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Florence Mihaela Singer *Editor*

Mathematical Creativity and Mathematical Giftedness

Enhancing Creative Capacities in Mathematically Promising Students

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Editor

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Introduction

What are the relationships between mathematical creativity and mathematical giftedness? How could mathematical creativity contribute to a balanced development of the child? What are the characteristics of mathematical giftedness in early ages? What about these characteristics at university level? What teaching strategies might enhance creative learning? How mathematical promise of young children can be maintained and extended towards a variety of professions? These are some of the questions addressed by this book. The book offers, among others: analyses of substantial learning environments for supporting creativity in mathematics lessons, discussions of a variety of strategies in problem-solving and posing, investigations of students' progress along years of training, and examinations of technological tools and virtual resources meant to enhance learning with understanding. Multiple perspectives in the interdisciplinary fields of mathematical creativity and giftedness are developed to offer a springboard for further research. The theoretical and empirical studies included in the book offer consistent data useful for researchers, as well as for the teachers of the gifted in specialized or inclusive settings, at various levels of education.

Chapter 1
Enhancing Creative Capacities in Mathematically-Promising Students. Challenges and Limits

Florence Mihaela Singer

Abstract The links between research in mathematics education, psychology of creativity and research in gifted education started to gain more attention in the last decade, from researchers and the large public as well. The paper is intended to provide a concise survey of these links, with a focus on: frameworks for studying students' creativity and giftedness in mathematics; domain specificity of creativity; some characteristics of mathematical creativity resulting from its specificity; relationships between mathematical giftedness and creativity from a mind-and-brain perspective; relationships between creativity, giftedness and social inclusion; underlying connections between mathematical creativity and innovation, creativity and metacognition, creativity, giftedness and expertise; and the teaching of mathematically-promising students with a focus on structuring their mathematical competencies. The paper offers also brief reviews of the chapters included in the book, stressing on the benefits of an integrated approach of creativity and giftedness in mathematics education.

Keywords Mathematical creativity · Mathematical giftedness
Mathematically-promising students · Problem solving · Problem posing
Domain-specific creativity · Expertise · Metacognition

1.1 Introduction: Setting the Context

A few years ago, in a comprehensive review study, Leikin (2011) identified a gap between research in mathematics education and research in gifted education. She noticed that in the first ten years of the twenty-first century, seven key journals in the fields of intelligence and giftedness (*American Psychologist, Creativity Research Journal, Gifted Child Quarterly, High Ability Studies, Journal for the*

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Education of the Gifted, The Journal of Secondary Gifted Education, and *Review of General Psychology*) published very few articles devoted directly to mathematical giftedness or creativity. In addition, a search of seven leading research journals in mathematics education (*Educational Studies in Mathematics, Focus on Learning Problems in Mathematics, For the Learning of Mathematics, Journal for Research in Mathematics Education, Journal of Mathematical Behavior, Mathematical Thinking and Learning,* and *ZDM Mathematics Education*) revealed that in that same decade again only few articles were explicitly devoted to mathematical giftedness. A small number of publications in other journals (such as *Journal of Educational Psychology*, *Psychological Science Journal,* and *Journal of Applied Psychology*) have been focused on specific issues related to problem solving and mathematical reasoning in the gifted population, usually with a focus on creativity (Leikin 2011). In addition, a few edited volumes had in focus these aspects (e.g. Sriraman 2008; Leikin et al. 2009).

Things seem to have changed, however, in the last years. Thus, recently, two new special issues of *ZDM Mathematics education* address the topics of mathematical giftedness (Singer et al. 2017a) and psychology of mathematical creativity (Sriraman 2017), completing the series initiated by the special issue devoted to mathematical creativity (Leikin and Pitta-Pantazi 2013). The change is not only in the focus, but it goes deeply into the relationship between psychology of creativity and the applied domain-specific study of creativity and giftedness in mathematics. Moreover, communities of practice contribute to developing these fields on a more systematic base, and their activities were reflected in 2016, during the 13th International Congress on Mathematical Education (ICME-13), through the Topic Study Group 4 (TSG 4) focused on *Activities for, and Research on, Mathematically Gifted Students* and TSG 29, which addressed *Mathematics and Creativity.* TSG 4 published a topical survey before the congress, and this current monograph builds on the work presented in both of those TSGs during the congress.

What looked like to be a far-off goal, i.e. situating mathematics education research within an existing canon of work in mainstream psychology (Sriraman et al. 2013) seems to be today more tangible than ever. Moreover, the call for interdisciplinary views appears to have become the rule rather than approaches that request narrow (frequently ill-defined) conceptualizations. For example, the recent developed model of the Active Concerned Citizenship and Ethical Leadership (ACCEL) for understanding giftedness (Sternberg 2017a) covers aspects related to: teaching for creativity; the role of science, technology, engineering, and mathematics (STEM) in teaching for wisdom; the developmental nature of giftedness; teacher education (Sternberg 2017a, b), and many other integrative perspectives. The present book puts together views underlying, beyond the variety of approaches, strong connections between the psychology of creativity, mathematics education, and the study of giftedness as an interdisciplinary field.

1.2 Frameworks for Studying Students' Creativity and Giftedness in Mathematics

Starting from the well-known work of Torrance (1974), researchers usually explore mathematical creativity through the following parameters: originality, fluency, and flexibility. Adaptations of this approach have been proposed by many researchers, who analyzed students' creativity in problem-posing and/or problem-solving; usually, these approaches are contextualized by the study design. For example, Kontorovich and Koichu suggested a framework based on four "facets": resources, heuristics, aptness, and social context in which problem posing occurs (Kontorovich and Koichu 2009). A refinement of this framework integrates task organization, knowledge base, problem posing heuristics and schemes, group dynamics and interactions, and individual considerations of aptness as parameters in analyzing creativity in problem-posing situations (Kontorovich et al. 2012).

Another set of studies investigated the relationship between students' cognitive flexibility and their abstract thinking using mathematics problem posing as a tool for detecting creative behavior in school children (Singer and Voica 2015; Singer et al. 2013a, b). Here, within an organizational-theory context, creativity has been discussed in terms of *cognitive flexibility*, which can be conceptualized as consisting of three primary constructs: *cognitive variety*, *cognitive novelty*, and *change in cognitive framing*. In problem-posing contexts, *cognitive novelty* was assessed by the "distance" from the initial problem. *Cognitive variety* was measured by the extension and the differences among the posed problems starting from a given one. *Cognitive framing* was assessed through the capacity of generating a pattern of thinking for a class of problems. In this case, creativity refers to the student's ability to change his/her cognitive frame, or even more, to his/her ability of reframing. This model proved effective for detecting some characteristics of mathematical creativity less visible when using other study frameworks.

The traditional psychometric approach seems not to offer today enough consistency for the study and understanding of nature and nurture of giftedness. Scrutinizing some of the deeply held assumptions about the nature of giftedness, Dai and Chen (2014) consider that a contextual developmental approach is more viable than simple psychometric tests, and highlight that understanding high potential and educating youths should be approached beyond the tensions between the gifted-child and talent-development movements, and between excellence and equity in a constructive and comprehensive way.

The issue of designing frameworks for the analysis and the development of students' creativity and students' giftedness raised the interest not only of researchers, but it was seen pragmatically as an issue with political-strategic implications. Thus, some countries and states developed general frameworks for designing differentiated learning experiences for gifted students having in view to complement the official standards and benchmarks. For example, in the US, Florida's Frameworks for K-12 Gifted Learners (2005–2007) stipulate that by graduation from K-12 education, the student identified as gifted will be able to:

critically examine the complexity and apply investigative methodologies; create, adapt, and assess multifaceted questions; conduct thoughtful research/exploration; and think creatively and critically to identify and solve real-world problems. In addition, the student identified as gifted should be trained so that become able to: assume leadership and participatory roles in both gifted and heterogeneous group learning situations; set and achieve personal, academic, and career goals; and develop and deliver a variety of authentic products/performances that demonstrate understanding (Florida's Frameworks for K-12 Gifted Learners (2005–2007).

In some countries, strong emphasis is put on training students for the International Mathematics Olympiad (e.g., China, Russian Federation, Korea), while special programs are devoted to (identified) gifted students in others (e.g. Germany, Israel, Singapore).

In other situations, communities of parents, teachers, and NGOs created groups that assumed theoretical views on giftedness and tried to influence the perception of the giftedness and made a plea for more attention granted to this special social group. The uniqueness of the gifted renders them particularly vulnerable and requires modifications in parenting, teaching and counseling in order to help these children to develop optimally (e.g. The Columbus Group, 1991). Concluding this section, from theoretical frameworks to ad hoc conceptual underpinnings, there is a large gamut of approaches in discussing giftedness in relation to creativity, and the involved special target population needs careful attention from both research and action.

1.3 Domain Specificity of Creativity

From a cultural-anthropological view, each domain of knowledge developed a way of thinking that is intrinsic to that domain. Therefore, it is meaningful to ask if being creative is a general trait or it is a domain-specific feature. Various observers of the theoretical and empirical creativity literature (e.g. Csikszentmihalyi 1988; Gardner 1993; Sternberg and Lubart 2000) assume that the debate might be settled in favor of domain specificity.

In fact, Baer (2010), among others, provided convincing evidence that creativity is not only content specific but is also task specific within content areas. Moreover, research has suggested that transfer across domains is both difficult to achieve and relatively rare (Willingham 2002, 2007). For example, a large-scale study that looked at the possibility of transfer of practiced intellectual skills came to a very negative conclusion. In this 6-week training study, 11,430 participants were trained several times each week on cognitive tasks designed to improve reasoning, memory, planning, visuospatial skills, or attention. “Although improvements were observed in every one of the cognitive tasks that were trained, no evidence was found for transfer effects to untrained tasks, even when those tasks were cognitively closely related” (Owen et al. 2010, p. 775).

If creativity is domain-specific, then the assessment of creativity must also be domain-specific. However, creativity assessment has often assumed domain generality. Thus, the most common tests of creativity have been divergent thinking tests, and the most widely used divergent thinking tests are the Torrance tests of creative thinking (TTCT), with its two components: figural and verbal, although both are used as general measures of creativity (e.g. Kaufman et al. 2012). This finding is in line with evidence Torrance himself offered showing that figural and verbal divergent thinking scores are not correlated, and are therefore measuring two essentially unrelated cognitive abilities. Divergent thinking may be important, but we may need multiple measures of it, domain by domain, for it to be useful (Baer 2012).

Recently, the relationships among domain-general divergent thinking ability, domain-specific scientific creativity, and mathematical creativity have been explored through a study that investigated the relative influences of domain knowledge and divergent thinking ability on scientific creativity and mathematical creativity (Huang et al. 2017). By exposing 187 primary school sixth-graders to The Mathematical Creativity Test (MCT) and the New Creativity Test to assess students' domain-general divergent thinking ability, they found that Mathematical Creativity Test is only modestly positively correlated with the general creativity test and mathematical achievement can effectively explain the variance in MCT performance, but the creativity test cannot. These results imply that there are diverse influences from domain knowledge and divergent thinking ability on creativity in different domains, which, again, supports the domain-specificity of creativity.

Moreover, evaluating potential creativity of 482 children and adolescents, Barbot et al. (2016) found that the contribution of each variance component (thinking-process general, thinking-process specific, domain-specific, task-specific, and measurement error) depends greatly on the task under consideration, and that the contribution of a general creative thinking-process factor is overall limited. Consequently, specialized thinking modalities might be the focus of identifying and developing creativity in a domain-specific approach.

1.4 Mathematical Creativity

As new evidence-based arguments to the debate regarding general versus domain-specific creativity incline the balance towards the second part, it makes sense to consistently discuss mathematical creativity. As highlighted above, there is cumulated evidence that mathematical creativity is of a special type, which distinguishes from other types of creativity. Large part of evidence comes from problem solving studies; the literature in this area is vast, in both time extension and quantity (e.g. Hadamard 1945; Krutetskii 1976; Leikin 2009).

Another set of evidence is brought by problem posing studies, which have gained more terrain in the last decade. Some of these studies investigated the relationship between students' cognitive flexibility and their abstract thinking using mathematics problem posing as a tool for detecting creative behavior (Singer et al. 2011, 2013a, b; Pelczer et al. 2011). In problem-posing contexts, creativity has been discussed in terms of *cognitive variety*, *cognitive novelty*, and *change in cognitive framing*. In a problem-posing context, an indicator of cognitive variety might be the number of different posed problems; *cognitive novelty* was assessed by the "distance" from the initial problem; *cognitive variety* was measured by the extension and the differences among the posed problems starting from a given one. *Cognitive framing* refers to the capacity of generating a pattern of thinking for a class of problems, accompanied by the ability to make changes into that cognitive frame. Singer and Voica (2013) found that, in PP contexts, high achieving students tend to make incremental changes to some parameters in order to arrive at simpler and essential forms needed in generalizing sets of data. It follows that mathematical creativity requires abstraction and generalization, which emerge from gradual and controlled incremental changes in cognitive framing (Singer 2012a, b). Mathematically promising students display a need for consistency that seems to limit their cognitive flexibility to a certain extent. A tension between the students' tendency to maintain a built-in cognitive frame, and the possibility to overcome it (Singer and Voica 2015; Voica and Singer 2012, 2013) is highly visible in the experimental data. These studies revealed that, in problem-posing situations, the students develop cognitive frames that make them cautious in changing the parameters of their posed problems, even when they made interesting generalizations, because of the constraint they self-impose to devise mathematical problems that are coherent and consistent.

Consequently, the training for the development of mathematical creativity should include features that distinguish it from training for creativity development in general. Briefly said, while in the latter, more general case, techniques are to be used for stimulating the free development of ideas, in mathematics learning the variation of parameters should be practiced within a variety of activities where the processes are mindfully controlled and oriented towards abstraction and generalization (e.g. Singer and Voica 2015). More research is needed to address the characteristics of mathematical creativity as compared to other types of creativity and the consequences such studies could have on research and practice.

