Chapter 9
Handling with Roll Motion

So far we have investigated the handling behavior of a vehicle under the assumption of negligible roll. Actually, we have not completely discarded roll angles, as they are absolutely necessary for evaluating, e.g., lateral load transfers. But we have not considered, for instance, the inertial effects of roll motion.

In this chapter, the roll motion is taken into account (Fig. 9.1). It is hard work, as the analysis becomes more involved [10]. However, it also sheds light onto one of the most controversial concepts in vehicle dynamics: the roll axis [1, 3, 5, 6, 11], in this book renamed no-roll axis. This concept has been already discussed in Sect. 3.10.9, but it will be considered again here.

We state from the very beginning what the outcome of our analysis will be: the roll axis, as that axis about which the vehicle rolls, does not exist. Or, in other words, the concept of an axis about which the vehicle rolls is meaningless (cf. [9, p. 115]). We understand it sounds harsh, but that is the way it is. There is no such thing as an axis about which the vehicle rolls, albeit the vehicle rolls indeed. A similar conclusion was obtained also in [7] and in [2, p. 400]. However, the no-roll axis, as defined here in Sect. 3.10.9, maintains its validity.

9.1 Vehicle Position and Orientation

Defining the position and orientation of a vehicle when roll is assumed to be zero is a simple matter. As shown in Fig. 3.3, the motion is two-dimensional and hence it suffices to know, with respect to a ground-fixed reference system, the two coordinates of the center of mass G and the yaw angle ψ.

Including roll (and, perhaps, also pitch) means having to deal with a full three-dimensional problem. Therefore, we must employ more sophisticated tools. Quite paradoxically, it turns out that it is easier to define unambiguously the *orientation* of the vehicle body, rather than the *position* of the vehicle. The reason is that the concept of "position of the vehicle" is not so clear anymore. As a matter of fact, roll causes point G to move sideways with respect to the wheels, but this movement does

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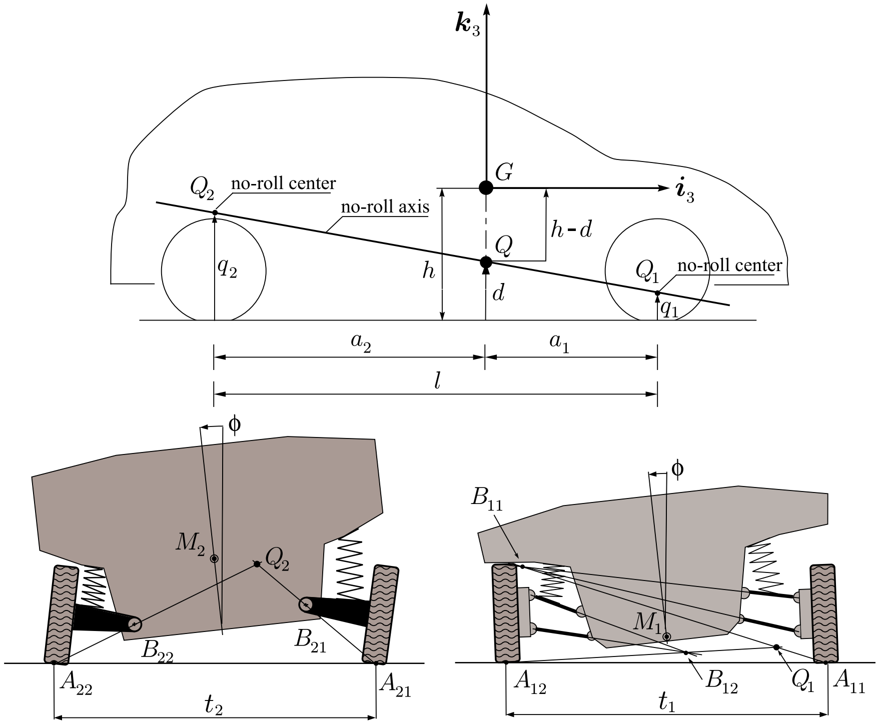

Fig. 9.1 Vehicle basic scheme including roll angle ϕ (lateral and front views)

not change the "position of the vehicle" directly. In other words, we pretend that the lateral velocity v of the vehicle does not contain any roll contribution. We will address this important aspect shortly. First, some other concepts need to be introduced.

9.2 Yaw, Pitch and Roll

Although everybody has an intuitive notion of roll, pitch and yaw of a vehicle, we need a more precise definition at this stage. The goal is to know the *orientation* of the vehicle body (assumed to be a rigid body) with respect to a ground-fixed reference system S_0. A typical approach is to give a *sequence* of three *elemental* rotations, that is rotations about the axes of a chain of coordinate systems.

The three elemental rotations must follow a definite order. In other words, the same rotations in a different order provide a different orientation. This aspect can be appreciated by a simple example. In Fig. 9.2a, a parallelepiped is rotated by 90° about the axis $\mathbf{i}$ and then by $-90°$ about the axis $\mathbf{j}$. In Fig. 9.2b, the same parallelepiped is

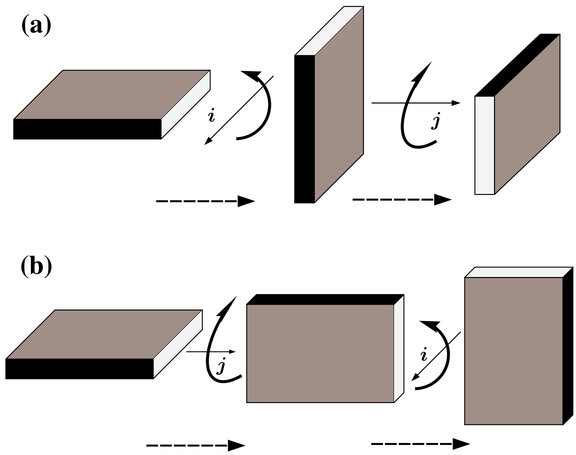

Fig. 9.2 Finite rotations are not commutative (i.e., their order is important)

subject to the same two rotations, but in reverse order. The final orientation is totally different, thus confirming that finite rotations are not commutative.[1]

Human beings are comfortable with two-dimensional rotations, and Euler was, perhaps, no exception when he invented the technique of three *elemental* rotations, often referred to as Euler angles. The basic idea is to generate a sequence of four Cartesian reference systems S_i, each one sharing one axis with the preceding system and another axis with the next one. Therefore, we can go from one system to the next by means of a two-dimensional rotation about their common axis.[2]

In vehicle dynamics it is convenient to use the following sequence of reference systems (Fig. 9.3)

$$(\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0) \xrightarrow[\mathbf{k}_0=\mathbf{k}_1]{\psi} (\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1) \xrightarrow[\mathbf{j}_1=\mathbf{j}_2]{\theta} (\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2) \xrightarrow[\mathbf{i}_2=\mathbf{i}_3]{\phi} (\mathbf{i}_3, \mathbf{j}_3, \mathbf{k}_3) \tag{9.1}$$

to go from the ground-fixed reference system S_0, with directions $(\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$, to the vehicle-fixed reference system S_3, with directions $(\mathbf{i}_3, \mathbf{j}_3, \mathbf{k}_3)$. This vehicle-fixed reference system has been already introduced in Fig. 1.4, although with a slightly different notation (no subscripts). When the vehicle is at rest, direction $\mathbf{k}_3 = \mathbf{k}$ is orthogonal to the road (hence directed like $\mathbf{k}_0$) and direction $\mathbf{i}_3 = \mathbf{i}$ is parallel to the road and pointing forward (hence like $\mathbf{i}_1$, Fig. 9.1).

