Chapter 3
Mathematical Giftedness and Creativity in Primary Grades

Daniela Assmus and Torsten Fritzlar

Abstract Creativity is often seen as a characteristic or a variety of content-specific giftedness, but also as an independent, more general kind of giftedness. In the first part of this article, we will discuss some key questions on mathematical giftedness, creativity and theoretical connections between the two constructs. Subsequently, we will specify these considerations with regard to primary students. The main question of the second part of the paper is how creativity can manifest itself in mathematical activities of gifted primary students. Generally, mathematical creativity is assumed to be closely linked to problem solving and problem posing; for mathematically experienced people both processes are embedded in theory building processes. Also primary students can vary given problems and solve problems that usually require only little mathematical knowledge. Moreover, mathematically gifted primary students are able to create new mathematical objects. We will describe types and examples of such invention processes in detail.

Keywords Content-specific giftedness · Embedded model of giftedness and creativity · Mathematical giftedness · Problem solving · Problem posing Theory building processes

3.1 Introduction

Creativity in the domain of mathematics has met with increasing interest in recent years. Nevertheless, primary school students have not been a focus of research up to now. This may be related to the fact that on the one hand, domain-related knowledge is regarded as an essential prerequisite for creative action (Silver 1997;

D. Assmus (✉) · T. Fritzlar
Martin Luther University of Halle-Wittenberg, Halle (Saale), Germany
e-mail: daniela.assmus@paedagogik.uni-halle.de

T. Fritzlar
e-mail: torsten.fritzlar@paedagogik.uni-halle.de

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Weisberg 1999), and that on the other hand, the mathematics-specific knowledge base in the primary school age is usually still very small. Mathematically gifted primary students may be an exception, as they are expected to have a higher level of experience and knowledge due to their special abilities and their frequently large specific interest. Therefore, we consider this special group to be particularly suitable for exploring mathematical creativity at primary school age and for investigating possible occasions for creative mathematical action.

In the scientific literature giftedness and creativity are seen as related in many different ways. Therefore, in the first part of this paper, we will initially consider both constructs and especially their connection in detail.

PART I

3.2 (Mathematical) Giftedness

In relevant literature, there is no standardised definition of *giftedness*; neither in general nor especially for the domain of mathematics. Not only is there unclarity regarding the definition of this term, but there are also a number of similar (e.g. *talent*, *expertise*) or connected terms (e.g. *special abilities*, *high achievement*, *creativity*). In this context, Ziegler (2008, p. 14) speaks of a "Babylonian language chaos" (translation by the authors), which makes a theoretical approach to mathematical giftedness extremely difficult.

From a superordinate perspective, there are three key questions regarding a construct of *mathematical giftedness*:

- Does the construct of mathematical giftedness describe extraordinary mathematical achievements or rather just the potentials for especially valuable (subsequent) achievements in the field of mathematics?
- Is mathematical giftedness an expression of specific cognitive characteristics or is it, at least for the main part, a result of high general intelligence?
- Is mathematical giftedness a monolithic construct or are there different profiles of giftedness? (cf. Wieczerkowski et al. 2000).

3.2.1 Giftedness: Potentials and Achievements

It is possible to distinguish the numerous definition approaches for the construct of giftedness based on the roles ascribed to the extraordinary achievements accomplished by the individual. If these are considered preconditions for a person to be described as gifted, they are referred to as performance-oriented definitions. If giftedness, however, is conceptualised as a potential for superior performance, one can speak of competence-oriented definitions.

With regard to children and adolescents, competence-oriented definitions predominate; with adults, a greater priority is given to documented achievements. For instance, Mayer (2005, p. 439) understands “giftedness as an age-specific term that refers to potential for the beginning stage, achievement for the intermediate stage and eminence for the advanced stage.”

From our point of view, a competence-oriented definition of giftedness is also appropriate for the domain of mathematics and, specifically, for studying giftedness at primary school age, because especially young children cannot possess the knowledge and experiences needed for extraordinary mathematical achievements.

However, this perspective leads to significant (theoretical and practical) difficulties concerning the diagnosis of giftedness because only achievements can be measured empirically. That is partly why Sternberg (1998) came up with his concept of developing expertise, which was specified by Fritzlar (2015) for the domain of mathematics.

3.2.2 Mathematical Giftedness and General Intelligence

Since the beginning of the 20th century, there has been a more intense theoretical and empirical discussion on the construct of giftedness. Among others due to the Terman study, the view of giftedness has been widened and multi-dimensional models have been created, on the one hand based on the recognition of domain-specific talents [e.g. Munich Model of Giftedness (Heller 2010) or Differentiated Model of Giftedness and Talent (Gagné 1985, 2003)], and on the other hand based on the inclusion of further personal and contextual characteristics [e.g. Three-Rings Conception by Renzulli (1986)].

Since these kinds of models mostly only differentiate between intellectual and non-intellectual areas, it remains open at first to what extent an independent *mathematical* giftedness exists. The answer to this question depends on one’s understanding of mathematics in particular. If, for example, mathematical achievements are recorded using tasks which hardly differ from items used in intelligence tests and for which it is mainly essential to be fast and accurate (whereby, incidentally, the latter is exclusively measured by the test developer’s horizon of expectation, Kießwetter 1992), it is not surprising if the dimensions of mathematical achievements hardly differ from those of general (test) intelligence (Zimmermann 1992).

Tests on school achievements and study capability tend to be more subject-specific. In a study by Benbow, almost 300 mathematically gifted and a little more than 150 linguistically gifted thirteen-year-olds were first identified based on the SAT—they belonged to the best 0.01% of their age group. As a next step, their achievements were compared using different intelligence and ability tests. Only 16 boys and 2 girls belonged to both groups, the others showed significant group differences in almost all areas of ability, with group membership and not gender having the biggest influence on the test results (Benbow and Minor 1990).

On this basis, the idea of a *general* intellectual giftedness that includes a mathematical giftedness cannot be kept.

Especially interesting for studies focusing on primary school age seems a study by Nolte, where children of third grade work both on a specially developed mathematics test as well as on an intelligence test which correlates strongly with grades in mathematics. In this whole group of more than 1600 girls and boys of nine years, the results from the intelligence and mathematics tests correlate with −0.34. However, this relation decreased for children who obtained particularly good results in the mathematics test. The rather weak statistical correlation and its further decrease can partly be expected because of the (increasing) selectivity and the (decreasing) sample size. Nevertheless, it seems reasonable to assume intelligence test results and mathematical potential correlate based on the total population, but a special mathematical giftedness cannot be derived from the IQ (Nolte 2011, 2013).