1.5 Relationships Between Mathematical Giftedness and Creativity—A Mind-and-Brain Perspective

Traditionally, giftedness has been related to a high IQ. However, the IQ concept becomes more and more controversial. Dweck (2006) supports this statement with her research on a growth vs. a fixed mindset and its effect on student learning. In addition, IQs do not remain stable over time. Briefly, a high IQ can be considered necessary but not sufficient for high achievement (Nolte 2012). As Cross and Riedl (2017) underlined, IQ testing should be reserved for finding specific forms of high ability and as a diagnostic tool, not as a gatekeeper that continues to perpetuate the underrepresentation of some groups.

A generous direction in cognitive science and neuroscience brings new data to understanding the development of mathematical talent and innovation in students of all ages and from all backgrounds. Mathematical giftedness started to be conceptualized in recent decades within a context that is sensitive to modern biology, based on studies of cognition within the discipline of educational neuroscience. Thus for example, Woolcott (2011) discussed exceptional performance in mathematics in relation to cognition and performance as a product of internal processing and environmental connectivity of the human organism.

Research that has examined the neuropsychological processes engaged by gifted and talented learners provides insights into how they process information, convert it to knowledge and make connections. It also assists in understanding the creative activity they display. These learners understand, think and know in ways that differ qualitatively from how regular learners perform these activities (Munro 2005).

Data derived from several psychophysiological studies support an important relationship between the specialized capacities of the right hemisphere and mathematical ability. Commonly, associated with giftedness is right hemisphere dominance (Jin et al. 2007), with frontal asymmetry in the right cortical area as a possible physiological marker of giftedness (Fingelkurts and Fingelkurts 2002).

The discussion is sometimes in terms of 'neural efficiency', where gifted functioning involves a more integrated brain with greater cooperation between the hemispheres (O'Boyle 2008), with reduced activity in certain areas as compared with average brains when performing similar tasks—possibly implying that gifted brains spend less time on such tasks. A more recent functional magnetic resonance imaging (fMRI) study using mental rotation to analyze mental capacities of mathematically gifted adolescents (Prescott et al. 2010) seems to confirm this.

Case studies of extremely gifted individuals often reveal unique patterns of intellectual precocity and associated abnormalities in development and behavior. The bulk of scientific inquiries provide evidence of unique patterns of right prefrontal cortex and inferior frontal activation implicated in gifted intelligence, although additional studies suggest enhanced neural processing and cerebral bilateralism (Mrazik and Dombrowski 2010).

Research literature in the area of mathematical ability at a very early age describes various early signs of mathematical giftedness in children (e.g. Diezmann and Watters 2000; Winner 1996). For example, Straker (1983) observed that mathematically gifted preschoolers generally show:

> …a liking for numbers including use of them in stories and rhymes; an ability to argue, question and reason using logical connectives: if, then, so, because, either, or…; pattern-making revealing balance or symmetry; precision in positioning toys, e.g. cars set out in ordered rows, dolls arranged in order of size; use of sophisticated criteria for sorting and classification; pleasure in jig-saws and other constructional toys. (Straker 1983, p. 17)

These signs, recognized in a large gamut of studies, can become abilities in adults developing a career that value them, or not. Still, the connection between mathematical creativity and giftedness can be reconsidered from the perspective of *mathematical promise*. The concept has been developed by the National Council of Teachers of Mathematics (NCTM) with the purpose of maximizing variables such as abilities, motivation, beliefs, and experiences or opportunities (Sheffield et al. 1999) among all students.

The question if giftedness and creativity are synonyms in mathematics has been addressed by Sriraman (2005), and his answer discusses the professional and school realms. The role of creativity in the education of the gifted has been analyzed in numerous studies (e.g. Mann 2006; Koichu and Berman 2005; Sriraman 2003; Chamberlin and Moon 2005; Reed 2004). Many of these studies emphasize the interactions between creativity and the development of mathematical expertise (e.g. Singer and Voica 2016).

Much of the empirical research explores the learning processes of mathematically talented students through problem-solving strategies, noticing their creative approaches (e.g. Amit and Neria 2008). Equally important is the connection between mathematical creativity and giftedness identification in relation to problem posing (e.g. Singer et al. 2015). Problem-posing sessions have shown effective for identifying gifted students. Voica and Singer (2014) found three characteristics that can offer an indication of mathematical giftedness: a thorough understanding of conveyed mathematical concepts, an ability to generalize reasoning moving towards abstractions, and a capacity to frame and reframe content while keeping consistency of new-created problems. In addition, the level of abstraction used to solve a given problem was correlated with the novelty of the newly posed problems starting from the given one; therefore, the abstraction level spontaneously used by a child might be a good predictor of the child‘s creative potential (e.g. Pelczer et al. 2015). More research is needed to see how mathematical giftedness and creativity mutually assist each other in children of various ages.

1.6 Creativity, Giftedness and Social Inclusion

Silverman (2013) has suggested that certain affective traits such as heightened sensitivity, early concern with moral issues, empathy, perfectionism, social maturity, and aesthetic appreciation are evident in gifted children. Beyond considering these as features allowing giftedness identification, these traits may, in a combination of internal and situational factors, put children at psychological risk, leading to interpersonal and psychological problems. Among the issues that may affect gifted children are their asynchronous development, their difficulties of socializing with peers and adults, and their own problems with self-learning (e.g. Singer et al. 2016).

There is also a special category of vulnerable students—twice-exceptional children. These are those who possess giftedness or exceptional ability in one or more areas in combination with special needs, a learning disability or a handicap in other areas. They may achieve high scores on certain intelligence tests but may not do well in school. They may have giftedness in combination with autism, emotional and behavioral disorders, or learning disabilities (dyscalculia, dyslexia, dysgraphia), ADD or ADHD, visual and auditory processing anomalies, or sensory integration and modulation disorders (Chamberlin et al. 2007).

The need for social integration and acceptance is very strong in gifted and talented children, although frequently they seem to isolate themselves from peers. This was identified also in many studies focusing on gifted children social interactions, but also indirectly on problem-posing contexts, where students' posed problems reflect not only a mathematical content, but also proposer's attitudes and affects. Singer and Voica (2015) noticed that both the posed problems and the behaviors displayed by the students in these studies highlighted a social dimension (Singer and Voica 2015; Pelczer et al. 2015).

A review made by Kurup et al. (2013) suggests addressing the following aspects in helping mathematically promising children develop their full potential: the need for talent-appropriate stimulation that is not restricted by the chronological age of the child (Roedell 1989); the need for counseling, acceptance and recognition of talents by peers, parents and teachers (Gross 1998; Silverman 2002); and programs and encouragement to aid the growth and blossoming of their special abilities (Reis et al. 1998).

1.7 Creativity and Innovation

As noted by the US National Science Board (NSB) report, *Preparing the Next Generation of STEM Innovators*, giving every student the opportunity to achieve his or her full potential is critical as we "will increasingly rely on talented and motivated individuals who will comprise the vanguard of scientific and technological innovation" (NSB 2010).

Even for moderately gifted students, research shows that approximately 40–50% of traditional classroom material could be eliminated for targeted gifted students in one or more of content areas, among which is mathematics (Reis et al. 1998). Care must be taken not to skip critical material, however, but to ensure that students are engaged and passionate about the mathematics they are learning, and are not simply memorizing algorithms or accelerating so they can finish taking required mathematics classes early.

The joint NAGC/NCTM/NCSM publication: *Using the Common Core State Standards for Mathematics with Gifted and Advanced Learners* has the following recommendation:

> In order to support mathematically advanced students and to develop students who have the expertise, perseverance, creativity and willingness to take risks and recover from failure, which is necessary for them to become mathematics innovators, we propose that a ninth Standard for Mathematical Practice be added for the development of promising mathematics students – a standard on mathematical creativity and innovation: Solve problems in novel ways and pose new mathematical questions of interest to investigate. The

> characteristics of the new proposed standard would be that students are encouraged and supported in taking risks, embracing challenge, solving problems in a variety of ways, posing new mathematical questions of interest to investigate, and being passionate about mathematical investigations. (Johnsen and Sheffield 2012, pp. 15–16)

1.8 Creativity, Giftedness and Expertise

The fact that the interplay between interests, activities, the environment, and mathematical explorations affect students' mathematical achievement leads to question whether there is a need to distinguish between giftedness and expertise. The need for expertise is a useful analog for many seemingly domain-general theories of creativity. It is true that some degree of expertise is important in many domains—creativity requires some level of knowledge and skill in most domains—but the content of such expertise varies by domain.

The formal school learning aims to expose new generations to the knowledge domains as they have been developed along the cultural history of humanity, by large contributions of communities of practice. As a product of human culture, each domain of knowledge has structured a specific way of thinking and, therefore, a domain-specific training becomes necessary to foster specialized cognitive mechanisms that are commonly activated in the expert of a domain.

Glaser (1988) characterizes expertise on six cognitive dimensions: knowledge organization, complexity of problem-solving representation, changing thinking schemes, goal-oriented procedural knowledge, automatic procedures, and metacognition. In terms of knowledge structure and organization, the expert possesses knowledge organizations that can integrate and structure new information items so that they are rapidly selected from memory in structured units, while novices hold isolated, frequently disconnected elements of knowledge. Regarding the complexity of problem-solving representation, the novice approaches a problem starting from its surface features, while the expert makes inferences and identifies principles underlying the surface structures. In changing thinking schemes, the expert amends his/her own knowledge theories and develops schemes that facilitate more advanced thinking, while the novice manifests rigidity in changing a thinking scheme. In terms of goal-oriented procedural knowledge, the expert displays functional purpose-oriented knowledge, while a novice holds information without clearly understanding the applicability conditions. In terms of automation that reduces the concentration of attention, an expert can focus attention that alternates between basic capacity and higher levels of strategic thinking and understanding, using automated procedures to achieve good performance, while novices have difficulty in sharing attention, and they frequently get lost in details.

An important attribute of an expert in a domain is to identify problems and to tackle them in a knowledgeable manner. In a recent study, Singer and Voica (2016) found that expertise and creativity mutually support each other in the process of building a solution for a nonstandard problem. Consequently, in order to get individual relevant data (for example, to avoid the situation in which one solves a problem because he/she internalized automatized strategies for that category of problems and this is taken as giftedness) the identification of creativity should take place based on tasks situated in the proximal range of a person's expertise and exceeding his/her actual level of expertise, at the time of analysis.

1.9 Creativity and Metacognition

Various experiments that exposed students to problem-posing sessions have shown that participant children become able to: find alternative pathways, predict outcomes or generalizations, note failure in understanding, comeback or plan ahead in order to improve own knowledge. All these are parts of metacognitive capacity developed within the learning process. The metacognitive dimension manifested in a couple of ways. Thus, most of the students were able to analyze critically their own proposals and their own thinking mechanisms, which made them aware of their strengths, and to use these strengths to reinforce a well-defined cognitive frame for a problem. The results show that in the problem-posing process, students develop a genuine philosophy, which refers both to practical actions—embodied in their problem-posing strategies—and to the qualitative form of the posed problems. Typically, students start from a model to which they apply certain constraints based on the philosophy they developed, and they then spontaneously try to get a problem that is mathematically consistent and coherent (e.g. Voica and Singer 2012). Moreover, they get a sense of difficulty or beauty of a problem.

Having in view this capability for metacognitive approaches, Sheffield (1994) recommended that teachers of gifted and talented mathematics students should convey a sense of the beauty and wonder of mathematics in their enthusiasm for both mathematics and for teaching; have confidence in their own mathematical abilities; admit mistakes and enjoy learning along with the students; be continuously involved in professional development; and be willing to let students take over the direction and responsibility for their own learning.

1.10 Mathematical Competencies and Teaching Mathematically Promising Students

Research has shown that students have preferences for some sub-areas of mathematics, or for some problem-solving strategies, which can be relatively easy identified through problem-posing activities. Students' preferences reveal some students' strengths on which teachers can focus in order to develop their mathematical competencies (Singer 2012b; Voica and Singer 2012; Pelczer et al. 2015; Singer and Voica 2015).

In general terms, the mathematics-specific competencies to be developed in students along their school stages refer to: identifying relationships among mathematical concepts/objects; interpreting quantitative, qualitative, structural and contextual data included in mathematical statements; using algorithms and mathematical concepts to characterize a given situation locally or globally; expressing the quantitative or qualitative mathematical features of a contextual situation in order to model it mathematically; analyzing problem situations to discover strategies, to find and optimize solutions; and generalizing properties by modifying a given context or by improving or generalizing algorithms (Singer 2006). These formulations can be particularized, taking into account adequate mathematical content, for a specific age and curriculum trajectory. The competencies acquisition along schooling creates the premises for mathematically promising students to orient their potential towards more expertise in approaching problem situations.

Taking into account the specificity of mathematical creativity, the training for its development should include features that distinguish it from the development of creativity in general. Briefly said, while in the latter, more general case, techniques are to be used for stimulating the free development of ideas, in mathematics the variation of parameters should be practiced within a variety of activities where the processes are mindfully controlled and oriented towards abstraction and generalization (Singer and Voica 2015).

Thus, adequate tasks should develop a domain-specific intuition that favors expert behavior. Through multiple representations, students arrive at seeing their own mistakes, contextualizing them, and explaining them. A focus should be on developing metacognitive abilities in students. From this perspective, the use of problem-posing sessions in teaching is beneficial for students' personal development. Data show that mathematically promising students manifest a strong need for social interaction, which is frequently hidden in certain circumstances. Consequently, social interaction should be part of the teaching-learning process in the class in a consistent way, for example, by means of activities involving posing and solving problems organized in pairs or in small research-teams.

Sriraman (2017) noticed that advances in the study of the psychology of creativity can be consistently linked with mathematics education. Educating for a growth mindset in learning is crucial for the development of creativity.

