this indicates that the children's understandings were at an initial stage of development for these concepts.

Considering the coordination of semiotic means from the perspective of objectification, our findings from all three studies, suggest that there was a complex but fruitful interplay between the semiotic resources provided by the tools or artefacts used, by the teacher and peers and the semiotic resources produced by the children under study. For example, in the study which investigated how a girl solved a shape configuration problem in different SCRs we concluded that there was something to gain from the complex interplay between the geometrical figures, configurations and spatial transformations provided by the computer or on paper, and the verbal utterances and gestures produced by the girl herself while using them. This semiotic coordination of culturally developed and personally developed resources (Radford et al. 2005) may have helped the child to enter into a process of differentiating between critical and non-critical (position or orientation) attributes of shapes and thus into a process of objectification of geometrical figures.

Another commonality between the studies described in this chapter, is the semiotic contractions that were found for certain concepts, including the transformation of horizontal translation of shapes and the spatial concepts "in" and "out", in children's embodied and verbal forms of thinking, indicating their increased awareness of these mathematical meanings and a certain level of generality. Also, children's gestural production involved catchments which appeared either within a small period of time in the same activity or over a longer period of time in two distinct activities (e.g., for the rotation of a geometrical figure). This phenomenon can be considered as an indication of persistent consistency in the visuospatial imagery of the child's thinking for the specific geometry aspect. In the case of the rotation of geometrical figures, catchments could be explained by the use of the relevant features of the SCR on the computer.

Even though the studies' focus was on the children's acts, there were indications of the impact of the adults' (teacher or researcher) verbal and non-verbal behavior on the children's use of semiotic resources, their thinking and its evolution in making sense of spatial concepts. Also the interaction with peers and specifically the impact of peers' verbal behavior on children's gesture and meaning making for two-dimensional figures and spatial concepts cannot be overlooked. Additionally, the use of different artefacts was found to influence children's semiotic activity in the transformation of two-dimensional figures. All these are issues of great theoretical and practical importance which could be investigated more systematically in future research with more children, longer observations and a variety of geometrical problem solving tasks.

Finally, overall this work suggests that gestures along with speech and other semiotic resources serve as a window for identifying the children's progress and difficulties in spatial and plane geometry content of varying levels of complexity. Considering children's gestures and their interrelations with other semiotic resources while developing geometric awareness, in whole classroom interactions, in peer interactions and in teacher-child interactions is important for teaching. It enables the teachers to gain valuable insights into children's (implicit) learning processes, reasoning and understandings in geometry, e.g., children's constructed mental images for geometrical concepts, and thus shape their teaching approaches (purposeful instruction or focusing on spontaneous learning opportunities) in a way that matches the children's needs. What is necessary for teachers to fulfill this role is an issue for further research. Similarly for researchers in geometry education, it is suggested that investigating geometry learning in the early years would benefit by observing and analyzing children's embodied forms of knowing and knowledge, even though this is not an easy task considering the micro-analysis that is required.

Acknowledgements A part of the work reported in this chapter was carried out in the project "The contribution of gestures in geometrical thinking development in early childhood" (2014–today) that is supported by a Program Grant from the "A.G. Leventis Foundation".

References

Alibali, M. (2005). Gesture in spatial cognition: Expressing, communicating, and thinking about spatial information. *Spatial Cognition and Computation, 5*(4), 307–331.

Allen, G. L. (2003). Gestures accompanying verbal route directions: Do they point to a new avenue for examining spatial representations? *Spatial Cognition and Computation, 3,* 259–268.

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigacion en Mathematica Educativa*, Numero especial, 267–299.

Brousseau, G. (1983): Etude de questions d'enseignement, un exemple: la géométrie. *Séminaire de didactique des mathématiques et de l'informatique,* (pp. 183–226). Grenoble: IMAG.

Casey, B. M., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction, 26,* 269–309.

Clements, D. H. (2004). Geometric and spatial thinking in early childhood education. In D. H. Clements & J. Sarama (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education*. Mahwah, NJ: Lawrence Erlbaum Associates.

Dindyal, J. (2015). Geometry in the early years: A commentary. *ZDM Mathematics Education, 47* (3), 519–529.

Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 142–157). Germany: Springer.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 37–51). Dordrecht: Kluwer Academic.

Duval, R. (2014). The first crucial point in geometry learning: Visualization. *Mediterranean Journal for Research in Mathematics education, 13,* 1–28.

Ehrlich, S., Levine, S., & Goldin-Meadow, S. (2006). The importance of gesture in children's spatial reasoning. *Developmental Psychology, 42,* 1259–1268.

Elia, I., & Evangelou, K. (2014). Gesture in a kindergarten mathematics classroom. *European Early Childhood Education Research Journal, 22,* 45–66.

Elia, I., Evangelou, K., Hadjittoouli, K., & Van den Heuvel-Panhuizen M. (2014). A kindergartner's use of gestures when solving a geometrical problem in different spaces of constructed representation. *Revista Latinoamericana de Investigacion en Matematica Educativa, 17*(4–I), 199–220.

Emmorey, K., Tversky, B., & Taylor, H. A. (2000). Using space to describe space: Perspective in speech, sign and gesture. *Journal of Spatial Cognition & Computation, 2,* 157–180.

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology, 21,* 1–25.

Goldin-Meadow, S. (2000). Beyond words: The importance of gesture to researchers and learners. *Child Development, 71*(1), 231–239.

Graham, J. A., & Argyle, M. (1975). A cross-cultural study of the communication of extra-verbal meaning by gestures. *International Journal of Psychology, 10,* 57–67.

Henschen, E. (2016, July). *Mathematical content of play activities in kindergarten, exemplified on blockplay activities*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher, 74,* 11–18.

Karsli, E. (2016, July). *Young children's embodied mathematical practices in a pre-k classroom*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Kita, S., & Davies, T. S. (2009). Competing conceptual representations trigger co-speech representational gestures. *Language and Cognitive Processes, 24,* 761–775.

Kita, S., & Özyürek, A. (2003). What does cross-linguistic variation in semantic coordination of speech and gesture reveal? Evidence for an interface representation of spatial thinking and speaking. *Journal of Memory and Language, 48,* 16–32.

Krauss, R. M. (1998). Why do we gesture when we speak? *Current Directions in Psychological Science, 7,* 54–60.

Lavergne, S., & Maschietto, M. (2015). Articulation of spatial and geometrical knowledge in problem solving with technology at primary school. *ZDM Mathematics Education, 47,* 435–449.

Levenson, E., Tirosh, D., & Tsamir, P. (2011). *Preschool geometry*. Rotterdam: Sense Publishers.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: The University of Chicago Press.

McNeill, D. (2005). *Gesture and thought*. Chicago: The University of Chicago Press.

National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Newcombe, N. (2010). Picture this: Increasing math and science learning by improving spatial thinking. *American Educator, 34*(2), 29–35.

Newcombe, N., & Frick, A. (2010). Early education for spatial intelligence: Why, what, and how. *Mind, Brain and Education, 4*(3), 102–111.

Peirce, C. S. (1931/1958). *Collected papers of C. S. Peirce.* In C. Hartshorne, P. Weiss, & A. Burks (Eds.), (Vols. 1–8). Cambridge, MA: Harvard University Press.

Radford, L. (2002). The seen, the spoken and the written: A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics, 22*(2), 14–23.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37–70.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM—The International Journal on Mathematics Education, 40*(1), 83–96.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70,* 111–126.

Radford, L., Bardini, C., Sabena, C., Diallo, P., & Simbagoye, A. (2005). On embodiment, artifacts, and signs: A semiotic-cultural perspective on mathematical thinking. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education (PME)* (Vol. 4, pp. 113–120). Melbourne: PME.

Radford, L., & Sabena, C. (2015). The question of method in a Vygotskian semiotic approach. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 157–182). Dordrecht: Springer.

Sabena, C. (2007). *Body and signs: A multimodal semiotic approach to teaching-learning processes in early calculus*. Ph.D. Dissertation. Universita Degli Studi Di Torino.

Sabena, C. (2008). On the semiotics of gestures. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom and culture* (pp. 19–38). Rotterdam: Sense.
Sabena, C. (2017). Early child spatial development: a teaching experiment with programmable robots. In G. Aldon, F. Hitt, L. Bazzini and U. Gellert (Eds.), *Mathematics and Technology* (pp. 13–30). Springer International Publishing.
Sabena, C., Radford, L., & Bardini. C. (2005). Synchronizing gestures, words and actions in pattern generalizations. In H. L. Chick and J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education (PME)* (Vol. 4, pp. 129–136). Melbourne: PME.
Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge.
Siegler, R. S. (1995). How does change occur: A microgenetic study of number conservation. *Cognitive Psychology, 25,* 225–273.
Sinclair, N., & Bruce, C. (2015). New opportunities in geometry education at the primary school. *ZDM Mathematics Education, 47*(3), 319–329.
Thom, J. S. (2018). (Re)(con)figuring space: Three children's geometric reasonings. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank and C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education.* (This book).
Van den Heuvel-Panhuizen, M., & Buys, K. (Eds.). (2008). *Young children learn measurement and geometry*. Rotterdam: Sense Publishers.
Van Hiele, P. M. (1985). *The child's thought and geometry.* In D. Geddes & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele* (pp. 243–252). Brooklyn: Brooklyn College, School of Education (Original work published 1959).
Verdine, B. N., Golinkoff, R. M., Hirsh-Pasek, K., Newcombe, N. S., Filipowicz, A. T., & Chang, A. (2014). Deconstructing building blocks: Preschoolers' spatial assembly performance relates to early mathematical skills. *Child Development, 85*(3), 1062–1076.
Wagner, S., Nusbaum, H., & Goldin-Meadow, S. (2004). Probing the mental representation of gesture: Is handwaving spatial? *Journal of Memory and Language, 50,* 395–407.
Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology, 101*(4), 817–835.
Watanabe, S. (2016, July). *Research on young children's spatial recognition ability*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.
Yagi, S. (2016, July). *An exploration of first grade students' engagement in mathematical processes during whole group discussions*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Chapter 10
Math-in-Context: The Types of Math Preschoolers ‘Do’ at Home

Ann Anderson and Jim Anderson

Abstract While research shows diversity across families in terms of the frequency and types of mathematics prior to school, parent-child interactions during “naturally occurring” activities in the home remain understudied. This observational study investigated the types of mathematics that preschoolers engaged in with family members during activities which six middle class mothers identified as contexts for mathematics learning in their home. Across and within dyads, a range of mathematics concepts was found, with four of the families sharing more geometry-related activities. Furthermore, although contexts and mathematics appeared common across the families, the specific mathematics shared within particular activities often differed according to the semiotic nature of the specific materials, and/or the specific adult-child interactions.

Keywords Parent-child interactions · Mathematics in context · Preschoolers Adult-child shared activity · Mathematics at home

10.1 Introduction

Despite the recognition that many young children learn mathematical concepts and processes prior to school entry (e.g., Gifford 2004), there appears to be limited research into young children’s experiences with mathematics during ‘naturally occurring’ activities at home. Although researchers (e.g., Baroody et al. 2009;

A. Anderson (✉)
Department of Curriculum and Pedagogy, University of British Columbia, 2125 Main Mall, Vancouver, BC V6T 1Z4, Canada
e-mail: ann.anderson@ubc.ca

J. Anderson
Department of Language and Literacy Education, University of British Columbia, 2125 Main Mall, Vancouver, BC V6T 1Z4, Canada
e-mail: jim.anderson@ubc.ca

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_10

Ginsburg and Pappas 2004) have demonstrated young children's considerable understanding of number sense, presumably developed "naturally", it remains unclear what other types of mathematics occur during shared everyday experiences.

10.2 Theoretical Perspective and Background Literature

According to socio-historical theory (Vygotsky 1978; Wertsch 1998), adults and significant others provide support relative to children's current knowledge, and structure activities in ways that engage children in more complex behaviors than those they can do independently. In doing so, they extend children's learning within their *zone of proximal development*. As Rogoff reminds us, the ways in which learning is supported "cannot be understood out of the context of the immediate practical goals being sought and the enveloping socio-cultural goals into which they fit" (Rogoff 1990, p. 139). Similarly, Cobb (1986) theorized that what constitutes children's mathematical learning is constructed in context. It follows, then, that children's mathematics understanding in the early years of school (e.g., preK-Grade 1) builds from the types of mathematics-in-context, that occurs during parent-child shared activities at home.

Much of the research into young children's mathematics learning within the family context has relied on interviews and questionnaires, typically involving mothers (e.g., Lefevre et al. 2009, 2010; Starkey and Klein 2008). These researchers report and describe an array of number-related activities in which the children engage. Other researchers have observed parent-child interactions during tasks that approximate what occur in the home (Benigno and Ellis 2008; Solmaz 2015; Stiles 2010; Tiedemann and Brandt 2010; Vandermaas-Peeler et al. 2009; Vandermaas-Peeler 2008). These are similar to our studies (Anderson 1997; Anderson et al. 2004, 2005) of parent-child dyads engaging with researcher-provided materials (e.g., storybooks, board games) within the home context. To date, few studies (Anderson and Anderson 2014; Aubrey et al. 2003; Meaney 2010; Tudge and Doucet 2004) have documented parent-child interactions during "naturally occurring" events at home. Furthermore, many who have researched parent-child interactions have not disclosed the mathematics focus of their studies to participants while others have used secondary analysis of data collected for other purposes. For instance, in Walkerdine's (1988) foundational study of mathematical discourse at home, she analyzed audiotaped conversations of 30 mother-daughters (average age 3.9 years) collected for the *Language at Home and at School* Project and focused specifically on relational terms used (e.g., more), signifiers for size (e.g., big), and measurement of distance and time (e.g., long).

As noted, this body of research demonstrates considerable diversity in terms of the frequency and types of mathematics, in which young children and parents engage. Yet, the literature appears to point to an emphasis on counting and number in home environments, although measurement (size), space and shape are indeed evident. Is it possible that our reliance on parental self-reports and task-based observations has led inadvertently to our identifying the prevalence of number-related activities and events in the home? Are we missing day-to-day activities and events in the lives of young children and their families that provide contexts for learning other mathematical concepts due to narrow perceptions of what early mathematics might entail? Is it possible that by naming generic research goals (e.g., parent-child interactions) rather than disclose our focus on mathematics, we inadvertently lead participants away from mathematical talk? What remain underrepresented, it seems, is observational studies of parent-child interactions during more diverse everyday activity in the home, in which parents are aware that mathematics is a focus. In this chapter, we report on our investigation of the types of mathematics a family member and pre-school child share within at-home activities, which the mothers viewed as contexts for mathematics learning.

10.3 Method

Six mothers and their two and a half-year old preschoolers (5 girls and 1 boy) participated in the study for 2.5 years. The families lived in middle/upper-middle class neighborhoods, the parents were well educated, three families included siblings, and three were single child families. When recruiting, we explicitly asked the mothers to assist us in documenting adult-child activity in their homes in which the pre-school child engaged with mathematics. In four families, a research assistant video-recorded the joint activities, which the mothers chose to carry out during home visits every six to eight weeks. In the other two families, the mothers elected to videotape activities on their own schedule without researcher assistance. One of these two mothers captured her child interacting with various family members (e.g., mother, grandmother, older sibling) while the other consistently videotaped the father and preschooler interacting.

Thus, the data sources for the study reported here were the verbatim transcripts of video-recorded at-home activities of 6 families, identified by the pseudonyms Adam (mother-daughter dyad), Beet (mother-daughter dyad), Liu (father-daughter dyad), Pimm (family member-daughter dyad), Penn (mother-son dyad), and Star (mother-daughter dyad). A total of 33 video-recorded sessions, each lasting at least 15 minutes, were collected. Our analysis identified 44 different activities (See Anderson and Anderson (2014) for details and Appendix 1 for a summary table). To determine the types of mathematics evident during these activities, we examined each transcript for any mathematics-related references verbalized by child or adult and assigned descriptors such as "number recognition". We then grouped the mathematics we coded, according to curriculum organizers (BC Ministry of Education 2007), noting the dyads that engaged with each. Next, we re-examined the transcripts in concert with the videos to further delineate the types of mathematics within each activity. In addition, we re-viewed selected videos (i.e., eight activities) to analyze the mathematics in relation to the contexts.

10.4 Results

10.4.1 The Mathematics Involved in the Events Mothers Chose to Videotape

Across the dyads, various mathematics concepts were enacted involving number and geometry[1] (i.e., shape and space, measurement). While all six dyads engaged in counting and shape recognition, six other concepts—number names, number recognition, sorting, spatial terms, size, and capacity—arose in activities across four dyads (see Table 10.1). On the other hand, each of three number concepts—estimate, one less, and number word recognition—appeared in one dyad only.

Looking across the activities within each dyad, we found that the Adam family focused primarily on number, the Liu family focused equally on number and geometry, while the remaining four families shared slightly more geometry concepts than number concepts (see Table 10.2).

10.4.2 Trends: Types of Mathematics Within Contexts

As might be expected, when we examined the mathematics concepts within the contexts in which they arose (see Tables 10.3 and 10.4), the types of mathematics were differentiated further (e.g., object counting vs. rote counting). Accordingly, several trends emerged that were not readily apparent when we identified the concepts in a more decontextualized manner, as previously shown (Table 10.1).

10.4.3 A Closer Look at Number Concepts and Contexts

Although counting was found across all six families, rote counting arose on three occasions only, once in the context of a song (i.e., Adam dyad: One, two, buckle my shoe) and twice during child-initiated "count of 3" routines (i.e., Star daughter: One, two, three, four—swings necklace; Penn son: one, two, three—drops puzzle piece) (Table 10.3). In contrast, object counting was more prevalent, with children counting spaces on game boards, objects pictured on a puzzle, toy cars, fingers, pizza slices and puzzle pieces. Except for one dyad counting 32 puzzle pieces and another dyad "counting on" (i.e., 45, 46, 47, 48) in Snakes and Ladders, the number of objects the children counted ranged from two to twelve. Similarly, when parents

[1]Subsuming shape, space, and measurement under Geometry serves as a dichotomous category to Number, and aligns with the definition of Geometry as a "branch of mathematics that deals with the measurement, properties and relationships of … [shapes]" (Merriam-Webster 1828 [inserted by authors]).

Table 10.1 Summary of mathematics shared by dyads

Description	Family	Description	Family
Number		*Shape*	
Counting	Adam, Liu, Penn, Star, Beet, Pimm	Shape recognition	Adam, Liu, Penn, Star, Beet, Pimm
Number names	Penn, Star, Beet, Pimm	Sorting/ attributes	Penn, Star, Beet, Pimm
Number recognition	Adam, Penn, Beet, Pimm	Properties	Adam, Penn, Pimm
Numeral recognition	Adam, Liu, Penn	*Space*	
Number comparison	Liu, Star	Spatial terms	Liu, Penn, Beet, Pimm
Symbol recognition	Adam, Liu	Spatial awareness	Liu, Penn
Representation	Adam, Star	*Measurement*	
Fractions	Adam, Beet	Size	Liu, Star, Beet, Pimm
Operations	Adam, Liu	Capacity	Liu; Star; Beet; Pimm
Double digits	Adam, Pimm	Volume	Liu; Beet
Number word recognition	Adam	Time	Penn; Pimm
One less	Liu		
Estimate	Pimm		

Table 10.2 Summary of types of mathematics shared by each family

Family	Number	Geometry[a]
Adam	Counting, number, numeral, symbol, number-word recognition, representation, fractions, operations, double digits	Shape recognition, properties
Liu	Counting, numeral and symbol recognition, number comparison, operations, one less	Shape recognition, spatial terms, spatial awareness, size, capacity, volume
Penn	Counting, number names, numeral and number recognition	Shape recognition, sorting, properties, spatial terms, spatial awareness, time
Star	Counting, number names	Shape recognition, sorting, size, capacity
Beet	Counting, number names, number recognition	Shape recognition, sorting, spatial terms, size, capacity, volume
Pimm	Counting, number names, number recognition	Shape recognition, sorting, properties, spatial terms, size, capacity

[a]*NB* Types of math pertaining to shape, space or measurement are included under geometry

Table 10.3 Nature of and contexts for number concepts

Concept (dyads[a])	Nature	Context
Counting (A, L, Pe, S, B, Pi)	Rote, object counting	Song, lunch, puzzles, toy cars, boardgames, playdoh-pizza, word problems
Number names (Pe, S, B, Pi)	Says age, birthdate	Toy cars, lunch, storybook, viewing photos, yearbook
Number recognition (A, Pe, B, Pi)	Names number of objects	Toy cars, cards, dots, tea party, yearbook, pizza
Numeral recognition (A, L, Pe)	Names or points to numeral	Cards; calculator, book, computer, puzzles, videos, BINGO, Snakes and Ladders
Double digits (A, Pi)	Names two-digit numbers, numerals	Snakes and Ladders, puzzle
Fractions (A, B)	½, ¼ named and shown	Playdoh-pizza, meatballs, baking cookies
Symbol recognition (A, L)	+, −, =, decimal point	Calculator, computer, word problems
Number comparison (L, S)	Too many; not enough	Rods; Hungry Hippos game
Representation (A, S)	Show me "five", draw me a two	Word problems, lunch
Operations (A, L)	Addition, subtraction	Word problems, computer game
Number word recognition (A)	Reads number words	Card game
One less (L)	One item removed	Pop-up and stuffed toys
Estimate (Pi)	About 50, lots	Jigsaw puzzle

[a]First letter(s) of dyad's pseudonym is used here to identify each family

Table 10.4 Nature of and contexts for geometry concepts

Concept	Nature (example)	Context
Shape recognition (A, L, Pe, S, B, Pi)	Square, rectangle, oval triangle, circle, heart, hexagon, diamond	Books, train tracks, puzzles, stickers, checkers, playdoh, pegboard
Sorting/attributes (Pe, S, B, Pi)	Color; broken or not; magical or everyday	Puzzles, toy cars, book
Properties (A, Pe, Pi)	Long sides, equal sides, square corners, same area	Books, puzzles, playdoh-pizza
Spatial terms (L, Pe, B, Pi)	In, over, through, under, to the side	Train tracks, plastic food, follow the leader, puzzles
Spatial awareness (L, Pe)	Irregular figures	Train tracks, toy cars
Size (L, S, B, Pi)	Little, big, short, high, low, same	Puzzles, stickers, photos, tracks, sprinkler, pizza
Capacity (L, S, B, Pi)	Filling and emptying containers	Playdoh, baking, Macaroni game, tea party
Volume (L, B)	3 cups, ½ cup, ½ tsp	Macaroni game, baking
Time (Pe, Pi)	Hours or o'clock, years	Toy cars, viewing photos

of three families called attention to numerals in their environment, all of the children readily recognized the numerals 0–9 found on cards, in books, on calculator and computer keypads, on board games and on video boxes, while two of the children readily recognized double-digit numerals [e.g., 10–99 (Snakes and Ladders); 50 (puzzle box)].

Interestingly, fractions arose in two families in food related contexts, both pretend (i.e., playdoh pizza and meatballs) and real (baking cookies) as the Adam daughter cut circles (pizza) or spheres (meatballs) into halves and quarters and the Beet daughter filled measuring cups to the ½ or ¼ mark as she helped her mother mix cookie dough. Not surprisingly, children's recognition of symbols (e.g. =, +, −) arose in dyads where children used technology, such as the Adam daughter using a calculator when playing store, or the Liu daughter playing a computer game. An exception was the Adam mother's scribed equations (e.g., 2 + 4 = 6) expanding on her child's enacted solutions to oral problems (e.g., "If four friends came for Pizza, how many slices do we need?"). Finally, within some contexts (e.g., Adam dyad: number puzzles, books, card game; Liu dyad: number puzzles) concepts were explicitly interconnected, such as when counting, number, and numeral recognition (e.g., Liu Dyad: D: "one, two, three, four, five"; F: "Is that a five?"; D: "The five is right here") were simultaneously evident, as children matched numerals and pictures of objects or numerals and dot patterns.

10.4.4 A Closer Look at Geometry Concepts and Contexts

For all six families, shape recognition included plane figures with both straight and curved sides (Table 10.4). In addition to book illustrations, curved shapes, like hearts, ovals, and irregular figures, were attended to when dyads engaged with materials such as stickers, train tracks, and jigsaw puzzles. Interestingly, naming many-sided figures was supported through a storybook illustration (i.e., Adam dyad: octagon) and a hand-made light fixture (i.e., Beet dyad: hexagon). While all six dyads were recognizing and labeling conventional shapes (e.g., square, rectangle, triangle, circle), three of the mothers also described the shapes' properties (e.g., Beet mother: "yes, it (the square) is the one with the equal sides"). However, when four of the dyads sorted objects, mainly color was used, and when two of the children spontaneously sorted through objects to find what they wanted, dichotomous classifications (e.g., Penn son: broken or not broken toy cars; Pimm daughter: magical or everyday puzzle images) were verbally identified, using non-geometric properties as attributes.

With respect to spatial awareness, parents and children, in four of the dyads, mainly used directional language (i.e., spatial terms) during their play with objects (e.g., Penn mother: "this one (puzzle piece) is *under* this one"). In addition, children in two of those dyads also illustrated spatial awareness through the structures they created (i.e., Penn son constructs parking-lot-spaces; Liu daughter builds train track configurations). A third child did so through a movement activity (i.e., Pimm daughter and sibling go over, around and under obstacles as they play Follow the Leader).

Not surprising, the measurement seen across four families did not incorporate standardized tools or units (e.g., no rulers; no cm or ml). Rather children or parents used size to describe objects (e.g., stickers, train tracks, puzzle pieces), often relying on visual comparisons. For example, the Star daughter, who picked (and named) a big square sticker from among the many different-sized stickers available, visually compared her chosen sticker indirectly with the others. The Beet daughter, who lifted her foot to assess the height of the water flowing from a lawn sprinkler, directly compared the two heights, again visually. In addition, non-linear measurement arose when children explored capacity or volume by filling and emptying different containers (e.g., canisters, jug, teacups, measuring cups and spoons) with different materials (e.g., playdoh, macaroni, tea, flour and sugar respectively). On occasion, dyads quantified the amounts a container could hold, such as when the Liu child counted to 3 as she repeatedly poured a cup of macaroni into a jug or when the Beet mother confirmed "that's half a cup" as her child filled her graduated measuring cup with flour to the ½ mark. Finally, although time measurement arose minimally across dyads, the Pimm daughter's engagement with time while looking through family photos with her grandmother (e.g., Pimm grandma: "in 1934"; "I was 8 years old"; "your mother when she was little") warrants mention.

10.4.5 A Closer Look at "Common" Contexts: Examining Within a Category

Since the types of mathematics that arose in conversations in these homes occurred in such a range of activities across and within families, we now take a closer look at some of these contexts in order to elaborate further on the types of mathematics we found. As shown in Table 10.5 in Appendix 1, five dyads chose to have video-recorded (or to video-record themselves) their engagement with mathematics while playing with puzzles, a "common" adult-child joint activity in the early years. In these families, puzzles were a context in which object counting, numeral recognition including double-digits, estimation, shape recognition and properties, sorting, and spatial terms arose (see Tables 10.2 and 10.3). However, a closer look within this category indicates that two of the parent-child dyads engaged with number based puzzles. For the Adam and Liu families, the puzzles were such that, when each numeral 1–9 (either carved or written on a carved shape) was lifted from its position, a matching number of objects were shown in the space created. With these puzzles, the numerals were in sequence from left to right, across 2–3 rows and each carved piece typically fit only the matching space. For two other mother-child dyads, the Beet and Penn families, the puzzles were 'typical' jigsaw puzzles with differing numbers of interlocking pieces required to complete an image pictured on the cover of the puzzle box. For the fifth dyad, the Pimm family, the puzzle consisted of two scenes, one with images from a fairy tale, and the other with comparable images of "everyday" people and objects (e.g., a princess and a woman;

a unicorn and a horse). While these duplicated puzzle pieces, consisting of the carved "outline" of the person or object, lifted out, like the numerals in the number puzzles, the spaces created were blank.

Closer examination of the mathematics related to the number puzzle context indicates, not surprisingly, that each child counted or subitized the objects pictured prior to placing the puzzle piece with the corresponding numeral in the space. Although each puzzle piece was a unique shape, number and numeral recognition were typically used as a means to determine its placement. However, on occasion, shape and space were addressed, often when a chosen piece did not fit. To illustrate, we share an excerpt from the Adam dyad, showing the daughter subitizing [7–8], and counting [11–15] as well as the mother scaffolding shape and orientation [9; 18].

Adam dyad (Mother-Daughter) are sitting on the couch playing with a wooden number puzzle.

7 M: … What have we got here? [*D puts the puzzle piece into the puzzle.*] Oh a seven right off. Let's do some counting too. How many snakes? [*D holds up a puzzle piece for her mother to see.*] Good. You are doing pretty good eh? How many bunnies?
8 D: Two
9 M: Good. [*D selects a puzzle piece*] Can you find another two? You try it. That's not it sweetie, is it? Can you find number two? There! …
11 M: How many ducks?
12 D: One, two, three
13 M: Do you want mommy to help you count? Let's touch every one, one, two, three, four.
14 D: Four duckies—this many. [*D holds up four fingers*]
15 M: You are right, good for you. Can you find a four? …
18 M: … How many turtles? One, two, three, four, five, six. Can you find a six? That's a good try. You know why? Look at they are both nines aren't they? Because one of them is upside down … Try that one. Is that number six that fits in there? Yeah good—it's an upside down nine. …

In the jigsaw puzzle contexts, the opposite occured, whereby mothers suggested that children examine straight versus curved edges, concave or convex parts, or color and image attributes to find matching puzzle pieces to connect or fill an opening. On occasion, spatial awareness of where a puzzle piece was located (on the cover, or in a pile) was also used to determine where it might belong or be found. While number arose, it did so minimally. To illustrate we share an excerpt from the Penn family, where the properties [19, 21, 48] and the location [31–33] of the pieces, and the overall shape [68] of the puzzle as well as color [34], and image [21] are scaffolded, more frequently than number [64].

Penn Dyad (Mother-Son) are seated on the floor with a jigsaw puzzle; older sister sits nearby playing

19 M: Last night you did it pretty well and I just helped you. … Which pieces do you want to do first, the corner pieces or the flat pieces? … I bet you know where that piece goes?
21 M: … Good that looks like the corner piece even though the box doesn't show it that is the corner piece. What do you have to look for next, more flat pieces with flowers on?
22 S: Yeah …
26 M: Let's see if [you] can find it. I think that is pretty good. …
29 M: … what does that one look like on the picture?
30 S: Um, this one.
31 M: Very good, so probably it goes down here. So then what goes in the middle?
32 S: This one.
33 M: … What is in the middle on the picture? What's between here and here? (M points)
34 S: Blue, blue flowers.
45 M: Are you looking at the picture? … Where do you think this one goes?
47 S: Right here.
48 M: Do you think so? Remember about the flat at the end. … All the ones at the edge are flat—see. So this is flat here isn't it? Do you want to give it a try and see where that one goes? … All down here are going to be the flat ones.
64 M: There is one, two, three flowers …
68 M: Let's see. Flowers—look at this do you want to see where this piece goes. … OK, is that a flat piece? Remember the front it is going to be like a square—all flat. If that isn't a flat it is going to be in the middle not at the top—a square. …

Interestingly, in the lift-out puzzle context, while searching for particular images dominated the conversation, at times size and number arose, as illustrated here when the Pimm dyad included estimation [123–124], counting [132–138] and size [16] as well as geometric attributes (shape [38–40] and space [3; 55]) to complete their puzzle.

Pimm D and M are lying on the floor, playing with puzzles. Grandmother is videotaping.

3 M: … let's look on the box. [*shows D the puzzle box cover*] See on one picture there is the princess holding a basket and over there, there is a woman holding a basket. So now we need to find the princess holding a basket in here [*the box*] right?
16 M: …D, did you find the princess with the little crown on her head?

38 M: You can look over here and see what shape fits and put it to the one that fits …
40 M: Does it fit there? That looks good you are right, and here is the princess. …
41 D: This goes here.
55 M: … that looks great. Remember you look on the board and you see where it is in general and also you can see the shape. I think you know that already.
123 M: That's it, … You have a guess D, do you have any idea how many pieces this puzzle has?
124 D: 16
132 M: Should we count them?
133 D: Yeah. 1, 2, 3, … 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 21, 22, 23, 28, 29, 30, 31, 32, 32, 28.
136 M: … That is a lot of pieces. It has even more than 28 pieces. … Let's see if it says.

To summarize, then, while completing puzzles together appears to be a common context for most of these families to engage with mathematics, when the type of puzzle (e.g., number vs. jigsaw vs. lift-out) is taken into account, the activities appear less common (i.e., fewer families engage with a particular type of puzzle) and less similar (i.e., how one completes a jigsaw and lift-out puzzle differs) than the category, *puzzle*, might suggest. In addition, while some types of mathematics appear related to the type of puzzle (e.g., jigsaw puzzles afford shape-based mathematics), the features of the particular materials alone appear to be insufficient to determine what mathematics will arise. And while most of the types of mathematics (with "estimate" being the exception) were found across contexts, nuances particular to the activity each dyad constructed were found. Therefore, it seems that the (specific) types of mathematics in which these dyads engage differ according to the "semiotic" (Golden and Gerber 1990) nature of the contexts (i.e. the specific material, the specific child, the specific adult) in which the play takes place. That is, each child and mother (interpretants) bring their own background knowledge, experiences, personalities and proclivities to a particular mathematical sign (e.g., puzzle) and how they interact around and with the sign will inevitably vary from dyad to dyad. While further examination of the remaining 10 categories is pending, for the purpose of this chapter, these findings prompted us to explore our data from a slightly different perspective.

10.4.6 A Closer Look at "Unique" Contexts: Examining Across Categories

To complement our analysis of the types of mathematics arising during what appeared to be similar adult-child joint activities (i.e., common context: within a category), we turned our attention to the unique contexts, which these mothers chose to video-record. Looking across the categories (Table 10.5), we selected three activities (i.e., Playdoh (Pizza): ADAM M-D; Physical (Water Sprinkler): BEET M-D; Family (Photos): PIMM G-D), each of which involved a different dyad. As shown in Tables 10.3 and 10.4, the types of mathematics which arose in these contexts included counting, number recognition, fractions, shape recognition, properties (Pizza), time, number names (Photos), and size (Water Sprinkler, Photos, Pizza). As can be seen, then, each type of mathematics appears particular to a unique context, except for the concept of size. When we look closer, we find that in the Pizza context, the ADAM mother uses terms like "big" to describe the whole (Pizza) and "same or equal size" to describe portions (Pizza slices), implying and/or inviting a comparison of size based on area.

On the other hand, in the Photos context and in the Water Sprinkler context, size comparisons based on height are implied when the PIMM grandmother and child use terms like "little" and "short" to describe the people pictured, and the BEET mother and child describe the water level as "low" or "high" (see Appendix 2). Thus, although the usage of size-related terms appears common to all three unique contexts, the specific terms (and attributes) differ. In addition, when we look closer within the Pizza context, we find that number concepts arise on occasions, when the child or mother count the number of pizza slices, describe a number of people or slices (e.g. 4 or 6) or the mother speaks to "cutting it in half". While the mother's repeated emphasis on recognizing and creating equal portions indirectly points to properties of a circle (see Appendix 2), shape recognition, as shown in the following excerpt, arose early in the conversation and included naming familiar shapes, which were not present.

ADAM M-D dyad seated at child's table, playing with Playdoh.

5 M: Good girl, squish it. What shape is it going to be?
6 D: A heart.
7 M: A heart shaped pizza? That could be a little tough. What shapes are pizza usually?
8 D: Triangle.
9 M: Yes, when you cut the pieces they are. Let's see if we can make a regular shape. A big circle right? A big pizza.

In the Photos context, time and number names are threaded throughout the activity, with the PIMM grandmother's references to age (8 or 14 years old) or dates (1934) interconnecting both. Time also arose through references like "when she was a young woman" or "mommy came many years later". On one occasion the child named a number (e.g. two) when asking her grandmother to identify a pair of individuals in a photo (see Appendix 2).

To summarize, then, when dyads engage with mathematics within ‘unique’ contexts, both unique and common types of mathematics may arise. Considering the asymmetrical distribution of the types of mathematics across these particular contexts, however, further analysis is needed. In addition, types of mathematics, which appear common across contexts, hold context-specific characteristics. Hence, looking across categories via these three unique contexts, we again find the specific mathematics generated points to the nuanced nature (and hence the diversity) of the mathematics-in-context in which these pre-school children and their significant others engaged.

10.5 Conclusions

We believe these findings point to important considerations for researchers and educators interested in young children’s mathematics learning prior to school and the roles families play in that learning. As this study suggests, there are common, as well as unique activities, in which families engage to support their child’s mathematical development and learning. Indeed many of the activities these middle-class mothers chose to share have been reported in previous studies (e.g., Lefevre et al. 2009; Saxe et al. 1987). However, the study reported here clearly indicates that activities, which are typically deemed common because parents and/or others use the same generic descriptor (e.g., puzzles), differ in terms of the materials used, the type of mathematics supported and/or the ways in which each child and adult interact. Therefore, these results remind us that if we are to understand and build on children’s prior experiences, as teachers and researchers, we must inquire into the differences that reside in more detailed accounts regarding the mathematics-in-context. To do so, further analysis of this data, as well as future research into everyday practices, which attend to the nuances of parent-child mathematical engagement is needed to augment surveys and parental reports.

The breadth of mathematics concepts (e.g., number, shape, spatial awareness and measurement) **across** families found in this study is consistent with findings from previous studies (Anderson 1997; Anderson et al. 2004, 2008) where parents and children engaged with materials (i.e., blocks, story books) that the researchers provided. This convergence of findings suggests that when parents and children engage with the ‘same’ materials, whether an activity is of their own choosing or not, we can expect the types of mathematics to vary. In addition, because these families engaged in activities of their choosing, a larger variety of tasks were observed in the current study than in previous studies (e.g., Anderson 1997; Vandermaas-Peeler et al. 2009), resulting in a breadth of mathematics found **within** each dyad. When we extrapolate these findings to the myriad of daily activity in which parents and children routinely engage, the prevalence of number previously associated with mathematics at home, or the prevalence of geometry for some families suggested in the current study, needs further consideration and further research.

That said, the presence of geometry (shape, space, and measurement) found in the current study adds to the growing evidence of types of mathematics, other than number, with which parents engage their preschoolers and points to, and beyond, everyday practices of naming shapes, comparing sizes and navigating space. In addition, when identifying certain shape, space and measurement concepts with which these dyads engaged, our analysis of nonverbal interactions (e.g., gestures, movement) seemed particularly relevant. Indeed, when we considered our results in light of research on the embodied nature of young children's geometric and spatial reasoning (Elia 2018; Karsli 2016; Thom 2018) reported in this volume, we found several commonalities. More specifically, as was the case for the Kindergarten children in Elia's study and the grade 1 children in Thom's study, parents and children in the current study used gestures (finger, hand) to communicate, or augment their communication, of shape, location and size. While the Grade 1 children in Thom's study used body movement (arms, heads) to imagine a three-dimensional object, these preschoolers seemed to use body movement (i.e., arms, legs, torso) to explore space and measurement more directly. Indeed, the Beet daughter's engagement with height and distance when playing with a water sprinkler in the current study seems comparable to the pre-Kindergartener's engagement with speed and distance when playing with a hula hoop, in Karsli's study. Although our focus on types of mathematics provides only glimpses of such engagement with geometry, further in-depth analysis and description of the embodied ways in which preschoolers and their family members might 'naturally' engage with geometry seems warranted.