During the vehicle motion, S_3 moves accordingly. At any instant of time, the key step is the definition of the auxiliary direction $\mathbf{j}_1 = \mathbf{j}_2$

$$\mathbf{j}_1 = \mathbf{j}_2 = \frac{\mathbf{k}_0 \times \mathbf{i}_3}{|\mathbf{k}_0 \times \mathbf{i}_3|} = \frac{\mathbf{k}_1 \times \mathbf{i}_2}{|\mathbf{k}_1 \times \mathbf{i}_2|} \tag{9.2}$$

[1]Rotation matrices are a tool to represent finite rotation. As well known, the product of matrices is not commutative, in general.

[2]More precisely, the axis must share the same direction. The origin can be different.

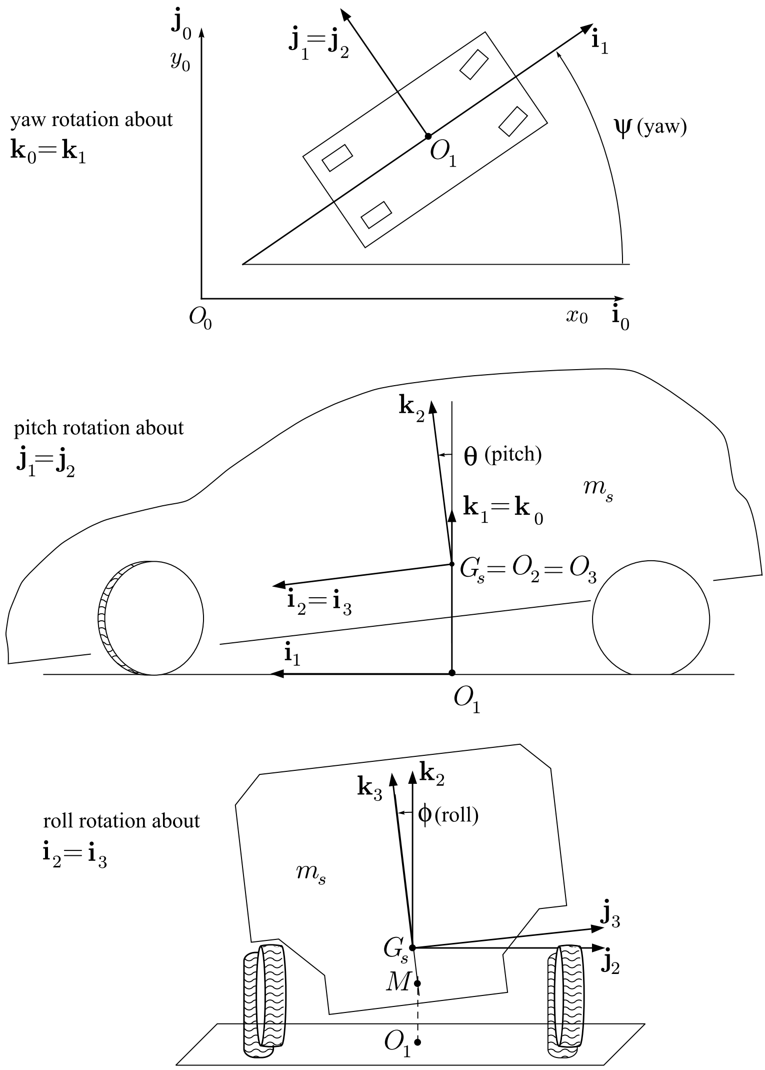

Fig. 9.3 Definition of yaw, pitch and roll

often called the *line of nodes*, which is *orthogonal* to both $\mathbf{k}_0 = \mathbf{k}_1$ and $\mathbf{i}_2 = \mathbf{i}_3$. This direction $\mathbf{j}_1 = \mathbf{j}_2$ is the link between the ground-fixed and the vehicle-fixed reference systems. This way, we have that we can go from S_0 to S_1 with an elemental rotation ψ about $\mathbf{k}_0 = \mathbf{k}_1$, and so on. Any two consecutive reference systems differ by a two-dimensional rotation, as shown in (9.1).

More precisely, as shown in Fig. 9.3, the first rotation ψ (yaw) is about the third axis $\mathbf{k}_0 = \mathbf{k}_1$, which S_0 and S_1 have in common, the second rotation θ (pitch) is about the second axis $\mathbf{j}_1 = \mathbf{j}_2$, shared by S_1 and S_2, and the third rotation ϕ (roll) is about the first common axis $\mathbf{i}_2 = \mathbf{i}_3$ of S_2 and S_3. This is why this sequence of elemental rotations is marked (3, 2, 1), or *yaw, pitch and roll*.[3] In vehicle dynamics, the pitch and roll angles are very small.

9.3 Angular Velocity

With this sequence of reference systems, the angular velocity of the vehicle body $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega} = \dot{\phi}\ \mathbf{i}_2(\psi, \theta) + \dot{\theta}\ \mathbf{j}_1(\psi) + \dot{\psi}\ \mathbf{k}_0 \tag{9.3}$$

This is a simple and intuitive equation, but it has the drawback that the three unit vectors are not mutually orthogonal (Fig. 9.3). Therefore, our goal is to obtain the following equation[4]

$$\boldsymbol{\Omega} = p\ \mathbf{i}_3 + q\ \mathbf{j}_3 + r\ \mathbf{k}_3 \tag{9.4}$$

where the vector $\boldsymbol{\Omega}$ is expressed in terms of its components in the vehicle-fixed reference system S_3.[5]

The expressions of p, q and r can be easily obtained by means of the rotation matrices

$$\begin{aligned}\begin{bmatrix} p \\ q \\ r \end{bmatrix} &= \mathbf{R}_1(\phi)\begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_1(\phi)\mathbf{R}_2(\theta)\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{R}_1(\phi)\mathbf{R}_2(\theta)\mathbf{R}_3(\psi)\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \\ &= \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_1(\phi)\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{R}_1(\phi)\mathbf{R}_2(\theta)\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}\end{aligned} \tag{9.5}$$

where, as well known, the rotation matrices for elemental rotations are as follows, for a generic angle α

– rotation around the first axis

$$\mathbf{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{9.6}$$

– rotation around the second axis

[3] Classical Euler angles use the sequence (3, 1, 3).

[4] In this chapter the symbol q is a component of $\boldsymbol{\Omega}$. Therefore, we use the symbol d for the height of the no-roll center Q (Fig. 9.1).

[5] The components p, q and r of $\boldsymbol{\Omega}$ cannot be given, in general, as time derivatives of an angle.

$$\mathbf{R}_2(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{9.7}$$

– rotation around the third axis

$$\mathbf{R}_3(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.8}$$

The final result is

$$\begin{aligned} p &= \dot\phi - \dot\psi\ \sin\theta \\ q &= \dot\theta\ \cos\phi + \dot\psi\ \sin\phi\cos\theta \\ r &= \dot\psi\ \cos\phi\cos\theta - \dot\theta\ \sin\phi \end{aligned} \tag{9.9}$$

which can be simplified in

$$\begin{aligned} p &\simeq \dot\phi - \dot\psi\,\theta \\ q &\simeq \dot\theta + \dot\psi\,\phi \\ r &\simeq \dot\psi \end{aligned} \tag{9.10}$$

because of the small values of pitch and roll. Therefore, the angular velocity of the vehicle body can be expressed as

$$\boldsymbol{\Omega} \simeq (\dot\phi - \dot\psi\,\theta)\,\mathbf{i}_3 + (\dot\theta + \dot\psi\,\phi)\,\mathbf{j}_3 + \dot\psi\,\mathbf{k}_3 \tag{9.11}$$

in the vehicle-fixed reference system.