Our contemporary dynamic society, exposed to unpredictable changes, needs better ways to train students for a changing world. As a practice of learning and thinking, problem posing may play an essential role in stimulating creative approaches and thus preparing students for more dynamic use of their knowledge. Considering that problem-posing research is an emerging force within mathematics education, Singer et al. (2013, 2015) advocate that the use of problem posing in classroom settings: can enhance students' engagement in authentic mathematical activity; develop students' creativity; may generate a positive effect on students' problem-solving achievement and/or their attitudes toward mathematics; and open students' thinking towards new ideas and approaches (Singer et al. 2011, 2013a, b; Ellerton et al. 2015).

The use of multiple representations together with strategies to move from one representation to another can provide explicit scaffolding for the transformation of students' intuitive ideas into domain-specific concepts and procedures, leading to an increase in expertise. These are more related to the cognitive part. Still, other aspects should be equally highlighted. Teachers need a clear goal for gifted education to act educating gifted students purposeful. Purpose is needed for transformational leadership, risk-taking capacity, and skills in ethical thinking. In the dynamic and inter-connected contemporary world, excellence and creativity should be combined with ethics.

The era of information and communication technology creates new social and physical environments and needs. Living in a world where interdependency and dynamics become main features of the global society, young generations have to face unpredictable changes they should learn to cope with. In these conditions, rethinking teaching effectiveness becomes a necessity. There are some capabilities that technology cannot provide yet, and which people need in the present and future society, briefly: higher order synthetic thinking, decision-making capacity under sometimes hardly-predictable situations, transfer capability for solving new problems in new contexts, and a set of values that orient personal behavior in social (and communication) contexts. Within this process, as Tirri (2017) underlined, teachers are seen as key agents in making a significant change in identifying and teaching the gifted. Researchers in gifted education should take the leadership in this change and commit to cooperation with schools.

1.11 Brief Overview of This Book

Within the above discussions, the present book synthesizes the developments presented during two topic-study groups at the 13th International Congress on Mathematics Education (ICME), which took place in Hamburg, Germany, in July 2016. The Topic Study Group 4 (TSG 4), which was focused on *Activities for, and Research on, Mathematically Gifted Students* and the Topic Study Group 29 (TSG 29), which addressed *Mathematics and Creativity*, put together their theories, research, policies and practices to generate a complex integrated approach addressing the issues of giftedness and creativity in mathematics.

The book is structured into four parts, advancing from theoretical underpinnings to practical matters: (1) Frameworks for studying mathematical creativity and giftedness; (2) Characteristics of students with exceptional mathematical promise; (3) Teaching strategies to foster creative learning; and (4) Tasks and techniques to enhance creative capacities.

A brief presentation of the book content follows, emphasizing some lines of thought in connection with the main ideas illustrated above.

An old issue is brought into the contemporary debate by Pitta-Pantazi, Kattou and Christou when discussing four components of Mathematical Creativity: Product, Person, Process and Press (Pitta-Pantazi et al. 2018, this volume). This chapter offers a broad view of various research studies conducted in the field of mathematical creativity which investigated the adaptation of the 1961 Rhodes' 4P model of creativity involving: person (mathematical ability, intelligence, general creative ability, age, gender, culture, personality traits and biographical experiences); product (a novel and useful idea or concept); process (the methodology, or the stages of the creative process); press (teaching environment and the teachers' role, activities and tasks triggering mathematical creativity, new technologies that support mathematical creativity, students interaction/communication). The authors stress that interconnections of the 4Ps are as important as its components. Although these strands can be studied in isolation, when their overlapping and interconnections are considered, the quality of the analysis is much higher and data interpretation leads to conclusions that are relevant for understanding creativity of various groups or individuals.

Under the sign of this conclusion, in the next paper of the volume, Assmus and Fritzlar (2018, this volume) investigate mathematically gifted primary students in their process of creation, in problem-solving and problem-posing contexts. They found that gifted second graders are able to create new mathematical objects. Even if their products are not necessarily true math objects, the chapter contributes with a new vision of young students' capacities. Reviewing a large gamut of situations, Assmus and Fritzlar discuss three instances of the relationship between creativity and giftedness: (mathematical) creativity as a precondition for (mathematical) giftedness, (mathematical) creativity as a possible component of mathematical giftedness, and (mathematical) creativity as a possible consequence of mathematical giftedness, illustrating each instance with examples and theoretical extrapolations. The authors opt for an embedded model of giftedness and creativity in which creativity and giftedness are seen as competence of a person, based on analyzing and confronting data resulting from two categories of samples: primary students gifted and not gifted, who have been exposed to the same types of tasks. Their study also emphasize the idea that tasks like the invention of new mathematical operations encourage primary school students in regular classes being creative with mathematical objects, making stronger an algebraic perspective, even in early grades.

To what extent parameters such as age and training are relevant for the quality of newly created products may seem to have a clear answer. However, it seems that things are not so obvious, and a careful analysis performed by Voica and Singer (2018, this volume) reveal a framework to study creativity by investigating cognitive variety in rich-challenging tasks. Groups of students of different ages and studies (from primary to university) selected based on their interest in mathematics (winners of mathematics competitions, students of faculty of mathematics, professional mathematicians) were asked to start from an image rich in mathematical properties, and generate as many problems related to the given input as possible. The authors found that cognitive variety seems randomly distributed among the tested groups and that, when talking about mathematical creativity, more sophisticated parameters, such as validity, complexity and topic variety, as well as the potential of respondents' products to break a well-internalized frame have to be taken into account. All those are to be balanced against the person's level of expertise in the specified domain.

Consequently, when dealing with concepts situated at the interaction between human knowledge and human psychology, many precautions and careful analysis are needed in order to formulate generalizable conclusions. A study investigating students' performances in multiple-solution tasks (MSTs) brings converging evidence for viewing domain-specificity of mathematical creativity as subdomain-specificity, e.g., in the contexts of geometry, algebra, or arithmetic separately (Joklitschke et al. 2018, this volume). Students' performances along three different MSTs from different mathematical domains such as geometry and algebra show that fluency, flexibility, and originality of the solutions differ consistently between the three subdomains of mathematics, and, therefore, precaution is needed when talking about general mathematical creativity.

The second part of the book starts with a discussion on the characteristics of mathematical giftedness in early primary school age. Here, Assmus (2018, this volume) proposes a comparative study that involves mathematically gifted children and those who are not. The results of the conducted study suggest that the cognitive abilities of mathematically gifted and non-gifted second graders differ in the examined areas. According to her conclusions, the following abilities represent characteristics of mathematical giftedness in early primary school children: ability to memorize mathematical issues by drawing on identified structures, ability to construct and use mathematical structures, ability to switch between modes of representation, ability to reverse lines of thought, ability to capture complex structures and work with them, and ability to use relational concepts and connections. A supportive environment can also have a favorable effect, and therefore, the construct of mathematical giftedness is not reducible to cognitive factors.

Characteristics of mathematically gifted students, such as: unusual quickness in learning; understanding, and applying mathematical ideas, even grasping new ideas before the teacher has finished explaining them; high capability for identifying regularities and complex structures in patterns, extracting them from empirical contexts, and characterizing them in general terms; ability to generalize and transfer mathematical ideas to detect general relationships when observing specific cases; ability to invert mental procedures of mathematical reasoning; flexibility to change from one problem-solving strategy to another if the new one seems to be more useful or easier; development of efficient strategies of problem-solving processes, such as efficiency in using analogical paths in solving various problems—these are characteristics recommended in the specific literature and identified by Gutierrez et al. (2018, this volume) in a gifted student. Through the case of a nine-year-old 5th grader in a primary school, who worked on an experimental pre-algebra teaching unit, the authors test the model of cognitive demand by measuring student's level of intellectual effort as the experiment advanced.

A comprehensive discussion, with intricacies strongly related to cognitive psychology, applied psychology and education is proposed by Nolte (2018, this volume). Her chapter discusses whether the special learning conditions of twice-exceptional students need a differentiated approach than what is usually applied. Furthermore, by means of examples of affected students, the implications for learning processes are illustrated. The paper extends beyond presenting empirical studies, by making a complex discussion of the interactions between mathematical giftedness occurring together with learning disabilities, attention deficit disorders (ADD), attention deficit disorders with hyperactivity (ADHD), and autism spectrum disorders (ASD). The chapter offers an overview of systemic approaches towards giftedness and learning disabilities and disorders, a model for acquiring mathematical competencies including barriers, a thorough discussion about twice exceptionality within the field of mathematics education, about underachievement as a collective term for different disorders and their implications in mathematically promising students, about learning disabilities related to reading, writing, and spelling which affect mathematics learning, including weaknesses in perception as a special learning disability, and explaining why each of them may cause problems to students. In addition, the study raises attention to care-givers and tutors about the masking effect used by these children to hide their giftedness and provides approaches to support students in these conditions. Using four exemplar cases, Nolte succeeds to illustrate the complex aspects presented above and discussed in a large gamut of psychological and educational literature. The study is a good example of how psychology and education can work together for the benefit of clarifying issues in both fields and for finding adequate solutions to cure/improve children's behaviors in a variety of situations.

The third part of the volume emphasizes some teaching strategies to foster creative learning. The research reports presented in this part investigate the effect of various tools on students' creative capacities.

Daher and Anabousy (2018, this volume) discuss flexibility of pre-service teachers in problem posing in different environments and conclude that technology, as well as what-if-not strategy, positively affect students' problem-posing products. However, the combination of technology and the what-if-not strategy positively affected the participants' flexibility in problem posing more than any one of the two tools alone. This finding, qualitatively as well as quantitatively checked, makes a plea for using technology mindfully for a true benefit for students.

Art can also be an important source for creative approaches, stimulating students in multicultural classrooms to engage in mathematics activities. The use of everyday objects like ornaments and the creation of ornaments make the students free to experiment and indulge their imagination (Moraová et al. 2018, this volume). The study found that if pre-service and in-service teachers face a culturally heterogeneous classroom, they tend to be very creative in planning their lessons and at the same time encourage creativity of their students. Thus, cultural heterogeneity may be perceived as an advantage as it may result in breaking out of stereotypes of mathematics classrooms. Moreover, the paper brings into discussion a contemporary issue of teaching: working with unmotivated students, in socially heterogeneous cultures, with migrant students and students from different socio-cultural backgrounds, and provides an effective solution to that. In these circumstances, teachers are naturally motivated to use their creative potential looking for the mathematics that can be discovered and taught in that particular environment and to create substantial learning environments in which the cultural background, the environment are brought into accessible mathematical expressions.

Still, mathematics itself can offer strong inputs for stimulating creative approaches of both teachers and students. This is demonstrated by the study proposed by Friedlander and Tabach (2018, this volume). The learning of algebraic procedures in middle-school algebra is usually perceived as an algorithmic activity, achieved by performing sequences of short drill-and-practice tasks, which have little to do with creative mathematical thinking. The authors provide five instances addressing procedural and conceptual learning, and examine methods of assessing their potential to induce higher-order, and creative thinking in all students. The occurrence of original thinking and students' fluency, originality and flexibility is related to the development of the following mathematical capacities: representing, modeling, interpreting, reversed thinking, generating examples, generalizing, justifying and proving, and thinking divergently.

The last part of the book contains chapters that highlight the benefits of an integrated approach towards creativity and giftedness. Although very different concerning structure and the target population taken into account for investigation, the papers included in this part contain relevant examples of tasks and techniques that can foster students' creativity from kindergarten to university.

The first study compares the methods three mathematically gifted university students used for the resolution of a problem, their strategies, and their transitions from geometrical to algebraic means and vice versa. Poulos and Mamona-Downs (2018, this volume) provided students with a problem to solve, which required the use of software for generating its solution. Observing the solvers' efforts and recording their conjectures in very detailed way, the authors succeeded not only to reveal the solving strategies of these gifted students, but to enrich our understanding of students' attitudes towards 'doing mathematics' in general.

Frequently, high achievers are confronting competitions that reveal their mathematical competencies. Veilande et al. (2018, this volume) propose a paper that analyzes the works of students who have participated in at least three Open Mathematical Olympiads in the 6th, 8th and 9th grades. A set of algebra and number theory problems, whose solving requires high levels of abstract thinking, algebraic reasoning, and an accurate use of the mathematical language were selected for this research. The data collected revealed an interesting result: although very competitive and theoretically well trained, these students showed deficiencies of algebra knowledge in a significant part of their works. Therefore a question is legitimate: Does repeated participation at mathematical Olympiads ensure students' progress in problem-solving? The authors conclude that students need mentors who would help them broaden their problem-solving competencies. In addition, teachers' professionalism is a key prerequisite for developing students' argumentation and justification skills.

If we think that the winners of mathematical Olympiads may become leaders in various domains of social-economical life as adults, then the adequate training of their capacities is more and more important concerning their structured actual specialized knowledge and the development of values and positive attitudes as well.

Special programs for gifted youth, student attendance in interest groups and in clubs, parental support for talent development and collaboration between parents and school are crucial to develop the mathematical abilities of gifted students.

Finally, we move further from Olympiads complex problems to complex open-ended tasks to enrich mathematical experiences of kindergarten students. The chapter proposed by Freiman (2018, this volume) stresses again that at a very young age, some children already manifest unusually strong precocious mathematical abilities that need to be fully developed and nurtured in school. This last chapter investigates in what way a kindergarten curriculum can offer all students a richer mathematical experience by means of open-ended and complex tasks. The data collected during the experiment show challenging situations in terms of the mathematics structures the kindergarten students create during such activities and the strategies they use. While some students struggle with increasing complexity of tasks but still remain engaged and try to overcome obstacles, others seem to exhibit more structured (in terms of mathematical relationships), systematic (in terms of problem-solving strategies), and abstract (in terms of mathematical symbolism) approaches. In addition, all students, even at a very young age, can benefit from a classroom culture of questioning, investigating, communicating, and reflecting on more advanced and meaningful mathematics that can help develop their mathematical minds.

This last paper urges us to think that using more systematic and efficient strategies and encouraging self-control and self-efficacy in young children can further contribute to high mathematics achievement in higher grades. For the readers, it helps increase understanding of the potential of open and complex tasks to enhance the development of mathematical high achievers from a very early age.