Finally, we reflect briefly on our design, as Tudge et al. (2008) suggested, "the methods used to assess children's involvement in everyday math heavily influence the apparent extent of their involvement" (p. 188). What distinguishes this study, we believe, is that we observed math-in-context that our mothers (and children) chose to share with us. We trusted that telling our parents about the math focus of the study would influence the amount, and type, of mathematics they demonstrated and we argue that placing mothers in an agentive role in terms of deciding what they considered mathematical offsets any perceived limitation. Of importance here is that our results stem from a combination of the mothers' identification of the activity, the adult-child performance of the mathematics while being video-recorded, and the researchers' identification of the types of mathematics we saw and heard. We do not claim that the types of mathematics identified herein necessarily match the mathematics that our parents themselves would identify and we remind our readers that these results are not meant to be generalizable. Instead, we argue, the results of the current study indicate the potential, and the importance, of designing future research in ways that include preschoolers and their families, from culturally and socially diverse backgrounds, engaging in mathematics as they go about their everyday affairs.

Appendix 1

Table 10.5 Activities mothers chose to videotape (Anderson and Anderson 2014)

Category	Each family's activity					
	Adam	Star	Penn	Pimm	Liu	Beet
Puzzles	Number		Jigsaw (2)	Lift-out	Number	Jigsaw
Play	Store	Stickers		Pegboard	Traintracks	Tea party
Board game	Snakes and Ladders	Hungry Hippos	BINGO			Checkers
Story time	Number and shapes	Felt story board		Sounds of world		Matching objects
Family time		Lunch		Photos	Videos	Baking
Toys		Traintracks	Cars	Food/dolls	Pop up	
Playdoh	Sharing pizza	Happy face				Making food
Physical games	Hopscotch			Follow the leader		Water sprinkler
Matching games	Cards: word numeral, dots			Cards: images	Rods: "Ten" family	
School like	Word problems			Yearbook entry	Computer game	
Songs	1, 2, buckle my shoe	Row, row your boat			ABCs	
Other games				Dreydel	Macaroni	
Miscellaneous				Coin trace		

Appendix 2

Playdoh (Pizza): Excerpts from transcript of Adam M-D dyad

M and D are playing with playdoh at a small table.

:

M: … OK you know what? Let's pretend we are going to have four people over for supper, four people are coming. We want to cut this pizza so everybody gets the same. So the first thing we need to do is cut it in half.

D: Like this? (*gestures several horizontal cuts*)

M: Yeah right down the middle so there is two pieces the same size.

:

M: Four people are coming for supper so can you cut it in half this way? Or this way? Yeah.
D: I am going to go this way (inaudible) [*gestures a horizontal cut towards the bottom of circle*]
M: OK but let's do this first—so you already cut it in half this way, so let's cut it in half this way. [*gestures perpendicular to the previous horizontal cut near the center*] Yeah. Let's see what happens. That's a girl. So we cut it in half that way. Now how many pieces do we have? One, two—
D: One, two, three, four. [*mother moves each piece to separate them slightly; pieces are unequal*]
M: Four.
D: Let's pretend it's six people.
:
M: OK. So how do you want to do it? [*D pretends to cut along two perpendicular lines similar to before and then repeats the cuts but slightly to the right and below previous gestures*] Ok, you show me. [*D cuts a horizontal line near the middle and then a parallel cut just slightly below it*]
D: (inaudible) [*offers her mother a tiny crumb of "pizza"*]
M: … Look at it, if one person gets this piece [*lifts a very thin strip of playdoh, leaving two semi-circle portions*] is that the same size as this piece?
D: No.
M: This guy will be so hungry. We need to make them so that they are the same size. So let's put it back together. How can we do that?
D: This is how you do it. I will show you the (inaudible). [*D pretends to make 2 horizontal parallel cuts (one in the lower half and another nearer the middle of the pizza)*]
M: Let's try it.
D: OK. [*she makes the parallel cuts with her knife*] Tic, Tack, Toe.
M: Tic, Tack, Toe. [*repeats jokingly*] First show mommy how you do it in half.
D: You know how (inaudible couple of words).
M: … Go right down the middle, right? [*uses index finger to trace the cut on playdoh; then child uses plastic knife to make the cut*) Good. OK. So now it is in half. [*lifts half circle nearest her to space the semi-circles of pizza*] So, let's try cutting this [*lifts the semi-circle near her*] into three pieces.
D: [*above the semi-circle on her left, the child moves her knife as if to cut at a 120° angle from center towards the curved edge and then another 'almost parallel' cut, slightly off cente*r]
M: [*nods towards child's actions*] yeah
D: no, we need six.
M: If we cut this one into three pieces [*using index finger she traces cuts on far semicircle*] and we cut this one into three pieces [*traces lines on other*] let's see how many we will have. So you are going to cut—mommy will do the cuts in this one. [*cuts the semicircle nearest her*]
D: (inaudible).

M: Do the cuts along the lines. Do you want mommy to be the cutter and you the counter? OK. [*cuts along the lines on other semicircle*] Let's count how many pieces do we have.
D: One, two, three, four, five, six.
:

Family (Photos): Excerpts from transcript of Pimm G-D dyad
Grandmother and pre-school daughter of Pimm family look at large sheets of photos

29D: Who are these two?
G: That is my mother after the war with me and that is my father.
D: You are so short.
G: Short? I was short, I was eight years old.
D: The dress is so nice.
G: Yes, that was my first new dress after the war.
D: Oh
:
G: … And here when she [*her mother*] was a young woman. And there she was—
D: Where you are?
G: Here she was an old lady. You see look at the (inaudible).
D: Where is you?
G: I'm not there any more.
D: Oh.
G: I am here [*points to specific photo*] … That is me when I was 14.
D: And this one is her.
:
D: Oh, oh.
G: … but this was the girls in my class.
D: Where is you?
70G: Here [*points to herself in the photo*]
:
73D: Where is Eema?
G: Oh, Eema wasn't even an idea yet?
D: Where is she?
G: Mommy came many years later.
D: Where is [*child's name*]?
G: [child's name], no where, no where my love.
D: Where is [*child's older sister's name*]?
80G: [*sister's name*], no, no, no. Let me see if there are more here. …
:
85D: Who is that?
G: Who is that?
D: I don't know.
G: Maybe could that be your mother when she was little? No.
D: No.

G: No, could that be [*sister's name*]?
D: No, [*sister's name*] isn't there.
G: That is me.
D: No
G: No, I think that might be [*child's name*].
D: No that is not me.
M: That is you.

Physical (Water sprinkler): Excerpt of transcript from Beet M-D dyad
Beet child plays with lawn sprinkler, while her Mother adjusts the water at daughter's request

:
87*D wearing swimsuit approaches the 'back' of the oscillating sprinkler, water trajectory is low*)
M: I need to fix the back [*of child's swimsuit*].
D: Yikes, [*skips backwards towards mother away from the sprinkler*] the water is coming [*the height of water is increasing*]
M: Unwind this part [*swimsuit strap*], there you go, you're perfect now. [*D returns to sprinkler*] Are you going to run through the bridge, through the tunnel?
Child watches the trajectory of the water as she skirts the periphery of the lawn and then
D: [*cautiously steps towards sprinkler from side, stands near sprinkler as water reaches peak just prior to tipping towards her*] I can't watch [*covers her eyes, moves closer and raises her knee so that it touches the water mid-stream*] Yeek! [*steps back slightly; squats down*] Can you turn the sprinkler [*squeals briefly as she runs away; high stream of water tips towards her*] a little bit—high?
M: Higher?
D: lower
M: lower, sure, do you want it more manageable? [*D watches from a path at the far left*]
D: [*as water is just above the ground*] oh, I don't want [*she runs towards the sprinkler*] now [*without the mother adjusting*], it's way too low for me; watch, watch this.
M: Okay, I see.
D: (*steps directly over the sprinkler and runs through the full length of the water stream…screams and laughs*) that was good, [*she walks on path, looking at sprinkler*] turn it lower.
M: okay
D: [*still on path*] no, no yikes [*runs further from sprinkler*]
M: how low, this low [*turns tap but height of water increases slightly*]
D: [*shouts*] No low
M: low, low, low [*water is "straight" above sprinkler and begins to lower*]
D: yeah that's how low [*water is about waist high, but quickly goes higher*] no, don't do it like that [*D has been raising arms up and down along with the changing*

water height] that low ok [*D approaches sprinkler; water is shoulder high; as she gets closer she stomps her foot in displeasure as water level increases*] that low
M: lower?
D: (*screams as mom changes water pressure*) no, low!!! Like that [*knee high*] that’s perfect. [*D screams as water changes*] You’re not staying low there.
102M: (*laughs*) okay, okay
D: I want you to stay there. [*knee high; she raises her foot and “rests” it at top of water stream*]
:
110M: Oh I’m sorry love; mommy was just adjusting and readjusting it
D: (*continues to place her foot in the stream; her hands in the stream; she jumps over it with two feet; she sits on it …*)
M: That’s a nice manageable height.
:

References

Anderson, A. (1997). Families and mathematics: a study of parent-child interactions. *Journal for Research in Mathematics Education, 28*(4), 484–511.

Anderson, A. & Anderson, J. (2014). Parent-child mathematics: A study of mothers’ choices. In C. Nicol, P. Liljedahl, S. Oesterle, D. Allan (Eds.), *Proceedings of 38th Conference of the International Group for the Psychology of Mathematics Education and 36th Conference of the North American Chapter of the Psychology of Mathematics Education*, (Vol. 2, pp. 33–38). Vancouver: International Group for the Psychology of Mathematics Education.

Anderson, A., Anderson, J., & Shapiro, J. (2004). Mathematical discourse in storybook reading. *Journal for Research in Mathematics Education, 35*(1), 5–33.

Anderson, A., Anderson, J., & Shapiro, J. (2005). Supporting multiliteracies: Parents’ and children’s talk within shared storybook reading. *Mathematics Education Research Journal, 16*(3), 5–26.

Anderson, A., Anderson, J., & Thauberger, C. (2008). Mathematics learning and teaching in the early years. In O. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 95–132). Charlotte, NC: Information Age Publishing Inc.

Aubrey, C., Bottle, G., & Godfrey, R. (2003). Early mathematics in the home and out-of-home contexts. *International Journal of Early Years Education, 11*(2), 91–103.

Baroody, A. J., Lai, M., Li, X., & Baroody, A. E. (2009). Preschoolers’ understanding of subtraction-related prinicples. *Mathematical Thinking and Learning: An International Journal, 11*(1–2), 41–60.

BC Ministry of Education. (2007). *Mathematics Kindergarten integrated resource package*. Victoria, BC, Province of British Columbia, Canada: Ministry of Education.

Benigno, J., & Ellis, S. (2008). Do parents count? The socializations of children’s numeracy. In O. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 291–308). Charlotte, NC: Information Age Publishing Inc.

Cobb, P. (1986). Contexts, goals and beliefs and learning mathematics. *For the Learning of Mathematics, 6*(2), 2–9.

Elia, I. (2018). Observing the use of gestures in young children’s geometric thinking. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education.* (This book).

Gifford, S. (2004). A new mathematics pedagogy for the early years: in search of principles for practice. *International Journal of Early Years Education, 12*(2), 99–115.

Ginsburg, H., & Pappas, S. (2004). SES, ethnic, and gender differences in young children's informal addition and subtraction: A clinical interview investigation. *Journal of Applied Developmental Psychology, 25*(2), 171–192.
Golden, J., & Gerber, A. (1990). A semiotic perspective of text: The storybook reading event. *Journal of Reading Behavior, 22*(3), 203–219.
Karsli, E. (2016, July) *Young children's embodied mathematical practices in a Pre-K classroom*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.
Lefevre, J., Polyzoib, E., Skwarchuk, S., Fast, L., & Sowinskia, C. (2010). Do home numeracy and literacy practices of Greek and Canadian parents predict the numeracy skills of kindergarten children? *International Journal of Early Years Education, 18*(1), 55–70.
Lefevre, J., Skwarchuk, S., Smith-Chant, B., Fast, L., Kamawar, D., & Bisanz, J. (2009). Home numeracy experiences and children's math performance in the early school years. *Canadian Journal of Behavioural Science, 41*(2), 55–66.
Meaney, T. (2010). Only two more sleeps until the school holidays: referring to quantities at home. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of CERME 6* (pp. 2617–2646), Lyon, France: Institut National de Recherche Pédagogique.
Merriam-Webster. (1828). Geometry. https://www.merriam-webster.com/dictionary/geometry. Accessed May 6, 2017.
Rogoff, B. (1990). *Apprenticeship in thinking*. Oxford: Oxford University Press.
Saxe, G., Guberman, S., & Gearhart, M. (1987). Social processes in early number development. *Monographs of the Society for Research in Child Development, 52*(2), 2–138.
Solmaz, G. (2015). Familial studies in early childhood that involve mathematical situations. In K. Krainer, N. Vondrovå (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 1996–2002). Prague, Czech Republic. https://hal.archives-ouvertes.fr/hal-01288506 Accessed June 30, 2017.
Starkey, P., & Klein, A. (2008). Sociocultural influences on young children's mathematical knowledge. In O. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 253–276). Charlotte, NC: Information Age Publishing Inc.
Stiles, T. (2010). *A study of parent-child numeracy interactions in families of English language learners*. (Unpublished Doctoral dissertation). Capella University, Minneapolis, MN.
Thom, J. (2018). (Re)(con)figuring space: Three children's geometric reasonings. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education.* (This book).
Tiedemann, K. & Brandt, B. (2010). Parent support in mathematical discourses. In U. Gellert, E. Jablonka, & C. Morgan (Eds.), *Proceedings of the Sixth International Mathematics Education and Society Conference,* (Vol. 2, pp. 457–466). Freie Universitat Berling, Germany.
Tudge, J., & Doucet, F. (2004). Early mathematical experiences: Observing young Black and White children's everyday activities. *Early Childhood Research Quarterly, 19*(1), 21–39.
Tudge, J., Li, L., & Stanley, T. (2008). Impact of method on assessing young children's everyday mathematical experiences. In O. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 187– 214). Charlotte, NC: Information Age Publishing Inc.
Vandermaas-Peeler, M. (2008). Parental guidance of numeracy development in early childhood. In O. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 277–290). Charlotte, NC: Information Age Publishing Inc.
Vandermaas-Peeler, M., Nelson, J., Bumpass, C., & Sassine, B. (2009). Numeracy-related exchanges in joint storybook reading and play. *International Journal of Early Years Education, 17*(1), 67–84.
Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
Walkerdine, V. (1988). *The Mastery of reason: Cognitive development and the production of rationality*. London: Routledge.
Wertsch, J. (1998). *Mind as action*. Oxford: Oxford University Press.

Part IV
Technology

Chapter 11
Time, Immersion and Articulation: Digital Technology for Early Childhood Mathematics

Nathalie Sinclair

Abstract Over the past five years, I have been involved in two major technology-based early childhood research projects, one focused on geometry (Geometry4yl) and the other on number sense (TouchCounts), both deeply rooted in a Papertian approach to technology design that seeks to transform mathematics and, potentially, the teaching and learning of mathematics. Three novel and significant themes have emerged in this work: the temporalizing of early childhood mathematics (time); the exposure of young children to advanced mathematics (immersion); and, the relations between digital technologies and paper-and-pencil technology (articulation). I also consider the challenges that teachers face in integrating new technologies that differ significantly from existing paper-and-pencil modalities and physical manipulatives.

Keywords Technology · Multi-touch · Time · Number · Geometry

11.1 Introduction

The past decade has yielded a proliferation of new digital technology for primary school mathematics education, which, especially at the grades K-2 level, had previously resisted the incursion of computers into the classroom. The dramatic shift coincides, it would seem, with the emergence of touchscreen tablets, which make interaction much easier for young children than did the keyboard, and which promote more mobility and flexible use in the classroom than did desktop computers or even laptops.

Research on the use of touchscreen devices at the primary school level is only now starting to become available. Given the growing number of apps (especially for the iPad), most researchers have focused on the challenge of evaluating their pedagogical potential in view to support teachers in choosing the most appropriate

N. Sinclair (✉)
Simon Fraser University, V5A1S6 Burnaby, Canada
e-mail: nathsinc@sfu.ca

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_11

apps for their practice. For example, Larkin (2016) evaluated 142 different apps designed for use at the primary school level and found that only three scored highly on all the measures he used, which included the Haugland Software Developmental Scale (Haugland 1999), Gee's (2003) Learning Principles and Atweh and Bland's (2005) Productive Pedagogies. Interestingly, these measures make very little reference to the long-standing research in mathematics education that relates to the middle and secondary school levels, perhaps because the nature of mathematics involved in each grade span is seen as being very different.

These measures can provide helpful guides for mathematics teachers, but they are less useful in helping researchers understand the specific ways in which mathematical concepts are being offered, and the new opportunities they provide learners and teachers for interacting with and thinking about these concepts. As a complement to macroscopic analyses conducted by Larkin and others, this paper examines very specific *mathematical* features of digital technology designed for primary school education and the impact that such features might have on the concepts at stake. By definition, then, criteria such as developmental appropriateness (Haugland 1999) or curricular fidelity (Dick 2008) may be disrupted, inasmuch as novel technology changes the pace, order and nature of what students can learn. This may make the integration of these technologies very challenging for teachers, a point I will return to later in the paper.

Over the past five years, I have been involved in two projects featuring the use of digital technology in the early years: Dynamic geometry for young learners (five to ten-years old) and *TouchCounts* (3–7 year old). These two projects focus on geometry and number, respectively. In this section, I will briefly outline each project and its relation to existing literature. Indeed, prior research, even at the middle and secondary school levels, can provide very useful insights for researchers working at the primary school level, even though the mathematical concepts at stake are different. In the next section, which is the heart of this paper, I will describe the three *mathematical* themes that have emerged as novel and significant throughout this work, and that I think are relevant both to the present context of technology use in early childhood education but also to future research and curriculum design.

11.1.1 Dynamic Geometry for Young Learners

The geometry project builds on decades-old research in the use of dynamic geometry environments (DGEs) (see, for example, Arzarello et al. 2002; Baccaglini-Frank and Mariotti 2010; Hollebrands et al. 2008; Laborde 2000), which were initially designed for high school mathematics, but have since percolated down—as well as up—the grades and also spread across to other topics such as algebra. In 2006, after finishing a project aimed at developing *Sketchpad*-based activities for the grades 3–8 level and realising how little geometry was actually done in most elementary classrooms in North America (see Clements and Sarama

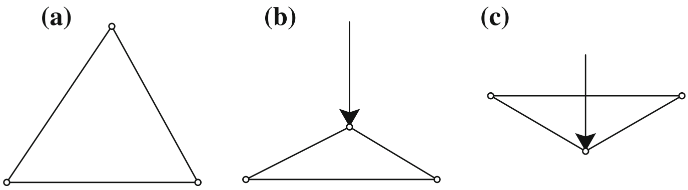


Fig. 11.1 **a** A triangle; **b** dragging a vertex down; **c** dragging the vertex further

2011), I decided to explore activities that might be feasible for even younger children, starting at the kindergarten level. Drawing on research conducted by Battista (2008), who showed how upper elementary children could develop more robust understanding of quadrilaterals, my first study involved investigating the use of dragging to learn about triangles.[1]

Dragging is a significant action in a DGE and so is worth explaining. Given a triangle with three vertices (see Fig. 11.1a), dragging the top vertex changes the shape of the triangle (making it "skinnier", as in Fig. 11.1b, or "upside down", as in Fig. 11.1c) but maintains its status as a three-sided polygon. The dragging is continuous (which is difficult to represent on static paper), which may help children see the three-sidedness of the shape as the invariance. Given extensive research showing that young children tend to reply on prototypical images (such as an equilateral triangle with its base parallel to the page, as in Fig. 11.1a) (see Battista 2007), the possibility of dragging is relevant because it quickly produces a family of non-prototypical triangles.

In the research at the secondary level, studies have focused primarily on how dragging can help students engage in geometric constructing, conjecturing and proving, activities that are less commonly associated with the primary school level. However, some researchers, such as Battista, have studied how students learn to identify quadrilaterals and, more specifically, develop a sense of the inclusive relations between various quadrilaterals. This research is closer to the primary school activities of identifying and comparing shapes and so is more directly relevant. In particular, it highlights two mathematical issues related to how learners perceive geometric objects. One is captured by Marrades and Gutiérrez (2000): "the main advantage of DGS learning environments" is that "students have access to a variety of examples that can hardly be matched by non-computational or static computational environments" (p. 95). The other issue, articulated in Laborde (1992) and Battista (2008), focuses on the continuous transformation of the draggable object rather than on the set of characteristics that can be abstracted from a given set

[1]It is worth noting that around the same time, the designers of Cabri-géomètre began development of their primary school software Cabri Elem, which also used dynamic geometry modalities for primary school geometrical concepts (Laborde and Laborde 2011).

of examples. This second issue underscores the way in which the object is an entity whose behaviour can be investigated and described—indeed, the object must be moved in order for it to be investigated. And, if we take Battista's hypothesis that people notice invariance—that what doesn't change under variation becomes salient, and can therefore be identified as a property—then dragging can be a powerful way of drawing learners' attention to geometric properties. Thus, DGEs are relevant to the primary school curriculum, which involves extensive work around the identification and comparison of geometric properties for both two- and three- dimensional shapes.

This has been borne out by several studies with which I've been involved in the past five years. In the earliest, which involved the use of desktop *Sketchpad* projected onto a screen, Sinclair and Moss (2012) describe how kindergarten children went from asserting that the triangle in Fig. 11.1c was *not* a triangle to explaining that it was, because it "always has three sides"—thus moving from reasoning based on a prototypical image, to reasoning based on noticing invariance over a certain kind of transformation, namely dragging. Developmental research, based on theories such as the van Hiele levels, maintain that young children do not engage in property-based reasoning, but such research is based on the use of physical manipulatives, which are rarely seen in non-prototypical forms and certainly not continuously transformable. Sinclair and Moss show, however, that children can begin describing and even defining triangles in kindergarten, when they have the opportunity to see and discuss a wide range of continuously varying three-sided shapes.

Later research within this project, but using desktop *Sketchpad* projected onto an interactive whiteboard, which enabled children to drag objects on the screen, has focused on comparing and identifying different types of triangles, such as isosceles, equilateral, right and scalene (Kaur 2013). These children from grades 1–2 were able to think of these triangles in terms of their inclusive relations (that an equilateral triangle is a special kind of isosceles triangle) and thus operate at the developmental van Hiele level of 3. Again, the digital technology, as well as the accompanying tasks, enabled the children to describe and think about these shapes in terms of their behaviour, so that instead of describing a scalene triangle as having three sides, the scalene triangle was seen as the most flexible of the different triangles since it could be dragged into any kind of triangle, including one with two or three equal sides.

Another concept that is rarely directly addressed in the primary school grades, even though it is informally relevant when children learn about squares and rectangles, is parallel and perpendicular lines, which grade 1 children were able to identify and describe (Sinclair et al. 2013). Likewise, the concept of angle was introduced to kindergarten children using the dynamic notion of angle-as-turn, which enabled children to compare and describe angles in qualitative terms, instead of using angle measure (Kaur 2013). Finally, the concept of reflectional symmetry was the target concept in a study with grade 2–3 children, where the dynamic movement of shapes and their images enabled children to develop a functional approach to symmetry and to describe symmetry in terms of the equidistance of the shape and its image to the line of symmetry (Ng and Sinclair 2015). Most recently, a project extension involving the use of the multi-touch Web Sketchpad (used with a classroom set of iPads), showed how various aspects of the concept of symmetry emerged in a classroom of grade 1 students (Chorney and Sinclair in press). Unlike the approach offered by Fletcher and Ginsburg (2016), which uses static reflections, our research has built on the dynamic approaches that have been shown to be effective at the secondary school level (see Hollebrands 2003).

These are all concepts that are rarely undertaken before grade 3 (with Fletcher and Ginsburg's work (2016) being a notable exception), but that have proven to be accessible and rich for younger learners, when supported. Without going into details for each study, the main feature of each is dragging: children drag or move objects to explore their behaviour. So, for example, instead of defining symmetry, children drag an object and see how its reflected image moves as a result. While the project began with the desktop version of *Sketchpad* (projected on to a whiteboard or an IWB), it has since moved to *Sketchpad Explorer* (on the iPad) and to Web-based Sketchpad (sketches can be found here: http://www.sfu.ca/geometry4yl.html). These two latter instantiations of *Sketchpad* are multi-touch, which means that users can drag multiple objects at the same time. In the example given above, with the triangle, for example, a young learner could drag all three vertices of the triangle; alternatively, three different children could each drag one vertex of the triangle, perhaps working together to fit it into a certain position (as can be seen in this activity: www.sfu.ca/content/dam/sfu/geometry4yl/sketchpadfiles/Triangle%20Designs/). Comparing Fig. 11.2a and b nicely highlights the difference between single-touch and multi-touch interactions; in the latter, many children can interact simultaneously, each potentially using more than one finger. The act of using three fingers to drag the triangle—which must be practiced, for many young learners—involves a gesture that reifies an essential property of the triangle (that is has three vertices). Multi-touch dynamic geometry thus offers both mathematical and pedagogical opportunities that have only recently been pursued (Jackiw 2013).

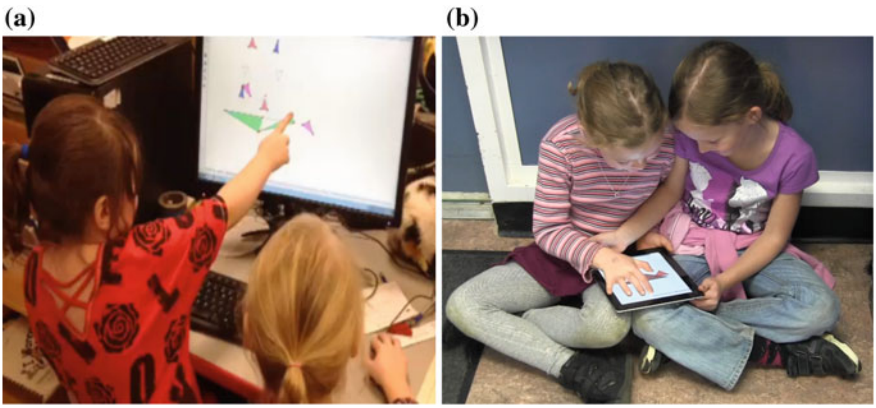


Fig. 11.2 **a** Single-touch desktop computer interaction; **b** multi-touch tablet interaction

Evaluating the multitouch digital technology—and the accompanying tasks—in terms of curriculum fidelity or developmental appropriateness would be problematic. However, for mathematics education researchers, there is much to learn about the nature of the concepts at play, how they are transformed by technology and potentially made more learnable and teachable. It is important to point out that this requires an ontological shift that rarely occurs in digital technologies that attempt to replicate existing technologies (such as virtual manipulatives) or existing school mathematics instantiations of geometric concepts, as with all the apps evaluated in Larkin (2016).

11.1.2 Early Number Sense with TouchCounts

In 2011, when the touchscreen tablets had recently come out, I led a team that developed a multi-touch app called *TouchCounts*, primarily for counting, but also for adding and subtracting (Jackiw and Sinclair 2014). Many apps for primary school number sense had already been created, but the large majority of them were single-touch (often because their design was based on existing single-touch—through the mouse of the keyboard—desktop software) and were level-driven and/or drill-based, often providing the users with only evaluative feedback (as in the interactive mathematics book described by Ginsburg et al. 2018). Sinclair and Baccaglini-Frank (2016) provide an overview of the basic component abilities that multi-touch technology can mediate, including: (1) subitizing; (2) one-to-one correspondence between numerosities in analogical form; (3) fine motor abilities; and, (4) part-whole relations. In her chapter, Baccaglini-Frank (2018) extends this list to include finger tapping (which is related to finger gnosia), estimation and four principles (other than one-to-one correspondence) considered necessary for children to master: the stable-order principle, the last-word rule that assigns the last said numeral not to the last counted object, but to the quantity as a whole, the principle of abstraction (objects of any nature can be counted) and order indifference.

In *TouchCounts*, counting happens in the Enumerating world, in which a user taps her fingers on the screen to summon numbered objects (yellow discs). The first tap produces a disc containing the numeral ‘1.’ Subsequent taps produce successively numbered discs. As each tap summons a new numbered disc, *TouchCounts* audibly speaks the number word for its number (“one,” “two,” …, if the language is set to English). Fingers can be placed on the screen one at a time or simultaneously. With five successive taps, for instance, five discs (numbered ‘1’ to ‘5’) appear sequentially on the screen, which are counted aloud one by one. However, if the user places two fingers on the screen simultaneously, two consecutively numbered discs appear at the same time, but only the higher-numbered one is named aloud (“two,” if these are the first two taps). The discs always arrive in order, with their symbolic names imprinted upon them.

If the ‘gravity’ option is turned on, then as long as the learner’s finger remains pressed to the screen, the numbered object holds its position beneath her

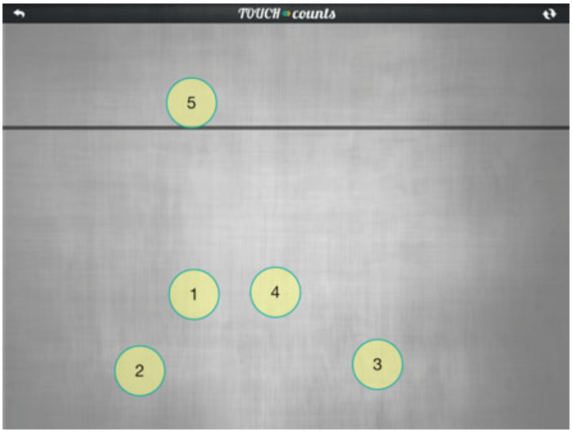


Fig. 11.3 The screen after four sequential taps below the shelf and a fifth tap above the shelf

fingertip. But as soon as she 'lets go' (by lifting that finger), the numbered object falls and then disappears "off" the bottom of the screen. With 'gravity' comes the option of a 'shelf,' a horizontal line across the screen. If a user releases her numbered object above the shelf, it falls only to the shelf and comes to rest there, rather than vanishing out of sight 'below.' (Fig. 11.3 depicts a situation in which there have been four taps below the shelf—these numbered objects were falling—and then a disc labelled '5' was placed above the shelf by tapping above it. See also https://www.youtube.com/watch?v=5PHzSlxSaWo)

Whilst tapping on the screen in the Enumerating World creates sequentially numbered objects, tapping on the screen in the Operating World creates autonomous numbered sets, which we refer to as herds. The user places one or several fingers simultaneously on the screen, which immediately creates a large disc that encompasses all the fingers. When the fingers are lifted off the screen, the numeral is spoken aloud and the smaller discs are then lassoed into a herd and arranged regularly around the inner circumference of the big disc (Fig. 11.4a shows herds of 3 and 4). After two or more such arrangements have been produced (as in Fig. 11.4b) they can either be pinched together (addition) or 'unpinched' (subtraction or partition). The pinching gesture draws on one of the four grounding

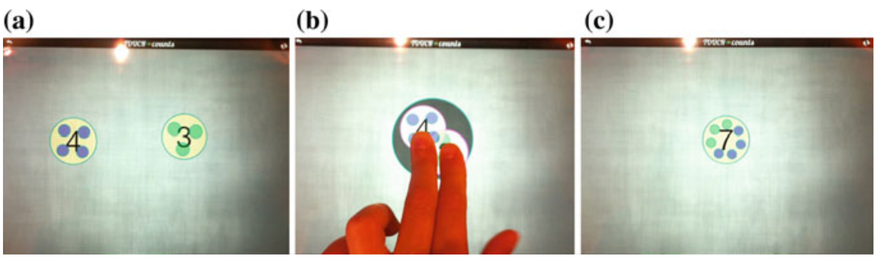


Fig. 11.4 **a** The herds; **b** pinching two herds together; **c** the sum of two herds

metaphors for addition, that of object collection (see Lakoff and Núñez 2000). It expresses the very idea of adding, and does so in a symmetric way so that, unlike symbolic expressions of adding (3 + 4 or three plus four), it does not imply a particular order. The new herd is labelled with the associated numeral of the sum (Fig. 11.4c), which *TouchCounts* announces aloud. Moreover, the new herd keeps a trace of the previous herds, which can be seen by means of the differentiated colours.

In our studies with three to five-year olds (pre-school), as well as five to seven-year olds (kindergarten and grade 1), we have been particularly interested in how the "shelf" provides children with an opportunity to move beyond the memorized number song and begin to attend to the relation between numbers (what comes before 10, what comes after 4, etc.), which is necessary part of developing number sense (see Sinclair and Heyd-Metzuyanim 2014). In our classroom experiments, we have also asked children to engage in skip counting using the shelf. This might involve, for example, placing four fingers below the shelf and then the fifth one above, and then repeating that sequence of actions several times in order to obtain 5, 10, 15, 20, etc. on the shelf (Sinclair et al. 2016). The four finger all-at-once tap provides a reification of the "jump" between consecutive numbers, one that is more gestural and even temporal in nature than the skip counting on a 100 s chart, for example.

We have also examined the following aspects of children's activity with *TouchCounts*: finger gnosis (Sinclair and Pimm 2015b), gestural subitizing (Sinclair and Pimm 2015a), attention to symbols and place value (Sinclair and Coles 2015). Indeed, by virtue of being able to so easily create very large numbers (children delight in making numbers such as 100, as well as numbers they have never even seen before, such as 479), the exclusive focus on the 1–20 range that is common in kindergarten and even grade 1 gives way to an immersion—both symbolic (123) and spoken (one hundred and twenty-three)—in the much more regularly spoken numbers (at least in English and French) of the 100s and beyond. We have noticed young children quickly gaining fluency in predicting what number comes after numbers such as 78 or 124, not because they necessarily have a sense of the actual size of those numbers (their cardinal value), but because they can work directly on the symbols. As Coles and Sinclair (2017) argue, working with these larger numbers can evoke the temporal and linguistic aspect of place value, which contrasts with the cardinal aspects that are emphasized in most school curricula and with most manipulatives (such as Dienes blocks).

However, the most surprising—and historically/philosophically interesting—feature of children's use of *TouchCounts* relates to ordinality. By ordinality, I follow Coles (2014) (who cites Tahta and Gattegno) in stressing not only number as sequence and order (in the sense of Peano or Dedekind), but also as the relation between symbols—which might involve knowing whether, for example, the numbers [4, 5, 6] are in order. A given ordinal number gets its meaning from the one that precedes or follows it. One does not need to know how big 1,000,001 really is to know that it is one more than 1,000,000 because 1,000,001 immediately succeeds in the whole number count list. One characteristic feature of ordinality is that it is not just spatial, but also temporal, which will be the first theme discussed in the next section. What *TouchCounts* changes to the usual technologies around ordinality is the multimodal connecting of touch, sight, sound and symbol, the gestural interaction, as well as the effortless production of numbers.

11.2 Themes

Whilst quite different in terms of content area and technology design, the two projects described above share some interesting similarities, which I describe below. These similarities are not so much about pedagogical approaches that the technologies support, but more about how the technology interacts with the relevant mathematical objects and relations.

11.2.1 Time

Despite the fact that humans live in time, much of modern mathematics can be characterized as being static and de-temporalized. This may be in part due to the way that it is written and drawn in textbooks and other paper media (Pimm 2006), but may also be a result of a long-standing suspicious attitude towards motion in mathematics, which began with the Ancient Greeks (Châtelet 2000). New digital technologies have challenged this static view of mathematics (Rotman 2008) and new theories of embodied cognition have argued that mathematical ideas emerge from our actions in the world (Lakoff and Núñez 2000). Indeed, researchers have shown that despite the de-temporalized nature of formal mathematics (i.e., written mathematics), mathematicians often think of mathematical concepts in temporal ways, as can be seen through their gestures (Núñez 2006; Sinclair and Gol Tabaghi 2010). An even greater number of studies have shown that this holds for mathematics learners as well. Many of these studies, however, whilst convinced that children's movements (gestures, body motions) can be significant in enabling them to develop mathematical understandings, remain agnostic about the ontological question of whether the mathematics itself—the objects and relations—are of a pure, immutable Platonic nature, or of this world. Most digital technologies for young learners follow suit, opting to put the non-mathematical components of the program in motion (hopping rabbits, twinkling starts, etc.), but to leave the mathematical ones static.

The possibility of a dynamic triangle, which can become any triangle—any shape, size, location—a triangle that *becomes* over time, rather than existing as a proposition accompanied by an illustrative diagram, invites learners and teachers to think of the triangle as a temporal concept. Thus far, our research has shown that working within this temporal discourse has pedagogical benefits in that it provides learners with a large example space (and thus decreases the negative impact of prototypical images), helps them attend to invariance (e.g., that an object and its reflected image are equidistant from the line of symmetry) and enables them to engage with sophisticated mathematical ideas at quite a young age. But it has also been interesting to see a shift in the mathematical discourse that has emerged as a result of their experiences. After dragging a triangle on the screen into various positions, and having to decide whether an upside down three-sided shape and a long-and-skinny one deserve to be called triangles, students have explained that these shapes are indeed triangles because "no matter how you move it, it just has three sides". Or when asked whether two angles are the same size, when one of them has longer arms than the other, children say "you have to turn them by the same amount". And when asked if a shape is symmetric, children have said "if one side moves away from the line when the other side does". These statements, I would argue, do not merely reflect the fact that children have moved their bodies or gestured in order to learn about geometry, but that they are learning about a moving geometry.

With *TouchCounts*, the ordinal aspect of number serves to revive a more temporal conceptualization of number than the cardinal aspect of number. It may well be that the cardinal aspect has become so dominant because of the general de-temporalization of mathematics, though several mathematicians have shown that one can get by without cardinality (see Rips 2015). It may well be, if we subscribe to the hypothesis of Seidenberg (1962), that the origins of counting are also temporal in nature, emerging out of rhythmic, ritual practices. In our work with young children, we have seen particularly temporal discourses emerging as well, such as "204 comes later than 200" (i.e., is said after when counting in the 'right' order). In contrast to the more cardinal expression "204 is bigger than 200", the temporal/ordinal statement is inscribed within the action of tapping on the screen, in which, indeed, 204 appears after 200 does. Statements such as "come after" and "come before" are also within the ordinal discourse.

As with the case of dynamic geometry, such statements reflect deeper ontological orientations about mathematics in that concepts such as numbers are things that occur in time, rather than objectified, cardinal quantities. While some mathematics education researchers have argued for a more ordinal approach to early number (such as Gattegno 1974), recent research in the neurosciences shows that what is significant in the learning of mathematics is not being able to link symbols to objects (which is the typical practice in early number) in a manner that is often considered accessible, but being able to link symbols to other symbols (Lyons and Beilock 2011). It also shows that skilled symbolic ordinal processing is correlated with success in higher levels of mathematics more generally (Lyons et al. 2014). Linking symbols to symbols can surely be done in a de-temporalized manner, but within the practices of *TouchCounts* counting, the symbols become linked *through* time. A question arises as to what related number concepts could also be re-temporalized. Addition? Place value?