Moreover, if there is no pitch, that is $\theta = \dot\theta = 0$, we have a further simplification

$$\begin{aligned} p &\simeq \dot\phi \\ q &\simeq \dot\psi\,\phi \\ r &\simeq \dot\psi \end{aligned} \tag{9.12}$$

A lot of work for getting such a simple result.

This definition of roll, pitch and yaw is quite general. It only needs the reasonable assumption that the vehicle body be considered as perfectly rigid. It is worth remarking that what matters in the definition of roll, pitch and yaw are only the *directions* of the axes of the four reference systems S_i. Their positions, that is the positions of their origins O_i, have no relevance at all.

It is useful to obtain the expressions of the unit vectors $(\mathbf{i}_3, \mathbf{j}_3, \mathbf{k}_3)$ in terms of $(\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1)$

$$\begin{aligned} \mathbf{i}_3 &= \cos(\theta)\,\mathbf{i}_1 - \sin(\theta)\,\mathbf{k}_1 \\ \mathbf{j}_3 &= \sin(\phi)[\sin(\theta)\,\mathbf{i}_1 + \cos(\theta)\,\mathbf{k}_1] + \cos(\phi)\,\mathbf{j}_1 \\ \mathbf{k}_3 &= \cos(\phi)[\sin(\theta)\,\mathbf{i}_1 + \cos(\theta)\,\mathbf{k}_1] - \sin(\phi)\,\mathbf{j}_1 \end{aligned} \tag{9.13}$$

which can be simplified into

$$
\begin{aligned}
\mathbf{i}_3 &\simeq \mathbf{i}_1 - \theta\,\mathbf{k}_1 \\
\mathbf{j}_3 &\simeq \mathbf{j}_1 + \phi\,\mathbf{k}_1 \\
\mathbf{k}_3 &\simeq \theta\,\mathbf{i}_1 - \phi\,\mathbf{j}_1 + \mathbf{k}_1
\end{aligned}
\tag{9.14}
$$

9.4 Angular Acceleration

The angular acceleration $\dot{\boldsymbol{\Omega}}$ is promptly obtained by differentiating (9.4) with respect to time

$$
\begin{aligned}
\dot{\boldsymbol{\Omega}} &= \dot{p}\,\mathbf{i}_3 + \dot{q}\,\mathbf{j}_3 + \dot{r}\,\mathbf{k}_3 + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \\
&= \dot{p}\,\mathbf{i}_3 + \dot{q}\,\mathbf{j}_3 + \dot{r}\,\mathbf{k}_3
\end{aligned}
\tag{9.15}
$$

where, according to (9.10)

$$
\begin{aligned}
\dot{p} &\simeq \ddot{\phi} - \ddot{\psi}\,\theta - \dot{\psi}\,\dot{\theta} \\
\dot{q} &\simeq \ddot{\theta} + \ddot{\psi}\,\phi + \dot{\psi}\,\dot{\phi} \\
\dot{r} &\simeq \ddot{\psi}
\end{aligned}
\tag{9.16}
$$

9.5 Vehicle Lateral Velocity

The vehicle lateral velocity v was introduced in (3.1) in the case of negligible roll motion. Now we need to extend that definition when the roll motion is taken into account. This task is not as simple as it may seem. Intuitively, we would like to obtain an expression of v independent of ϕ. Therefore, we are looking for a point which, broadly speaking, follows the vehicle motion, without being subject to roll. A point that is like G, except that it does not roll. More precisely, we are looking for the origin O_1 of the reference system S_1 in Fig. 9.3, that is a reference system which yaws, but does not pitch and roll.

For simplicity, we assume the tires are perfectly rigid in this chapter.

9.5.1 Track Invariant Points

Roll motion is part of vehicle dynamics. However, it is useful to start with a purely kinematic analysis to get an idea of the several effects of roll motion. This kinematic analysis should be seen as a primer for better investigating roll dynamics.

Figure 3.33 shows how to determine the no-roll centers Q_i for a swing arm suspension and a double wishbone suspension. The same method is applied in Fig. 3.35 to a MacPherson strut. In all these cases, the vehicle is in its reference configuration

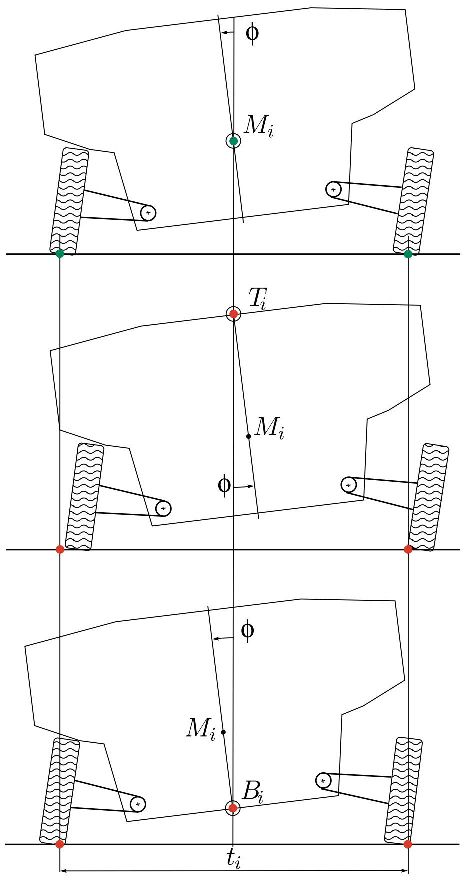

Fig. 9.4 Roll rotations about different points and comparison of the relative contact patch positions

(no roll). When the vehicle rolls, the no-roll centers Q_i migrate with respect to the vehicle body. They can be obtained, as shown in Fig. 9.1, using the same procedure of Fig. 3.33, i.e., as the intersection of the two lines passing through points A_{ij} and B_{ij}.

However, determining the current position of Q_i has little relevance in this context. Much more important are the following definitions.

We define point M_1 as the point *of the vehicle body* that coincides with Q_1 in the vehicle reference configuration (Fig. 9.1). The same idea, applied to the rear axle, leads to the definition of M_2. These points are called here *track invariant points*. Let us investigate their properties.

In Fig. 9.4, the vehicle body is rotated, in turn, by the same roll angle ϕ about three different points, namely M_i, T_i, and B_i. We see that in all cases the track length t_i is almost constant. However, in general, the two contact patches move sideways with respect to the point (see also [2, p. 400]). The only exception is with point M_i, which remains midway between the two contact patches (see also [4, p. 97]). This is the reason why it has been called *track invariant point*.

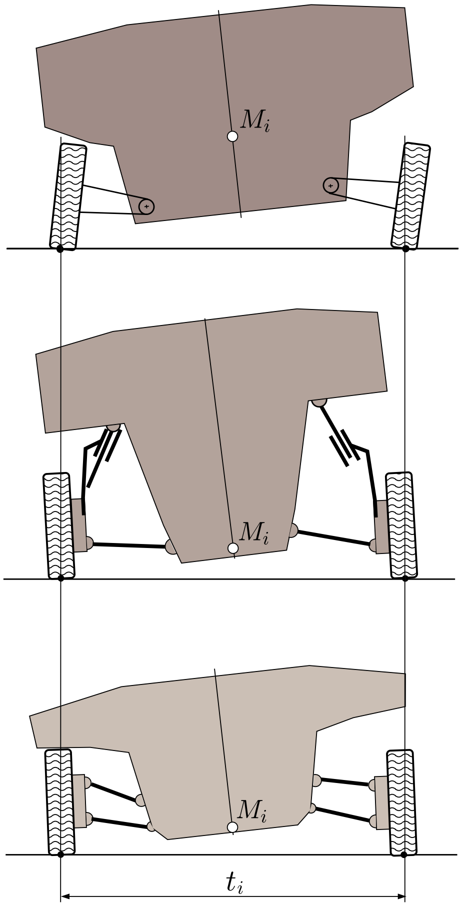

Fig. 9.5 Roll rotations about the track invariant point M_i for three different suspension layouts (top to bottom): swing arm, MacPherson strut, double wishbone

The property that a roll rotation about the track invariant point M_i does not affect the positions of the tire contact patches with respect to M_i itself holds true for any suspension type, as shown in Fig. 9.5.[6]

However, the vehicle does not care much about which point we applied the roll rotation. This is demonstrated in Fig. 9.6, where we superimposed the three vehicle rotations shown in Fig. 9.4. They are almost indistinguishable, suggesting that the notion of a roll axis about which the vehicle rolls is meaningless. For the vehicle, all points between, say, T_i and B_i are pretty much equivalent.