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Part I
Frameworks for Studying Mathematical Creativity and Giftedness

Chapter 2
Mathematical Creativity: Product, Person, Process and Press

Demetra Pitta-Pantazi, Maria Kattou and Constantinos Christou

Abstract In this chapter we provide an overview of the state-of-the-art in mathematical creativity. To do so, we will use as a road map the 4Ps theory proposed by Rhodes in which four strands are used to capture the definition of creativity. In particular, (1) product: the communication of a unique, novel and useful idea or concept; (2) person: cognitive abilities, personality traits and biographical experiences; (3) process: the methodology that produces a creative product; and (4) press: the environment where creative ideas are produced. In this chapter we will first discuss the four strands in the framework of general creativity and then transfer and adapt these considerations to the field of mathematics education. In an attempt to define and describe mathematical creativity we will present several examples drawn from various research studies, and highlight some of the main findings, hoping to offer a springboard for further developments. We suggest that although these strands can be studied in isolation, it is only when their overlap and interconnections are considered that we may get a clearer picture of the complex concept of creativity.

Keywords Mathematical creativity · Creative product · Creative person
Creative process · Creative press

2.1 Introduction

Although the issue of creativity is often addressed in an interdisciplinary way, interest in the field has increased especially in the last 30 years (Hersh and John-Steiner 2017). As Hersh and John-Steiner (2017) stated, research on creativity

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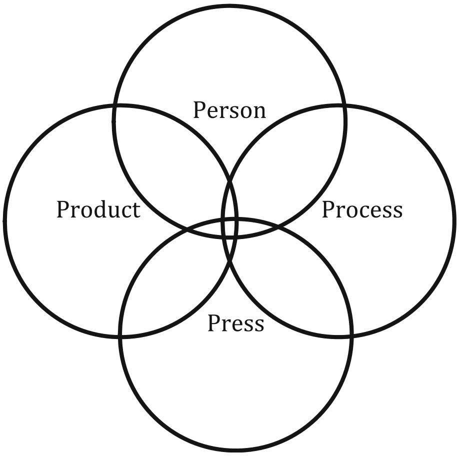
moved from the traditional perspective of genius, to wider themes of inquiry, such as creative behavior in everyday life. The same change is true in the field of mathematics education. Until about a decade ago, very little research had been undertaken in the field of mathematics education (Leikin and Sriraman 2017). In recent years, there has been renewed interest in creativity, which has brought about new and very significant knowledge. Part of the reason for this increased interest in the mathematics education community may have been the shift of interest that is observed in various mathematics curricula and international organizations towards creative and critical thinking (Australian Curriculum, Assessment and Reporting Authority 2010; Mann 2005; National Council of Teachers of Mathematics 2014; OECD 2014). Societies nowadays seem to realize that it is not enough for individuals to hold a vast amount of information, efficiently process it and deeply understand it; they need individuals who will also be highly creative, offer solutions to unsolved problems, and create advancements which will lead to a better life.

In order to move towards schools and educational systems that will be able to enhance individual students' mathematical creativity, we need to clarify what we mean by this term, understand its constituent factors, and have a clear idea of what we have learned from research so far. In the present chapter we do not attempt a summative review or a meta-synthesis of what is known; instead, we will present and discuss various studies which investigated general creativity and then transfer and adapt these considerations in the field of mathematics education through studies which investigated mathematical creativity. Of course, in such a short document we are not claiming that our text will be exhaustive or comprehensive. However, we feel that to make any significant advancements in the field we need some investigation of what we have learned so far about mathematical creativity. For the sake of clarity, we want to note that in the sections which follow, the term "creativity" is used as a synonym to "general creativity" and in the cases where we refer to "mathematical creativity" this term refers to studies which specifically investigated "mathematical creativity".

2.2 The Components of Creativity

Interestingly enough, even back in 1961, Rhodes stressed the necessity for precision in defining creativity. Speaking about this need, Rhodes (1961) collected different definitions of creativity and observed that the definitions were not mutually exclusive; rather they overlap and intertwine. He claimed that creativity is not simple nor unidimensional; hence, it cannot be interpreted as a single unity but rather as a composition of different factors. Thus, in his seminal paper, Rhodes (1961) concluded with a framework presented in a Venn diagram (Fig. 2.1), in which four strands exist—product, person, process and press—to conceptualize creativity.

Fig. 2.1 Rhodes' 4P model about creativity



Product: the communication of a unique, novel and useful idea or concept.
Person, as a human being: cognitive abilities, personality traits and biographical experiences.
Process that is operating in creating an idea: the methodology that produces a creative product.
Press: the relationship between human beings and their environment.

Although the four strands of the model have unique identity and exist separately, it is only in unity that the four strands operate functionally; hence, their interrelations and overlapping demand a more comprehensive examination of the concept (Isaksen et al. 1993). Rhodes' framework provides a simple way to organize our understanding of creativity and furthermore to visualize the whole system (Isaksen et al. 2011). Researchers were urged to distinguish the strands of creativity and then to carefully classify the knowledge that is built through years of research into the four strands.

Several theoretical models that followed were based on Rhodes' model but offered different terms to describe the 4Ps. For example, Mooney (1963) replaced the term "press" with the term "environment", whereas, Dellas and Gaier (1970) used the terms: "the nature of the individual", "the actual expression of the creative acts and continuing process during creation", "the nature and quality of product created", "the environmental factors and press that tend to initiate and foster creativity". The COCO model proposed by Treffinger (1991) is also a variation of the 4Ps model. In particular, it describes creative behavior as the result of interactions between the following elements: Characteristics, Operations, Context and Outcomes. Treffinger (1991) defined as characteristics, the personality of the creative person; as operations, the strategies and techniques that are used to solve problems and make decisions; as context, the cultural framework and the interactions of the environment, as well as the natural environment in which the individual acts; as outcomes, the products and concepts derived from a person's effort. In other words, Treffinger (1991), as well as using different terms for the 4Ps, gave a different orientation to their meaning; for instance, "Person" or "Characteristics" in this model, describe only an individual's personality without any reference to other abilities or aptitudes.

More than fifty years later, the 4Ps model—as well as its variations—is much appreciated by researchers in the area of creativity (Babij 2001), and we feel that it has a lot to offer if we apply it in the field of mathematics education. Indeed, the 4Ps model manages to connect different definitions of creativity, creating a flexible framework that gives meaning to the multidimensional nature of the concept (Isaksen et al. 1993).

It is important to clarify that for the purpose of the present paper we use the 4P model as the basis for conceptual clarity and the structure of an investigation of the state-of-the-art of research on mathematical creativity and not as a model/tool for identifying creative ability or creative individuals. Our attempt is to offer a broad view of research studies that have been done in these four strands which may serve as a springboard for further research in mathematical creativity which may also bring strongly together these four strands. We suggest that it is crucial in mathematics education to investigate the interrelationship of these four strands and for researchers to become more specific regarding the types of individuals and types of specific environments which may enhance their creative processes and products.

What follows is first a presentation of each of the 4Ps accompanied by theoretical and empirical pieces of evidence.

2.3 Product

The starting point of all research effort about creativity is analysis of the creative product, given that the creative product is the public face of creativity, the tangible form of the whole process (Cropley 2006). Therefore, two questions emerge: Which are the factors/characteristics that differentiate creative products from non-creative ones? How can we effectively evaluate how creative a product is?

In response to the above questions, several researchers focused on the production of definitions ("products definitions" or "product bias") and the establishment of criteria useful to assess the creative product (Runco 2007). The majority of definitions agreed that innovation, relevance and utility characterize the creative product (e.g. Plucker and Beghetto 2004; Sternberg and Lubart 2000). In particular, in the field of mathematics education, researchers stressed the importance of creative ideas being mathematically correct in order to fulfil their purpose and also be useful for the solution of mathematical problems (Chamberlin and Moon 2005). A number of researchers invested also in other elements that may characterize the creative product, such as elaborateness, appropriateness and the desire of mathematically creative people to dissent from commonly accepted principles and solutions (Klavir and Gorodetsky 2011).

Research studies which concentrated on the creative product dealt also with what was distinctive about it, and how it might be assessed. It appears that the majority of studies in mathematics education used Torrance's (1974) definition, who claims that creativity may be assessed based on originality, fluency, flexibility, and elaboration. Mathematics educators seem mainly to have used three of the four criteria: originality, fluency and flexibility as they are explained below (Klavir and Gorodetsky 2009; Leikin and Lev 2007; Pitta-Pantazi et al. 2011).

Originality of responses is probably the predominant characteristic to be found in most definitions of creativity, and this is often used as a synonym by those who are not familiar with research in the field (Mann et al. 2017). Leikin (2008) argued that mathematical responses should be original, rare and appropriate to the mathematical problem, while Shriki (2010) argued that the ability to generalize or to find an original proof or the discovery of new theorems are also creative products. Mann et al. (2017) added that "mathematical processes, procedures, and algorithms also can be highly original" (p. 61). Leikin and her colleagues (Leikin and Kloss 2011; Levav-Waynberg and Lekin 2012) suggested that originality determines creativity in a stronger way than fluency and flexibility. Leikin (2009b) also suggested that originality is rather an internal, unique characteristic of creativity.

Fluency captures the speed and accuracy of producing a large number of different responses (Klavir and Gorodetsky 2009; Mann et al. 2017). According to Mann et al. (2017), fluent thinkers are able to generate many ideas, possibilities and possible approaches to find solutions to a problem. Hence, several researchers feel that fluency is often the springboard for the production of an original response (Mumford 2003; Vidal 2005), given that the more ideas that are proposed, the more possibilities there are for an original one to emerge.

Flexibility is the ability to provide different responses to a question (Vidal 2005), by breaking a preconceived solution path and having the freedom to develop ideas and solutions (Mann et al. 2017). For this to be achieved, an individual needs to be able to look at the same thing from a different perspective, to transform representations, reverse procedures or even redefine ideas and transform the whole problem (or situation) in order to find a new different way of thinking (Klavir and Gorodetsky 2009; Mann et al. 2017; Sheffield 2009).

Even though the three components of the creative product are broadly used in the field of mathematical creativity, we feel that the existence of a measurement method or a tool that can measure them jointly without overlap or vagueness will be a great contribution. Such attempt was made by Leikin (2009b), who proposed a model that contains operational definitions and a corresponding scoring scheme for the evaluation of mathematical creativity.

2.4 Person

The question "When is a person mathematically creative?" is difficult to answer, and for this reason no model has been proposed for identifying a creative individual based solely on the person's characteristics without taking into consideration his/her performance in creative tasks. However, indications about cognitive characteristics, personality traits and biographical experiences are considered as pieces of evidence to characterize a person as creative (Davis 2004). In particular, Davis (2004) carried out a meta-analysis of over 200 adjectives and descriptions and reached 16 categories of frequently met traits in creative people.

In the following section, we present the most frequently discussed traits of mathematically creative people that we found in the literature: mathematical ability, intelligence, general creativity, age, gender, culture and personality traits. Of course, we do not suggest that an individual will possess all the characteristics that are suggested in the literature, or that if an individual who possesses one or more of these characteristics will be mathematically creative (Davis 2004).

2.4.1 Mathematical Ability

A number of studies investigated the relationship between mathematical ability and mathematical creativity (Haylock 1997; Kattou et al. 2013; Mann 2009).

Most researchers seem to believe that a strong mathematical background is related to mathematical creativity (Kattou et al. 2013; Mann 2009; Sheffield 2009). Using structural equation modeling analysis, we (Kattou et al. 2013) concluded that mathematical creativity is a subcomponent of mathematical ability. Mann (2009) found that mathematical achievement was the stronger predictor for mathematical creativity, accounting for 23% of variance. Similarly, Sak and Maker (2006) investigated the effect of knowledge on mathematical creativity's components. This research work concluded that students' knowledge made a significant contribution to the interpretation of their fluency, flexibility and originality (Sak and Maker 2006). In the same vein, Bahar and Maker (2011) attempted to find a correlation between students' performance in an achievement test and in a divergent thinking test. The results of this study revealed a strong correlation between the performances of students in the two tests. The strong relationship between mathematical ability and mathematical creativity could be explained by the fact that excellent content knowledge facilitates people's ability to recall and process information, and also to make connections between different concepts and representations (Sheffield 2009).

Moreover, creativity has been highlighted as an important factor in giftedness (e.g. Leikin 2009a) and has gained an important place in the context of gifted education (Kaufman et al. 2012). Kaufman et al. (2012) concluded that creativity should be included as part of a gifted assessment battery, while the measurement of mathematical creativity is often proposed in the instruments used to identify mathematical giftedness (Leikin 2009a).

Furthermore, some researchers warn that educators need to be cautious and not overemphasize mathematical procedures. For instance, Haylock (1997) claimed that too much emphasis on and exposure to specific algorithms may limit students' mathematical creativity, since it might guide students to well-practiced procedures. The assumption is that if a person knows well how things are working in a field, it will be difficult to escape from the already known and propose new ideas. This view is also supported by other researchers, who believe that knowledge might restrict individuals in stereotyped solutions, or to an erroneous transfer of knowledge to new situations (Weisberg 1999, 2006). However, these claims do not contradict the aforementioned research results about the connection between mathematical ability and mathematical creativity. It is not anticipated that a person who has substantial mathematical knowledge but is only able to follow procedures and algorithms will present creative behaviors. Similarly, individuals who can effectively handle and apply knowledge by breaking a preconceived solution path and develop ideas and solutions are those who are anticipated to be creative as well (Mann et al. 2017).

2.4.2 Intelligence

The results regarding the relationship between creativity and intelligence are conflicting (Leikin 2008) and the nature of their interaction is still debatable (Kaufman and Plucker 2011). Various investigations questioned the existence of this relationship and the level of their association, either concluding that there is a statistically significant relation between intelligence and creativity or finding that intelligence and creativity are unrelated.