From a cognitive-psychological and didactic perspective, different descriptions of mathematical giftedness have been developed based on mathematics-specific abilities, characteristics and patterns of action. In this context, the studies carried out by Käpnick (1998) were pivotal in Germany regarding primary school age. For him, mathematical giftedness is marked by the following characteristics and skills: *remembering mathematical facts, structuring mathematical facts, mathematical sensitivity and mathematical fantasy, transferring mathematical structures, intermodal transfer, reversing lines of thoughts* (cf. Benölken 2015). To what extent these features characterise mathematically gifted students depends on the mathematical richness of the tasks used to reveal them.

At a first glance, the specificity of some abilities may seem critical. However, in psychology the position is widespread that abilities do not exist by themselves but always in connection to specific contents to which they are inseparably related (Lompscher and Gullasch 1977).

All in all, it seems reasonable to assume that initially general cognitive abilities first generally develop and then become more specific during activities. In this regard, the accumulation of knowledge could play a vital role, because knowledge is on the one hand gained through abilities of the individual and on the other hand forms an important basis for the development and realisation of mental abilities. In this sense, abilities, knowledge and activities develop in close interaction and mutually reinforce.

3.2.3 Profiles of Mathematical Giftedness

If a list of specific abilities or action patterns is used to describe mathematical giftedness, the respective authors (e.g. Benölken 2015; Käpnick 1998; Nolte 2011, 2013) always emphasise that they can be evident to a various extent and not all of them are necessary for the presence of giftedness. Also Krutetskii, who as one of the first ones soundly studied special abilities of mathematically gifted students, emphasised that their composition to a structure of mathematical thinking can be *individually different*, whereby certain components can also be compensated by others. High mathematical achievements can be reached with different complexes of abilities or "mental specialities". As a result, there are different manifestations of mathematical giftedness, especially since, according to Krutetskii, further useful but not necessarily needed characteristics exist, like the speed of thinking processes, counting skills, a distinct memory for symbols, numbers and formulas, visual thinking as well as the ability to vividly imagine abstract mathematical relations and dependencies (Krutetskii 1976). For older students, he distinguished between a geometric, an analytic and a harmonic type based on the relation between visual and abstract-logical components. The last type, however, probably has the highest potential.

Qualitative research studies on mathematical giftedness in primary school have shown that different profiles of specific abilities already exist at this age (e.g. Fuchs 2006; Käpnick 1998). However, it is assumed that interindividual differences increase through growing domain-specific experience.

To sum up the discussion on the three key questions, mathematical giftedness at primary school age can, from our point of view, be understood as an extraordinary high potential to solve mathematically challenging questions and problems (compared to others of the same age). The various aspects of this potential can be differently pronounced, but in total it is mostly specific for the domain of mathematics. A detailed description of mathematical giftedness in early primary school age by means of specific abilities was recently developed by Assmus (in this volume).

3.3 Creativity

Since the 1950s, creativity research has continually been and is still being advanced. However, to this day there is neither a consistent definition of creativity nor a commonly acknowledged creativity theory. In scientific discourse, it is common to distinguish between creativity as a quality of a product, a person, a process or creativity-affecting environmental factors. In the English-speaking world this is also referred to as the "4P's of creativity" (product, person, process, press) based on the work of Rhodes (1961). Since in this article we will not discuss developmental aspects, neither in relation to giftedness nor to creativity, the fourth aspect (press) is not further considered.

What is normally considered the key criterion of a *creative product* is its "novelty". Since, however, objective novelty independent of space and time is extremely rare, some authors relativise this criterion in so far as an idea is seen as new (or unique) if it is rare among a particular population (e.g. Guilford 1967; Jackson and Messick 1965). In contrast to absolute creativity, we refer to relative creativity in this regard. Ideas that are new for an individual, but widely spread among the population considered (e.g. a school class) are not judged creative according to this definition. In pedagogic situations, however, an individual reference norm might be used as a basis for the novelty criterion (cf. e.g. Kießwetter 1977), which is then referred to as individual creativity.

Besides novelty, at least one further criterion is specified, which concerns the purpose of the product. In this regard, terms such as “meaningfulness”, “target-orientation”, “real-life relevance” and “usefulness” are used (Preiser 1976). It should, however, be mentioned that free creative processes would not be classified as creative according to this approach if the created products did not meet the criterion of usefulness. Since this would apply to many artistic products, the usefulness criterion might be seen as disproportionally constraining the kind and number of creative products.

Regarding *creativity as a quality of individuals,* the features based on the work of Guilford (1950) and their operationalization in the “Torrance Test of Creative Thinking” of Torrance (1966) are usually cited, namely fluency, flexibility, originality and elaboration. Fluency refers to the ability to produce as many associations, thoughts and ideas as possible on a content or problem within a short time. Flexibility can be described as the ability to think into different directions, to easily switch from one thinking category into another, and to look at a problem from different views. Originality is the ability to generate uncommon ideas and solution approaches. “Uncommonness”, “remoteness” and “cleverness” are mentioned as measuring criteria for originality. The ability to proceed from an idea to a definite plan and, thus enriching and developing the idea, is understood as elaboration.

These explanations show that creativity as characteristic of a person cannot be separated from the creative product. The product characteristic is needed to estimate the originality of a person. Also, the description of *creative processes*, for which in general multilevel phase models are used, like, e.g., that of Wallas (1926) and respectively Hadamard (1945), which propose the phases preparation, incubation, illumination and verification (e.g. Aldous 2007; Sriraman et al. 2013), cannot be made without considering the creative products.

3.4 Relations Between (Mathematical) Creativity and (Mathematical) Giftedness

Giftedness and creativity are often seen in close connection. However, the basic assumptions made in the scientific discourse differ concerning the relation between (mathematical) creativity and (mathematical) giftedness (cf. e.g. Singer et al. 2016 with many references). In our opinion, the different views can be classified as follows (cf. Aßmus 2017):

1. (mathematical) creativity as a precondition for (mathematical) giftedness
2. (mathematical) creativity as a possible component of mathematical giftedness
3. (mathematical) creativity as a possible consequence of mathematical giftedness
4. creativity as a (mostly) independent area of giftedness.

The single views are further explained below. Since we are mainly interested in the *relation* between these two constructs, we will not name and explain further influencing factors here. This does not imply that no further influencing factors exist.