11.2.2 Immersion

This second theme emerges from the way in which digital technology provides particular constraints that usually differ from those of other technologies. For example, with pattern block manipulatives there is a triangle shape that is green and small, isolated and singular, and equilateral, which constrains the set of triangles. It is also three-dimensional, which actually makes it a triangular prism. This pattern block tangram is usually sufficient for early geometry learning, in which the focus is often on naming and classifying shapes. On printed worksheets, the triangles will be more two-dimensional and will take on a slightly larger variety of colour, size and orientation. The mathematical coherence of a DGE such as *Sketchpad* enables users to make any triangle, or any colour, size, shape and orientation; the triangle can even be put into motion through the animation functionality. I view this as full immersion into the mathematical triangle in that children can make and encounter triangles they might not meet formally until later in their schooling, such as isosceles or obtuse-angled triangles, or even a triangle in which a vertex has been dragged onto the opposite side.

In a sense, such technologies are blurring distinctions between concrete and abstract, between physical and symbolic, between real and imaginary. Such blurring will likely have an effect on long-standing developmental stances, which assume a transition from the concrete to the abstract, for example. It may be, however, that in interacting with an object that is both concrete and abstract (concrete because you can touch, it, drag it; abstract because it is virtual, precise, and infinitely malleable) these developmental trajectories are disrupted.

In the case of *TouchCounts*, the immersion is into the world of all whole numbers, not just brackets of numbers identified in the curriculum. This means that young children will not infrequently create numbers that are very big, and will delight in repeating what they hear, "three hundred and seventy-four?" This is very different than a ten-frame, for example, in which the constraint is set to 10, or even with counters, where counting rarely exceeds 100. We have been asked several times whether we could produce a version of *TouchCounts* that does not go past 100, with the argument being that children will get confused by the bigger numbers.

In both cases, children are working within an environment that has not been constrained for psychological or historical reasons (or by the curriculum), but for mathematical ones (you cannot turn a triangle into a square and you can always keep adding one). It is certainly possible to argue that there are pedagogical benefits to working within such environments (for example, it is hard to appreciate the structure of numbers if you stop at 20), but the possibility offered within these new environments can also push us to reflect on our assumptions of how concepts should be sequenced. That said, the pedagogical benefits to opening up the curriculum into less constrained environments will have important consequences on the expectations and preparedness of teachers. Baccaglini-Frank (2018) highlights this challenge in pointing out that the educator who used the apps with young children often overruled the children's strategies and promoted a more narrow understanding of number (counting and/or using known finger configurations). Indeed, the importance of appropriate and extensive teacher professional development has been documented in many other situations involving the introduction of novel digital technologies.

11.2.3 Articulation

In a discussion about the challenges involved in teaching and learning arithmetic, Tahta (1998) distinguishes between metaphoric and metonymic ways of accessing number. The metaphoric ways might involve an abacus, a ten-frame, rods, or any other re-presentations of numbers (as well as pictures of apples and balls, as presented by Ong (2016). Metonomy involves a change in name, so the numeral (symbol) or the number-word is another name for number. Tahta argued that there is too much metaphoric work in early number, and not enough metonymic work. In this sense, I see *TouchCounts* as providing a metonymic way of accessing number since children are working directly on and with number names and symbols, through their finger tapping and gestures. What this means is that the distance to number (which some might call the transfer required) is lessened. A similar situation occurs with the example of triangles in that children are working directly with two-dimensional shapes, directly moving points or edges rather than manipulating something that represents a vertex of or segment. As Jackiw and Sinclair (2006) write, the "heart of such a 'dynamic geometry' experience lies in its idealized and friction-free mathematical physics" (p. 146). The dragging triangle is an actualization of the mathematical triangle in that it communicates precision and generality.

As environments of metonomy one can thus argue that they provide an experience of mathematics that is closer to mathematics, requiring less of a transfer or articulation. That is interesting in and of itself but it is also the case that in our research projects, inspired by the work of Bartolini Bussi and others, we have designed activities in which digital tool-based interactions are almost always followed up by paper-and-pencil diagramming. According to the theory of semiotic mediation (Bartolini Bussi and Mariotti 2008), the act of drawing offers a mechanism through which actions on the screen can become mathematical signs. For example, the drawing in Fig. 11.5a was produced by a Canadian 6 year-old after having dragged triangles on an iPad and Fig. 11.5b was drawn by an Italian 6 year-old who had created 5 in different ways using *TouchCounts*.

While the appearance of fingers/hands in these drawings does not happen all the time, their inclusion in these ones evokes nicely the way in which the hand can be seen as being part of the tool. In geometry, the connection with the hand is long-standing, as Catton and Montelle (2012) write: "in order truly to learn from Euclid, one needs one's hands, not only one's eyes and ones brain" (p. 27). And in number, fingers are thus simultaneously subject and object 'counting with my fingers' and 'counting on my fingers', with increasing neuro-scientific evidence

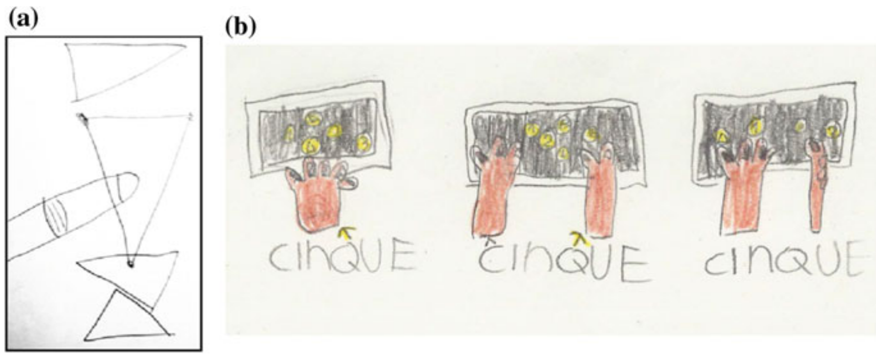


Fig. 11.5 **a** Drawing of a moving triangle, **b** drawing of making 5 in *TouchCounts*

showing the strong link between fingers and arithmetic. In both cases, the moving technology is perhaps bringing fingers back into play, drawing them back into mathematics.

11.3 Concluding Remarks

Digital technology for primary school education has quickly been populated by touchscreen, tablet-based applications, which are now available in many schools and relatively easy to use (in comparison with desktop software). The majority of the applications developed thus far fall into the category of “edutainment” (Larkin 2016), with the more pedagogically serious ones replicating existing physical manipulatives (Cuisinaire rods, geoboards, tens-charts, etc.). These latter applications tend to be rated highly by mathematics education researchers, especially for the developmental and curricular fidelity. Applications that depart significantly from physical technologies (paper-and-pencil, manipulatives) may present significant challenges for teachers.

Even at the secondary school level, for example, teachers can find the integration of DGEs very difficult since they require significant changes in practice. Indeed, Ruthven’s (2014) framework for analysing the expertise that underpins successful integration of digital technology highlights five significant changes in practice that are required, from the amount of time to spend on various concepts to the design of assessment tasks. Sinclair and Yurita (2008) underscore the significant change in discourse required for teachers shifting from a static geometry environment to a dynamic one. With respect to *TouchCounts*, the prevalence of symbol use, the lack of constraint on the size of numbers, as well as the emphasis on ordinality can also present challenges for teachers. As Coles and Sinclair (2017) report, teachers have expressed concern over the perceived lack of “meaning” in *TouchCounts*, where meaning is associated with cardinal quantities represented by physical manipulatives such as tens-charts and Dienes blocks.

There is thus a challenging choice to be made, both by teachers and researchers, about the extent to which new digital technologies—those that significantly change mathematics—can and should be integrated. Now, thirty years after their initial introduction, DGEs have gained such widespread approval in the mathematics education community that dynamic geometry objects are now being used in on-line textbooks and even in assessments. Indeed, dynamism has spread beyond geometry and is now also a feature of many software programs focused on the teaching and learning of algebra (such as Desmos). Most of the research at the secondary level focuses less on student learning and more on teacher integration, task design and assessment (see Sinclair and Yerushalmy 2016). Hopefully, the results of this research can be used to guide the professional development of primary school teachers who wish to use DGEs in their classrooms.

Based on research I have been doing for the past five years, involving the use of digital technology in early mathematics learning, I have identified three characteristic features that are common to two disparate digital technologies, each of which represents a major shift in relation to the nature of the mathematical concepts at play. The first two themes, in particular, arise from specific design decisions made within the digital technologies (DGE and *TouchCounts*), and may thus be much less relevant for other digital technologies that have been designed and researched. The first theme highlights the temporal aspect of mathematics, and its uneasy relationship to modern and/or formal mathematics, and emerges out of design decisions to enable mathematical objects and relations to be put in motion. The second theme also relates to design in the sense that it concerns the mathematical coherence of technology design and its consequences on the sequencing of topics in the curriculum. The third theme concerns the articulation process that is involved in working with models of or manipulatives (virtual or not) for mathematics. This process of articulation may be much more transparent in technology-based environments in which children are working with metonyms rather than metaphors, directly on numbers, vertices and edges, but the question of how such environments can be productively connected to existing resources still remains.

References

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in Cabri environments. *Zentralblatt für Didaktik der Mathematik, 34*(3), 66–72.

Atweh, B., & Bland, D. (2005). Mathematics through/for understanding social life: Productive pedagogies meets critical mathematics. In M Goos, C. Kanes & R. Brown (Eds.), *Proceedings of the 4th International Mathematics Education and Society Conference* (pp. 47–57). Brisbane: Centre for Learning Research, Griffith University.

Baccaglini-Frank, A. (2018). What schemes do preschoolers' develop when using multi-touch applications to foster number sense (and why)? In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank and C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Baccaglini-Frank, A., & Mariotti, M. A. (2010). Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning, 15*(3), 225–253.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. G. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (2nd ed.) (pp. 746–783). New York and London: Routledge.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Reston, VA: NCTM.

Battista, M. T. (2008). Development of shapemakers geometry microworld. In G. W. Blume, M. K Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Cases and perspectives* (Vol. 2, pp. 131–156). Information Age Publishing.

Catton, P., & Montelle, C. (2012). To diagram, to demonstrate: To do, to see, and to judge in greek geometry. *Philosophia Mathematica, 20*(1), 25–57.

Châtelet, G. (1993/2000). *Les enjeux du mobile*. Paris: Seuil. (Engl. transl., by R. Shore & M. Zagha: *Figuring space: Philosophy, Mathematics and Physics*). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Chorney, S., & Sinclair, N. (in press). Fingers on geometry: Primary school children using multi-touch dynamic geometry. In N. Calder, K. Larkin & N. Sinclair (Eds.), *Using mobile technologies in the learning of mathematics*. Heidelberg, Germany: Springer.

Clements, D. H., & Sarama, J. (2011). Early childhood teacher education: The case of geometry. *Journal of Mathematics Teacher Education, 14*(2), 133–148.

Coles, A. (2014). Ordinality, neuro-science and the early learning of number. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allen (Eds.). *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 329–336). Vancouver: PME 38.

Coles, A., & Sinclair, N. (2017). Re-thinking place value: From metaphor to metonomy. *For the Learning of Mathematics, 37*(1), 3–8.

Dick, T. P. (2008). Fidelity in technological tools for mathematics education. In G. W. Blume & M. K. Reid (Eds.), *Research on technology and the teaching and learning of mathematics: Cases and perspectives* (Vol. 2, pp. 333–339). Charlotte, N.C.: Information Age Publishing.

Fletcher, N., & Ginsburg, H. (2016, July). *Teaching symmetry in the early childhood classroom: using software to facilitate discourse and promote understanding of symmetric transformations*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Gattegno, C. (1974). *The common sense of teaching mathematics*. New York, NY: Educational Solutions.

Gee, J. P. (2003). *What video games have to teach us about learning and literacy*. New York: Palgrave Macmillan.

Ginsburg, H. P., Uscianowski, C., & Almeda Ma. V. (2018). Interactive mathematics storybooks and their friends. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Haugland, S. (1999). Computers and young children: The newest software that meets the developmental needs of young children. *Early Childhood Education Journal, 26*(4), 245–254.

Hollebrands, K. (2003). High school students' understanding of geometric transformations in the context of a technological environment. *Journal of Mathematical Behavior, 22,* 55–72.

Hollebrands, K., Laborde, C., & Sträßer, R. (2008). Technology and the learning of geometry at the secondary level. In K. Heid & G. Blume (Eds.), *Research in technology and the teaching and learning of mathematics: Research syntheses* (Vol. 1, pp. 155–203). Charlotte, NC: Information Age Publishing.

Jackiw, N. (2013). Touch and multitouch in dynamic geometry: Sketchpad explorer and "digital" mathematics. In Faggiano, E. & Montone, A. (Eds.), *Proceedings of the 11th International Conference on Technology in Mathematics Teaching* (pp. 149–155). Bari, Italy: Università degli studi di Bari Aldo Moro.

Jackiw, N., & Sinclair, N. (2006). Dynamic geometry design for elementary school mathematics. In C. Hoyles, J.-B. Lagrange, L. Hung Son, & N. Sinclair (Eds.), *Proceedings of 17th ICMI Study conference, Technology Revisited* (pp. 236–245). Hanoi: Hanoi University of Technology.

Jackiw, N., & Sinclair, N. (2014). *TouchCounts*. Application for the iPad.

Kaur, H. (2013). Children's dynamic thinking in angle comparison tasks. In A. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 145–152). Kiel, Germany: PME.

Laborde, C. (1992). Solving problems in computer based geometry environments: the influence of the features of the software. *Zentrablatt für Didactik des Mathematik, 92*(4), 128–135.

Laborde, C. (2000). Dynamical geometry environments as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics, 44*(1–2), 151–161.

Laborde, C., & Laborde, J.-M. (2011). Interactivity in dynamic mathematics environments: what does that mean? http://atcm.mathandtech.org/EP2011/invited_papers/3272011_19113.pdf. Accessed August 8, 2012.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.

Larkin, K. (2016). An App! An App! My kingdom for an App: An 18-month quest to determine whether Apps support mathematical knowledge building. In T. Lowrie & R. Jorgenson (Eds.), *Digital games and mathematics learning* (pp. 251–276). New York: Springer.

Lyons, I., & Beilock, S. (2011). Numerical ordering ability mediates the relation between number-sense and arithmetic competence. *Cognition, 121*(2), 256–261.

Lyons, I., Price, G., Vaessen, A., Blomert, L., & Ansari, D. (2014). Numerical predictors of arithmetic success in grades 1-6. *Developmental Science, 17*(5), 714–726.

Marrades, R., & Gutiérrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44,* 87–125.

Ng, O., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM—The International Journal on Mathematics Education, 47*(3), 421–434.

Núñez, R. (2006). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. In R. Hersh (Ed.), *18 Unconventional essays on the nature of mathematics* (pp. 160–181). New York: Springer.

Ong, D. J. (2016, July). *Simple augmented reality (ar) for early childhood mathematics*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Pimm, D. (2006). Drawing on the image in mathematics and art. In N. Sinclair, D. Pimm, & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 160–189). New York: Springer.

Rips, L. J. (2015). Beliefs about the nature of numbers. In E. Davis & P. J. Davis (Eds.) *Mathematics, substance and surmise: Views on the meaning and ontology of mathematics* (pp. 321–345). Springer International Publishing.

Rotman, B. (2008). *Becoming beside ourselves: The alphabet, ghosts, and distributed human beings*. Durham: Duke University Press.

Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 373–393). Netherlands: Springer.

Seidenberg, A. (1962). The ritual origin of counting. *Archive for History of Exact Sciences, 2*(1), 1–40.

Sinclair, N., & Baccaglini-Frank, A. (2016). Digital technologies in the early primary school classroom. In L. English & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (3rd ed., pp. 662–686). New York, NY: Taylor Francis/Routledge.

Sinclair, N., Chorney, S., & Rodney, S. (2016). Rhythm in number: Exploring the affective, social and mathematical dimensions of using TouchCounts. *Mathematics Education Research Journal, 28*(1), 31–51.

Sinclair, N., & Coles, A. (2015). 'A trillion is after one hundred': Early number and the development of symbolic awareness. In X. Sun, B. Kaur & J. Novotná (Eds.) *Proceedings of ICMI Study 23, Primary Mathematics Study on Whole Numbers* (pp. 251–259). Macau, China: University of Macau.

Sinclair, N., de Freitas, E., & Ferrara, F. (2013). Virtual encounters: The murky and furtive world of mathematical inventiveness. *ZDM—The International Journal on Mathematics Education, 45*(2), 239–252.

Sinclair, N., & Gol Tabaghi, S. (2010). Drawing space: Mathematicians' kinetic conceptions of eigenvectors. *Educational Studies in Mathematics, 74*(3), 223–240.

Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning number with *TouchCounts*: The role of emotions and the body in mathematical communication. *Technology, Knowledge and Learning, 19*(1–2), 81–99.

Sinclair, N., & Moss, J. (2012). The more it changes, the more it becomes the same: The development of the routine of shape identification in dynamic geometry environments. *International Journal of Education Research, 51 & 52,* 28–44.

Sinclair, N., & Pimm, D. (2015a). Whatever be their number: Counting on the visible, the audible, and the tangible. In. M. Meletiou-Mavrotheris, K. Mavrou, E. Paparistodemou (Eds.), *Integrating touch-enabled and mobile devices into contemporary mathematics education* (pp. 50–80). Hershey: IGI Global.

Sinclair, N., & Pimm, D. (2015b). Mathematics using multiple sense: Developing finger gnosis with three-and four-year-olds in an era of multi-touch technologies. *Asia-Pacific Journal of Research in Early Childhood Education, 9*(3), 99–109.

Sinclair, N., & Yerushalmy, M. (2016). Digital technology in mathematics teaching and learning: A decade focused on theorising and teaching. In A. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education* (pp. 235–274). Sense Publishers.

Sinclair, N., & Yurita, V. (2008). To be or to become: How dynamic geometry changes discourse. *Research in Mathematics Education, 10*(2), 135–150.

Tahta, D. (1998). Counting counts. *Mathematics Teaching, 163,* 4–11.

Chapter 12
What Schemes Do Preschoolers Develop When Using Multi-touch Applications to Foster Number Sense (and Why)?

Anna Baccaglini-Frank

Abstract As part of an educational project proposed in Italian pre-schools, an educator followed a protocol that had been used in a previous study (Baccaglini-Frank and Maracci in Digit Exp Math Educ 1:7–27, 2015) proposing two chosen iPad apps to children of ages five to six. This study investigates the schemes developed by the children in response to the apps, and the role the educator's interventions seemed to play in such development. Analyses of the data collected suggest that her interventions privileged and encouraged schemes involving counting, which limited the variety of schemes enacted and the aspects of number sense strengthened through the protocol.

Keywords Counting · iPad applications · Multi-touch technology
Numerical abilities · Representations · Pre-school

12.1 Using Multi-touch Technology in the Classroom

Modern multi-touch technology offers learners new affordances that include recognition of a range of touch and multi-touch gestures as well as voice as inputs. Some studies, though not yet many, have started to analyze these affordances in relation to students' mathematical development, in particular to their development of numerical abilities, or "number sense" (e.g., Baccaglini-Frank and Maracci 2015; Sinclair and Baccaglini-Frank 2016; Sinclair and Pimm 2015). Though it is still an elusive notion, different research communities agree that number sense is a necessary condition for learning formal arithmetic at the early elementary level and it is critical to early algebraic reasoning (English and Mulligan 2013). In particular, literature from different fields of research converges in suggesting that using fingers for counting and representing numbers (Brissiaud 1992), but also in more basic

A. Baccaglini-Frank (✉)
Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo, 5, 56127 Pisa, Italy
e-mail: anna.baccaglinifrank@unipi.it

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_12

ways (e.g., Gracia-Bafalluy and Noël 2008), can have a positive effect on the development of numerical abilities and of number sense. Indeed, neuroscience research has shown that there is a neurofunctional link between fingers and number processing. For example, Butterworth (1999) has hypothesized that numerical representations and processes are supported by several component abilities: the innate ability to recognize small numerosities without counting (*subitizing*), fine motor ability (e.g., *finger tapping*), and the ability to mentally represent one's fingers (*finger gnosia*). He argues that through our fingers we construct concrete and abstract representations of number, number words, and number symbols. Fingers are used in all cultures to represent numerosities, and this is why the author believes that finger gnosia is intrinsically linked to numerical representations. Moreover, fingers are always available and they can also be used as an aide in calculations, and therefore they can work as a bridge between concrete and abstract representations of the notions of "quantity" and "operations". In a study by Penner-Wilger et al. (2007), each component ability was in fact found to be a significant unique predictor of number system knowledge, which in turn was related to calculation skill. Noël has obtained consistent results (2005), and she has also demonstrated how consistent use of fingers positively affects the formation of number sense and thus also the development of calculation skills (Gracia-Bafalluy and Noël 2008). Other researchers have suggested that finger-based counting may facilitate the establishment of number practices (e.g., Sato et al. 2007; Thompson et al. 2004).

12.1.1 How Does Multi-touch Technology Have the Potential to Foster the Development of Certain Aspects of Students' Numerical Abilities?

Literature from the fields of cognitive psychology, neuroscience, and mathematics education has pointed out some fundamental aspects of numerical abilities; we have considered a set of these that can potentially be fostered by multi-touch technology (Baccaglini-Frank and Maracci 2015). Below we list this set and then provide a table with hypotheses on software affordances that might support the development of such aspects (Table 12.1).

Finger tapping (either simultaneously or sequentially) is considered to be a fine motor ability closely related to finger gnosia and to numerical abilities (Gracia-Baffaluy and Noël 2008).

Subitizing is the rapid, accurate and confident judgment of the number of items in small collections 'at a glance', without counting. In developmental psychology it is considered a pre-verbal ability, that is thought to emerge from the ability to allocate attention over multiple individual items in parallel; and it is considered one of the neurocognitive "start-up tools" on which numerical abilities are thought to be later founded (Piazza 2010).

Table 12.1 Aspects of number abilities associated to multi-touch software affordances that might support their development

Aspect of number abilities	Affordances of multi-touch technology with the potential of fostering development of the aspect
Finger tapping	• Detecting and differentiating rapid sequences of inputs from different areas of the screen • Accepting input in the form of sequences of rapid taps to identify a target numerosity • Detecting as different inputs the simultaneous presence at a given time (or small interval of time) of two or more fingers • Recording different gestures as separate inputs (swipe with 1 finger, swipe with 2 fingers, lasso, pinch, un-pinch/enlarge…) • Manipulating virtual hands by user (to answer questions) or by computer (in proposing questions) • Simulating pianos or string instruments
Subitizing	• Showing numerosities on the screen for very brief amounts of time (possibly even fractions of a second) • Returning immediate feedback in response to the input given by the user • The objects to be considered may appear still and placed randomly on the screen or in given arrangements (for example like dots on dice), or they can move all together or one with respect to the other • Input may be given not only as typed numbers (in Arabic code or letters) but also in terms of a number of fingers placed simultaneously on the screen, as a number of sequential taps (possibly on items in the stimulus), or as a “capture” gesture
Recognizing parts of a whole	• Detecting as different inputs the simultaneous presence at a given time (or small interval of time) of two or more fingers • Manipulating virtual objects
One-to-one correspondence (with fingers)	• Detecting as different inputs the simultaneous presence at a given time (or small interval of time) of two or more fingers • Accurate timing of the user’s performance
Estimation	• Providing stimuli with different (large) numerosities which may remain on the screen or disappear after a given time • Providing immediate feedback on the input received as a product of the estimation process
Counting principles	• Adding verbal feedback in the form of verbal symbolic number representations to sets of fingers placed on the screen, or to numbers represented in analogical form • Detecting gestures such as simultaneous taps and sequential taps, or their combination, and providing different feedback in response to each of them • Arranging objects on the screen through dragging

Recognizing parts of a whole allows to recognize the complementarity of two numbers with respect to a given one, and therefore to *compose* and *decompose* numbers, for example, "7 plus 8" can be seen in many different ways: $(5 + 2) + (5 + 3) = (5 + 5) + (2 + 3) = 10 + 5 = 15$; $(5 + 2) + 8 = 5 + (2 + 8) = 5 + 10 = 15$; $(8 - 1) + 8 = 8 \times 2 - 1 = 16 - 1 = 15$; etc. (Resnick et al. 1991). Indeed, a key milestone in children's numerical development is their understanding of how numbers can be decomposed: e.g., that the number 'seven' is not just a word in a sequence but a cardinal amount that can be decomposed into smaller numbers such as 2 and 5 (Fuson 1992).

One-to-one correspondence (with fingers) consists in establishing a one-to-one correspondence between numerosities in analogical form (e.g., dots on a screen) and fingers (not necessarily raised and placed simultaneously); though subitizing may be involved, it is not necessary and the ability lies in establishing correctly the one-to-one correspondence. Indeed, Margolinas and Wosniak (2012) stress the importance for developing numerical abilities of considering quantities independently of numbers. These processes are intertwined with development of the so-called "finger symbol sets" (Brissiaud 1992) that is the representation of numbers and numbers operations and relations through finger gestures. This ability seems to be an important stepping-stone for quickly representing numbers with fingers. Later, building on such ability, adding to analogical representations of numbers verbal or written symbolic representations of the same numbers may foster automatization of the linking between sets of fingers and numbers in symbolical form (Clements 2002; Ladel and Kortenkamp 2013).

Estimation is an ability that has been closely related to numerical abilities (e.g., Sowder 1992).

Counting principles consist of five principles considered to be necessary for children to master for developing number sense (Gelman and Gallistel 1978). These are (a) the one-one-principle that relates every single object to exactly one numeral; (b) the stable-order principle prescribing the correct order of numbers; (c) the last-word rule that assigns the last said numeral not to the last counted object, but to the quantity as a whole; (d) the principle of abstraction, according to which objects of any nature, also abstract, can be counted, and (e) and the order in which the objects are counted does not matter.

The identification of the aspects of numerical abilities described above, together with the hypotheses on software affordances that might support their development led us to a working hypothesis on what we called *multi-touch potential*:

> Multi-touch technology has the potential to foster important aspects of children's *development of number-sense*, including the ability to use fingers to represent numbers in an analogical format. We will call this the *multi-touch potential*. (Baccaglini-Frank and Maracci 2015, p. 6)

12.1.2 From an Initial Study to Investigate the Multi-touch Potential of Two IPad Apps to the Current Study

This hypothesis was explored in an initial study, with the goal of analyzing the multi-touch potential of two iPad apps for fostering preschoolers' development of number-sense, by (1) investigating the schemes that four-year old children develop in their interactions with the software, and how they use their fingers; (2) and relating the schemes enacted with the considered aspects of numerical abilities. The apps are *Ladybug Count* (LBC) and *Fingu* (F) (that we will introduce more in detail in a later section of this chapter) these are environments providing a stimulus (either dots on the back of a ladybug or fruit in groups floating on the screen) to which the user responds placing on the screen a number of fingers that corresponds to the numerosity of the stimulus. We considered the following aspects of numerical abilities: multiple fingers tapping (simultaneously or sequentially), subitizing (simple or double), recognizing parts of a whole, one-to-one correspondence, approximate estimation (of small or large quantities), and the counting principles (Baccaglini-Frank and Maracci 2015). Students' interactions with the apps were analyzed identifying subsets of these aspects that were present in the different strategies used. A total of 15 different strategies were recognized, and all aspects of numerical ability appeared in at least two strategies. Interestingly, counting strategies, or, more in general, verbal symbolic utterances, were used by very few children. An important finding was confirmation of how the multi-touch potential could be exploited to foster development of the ability to use fingers to represent numbers in an analogical format.

One year later, a research-to-practice group from a university in a different city in northern Italy decided to adopt the same protocol used in the initial study within an educational project aimed at strengthening preschoolers' numerical abilities, and asked for my supervision. In exchange, I was able to obtain consent forms from the parents of the children in one class to collect videos of the sessions. The 24 children of this class were in the last year of pre-school (five to six years old), from socioeconomic backgrounds comparable to those of the children in the initial study. The two major differences with respect to the initial study were: (1) the age of the children involved, and (2) the background of the educator. The educator in this study, who would be introducing the apps and working with the children, was an in-service pre-school teacher (not in the same school) with a degree in psychology, was presented as an "expert", and had planned to intervene during the preschoolers' activities with the iPad to "help them learn". On the contrary, in the initial study the pre-service teacher carrying out the protocol was trained by the research group and intervened minimally during each play session.

My main objective in analyzing these new data was to compare these children's schemes to those of the younger children in the first study, and gain insight into how they evolved, expecting that such evolution might depend on the interventions of the educator.

The research questions I sought to address were:

1. Are there (and if so which) recurrent behaviors in the enactment of the schemes developed by the two groups of children?
2. Are there differences (if so, which) between the enactments of the schemes developed by the two groups of children?

   2a. If so, what do these differences suggest in terms of aspects of number abilities developed by the two groups of children?
   2b. If so, how might these differences be explained in terms of the interventions of the educator?

This chapter addresses, in particular, question 2.

12.2 Conceptual Framework and Methodology

I make use of the notion of *scheme* as developed by Vergnaud (1990) to link children's actions to their goals and intentions in a given situation, and to certain characteristics of the situation itself. This will allow me to relate enactments of schemes to the aspects of number abilities introduced above, and to identify and compare the children's enactments of the schemes developed, and gain insight into different aspects of numerical abilities developed by the two groups of children. In this section the introduction of the notion of scheme is followed by a presentation of the apps used in the protocol accompanied by summaries of the enactments of schemes developed by four-year olds in the initial study.

12.2.1 The Notion of Scheme

The notion of scheme as developed by Vergnaud (1990, 2009) elaborates on the Piagetian notion of "scheme", and characterizes it as *an invariant organization of the activity for a given class of situations*. The main components of a scheme are: the goal and the anticipated outcomes; the rules of action, of gathering information, of control taking; and the operative invariants (implicit knowledge), including *concepts-in-action*, that is concepts that are implicitly considered as pertinent, and *theorems-in-action* that is, propositions believed to be true. We will refer to a (visible) recurring sequence of actions as the "enactment" of a scheme.

Even though all the components of a scheme are important, operational invariants have a prominent role. They consist of the implicit knowledge which structures the whole scheme: they drive the identification of the situation and of its relevant aspects, and allow selecting suitable goals and inferring the rules for generating appropriate sequences of actions for achieving those goals.

12.2.2 The Apps Used and the 4-Year-Old Children's Enactments of Schemes

The choice of the apps used in the protocol was constrained by limitations during the design of the protocol in the initial study (Baccaglini-Frank and Maracci 2015). In particular, we could use only already published, free or very cheap apps, easy for children to become familiar with, and presenting a strongly structured environment allowing primarily closed conversing-type interactions (Sedig and Sumner 2006; Sinclair and Baccaglini-Frank 2016). Within the constraints two multi-touch apps, which seemed to have some potential for fostering the development of children's number sense, were identified. They are: Ladybug Count and Fingu.

Ladybug Count (Finger Mode[1]): The layout of this app is the top view of a ladybug sitting on a leaf, and the aim of each playing turn is to make the ladybug walk off the leaf. This happens when the player places on the screen (in any position) as many fingers as the dots that are on the ladybug's back. Given a certain number, the dots appear on the ladybug's back always in the same pattern. As each finger is placed on the screen one of the dots on the ladybug's back is highlighted (Fig. 12.1), and the iPad makes a "pop" sound.

When all the dots are highlighted a sound is emitted preceding the announcement of the number of dots that are on the ladybug's back. At this time the ladybug walks off the screen and a new one appears. This process repeats as long as the player wants to play. If the player places more fingers on the screen than the dots on the ladybug's back, all the dots become highlighted, but the ladybug does not walk off the leaf and the sound: "Oops!" is emitted. If the player places on the screen fewer fingers than the dots on the ladybug's back, only a number of dots corresponding to the fingers on the screen are highlighted and nothing else happens. This app will be referred to as LBC.

In the initial study we identified 11 enactments used by the children in LBC. These were classified into "general" (6) that is enactments of schemes not apparently linked to a "small" or "large" number of dots on the ladybug's back, and "specific" (5) ones that were sensitive to the number of dots to "count". This was necessary because the children seemed to hold different schemes for a very small number of dots (1–3) or large numbers of dots (7–10). Moreover, in several cases the children reacted to the appearance of the ladybug with a large number of dots through verbal expressions such as: "How many!" "That's a lot!". This allowed us to infer that the two situations identified above were different for them, and thus we identified different schemes, possibly related to the different aspects of number sense. For example, the most common enactment of a scheme in the presence of a small number of dots involved the rapid recognition of the small number of dots,

[1]There is also another mode, called 'tap mode', in which success is reached when the user taps the screen (sequentially) as many times as the numerosity of the dots on the ladybug's back. This mode seems to mostly encourage students to use counting strategies, and to support only to a limited extent children's development of number abilities. This is why we chose not to use it.

Fig. 12.1 View of the LBC screen with a player that set three fingers on the screen

apparently through subitizing followed by the placement of the same number of fingers (in a variety of configurations) on the screen simultaneously. This enactment did not contain verbal utterances. The most common enactment in the presence of a large number of dots involved placing all fingers on the screen and then, possibly, removing fingers one at a time until positive feedback was received from the app.

In terms of the aspects of numerical abilities that are in relation with the schemes identified for LBC, we identified the following: multiple finger tapping—simultaneous (six schemes); multiple finger tapping—sequential (six schemes, but not all coincident with the six for simultaneous tapping); subitizing (for four schemes); recognizing parts of a whole (two schemes); the ability to match numbers of fingers (not necessarily instantaneously) to a number of objects, without counting (seven schemes); the ability to match numbers of fingers (not necessarily instantaneously) to a number of objects counting (two schemes); estimation (five schemes); counting principles (three schemes).

Fingu: The layout of this environment (Barendregt et al. 2012) looks like a room in which different kinds of floating fruits appear. The objects appear in one group or in two groups that float independently, but within each group the arrangement of the objects remains unvaried. The player has to place on the screen, simultaneously, as many fingers as the objects that are floating within a given amount of time (Fig. 12.2). If s/he succeeds the iPad emits a sound and shows a few dancing happy animations. Otherwise, if the number of fingers is incorrect or time runs out, a different sound is emitted and sad animations appear on the screen. Then the player can go to the next round, until s/he looses or passes the level. The game provides statistics on the performance of the player for each level attempted. For an analysis of cognitive abilities potentially stimulated by this app the reader can refer to pp. 670–671 of Sinclair and Baccaglini-Frank (2016). This app will be referred to as F.

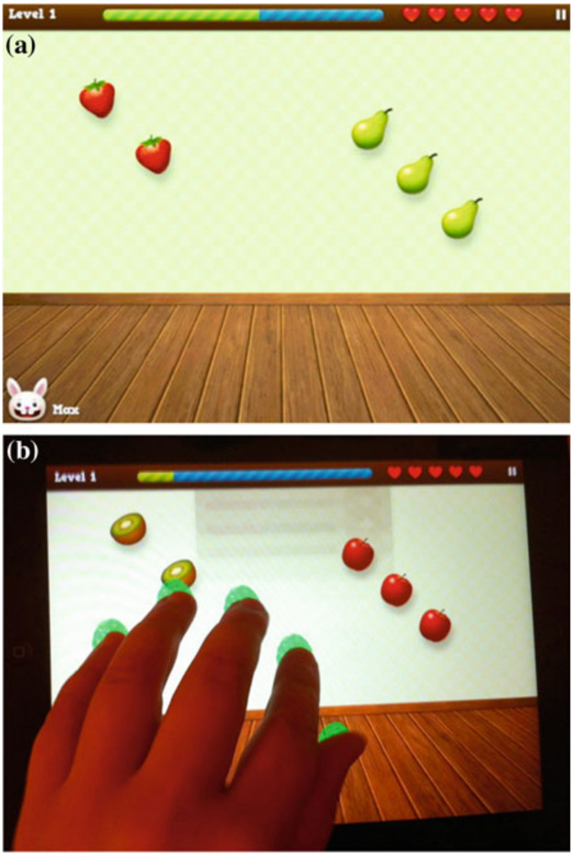


Fig. 12.2 **a** View of the screen of F with two groups of floating fruits in fixed arrangements; **b** view of the screen of F after the player has set five fingers on the screen, simultaneously

In the initial study we identified four enactments used by the children in F. In this environment, selecting the exact number of fingers and placing them simultaneously on the screen within a limited amount of time was a source of difficulties for most children: indeed, it requires, among other skills, the development of advanced fine motor abilities. Typically the children's tendency was to continue to use the first enactment that seemed to be effective in a few cases, despite possible successive failures. Four enactments were identified, two of which involved recognizing the number of objects (be they in a single group or in two groups) without verbally counting, and trying to place on the screen the corresponding number of fingers either of a same hand or of two hands, simultaneously. Many children tried to place their fingers as close as possible to the floating objects, so as to "catch" them, or to reproduce with their fingers the same spatial arrangement of the floating objects (enactment 3). Only a few children quickly counted the floating objects (pointing to them and counting aloud) and then placed a same quantity of fingers on the screen (enactment 4).

In terms of the aspects of numerical abilities that are in relation with the schemes identified for F, we identified the following: multiple finger tapping—simultaneous (3 schemes); multiple finger tapping—sequential (1 scheme); subitizing (1 scheme); recognizing parts of a whole (1 scheme); the ability to match numbers of fingers to a number of objects, without counting (1 scheme); the ability to match numbers of fingers to a number of objects counting (1 scheme); counting principles (1 scheme).

The protocol was administered as follows: the 24 children (ages 5 and 6) worked in groups of five and played 5 minutes a day for 2 weeks with the two iPad apps under the supervision of the educator, taking turns while the other children in the group watched. If prompted by the educator, a child from the group could give advice to the child playing, through verbal or gestural utterances. All activities with the apps were video-recorded and transcribed. Identification of students' recurrent behaviors identified as enactments schemes were flagged, as well as all interventions of the educator. Finally, at the end of her work with the children, I interviewed the educator in order to have an additional key of interpretation when analyzing the possible effects of her interventions.

Below we report on two cases that are representative of 20 of the children; then we report on the non-prototypical case of one of the four other children.

12.3 Results: The Analysis of Two Prototypical Cases and of One Non-prototypical Case

The schemes enacted by Giovanna and Sara were quite similar to those of most of the other children in the class. We analyze their development and highlight typical interventions of the educator. In this section the focus is on the students' schemes, while the interventions of the educator will be analyzed in the next section of the chapter.

12.3.1 Giovanna (Prototypical)

Giovanna (5 years, 3 months) starts her first interaction with LBC very hesitantly, barely showing the fingers she intends to put on the screen and placing them very close together. Every time she hesitates (even for only a few seconds) the educator asks her classmates to "show Giovanna how to do this with her fingers". After three iterations of this process, Giovanna immediately looks for hints from her classmates, from one in particular praised by the educator, checking the configuration of fingers he is raising and imitating it. Together with finger configurations, her classmates also shout out the numbers of dots. Giovanna pretends to count them, pointing to a few and then repeats the number pronounced by her classmates. The educator says nothing to stop the classmates from talking, glances at the screen to check it, and makes comments like:

Educator: Six! How do you do six with your fingers? Come on!
Educator: Come on, you know how to do nine [with your fingers]!