Getzels and Jackson (1962) were among the first researchers who claimed that creativity and intelligence are unconnected structures. In their work, the researchers compared two groups of middle-class students—the first group had scored well on intelligence tests, the second group on creativity tests—and they found that the differences between the two groups were negligible. Following a similar experimental procedure, Wallach and Kogan (1965) identified a low correlation index among participants' performance in a creative and an intelligence tool, repeating the conclusion that the two concepts are not related. Similar results were also obtained by Silvia (2008) and Kim (2008). In particular, Silvia (2008) meta-analyzed the data of Wallach and Kogan's research study and noted that the relationship between creativity and intelligence is low. As for Kim's work (2008), she made a meta-analysis of numerous studies and found a negligible correlation between creativity and intelligence, suggesting that students with low IQ can be creative as well, and vice versa. However, the relationship that she identified between the two constructs was positive ($r = .17$).

In contrast to the above results, that deal with the relationship between creativity and intelligence, contemporary studies of mathematical creativity verified the existence of a relationship between the two constructs. Through a confirmatory factor analysis which investigated factors that may predict mathematical creativity, we (Kattou et al. 2015) concluded that intelligence predicts mathematical creativity to a smaller extent than mathematical ability and general creativity. We explained that intelligence seems to be a necessary but not a sufficient condition for the emergence of mathematical creativity. Livne and Milgram (2006) also found that intelligence contributed in a small way to mathematical creativity.

Moreover, other researchers took the existence of a relationship between creativity and intelligence for granted, and focused on characterizing the way in which the two concepts are linked. For example, in the Structure of Intellect model proposed by Guilford (1967), creativity was considered as an element of intelligence, while in Sternberg's theory (Sternberg and Lubart 1996) intelligence is amongst the variables that are necessary for creativity to appear.

There have also been researchers who reached conclusions combining the above views. One such theory is the "Threshold theory of intelligence" (Torrance, 1962), which proposes that creativity and intelligence are separate entities, but there is a relationship between them. In particular, creativity and intelligence are related to each other under a cut-off point of intelligence (usually 120) (Runco 2007; Sternberg and O'Hara 1999). When a person's IQ is over 120, the relationship between creativity and intelligence is negligible, and both concepts seem to be independent mental abilities (Runco 2007).

Although we looked back on a scientific tradition of over 50 years of investigation of the relationship between creativity and intelligence, the value and significance of this relationship is still under debate. According to Kim (2008), the discrepancies in the results can be partially explained by the heterogeneity of the measures employed and the populations studied. Indeed, we can see from the aforementioned results that findings differ according to the type of creativity and intelligence that are measured: General creativity researchers (e.g. Getzels and Jackson 1962) concluded that there is no relationship between creativity and intelligence, whereas mathematical creativity researchers seem to agree that such a relationship exists, although its strength is low to moderate (e.g. Kattou et al. 2015). Furthermore, the design and type of study as well as the sampling procedures employed might have led to contradictory results. Additionally, the relationship between intelligence and creativity in different populations may also be different, giving rise to discrepancies that appear in literature.

2.4.3 General Creativity

With the term general creativity, researchers refer to the universal ability which contributes to all creative achievements in any cognitive area (Plucker and Zabelina 2009). Accordingly, a domain-specific creativity perspective assumes that creativity cannot be understood without reference to the domain in which it takes place (Plucker and Zabelina 2009). In the last decade, various scholarly attempts have been directed towards examining the extent to which creativity is a domain-general or a domain-specific ability. Again, it appears that there is no consensus amongst researchers (Kaufman, Cole and Baer 2009; Plucker and Zabelina 2009). Kaufman et al. (2009), while measuring individuals' self-reported creativity, found that the model that best described the relationship between seven general thematic area factors with general creativity was the "Amusement Park Theoretical Model". This was a hierarchical model which suggested some initial requirements common to all creative activity followed by seven different large domains (or General Thematic Areas). In this study, Math/Science and Problem solving had the lowest relationship with general creativity.

Regarding the relationship between general and specific creativity (in mathematics), recent studies (Hong and Milgram, 2010; Kattou et al. 2015; Livne and Milgram 2006) confirmed that the former ability may contribute to the interpretation of the latter. For instance, in one of our studies (Kattou et al. 2015) a mathematical creativity and a general creativity test was administered to students 10–12 years old. We (Kattou et al. 2015) aimed to investigate in which manner students' performance on a domain-general creativity test was related to their performance in a domain-specific creativity test. Results indicated that general creativity was an important prerequisite for the emergence of an individual's creative potential in the domain of mathematics. According to our results, general creativity enables individuals to combine ideas and consider alternative approaches to a situation in original ways (Kattou et al. 2015). Along the same lines were the results by Hong and Milgram (2010), who worked with preschoolers, aged 3–5 years. They found a statistically significant effect of general creativity on mathematical creativity. Milgram and Hong (2009) suggested that both general and specific creativity should be taken into account. In particular, they stated that general creativity cannot explain the kind of thinking that leads to the production of mathematical ideas, but that it is a prerequisite for the emergence of creativity in mathematics (Milgram and Hong 2009).

Despite these findings, it appears that mathematical creativity is not a part of general creativity; rather it is a specialized ability that is not transferable in other domains. In one of our studies (Kattou et al. 2015) we verified through the exploitation of different statistical approaches the domain-specificity of creativity. Through correlation analysis we found a low relationship between different creative instruments, whereas through crosstabs analysis we concluded that a person who is creative, using as indicator his/her performance in one of the instruments, is not necessarily identified as creative by another instrument. These results were also verified with a confirmatory factor analysis which indicated that the model that best fitted to the data was the one suggesting that different types of creativity exist. The interpretation of these results is that educators should not anticipate that a student who is creative in mathematics is necessarily creative in one or more other domains, and vice versa. Accordingly, low creativity in one field does not automatically exclude an individual from being creative in the subject of mathematics.

2.4.4 Age

Steinberg (2013) provided evidence that it is possible to identify mathematical creativity in young students as early as the age of four. One case study that she conducted showed that a 4-year-old child who was not only able to solve mathematical problems, but also to invent problems for himself and find mathematical situations in his environment to attend to. Similarly, in their research work with 3 to 5-year-old students, Hong and Milgram (2010) concluded that there are age differences. The older students were able to propose more divergent solutions than the younger ones, leading the researchers to conclude that preschoolers' experience (including schooling and culture) affect their mathematical creativity.

Although creative outcomes might be obvious from early years, the relationship between age and creativity is not clear. According to Kim and Pierce (2013), there are no large scale studies which allow us to reach solid conclusions about the relationship between creativity and age, across an entire lifespan. In addition to this, the results of these studies were often conflicting. Torrance (1968) claimed that the development of creativity follows a U shaped course. He claimed that although in early life children are creative, around the age of 10 years this creativity diminishes and reaches its lowest point, while later on, as individuals become older, creativity again increases. A similar conclusion was reached by Charles and Runco (2001), who asserted that students in fourth grade are at the top of their creative capacity, while there is a steady decline in the fluency, flexibility and originality of fifth graders. Smith and Carlsson (1983, 1985) concluded that creative development follows a linear trajectory, in other words, children become more creative as they get older. Kim's (2011) study was based on the results from a large number of individuals, from kindergarten through to grade 12 students and adults, who responded to the Torrance Tests of Creativity. She found that after secondary school, although individuals have enhanced cognitive capacities, social pressures such as the conformity of a profession or a convergent body of knowledge limits their creativity.

Sak and Maker (2006), when investigating mathematical creativity and its relation to students' age, found that the progression followed a plateau-hill-little top-plateau course. At the age of 8–9 years students' creativity appeared to plateau, from 9 to 10 years it increased, then from 10 to 11 years there was another plateau before it increased again. In one of our studies in which different age groups participated (Kattou et al. 2016) we found that as students became older, they had stronger mathematical background and were more creative than their younger peers. Although as a result it appeared that age significantly contributed to their mathematical creative potential, it is possible that this was not a direct effect but rather an indirect effect caused by the greater educational experience that the older children had.

2.4.5 Gender

"…gender differences in creativity has not become an important focus in either the creativity or psychology of women literatures" (Baer and Kaufman, 2008, p. 76). Nevertheless, these limited research attempts conducted using test scores, creative achievements and self-reported creativity questionnaires, produced conflicting results (Baer and Kaufman 2008). Baer and Kaufman (2008) wrote an interesting meta-analysis where they investigated the result of approximately 80 studies dealing with this topic. They found that some studies showed no gender differences while others had mixed results. They found that most studies did not find any differences between males and females, and that the cases where one sex outperformed the other were counter-balanced by studies showing the opposite. Their final conclusion was that it was unlikely that any meta-analysis would show any significant gender differences. The picture regarding specifically mathematics creativity appears to be equally hazy.

The whole issue of creativity and gender becomes even more complex when one tries to untangle the various factors that may come into play—biological differences, aptitudes, motivations and opportunities. We agree with Baer and Kaufman (2008) that, in order to get a clearer picture regarding gender and mathematical creativity, we would need more than mathematical creativity tests. We will need to use multiple sources of information regarding aptitude, motivation, environment and opportunities offered to the individuals under examination.

The issue of gender in mathematical creativity research is limited. Among the relevant research is some conducted over 50 years ago. In particular, Evans (1964), Jensen (1973) and Prouse (1964) reported that a majority of females outscored their male peers in a mathematical creative test. Evans (1964) found significant gender differences only in the seventh and eighth grades, whereas no gender differences were found in the fifth and sixth grades. As for Jensen's (1973) work, she found a significant difference favoring females in one of the three schools involved in her study. Similarly, Prouse (1964) concluded with a significant mean difference in creativity scores, favoring females.

In the last fifteen years research interest regarding gender has reawakened. Mann (2005) compared seventh graders on the "Creative Ability in Mathematics Test", and he reported that females scored 6.5 points higher than males. On the contrary, Walia (2012), Ganihar and Wajiha (2009) found no gender differences in relation to mathematical creativity. However, Walia (2012) did observe that girls were found to be better than boys with regard to flexibility.

2.4.6 Culture

According to Csikszentmihalyi (1999), creativity should be perceived as a cultural and social phenomenon and not simply as a mental process. Rudowicz (2003) stated that, although since the 1960s researchers acknowledged that creativity was not simply a mental process but was influenced by culture, they did not seem to appreciate the extent of this influence. The reason may have been that the majority of researchers investigating this topic were psychologists.

From an extensive review that Rudowicz (2003) conducted on creativity and culture, she reached the following conclusions: (a) culture has a significant influence on the conceptualization of creativity; (b) the relationship between creativity and cultural factors is very complex; (c) the relationship between culture and creativity is not only reciprocal but also involves other historical, societal and individual factors; (d) creativity is a universal human characteristic; and (e) creativity is a multifaceted phenomenon, therefore its manifestations need to be understood from the perspective of the individual involved.

Fewer studies have been conducted concentrating specifically on mathematical creativity and its connections to different cultures. In a study by Leikin et al. (2013), culturally-based aspects of the creative person were examined. One thousand one hundred teachers from six countries (Cyprus, India, Israel, Latvia, Mexico and Romania) responded to a questionnaire about issues regarding: (a) who is a creative person; (b) who is a creative student; (c) who is a creative mathematics teacher; and (d) how is creativity related to culture. The results of this study suggested that, although all countries acknowledge the importance of creativity in mathematics, there were differences in their perceptions and approaches to teaching creativity.

In another study, Ma and Rapee (2014), investigating the mathematical performance and mathematical creativity of students from different cultures, showed that although Chinese students had a better performance in mathematics, their Australian peers with an Anglo-Saxon background had a higher score in the creativity test. These Chinese students were born in Mainland China, had studied there for at least 10 years, and had been living in Australia for an average 1 year and 3 months.

2.4.7 Personality Traits

Most researchers agree that creative personalities are independent and autonomous and are most often aware of their capabilities (Selby et al. 2005). This independence allows them to find their own way of solving problems and go beyond the known and accepted methods of working (Hersh and John-Steiner 2017). They are also characterized by imagination, intuition, open mindedness and a desire to gain new experiences (Selby et al. 2005). Sternberg (2006) suggested that creative personalities are also characterized by persistence, eagerness to overcome obstacles, curiosity, self-regulation, imagination and the confidence to take risks.

In mathematics education, Klavir and Gorodetsky (2009) and Freiman and Sriraman (2011) suggested that creative individuals are interested in: deeper understanding of known results; discovery of new mathematical concepts; diverse methods of working; properties and connections between areas that at first glance appear to be completely independent from each other; inventive, practical and economical solutions. Therefore, open-mindedness, independence, curiosity, perseverance and conciseness were amongst the most frequently identified characteristics of creative individuals in mathematics. According to Mann et al. (2017), creative persons are also characterized by courage: “Without that courage, potentially creative mathematical ideas remain unknown and unexplored” (p. 59).

2.5 Process

"Researching the creative product may not provide full understanding of the development of creativity, or may not reflect the creativity used to reach that product" (Savic et al. 2017, p. 25). Indeed, when we judge the creativity of an individual we have to pay attention to the process by which this individual arrived at the results (Pelczer and Rodriguez 2011). The term "creative process" is used to describe stages, actions and behaviors that are active during the generation of an idea (Johnson and Carruthers 2006). However, the creative process is neither easily understandable nor searchable. Hence, questions like: "What makes a process creative? In what way does a creative process vary from non-creative processes?" still remain unanswered.

Attempting to understand mathematical creativity, several mathematics educators tried to describe the creative process in mathematics. Thus, theoretical models that describe stages of approaching a creative task and/or a problem have been proposed. The majority of these models are linear whereas some are non-linear. Additionally, some models are specifically for mathematical creativity and others have been adopted from general creativity. However, we feel that demystifying the creative process is not an easy target, due to the fact that it is an internalized procedure that is obvious through the actions and descriptions provided by the solver. Nevertheless, investigating the creative process allows us to identify ways and methods for its improvement (Kilgour 2006).