Whenever specific study results or theoretical approaches exist for the domain of mathematics, these are considered, even though mathematical creativity has not been regarded so far in this paper. As a detailed explanation of the term mathematical creativity is not needed yet at this point, we will provide an in-depth discussion of the concept in the second part of this paper.

3.4.1 (Mathematical) Creativity as a Precondition for (Mathematical) Giftedness

Based on this conception, well-developed creativity is seen as a necessary precondition for giftedness (Fig. 3.1). Renzulli's model (1986) can be cited as an example in the context of general giftedness. According to this model, the three factors "above average ability", "task commitment" and "creativity" are needed for developing gifted behaviours. Some of the characteristics of creativity as Renzulli understands it are fluency, flexibility, originality of thought, openness to experience and willingness to take risks in thought and action.

In the area of mathematical giftedness, this conception can be found e.g. in Leikin et al. (2009), who consider mathematical giftedness as special problem solving abilities and, referring to Renzulli, describe mathematical creativity as a needed component besides "problem solving effectiveness" and "task commitment" (Leikin et al. 2009). Here, the mathematical creativity is also characterised by the three subcomponents fluency, flexibility and originality.

3.4.2 (Mathematical) Creativity as a Possible Component of Mathematical Giftedness

For several models, creativity in the sense of creative abilities is not a precondition for giftedness, but is rather understood as part of the giftedness itself (Fig. 3.2).

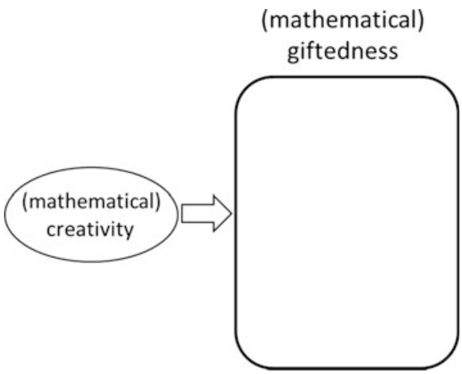


Fig. 3.1 (Mathematical) Creativity as precondition for (mathematical) giftedness

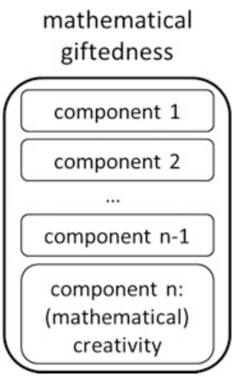


Fig. 3.2 (Mathematical) Creativity as a possible component of mathematical giftedness

For the domain of mathematics, further mathematics-specific abilities are named besides creative abilities. Since not all listed abilities are expected to become evident to an equal extent, different types of mathematical giftedness are possible (cf. our explanations concerning mathematical giftedness above). As a consequence, also a well-developed creativity is not mandatory for mathematical giftedness.

This understanding can, e.g., be found in Krutetskii (1976). He does not use the term “creativity” itself in his components model, but refers to it as “flexibility of mental processes in mathematical activity” (p. 350), which became apparent in his studies when participants managed to overcome fixations or break away from a stereotyped method of solution. Elsewhere, he explains that turning away from typical procedures as well as finding several different solutions is “the real appearance of mathematical creativity” (Krutetskii 1969, cited from Haylock 1984, p. 30).

Käpnick (1998) also mentions another component pertaining to the concept of mathematical creativity as one of seven mathematics-specific characteristics of mathematically gifted children in primary school. He describes it as “mathematical phantasy”, which he understands as the “most important main aspect of childlike creativity” (Käpnick 2013, p. 31; translation by the authors). According to him, the development of diverse imaginative patterns and respectively structures as well as the development and usage of creative solutions for demanding tasks belong to phantasy, just like (not necessarily target-oriented) playful actions with mathematical materials (Käpnick 1998).

Kontoyianni et al. (2013) distinguish between the two categories “mathematical ability” and “mathematical creativity”, which are additionally split up into sub-categories. For mathematical creativity, the sub-categories fluency, flexibility and originality are assumed. In their study with students from fourth to sixth grade they worked out mathematical giftedness as a multi-factorial construct which contains both special mathematical and creative abilities. However, it was also possible to conclude from their data that the importance of mathematical abilities for the construct of giftedness is higher than the importance of creative abilities. Additionally, they proved the relationship between mathematical abilities and mathematical creativity using statistical methods. Of the three approached models, the one that best explained the gathered data was the one that understood mathematical creativity as a sub-component of mathematical giftedness (Kattou et al. 2013). Even if the terms are used differently and the models' sub-components are not identical, general similarities to the characteristics lists of Käpnick (1998) and Krutetskii (1976) mentioned above can still be found in the results concerning the relation of creativity and giftedness.

3.4.3 (Mathematical) Creativity as a Possible Consequence of Mathematical Giftedness

From this perspective, mathematical creativity is understood as the ability to create creative products that contribute to a knowledge progress within mathematics as a science (Fig. 3.3). Creative achievements are therefore reserved for a small group of people. As a result, mathematical creativity implies mathematical giftedness while the inversion is not valid (Howe 1999; Sriraman 2005).

Exemplary for such an understanding is the hierarchy of mathematical talent by Usiskin (2000) (cf. Sriraman 2005). Usiskin proposes eight levels of mathematical talent starting with level 0, which covers adults who barely know something about mathematics. The two highest levels (level 6 and 7) are assigned to people who stand out due to especially creative achievements. The lower levels, too, attest a mathematical talent to people, but in this case this talent does not come along with extraordinarily creative achievements.

Also Sheffield (2009) shapes a "Continuum of mathematical proficiency" (innumerates—doers—computers—consumers—problem solvers—problem posers—creators) which considers the creative creation processes as the highest manifestation.

3.4.4 Creativity as a (Widely) Independent Area of Giftedness

Creativity is considered as a widely independent area of giftedness in several models (Fig. 3.4). A known example is the "Differentiated Model of Giftedness and

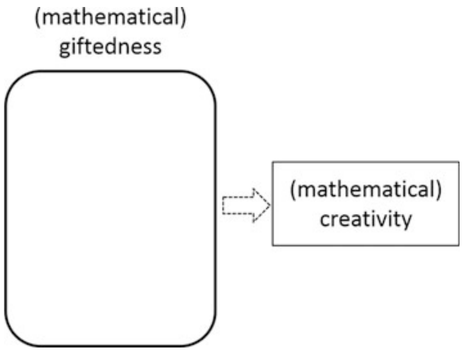


Fig. 3.3 (Mathematical) Creativity as a consequence of (mathematical) giftedness

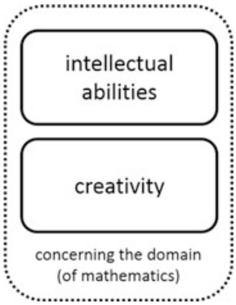


Fig. 3.4 Creativity as a (widely) independent area of giftedness

Talent" by Gagné (1985, 2003) where creativity is listed as one of four areas of giftedness. Others distinguish two fundamentally different forms of giftedness, namely educational or academic giftedness on the one hand, and creative giftedness on the other (cf. e.g. Renzulli and Reis 2003; Hong and Acqui 2004). While "schoolhouse giftedness" according to Renzulli is a specific giftedness concerning test and school achievements, the creative giftedness can be seen in real application situations. "Creative-productive giftedness describes human activity and involvement in which a premium is placed on the development of original material and products that are purposefully designed to have an impact on one or more target audience" (Renzulli and Reis 2003, p. 185).