The educator also highly praises children who “know”, as in the following example.

[A ladybug with 7 dots on her back appears]

Giovanna: [Counts up the dots pointing with her finger.] Seven [in a whisper].
Educator: Seven! How do you do seven with your fingers, Giovanna? Come on!
Giovanna: [She timidly raises the fingers on a hand and the index and middle finger of her other hand, imitating the fingers shown by a classmate.]
Educator: Right, very good! See, you *know*!

These interventions by the educator both in LBC and in F seem to lead Giovanna to develop two schemes such as the one described in Table 12.2, with the goal of doing what she thinks the educator (seen as a teacher) wants, and seemingly identifying two situations in LBC, based on whether she immediately recognizes the number of dots on the ladybug’s back (1a) or not (1b).

She seems to also develop what we could see as a concept-in-action: every number pronounced verbally corresponds to a *fixed* configuration of fingers.

Giovanna seems to inhibit enactments that involve configurations of fingers other than what she believes to be *the right one*. In fact, in one episode during her first playing session, Giovanna sees the dots on the ladybug’s back [there are two on each wing], and she timidly raises two fingers on each hand fingers (here she might have used one-to-one correspondence), waiting for feedback. She sees other children showing four fingers, raised on one hand, and copies their finger configuration. The educator says nothing to the other children and simply says “Good!” to Giovanna when she touches the screen and receives positive feedback. A short time after, a ladybug with 9 dots appears. Giovanna does not count, but she seems to over-estimate (here she seems to be using estimation), placing all fingers of both hands on the screen, a scheme that was frequently enacted also by children in the initial study. Instead of trying to adjust her fingers (as in schemes identified in the initial study), for example, by lifting one, she takes her hands off and looks for a configuration to copy. After these two episodes Giovanna’s behavior can be well described in light of the schemes we hypothesized above.

In general, when Giovanna cannot remember what she believes to be *the approve*d finger configuration, she depends on her classmates’ hints, and copies, seemingly with no control over the answer that she then gives; or, if she cannot

Table 12.2 Two initial schemes enacted by Giovanna while interacting with LBC and F

Two of Giovanna’s schemes: 1a–2–3; 1b–2–3
1. Figure out the number of dots/fruits, to do this either
1a. Recognize the number immediately
1b. Count them up from one
2. Say the number
3. Use *the* fixed configuration approved by the educator for that number

catch a hint quickly, she listens to the number pronounced by the class and counts up from “one”, raising her fingers one at a time and always in the same order. Clearly, remembering fixed configurations of fingers associated to a word requires a lot of (otherwise unnecessary) memory and it could even inhibit the development of fundamental aspects of numerical abilities: the child can be successful without ever putting in relation the dots and the fingers raised, other than through a verbal utterance (when a finger configuration is not shown directly), and without developing, for example, awareness of part-whole relations.

In F, Giovanna relies entirely on reaching (either by herself or hearing it from her classmates) a verbal pronunciation of the number of floating fruits from which she produces *the* configuration of fingers if she remembers it. She does not count up her fingers because the game does not give her the time. Her configurations of fingers do not seem to be directly related to the partitions of the fruits into smaller sets when there is more than one. For example, when two floating sets of two fruits appear, she hears a classmate say “four” and puts down the fingers of her right hand excluding her thumb. The same happens when sets of three and one appear on the screen. Giovanna seems to rely heavily on her schemes; so much that when possibly perturbing events occur, the scheme remains unchanged. For example, when four and one fruits appear at a certain point she hears a classmate say (erroneously) “four” and she puts down her usual “four” configuration. When three and two fruits appear a classmate shows three fingers on one hand and two on the other; Giovanna sees, but she hesitates and then says: “There are five” and places down all her fingers of a single hand.

Her schemes turn out to be successful in F and Giovanna passes to level II of the game. Now more than five fruits can appear. The other children no longer have time to figure out how many fruits are floating around on the screen before they disappear, so Giovanna counts them up each time, starting from “one” and saying the numbers aloud as she points to each fruit. She appears to not know (at least not quickly enough) the configurations for any of the quantities above 5, so she either puts down no fingers or she tries to put down some, frequently less than 5 and loses.

In F her enactments do not seem to change, however in LBC they do. During the later playing sessions, Giovanna proceeds more and more independently with respect to what her classmates say, and the form of her gestures change, as well. For example, she places her fingers on the screen spreading them out on the leaf rather than bunching them up, like during the initial episodes. During the last session with LBC Giovanna has learned to generate fixed configurations quickly for quantities below 5, and for larger quantities she modifies her behavior for figuring out how many fingers to raise. Interestingly, she never counts up her fingers when she recognizes the number of dots. These changes can be seen in terms of new schemes (1a–2a; 1b–2a; 1b–2b), as shown in Table 12.3.

For example, when a ladybug with 8 dots appears, Giovanna counts up the dots and counts her fingers, starting with “one” as the thumb of her right hand. The educator gives very positive feedback. The same happens when a ladybug with 10 dots appears, and then when a ladybug with 8 dots appears again. When the next ladybug appears and it presents, again, 8 dots, Giovanna seems to recognize the

Table 12.3 Schemes enacted by Giovanna while interacting with LBC at the end of the administration of the protocol

Giovanna's final LBC schemes: 1a–2a; 1b–2a; 1b–2b
1. Figure out the number of dots, to do this either
1a. Recognize the number immediately (if below five) and say it out loud
1b. (if 1a fails) count them up from one
2. Raise fingers by
2a. Using *the* fixed configuration of fingers, when known, corresponding to the number pronounced
2b. Otherwise counting up fingers (in a constant order) starting from one

configuration and remember the configuration for "eight" (as a hand and 3 fingers). This suggests that she still holds valid the concept-in-action hypothesized earlier.

This can also be seen in how Giovanna seems to privilege schemes 1a–2a and 1b–2a over ones using 2b. For example, when a ladybug with 4 dots appears she says "four" and in a seemingly automatic way counts up four fingers starting from her thumb on the right hand, but as soon as she sees the four fingers she changes the fingers to her *fixed configuration* for "four".

The aspects of number abilities that can be related to Giovanna's most frequently used schemes, which were found to be similar to those of most children in the class, are: multiple fingers tapping (simultaneous), subitizing, and use of the three counting principles (one-to-one correspondence, stable order and cardinality, but not order irrelevance or abstraction). We note that one-to-one correspondence does not seem to be directly involved (especially in the early enactments where it would have been important), because the figure configuration is always mediated by pronunciation of a verbal-symbolic numeral that Giovanna directly associates (either because she remembers it, or because she imitates a classmate, or because she carries out a new counting process) to it.

12.3.2 Sara (Prototypical)

Although Sara (5 years, 5 months) is not in Giovanna's group, with her the educator keeps on intervening in the same way as with Giovanna, proposing to count the dots or fingers immediately at the smallest hesitation, and praising her emphatically whenever she does count. During her initial interactions both with LBC and F, Sara is less insecure than Giovanna: for quantities of 1, 2, 3 or 4 she simply says aloud the number corresponding to the quantity and raises a known configuration of fingers. Sara uses constant configurations for "one", "two" and "three", while for "four" she seems to flexibly change the fingers raised and placed on the screen. The educator, in these cases, simply praises Sara for getting positive feedback from the software.

Table 12.4 Schemes enacted by Sara

Sara's final LBC schemes: 1a–2a; 1b–2a; 1b–2b
1. Figure out the number of dots/fruits, to do this either
1a. Recognize the number immediately (if below five) and say it out loud
1b. (If 1a fails) count them up from one, aloud, pointing to each
2. Raise fingers by
2a. Raising *any* known configuration of fingers corresponding to the number pronounced
2b. Otherwise counting up fingers (in a constant order) starting from one
3. In any case, count up fingers before placing any on the screen

Both for LBC and F the schemes developed by Sara seem to be very similar to Giovanna's (see Table 12.4).

Interestingly, if Sara has reached 2a in the enactment of her scheme, she still proceeds to step 3, that is, she counts up her raised fingers, seemingly to please the educator. Indeed, although she shows a bit more flexibility than Giovanna in making appropriate finger configurations for quantities up to 4, she, too, seems to be conditioned by the educator's insistence on counting. This results in episodes such as the following. In LBC a ladybug with 4 dots appears:

Sara: Four [and raises two fingers (index and middle) on both hands immediately]. One, two, three, four [she counts the fingers on one hand starting from the thumb].
Educator: Very good!
Sara: [She switches back to her initial configuration of 2 and 2 fingers and places them on the screen, receiving positive feedback from LBC].
Educator: Oh, that's OK, too. Good job!

In this case Sara seems to have recognized the number of dots (1a) and associate correctly a known finger configuration, without counting (2a); so it is surprising that she then counts to four on the fingers of her other hand, to then go back to the configuration with two hands to interact with the app. My conjecture is that she enacts the counting just to please the educator, having picked up on her "counting cues".

All the counting in the enactment of Sara's schemes is too time consuming to be effective in F, so Sara receives negative feedback almost every time. She soon asks to stop playing and to give another classmate a turn. The educator satisfies her request and calls a classmate to play.

The aspects of number abilities that can be related to Sara's most frequently used schemes, are very similar to Giovanna's and to the ones of most children (20 out of 24) in the class; they are: multiple fingers tapping (simultaneous), subitizing, and use of the three counting principles (one-to-one correspondence, stable order and cardinality, but not order irrelevance or abstraction). We note that one-to-one correspondence, as before, does not seem to be directly involved, however Sara seems to be aware of more than one finger configuration for different numbers pronounced verbally. This may be an effect of use of one-to-one correspondence between fingers and objects in experiences prior to her interaction with the apps.

Four (of the 24) children enact schemes that seem quite different from the ones used by the majority of students. Indeed these simply refuse to count, possibly because they have not sufficiently mastered the counting principles. Instead, they either imitate the configurations of fingers shown by their classmates, when they were able to, and failed otherwise, or they try to use enactments similar to those of children in the initial study.

12.3.3 Amanda (Non-prototypical)

One of these four students is Amanda (5 years, 2 months), who never says numbers aloud, but puts down precise numbers of fingers, in a variety of configurations, for quantities up to 4, both in LBC and in F, while for larger numbers she seems to estimate, quickly putting down a hand of fingers and some additional ones, a strategy used by various students in the initial study, as well; or placing all her fingers on the screen and lifting one at a time until she receives positive feedback from the app. This happens repeatedly in LBC for ladybugs with 7 or more dots on their backs. Each time Amanda receives negative feedback from the app and starts adjusting her fingers, the educator intervenes with comments like:

Educator: Remember how [another student who counted the dots and then his fingers] did it? Can you do that, too?

Educator: Sweety, you need to count…look at the dots. How many are there?

This sort of comment contains reference to a counting strategy, either implicitly as part of another child's enactment, or explicitly. These interventions would interrupt the enactment the Amanda's schemes, making her attempts look like a failure, which would trigger another "counting cue" from the educator, and the vicious cycle would continue, breaking only when the child would give into counting, or somehow be able to place the correct number of fingers on the screen.

While playing F, Amanda seems to be enacting different schemes: when two sets of fruits appear and are small (one, two, or three fruits) she tries to place on the screen the number of fingers, on each hand, corresponding to each floating set; or when up to four fruits appear, not necessarily in a same set, she puts down the corresponding number of fingers of her right hand. Amanda responds rather quickly, but frequently receives negative feedback, because she does not seem to double check her raised fingers before placing them on the screen. Sometimes the negative feedback is given by the app also because she does not wait to place down all fingers on the screen simultaneously. In these cases the educator makes statements like the following.

Educator: Slow down, you need to check your fingers!
Educator: Careful! Your fingers need to go down together!

By the end of the playing sessions Amanda has developed use of fixed configurations, like Giovanna, for small numbers (below 5); and recognizes fixed configurations of dots in LBC for larger numerosities (7, 8 and 10), stating the number aloud, verbally, but without reproducing the numerosity on her hands.

12.4 Results: What Is Behind the Interventions of the Educator?

From the representative excerpts above, and thorough analysis of the whole data set, it is possible to make more general inferences about characteristics of the educator's behavior. Her interventions turn out to be quite explicit about what the children should (or should not) be doing, and most of her interventions are triggered by the student's receiving negative feedback from the app. She tends to not take the time to discuss the children's schemes or enactments, neither collectively, nor individually. Moreover, although she accepts different finger configurations for a same quantity, she does not explicitly comment on how a same quantity can be represented through different finger configurations, for example, putting in one-to-one correspondence two different quantities of fingers for a given quantity of dots or fruits. The only sharing that the educator fosters is in cases in which the child playing appears to be hesitant: she calls on other children to "show your classmate how to do it". These characteristics are not at all aligned with those I had expected.

The educator did not seem to be trying to "access students' thinking", focusing on processes that might have led the children to the development of a certain strategy; instead she would "assess their thinking", in Crespo's words (2000), and act on the end product, the feedback received from the apps. I had expected that her guidance would have taken into consideration both successful and non-successful (in terms of receiving positive feedback from the apps) enactments, and that she would use these to foster students' sharing of their strategies, and therefore their talking about numbers in verbal and analogical form, perception of numerosity, or representation of numerosity through fingers.

As for the enactments the educator chose to foster, I was curious why she had valued so much, on the one hand, counting, and on the other hand immediate association of finger configurations to verbal-symbolic numerals. More specifically, I was curious why she seemed to be pushing children in two directions (she may have been seeing 2 as an "evolution" of 1). The two directions were: (1) counting the dots or fruits, counting the fingers, and placing the counted fingers on the screen simultaneously, and (2) counting the dots or fruits and immediately making a finger configuration that corresponded to the verbal-symbolic numeral pronounced. From the follow-up interview it became clear that indeed she did have a particular procedure in mind that she thought was best (and should be accomplished quickly), because applicable to all the situations generated by the apps. Below are various claims about this that she made in the interview.

Educator: You need to count up the dots or fruits and then count up your fingers, quickly, and put them down together. […] Once a kid knows how to do a certain number with his fingers, he doesn't have to count up his fingers any more, and he can use whatever way he wants. […] which fingers they raise is not important, but they should do it quickly and there are easier ways of doing it. […] This is how they can always experience success.

12.5 Answers to the Research Questions and Concluding Remarks

Once the schemes of the 24 students in the second study were identified, I compared their enactments to those of the children in the initial study. This comparison showed that the 24 children had behaviors similar to those of the younger children mostly in the first encounters with the apps. By the second activity session with the educator, most of the 24 children's schemes had started to transform into ones like Giovanna and Sara's.

There were significant differences between the enactments of the schemes developed by the two groups of children. The most prominent are: (1) the heavy reliance on (or presence of) counting in the schemes of the 24 students, while such presence was quite limited in the schemes of the students in the initial study (only 4 of the 15 schemes included counting), and (2) most students in the second study seemed to rely on memorized finger configurations corresponding to each verbal-symbolic numeral (with no apparent reference to one-to-one correspondence between the fingers and other analogical representations of the number). This behavior was not found in students of the initial study.

To me it was particularly surprising to see how almost all the students in the second study had incorporated counting processes into their schemes, especially because the counting processes always started from "one". Indeed knowing how to count from "one" is important, and possibly through these activities the children might have learned to count faster. However, it is not clear how much this enhances other numerical abilities (including those involving other counting strategies) in general: achieving mastery in counting does not simply mean learning to do it *faster*! For example, it is also important to learn to count on from a number greater than "one", to count backwards, and to learn to r*eplace* counting with more effective strategies (e.g., Gray and Tall 1994); but these strategies were not within the multi-touch potential of the apps considered.

Table 12.5 Relationship between students' enactments of schemes and aspects of number abilities

	Aspect of number sense involved (and no. of schemes)—initial study	Aspect of number sense involved in most common schemes—second study
Finger tapping		
Simultaneous	Yes (9)	Yes
Sequential	Yes (7)	No
Subitizing	Yes (5)	Yes
Recognizing parts of a whole	Yes (3)	No
One-to-one correspondence (with fingers)		
Not mediated by verbal-symbolic numerals	Yes (8)	No
Mediated by verbal-symbolic numerals	Yes (3)	Yes (cases like Sara)
Estimation	Yes (5)	No
Counting principles		
One-one	Yes (4)	Yes
Stable order	Yes (4)	Yes
Cardinality	Yes (4)	Yes
Abstraction	No	No
Order irrelevance	Yes (4)	No

These differences can further be analyzed in terms of aspects of number abilities they involve, and thus potentially strengthened in the two groups of children; these are summarized in Table 12.5.

Analyses leading to the construction of this table suggest that the schemes developed by the children in the initial study exploited the multi-touch potential of the apps, used to a greater extent than did the schemes that the children in the second study were led to develop. In particular, the abilities to recognize parts of a whole, estimate, or create one-to-one correspondences between fingers and sets of objects, did not seem to be promoted in the group of the 24 children.

Although other factors may have contributed (e.g., the age of the students, or previous classroom experiences), a factor that seemed to be quite influential in determining these differences is how the educator intervened. Indeed, the 24 children initially enacted schemes quite similar to those of the children in the first study, however the educator seemed to vigorously promote *counting* and/or use of *known finger configurations*, influencing the children's strategies. Her interventions were mostly consistent with the claims she made during the follow-up interview. Her main goal was to help children experience success in the apps and, with respect to strengthening numerical abilities she intended to help children learn to count and to represent numbers on their fingers. She did not mention, for example, recognition of part-whole relationships, subitizing, estimating, or even counting on strategies (indeed children's counting always started from "one").

In my opinion, the educator's short-term goal of helping the children experience success, and her narrow-sighted view of how to obtain this while fostering the development of numerical abilities, actually limited such development (at least during this experience) for many children. In most cases the children seemed to develop schemes like those of Sara and Giovanna, whose enactments included trying to memorize fixed configurations of fingers corresponding to verbal numbers (this seemed to be a common interpretation of the children of the educator's comments), or simply copying. The other abilities the educator explicitly intended to promote were to represent a same number in different ways (with fingers), and to count. She failed to promote the former for many children, possibly because she did not take the time to discuss any of the situations in which children used different representations of a same number.

This brings me to a consideration that goes beyond the scope of this study, but that is closely related to the findings. What might have happened if this educator (and the children she worked with) used a more open digital environment where many different tasks can be proposed and a greater variety of solutions can be given?

For example, let us consider TouchCounts, described in Sinclair (2018) (this book), by Sinclair and Jackiw (2011), a very interesting app which exploits the potential of multi-touch screens in innovative ways, offering a wide range of possible interactions (Sinclair and Baccaglini-Frank 2016), especially manipulative interactions (Sedig and Sumner 2006). By encouraging the user to associate specific gestures to numerical manipulation, children's meaning making is promoted (Goldin-Meadow 2004). TouchCounts seems to have a very high multi-touch potential. In particular, it recognizes a "pinch" gesture to add together sets of floating herds (represented in analogical and symbolical form), generating new larger herds. Such gesture can be seen to embody the fundamental metaphor of addition "collecting together" (Lakoff and Núñez 2000, in Sinclair and Sedaghat Jou 2013), and its symmetry incorporates the commutate property of addition. Another interesting gesture that one of the sub-environments of the software recognizes is a 5-finger-placement together with sequences of one-finger-taps that generate sets of 5 + 1 + 1 + 1 + 1… elements. This gesture is associated to the idea of constructing numbers (above 5) as successors of one another; to perform it the child can strengthen the 5-fingers to 5-objects correspondence and finger tapping.

Though with a seemingly very high potential with respect to fostering many aspects of children's numerical abilities, the app has no "built in" assignments, so when it is opened the user finds him/herself in a completely "open" situation. Of course very insightful and rich tasks can be designed and assigned (e.g., Sinclair and Zaskis 2017). However an educator such as the one in this study might find it difficult to come up with any. Further, even if s/he were given the tasks to assign the students in advance, an expectation to foster rigid interactions making use of a single "good" strategy would inhibit exploitation of the software's multi-touch potential.

Finally, I believe that recognizing and analyzing the role played by the educator in contexts where learning is fostered through software also has important implications for teacher education, as discussed also in Ginsburg et al. (2018). Indeed, such a finding can be used to help teachers (both pre-service and in-service) become more aware of how difficult it is to "*hear* what children are saying", an ability that "transcends disposition, aural acuity, and knowledge, although it also depends on all of these" (Ball 1993, p. 388). Teacher education, possibly through collective analysis of case studies (e.g., Levin 2002), should foster such ability to hear and respectfully interpret students' contributions, a kind of knowledge for teaching, that Ribeiro et al. (2016) call *interpretative knowledge.*

References

Baccaglini-Frank, A., & Maracci, M. (2015). Multi-touch technology and preschoolers' development of number-sense. *Digital Experiences in Mathematics Education, 1,* 7–27.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93,* 371–397.

Barendregt, W., Lindström, B., Rietz-Leppänen, E., Holgersson, I., & Ottosson, T. (2012). Development and evaluation of Fingu: A mathematics iPad game using multi-touch interaction. In D. Marjanovic, M. Storga, N. Pavkovic & N. Bojcetic (Eds.), *Proceedings of DESIGN 2010, the 11th International Design Conference* (pp. 204–207). Dubrovnik, Croatia: ACM.

Brissiaud, R. (1992). A toll for number construction: Finger symbol sets. In J. Bidaud, C. Meljac & J.-P. Fischer (Eds.), *Pathways to number. Children's developing numerical abilities*. New Jersey: Lawrence Erlbaum Associates.

Butterworth, B. (1999). *The mathematical brain*. London: Macmillan.

Clements, D. H. (2002). Computers in early childhood mathematics. *Contemporary Issues in Early Childhood, 3*(2), 160–181.

Crespo, S. (2000). Seeing more than right and wrong answers: Prospective teachers' interpretations of students' mathematical work. *Journal of Mathematics Teacher Education, 3,* 155–181.

English, L., & Mulligan, J. (2013). *Reconceptualizing early mathematics learning*. Dordrecht: Springer.

Fuson, K. C. (1992). Research on learning and teaching addition and subtraction of whole numbers. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 53–187). Hillsdale: Lawrence Erlbaum Associates.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge: Harvard University Press.

Ginsburg, H. P., Uscianowski, C., & Almeda Ma. V. (2018). Interactive mathematics storybooks and their friends. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank and C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Goldin-Meadow, S. (2004). Gesture's role in the learning process. *Theory into Practice, 43*(4), 314–321.

Gracia-Bafalluy, M. G., & Noël, M. P. (2008). Does finger training increase young children's numerical performance? *Cortex, 44,* 368–375.

Gray, E., & Tall, D. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 26*(2), 115–141.

Ladel, S., & Kortenkamp, U. (2013). An activity-theoretic approach to multi-touch tools in early maths learning. *The International Journal for Technology in Mathematics Education, 20*(1), 3–8.

Levin, B. B. (2002). Using the case method in teacher education: The role of discussion and experience in teachers' thinking about cases. *Teaching and Teacher Education, 1,* 63–79.

Margolinas, C., & Wosniak, F. (2012). *Le nombre à l'école maternelle. Une approche didactique*. Bruxelles: De Boeck.

Noël, M. P. (2005). Finger gnosia: A predictor of numerical abilities in children? *Child Neuropsychology, 11,* 1–18.

Penner-Wilger, M., Fast, L., LeFevre, J. A., Smith-Chant, B. L, Skwarchuk, S. L, Kamawar, D., & Bisanz, J. (2007). The foundations of numeracy: Subitizing, finger gnosia, and fine motor ability. *Proceedings of the 29th annual conference of the cognitive science society* (pp. 1385–1390). Austin, TX: Cognitive Science Society.

Piazza, M. (2010). Neurocognitive start-up tools for symbolic number representations. *Trends in Cognitive Sciences, 14*(12), 542–551.

Resnick, L. B., Bill, V., Lesgold, S., & Leer, M. (1991). Thinking in arithmetic class. In B. Means, C. Chelemer, & M. S. Knapp (Eds.), *Teaching advanced skills to at-risk students: Views from research and practice* (pp. 27–53). San Francisco: Jossey-Bass.

Ribeiro, M., Mellone, M., & Jakobsen, A. (2016). Interpreting students' non-standard reasoning: insights for mathematics teacher education. *For the Learning of Mathematics, 36*(2), 8–13.

Sato, M., Cattaneo, L., Rizzolatti, G., & Gallese, V. (2007). Numbers within our hands: Modulation of corticospinal excitability of hand muscles during numerical judgment. *Journal of Cognitive Neuroscience, 19*(4), 684–693.

Sedig, K., & Sumner, M. (2006). Characterizing interaction with visual mathematical representations. *International Journal of Computers for Mathematical Learning, 11*(1), 1–55.

Sinclair, N. (2018). Time, immersion and articulation: Digital technology for early childhood mathematics. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (This book).

Sinclair, N., & Baccaglini-Frank, A. (2016). Digital technologies in the early primary school classroom. In L. English & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (3rd ed., pp. 662–686). New York, NY: Taylor Francis/Routledge.

Sinclair, N., & Jackiw, N. (2011). *Touchcounts* [computer software]. Tangible mathematics project, Simon Fraser University.

Sinclair, N., & Pimm, D. (2015). Mathematics using multiple senses: Developing finger gnosis with three- and four-year-olds in an era of multi-touch technologies. *Asia-Pacific Journal of Research in Early Childhood Education, 9*(3), 99–109.

Sinclair, N., & Sedaghat Jou, V. (2013). Finger counting and adding on a touchscreen device. In B. Ubuz, C. Haser & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of European Society for Research in Mathematics Education* (CERME 8) (pp. 2198–3207). Ankara, Turkey: Middle East Technical University and ERME.

Sinclair, N., & Zaskis, R. (2017). Everybody counts: Designing tasks for TouchCounts. In A. Leung & A. Baccaglini-Frank (Eds.), *Digital technologies in designing mathematics education tasks. Mathematics education in the digital era (MEDE) book series* (Vol. 8, pp. 175–192). Cham: Springer.

Sowder, J. (1992). Estimation and number sense. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 371–389). New York: Macmillan.

Thompson, J. C., Abbott, D. F., Wheaton, K. J., Syngeniotis, A., & Puce, A. (2004). Digit representation is more than just hand waving. *Cognitive Brain Research, 21,* 412–417.

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques, 10,* 133–170.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94.

Chapter 13
Interactive Mathematics Storybooks and Their Friends

Herbert P. Ginsburg, Colleen Uscianowski and Ma. Victoria Almeda

Abstract Little is yet known about Interactive Mathematics Storybooks (IMS) enveloped in a digital surround of supporting materials—their "Friends"—designed to delight and educate young children as well as those who read with them. Clearly different from paper books and physical manipulatives, interactive books entail a special set of affordances that can promote young children's mathematics learning, and the surrounding Friends can help the adult understand the mathematics and the child. This chapter relies to the extent possible on existing research and theory, but goes beyond current knowledge to speculate, imagine, and dream about the potential of IMS for helping young children to learn mathematics at home. The chapter uses what is known to imagine what could be.

Keywords Interactive mathematics storybooks · Learning mathematics
Reading · Supporting parents · Technology

13.1 The Context

13.1.1 Early Storybook Reading

Before describing these potentially powerful Interactive Mathematics Storybooks (IMS) and Friends, we want to situate them in the context of parents' early book reading. Everyday experience suggests that reading can be a magical opportunity for parent and child to bond as they engage in a warm and protected exploration of

H. P. Ginsburg (✉) · C. Uscianowski · Ma.Victoria Almeda
Department of Human Development, Teachers College Columbia University,
525 West 120th Street, New York, NY 10027, USA
e-mail: ginsburg@tc.edu

C. Uscianowski
e-mail: cu2156@tc.columbia.edu

Ma. Victoria Almeda
e-mail: mqa2000@tc.columbia.edu

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_13

the everyday world and the world of fantasy. Books can immerse children's minds in nature, trucks and buses, imaginary characters, and exciting, scary, or funny narratives. Books allow children—and adults—to experience the constructed world of literature, language, and art. We read to children partly to teach them to love reading itself. This goal is an expression of our values.

In developing mathematics storybooks and related materials, we in the mathematics education community should always keep in mind the social-emotional and broader literacy context (Aram and Aviram 2009). We should not destroy literacy and love of reading by creating fiction or non-fiction mathematics books that are didactic, boring, dull and pedestrian.

13.1.2 Benefits of Reading with Children

One seldom-mentioned benefit of reading is an opportunity for parents to learn about their children's worlds. Young children (ages two to six) are often described as egocentric: they see the world from a limited perspective (Piaget 1955). But adults can be equally egocentric. They often do not understand what the world looks like from a child's point of view. During the course of reading, a parent may learn that his daughter interprets events differently than he; that she sees things in the story that he did not; that she does not see things that he does; and that she learns from the story in ways he did not expect. Another benefit of reading lies in the opportunity for a parent to converse and bond with the child, creating a warm environment that can support early social-emotional development (Aram and Aviram 2009). For instance, elaborating on the emotions of a character in the story has been shown to encourage children's ability to reflect on their own emotional experiences (Laible 2004). Thus, reading can provide both a window into the young child's mind and emotions, as well as ideas about how to nurture them.

Children, like adults, can learn a great deal from books. Children learn content: a truck is different from a car and Peter Pan can fly. The more children read, the more they understand the world around them, and conversely, as their knowledge of the world grows, they become better readers. A child who has never encountered a giraffe will learn about its long neck and funny ears from the story's illustrations. The next time the child visits the zoo and meets a giraffe up close, she will gain information about its height, gait, and eating habits. This knowledge will help the child better understand the next story she reads about giraffes.

As the adult turns the pages and points to the words being read aloud, children learn about the world of print—a book has pages that are read in order and letters that indicate words. These words are read from left to right and from top to bottom, in most languages. They learn that the sounds of words can rhyme and be silly. Not only are words made of sounds, but they're also made of letters that have special names and shapes. They learn that words have the essential function of describing, telling a story, and explaining.

As children listen to the story unfold, they learn new words that they have not heard before. In fact, children hear richer and more varied language from storybooks than from everyday conversations (Montag et al. 2015) or from talk during toy play, dressing, and meal times (Hoff-Ginsberg 1991). Storybooks written for young children often have repeated text: *I'll huff and I'll puff and I'll blow your house in.* Children hear the same text over and over again, soon predicting what the big bad wolf is about to say, until *huff* and *puff* become part of the child's own vocabulary. Parents also help children to learn new words. They point out and name parts of the illustrations and use the language from the text when asking questions about the story. Reading helps children become literate in the narrow and broad senses.

The words that adults use when talking with children influence how children learn and think. Building early mathematics vocabulary not only sharpens understanding of numbers, but also provides the opportunity and means to communicate mathematical thinking. Children whose parents use spatial language, such as describing the size and shape of objects, use more spatial words themselves (Pruden et al. 2011). In turn, these children solve spatial puzzles better than children whose parents provide little spatial input. Storybooks, full of rich text and interesting illustrations, are a convenient way for adults to expose children to words and ideas that may not otherwise be present in their environment. Most storybooks can also introduce mathematics ideas using a narrative appropriate for young children, providing a tool for facilitating early mathematics communication. Parents can attend to the mathematics language on each page of the book (e.g., words about number, size, and shape), and encourage young children to use these words when explaining their thinking.

13.1.3 Methods of Reading with Children

Given the benefits linked to reading, research also suggests productive methods for adult reading with children. All too often, the adult simply reads while the child passively listens. Dialogic reading involves a shift in these roles in that the adult encourages the child to contribute to the reading of the story. The goal is to stimulate the child's thinking, engagement, and language. For example, the adult can engage the child in conversation with open-ended prompts (e.g., "Tell me what's happening in this picture."), which prod the child to describe interesting and important ideas in the book (Lonigan and Whitehurst 1998). Dialogic reading has been extremely effective in achieving its goals (LaCour et al. 2013; Sim et al. 2014).

The level of complexity in parents' language has also been studied in the context of storybook reading. Research indicates that parent's use of higher, more abstract levels of language is positively associated with children's later abstract language abilities, corroborating the connection between abstract language and early literacy skills (van Kleeck et al. 1997). For example, a question such as "How did you know that?" produces more thought and language than the question, "What is that?" However, little research has studied how adults use levels of abstraction in storybook reading to support children's mathematical thinking in particular.

13.1.4 Storybooks for Mathematics Learning

We propose that books of several different kinds may help young children learn about mathematics. The first two types are well known. Some storybooks, with illustrations, are explicitly designed to teach mathematics. Mathematics storybooks are fictions that involve mathematical ideas. For example, the *Elevator Magic* (Murphy 1997) story revolves around figuring out how to subtract in order to make an elevator stop at different floors for different purposes.

A second kind of explicit mathematics picture book has minimal stories, and is mostly didactic non-fiction, as in the case of counting books. Books of this type tend not to offer an overall narrative, and instead to focus attention on specific mathematical task like counting collections of objects or naming and describing simple shapes.

A third type of storybook does not aim to teach mathematics explicitly but contains important mathematical ideas naturally embedded within the narrative (Van den Heuvel-Panhuizen et al. 2009). Goldilocks sees that Baby Bear's bed is the smallest, and that Mama's bed is bigger than Baby's but smaller than Papa's. The series of beds and bears are in one-to-one correspondence. As bears vary in size so do beds (and amounts of porridge too).

The first and third types of storybook incorporate the mathematics in the narrative, providing a context for children to connect the story to their everyday lives. The plot reveals mathematics problems that children can solve alongside the characters. In this way, mathematics problem solving becomes not only meaningful to advance the story, but fun and exciting. For example, in *Rooster is Off to See the World* (Carle 2013), children first learn Rooster's motivation for taking a trip and how he feels lonely before inviting two cats to join him. More animals join the group as his journey continues. The story-based context invites children to explore the increase in number of animals, predict how many and which type of animal they will encounter next, and decide if Rooster's loneliness is abated by sharing his journey with his new friends.

13.1.5 Analyzing Mathematics Storybooks

Although some books may promote mathematics learning (Van den Heuvel-Panhuizen et al. 2016), others may not. Indeed, books may convey misleading, confusing, or incorrect mathematics. One page from *Five Little Monkeys*

Bake a Birthday Cake (Christelow 2005) shows two monkeys holding eggs for the recipe while saying, "I have 2 eggs!" although each is holding three eggs. The disparity between the number shown in the illustration and the number written in the text can disrupt young children's developing understanding of quantity. Creating and reading *high quality* books should be the goal. To realize this goal, we created a system to evaluate and select picture books that have both literary and mathematical merit, based on both our observations of parents reading storybooks and on similar evaluation systems (Austin 1998; Hellwig et al. 2000; Hunsader 2004). We start with three broad questions: (1) What kind of story is this and how well is the story told? (2) How much and what kind of mathematics is presented in the text? and (3) How accurately is the mathematics represented in the text and illustrations?

The first question deals with the literary aspects of the picture book. Although our focus is mathematical picture books, a high quality story should stand on its own as a piece of literature. To choose stories with literary merit, we look for an interesting plot, relatable characters, and attractive illustrations. Children should find wonder in the story and want to turn the page to find out what happens next. The adult, too, should choose intriguing stories since they will be called upon to read favorite books time and time again.

The second question begins to tackle the mathematics aspects of the story and helps the evaluator to quantify the types of mathematics present in the picture book. Does the book primarily deal with number, shape, measurement, or another mathematics domain? If this is a number book, does the story contain ideas about cardinality, computation, or comparing sets? How many examples of mathematical symbols are shown in the text and illustrations? Answering these questions allows us to determine roughly how much of the story is mathematical in nature.

We consider the third question to be the most pertinent in determining whether the picture book presents a high quality mathematics story that should be read with children versus a poorly presented mathematics story that should be avoided altogether. Here we examine whether the mathematics is accurate, whether the mathematics content in the text closely aligns with how it is portrayed in the corresponding illustrations, and whether the story is likely to advance children's understanding of mathematics.

In our analysis of picture books, we have discovered that many books, even those not explicitly written with the intention to teach mathematics, implicitly contain many mathematics concepts. Yet our observations suggest that many parents fail to realize that the story contains fundamental mathematical ideas. Therefore, we have created guides to help parents first notice the mathematics in the story, and then ask open-ended questions that support their children's understanding of the mathematical content. Our reading guides come in two forms: First, specific guides that explain the mathematics in individual picture books and offer suggested questions to ask while reading. The other, a broader guide, lists general tips and questions that can be modified for a variety of mathematics picture books.

13.1.6 The Benefits and Limitations of Interactive Storybooks

Stories presented electronically on tablet devices are increasingly common in homes and schools, with 72.5% of middle-income families reporting the reading of electronic books with their two to six-year old children (Vaala and Takenchi 2012). IMS range in their complexity, with little uniformity in format (Guernsey et al. 2012). While some IMS contain few interactive elements, others have many digital affordances that make them quite unlike print books. Unfortunately, research on IMS is limited, probably because they are so novel. However, findings are beginning to emerge that describe the benefits and limitations of electronic storybooks that were not designed to teach mathematics. Much of this literature complements work on interactive games and software presented at ICME-13 in TSG 1, such as Sinclair's work on dynamic geometry (Sinclair 2018) and multi-touch apps fostering numeracy, Fletcher and Ginsburg's (2016) work on symmetry software, Baccaglini-Frank's (2018) work on students and teachers using multi-touch apps in pre-school, and Nivens and Geiken's (2016) work on computer science-based games.

We already know that print books can bolster children's literacy skills (Bus et al. 1995), such as their knowledge of word sounds and meanings, so the question is whether electronic storybooks can provide similar or even greater advantages. Electronic storybooks include an assortment of features, such as animation, sound effects, embedded games, and oral narration. These affordances may be better at attracting and maintaining children's attention than print books, leading to improved reading abilities over time. Animation, in particular, may help improve children's reading comprehension and vocabulary. Moving images have the potential to draw the reader's attention to particular aspects of the scene and highlight relevant parts of the illustration, while static images sit motionless on the page (Bus et al. 2015). Verhallen et al. (2006) presented five-year-old Dutch children at risk for language delay with either an animated or static electronic storybook. Over the course of four reading sessions, children who read the enhanced book understood more of the story and recalled a greater number of vocabulary words than children who read the static book. According to the cognitive theory of multimedia learning (Mayer 2005), the animation in electronic storybooks may be better suited than picture books to support children's ability to make a direct connection between verbal and visual information presented in the story.

Like animation, sound effects can support story comprehension by clarifying the meaning of unknown words. Hearing a crowd *cheer* as the word is narrated helps the child connect the word with its meaning. However, not all children benefit from added sounds and music. For example, children with severe language impairments learned more vocabulary words when reading a book without accompanying sounds and music (Smeets et al. 2014). These sounds may create noise that can hinder learning in children who struggle to perceive speech. While sounds and music can inhibit processing oral text and impede learning in certain populations, those without language delays may still benefit from the simultaneous presentation of text and sound. Bus et al. (2015) suggest that the animation and sound effects should connect closely to the story to prevent cognitive overload. When children simultaneously attend to both the plot of the story and various disparate electronic features, it can hinder their understanding of the story.