2.5.1 Stages of the Creative Process

Graham Wallas (1926) was one of the first researchers who attempted to model the creative process. Through seven discrete stages of encounter, preparation, concentration, incubation, illumination, verification and persuasion, Wallas (1926) offered a model for the creative process. At the first stage, that of encounter, the existence of a problematic situation is determined, while at the second stage, that of preparation, the solver tries to understand and explore the problematic situation (Johnson and Carruthers 2006). During concentration, the solver is working consciously in order to find a solution to the problem, whereas during incubation, work is taking place subconsciously (Johnson and Carruthers 2006). A promising idea suddenly comes into conscious awareness at the stage of illumination (Davis and Rimm 2004). At the stage of verification, tests, configurations and the development of ideas are taking place (Johnson and Carruthers 2006). At the last stage, persuasion, the solver is trying to convince others that the idea or solution he/she proposes is effective for the purpose it has been created for. A similar model was proposed by Osborn (1963). His stages were the following: orientation, preparation, analysis, ideation (in 1953, "hypothesis"), incubation, synthesis and evaluation (in 1953, "verification"). At the orientation stage, the person identifies the problem and analyzes it into sub-problems, in order to collect relevant data and information for solving it. At the analysis stage, the solver is retrieving relevant information, and during ideation alternative ideas are identified. At the fifth stage, that of incubation, the person stops consciously dealing with the target. During synthesis, the solver combines the elements he/she collects, and at the last stage he/she evaluates and verifies the ideas that have emerged, according to the initial aims. Osborn also claimed that the creative process "usually includes some or all" (p. 115) of the phases. Although there are some similarities between Wallas' (1926) and Osborn's (1963) stages, there are also some differences. In Osborn's version (1963) the stage of concentration does not appear, and he provides two other stages (analysis and ideation); he replaces illumination with synthesis, and the persuasion stage where one is trying to convince others is missing. In later years Wallas' (1926) model was narrowed down to four phases—preparation, incubation, illumination and verification—and was used as a baseline for similar attempts to conceptualize the creative process. Cropley and Urban (2000) claimed that the application of the creative idea was absent from Wallas' (1926) model. For this reason, they proposed a model of seven stages to incorporate application in the creative process: preparation, information (learning or reminding of expertise), incubation, illumination, verification, communication (presentation to other people, receiving feedback) and validation (assessing the relevance and effectiveness of solution by judges, e.g. teachers). During the two last stages, the requisite characteristic of application is taking place.

Wallas' (1926) short model was also adopted in the field of mathematics education (e.g. Liljedhal 2004; Sriraman 2004). In particular, Wallas' (1926) first stage, distinguished by hard, purposeful and conscious work, was characterized by Poincare (1948) as preliminary and by Hadamard (1945) as initiation, due to the fact that the person retrieves prior knowledge and experience in order to find the solution to a problematic situation. If the solver is unable to come up with a solution, then conscious work on the problem terminates. At this stage the solution of the problem is treated at an unconscious level (Hadamard 1945). This is the stage of incubation, which is directly connected to the conscious procedure of the previous stage. The third stage is characterized by the sudden appearance of the solution as a combination of conscious and unconscious mind function. Hadamard (1945) named this step illuminatory. The creative process is not completed at the third stage, but at a fourth stage, which follows. At the fourth stage, expression and communication of the result is taking place, through the verification of accuracy and utilization of the solution, either for the expansion or the exploitation of the idea (Sriraman 2008). Along the same lines, Sriraman (2004) found that mathematicians' creative processes follow Wallas' (1926) four-stage model of preparation-incubation-illumination-verification.

Liljedhal (2013) explored the nature of illumination in greater depth, and tried to distinguish it from other mathematical experiences. In doing so, he tried to compare and contrast the AHA! experiences of preservice teachers with those of prominent mathematicians. He found that although these two populations manifest creativity differently, the AHA! experiences are clearly related to affective aspects.

Although linear models have earned great acceptance in the field, some researchers are still not satisfied with them (Lubart 2001). In contrast to linear models that describe a fixed sequence of sub-processes, researchers recently suggested that this type of models is insufficient to represent the complexity of creative processes. On the contrary, non-linear models in which there is no fixed starting point and where the transition from one stage to another is according to the solver's needs are proposed.

A non-linear heuristic model was proposed by Sheffield (2009), in which five stages are presented: investigating, relating, creating, evaluating and communicating. In particular, the stage of investigation refers to an in-depth study of the available information and relevant mathematical concepts and ideas. The stage of relation is defined as the process of comparing ideas, identifying similarities and differences and combining information. At the creation stage, individuals find solutions or identify new ideas. During evaluation students are reflecting on the proposed solutions and confirming the success of the targets that were set in the first place. The communication stage refers to the description and explanation of ideas and strategies. According to Sheffield (2009), an individual may start from various points on this model and proceed in a non-linear way to reach a creative solution. For example an individual may relate the problem to previous solved problems, investigate possible approaches, reach a creative solution, evaluate this solution, communicate the results, create other related problems and communicate these to others.

2.6 Press

The creation of a creative product and the interaction of a creative person and a creative process do not occur in a vacuum. These interactions and the results of these interactions occur in a certain environment, which is defined as press. Csikszentmihalyi (1999) and Nuessel, Stewart and Cedeño (2001) explained that creativity appears in a social context, and is assessed by cultural and social criteria. Hence, it is impossible to separate creativity from the context in which it takes place (Basadur and Hausdorf 1996).

Goldin (2002) argued that for the development of a creative environment which supports higher order mathematical thinking, one needs to consider the design of a creative environment, the implementation of appropriate teaching interventions and the selection of suitable tasks. Yerushalmy (2009) also suggested that the use of new technologies might be another factor that may support mathematical creativity.

2.6.1 Teaching Environment and the Teachers' Role

As Gnedenko (1991, in Freiman and Sriraman 2011) said, everyone has innate creativity that seems to be restricted by the educational system, implying that the presence or absence of certain factors in school affects students' creativity. Wheeler et al. (2002) believe that teachers are the key to encouraging and developing creative thinking in school. A teacher might be able to encourage the development of mathematical creativity if he/she is able to recognize creative behavior and knows the way to cultivate it (Beghetto and Kaufman 2009). Moreover, it is important for the teacher to be persuaded about the importance of creativity in enhancing mathematical understanding (Leikin 2009a). Hence, teachers should be aware in which ways creativity is related to the mathematics curriculum (Boden 2001) and feel safe (mathematically and pedagogically) to implement corresponding activities in the classroom (Leikin 2009a).

Since creativity in mathematics can be improved through appropriate teaching methods (Hershkovitz et al. 2009), the teacher's role is important: during the selection of activities, in their implementation in the classroom, and in the organization of students' work (Freiman 2009). In particular, teachers should involve students in interesting, creative investigations that engage their interest and curiosity without limiting them in standard tasks with typical solutions, where they merely implement rules and algorithms (Mann 2006). At the same time, teachers should create an emotionally safe climate, where mistakes are not critisized (Goldin 2009; Koichu and Orey 2010; Sheffield 2009). Teaching environments should allow students to have the freedom to express their opinions and exchange ideas with their peers (Sriraman 2009). Furthermore, teachers should encourage all students to think (Freiman 2009), take risks in order to find solutions that are not directly perceived (Sriraman 2009), and look for different solutions (Presmeg 2003).

2.6.2 Activities and Tasks Triggering Mathematical Creativity

Exploring what sort of tasks trigger the emergence of a creative product has been one of the main lines of research regarding creativity. Among mathematics education researchers, there appears to be some consensus that inquiry-oriented instruction, exploratory learning and generally speaking problem solving environments support and increase creativity (Silver 1997). Researchers explored the impact of a variety of activities and found that the following types of activity have a positive impact on the development of mathematical creativity: mathematical investigations (Leikin 2014), open-ended approaches (Kwon et al. 2006), and modeling problems (Chamberlin and Moon 2005; Coxbill et al. 2013; Wessels 2014). Furthermore, daily life scenarios that allow students to decide on the way to work and how to present the results are considered as great opportunities for revealing creative ability (Palsdottir and Sriraman 2017). Leikin and Lev (2007) argued strongly for, and demonstrated convincingly, that multiple solution tasks offer ample opportunities to individuals to reach creative products.

Many researchers suggested that problem posing is a powerful tool for identifying and assessing mathematical creativity (Kontorovich et al. 2011). Voica and Singer (2013) built on this idea and put forward another type of task; they claimed that problem modification is an additional form of problem posing, offering individuals the opportunity to produce coherent and consistent, creative problems. Haylock (1997) also suggested that redefinition may be another type of task which offers the possibility to recognize students' mathematical creativity and also help them overcome fixation.

2.6.3 New Technologies that Support Mathematical Creativity

"Technology has always been part of the creative process. […] Supportive technologies can become the potter's wheel and mandolin of creativity—opening new media of expression and enabling compelling performances" (Shneiderman 1999, p. 119). Although numerous studies have explored the impact of technology on mathematical understanding, there are not that many which investigated the relationship between technology and mathematical creativity.

Instant feedback, speed, range of information, interactivity and personalization are some of the facilities that new technologies offer, motivating users to think creatively in a short time (Yang and Chin 1996). Furthermore, technology enables individuals to make a pool of ideas, discard the ones that did not work, edit or revise some of them and finally present the best ones, engaging learners in the creative process (Loveless et al. 2006). Moreover, technology offers the opportunity to shift between different perspectives; in other words, to exchange representations or views of the same construct. This opportunity enables the solver to redefine a situation, to see and give alternative interpretations of familiar objects, and reach a creative product (Guilford 1959). As technology provides the opportunity of testing-retesting a concept or an idea, it enables learners to construct their knowledge in meaningful ways through reflection, application and interaction (Jang 2006; Macdonald et al. 2001).

A few studies that we were able to trace seem to agree that technology can support the development of mathematical creativity (Yerushalmy 2009; Yushau et al. 2005). Writing about creativity in a technological environment, Yerushalmy (2009) argued that creativity is obvious through individuals' ability to conjecture, to go beyond the known, to explore situations, to take initiatives by asking, arguing, explaining, and disputing. In one of our studies we (Kattou et al. 2012) we found that technology provides learners with the opportunity to engage in activities that may otherwise be unattainable. For instance, learners can observe and interact with mathematical concepts which are difficult to visualize or understand without the use of technology (Idriset al. 2010). Moreover, technology enables learners to propose more solutions, using different mathematical ideas (Kattou et al. 2012).

We (Kattou et al. 2012) argued that, participants' fluency increased due to the reduction of the time needed to find, write or draw the solution. As for flexibility, the easy alternation of representations as well as the opportunity to elaborate and edit a mathematical idea motivated participants to think flexibly. Originality of ideas emerged through the fluent and flexible thinking, since quantity of solutions may embrace quality of solutions (Kattou et al. 2012). In addition to this, in a study by Sophocleous and Pitta-Pantazi (2011) we investigated whether primary school students' creative abilities improved while working with interactive 3D geometry software. Our results suggested that students' creative abilities improved, mainly due to the opportunities the software offered them to imagine, synthesize and elaborate. However, it needs to be stressed that the way in which technology is used, the mathematical concept it addresses, the way in which the lessons are conducted, and the type of participants involved are only some of the factors that affect the impact of technology on students' creativity. Therefore, there is no simple or straightforward answer as to the impact of technology.

2.6.4 Students Interaction/Communication

Students' interactions are especially important in the development of creativity (Selby et al. 2005). Indeed, John-Steiner (2000) and Neumann (2007), through their observations of creative individual's work in various fields, concluded that cooperation and social interactions affected an individual's creative ability. Similar conclusions were reached both by Sriraman (2009) and Shriki (2010). In particular, Sriraman (2009) conducted interviews with five eminent mathematicians, who mentioned the role of social interactions in enhancing creative work. Shriki (2010) reported that human interactions support the development of creativity, due to the fact that creative ideas are developed mainly through the exchange of ideas. Interactions between people with common interests or motivation, and communication and discussion of mathematical ideas might inspire students to reflect and organize their thinking (NCTM 2000; Shriki 2010). This is possible because people organized in groups might build on each other's ideas or expand seemingly insignificant ideas in more creative ways (Makel and Plucker 2007). Therefore, it is proposed that leaners should be given opportunities to communicate and discuss mathematical concepts and ideas (NCTM 2000; Shriki 2010; Sriraman 2005).

2.7 Epilogue

The discussion above has revealed that over the past decades, significant progress has been made in the investigation of mathematical creativity. Rhode's 4Ps framework provided a simple way to organize our presentation of the research efforts in the domain of mathematical creativity. In particular, research efforts in the domain of mathematical creativity have focused: on the identification of cognitive or personality characteristics (e.g. Freiman and Sriraman 2011; Klavir and Gorodetsky 2009); on the description of stages that define the creative process (e.g. Sheffield 2009); on the description of the creative outcome (e.g. Chamberlin and Moon 2005); and on the identification of environments that encourage creativity (e.g. Goldin 2002; Kleiman 2005; Yerushalmy 2009). Through this review, factors that might enhance mathematical creativity have been revealed. In particular, the awareness of cognitive characteristics that might empower the creative ability, as well as the sub-processes students follow in order to find different and original solutions, provides teachers and educators with "educational equipment" regarding the dimensions they should invest in during their lessons.

Although, in our discussion we reviewed the four facets of creativity—person, product, process and press - separately, the real challenge is to consider the 4Ps as a whole. Taking into consideration the complexity and complicity of the concept of mathematical creativity, unidimensional approaches should be avoided since they are not giving a coherent picture of the concept (Batey 2012). Until now, most research studies concentrated on one or two of the four facets of creativity. For example, in order to examine the characteristics of the creative person, researchers identified their creativity through their products (such as solutions in a test) (e.g. Kattou et al. 2016). Hence, we feel that questions remain open, and we cannot fully fathom creativity unless we bring the 4Ps together. The creative process cannot be addressed without reference to the creative person, as in his/her mind a complex system of cognitive skills, personal factors, motivation, cognitive style, strategies and metacognitive skills, work together to lead to creative behaviors. At the same time, the process cannot be seen independently of the outcome, since the latter is what will be judged and used to assess the success or failure of the creative effort. Along the same line of argument, the environment will determine the relevance and effectiveness of the process or its outcome. According to Batey (2012), the investigation of each separate facet might provide indications about the nature of creativity, however, if we take into account the interactions between its components, we will have additional clues for a comprehensive definition (Batey, 2012). Given the importance of considering the 4Ps as a whole, a key priority for research in the next decade must be to search for overlaps, interconnections and synergies between these 4Ps. In particular, we need more research studies which will describe the students' profile and the impact the specific learning environments have on their creative processes and products. Not all individuals are the same and one size does not fit all, thus, we need to identify the differences in individuals and explore the impact of carefully designed interventions have on their way of thinking, approaches and products.