In their "Comprehensive Model of Giftedness and Talent" Milgram and Hong (2009) distinguish between "analytical-thinking ability" and "creative-thinking ability", which can manifest themselves in "expert talent" or "creative talent". Both forms of talent require both abilities, however, analytic abilities predominate with regard to "expert talent" and creative abilities predominate with regard to "creative talent". "Creative Talent" according to Milgram and Hong is, besides profound specialist knowledge in the respective domain, characterised by the creation of creative and useful products.

Subotnik et al. (2009) examined requirements for the evolution of mathematical talent. They were able to identify different influence factors; mathematical creativity however was not among them. "It seems that a number of variables other than innate mathematical creativity shape the development of talent and ensure a successful career trajectory" (p. 177). This could also be indicative for the fact that mathematical creativity represents a separate area of giftedness. However, the authors are not clear concerning the term "creativity". In the above stated quote, they seem to refer to creativity as ability, but they do not specify it. Furthermore, they use the term "creativity" in relation to the term "talent", which suggests that they see the two concepts as equivalent. The explanations can therefore be interpreted as follows: Creative abilities do not play a role in the development of mathematical talent, but creative products as "output" are closely related to mathematical talent.

Haylock (1997) detected that mathematically efficient students are highly diverse concerning their mathematical creativity. This could also be an indicator of the autonomy of both constructs. Maybe, however, these differences can also be explained by differently evolved personality traits: Students with very good mathematical, but only relatively poor creative achievements increasingly developed negative associations with the subject of mathematics, had quite a low self-esteem and were hardly prepared to take risks when solving mathematical problems.

All in all, there are many different perspectives on the relations between (mathematical) giftedness and (mathematical) creativity. In our view, however, these perspectives are not necessarily contradictory, but they rather result from *different understandings of giftedness and creativity which are not independent from each other*. While giftedness is rather defined as a potential for extraordinary achievements (and characterised by special abilities), creativity is frequently also seen as a person's individual characteristic and the creation of creative products does not have priority (like in perspective 2). If the construction of creative products is focussed, this results in perspective 3. In contrast, in a performance-oriented understanding of giftedness, creative abilities are also seen as a requirement for special mathematical achievements (perspective 1). Perspectives 1–3 on giftedness as competence or performance (always related to mathematical giftedness) can be summarised as follows.

Concerning perspective 4, it can be said that satisfactory evidence on creativity as an independent area of giftedness has not yet been provided. Related observed phenomena can possibly also be explained via different forms of giftedness in perspective 2. Also, in this view, types of giftedness might exist where creativity is more or less extensively developed.

For these reasons and based on above explanations, the best model for our purposes is the left one in Fig. 3.5, where giftedness is understood as potential for extraordinary achievements and creativity is understood as an optional sub-component of giftedness. This does not exclude overlaps of the component "creative abilities" and other components.

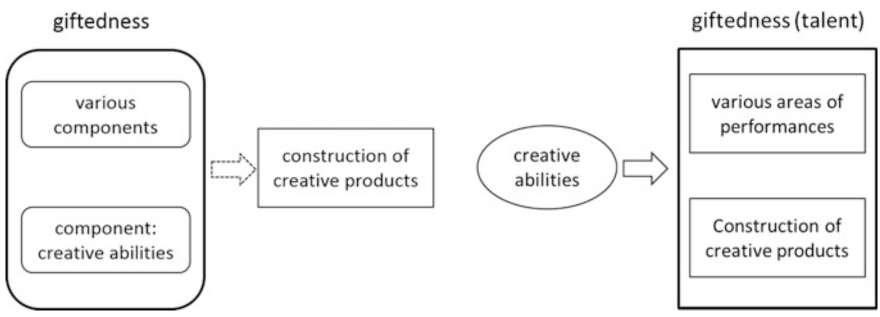


Fig. 3.5 Relations between creativity and giftedness as competence (left) or performance (right)

PART II

3.5 Mathematical Creativity in Primary School Age

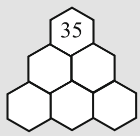
The *embedded model of giftedness and creativity* in the left part of Fig. 3.5 was more or less theoretically assumed. Prerequisite for a stronger empirical support—especially with regard to mathematics at primary school age—is, first of all, a sharpening of underlying concepts. This particularly applies to the concept of mathematical creativity.

As Mann (2006) has pointed out, there is a lack of a widely-accepted definition of mathematical creativity. In the literature, mathematical creativity is, for instance, described as the ability to choose (sensu Poincaré) or to engage in non-algorithmic decision-making. From a common didactical perspective, mathematical creativity can be seen as the ability to generate novel and useful solutions to problems (cf. Sriraman 2009). For primary school age, this definition approach can be combined with the above depicted approach of research on creativity, but specifications and extensions are needed.

At first it should be discussed to which extent criteria (for products, persons or processes) from general creativity research could also be suitable for describing mathematical creativity at primary school age.

(a) *Fluency* can be expressed by primary school children in the area of mathematics, e.g. when solving a problem in several ways, finding several solutions for an open problem (field), continuing a pattern in several ways, finding several representatives for a mathematical characteristic. Such actions demonstrate that children can vary their approaches to doing mathematics.
However, from our point of view, it should be critically questioned if the number of created answers, especially in mathematics, could be indicative of creativity. A large number of *similar* solutions can be quickly and systematically developed for many open tasks by creating structures and ordering principles. The following task to measure fluency used by Kattou et al. (2013, p. 172) serves as an example in this regard:

"Look at this number pyramid. All the cells must contain one number. Each number in the pyramid can be computed by performing always the same operation with the two numbers that appear underneath it. Fill in the pyramid, by keeping on the top the number 35. Try to find as many solutions as possible."