In general, in addition to roll, there may be some suspension jacking, which results in a vehicle vertical displacement z_i, as discussed in Sect. 3.10.10. Figure 9.7 shows the same axle with and without suspension jacking. The roll angle is the same. It is evident, particularly when comparing the two cases, as it is done in

[6]In Fig. 9.5 it is also quite interesting to note the camber variations due to pure roll in each type of suspension. This topic has been addressed in Sect. 3.10.3.

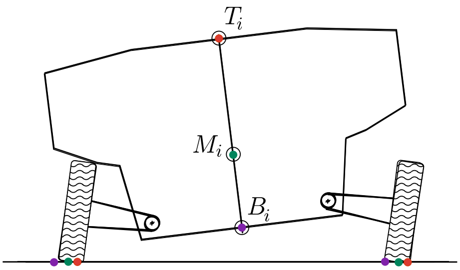

Fig. 9.6 Comparison of roll rotations about different points: they have almost the same effect on the vehicle

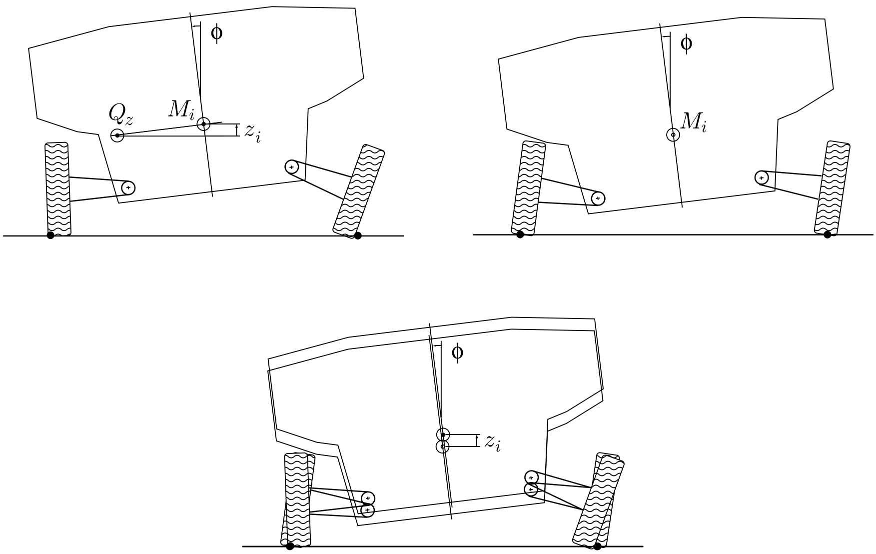

Fig. 9.7 Roll rotations with and without suspension jacking

Fig. 9.7-bottom, that the combination of roll and suspension jacking is like a rotation about a point Q_z.

We recall that suspension jacking occurs whenever the lateral forces exerted by the two tires of the same axle are not equal, which is always the case, indeed. However, it is a small effect that can be safely neglected, particularly in more sophisticated suspensions, like the double wishbone or the MacPherson strut.

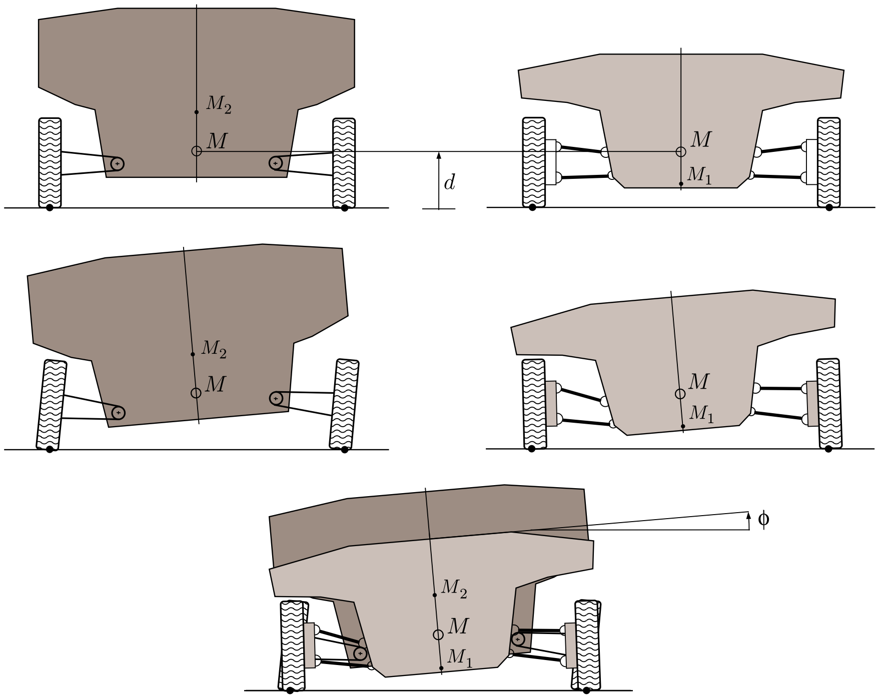

Fig. 9.8 Roll motion explained without the recourse to any roll axis, and definition of the vehicle invariant point M

9.5.2 Vehicle Invariant Point (VIP)

Now let us look at both axles together, that is at the vehicle as a whole, as done in Fig. 9.8. For simplicity, let us assume the front and rear tracks to be equal to each other, that is $t_1 = t_2$, and that they are not affected by roll (no suspension jacking).

Points M_1 and M_2 have, in general, different heights. Therefore, roll motion makes the front and rear tracks "slide" a little bit with respect to each other (Fig. 9.8). We remark that we know the *direction* $\mathbf{i}_3$ about which the vehicle (by definition) rolls, but we cannot say anything about an elusive *axis* about which the vehicle rolls.

We are looking for a point of the vehicle body that, regardless of the roll angle ϕ, remains most centered with respect to the four contact patches. Figure 9.8 suggests that the point that is less sensitive to roll is indeed a point M between M_1 and M_2.

Therefore, we define point M as the point *of the vehicle body* that, in the reference configuration, coincides with the no-roll center Q. We call M *vehicle invariant point* (VIP). Point O_1 is the point on the ground always below M, as shown in Fig. 9.9.