Studies testing the effects of games and other "hotspots" on learning have yielded both positive and negative results. Moody et al. (2010) observed adult-guided reading sessions with both print and electronic storybooks. Their results suggest that children exhibit greater levels of persistence when reading the electronic storybook rather than print storybooks. However, other studies have found that game-like features embedded in the story can distract from comprehension and lead to incorrect recall of the plot (Courage et al. 2015; Trushell et al. 2003).

In addition to the specific features of electronic storybooks, researchers have studied the role that adults play in shared reading with digital devices (Chiong et al. 2012; Lauricella et al. 2014; Robb 2010). Unlike print books, many electronic books have a feature that narrates the text aloud. The device assumes the responsibility for reading the text and controlling the pace of the story. The parent then assumes a new role in mediating the relationship between the story and the child, although it is not clear how the parent can best support the child's learning amid multimedia features such as sound and animation. Some research suggests that interactive books can increase parent involvement in reading, but are otherwise similar to paper books (Lauricella et al. 2014). Robb (2010) found that parents play an important role in guiding children's understanding of the story, but only when reading electronic books with fewer digital features. Chiong et al. (2012) recommend that designers create electronic storybooks that allow parents to customize the settings to give them greater control over the shared reading experience with their children.

We still have a great deal to learn, particularly about designing IMS to promote learning; about effective methods for reading IMS to children; about parents' understanding of IMS and attitudes towards them; about what children can learn from IMS; about the surrounding Friends that can support and extend parents' reading of IMS and children's learning from them; and about evaluating IMS quality.

13.2 Monster Music Factory

To illustrate the potential and limitations of IMS, we next present a description and analysis of a carefully designed interactive storybook, The *Monster Music Factory*, along with its surround, digital and non-digital (the Friends). The goal of the story and supporting material is to promote children's meaningful mathematics learning as well as parents' understanding of it. The target audience is children in the three to five-year old range, especially low-income, minority children, who often receive inferior education, at least in the U.S. (Ginsburg et al. 2008).

Monster Music Factory (Ginsburg et al. 2016), operating on a touch screen device, tells the story of several monsters engaged in determining the number of instruments—drums, trumpets, kazoos, and ukuleles—needed for a concert to be performed by a famous band, the Whirling Wailers. Each "page" of the storybook is like a scene in a play. The page presents various actions of the characters, not just static illustrations with text, and also gives the child the opportunity to interact with the characters and objects in the story.

In the first scene (see Fig. 13.1), the monsters (Oona with one eye, Marluk with two and Tigga with three) learn from the adult-like Zoller (with eye-glasses) that their task is to fill boxes with specified numbers of drums. If there are 4 drums on the screen, the monsters must put them in the numeral 4-box. The monsters can also check their response by touching the box, whereupon the drums pop up long enough to be counted.

At the beginning of each subsequent scene, one of the monsters uses a sensible strategy to solve a problem that the story presents. For example, in the second scene, after four tambourines emerge onto a conveyor belt, Oona rashly proclaims that there are three (see Fig. 13.2), but Tigga admonishes her: "Not so fast, Oona. Let's count them." Tigga then uses the strategy of touching each tambourine very carefully, saying, "1, 2, 3…4! There are four tambourines altogether." Next Oona says, "Whoops, my bad. I should have counted."

The child is then asked to push the correct number on the machine so that the 4-box can arrive to receive the corresponding number of tambourines. If the child gets the answer wrong, Tigga says that because there are four tambourines, the 4-button needs to be touched. When the child does so, the 4-box appears and the

Fig. 13.1 The factory

Fig. 13.2 Tambourines

tambourines go into it. What if the child cannot read the numeral 4? The parent can help, in this case by reading the numeral for the child.

Then Tigga says, "Let's try it again," whereupon 3 tambourines appear on the conveyor belt. If the child touches the tambourines, each shakes, makes a tambourine sound, lights up, and stays lit as other tambourines are touched. If the child presses the wrong button, Marluk says, "Let's count together. One, two…oh… three. It should be three." After the child presses the correct number, a 3-box pops up and the tambourines fly into it. After this exchange, the child has a chance to solve several more problems of the same type.

The third scene involves trumpets, appearing as a visual mess in which some trumpets obscure other trumpets behind them. The suggested method of solution is to carefully push each trumpet aside into a line. The fourth scene involves "subitizing" kazoos—that is learning to see the number when objects appear quickly and then disappear into a box. Here strategy is not an issue: children need to develop the perceptual skill of subitizing through practice with the problems presented in this scene (Clements 1999). This experience will eventually help the child to see small numbers up to about 5. The fifth scene introduces numbers between 5 and 10. Here the method of solution is to group by five and then count on to get the correct number. The next scene involves lining up the numerals in order, from 1 to 10, after which, you will be happy to learn, the monsters get to attend the concert and hear original music about numbers.

Several principles and goals guided our design of *Monster Music Factory*. First, we wanted to create an enjoyable, funny and well-written story. Monsters are a big hit with young children; the artwork of the book is attractive; and the book ends with perky music during the concluding concert. Literacy and artistic quality were our first criteria.

Second, we wanted to present children with challenging problems appropriate for their age level. Using the research literature as a guide, we decided that the skills and concepts of enumeration are fundamental and challenging for young children (Gelman and Gallistel 1986). Many adults seem to be unaware of these concepts or of the need to teach them. The same seems to be true of at least some developers who create children's apps and storybooks, both paper and interactive. Moreover, the research literature identifies components of enumeration that appear in a developmental progression (Gelman and Gallistel 1986). We therefore arranged the scenes—that is, the action and interaction involving each component—to mirror this progression. For example, the first challenge involves counting carefully by ones, the second counting by moving aside, the third subitizing and fourth grouping by 5 and counting on to get the number. Having used the research literature to determine the content of each scene, we then wrote a narrative that we hoped would present these ideas in as engaging a manner as possible. Our informal observations suggest that this IMS attracts the attention and interest of young children, roughly from 3 to 5 years of age. The younger children can understand and engage with the relatively easy scenes, and the older children with the more difficult material.

Third, we wanted each scene to offer guidance in use of appropriate methods of solution and understanding the mathematical ideas. Consequently, each scene first shows characters demonstrating the relevant strategies and talking about them, and second, encourages the child's use of strategies to solve the relevant problems.

Fourth, we wanted the visual and sound design to reflect and reinforce the various mathematical ideas. For example, in the scene involving four tambourines, Tigga touches each very carefully in sequence. When he touches the first, saying "one," it glows, then makes a rattling sound, and continues to glow. The same occurs for the third and fourth tambourines, until at the end all four glow. When Tigga then says, "There are four tambourines altogether," they shake in unison and make the rattling sound.

This sequence of events was carefully designed to show that when the first and second objects are touched, they are a set, which expands when the third object and then the fourth are touched. We wanted to avoid the common misunderstanding that the second object itself is 2 (Fuson et al. 1985). It is not: it is a member of the set of two. Also, at the end, Tigga's statement, "There are four tambourines altogether," emphasizes that the tambourines form a set with the cardinal value four. This approach is necessary because many children at this age count to four but do not know that the last count word indicates the number of the set as a whole. Asked how many are there, these children simply count the set again. The answer to "How many?" is simply to repeat the counting words.

Fifth, we wanted to make the symbolism meaningful. Our general approach was to link symbols with illustrations and actions. In the first scene, the child sees boxes with numerals on each. When the 4-box is touched, four drums jump up from within the box and remain suspended in the air. As Tigga counts them, each drum lights in sequence, shakes, and makes drum sounds. At the end, as Tigga says, "Four drums altogether," the numeral 4 grows larger and then returns to its regular size, after which the four drums shake and sound.

IMS are able not only to tell a story, but can also write a “story” about the child, a report about the child’s performance. The IMS software can track the child’s accuracy, the strategies employed under certain conditions, and the trajectory of performance within the reading session. Thus the report might indicate that the child began by producing incorrect answers in scene 2, but concluded it with consistent accuracy. The report might indicate that the child responded so quickly in a particular scene that she was unlikely to have used any strategy. The report might indicate that the child did indeed use the pushing aside strategy at the appropriate point to achieve a correct answer. We explore how to use the touches and movements recorded in the software’s data log to create indices of accuracy and strategy. Our long-term goal is using this information to create reports that can guide parents’ reading and other efforts designed to promote children’s mathematics learning.

In brief, we designed an interactive storybook intended to serve as a model for other efforts in this area. The IMS offers a story that children enjoy; presents challenging mathematical problems; offers guidance on strategy and concepts; uses visual and sound designs that reflect and reinforce the mathematics; links mathematical symbols to actions and visualizations; and is capable of providing reports on accuracy and strategy.

13.3 The Friends

When we exhibited Monster Music Factory at the U.S. National Mathematics Festival, several parents came to our table and left the children to work with the IMS by themselves. The parents seemed to feel that Monster Music Factory is a game, an app like *Angry Birds*, which children can use independently, without much or any supervision. In our view, this approach is inadequate. IMS are not apps! They are books that should be read with a knowledgeable and strategic reader, usually an adult—most likely a parent, caretaker, or teacher. But most adults are not as familiar as they should be with IMS, mathematics, methods of reading, and children’s learning of mathematics.

How can we help adults to read IMS in an informed and effective manner? Consider how several supports, which we call “Friends,” might provide the necessary guidance. Everything we describe below is technologically possible, even though much has not yet been implemented. Imagine a situation in which both the adult reader and the child have tablets. The child’s tablet is used for reading and interacting with the IMS. The adult’s tablet has not only the IMS, but also several Friends, including PDF documents, learning software, and videos, as follows.

13.3.1 The Guide

One support residing on the adult's tablet is the Guide, including text, embedded illustrations, and videos. The Guide offers several different types of assistance. For example, in the tambourine scene, shown earlier in Fig. 13.2, Oona looks at the instruments and says, impulsively, "I know! I know! Three tambourines." The Guide offers a short video of this event and then advises how to deal with it.

> In this scene, you may want to pause the action and ask your child about whether Oona was right. Then let your child watch Tigga teach Oona how to check her counting. Asking a question at this point may distract your child from understanding the story.

The scene continues to the point where Tigga shows Oona how to count the tambourines. The Guide advises:

> Here, Tigga is showing your child one-to-one correspondence. He counts as he touches each tambourine once and only once. Notice that the tambourines shake and light up all together to show that Tigga is counting a collection of tambourines to get its number.

Later, when the child has a chance to count different numbers of tambourines, the Guide advises:

> If your child blurts out the number of tambourines without obvious counting, you can say, "How did you know? Let's check! Why don't you touch and count each one just like Tigga!" If your child is having a hard time remembering what Tigga did, you can model one-to-one correspondence. Point to each tambourine and count, "One, two, three!"

In addition to providing advice about responding to each scene, the Guide provides access to other resources. Touching an icon on the screen opens a video of parents who demonstrate important strategies for reading the *Monster Music Factory* to their children. For example, one clip shows how a parent encourages his child to check his answer. As shown in Fig. 13.3, the Guide briefly describes the parent's strategies to help his child attend to the mathematical idea in this scene. Another shows a parent modeling a useful solution strategy. A third shows a parent directing the child's attention to a key element of the story. We believe that video demonstrations are particularly useful for parents who are anxious about helping their children learn mathematics. We plan to conduct research both on the effectiveness of the examples and parents' use of them.

13.3.2 Professor Ginsboo

A second Friend on the tablet is entitled *Professor Ginsboo Explains Everything You Always Wanted to Know About Math.* This distinguished Professor presents amusing illustrated accounts of the meaning of counting and other concepts covered in the IMS. The stories of Professor Ginsboo can help parents comprehend the basic mathematics ideas that young children should be learning. For example,

In this video, the parent asks the child, “Want to touch them to see if they make a sound?” to gently remind him to check and touch each of the tambourine. When the child says there are three tambourines, the parent encourages the child to notice other things that sum up to three. He comments that the number of instruments matches the number of Tigga’s eyes!

Fig. 13.3 An example of a parent encouraging careful counting

Professor Ginsboo’s *Story of How Many* explains the idea of enumeration through a conversation between Professor Ginsboo and a curious young student, Menette.

In the first part of the story, Menette is introduced to Trumpet and Elephant. She notes that there is one Trumpet and one Elephant. But Trumpet and Elephant argue and both exclaim, “I am one!” Each feels that the other does deserve the exalted position of number one.

Professor Ginsboo offers some help and explains the following to Menette and the others. “First of all, you are all one. Each and every one of you is one. All for one and one for all. Here’s a one, there’s a one, everywhere a one, one.

> “But. Watch this. Elephant, say, ‘I am one.’” He said it. “Trumpet, say, ‘I am one.’” He did. “Now, Elephant, hold hands with Trumpet.”

This was not so easy because Elephant had such big hands and Trumpet didn’t want to hold hands with anyone. But finally they succeeded, and when each hand was holding the other, Elephant and Trumpet sang out, “We are two!” They did not intend to do it, but when holding hands could not help themselves. They sang it a second time, “We are two!” They sang, “We are two!” two times. They thus learn that although each is one, the combination of them is a different larger number.

The story is intended to help adults realize that the basic idea of one is not simple and that all numbers are constructed from units and thus achieve new “identities.” When we count, “One, two,” we are not referring to the second object counted as “two.” The second object is “one,” and the combination of the units is “two,” an entirely different number from the units that are subsets of it.

Will adults learn the mathematics presented in Professor Ginsboo and find it useful? We are conducting research to find out.

13.3.3 Mathematics Thinking Stories

A third Friend offers Mathematics Thinking Stories, which illustrate children's understanding of the mathematical ideas entailed in Monster Music Factory. These "stories," somewhat like case studies of individuals, bring to life children's engagement with mathematical ideas. Embedded in the text are videos of actual children doing various mathematical tasks.

Ben Learns How Many is one example of a Mathematics Thinking Story that describes a child's path from reciting the number words to enumerating the number of objects before him. The story begins with a video clip of the cheerful three-year-old Ben, who rattles off the numbers one through ten with ease but cannot enumerate a set of four bears. After establishing that counting (One, two, three, four) is different from enumerating (There are four bears), the story continues to deconstruct the concept of how many. We as adults can easily count how many pennies are in our wallet or candies are left in the jar, forgetting what a complex notion this is for children. Number words are quite special and different from other words that name or describe objects. The confusing complexities of counting, such as understanding the cardinal value of a set, are described from the perspective of the developing child.

Ben Learns How Many ends with concrete suggestions about how to teach how many. Key concepts, such as enumeration and cardinal value, are not only defined but also illustrated in words, images, and videos throughout this Mathematics Thinking story. The story of Ben explains these key concepts in narrative form, which helps parents relate Ben's understanding of numbers to their own children. The goal of the mathematics thinking stories is not only to interpret the child's thinking, but also to develop a deeper understanding of the mathematics content.

13.3.4 Assessment

A fourth friend, *Assessment*, offers various forms for evaluating children's mathematical thinking. One way to assess a child's thinking is to collect and analyze log data from the IMS software. We have not yet implemented this but it can easily be done. Research in the field of learning analytics suggests that we can collect data on children's online interactions with the digital environment, and conduct micro-level analyses on how they learn using the software (Baker and Yacef 2009). In our case, children's online behaviors when using the software can provide information about their mathematics knowledge. For example, the number of attempts at getting to the right answer could be indicative of the mathematics concepts children find difficult to understand. These results could potentially be integrated on the second tablet used by adults, so that they can be notified if a child is struggling and provide support in real time. We have not yet created this kind of online assessment of a child's performance on our IMS, but it technically possible to do so. There is great promise in utilizing log data to create useful measures of children's mathematical thinking.

Another way to assess how a child thinks is through clinical interviewing, in which an adult attempts to gain a deeper understanding about the child. The *Do's and Don'ts of Clinical Interviewing* consists of guidelines for conducting an interview with an individual child. It specifies the process of planning a concrete task for the child to do, observing and listening to the child's responses, and probing the child with open-ended questions. The interviewer is likely to gain more insight into the child's reasoning by encouraging the child to explain his or her answers. The adult can do this by asking the following questions, "How did you do that? How could you explain it to a friend? Can you show me how you did it?" These questions allow the child to elaborate and demonstrate his or her thinking, potentially revealing surprising competence as well as some misconceptions or ineffective strategies. Use of informal questioning can provide parents with very surprising insights into their children's thinking. In our experience, parents (and other adults, including teachers) are often quite surprised to learn what their children do and do not know.

In addition to these guidelines, adults can watch and learn from a voice over video of an exemplary clinical interview of a four-year old child. The video contains a comprehensive analysis of the interview, with detailed commentary and interpretations about the child's mathematical knowledge.

Clinical interviewing has traditionally been the preserve of research psychologists. But we believe that parents can learn a version appropriate for their everyday interactions with children. Again, research on this issue needs to be conducted.

13.3.5 Mathematics Activities

A fifth Friend is *Mathematics Activities to Do at Home*, a brief list of home activities (with real objects!) that reinforce and extend the different mathematical challenges in the IMS. Four of the major mathematics themes in *Monster Music Factory* are cardinality, one-to-one correspondence, subitizing, and grouping by 5s. Each activity corresponds to one of these four themes, giving the child the opportunity to further explore number beyond the storybook.

We do not expect the child to master counting and become an expert at choosing appropriate counting strategies after reading *Monster Music Factory*. Rather, this IMS, and others, should serve as one tool in a larger toolbox of instructional ideas. A parent should engage a child in mathematical conversations during both planned activities (e.g., reading storybooks, intentionally playing with blocks) and during unplanned activities (e.g., on the playground, while cooking dinner). Our *Mathematics Activities* packet suggests some planned activities that extend learning beyond the IMS. The activities are taken from a number of research-based mathematics programs and curriculums, such as *Big Math for Little Kids* (Ginsburg et al. 2003).

13.3.6 MathemAntics

A sixth friend is new mathematics software, *MathemAntics* (that is, antic mathematics), specially designed to extend children's work with the mathematics concepts embedded in the story (Ginsburg et al. 2015). There are seven activities children can explore on the tablet device. Each activity closely connects to the IMS by reinforcing the same basic number concepts from the story and using the same instruments, characters, and background illustrations.

Explore provides a free play environment for children to become accustomed to the software. They learn how to use the various tools in the program, such as scatter around, line-up, and box.

In **subitizing**, instruments fall quickly from a chute before going into a box below. Children need to select a number on the number line that corresponds to the amount of instruments they saw on-screen.

Counting provides practice with enumeration. Children need to count the number of instruments that drop from the chute and land on the screen. They can use the highlight, line-up, pair-up, and scatter around tools to help them solve the problem.

Pond encourages counting forwards and backwards, along with basic adding and subtracting by one. Children drag frogs into and out of the pond as the program narrates the number of frogs.

In **visible addition and subtraction**, children drag a specified number of frogs into the pond. Then a few more frogs jump into the pond and children must determine how many there are altogether. Similarly, frogs can jump out of the pond, providing subtraction practice.

In **addition/subtraction**, children practice adding and subtracting when the objects are not visible. Instruments are dragged into boxes, and then more instruments appear on screen. Children add these instruments to the box, where they are hidden from sight. Then children must predict how many instruments reside within the box.

> = < is an equivalence activity. Children compare the number of animals on either side of the screen and determine which side has more animals or whether both sides have the same number. They can use the line-up and pair-up tools to help them organize the animals as they are counted.

Overall, *MathemAntics* can be used to help children practice finding the cardinal value of a set, explore addition and subtraction, read numerals, use strategies such as lining up objects to help them count, and learn to check if their answer is correct.

13.3.7 Video Dictionary

A seventh Friend is a video dictionary of mathematical words used in the IMS. *Professor Ginsboo's Excellent Mathematics Explainer* uses story characters from the IMS to provide amusing short definitions of the concepts underlying the words and thus reinforce children's understanding and usage of mathematics terms. For example, to explain the strategy of *Counting On from 5*, a scene with Tigga from

Monster Music Factory is presented to the child. Tigga demonstrates the technique of counting on. As Tigga explains, "This trick makes counting easier, if you know there are 5," five guitars on the first row play a tune all at once. Tigga continues, "You can start counting with 6", and the 6th guitar on the 2nd row plays a tune.

In general, research has shown that reading storybooks increases children's acquisition of new vocabulary words (Sénéchal 1997). The animated *Explainer* leverages children's experience with an IMS to enable them to learn the meaning of mathematics terms and phrases.

13.4 Conclusion

We have created IMS and Friends to help children learn basic mathematical ideas and to help adults understand their children and how to read the stories. The enjoyable narrative encourages exploration of the ideas in a sequence guided by cognitive research. The story offers challenging interactions that deepen mathematical understanding and promote useful strategies. Further, surrounding Friends include video and other materials, linked to and deriving from the story, that promote adults' understanding of storybook reading and children's learning. In providing a diverse set of supporting materials for the IMS, different readers can potentially benefit from different Friends.

Our work is in its infancy and we are only beginning to conduct research on the effectiveness of our IMS and Friends. We do not know yet whether our materials all "work" in the ways intended. But we hope that this paper demonstrates the potential of our design principles and approach, and encourages others to engage in development and research in this promising area of mathematics education. Advances in computer technology provide the possibility of creating complex systems designed to help children learn mathematics from narratives and to help adults to understand and promote children's mathematical thinking and learning.

Acknowledgements We wish to acknowledge the invaluable assistance of our research and development team, including Cassie Freeman, Bona Lee, Eileen Wu, Mei Tan, and Jessica Wille. We are grateful to the Heising-Simons Foundation for providing generous support for the project.

References

Aram, D., & Aviram, S. (2009). Mothers' storybook reading and kindergartners' socioemotional and literacy development. *Reading Psychology, 30*(2), 175–194.

Austin, P. (1998). Math books as literature: Which ones measure up? *New Advocate, 11*(2), 119–133.

Baccaglini-Frank, A. (2018). What schemes do preschoolers' develop when using multi-touch applications to foster number sense (and why)? In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Baker, R. S., & Yacef, K. (2009). The state of educational data mining in 2009: A review and future visions. *JEDM-Journal of Educational Data Mining, 1*(1), 3–17.

Bus, A. G., van Ijzendoorn, M. H., & Pellegrini, A. D. (1995). Joint book reading makes for success in learning to read: A meta-analysis on intergenerational transmission of literacy. *Review of Educational Research, 65*(1), 1–21.

Bus, A. G., Takacs, Z. K., & Kegel, C. A. T. (2015). Affordances and limitations of electronic storybooks for young children's emergent literacy. *Developmental Review, 35,* 79–97.

Carle, E. (2013). *Rooster is off to see the world* (1st Simon Spotlight Ready-to-Read ed.). New York, NY: Simon Spotlight.

Chiong, C., Ree, J., Takeuchi, L., & Erickson, I. (2012). Comparing parent-child co-reading on print, basic, and enhanced e-book platforms. Resource document. Joan Ganz Coony Center. http://www.joanganzcooneycenter.org/wp-content/uploads/2012/07/jgcc_ebooks_quickreport.pdf. Accessed June 24, 2017.

Christelow, E. (2005). *Five little monkeys bake a birthday cake*. New York, NY: Clarion Books.

Clements, D. H. (1999). Subitizing: What is it? Why teach it? *Teaching Children Mathematics, 5*(7), 400–405.

Courage, M. L., Bakhtiar, A., Fitzpatrick, C., Kenny, S., & Brandeau, K. (2015). Growing up multitasking: The costs and benefits for cognitive development. *Developmental Review, 35,* 5–41.

Fletcher, N., & Ginsburg, H. (2016, July). *Teaching symmetry in the early childhood classroom: Using software to facilitate discourse and promote understanding of symmetric transformations*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Fuson, K. C., Pergament, G. G., Lyons, B. G., & Hall, J. W. (1985). Children's conformity to the cardinality rule as a function of set size and counting accuracy. *Child Development, 56*(6), 1429–1436. https://doi.org/10.2307/1130462.

Gelman, R., & Gallistel, C. R. (1986). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Ginsburg, H. P., Cerf, C., & Creighan, S. (2016). Monster Music Factory (version 3.5.3) [Application software]. http://www.speakaboos.com/stories/monster-music/factor. Accessed June 24, 2017.

Ginsburg, H. P., Greens, C., & Balfanz, R. (2003). *Big math for little kids*. Parsippany, NJ: Dale Seymour Publications.

Ginsburg, H. P., Labrecque, R., Carpenter, K., & Pagar, D. (2015). New possibilities for early mathematics education: Cognitive guidelines for designing high-quality software to promote young children's meaningful mathematics learning. In A. Dowker & R. C. Kadosh (Eds.), *Oxford handbook of mathematical cognition*. Oxford, England: Oxford University Press.

Ginsburg, H. P., Lee, J. S., & Boyd, J. S. (2008). Mathematics education for young children: What it is and how to promote it. *Society for Research in Child Development Social Policy Report-Giving Child and Youth Development Knowledge Away, 22*(1), 1–24.

Guernsey, L., Levine, M. H., Chiong, C., & Severns, M. (2012). Pioneering literacy in the digital wild west: Empowering parents and educators. Resource document. Campaign for Grade-Level Reading. http://www.joanganzcooneycenter.org/wp-content/uploads/2012/12/GLR_TechnologyGuide_final.pdf. Accessed August 8, 2017.

Hellwig, S. J., Monroe, E. E., & Jacobs, J. S. (2000). Making informed choices: Selecting children's trade books for mathematics instruction. *Teaching Children Mathematics, 7*(3), 138–143.

Hoff-Ginsberg, E. (1991). Mother-child conversation in different social classes and communicative settings. *Child Development, 62*(4), 782–796.

Hunsader, P. D. (2004). Mathematics trade books: Establishing their value and assessing their quality. *The Reading Teacher, 57*(7), 618–629.

LaCour, M. M., McDonald, C., Tissington, L. D., & Thomason, G. (2013). Improving pre-kindergarten children's attitude and interest in reading through a parent workshop on the use of dialogic reading techniques. *Reading Improvement, 50*(1), 1–10.

Laible, D. (2004). Mother-child discourse in two contexts: Links with child temperament, attachment security, and socioemotional competence. *Developmental Psychology, 40*(6), 979.

Lauricella, A. R., Barr, R., & Calvert, S. L. (2014). Parent-child interactions during traditional and computer storybook reading for children's comprehension: Implications for electronic storybook design. *International Journal of Child-Computer Interaction, 2*(1), 17–25.
Lonigan, C. J., & Whitehurst, G. J. (1998). Relative efficacy of parent and teacher involvement in a shared-reading intervention for preschool children from low-income backgrounds. *Early Childhood Research Quarterly, 13*(2), 263–290.
Mayer, R. E. (2005). Cognitive theory of multimedia learning. In R. E. Mayer (Ed.), *The Cambridge handbook of multimedia learning* (pp. 31–48). Cambridge, UK: Cambridge University Press.
Montag, J. L., Jones, M. N., & Smith, L. B. (2015). The words children hear: Picture books and the statistics for language learning. *Psychological Science, 26*(9), 1489–1496.
Moody, A. K., Justice, L. M., & Cabell, S. Q. (2010). Electronic versus traditional storybooks: Relative influence on preschool children's engagement and communication. *Journal of Early Childhood Literacy, 10*(3), 294–313.
Murphy, S. J. (1997). *Elevator magic*. New York, NY: HarperCollins.
Nivens, R. A. & Geiken, R. (2016, July). *Using a computer science-based board game to develop preschoolers' mathematics*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.
Piaget, J. (1955). *The language and thought of the child*. Cleveland, OH: World Publishing.
Pruden, S. M., Levine, S. C., & Huttenlocher, J. (2011). Children's spatial thinking: Does talk about the spatial world matter? *Developmental Science, 14*(6), 1417–1430.
Robb, M. B. (2010). *New ways of reading: The impact of an interactive book on young children's story comprehension and parent-child dialogic reading behaviors* (Unpublished dissertation). Riverside: University of California.
Sénéchal, M. (1997). The differential effect of storybook reading on preschoolers' acquisition of expressive and receptive vocabulary. *Journal of Child Language, 24*(01), 123–138.
Sim, S. S. H., Berthelsen, D., Walker, S., Nicholson, J. M., & Fielding-Barnsley, R. (2014). A shared reading intervention with parents to enhance young children's early literacy skills. *Early Child Development and Care, 184*(11), 1531–1549.
Sinclair, N. (2018). Time, immersion and articulation: Digital technology for early childhood mathematics. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).
Smeets, D. J. H., van Dijken, M. J., & Bus, A. G. (2014). Using electronic storybooks to support word learning in children with severe language impairments. *Journal of Learning Disabilities, 47*(5), 435–449.
Trushell, J., Maitland, A., & Burrell, C. (2003). Pupils' recall of an interactive storybook on CD-ROM: Pupils' recall of an interactive storybook. *Journal of Computer Assisted learning, 19*(1), 80–89.
Vaala, S., & Takenchi, L. (2012). *The parent co-reading survey: Co-reading with children on iPads, parent's perceptions and practices*. Resource document. Joan Ganz Cooney Center. http://www.joanganzcooneycenter.org/wp-content/uploads/2012/11/jgcc_ereader_parentsurvey_quickreport.pdf. Accessed June 24, 2017.
Van den Heuvel-Panhuizen, M., van den Boogard, S., & Doig, B. (2009). Picture books stimulate the learning of mathematics. *Australasian Journal of Early Childhood, 34*(3), 30–39.
Van den Heuvel-Panhuizen, M., Elia, I., & Robitzsch, A. (2016). Effects of reading picture books on kindergarteners' mathematics performance. *Educational Psychology, 36*(2), 323–346.
Van Kleeck, A., Gillam, R. B., Hamilton, L., & McGrath, C. (1997). The relationship between middle-class parents' book-sharing discussion and their preschoolers' abstract language development. *Journal of Speech, Language, and Hearing Research, 40*(6), 1261–1271.
Verhallen, M. J. A. J., Bus, A. G., & de Jong, M. T. (2006). The promise of multimedia stories for kindergarten children at risk. *Journal of Educational Psychology, 98*(2), 410–419.

Part V
Early Childhood Educators' Professional Issues and Education

Chapter 14
Early Childhood Educators' Issues and Perspectives in Mathematics Education

Audrey Cooke and Julia Bruns

Abstract This chapter presents an overview of the contributions in TSG 1: Early Childhood Mathematics Education at ICME-13 which focused on issues from the perspective of early childhood educators. A basic assumption of this chapter is that the opportunities for young children to develop mathematical understandings and skills are influenced by several conditions at a macro, meso and micro level. First, curricula provide a framework for early mathematics teaching and learning with varying expectations (by teachers) about what can occur in the pre-school environment—informal learning (such as through play), content to learn and activities to experience. Second, early childhood educators' mathematical knowledge, pedagogical knowledge, understandings, beliefs, and perceptions influence how they enact these expectations. These competencies can be developed and supported by professional learning. Third, educational programs, resources and activities used in the pre-school environment impact on the mathematical opportunities children engage in at the micro level.

Keywords Early childhood · Mathematics education · Competency
Curricula · Professional learning

14.1 Introduction

This chapter presents an overview and discussion of a range of issues raised by a wide variety of papers about (i) children's mathematics learning and thinking, and pedagogical and assessment approaches to improve learning, and (ii) the perspective

A. Cooke (✉)
School of Education Humanities, Curtin University, GPO Box U1987,
Perth 6845, Australia
e-mail: Audrey.Cooke@curtin.edu.au

J. Bruns
Universität Osnabrück, Mathematikdidaktik Albrechtstr. 28 a,
49076 Osnabrück, Germany
e-mail: julia.bruns@uni-osnabrueck.de

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_14

of the educators in early childhood mathematics education at a macro-, micro- and meso-level that influence early mathematical learning opportunities. Table 14.1 provides an overview of all the presentations and the recurring themes. Many presentations reflected papers focused on the influences of curricula, intervention programs, and professional learning initiatives on early mathematics education. Other papers focused on the knowledge and beliefs of in-service and pre-service early childhood teachers (see Table 14.1). How these themes are integrated within the presentations vary, but each considers the educators' role in developing children's mathematical understandings.

Table 14.1 Themes addressed in the presentations

	Presentations	Authors	Themes
Macro-level	Norwegian Kindergarten teacher's work with mathematics	Fosse, Lossius	Curriculum cultural considerations
	Swedish pre-school teachers' views of children's socialisation	Lembrér, Johansson	Curriculum views of the child as mathematician
	Comparison of three early childhood curricula from the perspective of mathematics education	Lao	Curriculum cultural considerations
Meso-level	Development and validation of a test battery assessing pre-school teachers' professional competence in the field of mathematics	Jenßen, Eilerts, Koinzer, Schmude, Blömeke	Mathematics-related knowledge General pedagogical knowledge
	Pre-school teachers' mathematics-related competencies	Bruns, Eichen, Blömeke	Mathematics-related knowledge Mathematical beliefs
	Pre-school teachers' responses to repeating pattern tasks	Tsamir, Tirosh, Levenson, Barkai, Tabach	Mathematics-related knowledge Mathematical self-efficacy
	Pre-service teacher relationships with mathematics-creative? anxious? competent?	Cooke	Mathematics-related knowledge Mathematical beliefs
	Reform of the Kindergarten teachers and child care workers in day-care center training curriculum in Japan	Goto	Professional learning Cultural considerations
	"SENSO-MATH" pre-school program facilitators contribute to mathematics education in the pre-school	Hassidov, Ilany	Professional learning
	South African educator's mathematics teaching journey: a case of 5–6 year old educator practices	Feza, Bambiso	Professional learning Cultural considerations

(continued)

Table 14.1 (continued)

	Presentations	Authors	Themes
	Relations of affective, cognitive and situation-specific facets of pre-school teachers' professional competence	Dunekacke, Eilerts, Jenßen	Mathematics-related knowledge Mathematics anxiety and perceptions
	Pro-KomMa: effectiveness of pre-school teacher education in the field of mathematics	Rasche, Eilerts, Jenßen, Eid, Schmude, Koinzer, Blömeke	Mathematics-related knowledge Mathematical beliefs and perceptions
Micro-level	From thought to reality—implementation of an in-school mathematics training in South Africa	Fritz-Stratmann, Ehlert, Herzog	Intervention programs

The chapter begins with a wider institutional view provided by curricula and educational guidelines (macro-level). It then focuses on educator issues (meso-level) and issues concerning children (micro-level). The discussion centres on how these presentations, together with recent research, can provide a better understanding of how to create conditions for high-quality mathematics learning environments. Implications for practice, further considerations and final remarks are provided at the conclusion of the chapter.

14.2 The Macro-level: The Influence of Curricula and Guidelines

Early childhood curricula and guidelines are intended to impact on the pre-school and kindergarten learning environments in order to enrich learning opportunities for children (Department of Employment, Education and Workplace Relations [DEEWR] 2009). This impact can be positive for the child, both directly, in terms of their learning, and indirectly via the educator (Brodin and Renblad 2015). However, it will depend on how the content of the curricula and guidelines are addressed, how they are interpreted by educators, and how the educator views them when framing their work with children (Brodin and Renblad 2015).

14.2.1 Positioning Children's Mathematical Learning Within Norwegian Kindergarten (Fosse and Lossius 2016)

In their research, Fosse and Lossius (2016) examine how the Norwegian Framework plan is incorporated in Norwegian kindergartens. The Norwegian

Ministry of Education and Research (NMER) introduced the Framework Plan for the Content and Tasks of Kindergartens in 2006 and revised the Framework Plan in 2011 (NMER 2011).

Fosse and Lossius (2016) investigated 160 Norwegian kindergarten teachers using self-reports on what mathematical activities they were doing in their kindergartens and the reasons behind why they were doing them. Specifically, they were interested in determining how the learning area outlined by NMER (2011)—Number, space and shape—was being implemented. They were also interested in whether the schoolification of kindergarten, as outlined by Lembrér and Meaney (2014) was occurring, particularly the ideas of 'being and becoming' and how these align with schoolification.

When situating their work, Fosse and Lossius (2016) referred to the work of Benz (2012a), who conducted a study of German in-service and pre-service kindergarten teachers. The research investigated the teachers' expectations regarding mathematical learning goals for kindergarten, their emotions regarding mathematics, attitudes towards mathematics itself and the learning and teaching of mathematics, and actual practices of the kindergarten environment. Her findings indicated that kindergarten teachers engaged more in counting and sets activities. Although a broad range of mathematical areas were provided (with some more appropriate for older children in formal schooling), the domains that were most frequently referred to as learning goals were the topics of counting and sets.

Fosse and Lossius' (2016) research fits only partially to the German results. They found that a high proportion (94%) of respondents considered counting to be the most important area, followed by shapes (88%), locating (63%), measuring (65%), and patterns (60%). These results were considered to conflict with the emphasis on everyday activities within the Norwegian Framework Plan, as this would imply more locating and measuring activities. The findings about teachers' reasons for completing mathematical activities in the kindergarten reflected their views that children should be given opportunities to be active agents in their mathematical learning. They also believed that preparing the child for the mathematics they will encounter once they begin their formal schooling was important. Fosse and Lossius (2016) linked these ideas to those of *being* and *becoming*, respectively, as proposed earlier by Lembrér and Meaney (2014). Fosse and Lossius (2016) also found that although the Norwegian Framework Plan emphasizes play, this emphasis was not evident in the teachers' responses. Based on this result they inferred that kindergarten teachers interpret and implement the Framework Plan differently and they need more support to strengthen their professional reflective capacities.

14.2.2 Mathematics Learning as Being and Becoming (Lembrér and Johansson 2016)

Lembrér and Johansson's (2016) paper presented a case study investigating pre-school teachers' views of the contrast between children's present interests and the knowledge and understandings needed for their future. These competing needs were viewed as *being*, which focused on the child's interests, and *becoming*, which focused on what the child needs to experience to fulfil his or her potential. As Lembrér and Johansson noted, policies are being enacted that impact on mathematics education in learning environments prior to formal schooling.

In 2010, the pre-school curriculum in Sweden was revised (Skolverket 2011). In light of the ideas of *being* and *becoming*, the curriculum refers to the creation of opportunities for children to develop the learning disposition, skills, and knowledge children need now and as a foundation for their future learning. These opportunities were described as needing to arise from interactions with the environment, educators, other children and their families.

Lembrér and Johansson (2016) discuss the tension in the pre-school environment that exists between the child *being* and *becoming* a mathematician. The tension is highlighted by Lembrér and Meaney (2014), who stated that the different sections within the Swedish pre-school curriculum conflict within guidelines and goals, potentially with "the increase in the 'schoolification' of the formal pre-school curriculum" (p. 83). This might encourage the educator to focus on creating opportunities for children to develop understanding or skills that they do not currently possess, described by Lembrér and Meaney (2014) as *becoming*, and less opportunities for children to demonstrate their existing understandings and skills, described as *being*. However, Lembrér and Meaney found the Swedish pre-school curriculum does present opportunities where both *being* and *becoming* can co-exist, although they argue that it would be necessary to see how educators enacted the curriculum to see if these opportunities eventuated.

Lembrér and Johansson (2016) interviewed two pre-school teachers to determine their views on children *being* mathematicians, where the children were seen as having mathematical skills and understandings, or *becoming* mathematicians, and where the children were seen as needing to develop the skills and understandings. Themes identified from the interviews reflected both *being* and *becoming*, although the schoolification of pre-primary curriculum seemed to favour the process of *becoming*. Educators' freedom to create opportunities based on what the children knew would enable children to use their existing mathematical understandings and these could provide opportunities for being mathematicians. But these experiences were under pressure in a shift to *becoming* mathematicians. The impact of regulatory processes provided some explanation of the constraints of the pre-school environment such as documentation and evaluation, and time limitations that reduced opportunities for the educator to engage in observation and reflection. Teachers considered the potential of the institutional environment to provide instances where both *being* and *becoming* mathematicians are reflected in opportunities for children to demonstrate their existing knowledge but also to develop new knowledge. The notion of *becoming* mathematicians was more likely to be emphasized.