A high number of solutions can already be created through additive partition of the numbers. Creativity is not necessary in this procedure. On the contrary, abilities to identify and use mathematical structures are needed.

Quickly creating many answers to a mathematical stimulus can therefore give a first indication of mathematical creativity, but might as well be based on other, convergent abilities.

(b) While with "fluency", the created solutions/products are considered in terms of quantity, *flexibility* is mainly about the diversity of the products (Neuhaus 2001). A particular flexibility can also become apparent in the above mentioned mathematical actions, if the solutions and ideas fundamentally differ from each other. Additionally, flexibility in doing mathematics can be expressed when the perspectives are changed or when successfully dealing with adapted processing aspects. The following aspects of change are possible at primary school age: representations of a mathematical content, contexts, perspectives on a mathematical content/a mathematical problem, processing directions (e.g. direct vs. converse lines of thoughts), use of given task elements (e.g. switched given and searched elements).[1]

(c) Like the "novelty" of a product, *originality* can only be evaluated using a reference group. In relation to this, the individual reference norm is particularly appropriate for primary school age. For mathematically gifted students, however, it can be assumed that some of them create extraordinary mathematical products, find procedures etc. in relation to the peer group or the group of similarly mathematically experienced students. In these cases, one can assume a high (relative) creativity.

(d) Concerning "real-life relevance" or "usefulness", we agree with Sriraman's statement (2009, p. 15) that the results of creative mathematical processes do not always have to be applicable, because mathematics is also a world with its own value. Consequently, it seems "[…] sufficient to define creativity as the ability to produce novel or original work." This is especially valid for primary school age.

So, from a psychological perspective only the criteria flexibility—concerning mathematical products and in particular also processes of doing mathematics—and originality on an individual level seem to be adaptable for describing mathematical creativity at primary school age.

To continue with the common didactic perspective on creativity already mentioned above, we initially have to go a little further.

According to many researchers, creative mathematical processes particularly occur at *problem solving* (Chamberlin and Moon 2005; Leikin and Lev 2013; Pehkonen 1997). In recent years, the importance of *problem posing* has also been emphasised. The proximity of creativity and *problem solving* is, according to Guilford (1977), already implied by the similar understanding of the two terms: "Creative thinking produces novel outcomes and problem solving involves

[1]Changing aspects 1 and 4 are described as creative abilities of mathematically promising students e.g. by Sheffield (2003).

producing a new response to a new situation, which is a novel outcome" (Guilford 1977, p. 161). He therefore concludes that problem solving involves creative processes. Since problem solving constitutes an essential part of mathematical activities, creativity necessarily plays an important role in mathematics. However, the problem solving process is not always considered as creative. Kießwetter (1977) and Haylock (1984) only judge solution processes as creative which involve divergent thinking, like e.g. the association of distant things, the creation of new means or the novel usage of known/existing means. By contrast, a process which solely consists of applying solution schemes, logical reasoning or systematic sorting is not seen as creative.

Guilford already cited problem sensitivity as a special characteristic of a creative person. What he understands by problem sensitivity is the ability to approach the material and social environment with an open, critical attitude and to discover problems and opportunities for improvement, contradictions, inconsistencies and novelties (Preiser 1976), which is linked to *problem posing*. In mathematics, as a constantly broadening science of self-created abstract structures, the identification, extension, narrowing or widening and transferring of (new) scientific questions plays a vital role. "Problem finding" or "problem posing" per se is seen as creative act by some authors (Leung 1997; Silver 1994). It is partly also viewed as impulse for especially creative performances (Sheffield 2009). In school situations however, problem posing is often not a consequence of a genuine impulse to discover, but specifically initiated by the teacher. The specifications here may vary: based on an already solved problem, related problems can be identified and follow-up questions can be raised. Independent of a concrete problem, questions may be developed, for example, concerning a specific mathematical content or context, specific terms or numbers, specific solution strategies, but also entirely without predefined specifications. In any case, problem posing is closely related to problem solving. For a detailed overview of problem posing in mathematics learning see the report by Singer et al. (2011).

Problem solving is already an important part of mathematics classes in primary school, where mainly problems are dealt with that can be solved without a broad mathematical knowledge base. Since it is particularly challenging for young students to formulate their own questions, creativity in connection with problem posing might mainly manifest itself in the design of diverse variations of given, perhaps even partly solved problems.

With mathematically interested and experienced older students and even more with adult researchers in mathematics, problem solving and problem posing are usually embedded in more comprehensive *theory building processes*. Here, the handling of an initial problem becomes part of a circular process of problem solving and problem posing through variation and expansion and the subsequent analysis of this circle. The results and methods as well as the newly developed terms and logical relations and, respectively, the novel strategies and tools emerging from this process form a "theoretical fabric", which is then optimised, preserved and integrated into the existing knowledge base (Fritzlar 2008).

In such theory building processes, creative acts, the invention of (subjectively) new mathematical objects and structures as well as new mathematical methods play an important role. They also specifically highlight the interplay between divergent thinking, i.e. the ability to develop and elaborate diverse and original ideas with fluency, and convergent, i.e. logical and evaluative, thinking.

It can hardly be expected that young students with little mathematical experience are already capable of such theory building processes. However, concerning mathematical creativity, it could be possible that primary school children are already able to create subjectively new mathematical objects and relations and gain mathematical experiences in investigating these (Fig. 3.6). Thereby the student's invention can either be rather target-oriented, especially when they are dealing with a superordinate problem, or relatively free.

Overall, mathematical creativity in primary school age appears when students work on low-knowledge problems, vary given problems and create mathematical objects. This creativity can be especially high when students work flexibly and/or invent original products.

The close connection between creativity and problem solving or problem posing has been proposed many times (Chamberlin and Moon 2005; Haylock 1987; Leikin and Lev 2013; Leung 1997; Pehkonen 1997; Silver 1994; Sriraman 2009; Yuan and Sriraman 2011). In a case study we therefore want to explore in how far students of the fourth, fifth and sixth grade are already capable of creating *subjectively new mathematical objects*. Focusing on arithmetic, which is of particular significance for mathematical education at primary school level, possible novel mathematical objects are, for example, numbers, sequences of numbers, relations, operations and algorithms.