The conduction of meta-analysis studies may also contribute towards the better visualisation of the interconnections of the 4Ps. The investigation of what we have learned and what we need to explore and work on in the future may be revealed. Having as a guideline the connections that have already been made between mathematical creativity and its various factors, researchers can avoid repetition of similar studies and concentrate on finding new connections and extensions of theoretical and empirical information. Thus, different perspectives of the concept of mathematical creativity might offer a springboard for further development.

Furthermore, in our venture to explore interconnections and synergies between these 4Ps new research designs, methodologies and tools may also be needed. It is encouraging that in recent years we have seen the topic of creativity being explored through new methodologies and tools, such as neuropsychological (Cropley et al. 2017; Lev and Leikin 2017) and eye tracking (Schindler et al. 2016). These new methodological approaches seem to open new, promising perspectives for the exploration of the creative person, press, process and product. They allow us to zoom-in and explore interrelationships and connections which were inaccessible to us in the past.

This chapter offered a broad view of various research studies conducted in the field of mathematical creativity which investigated the person, process, product, and press. We believe the interconnections of the 4Ps open new avenues for research studies in mathematical creativity and we hope that this chapter is a small step towards this direction.

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Chapter 3
Mathematical Giftedness and Creativity in Primary Grades

Daniela Assmus and Torsten Fritzlar

Abstract Creativity is often seen as a characteristic or a variety of content-specific giftedness, but also as an independent, more general kind of giftedness. In the first part of this article, we will discuss some key questions on mathematical giftedness, creativity and theoretical connections between the two constructs. Subsequently, we will specify these considerations with regard to primary students. The main question of the second part of the paper is how creativity can manifest itself in mathematical activities of gifted primary students. Generally, mathematical creativity is assumed to be closely linked to problem solving and problem posing; for mathematically experienced people both processes are embedded in theory building processes. Also primary students can vary given problems and solve problems that usually require only little mathematical knowledge. Moreover, mathematically gifted primary students are able to create new mathematical objects. We will describe types and examples of such invention processes in detail.

Keywords Content-specific giftedness · Embedded model of giftedness and creativity · Mathematical giftedness · Problem solving · Problem posing
Theory building processes

3.1 Introduction

Creativity in the domain of mathematics has met with increasing interest in recent years. Nevertheless, primary school students have not been a focus of research up to now. This may be related to the fact that on the one hand, domain-related knowledge is regarded as an essential prerequisite for creative action (Silver 1997;

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Weisberg 1999), and that on the other hand, the mathematics-specific knowledge base in the primary school age is usually still very small. Mathematically gifted primary students may be an exception, as they are expected to have a higher level of experience and knowledge due to their special abilities and their frequently large specific interest. Therefore, we consider this special group to be particularly suitable for exploring mathematical creativity at primary school age and for investigating possible occasions for creative mathematical action.

In the scientific literature giftedness and creativity are seen as related in many different ways. Therefore, in the first part of this paper, we will initially consider both constructs and especially their connection in detail.

PART I

3.2 (Mathematical) Giftedness

In relevant literature, there is no standardised definition of *giftedness*; neither in general nor especially for the domain of mathematics. Not only is there unclarity regarding the definition of this term, but there are also a number of similar (e.g. *talent*, *expertise*) or connected terms (e.g. *special abilities*, *high achievement*, *creativity*). In this context, Ziegler (2008, p. 14) speaks of a "Babylonian language chaos" (translation by the authors), which makes a theoretical approach to mathematical giftedness extremely difficult.

From a superordinate perspective, there are three key questions regarding a construct of *mathematical giftedness*:

- Does the construct of mathematical giftedness describe extraordinary mathematical achievements or rather just the potentials for especially valuable (subsequent) achievements in the field of mathematics?
- Is mathematical giftedness an expression of specific cognitive characteristics or is it, at least for the main part, a result of high general intelligence?
- Is mathematical giftedness a monolithic construct or are there different profiles of giftedness? (cf. Wieczerkowski et al. 2000).

3.2.1 Giftedness: Potentials and Achievements

It is possible to distinguish the numerous definition approaches for the construct of giftedness based on the roles ascribed to the extraordinary achievements accomplished by the individual. If these are considered preconditions for a person to be described as gifted, they are referred to as performance-oriented definitions. If giftedness, however, is conceptualised as a potential for superior performance, one can speak of competence-oriented definitions.

With regard to children and adolescents, competence-oriented definitions predominate; with adults, a greater priority is given to documented achievements. For instance, Mayer (2005, p. 439) understands "giftedness as an age-specific term that refers to potential for the beginning stage, achievement for the intermediate stage and eminence for the advanced stage."

From our point of view, a competence-oriented definition of giftedness is also appropriate for the domain of mathematics and, specifically, for studying giftedness at primary school age, because especially young children cannot possess the knowledge and experiences needed for extraordinary mathematical achievements.

However, this perspective leads to significant (theoretical and practical) difficulties concerning the diagnosis of giftedness because only achievements can be measured empirically. That is partly why Sternberg (1998) came up with his concept of developing expertise, which was specified by Fritzlar (2015) for the domain of mathematics.

3.2.2 Mathematical Giftedness and General Intelligence

Since the beginning of the 20th century, there has been a more intense theoretical and empirical discussion on the construct of giftedness. Among others due to the Terman study, the view of giftedness has been widened and multi-dimensional models have been created, on the one hand based on the recognition of domain-specific talents [e.g. Munich Model of Giftedness (Heller 2010) or Differentiated Model of Giftedness and Talent (Gagné 1985, 2003)], and on the other hand based on the inclusion of further personal and contextual characteristics [e.g. Three-Rings Conception by Renzulli (1986)].

Since these kinds of models mostly only differentiate between intellectual and non-intellectual areas, it remains open at first to what extent an independent *mathematical* giftedness exists. The answer to this question depends on one's understanding of mathematics in particular. If, for example, mathematical achievements are recorded using tasks which hardly differ from items used in intelligence tests and for which it is mainly essential to be fast and accurate (whereby, incidentally, the latter is exclusively measured by the test developer's horizon of expectation, Kießwetter 1992), it is not surprising if the dimensions of mathematical achievements hardly differ from those of general (test) intelligence (Zimmermann 1992).

Tests on school achievements and study capability tend to be more subject-specific. In a study by Benbow, almost 300 mathematically gifted and a little more than 150 linguistically gifted thirteen-year-olds were first identified based on the SAT—they belonged to the best 0.01% of their age group. As a next step, their achievements were compared using different intelligence and ability tests. Only 16 boys and 2 girls belonged to both groups, the others showed significant group differences in almost all areas of ability, with group membership and not gender having the biggest influence on the test results (Benbow and Minor 1990).

On this basis, the idea of a *general* intellectual giftedness that includes a mathematical giftedness cannot be kept.

Especially interesting for studies focusing on primary school age seems a study by Nolte, where children of third grade work both on a specially developed mathematics test as well as on an intelligence test which correlates strongly with grades in mathematics. In this whole group of more than 1600 girls and boys of nine years, the results from the intelligence and mathematics tests correlate with −0.34. However, this relation decreased for children who obtained particularly good results in the mathematics test. The rather weak statistical correlation and its further decrease can partly be expected because of the (increasing) selectivity and the (decreasing) sample size. Nevertheless, it seems reasonable to assume intelligence test results and mathematical potential correlate based on the total population, but a special mathematical giftedness cannot be derived from the IQ (Nolte 2011, 2013).

From a cognitive-psychological and didactic perspective, different descriptions of mathematical giftedness have been developed based on mathematics-specific abilities, characteristics and patterns of action. In this context, the studies carried out by Käpnick (1998) were pivotal in Germany regarding primary school age. For him, mathematical giftedness is marked by the following characteristics and skills: *remembering mathematical facts, structuring mathematical facts, mathematical sensitivity and mathematical fantasy, transferring mathematical structures, intermodal transfer, reversing lines of thoughts* (cf. Benölken 2015). To what extent these features characterise mathematically gifted students depends on the mathematical richness of the tasks used to reveal them.

At a first glance, the specificity of some abilities may seem critical. However, in psychology the position is widespread that abilities do not exist by themselves but always in connection to specific contents to which they are inseparably related (Lompscher and Gullasch 1977).

All in all, it seems reasonable to assume that initially general cognitive abilities first generally develop and then become more specific during activities. In this regard, the accumulation of knowledge could play a vital role, because knowledge is on the one hand gained through abilities of the individual and on the other hand forms an important basis for the development and realisation of mental abilities. In this sense, abilities, knowledge and activities develop in close interaction and mutually reinforce.

3.2.3 Profiles of Mathematical Giftedness

If a list of specific abilities or action patterns is used to describe mathematical giftedness, the respective authors (e.g. Benölken 2015; Käpnick 1998; Nolte 2011, 2013) always emphasise that they can be evident to a various extent and not all of them are necessary for the presence of giftedness. Also Krutetskii, who as one of the first ones soundly studied special abilities of mathematically gifted students, emphasised that their composition to a structure of mathematical thinking can be *individually different*, whereby certain components can also be compensated by others. High mathematical achievements can be reached with different complexes of abilities or "mental specialities". As a result, there are different manifestations of mathematical giftedness, especially since, according to Krutetskii, further useful but not necessarily needed characteristics exist, like the speed of thinking processes, counting skills, a distinct memory for symbols, numbers and formulas, visual thinking as well as the ability to vividly imagine abstract mathematical relations and dependencies (Krutetskii 1976). For older students, he distinguished between a geometric, an analytic and a harmonic type based on the relation between visual and abstract-logical components. The last type, however, probably has the highest potential.

Qualitative research studies on mathematical giftedness in primary school have shown that different profiles of specific abilities already exist at this age (e.g. Fuchs 2006; Käpnick 1998). However, it is assumed that interindividual differences increase through growing domain-specific experience.

To sum up the discussion on the three key questions, mathematical giftedness at primary school age can, from our point of view, be understood as an extraordinary high potential to solve mathematically challenging questions and problems (compared to others of the same age). The various aspects of this potential can be differently pronounced, but in total it is mostly specific for the domain of mathematics. A detailed description of mathematical giftedness in early primary school age by means of specific abilities was recently developed by Assmus (in this volume).

3.3 Creativity

Since the 1950s, creativity research has continually been and is still being advanced. However, to this day there is neither a consistent definition of creativity nor a commonly acknowledged creativity theory. In scientific discourse, it is common to distinguish between creativity as a quality of a product, a person, a process or creativity-affecting environmental factors. In the English-speaking world this is also referred to as the "4P's of creativity" (product, person, process, press) based on the work of Rhodes (1961). Since in this article we will not discuss developmental aspects, neither in relation to giftedness nor to creativity, the fourth aspect (press) is not further considered.

What is normally considered the key criterion of a *creative product* is its "novelty". Since, however, objective novelty independent of space and time is extremely rare, some authors relativise this criterion in so far as an idea is seen as new (or unique) if it is rare among a particular population (e.g. Guilford 1967; Jackson and Messick 1965). In contrast to absolute creativity, we refer to relative creativity in this regard. Ideas that are new for an individual, but widely spread among the population considered (e.g. a school class) are not judged creative according to this definition. In pedagogic situations, however, an individual reference norm might be used as a basis for the novelty criterion (cf. e.g. Kießwetter 1977), which is then referred to as individual creativity.

Besides novelty, at least one further criterion is specified, which concerns the purpose of the product. In this regard, terms such as “meaningfulness”, “target-orientation”, “real-life relevance” and “usefulness” are used (Preiser 1976). It should, however, be mentioned that free creative processes would not be classified as creative according to this approach if the created products did not meet the criterion of usefulness. Since this would apply to many artistic products, the usefulness criterion might be seen as disproportionally constraining the kind and number of creative products.

Regarding *creativity as a quality of individuals,* the features based on the work of Guilford (1950) and their operationalization in the “Torrance Test of Creative Thinking” of Torrance (1966) are usually cited, namely fluency, flexibility, originality and elaboration. Fluency refers to the ability to produce as many associations, thoughts and ideas as possible on a content or problem within a short time. Flexibility can be described as the ability to think into different directions, to easily switch from one thinking category into another, and to look at a problem from different views. Originality is the ability to generate uncommon ideas and solution approaches. “Uncommonness”, “remoteness” and “cleverness” are mentioned as measuring criteria for originality. The ability to proceed from an idea to a definite plan and, thus enriching and developing the idea, is understood as elaboration.

These explanations show that creativity as characteristic of a person cannot be separated from the creative product. The product characteristic is needed to estimate the originality of a person. Also, the description of *creative processes*, for which in general multilevel phase models are used, like, e.g., that of Wallas (1926) and respectively Hadamard (1945), which propose the phases preparation, incubation, illumination and verification (e.g. Aldous 2007; Sriraman et al. 2013), cannot be made without considering the creative products.

3.4 Relations Between (Mathematical) Creativity and (Mathematical) Giftedness

Giftedness and creativity are often seen in close connection. However, the basic assumptions made in the scientific discourse differ concerning the relation between (mathematical) creativity and (mathematical) giftedness (cf. e.g. Singer et al. 2016 with many references). In our opinion, the different views can be classified as follows (cf. Aßmus 2017):

1. (mathematical) creativity as a precondition for (mathematical) giftedness
2. (mathematical) creativity as a possible component of mathematical giftedness
3. (mathematical) creativity as a possible consequence of mathematical giftedness
4. creativity as a (mostly) independent area of giftedness.