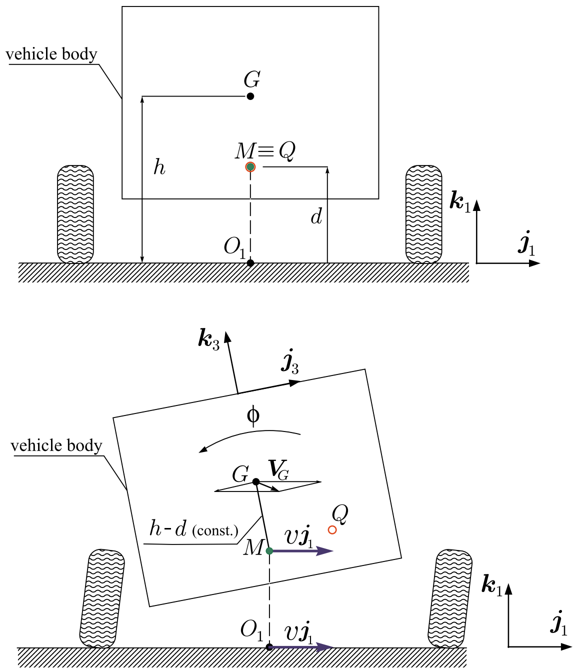

Fig. 9.9 Definition of the lateral velocity $v\,\mathbf{j}_1$ of the vehicle (front view)

The selection of point M as the best suited to represent the vehicle position purged by the roll motion, is reasonable (we believe), but nonetheless arbitrary.[7] However, this is what is commonly done in vehicle dynamics, although often without providing an explicit explanation. We repeat that point M, and hence also O_1, are basically in the middle of the vehicle, even when it rolls. This is the reason that makes them the best option to monitor the vehicle position.

9.5.3 Lateral Velocity and Acceleration

The vehicle velocity is, by definition, that of the *vehicle invariant point M*. Therefore, pretty much like in (3.1)

$$\mathbf{V}_M = u\ \mathbf{i}_1 + v\ \mathbf{j}_1 \tag{9.17}$$

where u is the forward velocity and v is the lateral velocity. We recall that we have assumed the tires to be rigid, and hence there is no roll motion due to tire deformation.

The vehicle acceleration is given by a formula identical to (3.24)

---

[7]The use of the center of mass G to represent the vehicle position in Chaps. 3–7 was arbitrary as well.

$$
\begin{aligned}
\mathbf{a}_M &= (\dot{u} - v\dot{\psi})\ \mathbf{i}_1 + (\dot{v} + u\dot{\psi})\ \mathbf{j}_1 \\
&\simeq (\dot{u} - vr)\ \mathbf{i}_1 + (\dot{v} + ur)\ \mathbf{j}_1
\end{aligned}
\tag{9.18}
$$

Actually, point M may also have a vertical velocity, due to uneven road or suspension jacking. Here we assume the road to be perfectly flat and suspension jacking to be negligible.

Point M inherits almost everything that was obtained for G in Chaps. 3–7, in the sense that now we have to use M (or O_1) to define the vehicle slip angle $\hat{\beta}$, trajectory, etc.

9.6 Three-Dimensional Vehicle Dynamics

We have assumed the vehicle sprung mass m_s to be a rigid body. If roll motion is taken into account, it has a three-dimensional dynamics. For simplicity, at least at the beginning, it is useful to suppose the unsprung mass m_n to be negligible (i.e., $m = m_s$).

Like in (3.72), the classical force and torque equations for the dynamics of a single rigid body are [8]

$$
\begin{aligned}
m\,\mathbf{a}_G &= \mathbf{F} \\
\dot{\mathbf{K}}_G^r &= \mathbf{M}_G
\end{aligned}
\tag{9.19}
$$

where $m = m_s$ is the total mass of the vehicle, $\mathbf{a}_G$ is the acceleration of its center of mass, $\mathbf{F}$ is the resultant of all forces applied to the vehicle body, $\dot{\mathbf{K}}_G^r$ is the rate of change of the angular momentum of the vehicle body with respect to $G = G_s$, and $\mathbf{M}_G$ is the global moment (torque) of all forces, again with respect to G.

If the second equation is written with respect to any other point, like, e.g., the freshly defined vehicle invariant point M, it generalizes into

$$
\dot{\mathbf{K}}_G^r + MG \times m\,\mathbf{a}_G = \mathbf{M}_M \tag{9.20}
$$

9.6.1 *Velocity and Acceleration of G*

Dynamics cannot get rid of G. We have to compute its velocity and acceleration.

Both points M and G belong to the same rigid body. Therefore, we can use again the fundamental formula (5.1) to relate the velocity of G to the velocity of M, plus the roll contribution

$$
\mathbf{V}_G = \mathbf{V}_M + \boldsymbol{\Omega} \times MG \tag{9.21}
$$

where, by definition

$$\begin{aligned} MG &= (h-d)\,\mathbf{k}_3 \\ &\simeq (h-d)(\theta\,\mathbf{i}_1 - \phi\,\mathbf{j}_1 + \mathbf{k}_1) \end{aligned} \tag{9.22}$$

Therefore

$$\begin{aligned} \mathbf{V}_G &= u\ \mathbf{i}_1 + v\ \mathbf{j}_1 - p(h-d)\,\mathbf{j}_3 + q(h-d)\,\mathbf{i}_3 \\ &= u\ \mathbf{i}_1 + v\ \mathbf{j}_1 - (\dot{\phi} - \dot{\psi}\theta)(h-d)\,\mathbf{j}_3 + (\dot{\theta} + \dot{\psi}\phi)(h-d)\,\mathbf{i}_3 \end{aligned} \tag{9.23}$$

where in the last equation we employed the approximate expression (9.11).

We can proceed in a similar way for accelerations, that is using the fundamental formula (5.4)

$$\mathbf{a}_G = \mathbf{a}_M + \dot{\mathbf{\Omega}} \times MG + \mathbf{\Omega} \times (\mathbf{\Omega} \times MG) \tag{9.24}$$

that is

$$\begin{aligned} \mathbf{a}_G = \;& (\dot{u} - v\dot{\psi})\ \mathbf{i}_1 + (\dot{v} + u\dot{\psi})\ \mathbf{j}_1 \\ & - \dot{p}(h-d)\,\mathbf{j}_3 + \dot{q}(h-d)\,\mathbf{i}_3 \\ & + (h-d)[-p(p\,\mathbf{k}_3 - r\,\mathbf{i}_3) + q(r\,\mathbf{j}_3 - q\,\mathbf{k}_3)] \end{aligned} \tag{9.25}$$

which can be rewritten as

$$\begin{aligned} \mathbf{a}_G = \;& (\dot{u} - v\dot{\psi})\ \mathbf{i}_1 + (\dot{v} + u\dot{\psi})\ \mathbf{j}_1 \\ & + (h-d)[-\dot{p}\,\mathbf{j}_3 + \dot{q}\,\mathbf{i}_3] \\ & + (h-d)[r(p\,\mathbf{i}_3 + q\,\mathbf{j}_3) - (p^2+q^2)\,\mathbf{k}_3] \end{aligned} \tag{9.26}$$

Each term has a clear physical meaning. The acceleration $\mathbf{a}_G$ is one of the fundamental bricks in the force equation in (9.19).