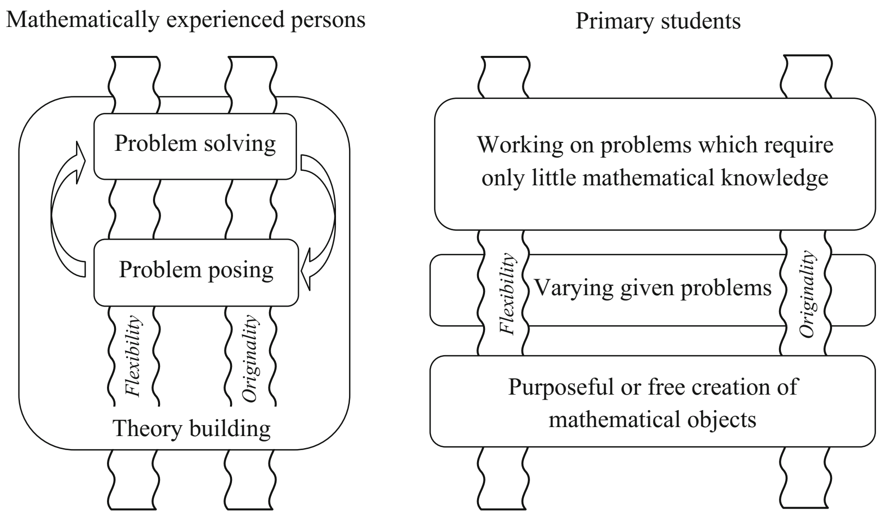


Fig. 3.6 Initiating mathematical creativity

3.6 Creativity as Inventing Mathematical Objects in Primary School

In recent literature, most researchers suggest that creativity is the result of confluence of different individual and environmental factors. According to Sternberg and Lubart (1996, 1999), for example, creativity results from the interplay between six different, but not independent resources: intellectual abilities, knowledge, styles of thinking, personality, motivation and environment. At this point, we cannot deal with all mentioned resources in depth, but for the implementation of our case study this "investment theory" approach had the following consequences: in a first "round" we worked with children participating in talent programs or special classes for mathematically gifted students because we could assume that these children would be intrinsically motivated to deal with mathematical problems and possess a broader mathematical knowledge base (relative to their age group). In addition, we have developed a collection of problems which inspire the creation of mathematical objects and relations. Finally, it was important for us to create an environment in which students can propose their ideas without the pressure of being assessed.

3.6.1 Inventing Mathematical Operations

The aim of the investigations described below was to encourage the children to invent subjectively new computing operations. The investigations were carried out with 127 fourth-graders and 33 fifth-graders. 35 of the fourth-graders participated at a fostering project for mathematically gifted and interested children. They were proposed for participation by their teachers. In addition, they had to pass an entrance test including tasks that test essential characteristics of mathematical giftedness (cf. Assmus, this volume; Käpnick 1998). The other fourth-graders took part at a mathematical correspondence circle for mathematically interested children and were also chosen by teachers. The fifth-graders attended a special school for mathematics. An entrance test had to be successfully completed for admission to this school.

To become familiar with this type of task, the students should first decrypt predefined operations. The next problem was to invent new arithmetic operations (to design appropriate tasks for their classmates to solve) and to explain how the calculation works. For example, the following is a concrete task used in the correspondence circle (original German):

"Marc finds the calculation modes addition, subtraction, multiplication and division boring. He has designed a new type of arithmetic operation and for this purpose invented this operation sign: $\diamondsuit$

He calculates: $1 \diamondsuit 3 = 5$ $6 \diamondsuit 4 = 16$ $3 \diamondsuit 1 = 7$ $0 \diamondsuit 5 = 5$

(a) Explain how Marc's type of calculation works.
(b) Find another new type of calculation. Invent an operation sign, give 4 examples and explain how your calculation works. Think of a name for your type of calculation."

For the most part, the students worked alone, partly in pairs. They received no assistance by the researchers. The students found different ways of creating new arithmetic operations. Similarly to place-value systems, they for example combined numbers into new numbers (Fig. 3.7).

In all student groups, familiar arithmetic operations were combined into new operations in a wide variety of ways. The Figs. 3.8 and 3.9 show some examples. For a better understanding, we provide the mathematical formula for describing the operation.

We could also observe combinations of known operations and transformations of numbers, for instance by rounding or summing their digits. Figure 3.10 shows an example developed by a fourth-grader. Another member of the correspondence circle had the idea to change calculation rules (Fig. 3.11).

With fifth-graders, combinations of the already presented invention strategies could be observed as well (Fig. 3.12).

Very interesting seems the operation on (Fig. 3.13) known operations invented by a fifth-grader.

We also asked the fifth-graders to investigate their new operations and look for interesting characteristics. Figure 3.14 shows some examples.

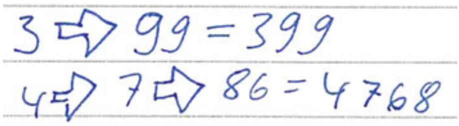
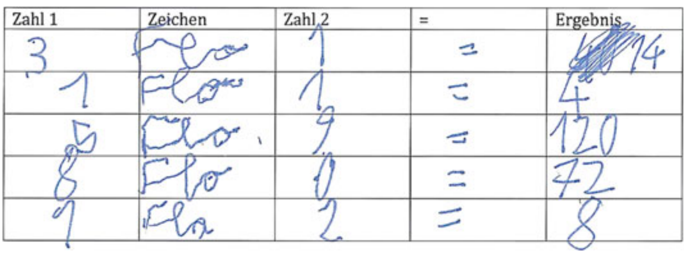
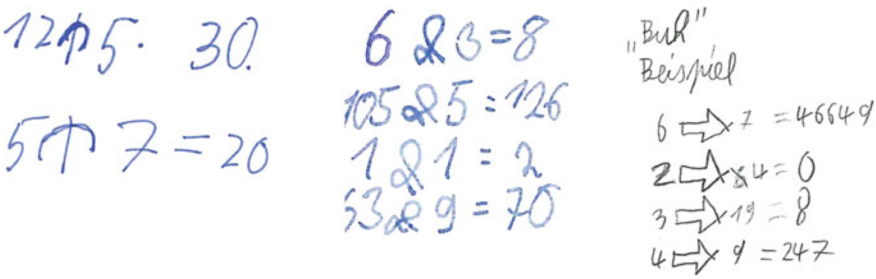


Fig. 3.7 Composing numbers (5th-grader)



$$a^2 + b^2 + a + b$$

Fig. 3.8 Creating a new operation (4th-grader)



$$\frac{(a-1)+(b-1)}{2}\cdot 4$$

(4th-grader)