14.2.3 The Context Surrounding Early Childhood Curricula and Mathematics Education (Lao 2016)

Lao (2016) emphasized the importance of the curricula by connecting mathematical learning opportunities in early childhood to later achievement. She supported this by referring to the findings of Watts et al. (2014), specifically that children's mathematics achievement at 4.5 years of age can be a predictor of their mathematical achievement at 15 years of age. Further, they indicate that children making substantial gains in mathematics during their first year in formal school would continue to make gains in mathematics as they progressed through their school years. Considering the long-term impact of early childhood mathematical achievement found in this study provided impetus for Lao to investigate early childhood mathematics curricula, especially in locations that perform well in international tests.

Lao (2016) planned a comparison of the early childhood curricula for mathematics from three locations, Shanghai, Hong Kong, and Chinese Taipei. Each of these locations were considered as part of East Asia, which Leung et al. (2006) described as "Chinese/Confucian" (p. 4). This was considered a cultural and social factor that may potentially impact on educational traditions that, in turn, contribute towards achievement in the Trends in International Mathematics and Science Study [TIMSS] (TIMSS and PIRLS International Study Centre, n.d.) and the Programme for International Student Achievement [PISA] (Organisation for Economic Cooperation and Development [OECD], n.d.).

Lao (2016) proposed three aspects of the curricula be compared—how the curriculum is organised, considering how aims, objectives and mathematical content are connected; the intentions of the curriculum in relation to early mathematics; and the areas of focus for early mathematics. Lao (2016) stated that the last aspect should consider both content domains (subject matter such as number, geometric shape, measurement, and data) and cognitive domains (thinking processes, including knowing, applying, and reasoning).

14.3 The Meso-level: Early Childhood Educators' Mathematics-Related Competencies

Early childhood educators' mathematical content and pedagogical knowledge, and their beliefs about mathematics learning impact on the environment/contexts for learning and the experiences they promote for young children (e.g. Klibanoff et al. 2006; Lehrl et al. 2016). As Björklund (2015) proposed, the pre-school teacher is one of the most important factors that will impact on the opportunities for children to engage with mathematics in the pre-school environment.

14.3.1 Measuring Mathematics-Related Competence of Pre-service Pre-school Teachers—KomMa Project (Dunekacke et al. 2016; Jenßen et al. 2016; Rasche et al. 2016)

14.3.1.1 Background

The KomMa project, conducted in Germany, investigated the mathematical professional competencies of kindergarten teachers (http://www.kompetenzen-im-hochschulsektor.de/168_ENG_HTML.php). The project was based on Shulman's (1986) seminal work and his three types of knowledge—mathematical content knowledge (MCK), mathematical pedagogical content knowledge (MCPK), and generalised pedagogical knowledge (GPK) (Dunekacke et al. 2015). Based on Shulman's (1986) notion of teacher knowledge, the project's focus was on generating a model that can describe mathematics-related competencies, particularly in terms of the less formal kindergarten environment required for both teaching and learning mathematics.

14.3.1.2 Validating the KomMa Test Instrument

The paper by Jenßen et al. (2016) outlined how the KomMa test instrument was designed to measure the MCK, MPCK, and GPK competence of pre-service pre-school teachers. The KomMa test battery used in their study of 1851 pre-service pre-school teachers contained three test instruments. The MCK test consists of 24 items and addresses the mathematical domains and mathematical processes. The domains included four areas—number and operations; quantity and relations; geometry; and data, combinatorics, and chance. The mathematical processes comprised problem solving, modelling, communicating, representing, reasoning, and patterns and structure. The MPCK test with 28 items addresses knowledge on the development of children's mathematical understandings in the informal and formal setting in pre-school as well as knowledge about the development of mathematical literacy and the diagnosis of early mathematical skills. The GPK test with 18 items addresses the underlying understandings of the psychology of learning and pedagogical approaches.

Two research questions were investigated:

(i) How appropriate is a three-dimensional model of knowledge (MCK, MPCK and GPK) than a one-dimensional model (professional knowledge) to describe pre-school teacher knowledge of mathematics? and,
(ii) Do the three dimensions of knowledge yield correlations of varying strengths?

Two samples were used, one for the main study testing these hypotheses and the other for the validation study of the KomMa tests. The first sample of 1851 pre-service students included 881 students beginning their studies and a cohort of

970 students at the end of their studies in either vocational (secondary level) or university (tertiary level) pathways for pre-school teacher education. The second sample contained 354 students from Berlin and Lower Saxony who were completing pre-school pre-service teacher education courses at vocational schools.

The factorial structure indicated that the hypothesis—that the proposed three-dimensional model was better than a one-dimensional model—fit, with latent correlations strongest between MPCK and GPK, followed by MPCK and MCK, and then by MCK and GPK. Convergent and discriminant validity was explored by correlating the MPCK and MCK test scores with relevant constructs regarding beliefs, affect and motivation for the validation study. Significant correlations were found between the test score for MPCK and mathematical beliefs and between the test score for MCK and affective-motivational constructs.

14.3.1.3 Relationships Between Knowledge, Mathematics Anxiety and Perceptions of Mathematics Situations

Dunekacke et al. (2016) investigated the relationships between MPCK, perceptions of mathematics situations in kindergarten, and pre-service kindergarten teacher mathematics anxiety using confirmatory factor analysis. They found that there were strong and significant correlations between MPCK and perceptions of mathematics situations ($r = 0.65$, $p < 0.01$), but no significant correlations between mathematics anxiety and either MPCK or perceptions of mathematics situations. The authors interpreted their results to indicate a lack of awareness or recognition of mathematics in learning situations (which may have involved mathematics), which may explain their lack of mathematics anxiety (as mathematics was not 'present'). They refer to the findings of Bates et al. (2013) that indicated early childhood pre-service teachers were anxious about teaching mathematics or fearful of mathematics, particularly in terms of their mathematical content knowledge, to support their interpretation.

14.3.1.4 Relationship Between Dispositional Competence Facets, Situation-Specific Skills, Performance and Children's Mathematical Development

The poster by Rasche et al. (2016) proposed the continuation of the use of the KomMa test instruments in a subsequent Pro-KomMa project (Jenßen et al. 2016). Specifically, after the validation of the test instrument, opportunities exist to compare results from the KomMa test instruments to the in-service pre-school teacher responses to situations set up to examine their situation specific skills (perceptions and interpretation) and the activities they created to address their children's needs in a learning environment (activities). This could serve as an evaluation of the effectiveness of the education provided to in-service pre-school teachers and a potential indicator of the impact of the pre-school educators on the children's achievements.

14.3.2 The Structure of Mathematics-Related Competencies: Focusing on In-Service Pre-school Teachers (Bruns et al. 2016)

Bruns et al. (2016) focused on the competence structure of in-service pre-school teachers. The paper builds on earlier research about the competence structure found by the KomMa project and another project reported by Anders and Rossbach (2015). Both projects come to different results concerning the relationship of MCK, MPCK and beliefs, using different samples (KomMa: pre-service versus Anders and Rossbach: in-service teachers) and different instruments. In order to clarify if the different results regarding this relationship are related to the samples or the instruments, Bruns et al. used the instruments of the KomMa project with a sample of in-service teachers.

To assess the beliefs of the pre-school teachers concerning mathematics in general Bruns et al. used a questionnaire developed in the KomMa project. The questionnaire was based on the earlier research about beliefs about mathematics generated from both Grigutsch et al. (1998) and Benz (2012b). Grigutsch et al. (1998) referred to four aspects—schema, formalisation, process, and application—with formalisation and schema connected to the static view of mathematics, and process connected to the dynamic view of mathematics. Grigutsch et al. (1998) also linked these two views of mathematics with algorithms and computations (the static view) and understanding and problem solving (the dynamic view). Benz (2012b) firstly used the questionnaire developed by Grigutsch et al. (1998) for pre-school teachers; she revised it reducing the number of items, and combining the formalisation and schema aspects.

The KomMa questionnaire contained 17 items grouped within three descriptions of mathematics beliefs—static (combining formalisation and schema aspects like Benz (2012b), 7 items), process (4 items), and application (6 items) (Dunekacke et al. 2016). Like Benz (2012b), the KomMa questionnaire used Likert-style questions, but had a 6-point rating scale instead of a 4-point rating scale as used by Benz (2012b).

To study the competence structure of in-service pre-school teachers the KomMa test instrument (see above) and questionnaire were administered to 95 pre-school teachers. The data was analysed using confirmatory factor analysis. Significant positive correlations were identified between MPCK and MCK and between each of the pairings between the three mathematical beliefs (static, process, and application). The only positive correlations found between mathematical knowledge and beliefs were for the application-orientation beliefs, which were positively correlated to both MPCK and MCK. Overall, these findings were more in line with the findings of the KomMa project than the findings of Anders and colleagues (2015). This finding, however, differed marginally from that of Dunekacke et al. (2016), who found significant correlations between MPCK and MCK and between application-orientation beliefs and both MPCK and MCK, but also between process-orientation beliefs and both MPCK and MCK. This difference may have been a reflection of the difference in participants (in-service compared to pre-service). Pre-service pre-school teachers relied more on their own experiences as students due to limited, or no experience as teachers.

14.3.3 Application of Teacher Knowledge: Pre-school Teachers' Responses to Repeating Pattern Tasks (Tsamir et al. 2016)

Tsamir et al. (2016) presented a paper investigating pre-school teachers' knowledge of patterns, specifically, the capacity to identify repeating patterns, and mistakes and continuations of repeating patterns. Chapter 15 following (Tsamir et al. 2018), provides a comprehensive overview of the project. The research used the Cognitive Affective Mathematics Teachers Education (CAMTE) framework (Tsamir et al. 2012). As with the papers in the section above on the KomMa project, the framework draws on the work of Shulman (1986) as well as refinements of Shulman by Ball et al. (as cited by Tsamir et al. 2012, p. 2). Self-efficacy is also part of the CAMTE framework, drawing from social cognitive theory from Bandura (as cited by Tsamir et al. 2012, p. 2) and the more focused mathematics self-efficacy of Hackett and Betz (as cited by Tsamir et al. 2012, p. 2).

To demonstrate the importance of pre-school teachers' knowledge of patterns, the authors refer to Papic et al. (2011) and Rittle-Johnson et al. (2013), who had conducted research on children's capacity to replicate patterns. Both research projects worked with children aged 4 and 5 years old and attending pre-school. Papic et al. (2011) also investigated the impact of an intervention designed to improve pattern understandings, suggesting that professional learning provided to educators contributed to gains in children's mathematical understandings.

Tsamir et al. (2016) found that the 51 pre-school teachers participating in the research could identify repeating patterns shown in drawings and indicated errors that stopped the drawings from representing repeating patterns, which matched their high levels of self-efficacy with these aspects. Pre-school teacher difficulties were evident when identifying appropriate continuations of repeating patterns, primarily in patterns and continuations that ended mid-cycle (incomplete unit of repeat), contrasting with the high levels of self-efficacy. The authors cautioned that teachers with a high self-efficacy might lead these teachers to believe they do not need professional learning.

14.3.4 Pre-service Teachers' Views of Mathematics: Mathematics as Creative, Mathematical Competency, and Mathematics Anxiety (Cooke 2016)

Cooke (2016) presented a study that investigated pre-service teachers' perceptions of mathematics and whether their perceptions of mathematics were related to their mathematical competency or self-reported mathematics anxiety. Cooke (2016) uses Ernest's (1989) three views of mathematics:

(i) The instrumentalist view of mathematics as rules and facts that were unrelated and all-encompassing for the mathematics completed;
(ii) The Platonist view of mathematics as static rules and knowledge that could be discovered and connected but not created, and
(iii) The problem-solving view of mathematics as dynamic, revisable and impacted by culture and thinking.

Ernest (1989) proposed that the philosophical view held by the educator would impact on how they viewed and addressed teaching and learning mathematics. He linked the instrumentalist view of mathematics with the educator adopting an instructor role for teaching. The Platonist view of mathematics was linked to the teacher taking on a role of explaining. The problem-solving view of mathematics was linked to the teacher taking on the role of facilitator.

The aim of the study presented by Cooke (2016) was to ascertain how pre-service teachers conceptualized mathematics, specifically, their agreement with the idea that mathematics is creative. The instrument developed by Cooke et al. (2011) included three sections that addressed thinking about using mathematics in three different situations—working on mathematics in a group situation, working on mathematics in a test situation, and teaching mathematics. Each of the three sections used the same 22 items, with responses on 4-point Likert-style scales. Participants were asked to indicate the level of agreement with each statement when thinking of using mathematics in the specified situation. Conceptualisation of mathematics was measured using an instrument with 20 statements (Cooke 2015), although responses to only one statement, "Mathematics is creative", were used in this paper. The instrument used the same 4-point Likert-style scale. Mathematics competency was measured using a Mathematics Competency Test (MCT) that was available from a commercial online platform and contained a total of 50 multiple choice and short answer questions that were to be completed in 60 min (Cooke and Sparrow 2012).

The survey was conducted with 698 Australian pre-service teachers in their first year of study, enrolled in either an early childhood bachelor degree (who would qualify to work with children from birth to 8 years of age), or a primary bachelor degree (who would qualify to work with work with children from grade 1 to grade 3 in primary school as well as later primary years).

Overall, 71.4% of pre-service teachers (n = 671) responded affirmatively to the statement "Mathematics is creative". The results indicated that pre-service teachers who reported low anxiety when thinking about mathematics in all three situations (working on mathematics in a group situation, working on mathematics in a test situation, and teaching mathematics) were more likely to agree that mathematics is creative. It was inferred from these data that being comfortable with mathematics (that is, not anxious) may be linked to a willingness to use mathematics in the three specified situations (Brady and Bowd 2005; Isiksal et al. 2009).

14.4 The Meso-level: Professional Learning of Early Childhood Teachers

Professional learning programs are critical to developing early childhood teachers' mathematical understandings, their awareness of the importance of mathematics in early childhood settings, and their confidence to teach mathematics. MacDonald et al. (2016) highlight the importance of research into professional learning for early childhood educators who are engaged in creating mathematical understanding with young children.

14.4.1 Educators' Mathematical Understandings in South Africa (Feza and Bambiso 2016)

Feza and Bambiso (2016) presented a case study from South Africa outlining an intervention to help educators develop mathematical understandings of children aged 5–6 years. She referred to the analysis of the Southern and East African Consortium for Monitoring Educational Quality (SACMEQ) by Venkat and Spaull (2015) to support the need for intervening in educators' mathematical content knowledge. Feza and Bambiso found that four types of lessons were evident in the videos provided by the participants:

- Pre-readiness, where educators could not engage the children with the activity or the mathematical ideas;
- Knowledge of number (mathematical content) and development, which enabled educators to provide structure in the experience, use goal-oriented practice, and interactions with the learners that progressed through mathematical understandings;
- Knowledge of how young learners learn, including selection of appropriate tools to use in the experience, and
- Purposeful practice not developed, in which children's meaningful mathematical interactions with each other and educator assessment of learning were not evident.

The professional learning was deemed to have succeeded in providing developmental opportunities for the educators' mathematical content knowledge, specifically number sense. It was promising to note that identified gaps in content knowledge could be addressed by professional learning, provided it was strongly connected to the classroom. However, Feza and Bambiso (2016) proposed that greater consideration of reflecting and planning was needed to assist the educators in developing their children's understanding of number knowledge and counting.

14.4.2 Empowering Pre-school Teachers to Teach Mathematics (Hassidov and Ilany 2016)

Hassidov and Ilany (2016) argued, in reference to Philippou and Christou (1998), that the teacher's attitude is a central factor influencing the mathematical development of children. As a result, Hassidov and Ilany proposed that teachers should be supported through teaching and learning programs and professional learning. They developed a coaching program to foster pre-school teacher attitudes to mathematics and learning and teaching ("Senso-Math Pre-school"). Following from Lloyd and Modlin's (2012) recommendations, pre-school teachers received training on integrating facilitators into pre-school environments, support with mathematical content required for early childhood education, and working in one-to-one mentoring relationships. They were also trained in implementing the Senso-Math Pre-school program, and an accompanying resources kit. Hassidov and Ilany (2016) trained 500 facilitators in the pilot program to teach the Senso-Math Pre-school program in schools. The focus of the research was on whether the program would increase the facilitator's realisation of the need for teaching mathematics in pre-school, their self-confidence for teaching mathematics, and their awareness of developing their careers, as well as whether the program provided appropriate tools and support.

Both quantitative and qualitative research methods via a 22-item questionnaire and interviews were used by Hassidov and Ilany (2016) to investigate the impact of a professional learning program for becoming mathematics specialist teachers so that they could assume the role of facilitators within pre-school classrooms. Results from the 49 participants randomly selected from the 500 facilitators in the pilot program indicated that the course was positively viewed by all participants, although the lowest score related to changing careers to mathematics teaching. The Senso-Math program and resources kit were perceived favourably and the program was regarded as effective in developing mathematical understandings, teaching strategies for mathematics in pre-school, and self-confidence in teaching mathematics. The presence of the facilitators in the pre-primary classrooms was reported to have made the mathematical experiences interesting and challenging. These findings highlighted the importance of professional learning for teaching mathematics in pre-school.

14.4.3 Addressing Mathematical Knowledge for Early Childhood Educators in Japan (Goto 2016)

Goto (2016) analyzed the differences in the mathematical education of early childhood educators in Japan. Japan has two historical forms of early childhood services, *hoikuen* and *yochien*. There are distinct differences in how these forms are viewed by the community, with *hoikuen* seen more as long daycare and *yochien* seen as more educative (Hayashi and Tobin 2013). In 2006, the Ministry of Education, Culture, Sports, Science and Technology and the Ministry of Health, Labour and Welfare (as cited in Hayashi and Tobin 2013, p. 36) created a service that combined aspects of both *hoikuen* and *yochien*, *nintei kodomo-en*, with the educational focus of *yochien* and the early age range of *hoikuen* (Chesky 2011; Hayashi and Tobin 2013). This is reflected in the standards for content and method, where the *nintei kodomo-en* had to address those for both *hoikuen*, the Guidelines for Nursery Care at Day Nurseries, and *yochien*, the National Curriculum Standard for Kindergartens (Abumiya n.d.). There are, however, differences in the training and licencing of the educators at these two services (Hedge et al. 2014). Educators at *hoikuen* can either complete courses at college or university that address early childhood care and education or can work at a *hoikuen* for two years then pass a nationally-set exam; educators at *yochien* must complete university or college courses that provide a specialisation (which are set by the Ministry of Education, Culture, Sports, Science, and Technology 2008) and complete practicum hours to obtain a licence.

Goto's (2016) poster focused also on the issues of qualifications; that the educators working with children in early childhood should be qualified, regardless of whether this is at *hoikuen*, *yochien* or *ninte kodomo-en*. It was considered central for educators to have an understanding of the mathematics children encounter in formal schooling (that is, from grade 1) and to have the mathematical understandings themselves. Goto reasoned that these are both needed to ensure children experience activities that will provide a foundation for the mathematics they will encounter in formal schooling. However, the curriculum content in the National Curriculum Standard for Kindergarten was described by Goto as lacking mathematics, with the example provided from the curriculum area of environment including aspects of mathematical knowledge involved with numbers and shapes. As a consequence, it is proposed that early childhood educators need to be educated to preview the curriculum for formal schooling; to identify mathematical understandings and skills and then use these to determine what mathematical experiences should be provided in early childhood.

14.5 The Micro-level: Interventions Programs in Mathematics for Pre-school Children

In their review of the effectiveness of programs designed as interventions for mathematics in early childhood, Wang et al. (2016) found that the intervention programs did impact on children's learning. Interventions programs in mathematics in early childhood have been found to support young children who have not had opportunities to develop their mathematical understandings and skills in the early years (Clements et al. 2013, p. 2).

14.5.1 From Thought to Reality: Implementation of an In-school Mathematics Training in South Africa (Fritz-Stratmann et al. 2016)

The presentation focused on an intervention designed to be used with young children. Fritz-Stratmann et al. (2016) describe the enactment of the intervention to assist children in early childhood centres to develop early arithmetic concepts. The rationale for this program was supported by reference to Aunola et al. (2004), who found that the growth of mathematical performance was cumulative in grades 1 and 2. Krajewski and Schneider (2009) found that nonverbal intelligence influenced mathematical performance in pre-primary, and pre-primary mathematical performance related to early quantity-number competencies could predict grade 4 mathematical performance. The findings of both studies lead to further research to support numeracy and general metacognitive skills with young children.

Fritz-Stratmann et al. (2016) examined the use of the mathematical program, Calculia, in South Africa. The program used a test assessing early number development and calculation that was developed in Germany and translated into four South African languages (see Fritz et al. 2014). The program was based on the arithmetic competence level model described by Fritz et al. (2013). The model contained six levels of understandings that children aged 4–8 years progressed through in their development of arithmetic. The aspects of the Calculia grade-R program comprised three modules. The first module contained pre-numeric concepts and was provided to ensure all children had the opportunity to develop appropriate arithmetic concepts. The second and third modules addressed number word chain, ordinal number line and cardinality and decomposability (Fritz et al. 2013). Findings indicated that special consideration was needed to provide cultural contexts appropriate for the children and sufficient training for the teachers was essential for the effective use of the program. The implementation of the program needed to be iterative to feed back into the research to improve the program and subsequent implementations.

14.6 Discussion

14.6.1 The Macro-level: Curriculum

Three presentations of the TSG1 focused on different aspects at the macro-level which influenced the opportunities for mathematical learning in pre-school. Cultural context can impact on early childhood curricula, as outlined by Lao (2016) in the comparison of curricula within different locations in East Asia. Lao proposed that this, in turn, could then impact on the potential mathematical achievement of the children through the organisation, aims, objectives and mathematical content contained in the curricula. The presentation by Fosse and Lossius (2016), however, showed that this might not be a direct effect but, rather, influenced by the educators' interpretation of the curricula. They considered opportunities for children to be active agents in their mathematical learning as well as preparating them for the mathematics they will encounter once they attend formal schooling. These aspects were linked to Lembrér and Meaney's (2014) ideas of *being* and *becoming*.

Viewing children entering early childhood learning environments as not capable of mathematics risks damaging their later educational opportunities (Baroody et al. 2006). If opportunities are not created for children to experience mathematics, their development of mathematical skills and understandings may be limited and can impact on their later development (Watts et al. 2014). Opportunities for play need careful consideration in the pre-school environment, particularly as the educator controls the space created, access to resources, the structure of the environment (Martinsen 2015) and establishes a relationship between the resources used and the pedagogical purpose (Björklund 2014). The mathematical focus of activities could be limited (Fosse and Lossius 2016), and this can be considered detrimental to learning (Lewis et al. 2015). This issue was raised in the presentation from Lembrér and Johansson (2016), particularly as regulatory processes might constrain the pre-school environment. These constraints could also lead to a greater focus on the child developing formal mathematical understandings (*becoming*), rather than on who they are (*being*). However, they noted that it would be the educator who determines the balance between the children *being* and *becoming*.

14.6.2 The Meso-level: Educators' Mathematical Competence

Early childhood educators' competence was the central topic of the presentations relating to the meso-level. All authors argued that a fundamental understanding of mathematics is the basis for high-quality early mathematics education. However, different authors used different conceptualizations to measure competence.

Several papers used the Shulman framework (Bruns et al. 2016; Dunekacke et al. 2016; Jenßen et al. 2016; Tsamir et al. 2016) and found that pre-school educators' competence incorporated aspects of mathematical content knowledge, mathematical pedagogical content knowledge, general pedagogical content knowledge and affective-motivational aspects as well as the perception of mathematics situations. Although a different theoretical framework was utilized, Cooke (2015) identified and measured the same facets of educator's math-related competence, indicating that these facets are quite stable from different research approaches.

Bruns et al. (2016) point out that different projects produced different findings about the relationship between these competency facets. This was seen in the TSG1 presentations: For example Dunekacke et al. (2016) found no significant correlations between mathematics anxiety and either MPCK or perceptions of mathematics situations, whereas Tsamir et al. (2016) found that self-efficacy for identifying repeating patterns and errors in repeating patterns was well-matched to the pre-school teacher understandings. However, this was not the case for continuations of repeating patterns, with high levels of self-efficacy not supported by demonstrated knowledge. Furthermore, earlier work by Jenßen et al. (2015) found a negative relationship between mathematics anxiety and mathematical content knowledge as measured in the mathematical domains of number and operations, quantity and relations, geometry, and data, combinatorics, and chance. Cooke (2016) did not find significant relationships between math-anxiety and mathematical content knowledge. These differences indicate that further research should address the dispositional facets more closely by using different measures and more representative samples; and examine their relationship to early childhood educators' performance, and the quality of the experiences they create to engage children in mathematics. Rasche et al. (2016) already outlined one future study in this direction and describe how the Pro-KomMa project will compare results from the KomMa test instruments to the in-service teacher responses to situations set up to examine their situation specific skills (perception and interpretation) and the activities they created to address their children's needs in a learning environment (activities).

Another line of research on the meso-level focused on the development of educators' competence in professional learning courses. Feza and Bambiso (2016) and Hassidov and Ilany (2016) both reported successful approaches that provided opportunities for educators to address these competencies. The changes Hassidov and Ilany identified in the attitudes of the facilitators in regard to the importance of mathematics and confidence in teaching mathematics both have the capacity to impact on the children, the educators' work and the mathematical experiences the educators create (MacDonald 2015). The attitudinal changes resulting from professional learning for educators can also impact positively on children's mathematical learning and, potentially, the parents within that early childhood setting (Perry and MacDonald 2015).

14.6.3 The Micro-level: Mathematical Interventions for Children's Learning

Fritz-Stratmann et al. (2016) demonstrated that developmental-model based programs could be used in early childhood settings but noted the high requirement for the training to be effective. Focusing on one content area (in this case, arithmetic) was one of the factors that Wang et al. (2016) found increased the likely effectiveness of an intervention program. Clements et al. (2013) reiterated the importance of the use of research-based programs, stating that developmentally sequenced activities can enable teachers to "become aware of, assess, and remediate" (p. 10). Targeting children early may improve both their first years of formal schooling and their later mathematical achievement (Watts et al. 2014), and development-oriented interventions can help children develop mathematical understandings (Fritz et al. 2013). However, there needs to be careful monitoring to ensure the impact of the intervention does not fade away (Sarama and Clements 2015).

14.6.4 The Educator—The Common Focus

Although opportunities for children to develop mathematical understandings and skills are influenced by several conditions discussed in this chapter, the early childhood educator is the common feature across the three levels. At the macro-level, direction is provided by curricula and its context (Lao 2016), together with the educator's enactment (Fosse and Lossius 2016) and interpretation (Lembrér and Johansson 2016) of the curricula. At the meso-level, it is the educator's mathematical knowledge (Bruns et al. 2016), pedagogical knowledge (Jenßen et al. 2016), beliefs (Cooke 2016), and perceptions, and combinations of these (Dunekacke et al. 2016). In addition, the opportunities provided through professional learning to improve educator content knowledge and self-efficacy (Feza and Bambiso 2016), their attitudes towards mathematics (Hassidov and Ilany 2016), and their understanding of the mathematics, will enable children to learn in formal schooling (Goto 2016). At the micro-level, it is the capacity of educators to effectively select and implement intervention programs that address children's mathematical needs (Fritz-Stratmann et al. 2016).

14.7 Conclusion

In the early childhood setting, the primary influence is the early childhood educator who largely determines what occurs in their classroom. However, there are several different factors that contribute to the decisions of the educator at a macro-, meso-, and micro-level. At a macro-level, curricula inform educators about their role of enabling children to develop mathematical understandings. These influences can be considered in terms of comparisons between different curricula internationally, or as tensions between concurrent curricula and regulation. Thereby regulation at the macro-level where the curriculum is implemented through the education system, plays a central role in early mathematics education. At the meso-level, many studies have investigated the impact of the educators' skills, understandings and beliefs about older children's development of mathematics (e.g., Bates et al. 2013; Hill and Ball 2009). Nonetheless there is insufficient research addressing what the educator brings to the early childhood learning environment (Dunekacke et al. 2015) and how the different competence facets are related. Professional learning programs that support educators' teaching of mathematics in pre-schools can bring research into the classroom, and can influence the scope and quality of early mathematics learning (Clements et al. 2013). At the micro level, intervention programs can provide an innovative and systematic approach to developing children's mathematical understandings.

The existence of the role of the early childhood educator within these three levels makes the role a complex and challenging one that is constantly evolving and stimulating. It seems convincing that efforts to support early mathematical education cannot be successful by just changing curricula or implementing intervention programs; but most importantly we must have to consider the pivotal role of early childhood educators in all three levels. The research presentations and discussions of the TSG1 have provided a broad range of studies that are providing new insights into different approaches, but further longitudinal and comparative research is necessary to support this idea.

References

Abumiya, M. I. (n.d.). *Preschool education and care in Japan*. National Institute for Educational Policy Research. https://www.nier.go.jp/English/educationjapan/pdf/201109ECEC.pdf. Accessed January 9, 2017.

Anders, Y., & Rossbach, H. G. (2015). Pre-school teachers' sensitivity to mathematics in children's play: The influence of math-related school experiences, emotional attitudes, and pedagogical beliefs. *Journal of Research in Childhood Education, 29*(3), 305–322.

Aunola, K., Leskinen, E., Lerkkanen, M.-J., & Nurmi, J.-E. (2004). Developmental dynamics of math performance from preschool to grade 2. *Journal of Educational Psychology, 96*(4), 699–713.

Baroody, A. J., Lai, M. L., & Mix, K. S. (2006). The development of young children's early number and operation sense and its implications for early childhood education. In S. Bernard & O. N. Saracho (Eds.), *Handbook of research on the education of young children* (2nd ed., pp. 187–221). New York, NY: Routledge.

Bates, A. B., Latham, N. I., & Kim, J. A. (2013). Do I have to teach math? Early childhood pre-service teachers' fears of teaching mathematics. *Issues in the undergraduate mathematics preparation of school teachers*, *5*. http://files.eric.ed.gov/fulltext/EJ1061105.pdf. Accessed February 10, 2017.

Benz, C. (2012a). Maths is not dangerous—Attitudes of people working in German kindergarten about mathematics in kindergarten. *European Early Childhood Education Research Journal, 20*(2), 249–261.
Benz, C. (2012b). Attitudes of kindergarten educators about math. *Journal für Mathematik-Didaktik, 33*(2), 203–232.
Björklund, C. (2014). Less is more-mathematical manipulatives in early childhood education. *Early Child Development and Care, 184*(3), 469–485.
Björklund, C. (2015). Pre-primary school teachers' approaches to mathematics education in Finland. *Journal of Early Childhood Education Research, 4*(2), 69–92.
Brady, P., & Bowd, A. (2005). Mathematics anxiety, prior experience and confidence to teach mathematics among pre-service education students. *Teachers and Teaching, 11*(1), 37–46.
Brodin, J., & Renblad, K. (2015). Early childhood educators' perspectives of the Swedish National Curriculum for Preschool and Quality Work. *Early Childhood Education Journal, 43*(5), 347–355.
Bruns, J., Eichen, L., & Blömeke, S. (2016, July). *Preschool teachers' mathematics-related competencies*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.
Chesky, A. K. (2011). Hoikuen or Yochien: Past, present, and future of Japanese early childhood education. *Childhood Education, 87*(4), 239–243.
Clements, D. H., Baroody, A. J., & Sarama, J. (2013). *Background research on early mathematics*. National Governor's Association, Center Project on Early Mathematics. https://www.nga.org/files/live/sites/NGA/files/pdf/2013/1311SEME-Background.pdf. Accessed December 31, 2016.
Cooke, A. (2015). Pre-service teacher conceptualisation of mathematics. *The International Commission for the Study and Improvement of Mathematics Teaching Conference (CIEAEM67) Teaching and Learning Mathematics: Resources and Obstacles*, Aosta, Italy. Published in Quaderni Di Ricerca in Didattica/Mathematics (QRDM), Supplementary 2, http://math.unipa.it/~grim/quaderno25_suppl_2.htm. Accessed January 12, 2017.
Cooke, A. (2016, July). *Pre-service teacher relationships with mathematics—Creative? Anxious? Competent?* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.
Cooke, A., Cavanagh, R., Hurst, C., & Sparrow, L. (2011, November–December). *Situational effects of mathematics anxiety in pre-service teacher education*. Paper presented at the Australian Association for Research in Education (AARE) Annual Conference, Hobart, Australia. http://www.aare.edu.au/data/publications/2011/aarefinal00501.pdf. Accessed January 7, 2016.
Cooke, A. & Sparrow, L. (2012, July). *Anxiety, awareness and action: Mathematical knowledge for teaching*. Paper presented at the 12th International Congress on Mathematical Education, Seoul, South Korea.
Department of Education, Employment and Workplace Relations [DEEWR]. (2009). *Belonging, being, becoming*. Commonwealth of Australia. https://docs.education.gov.au/system/files/doc/other/belonging_being_and_becoming_the_early_years_learning_framework_for_australia.pdf. Accessed February 3, 2015.
Dunekacke, S., Eilerts, K., & Jenßen, L. (2016, July). *Relations of affective, cognitive and situations-specific facets of preschool teachers' professional competence*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.
Dunekacke, S., Jenßen, L., & Blömeke, S. (2015). Effects of mathematics content knowledge on pre-school teachers' performance: A video-based assessment of perception and planning abilities in informal learning situations. *International Journal of Science and Mathematics Education, 13*(2), 267–286.
Dunekacke, S., Jenßen, L., Eilerts, K., & Blömeke, S. (2016b). Epistemological beliefs of prospective pre-school teachers and their relation to knowledge, perception and planning abilities in the field of mathematics: A process-model. *ZDM Mathematics Education, 48*(1–2), 125–137.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In C. Keitel, P. Damerow, A. Bishop, & P. Gerdes, (Eds.), *Mathematics, education, and society* (pp. 99–101). http://unesdoc.unesco.org/images/0008/000850/085082eo.pdf. Accessed February 14, 2014.

Feza, N., & Bambiso, N. (2016, July). *South Africa's educators' mathematics teaching journey: A case of 5-6 years old educator practices*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Fosse, T., & Lossius, M. H. (2016, July). *Norwegian Kindergarten teachers' work with mathematics*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Fritz, A., Balzer, L., Herholdt, R., Ragpot, L., & Ehlert, E. (2014). A mathematics competence test for grade 1 children migrates from Germany to South Africa. *South African Journal of Childhood Education, 4*(2), 114–133.

Fritz, A., Ehlert, A., & Balzer, L. (2013). Development of mathematical concepts as basis for an elaborated mathematical understanding. *South African Journal for Childhood Education, 3*(1), 38–67.

Fritz-Stratmann, A., Ehlert, A., & Herzog, M. (2016, July). *From thought to reality—Implementation of an in-school mathematics training in South Africa*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Goto, M. (2016, July). *Reform of the kindergarten teachers and child care workers in day-care workers training curriculum in Japan*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Grigutsch, S., Raatz, U., & Törner, G. (1998). Einstellungen gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematik-Didaktik, 19*(1), 3–45.

Hassidov, D., & Ilany, B. (2016, July). *"Senso-Math" preschool program facilitators contributions to mathematics education in the preschool*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Hayashi, A., & Tobin, J. (2013). Continuity and change in Japanese pre-school education. In G. DeCoker & C. Bjork (Eds.), *Japanese education in an era of globalization: Culture, politics and equity* (pp. 33–47). New York, NY: Teachers College Press.

Hedge, A. V., Sugita, C., Crane-Mitchell, L., & Averett, P. (2014). Japanese nursery and kindergarten teachers' beliefs and practices regarding developmentally appropriate practices. *International Journal of Early Years Education, 22*(3), 301–314.

Hill, H., & Ball, D. L. (2009). The curious—And crucial—Case of mathematical knowledge for teaching. *Kappan, 91*(2), 68–71.

Isiksal, M., Curran, J. M., Koc, Y., & Askun, C. S. (2009). Mathematics anxiety and mathematical self-concept: Considerations in preparing elementary-school teachers. *Social Behavior and Personality, 37*(5), 631–644.

Jenßen, L., Dunekacke, S., Eid, M., & Blömeke, S. (2015). The relationship of mathematical competence and mathematics anxiety: An application of latent state-trait theory. *Zeitschrift für Psychologie, 223*(1), 31–38.

Jenßen, L., Eilerts, K., Koinzer, T., Schmude, C., & Blömeke, S. (2016, July). *Development and validation of a test battery assessing preschool teachers' professional competence in the field of mathematics*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Jenßen, L., Jegodtka, A., Eilerts, K., Eid, M., Koinzer, T., Schmude, C., … Blömeke, S. (2016). Pro-KomMa—Professionalization of early childhood teacher education: Convergent, discriminant, and prognostic validation of the KomMa models and tests. In H. A. Pant, O. Zlatkin-Troitschanskaia, C. Lautenbach, M. Toepper, & D. Molerov (Eds.), *Modeling and measuring competencies in higher education—Validation and methodological innovations (KoKoHs) overview of the research projects* (pp. 39–43). http://www.kompetenzen-im-hochschulsektor.de/Dateien/KoKoHs_Working_Papers_No.11_Final_04.07.pdf. Accessed January 3, 2017.

Klibanoff, R. S., Levine, S. C., Huttenlocher, J., Vasilyeva, M., & Hedges, L. V. (2006). Pre-school children's mathematical knowledge: The effect of teacher "math talk". *Developmental Psychology, 42*(1), 59–69.

Krajewski, K., & Schneider, W. (2009). Early development of quantity to number word linkage as a precursor of mathematical school achievement and mathematical difficulties: Findings from a four-year longitudinal study. *Learning and Instruction, 19*(6), 513–526.

Lao, K. L. (2016, July). *Comparison of three early childhood curricula from the perspective of mathematics education*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Lehrl, S., Kluczniok, K., & Rossbach, H.-G. (2016). Longer-term associations of pre-school education: The predictive role of pre-school quality for the development of mathematical skills through elementary school. *Early Childhood Research Quarterly, 36,* 475–488.

Lembrér, D., & Johansson, M. C. (2016, July). *Swedish preschool teachers' views of children's socialisation*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Lembrér, D., & Meaney, T. (2014). Socialisation tensions in the Swedish pre-school curriculum: The case of mathematics. In L. Bergman, I. Ericsson, N. Hartsmar, L. Lang, C. Ljungberg, T. Småberg, & J. Söderman (Eds.), *EDUCARE 2014:4 childhood, learning and didactics* (pp. 89–106). Malmö: Malmö Högskola. http://dspace.mah.se/bitstream/handle/2043/17774/Educare%202014.2%20muep.pdf. Accessed January 7, 2017.

Leung, F. K. S., Graf, K.-D., & Lopez-Real, F. J. (2006). Mathematics education in different cultural traditions: A comparative study of East Asia and the West. In F. K. S. Leung, K.-D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics education in different cultural traditions—A comparative study of East Asia and the West: The 13th ICMI Study* (pp. 1–20). Boston, MA: Springer.