The acceleration $\mathbf{a}_G$ can be expressed in S_1

$$\begin{aligned} \mathbf{a}_G = \;& (\dot{u} - v\dot{\psi})\ \mathbf{i}_1 + (\dot{v} + u\dot{\psi})\ \mathbf{j}_1 \\ & + (h-d)[-\dot{p}(\mathbf{j}_1 + \phi\,\mathbf{k}_1) + \dot{q}(\mathbf{i}_1 - \theta\,\mathbf{k}_1)] \\ & + (h-d)\{r[p(\mathbf{i}_1 - \theta\,\mathbf{k}_1) + q(\mathbf{j}_1 + \phi\,\mathbf{k}_1)] - (p^2+q^2)(\theta\,\mathbf{i}_1 - \phi\,\mathbf{j}_1 + \mathbf{k}_1)\} \end{aligned} \tag{9.27}$$

which can be rearranged as

$$\begin{aligned} \mathbf{a}_G = \;& (\dot{u} - v\dot{\psi})\ \mathbf{i}_1 + (\dot{v} + u\dot{\psi})\ \mathbf{j}_1 \\ & + (h-d)[\dot{q} + rp - (p^2+q^2)\theta]\ \mathbf{i}_1 \\ & + (h-d)[-\dot{p} + rq + (p^2+q^2)\phi]\ \mathbf{j}_1 \\ & + (h-d)[-\dot{p}\phi - \dot{q}\theta - rp\theta + rq\phi - (p^2+q^2)]\,\mathbf{k}_1 \end{aligned} \tag{9.28}$$

Taking (9.16) into account, and discarding the small terms, we get

$$\begin{aligned}\mathbf{a}_G \simeq\ & (\dot{u} - v\dot{\psi})\,\mathbf{i}_1 + (\dot{v} + u\dot{\psi})\,\mathbf{j}_1 \\ & + (h-d)[(\ddot{\theta} + \ddot{\psi}\,\phi + \dot{\psi}\,\dot{\phi}) + \dot{\psi}(\dot{\phi} - \dot{\psi}\theta)]\,\mathbf{i}_1 \\ & + (h-d)[-(\ddot{\phi} - \ddot{\psi}\,\theta - \dot{\psi}\,\dot{\theta}) + \dot{\psi}(\dot{\theta} + \dot{\psi}\phi)]\,\mathbf{j}_1\end{aligned} \tag{9.29}$$

If also $\dot{\psi}$ and $\ddot{\psi}$ are small

$$\mathbf{a}_G \simeq (\dot{u} - v\dot{\psi})\,\mathbf{i}_1 + (\dot{v} + u\dot{\psi})\,\mathbf{j}_1 + (h-d)[\ddot{\theta}\,\mathbf{i}_1 - \ddot{\phi}\,\mathbf{j}_1] \tag{9.30}$$

9.6.2 Rate of Change of the Angular Momentum

It is very convenient to use, as already done in Sect. 9.2, a reference system S_3 attached to the vehicle body and with its origin in the center of gravity of the sprung mass G_s.

As already stated, when the vehicle is at rest, direction $\mathbf{k}_3$ of S_3 is orthogonal to the road and direction $\mathbf{i}_3$ is parallel to the road pointing forward (like in Fig. 1.4, where the body-fixed axes do not have the subscript 3, or in Fig. 9.1). Therefore, in general, S_3 is not directed as the principal axes of inertia, and the inertia tensor

$$\mathbf{J} = \begin{bmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{yx} & J_y & -J_{yz} \\ -J_{zx} & -J_{zy} & J_z \end{bmatrix} \tag{9.31}$$

is not diagonal.

Consequently, the expression of $\dot{\mathbf{K}}^r_G$ is a little involved (see also (3.74)

$$\begin{aligned}\dot{\mathbf{K}}^r_G =\ & [J_x\dot{p} - (J_y - J_z)qr - J_{xy}(\dot{q} - rp) - J_{yz}(q^2 - r^2) - J_{zx}(\dot{r} + pq)]\,\mathbf{i}_3 \\ & + [J_y\dot{q} - (J_z - J_x)rp - J_{yz}(\dot{r} - pq) - J_{zx}(r^2 - p^2) - J_{xy}(\dot{p} + qr)]\,\mathbf{j}_3 \\ & + [J_z\dot{r} - (J_x - J_y)pq - J_{zx}(\dot{p} - qr) - J_{xy}(p^2 - q^2) - J_{yz}(\dot{q} + rp)]\,\mathbf{k}_3\end{aligned} \tag{9.32}$$

Actually, most vehicles have $(J_{xy} = J_{yz}) \simeq 0$, and hence we can use the simplified expression

$$\begin{aligned}\dot{\mathbf{K}}^r_G =\ & [J_x\dot{p} - (J_y - J_z)qr - J_{zx}(\dot{r} + pq)]\,\mathbf{i}_3 \\ & + [J_y\dot{q} - (J_z - J_x)rp - J_{zx}(r^2 - p^2)]\,\mathbf{j}_3 \\ & + [J_z\dot{r} - (J_x - J_y)pq - J_{zx}(\dot{p} - qr)]\,\mathbf{k}_3\end{aligned} \tag{9.33}$$

This very same quantity can be expressed in S_1, if (9.14) is taken into account

$$\begin{aligned}\dot{\mathbf{K}}^r_G =\ & [J_x\dot{p} - (J_y - J_z)qr - J_{zx}(\dot{r} - pq)](\mathbf{i}_1 - \theta\,\mathbf{k}_1) \\ & + [J_y\dot{q} - (J_z - J_x)rp - J_{zx}(r^2 - p^2)](\mathbf{j}_1 + \phi\,\mathbf{k}_1) \\ & + [J_z\dot{r} - (J_x - J_y)pq - J_{zx}(\dot{p} - qr)](\theta\,\mathbf{i}_1 - \phi\,\mathbf{j}_1 + \mathbf{k}_1)\end{aligned} \tag{9.34}$$

That is, with some further simplifications because θ, ϕ, p and q are small

$$\begin{aligned}\dot{\mathbf{K}}_G^r = \; & [J_x\dot{p} - (J_y - J_z)qr - J_{zx}\dot{r} + J_z\dot{r}\theta]\,\mathbf{i}_1 \\ & + [J_y\dot{q} - (J_z - J_x)rp - J_{zx}r^2 - J_z\dot{r}\phi]\,\mathbf{j}_1 \\ & + [J_z\dot{r} + J_{zx}(\dot{r}\theta - r^2\phi - \dot{p} + qr)]\,\mathbf{k}_1\end{aligned} \tag{9.35}$$

And finally, taking (9.16) into account (cf. (3.74))

$$\begin{aligned}\dot{\mathbf{K}}_G^r = \; & [J_x(\ddot{\phi} - \ddot{\psi}\,\theta - \dot{\psi}\,\dot{\theta}) - (J_y - J_z)(\dot{\theta} + \dot{\psi}\,\phi)\dot{\psi} - J_{zx}\ddot{\psi} + J_z\ddot{\psi}\theta]\,\mathbf{i}_1 \\ & + [J_y(\ddot{\theta} + \ddot{\psi}\,\phi + \dot{\psi}\,\dot{\phi}) - (J_z - J_x)\dot{\psi}(\dot{\phi} - \dot{\psi}\,\theta) - J_{zx}\dot{\psi}^2 - J_z\ddot{\psi}\phi]\,\mathbf{j}_1 \\ & + [J_z\ddot{\psi} + J_{zx}(2\ddot{\psi}\theta - \ddot{\phi} + 2\dot{\psi}\dot{\theta})]\,\mathbf{k}_1\end{aligned} \tag{9.36}$$

If also $\dot{\psi}$ and $\ddot{\psi}$ are small (obviously, ψ is not small)

$$\dot{\mathbf{K}}_G^r = (J_x\ddot{\phi} - J_{zx}\ddot{\psi})\,\mathbf{i}_1 + J_y\ddot{\theta}\,\mathbf{j}_1 + (J_z\ddot{\psi} - J_{zx}\ddot{\phi})\,\mathbf{k}_1 \tag{9.37}$$

Of course, all inertia terms J_x, J_{xz}, etc. are constant because the reference system S_3 is fixed to the vehicle body. We see that the definition of roll, pitch and yaw is crucial in these equations.