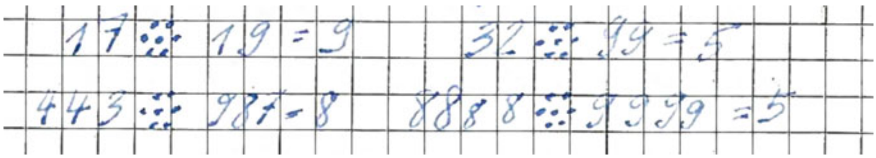
$$\frac{a}{b}+a$$

(5th-grader)

$$a^a - b$$

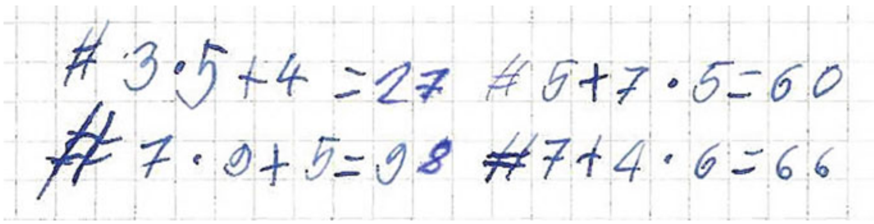
(5th-grader)

Fig. 3.9 Creating new operations



Repeatedly compute the sum of digits of both numbers until they are single figures; then add these figures.

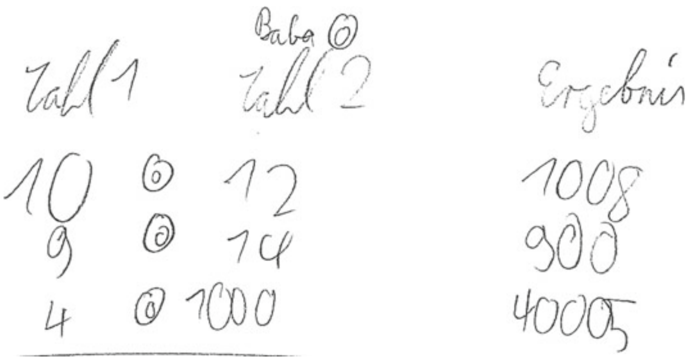
Fig. 3.10 Combining known operation and transformation of numbers (4th-grader)



If # is placed before the arithmetical task, the order of operation rule changes.

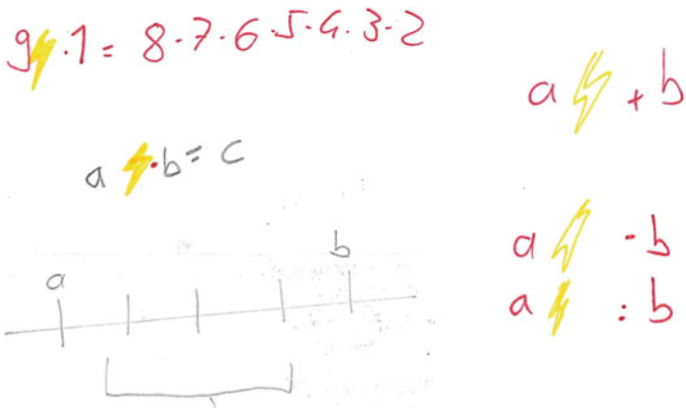
Fig. 3.11 Changing calculation rules (4th-grader)

At least in the last examples, the students' willingness and commitment for investigating their new operations demonstrate that they do not only use new names or signs but actually invent (subjectively) new mathematical *objects*.



Compose both numbers and subtract the sum of digits.

Fig. 3.12 Combining invention strategies (5th-grader)

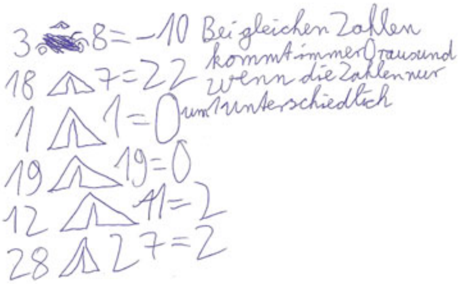


All natural numbers between two numbers *a* and *b* will be linked by the specified operation.

Fig. 3.13 Operation on known operations (5th-grader)

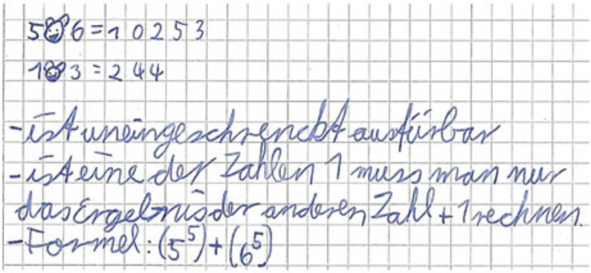
3.6.2 A Purposeful Invention During Problem Solving

The inventions of arithmetic operations described above can be seen as free creations; the following example is a purposeful creation during problem solving. In a math circle for sixth grade students, we used the following problem (cf. Kuzman 2016) for the solution of which the students invented different encryption methods.



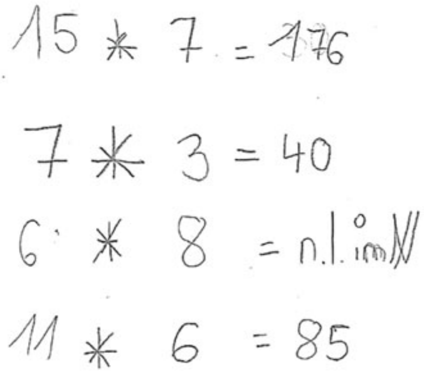
If the numbers are the same, the result is always 0.

$$\mathbf{2} \cdot (\boldsymbol{a} - \boldsymbol{b})$$



- Can be executed without restrictions
- If one of the numbers is 1, you only have to calculate the result of the other number plus 1

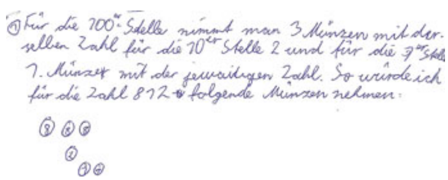
$$\boldsymbol{a}^{\mathbf{5}} + \boldsymbol{b}^{\mathbf{5}}$$



- If the first number is less than the second number, it is not solvable in $\mathbb{N}$.
- If there are two equal numbers, the result is 0.
- If the first number is 1 greater than the second, the result is exactly the same as for the addition task.