Lewis Presser, A., Clements, M., Ginsburg, H., & Ertle, B. (2015). Big math for little kids: The effectiveness of a pre-school and kindergarten mathematics curriculum. *Early Education and Development, 26*(3), 399–426.

Lloyd, C. M., & Modlin, E. L. (2012). Coaching as a key component in teachers' professional development: Improving classroom practices in head start settings. *OPRE Report 2012-4*. Washington, DC: OPRE, U.S. Department of Health and Human Services. https://www.acf.hhs.gov/sites/default/files/opre/coaching_key.pdf. Accessed January 9, 2017.

MacDonald, A. (2015). Let's count: Early childhood educators and families working in partnership to support young children's transitions in mathematics education. In R. Perry, A. MacDonald, & A. Gervasoni (Eds.), *Mathematics and transition to school* (pp. 85–102). Singapore: Springer Singapore.

MacDonald, A., Goff, W., Dockett, S., & Perry, B. (2016). Mathematics education in the early years. In K. Makar, S. Dole, J. Visnovska, M. Goos, A. Bennison, & K. Fry (Eds.), *Research in mathematics education in Australasia 2012-2015* (pp. 165–186).

Martinsen, M. T. (2015). Structural conditions for children's play in kindergarten. *Nordisk Barnehageforskning, 10*(1), 1–18.

Ministry of Education, Culture, Sports, Science and Technology. (2008). *Course of study for Kindergarten*. http://www.mext.go.jp/component/english/__icsFiles/afieldfile/2011/04/07/1303755_002.pdf. Accessed January 9, 2017.

Norwegian Ministry of Education and Research [NMER]. (2011). *Framework plan for the content and tasks of kindergartens*. https://www.udir.no/Upload/barnehage/Rammeplan/Framework_Plan_for_the_Content_and_Tasks_of_Kindergartens_2011_rammeplan_engelsk.pdf. Accessed January 10, 2017.

Organisation for Economic Cooperation and Development. (n.d.). *About: What is PISA?* https://www.oecd.org/pisa/aboutpisa/. Accessed January 17, 2017.

Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of pre-schoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42*(3), 237–269.

Perry, B., & MacDonald, A. (2015). Educators' expectations and aspirations around young children's mathematical knowledge. *Professional Development in Education, 41*(2), 366–381.

Philippou, G. N., & Christou, C. (1998). The effects of a preparatory mathematics program in changing prospective teachers' attitudes towards mathematics. *Educational Studies in Mathematics, 35*(2), 189–206.

Rasche, J., Eilerts, K., Jenßen, L., Eid, M., Schmude, C., Koinzer, T., & Blömeke, S. (2016, July). *Pro-KomMa: Effectiveness of preschool teacher education in the field of mathematics*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Rittle-Johnson, B., Fyfe, E. R., McLean, L. E., & McEldoon, K. L. (2013). Emerging understanding of patterning in 4-year-olds. *Journal of Cognition and Development, 14*(3), 376–396.

Sarama, J., & Clements, D. H. (2015). Scaling up early mathematics interventions: Transitioning with trajectories and technologies. In R. Perry, A. MacDonald, & A. Gervasoni (Eds.), *Mathematics and Transition to School* (pp. 153–169). Singapore: Springer Singapore.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Skolverket. (2011). *Curriculum for the pre-school Lpfö 98 Revised 2010*. https://www.skolverket.se/publikationer?id=2704 Accessed January 6, 2017.

Trends in International Mathematics and Science Study (TIMSS) & Progress in International Reading Literacy Study (PIRLS) International Study Centre. (n.d.). *About TIMSS & PIRLS International Study Centre*. http://timssandpirls.bc.edu/about.html. Accessed January 11, 2017.

Tsamir, P., Tirosh, D., Levenson, E., & Barkai, R. (2018). Early childhood teachers' knowledge and self-efficacy for evaluating solutions to repeating pattern tasks. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Tsamir, P., Tirosh, D., Levenson, E. S., Barkai, R., & Tabach, M. (2016, July). *Preschool teachers' responses to repeating pattern tasks*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Tsamir, P., Tirosh, D., Levenson, E., Tabach, M., & Barkai, R. (2012, February). *Conceptualizing pre-school teachers' knowledge and self-efficacy for teaching mathematics: The CAMTE framework*. Paper presented at the Perspectives on Early Mathematics Conference (POEM), Frankfurt, Germany, 27–29 February. http://cermat.org/poem2012/main/proceedings.html. Accessed March 22, 2017.

Venkat, H., & Spaull, N. (2015). What do we know about primary teachers' mathematical content knowledge in South Africa? An analysis of SACMEQ 2007. *International Journal of Educational Development, 41,* 121–130.

Wang, A. H., Firmender, J. M., Power, J. R., & Byrnes, J. P. (2016). Understanding the program effectiveness of early mathematics interventions for prekindergarten and kindergarten environments: A meta-analytic review. *Early Education and Development, 27*(5), 692–713.

Watts, T. W., Duncan, G. J., Siegler, R. S., & Davis-Kean, P. E. (2014). What's past is prologue: Relations between early mathematics knowledge and high school achievement. *Educational Researcher, 43*(7), 352–360.

Chapter 15
Early Childhood Teachers' Knowledge and Self-efficacy for Evaluating Solutions to Repeating Pattern Tasks

Pessia Tsamir, Dina Tirosh, Esther S. Levenson and Ruthi Barkai

Abstract This study examines three aspects of early childhood teachers' patterning knowledge: identifying features, errors and appropriate continuations of repeating patterns. Fifty-one practicing early childhood teachers' self-efficacy is investigated in relation to performance on patterning tasks. Results indicated that teachers held high self-efficacy beliefs about solving patterning tasks correctly. Regarding performance, teachers were able to identify repeating patterns and errors in those patterns. However, when evaluating ways in which a repeating pattern may be continued, teachers found it more difficult to choose correct continuations for patterns that did not end with a complete unit of repeat than for those patterns that did. They tended to only choose continuations which would end the pattern with a complete unit of repeat. These results are discussed in light of findings from related previous studies.

Keywords Repeating patterns · Early childhood teachers · Self-efficacy
Pattern tasks · Unit of repeat

P. Tsamir · D. Tirosh · E. S. Levenson (✉) · R. Barkai
Tel Aviv University, P.O. Box 39040, 6997801 Tel Aviv, Israel
e-mail: levenso@post.tau.ac.il

P. Tsamir
e-mail: pessia@post.tau.ac.il

D. Tirosh
e-mail: dina@post.tau.ac.il

R. Barkai
e-mail: ruthi11@013net.net

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_15

15.1 Introduction

15.1.1 Knowledge and Teaching of Early Childhood Mathematics

It is well known that teachers' mathematical knowledge has an impact on their teaching of mathematics (e.g., Shulman 1986). For early childhood teachers, this knowledge is no less important. A joint position paper published in the United States by the National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM) advocates that high quality, challenging, and accessible mathematics education for children from three to six years is a vital foundation for future mathematics learning (NAEYC and NCTM 2010; NCTM 2013). As such, they recommend that teachers of young children should learn the mathematics content that is directly relevant to their professional role. Similarly, the Australian Association of Mathematics Teachers (AAMT) and Early Childhood Australia (ECA) published a joint position paper calling for the adoption of "pedagogical practices that encourage young children to see themselves as mathematicians" (AAMT/ECA 2006, p. 2). They too recommended that early childhood staff be provided with "ongoing professional learning that develops their knowledge, skills and confidence in early childhood mathematics" (p. 4). One study found that mathematical content knowledge is a significant predictor of an early childhood teacher's ability to perceive learning situations and to plan educational actions that foster learning (Dunekacke et al. 2015). Likewise, Bruns et al. (2016) found that early childhood teachers' mathematical content knowledge is correlated with their mathematical pedagogical content knowledge. Based on an analysis of early childhood teacher education curricula in Germany, mathematical content knowledge for early childhood teachers includes mathematical domains (e.g., number and operations, quantity and relations, geometry, data, combinatorics and chance), as well as mathematical processes (e.g., problem solving, modeling, communicating, representing, reasoning, and patterns and structuring) (Jenßen et al. 2016). This present study focuses on early childhood teachers' knowledge for teaching patterning.

15.1.2 Self-efficacy and Teaching Mathematics

Another factor related to teachers' classroom actions is teachers' self-efficacy. Bandura defined self-efficacy as "people's judgments of their capabilities to organize and execute a course of action required to attain designated types of performances" (1986, p. 391). Bates et al. (2011) found that teachers who reported higher mathematics self-efficacy were more confident in their ability to teach mathematics than teachers with a lower mathematics self-efficacy. Hackett and Betz (1989) defined mathematics self-efficacy as, "a situational or problem-specific assessment of an individual's confidence in her or his ability to successfully perform or accomplish a particular [mathematics] task or problem" (p. 262). Teacher self-efficacy may be conceptualized as "a teacher's individual beliefs in their capabilities to perform specific teaching tasks at a specified level of quality in a specified situation" (Dellinger et al. 2008, p. 752). Teacher self-efficacy has been related to teachers' effort in teaching, persistence and resilience in the face of difficulties with students, enthusiasm, and commitment (e.g., Allinder 1994).

For early childhood teachers, it has been shown that their instructional self-efficacy and efficacy towards creating a positive climate was related to the quality of instructional support for concept development and emotional support for children. In turn, this was found to be related to language development (Guo et al. 2010). Regarding early childhood teachers' self-efficacy and the teaching of mathematics, most studies investigated teachers' beliefs regarding their general ability to teach and not their specific ability to teach mathematics (McMullen 1997). In one study (Bates et al. 2011), early childhood prospective teachers' mathematics teaching self-efficacy was studied. It was shown that prospective teachers who felt good about their mathematical abilities were more likely to also feel good about their ability to teach mathematics than pre-service teachers who did not feel good about their mathematical abilities. However, the questionnaire in that study included very general statements such as "I will continually find better ways to teach mathematics". The study did not focus on specific mathematical domains or on specific mathematical tasks. Yet, levels of self-efficacy are not necessarily equal in all domains and tasks (Bandura 1997). Hence, there is a need to devise instruments and interventions that will explicitly address teachers' self-efficacy. This present study investigates early childhood teachers' self-efficacy related to patterning activities and the relationship between their knowledge and self-efficacy beliefs.

15.2 Mathematical Patterning in Early Childhood

The importance of engaging young children in patterning activities is supported by mathematicians, mathematics education researchers and curriculum developers (Sarama and Clements 2009). There are several reasons for this support. First, pattern exploration and recognition may support children as they learn a variety of mathematical skills developed at this age. For example, recognizing repeating patterns may help children develop skip counting, such as 5, 10, 15, 20, 25, 30… where the ones digit forms the pattern 5, 0, 5, 0, … Mulligan and Mitchelmore (2018) found that recognizing a series of objects or symbols arranged in a definite order (e.g., repeating patterns), and being able to count in groups (e.g., counting by 2 s), may be part of a broader construct called Awareness of Mathematical Pattern and Structure (AMPS). In other words, recognition and analysis of patterns can provide a foundation for the development of algebraic thinking and provide children with the opportunity to observe and verbalize generalizations as well as to record them symbolically (Threlfall 1999). In fact, in the introduction to the

ICME-13 topical survey of the Topic Study Group on Teaching and Learning of Early Algebra it states that, “Mathematical relations, patterns, and arithmetical structures lie at the heart of early algebraic activity” (Kieran et al. 2016, p. 1). Mason (2016), in his presentation for this Topic Study Group, suggested that it is never too early to begin thinking algebraically. He suggested several patterning activities for young children that provide opportunities for generalization and abstraction.

A third reason for promoting young children’s pattern awareness is the possible relationship between structural awareness and other mathematical competencies. Makar (2016) suggested a connection between patterning and inferential practices in statistics. Just as patterning requires the ability to see beyond individual elements in a pattern and to focus on the ‘unit’ which repeats, making predictions based on data requires thinking of data as an aggregate, rather than as individual points. Lüken and Kampmann (2018) found that first-grade students who had participated in a program for promoting pattern and structure abilities improved their arithmetic skills to a greater degree than students who did not participate in the program. Schmerold et al. (2017) also found that patterning was related to working memory and cognitive flexibility (i.e., recognizing that different rules are appropriate for different tasks and being able to change the basis for one’s responses accordingly). In other words, patterning was found to be related to executive function.

15.2.1 Repeating Patterns

While there are several types of patterns, this study focuses on repeating patterns. Repeating patterns are patterns with a cyclical repetition of an identifiable ‘unit of repeat’. For example, a pattern of the form ABBABBABB… has a (minimal) unit of repeat of length three. Several studies have investigated ways in which young children engage with repeating patterns. For example, Seo and Ginsburg (2004) found that young children build block towers with an ABAB pattern. In another study, Rittle-Johnson et al. (2013) found that when young children were asked to duplicate or extend an ABB pattern, some could not produce more than one unit of repeat correctly while others reverted to producing an ABAB pattern. Papic et al. (2011) found that some children may be able to draw an ABABAB pattern from memory by recalling the pattern as single alternating colors of red, blue, red, blue, but this was basically recalling that after red came blue and after blue came red. However, when shown a more complicated pattern such as ABBC, they could not replicate the pattern.

Some researchers have described a possible progression of patterning competencies. Lüken (2018) investigated young children’s engagement with patterns. She found that three-year olds could not copy a simple pattern, even when it was visible and placed in front of them. When asked to compare patterns, they noticed only length and color. After one year of kindergarten, at age four, children were able to correctly solve more tasks and more complex patterns, although patterns with a double element (ABB) were still difficult.

Mulligan and Mitchelmore (2018) described four levels of structural awareness connected to patterns: those who struggle (e.g., they copy block patterns by matching one by one), those who easily recognize simple patterns, those who are aware of fundamental structures, and those who are aware of the generality of fundamental structures. Recognizing and identifying the unit of repeat in a repeating pattern is a step in both the second and third levels of structural awareness. The third level also includes being able to see relations to other patterns. In addition, Mulligan and Mitchelmore described five structural groupings, among which the first is sequences. Sequences include recognizing a series of objects or symbols arranged in a definite order or using repetitions, such as repeating patterns.

The transition between patterning and algebra is not necessarily simple. According to McGarvey (2012), the tendency to focus on successive elements and recursive reasoning when extending a pattern detracts from algebraic thinking and from finding a general rule. She found that when identifying, describing and justifying patterns, the most common approach for both students and teachers was to point and say aloud sequential elements or attributes in an image such as "white-black-white-black." At times, this led to incorrect identifications, such as when a child labeled a picture as representing a pattern saying, "small-big-small-small-medium." McGarvey (2012) advocated helping children focus on the unit of repeat.

In our own studies of young children (Tsamir et al. 2015), we found that when children were requested to choose possible ways to continue repeating patterns, more children were able to continue a pattern which ended with a complete unit of repeat than a pattern which ended with a partial unit. When deciding whether or not to choose some continuation, some children merely seemed to guess, while others exhibited some strategy. One strategy was to align possible continuation with the beginning of the pattern to see if it matched. Another was to physically move each continuation to the end, trying it out before deciding whether or not it was appropriate. One child chose continuations based on the last element of the pattern, claiming that the next element cannot be the same as the last element of the given patterns. It was suggested that in addition to promoting children's recognition of the unit of repeat, we should encourage children to recognize the sequencing aspect of the pattern and how to continue a pattern from any point.

In the above studies, children were observed without adult intervention. However, adult guidance could help children benefit further from engaging with pattern activities. Previous studies found that some teachers lack both knowledge and self-confidence to teach patterning (Papic 2007; Papic and Mulligan 2007). Waters (2004) found that teachers provided limited worthwhile patterning opportunities for children, and that even when children engaged spontaneously in patterning, teachers may have failed to capitalize on the child's interest, missing out on opportunities to extend their knowledge in patterning (Fox 2005). There is clearly a need for systematically studying early childhood teachers' knowledge for teaching patterns, as well as a need for providing professional development for early childhood teachers related to patterning. This study investigates early childhood teachers' knowledge for teaching repeating patterns.

15.3 The Cognitive Affective Mathematics Teachers Education (CAMTE) Framework and Related Studies

For the past several years we have been investigating early childhood teachers' knowledge and self-efficacy for teaching number concepts and geometry using the Cognitive Affective Mathematics Teachers Education (CAMTE) framework (e.g., Tirosh et al. 2014). Table 15.1 provides the components grouped by knowledge and self-efficacy. Cell 1 attends to teachers' knowledge for producing solutions. For example, within the context of number concepts, Cell 1 includes being able to compare the number of elements in two sets using a variety of strategies. Cell 2 attends to teachers' knowledge for evaluating solutions, such as evaluating justifications for why one set has more elements than another set. Cell 3 attends to knowledge of students' conceptions such as which number symbols are more difficult for children to learn, and what are children's common mistakes related to the counting sequence. Cell 4 attends to knowledge for designing and evaluating tasks, such as knowing which tasks have the potential to foster children's acceptance of the one-to-one principle. Cells 5–8 address teachers' self-efficacy related to each of the knowledge Cells 1–4 respectively.

The CAMTE framework draws on Shulman (1986), who identified subject-matter knowledge (SMK) and pedagogical content knowledge (PCK) as two major components of teachers' knowledge necessary for teaching. As can be seen, we differentiated between two components of teachers' SMK: being able to produce solutions, strategies, and explanations, and being able to evaluate given solutions, strategies, and explanations. Regarding PCK, and in line with Ball et al. (2008) we differentiated between knowledge about students and knowledge of designing and evaluating tasks. The CAMTE framework also drew upon the work of Bandura (1986) and took into consideration that self-efficacy beliefs may impact on teaching. Thus, for each knowledge cell in the framework, there is a related self-efficacy cell.

15.3.1 First Studies Involving the CAMTE Framework

Our first studies involving the CAMTE framework investigated early childhood teachers' knowledge and self-efficacy for teaching number concepts (e.g., verbally counting forwards and backwards, and enumerating objects) (Tirosh et al. 2012) and for teaching geometrical figures (e.g., triangles, pentagons, and circles)

Table 15.1 The cognitive affective mathematics teacher education framework

	Subject-matter		Pedagogical-content	
	Solving	Evaluating	Students	Tasks
Knowledge	Cell 1: producing solutions	Cell 2: evaluating solutions	Cell 3: knowledge of students' conceptions	Cell 4: designing and evaluating tasks
Self-efficacy	Cell 5: mathematics self-efficacy related to producing solutions	Cell 6: mathematics self-efficacy related to evaluating solutions	Cell 7: pedagogical-mathematics self-efficacy related to children's conceptions	Cell 8: pedagogical-mathematics self-efficacy related to designing and evaluating tasks

(Tirosh et al. 2011). For example, we found that when investigating teachers' knowledge and self-efficacy for identifying two-dimensional figures, teachers had a higher regard for their ability to identify triangles and circles, than their ability to identify pentagons. This limitation in self-efficacy was matched on their performance in identification tasks. When asked to identify several examples and non-examples of these figures, teachers' score for identifying pentagons was lower than their scores for identifying triangles and circles. This was mostly due to their incorrect identification of a curved-sided 'pentagon' as a pentagon.

In a related study, Tirosh et al. (2014) investigated early childhood teachers' knowledge and self-efficacy for defining triangles and circles. Teachers were requested to write a definition for a triangle and for a circle and to then identify various figures as examples or non-examples. Definitions were analyzed in terms of critical attributes, whether those attributes were sufficient to define the targeted concept, whether the definition was sufficiently detailed, or whether extra non-critical attributes constricted the targeted concept. In addition, the use of precise mathematical language versus everyday terminology was assessed. For the triangle, teachers' high self-efficacy for defining triangles corresponded with their acceptable triangle definitions. However, teachers' relatively high self-efficacy for defining circles, did not correspond with their knowledge of circle definitions; many of the teachers' circle definitions were missing references to critical attributes. Two possible reasons were given for this dissonance. First, it could be that teachers had a clear concept image of circles and felt that this would enable them to define circles as well. Another reason could be that teachers equated definitions of geometrical figures with descriptions of geometrical figures, and thought that if they described a circle it could be considered as a definition. To summarize, the above studies showed that even within the same mathematical domain (two-dimensional geometric figures), for some tasks there was a strong correspondence between self-efficacy and knowledge levels, but this was not consistent for other tasks.

15.3.2 The Study of Teacher Knowledge and Self-efficacy Related to Patterning

According to the Israel National Mathematics Early Childhood Curriculum (INMECC 2008) before entering first grade, children should be able to identify, draw, and continue repeating patterns as well as use mathematical language to describe these patterns. In order to achieve this aim, the teacher has several tasks, among them making patterning activities available to children and demonstrating to children how, for example, a pattern may be continued. Thus, it is critical that teachers know how to draw, identify, and continue a repeating pattern.

Recently, we have begun to use the CAMTE framework to investigate early childhood teachers' knowledge and self-efficacy for teaching repeating patterns. Specifically, related to Cells 1 and 5 of the CAMTE framework (see Table 15.1), we investigated early childhood teachers' knowledge and self-efficacy regarding three different patterning tasks—defining repeating patterns, drawing repeating patterns, and continuing repeating patterns (Tirosh et al. 2015). In that study, teachers were asked to write a definition for repeating patterns (i.e., to answer the question: what is a repeating pattern?), draw a repeating pattern, and to continue six different repeating patterns with different structures. In general, we found that teachers had a high level of self-efficacy for drawing and continuing repeating patterns which matched their performance on drawing and continuing tasks. Another finding was their strong tendency to end patterns with a complete unit of repeat. As for the task of defining repeating patterns, the picture was more complex. On the one hand, teachers indicated lower self-efficacy for defining than they had for drawing and continuing, which coincided with the difficulties they had in actually writing definitions for repeating patterns. On the other hand, their self-efficacy was still relatively high. Once again, self-efficacy and knowledge levels did not always match. To summarize, teachers' knowledge for drawing and continuing repeating patterns matched their self-efficacy but less of a match was found when defining repeating patterns. While that study focused on how teachers actually solved pattern tasks and their related self-efficacy, this present study focuses on teachers' knowledge for evaluating the solutions of various repeating pattern asks, and their reported self-efficacy (Cells 2 and 6 of the CAMTE framework).

This study complements the previous study by examining early childhood teachers' evaluations of possible ways for continuing repeating patterns. This is an important skill for teachers. As children begin to explore patterns and engage with patterning activities, it is the teacher's job to observe and evaluate the children's solutions. Although teachers may have a certain tendency to end patterns with a complete unit of repeat, it does not mean that they are unaware of other acceptable ways. Taken together, the two studies exemplify the use of the CAMTE framework as a research tool and the importance of investigating teachers' knowledge of producing solutions (such as actually drawing continuations of patterns) along with their knowledge of evaluating solutions (such as evaluating the appropriateness of different continuations).

15.4 Research Aim and Questions

The aim of this study was to investigate early childhood teachers' knowledge and self-efficacy for teaching repeating patterns, focusing on early childhood teachers' SMK and self-efficacy related to evaluating solutions (Cells 2 and 6 of the CAMTE framework). Specifically, the research questions were: (1) Are early childhood teachers able to identify examples and non-examples of repeating patterns, identify mistakes in given repeating patterns, and identify appropriate continuations of repeating patterns? (2) What are early childhood teachers' self-efficacy beliefs regarding their ability to identify examples and non-examples of repeating patterns, mistakes in given repeating patterns, and appropriate continuations of repeating patterns? (3) What are the relationships between teachers' knowledge and self-efficacy for three tasks: (i) identifying examples and non-examples of repeating patterns, (ii) identifying mistakes in given repeating patterns, and (iii) identifying appropriate continuations of repeating patterns?

15.5 Method

15.5.1 Context and Participants

Participants in this study were 51 early childhood teachers of children aged four to six years who were enrolled in municipal programs, and who attended one of the CAMTE professional development programs. All teachers had a first degree in education. Many prospective early childhood teachers in Israel attend only two mathematics education courses during their four-year education degree. These courses sometimes include one semester for learning about the development of number concepts and one semester for the development of geometrical concepts. Thus, providing ongoing professional development focused on early childhood mathematics education is imperative. Yet, while professional development is strongly recommended, and teachers are given credit for courses taken, the choice between programs is varied and teachers are not necessarily mandated to specifically enroll in mathematics education programs.

15.5.2 Instruments and Procedure

At the beginning of the program, teachers were asked to fill out a two-part questionnaire which began with self-efficacy statements and continued with knowledge questions. The self-efficacy statements were as follows:

- I am able to identify a drawing of a repeating pattern.
- If I am shown a repeating pattern with a mistake, I am able to identify the mistake.
- If I am shown a repeating pattern with a few possible ways of continuing that pattern, I am able to identify which continuations are acceptable.

A four-point Likert scale was used to rate participants' agreements with self-efficacy statements:

1. I do not agree that I am capable.
2. I somewhat agree that I am capable.
3. I agree that I am capable.
4. I strongly agree that I am capable.

After completing the first part of the questionnaire, participants returned it to the researcher who then gave them the second part of the questionnaire. The second part had three tasks. For the first task, participants were presented with four drawings and were told that each drawing was supposed to represent a repeating pattern, but that a mistake in the drawing needed to be corrected in order for the drawing to be a repeating pattern. Teachers were asked to circle the mistake (see Fig. 15.1 for an example). The second task on the questionnaire consisted of four drawings and participants were asked to write down for each drawing if it did or did not represent a repeating pattern (see Fig. 15.2 for an example).

The third task was a series of four mini-tasks (see Figs. 15.3, 15.4, 15.5 and 15.6). Each mini-task included one repeating pattern and four suggested ways of continuing the pattern. Patterns and continuations were chosen in order to vary between patterns and continuations that ended in a complete unit of repeat and those that ended mid-cycle. In addition, inappropriate continuations were chosen based on children's common responses found in previous studies, such as choosing a complete unit of repeat as a continuation, even when it is inappropriate (Tsamir et al. 2015). As can be seen below, all of the presented patterns included a minimum of three repetitions of the minimal unit of repeat.

Fig. 15.1 Find the mistake in this repeating pattern

Fig. 15.2 Does this drawing represent a repeating pattern?

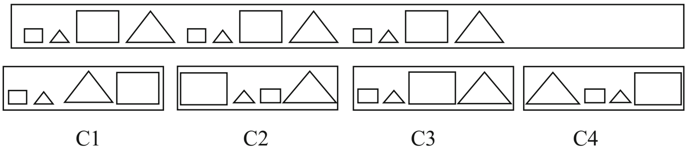


Fig. 15.3 Pattern 1 (P1) and four continuations

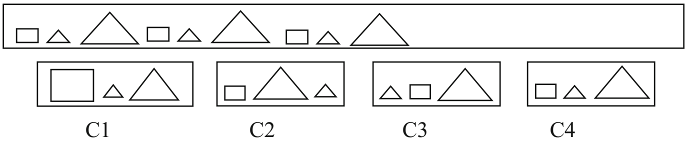


Fig. 15.4 Pattern 2 (P2) and four continuations

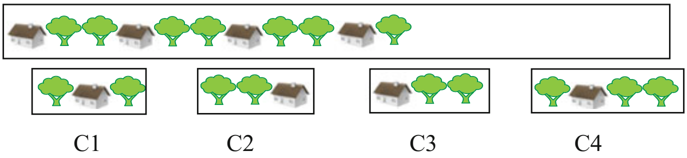


Fig. 15.5 Pattern 3 (P3) and four continuations

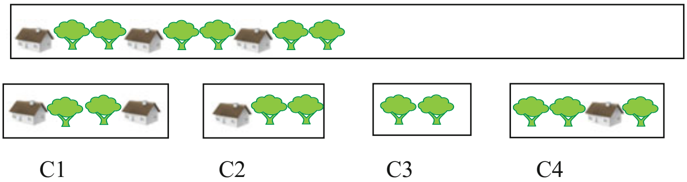


Fig. 15.6 Pattern 4 (P4) and four continuations

For each pattern and four suggested continuations, the following instructions were given: Circle all acceptable continuations for the given pattern. Table 15.2 offers an analysis of the given patterns and their corresponding appropriate continuations. Notice that the first and second patterns (P1 and P2) have only one acceptable way for continuing the pattern, while the third and fourth patterns (P3 and P4) have two acceptable ways for continuing that pattern.

Table 15.2 Task 3—analysis of patterns

Pattern	Structure	Ends with	Correct continuations
P1	ABCD	A complete cycle	One correct continuation (C3) with a complete cycle
P2	ABC	A complete cycle	One correct continuation (C4) with a complete cycle
P3	ABB	Mid-cycle	One correct continuation ending mid-cycle (C1) and one correct continuation ending with a complete cycle (C4)
P4	ABB	A complete cycle	One correct continuation ending mid-cycle (C1) and one correct continuation ending with a complete cycle (C2)

15.6 Results

15.6.1 Self-efficacy Beliefs

Table 15.3 presents the relative frequencies (as a percentage) of the teachers' levels of self-efficacy for the three evaluation tasks. Recall that a scale of 1–4 (4 being the strongest agreement) was used to score teachers' self-efficacy beliefs. As shown in Table 15.3, at least half of the teachers tended to strongly believe in their ability to identify examples of repeating patterns, to identify mistakes in repeating patterns, and to identify possible acceptable ways of continuing patterns.

The means and standard deviations for each of the three self-efficacy questions were configured. Table 15.4 summarizes these scores by task. The means were compared using an analysis of variance with repeated measures. No significant differences at the 0.05 significance level were found between the overall self-efficacy scores for each task.

15.6.2 Teacher Knowledge

In general, teachers were able to find the error in a given 'repeating pattern' and were able to evaluate whether or not a given figure represented a valid repeating pattern (see Table 15.5).

Table 15.3 Frequency in % of teachers' self-efficacy scores by task

Task-identifying …	I do not agree that I am capable	I somewhat agree that I am capable	I agree that I am capable	I strongly agree that I am capable
A drawing of a repeating pattern	–	3	41	56
A mistake in a repeating pattern	–	5	44	51
An acceptable continuation for a repeating pattern	–	5	34	61

Table 15.4 Mean self-efficacy scores per task

Task	Mean	SD
Identifying a drawing of a repeating pattern	3.54	0.55
Identifying a mistake in a repeating pattern	3.46	0.60
Identifying an acceptable continuation for a repeating pattern	3.56	0.59

Table 15.5 Frequencies of teachers' correct responses to Tasks 1 and 2

	Find the mistake				Is this a repeating pattern?			
Drawing	1	2	3	4	1	2	3	4
% Correct	97	100	94	94	100	100	95	95

Table 15.6 Frequency in % of correct responses by continuation by pattern

Continuation	1	2	3	4	Mean
Pattern 1 (see Fig. 15.3)	95	100	98*	89	95.5
Pattern 2 (see Fig. 15.4)	97	97	97	100*	97.75
Pattern 3 (see Fig. 15.5)	30**	92	73	78*	68.25
Pattern 4 (see Fig. 15.6)	62**	95*	100	97	88.5

*Appropriate continuation with a complete cycle; **Appropriate continuation ending mid-cycle

Results were more varied for the tasks that required the teacher to assess a given repeating pattern and then choose which of the presented continuations were appropriate (see Table 15.6).

Regarding the first two patterns that ended with a complete cycle (P1 and P2), the vast majority of teachers successfully chose the appropriate continuations. However, note that in each case the appropriate continuation was a complete unit of repeat. For P4, which also ended in a complete unit of repeat, nearly all teachers recognized the appropriate continuation when it was a complete unit, but a little over half of the teachers recognized the appropriate continuation when it did not end with a complete unit of repeat. The basic unit of repeat for P4 was  and the first suggested continuation was .

For P3, (see Fig. 15.5) results in general indicated that this mini-task was more difficult than the other three. Taking a closer look, only C2 was easily recognized as inappropriate, possibly because placing it at the end of the presented drawing would have caused three trees to be together, a blatant visual violation of the pattern. Interestingly, C3, also an inappropriate continuation, was recognized as inappropriate by approximately three-quarters of the participants. It could be that because this continuation was also the basic unit of repeat, teachers chose it as a continuation. Between the two appropriate continuations (C1 and C4), more teachers recognized C4 as appropriate, ending the pattern with a complete cycle, than C1, ending the pattern mid-cycle. Finally, for all four patterns, we note that six teachers (12%) chose only the continuations where the patterns ended in a complete unit of repeat.

15.6.3 Comparing Teacher Knowledge and Self-efficacy

As the results above indicate, teachers' high self-efficacy beliefs for identifying mistakes in given repeating patterns and for identifying drawings that were examples of repeating patterns, matched their actual performance on those tasks.

However, teachers' high self-efficacy beliefs for identifying possible correct continuations of given patterns did not always match their actual performance on those tasks. For example, taking a closer look at Pattern 3, 30% of the teachers identified the first continuation as acceptable. Yet, 95% of the teachers agreed that they were capable or even very capable of identifying acceptable continuations. In the following section we discuss possible reasons for this result and, in general, how this study fits in with our previous studies of teachers' knowledge and self-efficacy for teaching mathematics in early childhood.

15.7 Discussion

This study contributes to the theme of early childhood educators' beliefs and understandings of mathematical learning. While some studies focused on teachers' mathematics related competencies (e.g., Jenßen et al. 2016) and others focused on teachers' views of mathematics (e.g., Cooke 2016), this study combines knowledge and beliefs by studying the relationship between teachers' knowledge and their self-efficacy. In addition, the CAMTE framework provides a way for examining specific aspects of knowledge and teachers' self-efficacy related to those specific aspects. Like our previous study concerning early childhood teachers and repeating patterns, this study found that teachers had relatively high self-efficacy beliefs regarding their ability to perform various patterning tasks. This is an important finding. In a prior study, we also found that early childhood teachers held high self-efficacy beliefs for identifying two-dimensional geometric figures (Tirosh et al. 2011). However, our research question addressed the issue that mathematics self-efficacy is content-specific (Bandura 1986; Hackett and Betz 1989). Thus, we could not assume that we would obtain similar findings regarding self-efficacy beliefs for these patterning tasks.

Consistent with our earlier studies (e.g. Tirosh et al. 2014), we found in this study that knowledge and self-efficacy beliefs did not always correspond. Some discrepancy was found between teachers' beliefs in their ability to identify acceptable pattern continuations and their ability to recognize acceptable continuations. It could be that some teachers interpreted the instruction—"Circle all acceptable continuations for the given pattern", to mean "circle all continuations that would complete the pattern." In this case, teachers might hesitate to accept a continuation which did not "complete" the pattern. Another reason could be that the self-efficacy questions were not sufficiently explicit (e.g., the questions did not mention the situations in which teachers might be required to identify acceptable pattern continuations). Also, the self-efficacy scale was not validated. Perhaps it was not sensitive enough. A wider scale might have captured more fluctuations in self-efficacy.

Another possible reason for the discrepancy between knowledge and self-efficacy could stem from the teachers' concept image of what it means to continue a pattern. In the literature, we found that most activities that called for extending a pattern, presented a pattern that ended with a complete unit of repeat (e.g., Rittle-Johnson et al. 2013). In addition, most pattern extension activities request the participant to extend the pattern one element at a time. It could be that this is what teachers envisioned when they were requested to complete the tasks. According to Bandura (1986) performance attainments are an important source of self-efficacy; successes raise self-efficacy while repeated failures lower them. If teachers' experiences with success came mostly from extending repeating patterns presented with a complete unit of repeat, then their self-efficacy beliefs in this regard are understandable. However, it does point out a need to offer teachers additional experience with variations in patterning tasks, a need that could be met by professional development.

15.8 Implications for Professional Development

In this section, we synthesize the results from our previous study (Tirosh et al. 2015) along with this study, and discuss implications for professional development. Using the CAMTE framework as a reference, we investigated two aspects of early childhood teachers' knowledge for teaching patterning: solving patterning tasks (Cell 1) and evaluating solutions to patterning tasks (Cell 2). This knowledge, of course, is intertwined. When defining repeating patterns, several teachers noted that a repetition must appear, but did not specify what is repeated. Some wrote that a number of shapes must be repeated, without indicating that those shapes must appear in a consistent sequence (i.e., structure). In the present study teachers were able to draw repeating patterns. Likewise, they correctly recognized drawings as examples or non-examples of repeating patterns. They also recognized mistakes in patterns. All of the continuations teachers drew were acceptable continuations, yet they did not always recognize acceptable continuations of given patterns.

Professional development could build on what teachers do know, and then fill in the gaps. For example, although children may not need to learn a formal definition for repeating patterns, it is still important for teachers to know the attributes of repeating patterns as well as to accurately use words to describe mathematical concepts related to patterning. For example, the link between patterning and multiplicative thinking and early algebra could be highlighted in relation to the unit of repeat (Mulligan and Mitchelmore 2018).

Another study, for example, found that the amount of early childhood teachers' mathematics-related talk was significantly associated with the growth of children's conventional mathematical knowledge over the school year (Klibanoff et al. 2006). In designing professional development for early childhood teachers, we use patterning tasks, such as those presented in this study, to talk about patterns and promote the use of precise mathematical terminology, such as minimal unit of repeat (Tirosh et al. in press). As we discuss acceptable and unacceptable continuations, we also promote teachers' knowledge of students' ways of solving patterning tasks (Cell 3 of the CAMTE framework). Teacher educators can also build on teachers' knowledge to plan professional learning designed to strengthen teachers' knowledge of tasks (Cell 4).

The CAMTE framework, along with results of this study, remind us that as we promote teachers' knowledge, we should take care to promote their self-efficacy as well (Cells 5 and 6 of the framework). Both studies showed that in general, teachers have high self-efficacy with regard to patterning tasks. However, developing high self-efficacy with corresponding content knowledge is our aim. The downside of an inflated high self-efficacy could be that teachers may not feel the need for professional development. The downside of low self-efficacy, especially when teachers are actually quite knowledgeable, might be a lack of motivation and confidence in implementing mathematical activities in their classrooms. Further research is needed to understand the implications for teaching and learning when there is a discrepancy between self-efficacy and knowledge.

Professional development offers teachers a chance to look inside themselves. "Reflection is the ultimate key to one's professional growth as a teacher" (Schoenfeld and Kilpatrick 2008, p. 348). In our program we invite teachers to reflect on their self-efficacy beliefs as well as reflect on mathematical knowledge and ideas. Although engaging them with challenging mathematical activities may risk lowering self-efficacy beliefs, other studies have shown that a feeling of disequilibrium may actually foster teacher learning (Wheatley 2002). Bruns et al. (2016) have also suggested that professional development might provide early childhood teachers with an opportunity to gain positive mathematical experience, by doing mathematics. Thus, another implication of this study might be for teacher educators to allow teachers the time to reflect on their self-efficacy beliefs, not only in the beginning of a professional learning program, but at various stages, while at the same time supporting the teachers as they face new challenges. At the beginning of our program, informal interviews with the teachers found that most patterning activities in their early childhood classrooms consisted of drawing borders around pictures. For early childhood teachers who might consider patterning more of an art activity than a mathematical activity, promoting mathematical knowledge of repeating patterns, along with high self-efficacy beliefs for teaching repeating patterns, is especially important.

Acknowledgements This research was supported by The Israel Science Foundation (grant No. 1270/14).