9.6.3 Completing the Torque Equation

Once that $\mathbf{a}_G$ has been obtained, we can also compute the term $MG \times m\,\mathbf{a}_G$ in the torque equation (9.20). To keep the analysis fairly simple, we employ the simplified expressions (9.22) and (9.30)

$$\begin{aligned}& MG \times m\,\mathbf{a}_G \simeq \\ & m\big\{[(h-d)(\theta\,\mathbf{i}_1 - \phi\,\mathbf{j}_1 + \mathbf{k}_1)] \times [(\dot{u} - v\dot{\psi})\,\mathbf{i}_1 + (\dot{v} + u\dot{\psi})\,\mathbf{j}_1 \\ & \quad + (h-d)(\ddot{\theta}\,\mathbf{i}_1 - \ddot{\phi}\,\mathbf{j}_1)]\big\}\end{aligned} \tag{9.38}$$

which provides

$$\begin{aligned}MG \times m\,\mathbf{a}_G \simeq m\big\{ & [(h-d)^2\ddot{\phi} - (h-d)(\dot{v} + u\dot{\psi})]\,\mathbf{i}_1 \\ & + [(h-d)^2\ddot{\theta} + (h-d)(\dot{u} - v\dot{\psi})]\,\mathbf{j}_1 \\ & + (h-d)\dot{u}\phi\;\mathbf{k}_1\big\}\end{aligned} \tag{9.39}$$

9.6.4 Equilibrium Equations

We have obtained all inertia terms of the force and torque equations (left hand side terms). Considering (9.30), (9.37), and (9.39), we get the following explicit (linearized) form of the equilibrium equations (9.19) and (9.20) for a vehicle that can roll and pitch

$$\begin{aligned}
&m[(\dot{u} - vr) + (h - d)\ddot{\theta}] = ma_x = X \\
&m[(\dot{v} + ur) - (h - d)\ddot{\phi}] = ma_y = Y \\
&0 = Z \\
&[J_x + m(h - d)^2]\ddot{\phi} - J_{zx}\dot{r} - m(h - d)(\dot{v} + ur) = L_M \\
&[J_y + m(h - d)^2]\ddot{\theta} + m(h - d)(\dot{u} - vr) = M_M \\
&J_z\dot{r} - J_{zx}\ddot{\phi} + m(h - d)\dot{u}\phi = N_M = N
\end{aligned} \tag{9.40}$$

where, according to (9.10), we set $r = \dot{\psi}$, and (L_M, M_M, N_M) are the three components of the torque acting with respect to point M. It is useful to compare these equations with (3.91) and (3.92), that is with the equilibrium equations obtained when the inertial effects of pitch and roll are neglected.

Interestingly enough, the last three equations in (9.40) can be rewritten as

$$\begin{aligned}
&J_x\ddot{\phi} - J_{zx}\dot{r} - ma_y(h - d) = L_M \\
&J_y\ddot{\theta} + ma_x(h - d) = M_M \\
&J_z\dot{r} - J_{zx}\ddot{\phi} + ma_x(h - d)\phi = N_M = N
\end{aligned} \tag{9.41}$$

Of course, everything looks like the car rolls about point M, but it is not so. Actually, the car rolls about the point M as it does with respect to *any other of its points* (Fig. 9.6). It is just the fundamental law (9.21) of the kinematics of rigid bodies. Therefore, we should avoid sentences like “the car rolls about the roll axis”, simply because they have no physical meaning at all.

9.6.5 Including the Unsprung Mass

If the unsprung mass m_n cannot be neglected, Eq. (9.40) become

$$\begin{aligned}
&m(\dot{u} - vr) + m_s(h - d)\ddot{\theta} = X \\
&m(\dot{v} + ur) - m_s(h - d)\ddot{\phi} = Y \\
&0 = Z \\
&[J_x + m_s(h - d)^2]\ddot{\phi} - \tilde{J}_{zx}\dot{r} - m_s(h - d)(\dot{v} + ur) = L_M \\
&[J_y + m_s(h - d)^2]\ddot{\theta} + m_s(h - d)(\dot{u} - v\dot{\psi}) = M_M \\
&\tilde{J}_z\dot{r} - J_{zx}\ddot{\phi} + m_s(h - d)\dot{u}\phi = N
\end{aligned} \tag{9.42}$$

where $\tilde{J}_z$ and $\tilde{J}_{zx}$ take into account both m_s and m_n.

9.7 Handling with Roll Motion

The analysis carried out in Chap. 3 can now be extended taking roll and pitch into account. However, as already stated, we assume here that the tires are rigid, as in Sect. 3.10.14. Otherwise, the theory would become too involved, and some physical aspects would not be clear enough.

9.7.1 Equilibrium Equations

The inertia terms of the equilibrium equations have been already obtained in (9.40), and rewritten in an alternative form in (9.41). Therefore, we have to complete the equilibrium equations by including the resultant $\mathbf{F}$ and the moment $\mathbf{M}_M$ (right-hand side terms). Of course, now we have to include the effects of the dampers, which are sensitive to the roll time rate $\dot{\phi}$.

We call c_ϕ the global damping coefficients with respect to roll, much like k_ϕ is the global stiffness with respect to roll. More precisely, as in (3.114), we have

$$k_\phi = k_{\phi_1} + k_{\phi_2} \qquad \text{and} \qquad c_\phi = c_{\phi_1} + c_{\phi_2} \tag{9.43}$$

Similarly, according to (10.55), we have the following global stiffness and global damping coefficient with respect to pitch

$$k_\theta = k_1 a_1^2 + k_2 a_2^2 \qquad \text{and} \qquad c_\theta = c_1 a_1^2 + c_2 a_2^2 \tag{9.44}$$

Therefore, the right-hand side terms to be inserted into the equilibrium equations (9.40) are as follows (cf. (3.91) and (3.92))

$$\begin{aligned}
X &= X_1 + X_2 - X_a \\
Y &= Y_1 + Y_2 \\
Z &= Z_1 + Z_2 - (mg + Z_1^a + Z_2^a) \\
L_M &= -k_\phi\,\phi - c_\phi\,\dot{\phi} + mg(h-d)\phi \\
M_M &= -k_\theta\,\theta - c_\theta\,\dot{\theta} \\
N_M &= N = N_Y + N_X = (Y_1 a_1 - Y_2 a_2) + (\Delta X_1 t_1 + \Delta X_2 t_2)
\end{aligned} \tag{9.45}$$

9.7.2 Load Transfers

Having roll $\phi(t)$ and $\theta(t)$ as functions of time requires some other equations of the vehicle model developed in Chap. 3 to be updated. More precisely, we have to take dampers and inertia terms into account.

The lateral load transfers (3.141) now become

$$\begin{aligned} \Delta Z_1 t_1 &= Y_1 q_1 + k_{\phi_1}\phi + c_{\phi_1}\dot{\phi} \\ \Delta Z_2 t_2 &= Y_2 q_2 + k_{\phi_2}\phi + c_{\phi_2}\dot{\phi} \end{aligned} \tag{9.46}$$

which, if added, provide

$$\begin{aligned} \Delta Z_1 t_1 + \Delta Z_2 t_2 &= (k_{\phi_1} + k_{\phi_2})\phi + (c_{\phi_1} + c_{\phi_2})\dot{\phi} + Y_1 q_1 + Y_2 q_2 \\ &= k_\phi \phi + c_\phi \dot{\phi} + Yd = k_\phi \phi + c_\phi \dot{\phi} + m a_y d \end{aligned} \tag{9.47}$$

since, as in (3.138), $Yd = Y_1 q_1 + Y_2 q_2$.

Combining (9.41), (9.45) and (9.47), we obtain

$$Y(h-d) + mg(h-d)\phi = k_\phi \phi + c_\phi \dot{\phi} + J_x \ddot{\phi} - J_{zx}\dot{r} \tag{9.48}$$

which generalizes (3.154). With a little algebra, we can obtain also

$$\Delta Z_1 t_1 + \Delta Z_2 t_2 = m a_y h + mg(h-d)\phi - (J_x \ddot{\phi} - J_{zx}\dot{r}) \tag{9.49}$$

which generalizes (3.103).