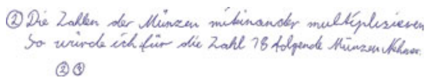
$$(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$$

Fig. 3.14 Invented operations with identified characteristics (5th-grader)



For the hundreds digit you take 3 coins with the same number, for the tens digit 2 and for the units digit 1 coin with the respective number. Thus, I would take the following coins for the number 812:

Fig. 3.15 Tim’s first idea



Multiply the numbers of the coins by each other. Thus, for the number 18, I would use the following coins:

Fig. 3.16 Tim's second idea

Take as many coins as is the numerical value.

Fig. 3.17 Tim's third idea

People have always tried to exchange secret messages in a form that cannot be read by outsiders. Today, it is up to you to invent an encryption method which could already have been used in ancient times when messages were still sent by courier. Only those who know the encryption method should be able to encode and decode the message.

The message can consist of a natural number between 1 and 999.

It should be encrypted by coins, each of which is labelled with a number between 0 and 9. The courier should be able to carry the coins in a small bag, which of course is thoroughly shaken during transportation. There are many coins of each kind available for encoding.

Your task is to invent a really good encryption method!

The six participating students attend the special school for mathematics mentioned earlier.

First of all, each student developed their own suggestions for the encryption, which they then presented to each other. Tim presented three ideas (Figs. 3.15, 3.16 and 3.17).

Here, the hundreds, tens and units digits are coded by three, two and respectively one coin of the corresponding sort. This code for coins, however, cannot always be definitely decoded, because for instance 811 and 188 would result in the same coin representation. (All in all, the algorithm fails at 90 numbers.) In a later discussion, there was the idea to "repair" this method, namely by using the respective coin four times in the case of identical units and tens digit number.[2]

Numbers should be encoded as a product of the coin values. However, this method fails with multi-digit prime factors (e.g. 23, 51, 91).

Here, numbers are coded by the number of coins. With this method, Tim has invented a simple procedure which however is easy to see through and not very practical: on average, 500 coins are needed for encoding one number.

[2]Having the same units and hundreds digit is also unproblematic with this improved method.

Tim's first coding idea can to a certain extent also be reversed. Jason, for example, suggested to first choose three kinds of coins and code the units digit with the corresponding number of coins with the lowest par value, the tens digit with the appropriate number of coins with medium par value and to proceed analogically for the hundreds digit. During the discussion, the students themselves realised that it is not possible to decode numbers with the digit 0 and, respectively, one- and two-digit numbers. This method, too, can be "repaired" if sender and receiver agree upon three kinds of coins. Thus, a maximum of 27 coins or, respectively, an average of 13.5 coins are needed.

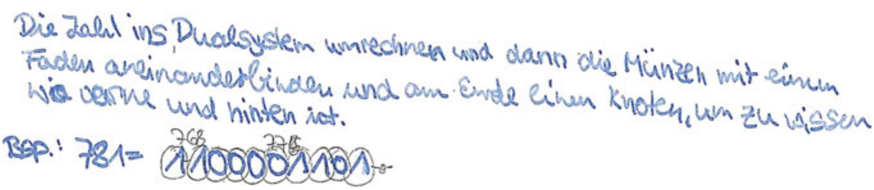
Anna presented two options (Figs. 3.18 and 3.19).

Through division, the number to be encoded can first be represented by smaller numbers, which can then be coded by coins. In the example, 781 would be represented by 86 coins with a value of 9 and one with a value of 7.

The second suggestion uses the conversion into the dual system. The coins should be tied up on a string with a knot, which makes it possible to keep the order of zeros and ones. In the follow-up discussion, the students themselves came up with the idea of associating the 2^n-digit of the binary number with the n-coin type and of putting a corresponding coin into the bag for each digit 1. Thus, the number 781 would be coded by the coins (9)(8)(3)(2)(0). Since $2^{10} - 1 > 999$ applies, all numbers required by the problem can be unambiguously encrypted with a maximum of nine coins.

…, i.e. 86 coins with a value of 9 and one with a value of 7.

Fig. 3.18 Anna's first idea



Convert the number into the dual system; tie the coins together with a string that has a knot marking its beginning.

Fig. 3.19 Anna's second idea

3.7 Conclusion and Outlook

Both examples firstly show that mathematically gifted primary school children are already capable of being mathematically creative and of developing and investigating subjectively new mathematical objects. With these results, we have achieved a main objective of our study. It also indicates the wide variety of creative products even among the small group of students participating in this experiment. A comparative observation furthermore shows that some products, e.g. in Figs. 3.11, 3.12, 3.13 or 3.19, are very rare, so in comparison to the others these products can be judged as very creative. Moreover, the students' ideas for maximally accurate and efficient encryption algorithms can be seen as seeds of theory building processes. It is, of course, also possible to follow up on the idea presented in the first example, e.g. regarding algebraic properties. Therefore, the described results and procedures also indicate further mathematical potentials beyond creativity.

As further investigations have shown, tasks like the invention of new mathematical operations may also encourage primary school students in regular classes to be creative with mathematical objects. For instance, we asked fourth-graders in two regular classes to invent new arithmetic operations like the ones presented above. Almost all of them were able to meet these requirements. Mostly they combined familiar arithmetic operations into new operations. Although the majority of these inventions were not as complex and diverse as many inventions in the gifted group, creative approaches were apparent. Thus, we think the tasks are suitable for initiating mathematical creativity (on different levels) in almost all primary school children. While in the free creation of mathematical objects, differences between gifted and non-gifted students appear in the different mathematical complexity of the invented objects, it could be possible that the purposeful creation of objects during problem solving is only achievable for gifted students. This hypothesis should be scrutinised in further studies.

The creation of mathematical objects at primary school age is not reducible to the creation of mathematical operations and algorithms. Furthermore, children can invent their own mathematical terms (e.g. special numbers like MUM- and DAD-numbers; special geometrical shapes) and formulate their characteristics. The area of geometrical patterns also offers numerous opportunities to create new figure patterns, geometrical ornaments or tilings. This means that also in primary grades and for almost all primary school students, there are many chances to be mathematically creative in the sense of creating (subjectively) new mathematical objects. Further studies with gifted students should investigate their abilities to invent different purposeful mathematical objects to review the embedded model of giftedness and creativity (left part of Fig. 3.5).

The type of tasks described in this paper does not only allow for creative mathematical action. It also creates the possibility of detailed investigations of mathematical characteristics of the created objects. For instance, the computing operations can be examined with regard to group axioms, commutativity or other algebraic features. It is also possible to investigate the relations between details of the definition of a new operation and its algebraic characteristics. In this way, students will be encouraged to take a stronger algebraic perspective, also on known operations and other mathematical objects. Our studies indicate that this is possible already for younger students.

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