References

Allinder, R. M. (1994). The relationships between efficacy and instructional practices of special education teachers and consultants. *Teacher Education and Special Education, 17*(2), 86–95.

Australian Association of Mathematics Teachers and Early Childhood Australia (AAMT/ECA). (2006). *Position paper on early childhood mathematics.* http://www.aamt.edu.au/documentation/statements. Accessed July 04, 2017.

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Bandura, A. (1986). *Social foundations of thought and action: A social cognitive theory*. Englewood Cliffs, NJ: Prentice Hall.

Bandura, A. (1997). *Self-efficacy: The exercise of self-control*. New York: Freeman.

Bates, A. B., Latham, N., & Kim, J. (2011). Linking preservice teachers' mathematics self-efficacy and mathematics teaching efficacy to their mathematical performance. *School Science and Mathematics, 111*(7), 325–333.

Bruns, J., Eichen, L., & Blömeke, S. (2016, July). *Pre-school teachers' mathematics-related competencies*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Cooke, A. (2016, July). *Pre-service teacher relationships with mathematics—Creative? Anxious? Competent?* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Dellinger, A., Bobbett, J., Livier, D., & Ellett, C. (2008). Measuring teachers' self-efficacy beliefs: Development and use of the TEBS-Self. *Teaching and Teacher Education, 24,* 751–766.

Dunekacke, S., Jenßen, L., & Blömeke, S. (2015). Effects of mathematics content knowledge on pre-school teachers' performance: A video-based assessment of perception and planning abilities in informal learning situations. *International Journal of Science and Mathematics Education, 13*(2), 267–286.

Fox, J. (2005). Child-initiated mathematical patterning in the pre-compulsory years. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 313–320). Melbourne, Australia: PME.

Guo, Y., Piasta, S. B., Justice, L. M., & Kaderavek, J. N. (2010). Relations among early childhood teachers' self-efficacy, classroom quality, and children's language and literacy gains. *Teaching and Teacher Education, 26*(4), 1094–1103.

Hackett, G., & Betz, N. (1989). An exploration of the mathematics self-efficacy/mathematics performance correspondence. *Journal for Research in Mathematics Education, 20,* 261–273.

Israel National Mathematics Early Childhood Curriculum (INMECC). (2008). http://meyda.education.gov.il/files/Tochniyot_Limudim/KdamYesodi/Math1.pdf. Accessed August 11, 2017.

Jenßen, L., Eilerts, K., Koinzer, T., Schmude, C., & Blömeke, S. (2016, July). *Development and validation of a test battery assessing early childhood teachers' professional competence in the field of mathematics*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Kieran, C., Pang, J., Schifter, D., & Ng, S. F. (2016). *Early algebra: Research into its nature, its learning, its teacher*. Springer Open.

Klibanoff, R. S., Levine, S. C., Huttenlocher, J., Vasilyeva, M., & Hedges, L. V. (2006). Preschool children's mathematical knowledge: The effect of teacher's math talk. *Developmental Psychology, 42*(1), 59.

Lüken, M. (2018). Repeating pattern competencies in three- to five-year old kindergartners: A closer look at strategies. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Lüken, M., & Kampmann, R. (2018). The influence of fostering children's patterning abilities on their arithmetic skills in Grade 1. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Makar, K. (2016). Developing young children's emergent inferential practices in statistics. *Mathematical Thinking and Learning, 18*(1), 1–24.

Mason, J. (2016, July). *How early is too early for thinking algebraically?* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

McGarvey, L. M. (2012). What is a pattern? Criteria used by teachers and young children. *Mathematical Thinking and Learning, 14*(4), 310–337.

McMullen, M. (1997). The effects of early childhood academic and professional experience on self-perceptions and beliefs about developmentally appropriate practices. *Journal of Early Childhood Teacher Education, 18*(3), 55–68.

Mulligan, J. T., & Mitchelmore, M. (2018). Promoting early mathematical structural development through an integrated assessment and pedagogical program. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

National Association for the Education of Young Children (NAEYC) and the National Council of Teachers of Mathematics (NCTM). (2010). *Early childhood mathematics: Promoting good beginnings.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM). (2013). *Mathematics in early childhood learning: A position of the national council of teachers of mathematics*. Reston, VA: NCTM.

Papic, M. (2007). Promoting repeating patterns with young children-more than just alternating colours! *Australian Primary Mathematics Classroom, 12*(3), 8–13.

Papic, M., & Mulligan, J. T. (2007). The growth of early mathematical patterning: An intervention study. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice.* Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia (pp. 591–600). Adelaide: MERGA.

Papic, M., Mulligan, J., & Mitchelmore, M. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42*(3), 237–269.

Rittle-Johnson, B., Fyfe, E. R., McLean, L. E., & McEldoon, K. L. (2013). Emerging understanding of patterning in 4-year-olds. *Journal of Cognition and Development, 14*(3), 376–396.

Sarama, J., & Clements, D. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. London, England: Routledge.

Schmerold, K., Bock, A., Peterson, M., Leaf, B., Vennergrund, K., & Pasnak, R. (2017). The relations between patterning, executive function, and mathematics. *The Journal of Psychology, 151*(2), 207–228.

Schoenfeld, A. H., & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching mathematics. In D. Tirosh & T. Wood (Eds.), *The International handbook of mathematics teacher education: Tools and processes in mathematics teacher education* (Vol. 2, pp. 321–354). Rotterdam: Sense Publishers.

Seo, K. H., & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. In D. H. Clements, J. Sarama, & A. M. DiBiase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 91–104). NJ: Erlbaum.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Threlfall, J. (1999). Repeating patterns in the primary years. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 18–30). London: Cassell.

Tirosh, D., Tsamir, P., Barkai, R., & Levenson, E. (in press). Using children's patterning tasks for professional development for preschool teachers. In C. Benz, H. Gasteiger, A. S. Steinweg, P. Schöner, H. Vollmuth, & J. Zöllner (Eds.), *Mathematics Education in the Early Years—Results from the POEM3 Conference, 2016*. New York: Springer.

Tirosh, D., Tsamir, P., Levenson, E., Barkai, R., & Tabach, M. (2015, November). *Early childhood teachers' self-efficacy and knowledge for defining, drawing, and continuing repeating patterns*. Paper presented at the 21st MAVI (Mathematical Views) Conference, Italy.

Tirosh, D., Tsamir, P., Levenson, E., Tabach, M., & Barkai, R. (2011). *Prospective and practicing early childhood teachers' mathematics knowledge and self-efficacy: Identifying two- and three-dimensional figures*. 17th MAVI (Mathematical Views) Conference. Bochum, Germany. http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Roesken/maviarticles.html. Accessed May 7, 2012.

Tirosh, D., Tsamir, P., Levenson, E., Tabach, M., & Barkai, R. (2012). Early childhood teachers' knowledge and self-efficacy regarding counting and enumerating tasks. *POEM (Perspectives on Early Mathematics) Conference,* Germany. http://cermat.org/poem2012/main/proceedings_files/Tirosh-POEM2012.pdf. Accessed August 19, 2012.

Tirosh, D., Tsamir, P., Levenson, E., Tabach, M., & Barkai, R. (2014, September). *Early childhood teachers' self-efficacy and knowledge for defining and identifying triangles and circles*. Paper presented at the 20th MAVI (Mathematics Views) Conference, Sweden.

Tsamir, P., Tirosh, D., Barkai, R., Levenson, E., & Tabach, M. (2015). Which continuation is appropriate? Kindergarten children's knowledge of repeating patterns. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Annual Conference for the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 249–256). Hobart, Australia: PME.

Waters, J. (2004). Mathematical patterning in early childhood settings. In I. Putt & M. McLean (Eds.), *Mathematics education for the third millennium* (pp. 565–572). Townsville: Mathematics Education Research Group of Australia.

Wheatley, K. (2002). The potential benefits of teacher efficacy doubts for educational reform. *Teaching and Teacher Education, 18*(1), 5–22.

Part VI
Conclusions

Chapter 16
Early Childhood Mathematics Education: Reflections and Moving Forward

Iliada Elia, Joanne Mulligan, Ann Anderson, Anna Baccaglini-Frank and Christiane Benz

Abstract This book brings together creative and insightful current research studies on teaching and learning mathematics in early childhood by scholars from different parts of the world. In this chapter we reflect on this work and discuss the major insights, conclusions, implications and future research directions for early childhood mathematics education. We focus on each of the five key themes of the book: pattern and structure, number sense, embodied action and context, technology and early childhood educators' professional issues and education.

Keywords Early childhood · Mathematics education · Reflections Implications · Future research

I. Elia (✉)
Department of Education, University of Cyprus, P.O. Box. 20537,
1678 Nicosia, Cyprus
e-mail: elia.iliada@ucy.ac.cy

J. Mulligan
Department of Educational Studies, Faculty of Human Sciences,
Macquarie University, North Ryde, Sydney 2109, Australia
e-mail: joanne.mulligan@mq.edu.au

A. Anderson
Department of Curriculum and Pedagogy, University of British Columbia,
2125 Main Mall, Vancouver, BC V6T 1Z4, Canada
e-mail: ann.anderson@ubc.ca

A. Baccaglini-Frank
Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo, 5,
53127 Pisa, Italy
e-mail: anna.baccaglinifrank@unipi.it

C. Benz
Department of Mathematics and Computer Science, University of Education Karlsruhe,
Bismarckstr. 10, 76133 Karlsruhe, Germany
e-mail: benz@ph-karlsruhe.de

I. Elia et al. (eds.), *Contemporary Research and Perspectives on Early Childhood Mathematics Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-73432-3_16

The early childhood period (up to 7 years of age) is probably the most crucial time for developing children's capacities and dispositions for learning. Consequently, high-quality early childhood education is essential in any educational system and research in this field can have a strong educational and social impact. It is well documented that children are capable of, and should engage in mathematics learning in the earliest years of their life (Sarama and Clements 2009), thus mathematics education should be a key component of high-quality early childhood education. These considerations have induced a growing impetus of research in early childhood mathematics internationally over the last years.

The work reported in this book records the advancement in international research and perspectives in early childhood mathematics education based on the work of TSG 1 at ICME-13, aiming to provide further and new insights on how to enhance children's mathematics learning and development and help support and improve pedagogy in early childhood mathematics.

From our perspective the book addresses significant and timely issues in the field of early childhood mathematics education which correspond to five key themes. In particular, the book starts with three chapters focusing on the meaning and importance of developing mathematical pattern and structure and how it can be promoted in early childhood (Theme 1). The next three chapters address the development of young learners' number sense competences in relation to various contributing factors, including picture book reading, the quality of early mathematics classroom practice and a measurement-oriented curriculum (Theme 2). The study of the vital roles of embodied action and mathematics-in-context within early childhood mathematics education, and particularly in geometry learning, is the central theme of the three chapters that follow (Theme 3). In the next three chapters the impact of, and opportunities provided by technology in early mathematics learning and teaching are examined and discussed (Theme 4). The following two chapters consider the critical role of early childhood educators, their roles and aspirations and the importance of professional development, as well as the challenges they encounter in engaging children in mathematics, and in light of new curricula, building effective environments for mathematics learning (Theme 5).

In what follows, a summary of the major insights as well as our conclusions, reflections, implications and questions for further research are discussed, based on the research and related studies reported in the preceding chapters.

16.1 Theme 1 Pattern and Structure

The studies on pattern and structure and patterning abilities were focused on both pre-school (the German kindergarten) and the early years of formal schooling. In the tradition of much research on early assessment of mathematical development these studies included both the assessment of individual's abilities and strategies and the impact of intervention programs situated in authentic classroom settings. An important outcome of the studies highlighting patterning and structural relationships is that the assessment tasks and forms of analysis, and the intervention tasks can be accessed, interpreted and adapted in the regular classroom. In most cases this could be achieved without a research team or specialist researcher conditions.

The detailed sequential account of the development of patterning strategies with pre-schoolers by Lüken (2018, this book) extends the research field by providing a new analysis of the very early stages of patterning concepts (Papic et al. 2011). Although Lüken's study is limited to six cases it does provide insight into the importance of tracking children's learning in explicit ways and providing fine-grained analysis of such learning. The results of the classroom intervention with first graders in the study by Lüken and Kampmann (2018, this book) may describe some limitations but again this is promising evidence that can be traced to the intervention strategies. The assessment and teaching strategies are at least provided with clarity and sufficient detail to be embedded in more effectively controlled studies with larger and more diverse samples of children. The impact on mathematics learning could then be ascertained through further evaluation studies.

Mulligan and Mitchelmore (2018, this book) describe the important interrelationships between structural groupings common to The Pattern and Structure Mathematics Awareness Program (PASMAP) approach: sequences, structured counting, shape and alignment, equal spacing and partitioning. An explicit sequence of pedagogical strategies scaffold these interrelationships with a view to promoting visualization, abstraction and emergent generalization. Further recommendations from this study and also from Lüken's and Lüken and Kampmann's studies might also provide guidance about how teachers can scaffold children's learning to make connections within and between concepts. At international level mathematics education curricula often silo concepts and processes which encourages teachers to disregard interrelationships.

Although there has been increased attention on the influence of teacher mathematical content knowledge and pedagogical knowledge, the research reported in this theme was limited to a focus on the child. What might have provided a broader view of the research context would be complementary analyses of the teachers' knowledge about patterns and structural relationships in mathematics. A structural approach to teaching mathematics may require teachers to first develop an Awareness of Mathematical Pattern and Structure (AMPS) in their own mathematical thinking. In the pre-school setting the inclusion of patterning in children's play and structured activities should not be assumed. The effective implementation of patterning tasks such as those developed by the Lüken studies points the need for teachers to gain insight into their own knowledge of patterning as well as their practice being informed by current research.

16.2 Theme 2 Number Sense

Number and arithmetic operations are arguably of primary importance for children's mathematics learning and development (Torbeyns et al. 2015; Verschaffel et al. 2007). Furthermore, mathematical development in this domain may be one of the most extensively researched fields in mathematics education, especially in the early primary grades and also in the pre-school years (4–6 years of age).

The two chapters related to number development turn attention to a new construct, the Spontaneous Focusing On Numerosity (SFON) showing that although numerical abilities are still in the focus of research concerning number sense, there is also an increasing interest into the dispositions for numerical competence. In the chapter by Rathé et al. (2018) a research review about SFON traces the development of this construct to the Finnish researcher Hannula-Sormunen (see Hannula-Sormunen et al. 2015). SFON refers to children's spontaneous (i.e., self-initiated) focusing of attention on the number of a set of items or incidents and using this numerosity in their actions. An important finding of research on SFON, which was mostly conducted in European and Western countries, is that young children's SFON tendency could be identified as a predictor of later mathematics achievement in primary school.

Rathé et al. report in their chapter that the research findings on the relation between SFON in experimental tasks and everyday situations for kindergarten children are inconsistent and contradictory. Although the chapter by Rathé et al. gives new insights into the relation between SFON in experimental tasks and everyday activities, it should be taken into consideration that some data revealed only partial empirical evidence for a possible association.

The study of Bojorque et al. (2018, this book) also provides new insights into the relation of SFON and children-related or classroom-related factors. In particular, children's early numerical abilities measured by a standardized numerical test in Ecuador is a predictive factor of children's SFON development, whereas the quality of early mathematics education, as measured by a standardized instrument, did not contribute to children's SFON development. Both chapters highlight the difficulty of measuring and analyzing children's mathematics competences in early childhood education in daily-life situations as they are often more spontaneous than in school. This will be an ongoing challenge for research in mathematics education.

The chapter by Cheeseman et al. (2018) shows that measurement could be the focus of research concerning number sense, rather than explicit numerical abilities. Here an approach of the Russian tradition of Davydov (1975), adopted earlier in the Measure Up project in Hawaii (Dougherty and Zilliox 2003), was enacted for a design research project in Victoria, Australia. In this project children started formal schooling at a school with a Reggio Emilia and socio-cultural approach to learning. The children did not start with a typical number-focused curriculum but with a measurement-focused curriculum which includes numbers. The case studies in this chapter revealed that when young children measure, they use numbers and can acquire number competencies. The qualitative analyses also revealed the richness of other learning possibilities in the project.

Here again a methodological challenge is raised about the use of reliable and valid instruments in early childhood education for analyzing educational and psychological aspects of mathematics learning. By using standardized assessment instruments often the children's competencies are only partly surveyed. Therefore especially in early childhood education, where a great amount of learning takes place in informal situations, even the well-explored domain of number sense remains an area for in-depth and explicit empirical research.

In this book only a few new aspects of research concerning number sense are addressed in the three chapters. But it is obvious that the development of number sense is influenced by various factors and connected to many other mathematical domains. Therefore there are chapters in other parts of the book, i.e., the themes of pattern and structure, the role of technology and embodied actions and context, which are connected to the research on number sense. A more holistic view of the contributions of these studies can contribute to enhancing a more coherent body of knowledge in this domain.

16.3 Theme 3 Embodied Action and Context

Each of the three chapters in Theme 3 (Anderson and Anderson 2018; Elia 2018; Thom 2018) offers important insights into the roles that embodied action and context play within early childhood mathematics education, and more specifically in geometry. As Elia's study readily indicates, hand and finger gestures do more than complement the kindergarten children's verbal descriptions, serving as a means by which children develop and communicate geometrical thinking under varied conditions. In turn, Thom's grade 1 children similarly build their understandings, in particular their spatial-geometric reasoning with two-dimensional and three-dimensional, from the hand and body movements they invoke when explaining and conjecturing about "what they see" in a photograph. Likewise, for Anderson and Anderson's families, the parent-child dyadic interactions (non-verbal and verbal) point to children enacting mathematics-in-context, including geometry, space and measurement, prior to formal school. In all three studies, it is evident that these young children's experiences of geometry are multimodal, are embedded within a vast array of activities and settings, are connected with other semiotic resources, and are inherently creative sense making. Likewise, while there is evidence in all three studies that acting-thinking-talking are tightly interconnected for many of these 4–7 year olds, we are also reminded that this is not always the case. For example, a child's embodied measurement was visible in the Water Sprinkler activity and was accompanied by minimal talk (Anderson and Anderson 2018). In Elia's (2018) study the children's 'silent' gestures appear to be interpersonally synchronized with their peers' talk. Both these examples remind us that children engaging with mathematics-in-context or embodied mathematics (acting-thinking) without speaking deserve our closer attention. And yet, as Thom indicated, not only is geometry underrepresented in most early years curriculum, the limited ways in which we invite children to engage with geometrical, spatial and measurement concepts in many classrooms undervalues the embodied, gestural, in-context nature of young children's engagement with mathematics.

What insights, then, might these case studies, both through their design and their findings, provide early childhood educators? First and foremost, in all three studies, young children's geometric and spatial reasoning comes into focus as they share their thinking with others. Indeed, while the contexts and concepts varied across the three studies, each activity provided space, time, and we would argue, motivation for children to overtly share their ideas. In turn, each child in these studies, rather than withdrawing from what some might seem challenging mathematics, is engaged in moment-to-moment in-context action. In this process the individual is thereby making sense of geometry (mathematics) through body and mind. Furthermore, each child enacted the mathematics using gestures and body movement voluntarily, e.g., pointing to shapes and where to place them, or moving them within a composite figure. Other pertinent examples were the children's constructing of an imagined cylindrical shape through hand and arm movements associated with a circle in a photo or positioning one's body—raised knee or bended torso—to measure changing water levels from a sprinkler, without being explicitly asked to do so. On one level, then, since all three studies report on data gathered in natural settings, the ecological validity of the actual activity descriptions suggest that literal translation of these 'research' tasks into early years' settings may prove unproblematic. That said, keeping in mind the nuanced nature of such activities (Anderson and Anderson 2018), what such case study work provides is not necessarily contexts or activities to be emulated but compelling evidence of children's funds of knowledge and ways of knowing (James et al. 1998) that we, as educators, need to leverage as we plan and carry out mathematics-related activities in our pre-school and primary classrooms.

While these authors (Anderson and Anderson 2018; Elia 2018; Thom 2018) outline important directions for further research based on their individual study, when we consider the three chapters together, several research directions are strengthened while other avenues for future research also become apparent. For instance, we see evidence that the adults (e.g., teacher, interviewer, parent) and peers (e.g., same age classmates, older siblings) are implicated in the ways in which individual children's gestures and bodies bring forth the mathematics (geometry). Such convergence adds strength to Elia's call for further research into the "others" role(s) in provoking, sustaining or diminishing children's embodied mathematical experiences. For example, as Thom suggests, how might the nature of the adult's initial prompt (e.g. "what is this?" versus "what do you see?") or the questions that follow support young children's generative responses and/or provide young children opportunities to explore possibilities in ways similar to those illustrated in these studies? Whether due to coincidence or implicated in classrooms where children are often seated when engaging in mathematics, most of the gestures and movement in the three studies were limited to finger, hand and arm movements (upper body). Further research seems warranted to explore what role the learning space and positioning of children plays with respect to their embodied ways of knowing. What occasions promote whole body action, lower body gestures, or gross motor movement when young children are trying to understand geometrical ideas? What might young children's facial and head gestures, as well as gaze, tell us about their mathematical thinking? Although not specified in the three studies, European and Euro-Canadian cultures seem prominent in this field of research. Thus as we continue to explore the affordances of embodied action and context in relation to mathematical teaching and learning in the early years, it is vital that we do so within contexts with children and teachers/parents from diverse cultural, linguistic, socio-economic backgrounds, inclusive and respectful of both indigenous and immigrant populations. Finally, while these three studies and those in the remainder of this book situate their research in early childhood mathematics, most do so with children aged 4–7 years. However, the study of embodied action and mathematics-in-context seems to offer ways of engaging with and learning about children's mathematics that more language-dependent analysis (e.g., Sfard 2008) does not, and thus seems particularly suited to future research with toddlers and younger (nonverbal) children (ages 1–3 years).

16.4 Theme 4 Technology

Theme 4 discusses the important issue of using technology in early childhood mathematics education, and, specifically addresses issues regarding its integration into mathematics teaching and learning that may take place both within schools and at home. First of all, in tackling this complex issue, we remind the reader of the many different types of digital technology for early mathematics education that have developed quickly thanks to touchscreen tablet-based applications, now available in many pre-schools and schools and relatively easy to use (in comparison with desktop software). Indeed the contributions to Theme 4 mostly focused on touch-screen tablet-based applications.

Integrating this kind of technology into the teaching and learning of mathematics can be rather straightforward if the applications replicate physical manipulatives (e.g., Cuisenaire rods, geoboards or tens-charts) and are designed with high developmental and curricular fidelity. However, as suggested by Sinclair (2018, this book), there are also applications that depart significantly from physical technologies (e.g., paper-and-pencil or manipulatives), and which may present significant challenges for teachers. In this respect, we can look at what has happened over the last thirty years to Dynamic Geometry Software (DGS). DGS has gained such widespread approval in the mathematics education community, however its integration into mathematics curricula around the world has taken many years and is not yet completely achieved. Moreover, most research has been carried out at the secondary level, focusing especially on teacher integration, task design and assessment (Sinclair and Yerushalmy 2016), while more research, some of which is provided in this book (Sinclair 2018; Fletcher and Ginsburg 2016; Sinclair and Baccaglini-Frank 2016), is needed at the primary and pre-school levels. Our intention is that the results of the research presented in this book can be used to guide the professional development of early childhood and primary school teachers who wish to use DGS in their classrooms. If it has taken this long for DGS to become integrated into mathematics curricula, we imagine that integrating new applications that share with DGS the characteristic of departing significantly from physical technologies, such as TouchCounts (see Sinclair 2018; Baccaglini-Frank 2018, this book) will be a long and non-linear process. Possibly, new forms of professional development could aid the process, helping educators identify and implement effective uses of such new applications, especially when there are no pre-designed tasks introduced by the software and the interactions afforded are not highly constrained.

The chapters discussed in Theme 4 also highlight the fact that even if the software in focus is not extremely distant from physical manipulatives or the paper-and-pencil environment (at least not to the extent of DGS or TouchCounts) there are still important issues that need to be further investigated. A first issue is the role of the educator (a teacher or parent) during (or in between) the child's interactions with the software. Indeed, analyses of the student-software-teacher interactions in Baccaglini-Frank's chapter (2018) shed light onto how the educator's short-term goal of helping the children experience success, and her narrow-sighted view of how to obtain this in the domain of numbers, actually limited the development of numerical abilities for many children in the study.

Also Ginsburg et al. (2018, this book) highlight the importance of parents being involved, as educators, in understanding and promoting children's mathematical thinking and learning in the context of Interactive Mathematics Storybooks (IMS), stressing the need for further research to explore the interactions between the children, the educators involved, and the IMS used. Indeed, for all these types of technology designed for early childhood mathematics education it makes sense to ask: What might effective forms of professional development be if the goal is to promote proper integration of such technology into early childhood mathematics education?

A second issue is the possibility offered today by more and more software that exposes young children to advanced mathematics early on. This issue is addressed both by Fletcher and Ginsburg (2016) and by Sinclair (2018). Indeed, Fletcher and Ginsburg find that through appropriate technology young children can learn more about symmetry than what they acquire naturally and what is currently taught in US schools. They conclude that teaching symmetry earlier than designated by the US's Common Core State Standards for Mathematics (NGACBP and CCSSO 2010) can provide a unique opportunity to utilize and build upon the skills and interests that young children bring to the classroom. Analogously, Sinclair reflects upon the ease of children's earlier exposure to advanced mathematical concepts through new multi-touch tablet-based applications. For example, interacting with TouchCounts allows children to encounter large numbers, also in symbolic form; moreover this application puts an emphasis on ordinality. This kind of interaction with numbers is not typical in classrooms where the meaning of number is usually associated with cardinal quantities represented by physical manipulatives such as tens' charts and Dienes' blocks, and the numbers encountered are typically below 20. Therefore, as Sinclair states, there is a challenging choice to be made, both by teachers and researchers, about the extent to which new digital technologies—those that significantly change mathematics—can and should be integrated.

We hope that the issues and perspectives introduced in this theme will help to develop positive and constructive, though critical approaches to the issue of using technology to promote mathematical learning in early childhood education.

16.5 Theme 5 Early Childhood Educators' Professional Issues and Education

Theme 5 focuses on three distinct but interrelated areas of research impacting on change in practice for early childhood educators influenced by several conditions at a macro, meso and micro level. First, new curricula and frameworks for early mathematics teaching and learning provide varying expectations about the scope and depth of the pre-school environment—informal learning (such as through play), content to learn and activities to experience. Second, early childhood educators' mathematical knowledge, pedagogical knowledge, understandings, beliefs, and perceptions influence how they enact these expectations. The importance of professional learning and the assessment of pre-service teacher mathematical competencies are discussed. Third, educational programs, resources and activities implemented in the pre-school environment impact on the mathematical opportunities children engage in at the micro level. The discussion at these three levels consider both the child's engagement with mathematics and the impact of the professional on this learning based on relevant contributions in TSG 1.

At a macro-level, the introduction of pre-school mathematics curricula or frameworks in several countries raises fundamental questions about the views of the early childhood educator. Cooke and Bruns (2018, this book) highlight the tensions raised by several papers about new curricula and frameworks that may impose mathematical content rather than allowing the child to develop mathematical concepts through play. Some research supports new directions on curriculum development for early childhood mathematics internationally. What could follow from these studies is further articulation of how the elements of each curriculum or framework are interpreted, implemented and evaluated within pre-school mathematics learning environments. Longitudinal research could consider how these elements may direct broader change over time such as how the curriculum is enacted, the need for increased mathematical content and pedagogical knowledge of the educator, and the need for tailored and increased professional development. The impact these policy documents and changes have on professional practice and the views of early childhood mathematical learning is another important issue for further research.

At the meso-level the focus is the early childhood educators' competence. All of the relevant studies in TSG 1 show consensus about the importance of fundamental understanding of mathematics by the educator as the basis for high-quality early mathematics education. However, different studies used different conceptualizations and instruments to measure the mathematical competence of educators, including pre-service teachers. Several papers used the Shulman framework (Bruns et al. 2016; Dunekacke et al. 2016; Jenßen et al. 2016; Tsamir et al. 2018). These studies found that pre-school educators' competence is related to a range of complex factors that incorporate mathematical content and pedagogical content knowledge, general pedagogical content knowledge and affective-motivational aspects as well as their perception of mathematical situations. Cooke (2016) identifies and measures similar facets of educators' math-related competence, indicating that these facets are subject to sustained investigation by different groups of researchers focused on improving teacher competence and confidence.

However, Cooke and Bruns (2018) point to a lack of consensus among the findings of many studies about these competency facets. This was seen in TSG 1 presentations: For example, a number of studies found no significant correlations between mathematics anxiety and mathematical content knowledge (Cooke 2016), mathematics pedagogical content knowledge or perceptions of mathematics situations (Dunekacke et al. 2016), whereas Tsamir et al. (2018, this book) found that self-efficacy for identifying repeating patterns and errors in repeating patterns was well-matched to the pre-school teacher understandings.

In particular, the chapter by Tsamir and colleagues draws attention to the importance of the educators' ability to identify and continue repeating patterns. Their study found that pre-school teachers were able to identify drawings which represent repeating patterns and identify the errors which preclude a drawing from actually being a repeating pattern. However, identifying appropriate continuations proved more difficult. The chapter provides an analysis of early-childhood educators' responses to conceptually-oriented tasks. This study could serve as an example of the need to develop rigorous assessment tasks that can identify and remedy misconceptions of early childhood educators from the outset.

While the contributions in TSG 1 on teacher-related issues incorporated a wide range of studies, and many reported on large scale projects such as KomMa, in light of the above, much more longitudinal and systematic research is needed as well as studies that provide internationally comparative insights. In line with this, Cooke and Bruns (2018) conclude that there is a need for further studies that use different measures and more representative samples, and studies that examine the relationship between early childhood educators' competences and the quality of the experiences they create to engage children in mathematics.

At the micro-level the focus is on the child as the centre of mathematics learning, although the research reported in the chapter by Cooke and Bruns (2018) address to some extent the effectiveness of program intervention and teacher knowledge. Several intervention programs are outlined, yet questions are raised about the practicality of their implementation processes. Fritz-Stratmann et al. (2016) demonstrate the need for high level professional training for a program to be effective. Another approach is to focus intervention programs on one aspect of mathematics such as number learning (in this case, arithmetic) that Wang et al. (2016) found to increase the likely effectiveness of their program. There are few other examples to draw upon within Theme 5, but there are other relevant examples presented through other chapters in this book. What could be gleaned from the discussion on intervention programs is the importance of designing and implementing carefully designed activities that are appropriate to the wide and developing needs of the children. However, it is very difficult to match these requirements to every learning context such that the creative mathematics learning of the students is not stifled.

Clements et al. (2013) reiterated the importance of the use of research-based programs, stating that developmentally-sequenced activities can enable teachers to "become aware of, assess, and remediate" (p. 10). Targeting children early may improve learning in both their first years of formal schooling and their later mathematical achievement (Watts et al. 2014), and development-oriented interventions can help children develop mathematical understandings (Fritz et al. 2013). However, there needs to be careful monitoring to ensure that the impact of the interventions does not fade away (Sarama and Clements 2015).

16.6 Further Perspectives and Concluding Comments

The international scope of the contributions in this book highlights the need to consider the above discussion by taking into account the diversity of early childhood mathematics education across different countries. All over the world pre-school educational systems and traditions in different countries vary and children begin primary school at different ages. Also the varying philosophies of how learning in early childhood education is supported should be acknowledged. Moreover, analogous variations apply for the education of prospective early childhood educators primarily regarding mathematics pedagogy. The very different conditions for early childhood mathematics education, from the perspective of the children or the educators all over the world, and the lack of international studies at this level highlight the need to design, implement and evaluate cross-cultural and comparative studies in this field.

The themes addressed in the chapters of this book reflect an impetus to develop broader and more integrated views of early mathematics learning rather than a traditional focus on counting and number concepts. Thus the role of early childhood mathematics education, as it is approached in this book, can be regarded from a new broader perspective, for example, as an opportunity to reflect on the contribution of this field to a recent educational direction, that is, early STEM (Science, Technology, Engineering and Mathematics) learning (McClure et al. 2017). Although this latter new approach to early learning is closely related to how young children explore and make sense of the world, very little research has been carried out on how concepts in STEM learning in early childhood are associated with young children's mathematical learning and development. In this book, research reported in many chapters concentrates on types and processes of mathematics which involve reasoning, structure, interconnections between topics, integration of technology, multimodal and embodied aspects of communication, thinking and learning, maths-in-context and in everyday activities. These features of early mathematics learning and teaching have a significant place in STEM topics and may contribute to a deeper understanding of relevant concepts, foster problem solving, and enhance the understanding of the application of concepts in real life. Thus, we hope that, even in subtle ways, this book may inspire reflective insights and new ideas for further investigation about how early childhood mathematics education may contribute to interdisciplinary research important for early and later STEM learning and development. Furthermore, investigating early childhood educators' roles and their knowledge for teaching mathematics, as well as the impact of the curriculum, from such an integrated approach could fuel new and critically important research directions.

Overall, we hope that the diverse international collection of studies in this book will provide powerful foundations for future research, professional learning and curriculum development in early childhood mathematics education.

References

Anderson, A., & Anderson, J. (2018). Math-in-context: The types of math preschoolers 'do' at home. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Baccaglini-Frank, A. (2018). What schemes do preschoolers' develop when using multi-touch applications to foster number sense (and why)? In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Bojorque, G., Torbeyns, J., Hannula-Sormunen, M. M., Van Nijlen D., & Verschaffel, L. (2018). Ecuadorian kindergartners' spontaneous focusing on numerosity development: Contribution of numerical abilities and quality of mathematics education. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Bruns, J., Eichen, L., & Blömeke, S. (2016, July). *Preschool teachers' mathematics-related competencies*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Cheeseman, J., Benz, C., & Pullen, Y. (2018). Number sense: The impact of a measurement-focused program on young children's number learning. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Clements, D. H., Baroody, A. J., & Sarama, J. (2013). *Background research on early mathematics.* National Governor's Association, Center Project on Early Mathematics. https://www.nga.org/files/live/sites/NGA/files/pdf/2013/1311SEME-Background.pdf. Accessed December 31, 2016.

Cooke, A. (2016, July). *Pre-service teacher relationships with mathematics—creative? Anxious? Competent?* Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Cooke, A., & Bruns, J. (2018). Early childhood educators' issues and perspectives in mathematics education. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Davydov, V. (1975). Logical and psychological problems of elementary mathematics as an academic subject. In L. P. Steffe (Ed.), *Children's capacity for learning mathematics: Soviet studies in the psychology of learning and teaching mathematics* (Vol. VII, pp. 55–107). Chicago: University of Chicago.

Dougherty, B. J., & Zilliox, J. (2003). Voyaging from theory to practice in learning: Measure up. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education (PME)*, (Vol. 1, pp. 17–22). Honolulu: PME Committee.

Dunekacke, S., Eilerts, K., & Jenßen, L. (2016, July). *Relations of affective, cognitive and situations-specific facets of preschool teachers' professional competence*. Poster presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Elia, I. (2018). Observing the use of gestures in young children's geometric thinking. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Fletcher, N., & Ginsburg, H. (2016, July). *Teaching symmetry in the early childhood classroom: Using software to facilitate discourse and promote understanding of symmetric transformations*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Fritz, A., Ehlert, A., & Balzer, L. (2013). Development of mathematical concepts as basis for an elaborated mathematical understanding. *South African Journal for Childhood Education, 3*(1), 38–67.

Fritz-Stratmann, A., Ehlert, A., & Herzog, M. (2016, July). *From thought to reality—implementation of an in-school mathematics training in South Africa*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Ginsburg, H. P., Uscianowski, C., & Almeda Ma, V. (2018). Interactive mathematics storybooks and their friends. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Hannula-Sormunen, M. M., Lehtinen, E., & Räsänen, P. (2015). Preschool children's spontaneous focusing on numerosity, subitizing, and counting skills as predictors of their mathematical performance seven years later at school. *Mathematical Thinking and Learning, 17,* 155–177.

James, A., Jenks, C., & Prout, A. (1998). *Theorizing childhood*. Cambridge: Polity Press.

Jenßen, L., Eilerts, K., Koinzer, T., Schmude, C., & Blömeke, S. (2016, July). *Development and validation of a test battery assessing preschool teachers' professional competence in the field of mathematics*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Lüken, M. (2018). Repeating pattern competencies in three- to five-year old kindergartners: A closer look at strategies. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Lüken, M. & Kampmann, R. (2018). The influence of fostering children's patterning abilities on their arithmetic skills in Grade 1. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

McClure, E. R., Guernsey, L., Clements, D. H., Bales, S. N., Nichols, J., Kendall-Taylor, N., et al. (2017). *STEM starts early: Grounding science, technology, engineering, and math education in early childhood*. New York: The Joan Ganz Cooney Center at Sesame Workshop.

Mulligan, J. T., & Mitchelmore, M. C. (2018). Promoting early mathematical structural development through an integrated assessment and pedagogical program. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

National Governors Association Center for Best Practices & Council of Chief State School Officers (NGACBP and CCSSO). (2010). Common core state standards for mathematics. http://corestandards.org/assets/CCSSI_Math%20Standards.pdf. Accessed July 16, 2017.

Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42,* 237–268.

Rathé, S., Torbeyns, J., de Smedt, B., Hannula-Sormunen, M. M., & Verschaffel, L. (2018). Kindergartners' spontaneous focus on number during picture book reading. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge.

Sarama, J., & Clements, D. H. (2015). Scaling up early mathematics interventions: Transitioning with trajectories and technologies. In R. Perry, A. MacDonald, & A. Gervasoni (Eds.), *Mathematics and transition to school* (pp. 153–169). Singapore: Springer.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge, England: Cambridge University Press.

Sinclair, N. (2018). Time, immersion and articulation: Digital technology for early childhood mathematics. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Sinclair, N., & Baccaglini-Frank, A. (2016). Digital technologies in the early primary school classroom. In L. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 662–686). New York, NY: Taylor Francis/Routledge.

Sinclair, N., & Yerushalmy, M. (2016). Digital technology in mathematics teaching and learning: A decade focused on theorising and teaching. In A. Gutiérrez, G. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education: The journey continues* (pp. 235–274). Rotterdam, The Netherlands: Sense Publishers.

Thom, J. S. (2018). (Re)(con)figuring space: Three children's geometric reasonings. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Torbeyns, J., Gilmore, C., & Verschaffel, L. (2015). The acquisition of preschool mathematical abilities: Theoretical, methodological and educational considerations. *Mathematical Thinking and Learning, 17*(2–3), 99–115.

Tsamir, P., Tirosh, D., Levenson, E., & Barkai, R. (2018). Early childhood teachers' knowledge and self-efficacy for evaluating solutions to repeating pattern tasks. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (this volume).

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557–628). Greenwich, CT: Information Age.

Wang, A. H., Firmender, J. M., Power, J. R., & Byrnes, J. P. (2016). Understanding the program effectiveness of early mathematics interventions for prekindergarten and kindergarten environments: A meta-analytic review. *Early Education and Development, 27*(5), 692–713.

Watts, T. W., Duncan, G. J., Siegler, R. S., & Davis-Kean, P. E. (2014). What's past is prologue: Relations between early mathematics knowledge and high school achievement. *Educational Researcher, 43*(7), 352–360.