For the longitudinal load transfer ΔZ we can follow a similar line of reasoning, thus obtaining

$$\Delta Z = -\frac{Xh + J_y \ddot{\theta}}{l} = -\frac{m a_x h + J_y \ddot{\theta}}{l} \tag{9.50}$$

which generalizes (3.101).

The main difference with respect to the model developed in Chap. 3, and summarized on p. 15, is that load transfers now depend explicitly on the angular accelerations of the vehicle body.

9.7.3 Constitutive (Tire) Equations

Taking explicitly into account the roll and pitch motions does not affect directly the tire equations. Therefore, the analysis developed in Chap. 3 applies entirely.

9.7.4 Congruence (Kinematic) Equations

The congruence equations listed in Sect. 3.15.5 can be employed even when the vehicle model has the roll and pitch degrees of freedom. Actually, according to Fig. 9.8, the lateral velocities of the front and rear axles should be, respectively

$$v_1 = v + ra_1 + (d - q_1)\dot{\phi} \qquad \text{and} \qquad v_2 = v - ra_2 + (d - q_2)\dot{\phi} \tag{9.51}$$

that is they include small contributions due to the different heights of the vehicle invariant point M and the two track invariant points M_1 and M_2. However, the additional terms are really very small, and hence can be neglected.

9.8 Steady-State and Transient Analysis

Obviously, including the roll and pitch motions into the vehicle model has very little, if any, influence on the vehicle steady-state behavior. We should not forget that the steady-state roll angle was part of the analyses carried out in Chaps. 3–7. On the other hand, the transient behavior, in particular when entering or exiting a curve, can be rather different.

9.9 Exercise

9.9.1 Roll Motion and Camber Variation

Camber variations $\Delta\gamma_i$ are strictly related to roll motion ϕ, and affect quite a bit the vehicle handling behavior. This topic was addressed in Sect. 3.10.3, *where the first order relationships were provided.*

In particular, here we are interested in the equation

$$\Delta\gamma_i \simeq -\left(\frac{q_i - b_i}{b_i}\right)\phi \tag{9.52}$$

With the aid of a ruler and a protractor, check this equation for the three cases of Fig. 9.10.

Solution

In all three cases, the roll angle $\phi = 6°$. Top to bottom, we have $(q_i - b_i)/b_i$ equal to about -1.14, 0.6, and 0.36, respectively. Therefore, the corresponding camber angles should be $-6.8°$, $3.6°$, and $2.1°$. Indeed, direct measurements confirm these results.

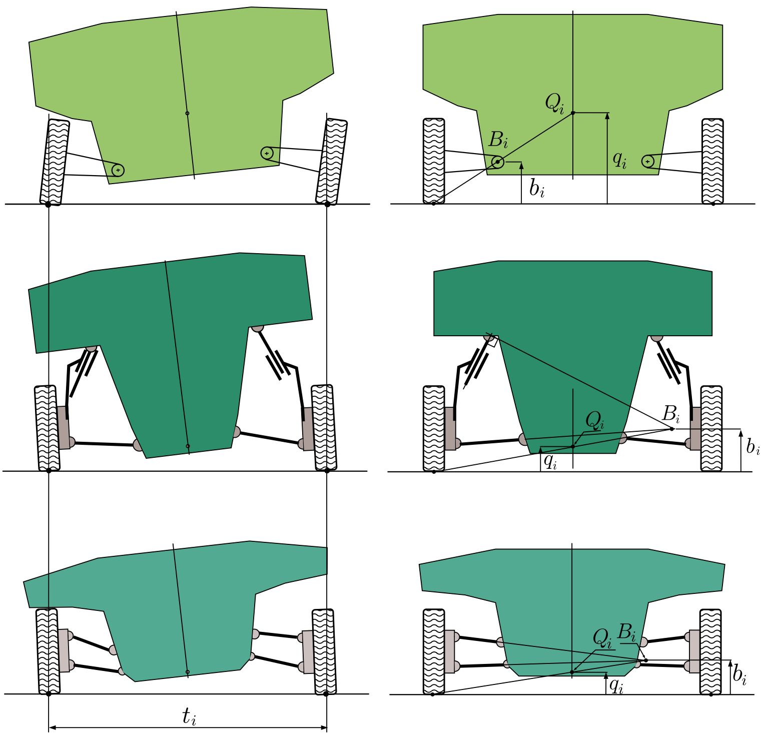

Fig. 9.10 Roll motion and camber variations (front view)

9.10 Summary

The vehicle orientation has been defined by means of the yaw-pitch-roll elemental rotations. Then, to define the vehicle position, a careful analysis of what happens when the vehicle rolls has been performed. The key result is the definition of the Vehicle Invariant Point (VIP) as the best option for monitoring the vehicle position, and also for defining the lateral velocity and acceleration.

VIP allows for a simple and systematic analysis of the vehicle three-dimensional dynamics. Among other things, it has been shown that the well known roll-axis, as the axis about which the vehicle rolls, is nonsense.

9.11 List of Some Relevant Concepts

p. 395 — finite rotations are not commutative;
p. 395 — yaw, pitch, and roll are the three elemental rotations commonly and conveniently employed in vehicles;

p. 399 — track invariant points belong to the vehicle body;
p. 402 — vehicle invariant point (VIP) belongs to the vehicle body and it is the point best suited to represent the vehicle position, lateral velocity, and lateral acceleration;
p. 402 — roll motion is better explained without recourse to the roll axis;
p. 411 — load transfers depend also on angular accelerations.

9.12 Key Symbols

a_1	distance of G from the front axle
a_2	distance of G from the rear axle
$\mathbf{a}_G$	acceleration of G
$\mathbf{a}_M$	acceleration of M
d	height of the no-roll axis below G
g	gravity acceleration
G	center of mass
h	height of G
L_M	first component of the torque with respect to M
$\mathbf{J}$	inertia tensor
J_x, J_y, J_z	moments of inertia
$\mathbf{K}_G^r$	angular momentum with respect to G
k_θ	total pitch stiffness
k_ϕ	total roll stiffness
k_{ϕ_i}	global roll stiffness of i-th axle
l	wheelbase
m	mass
m_n	unsprung mass
m_s	sprung mass
M	vehicle invariant point (VIP)
M_i	track invariant point (TIP)
M_M	second component of the torque with respect to M
N	yaw moment
N_M	third component of the torque with respect to M
p	first component of $\mathbf{\Omega}$ in S_3
q	second component of $\mathbf{\Omega}$ in S_3
Q	no-roll center
q_1	height of the front no-roll center
Q_1	front no-roll center
q_2	height of the rear no-roll center
Q_2	rear no-roll center
r	third component of $\mathbf{\Omega}$ in S_3
$\mathbf{R}_i$	rotation matrix
S_i	Cartesian reference system

t_1	front track
t_2	rear track
u	longitudinal velocity
v	lateral velocity
$\mathbf{V}_G$	velocity of G
X	longitudinal force
Y	lateral force
Y_i	lateral force on the i-th axle
Z	vertical force
Z_i	vertical load on i-th axle
Z_i^0	static vertical load on i-th axle
Z_i^a	aerodynamic vertical load on i-th axle
ΔZ	longitudinal load transfer
ΔZ_i	lateral load transfer on i-th axle
θ	pitch angle
ϕ	roll angle
ψ	yaw angle
$\mathbf{\Omega}$	angular velocity of the vehicle body

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