The single views are further explained below. Since we are mainly interested in the *relation* between these two constructs, we will not name and explain further influencing factors here. This does not imply that no further influencing factors exist.

Whenever specific study results or theoretical approaches exist for the domain of mathematics, these are considered, even though mathematical creativity has not been regarded so far in this paper. As a detailed explanation of the term mathematical creativity is not needed yet at this point, we will provide an in-depth discussion of the concept in the second part of this paper.

3.4.1 (Mathematical) Creativity as a Precondition for (Mathematical) Giftedness

Based on this conception, well-developed creativity is seen as a necessary precondition for giftedness (Fig. 3.1). Renzulli's model (1986) can be cited as an example in the context of general giftedness. According to this model, the three factors "above average ability", "task commitment" and "creativity" are needed for developing gifted behaviours. Some of the characteristics of creativity as Renzulli understands it are fluency, flexibility, originality of thought, openness to experience and willingness to take risks in thought and action.

In the area of mathematical giftedness, this conception can be found e.g. in Leikin et al. (2009), who consider mathematical giftedness as special problem solving abilities and, referring to Renzulli, describe mathematical creativity as a needed component besides "problem solving effectiveness" and "task commitment" (Leikin et al. 2009). Here, the mathematical creativity is also characterised by the three subcomponents fluency, flexibility and originality.

3.4.2 (Mathematical) Creativity as a Possible Component of Mathematical Giftedness

For several models, creativity in the sense of creative abilities is not a precondition for giftedness, but is rather understood as part of the giftedness itself (Fig. 3.2).

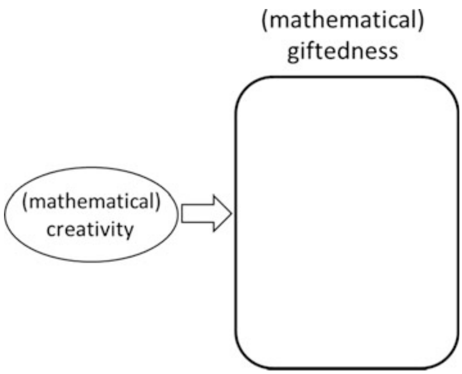


Fig. 3.1 (Mathematical) Creativity as precondition for (mathematical) giftedness

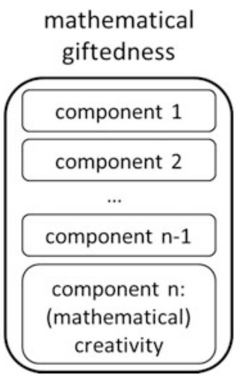


Fig. 3.2 (Mathematical) Creativity as a possible component of mathematical giftedness

For the domain of mathematics, further mathematics-specific abilities are named besides creative abilities. Since not all listed abilities are expected to become evident to an equal extent, different types of mathematical giftedness are possible (cf. our explanations concerning mathematical giftedness above). As a consequence, also a well-developed creativity is not mandatory for mathematical giftedness.

This understanding can, e.g., be found in Krutetskii (1976). He does not use the term "creativity" itself in his components model, but refers to it as "flexibility of mental processes in mathematical activity" (p. 350), which became apparent in his studies when participants managed to overcome fixations or break away from a stereotyped method of solution. Elsewhere, he explains that turning away from typical procedures as well as finding several different solutions is "the real appearance of mathematical creativity" (Krutetskii 1969, cited from Haylock 1984, p. 30).

Käpnick (1998) also mentions another component pertaining to the concept of mathematical creativity as one of seven mathematics-specific characteristics of mathematically gifted children in primary school. He describes it as "mathematical phantasy", which he understands as the "most important main aspect of childlike creativity" (Käpnick 2013, p. 31; translation by the authors). According to him, the development of diverse imaginative patterns and respectively structures as well as the development and usage of creative solutions for demanding tasks belong to phantasy, just like (not necessarily target-oriented) playful actions with mathematical materials (Käpnick 1998).

Kontoyianni et al. (2013) distinguish between the two categories "mathematical ability" and "mathematical creativity", which are additionally split up into sub-categories. For mathematical creativity, the sub-categories fluency, flexibility and originality are assumed. In their study with students from fourth to sixth grade they worked out mathematical giftedness as a multi-factorial construct which contains both special mathematical and creative abilities. However, it was also possible to conclude from their data that the importance of mathematical abilities for the construct of giftedness is higher than the importance of creative abilities. Additionally, they proved the relationship between mathematical abilities and mathematical creativity using statistical methods. Of the three approached models, the one that best explained the gathered data was the one that understood mathematical creativity as a sub-component of mathematical giftedness (Kattou et al. 2013). Even if the terms are used differently and the models' sub-components are not identical, general similarities to the characteristics lists of Käpnick (1998) and Krutetskii (1976) mentioned above can still be found in the results concerning the relation of creativity and giftedness.

3.4.3 (Mathematical) Creativity as a Possible Consequence of Mathematical Giftedness

From this perspective, mathematical creativity is understood as the ability to create creative products that contribute to a knowledge progress within mathematics as a science (Fig. 3.3). Creative achievements are therefore reserved for a small group of people. As a result, mathematical creativity implies mathematical giftedness while the inversion is not valid (Howe 1999; Sriraman 2005).

Exemplary for such an understanding is the hierarchy of mathematical talent by Usiskin (2000) (cf. Sriraman 2005). Usiskin proposes eight levels of mathematical talent starting with level 0, which covers adults who barely know something about mathematics. The two highest levels (level 6 and 7) are assigned to people who stand out due to especially creative achievements. The lower levels, too, attest a mathematical talent to people, but in this case this talent does not come along with extraordinarily creative achievements.

Also Sheffield (2009) shapes a "Continuum of mathematical proficiency" (innumerates—doers—computers—consumers—problem solvers—problem posers—creators) which considers the creative creation processes as the highest manifestation.

3.4.4 Creativity as a (Widely) Independent Area of Giftedness

Creativity is considered as a widely independent area of giftedness in several models (Fig. 3.4). A known example is the "Differentiated Model of Giftedness and

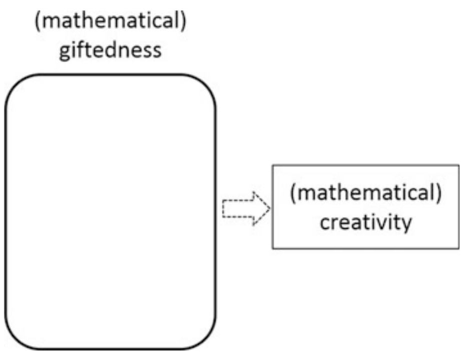
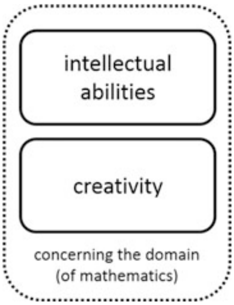


Fig. 3.3 (Mathematical) Creativity as a consequence of (mathematical) giftedness

Fig. 3.4 Creativity as a (widely) independent area of giftedness



Talent" by Gagné (1985, 2003) where creativity is listed as one of four areas of giftedness. Others distinguish two fundamentally different forms of giftedness, namely educational or academic giftedness on the one hand, and creative giftedness on the other (cf. e.g. Renzulli and Reis 2003; Hong and Acqui 2004). While "schoolhouse giftedness" according to Renzulli is a specific giftedness concerning test and school achievements, the creative giftedness can be seen in real application situations. "Creative-productive giftedness describes human activity and involvement in which a premium is placed on the development of original material and products that are purposefully designed to have an impact on one or more target audience" (Renzulli and Reis 2003, p. 185).

In their "Comprehensive Model of Giftedness and Talent" Milgram and Hong (2009) distinguish between "analytical-thinking ability" and "creative-thinking ability", which can manifest themselves in "expert talent" or "creative talent". Both forms of talent require both abilities, however, analytic abilities predominate with regard to "expert talent" and creative abilities predominate with regard to "creative talent". "Creative Talent" according to Milgram and Hong is, besides profound specialist knowledge in the respective domain, characterised by the creation of creative and useful products.

Subotnik et al. (2009) examined requirements for the evolution of mathematical talent. They were able to identify different influence factors; mathematical creativity however was not among them. "It seems that a number of variables other than innate mathematical creativity shape the development of talent and ensure a successful career trajectory" (p. 177). This could also be indicative for the fact that mathematical creativity represents a separate area of giftedness. However, the authors are not clear concerning the term "creativity". In the above stated quote, they seem to refer to creativity as ability, but they do not specify it. Furthermore, they use the term "creativity" in relation to the term "talent", which suggests that they see the two concepts as equivalent. The explanations can therefore be interpreted as follows: Creative abilities do not play a role in the development of mathematical talent, but creative products as "output" are closely related to mathematical talent.

Haylock (1997) detected that mathematically efficient students are highly diverse concerning their mathematical creativity. This could also be an indicator of the autonomy of both constructs. Maybe, however, these differences can also be explained by differently evolved personality traits: Students with very good mathematical, but only relatively poor creative achievements increasingly developed negative associations with the subject of mathematics, had quite a low self-esteem and were hardly prepared to take risks when solving mathematical problems.

All in all, there are many different perspectives on the relations between (mathematical) giftedness and (mathematical) creativity. In our view, however, these perspectives are not necessarily contradictory, but they rather result from *different understandings of giftedness and creativity which are not independent from each other*. While giftedness is rather defined as a potential for extraordinary achievements (and characterised by special abilities), creativity is frequently also seen as a person's individual characteristic and the creation of creative products does not have priority (like in perspective 2). If the construction of creative products is focussed, this results in perspective 3. In contrast, in a performance-oriented understanding of giftedness, creative abilities are also seen as a requirement for special mathematical achievements (perspective 1). Perspectives 1–3 on giftedness as competence or performance (always related to mathematical giftedness) can be summarised as follows.

Concerning perspective 4, it can be said that satisfactory evidence on creativity as an independent area of giftedness has not yet been provided. Related observed phenomena can possibly also be explained via different forms of giftedness in perspective 2. Also, in this view, types of giftedness might exist where creativity is more or less extensively developed.

For these reasons and based on above explanations, the best model for our purposes is the left one in Fig. 3.5, where giftedness is understood as potential for extraordinary achievements and creativity is understood as an optional sub-component of giftedness. This does not exclude overlaps of the component "creative abilities" and other components.

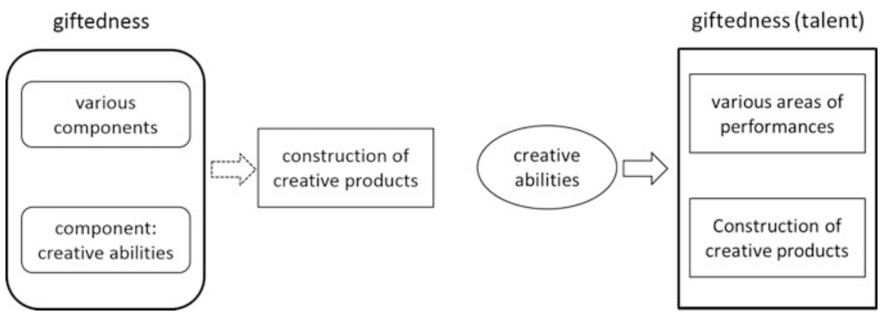


Fig. 3.5 Relations between creativity and giftedness as competence (left) or performance (right)

PART II

3.5 Mathematical Creativity in Primary School Age

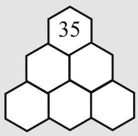
The *embedded model of giftedness and creativity* in the left part of Fig. 3.5 was more or less theoretically assumed. Prerequisite for a stronger empirical support—especially with regard to mathematics at primary school age—is, first of all, a sharpening of underlying concepts. This particularly applies to the concept of mathematical creativity.

As Mann (2006) has pointed out, there is a lack of a widely-accepted definition of mathematical creativity. In the literature, mathematical creativity is, for instance, described as the ability to choose (sensu Poincaré) or to engage in non-algorithmic decision-making. From a common didactical perspective, mathematical creativity can be seen as the ability to generate novel and useful solutions to problems (cf. Sriraman 2009). For primary school age, this definition approach can be combined with the above depicted approach of research on creativity, but specifications and extensions are needed.

At first it should be discussed to which extent criteria (for products, persons or processes) from general creativity research could also be suitable for describing mathematical creativity at primary school age.

(a) *Fluency* can be expressed by primary school children in the area of mathematics, e.g. when solving a problem in several ways, finding several solutions for an open problem (field), continuing a pattern in several ways, finding several representatives for a mathematical characteristic. Such actions demonstrate that children can vary their approaches to doing mathematics.
However, from our point of view, it should be critically questioned if the number of created answers, especially in mathematics, could be indicative of creativity. A large number of *similar* solutions can be quickly and systematically developed for many open tasks by creating structures and ordering principles. The following task to measure fluency used by Kattou et al. (2013, p. 172) serves as an example in this regard:

"Look at this number pyramid. All the cells must contain one number. Each number in the pyramid can be computed by performing always the same operation with the two numbers that appear underneath it. Fill in the pyramid, by keeping on the top the number 35. Try to find as many solutions as possible."



A high number of solutions can already be created through additive partition of the numbers. Creativity is not necessary in this procedure. On the contrary, abilities to identify and use mathematical structures are needed.

Quickly creating many answers to a mathematical stimulus can therefore give a first indication of mathematical creativity, but might as well be based on other, convergent abilities.

(b) While with "fluency", the created solutions/products are considered in terms of quantity, *flexibility* is mainly about the diversity of the products (Neuhaus 2001). A particular flexibility can also become apparent in the above mentioned mathematical actions, if the solutions and ideas fundamentally differ from each other. Additionally, flexibility in doing mathematics can be expressed when the perspectives are changed or when successfully dealing with adapted processing aspects. The following aspects of change are possible at primary school age: representations of a mathematical content, contexts, perspectives on a mathematical content/a mathematical problem, processing directions (e.g. direct vs. converse lines of thoughts), use of given task elements (e.g. switched given and searched elements).[1]

(c) Like the "novelty" of a product, *originality* can only be evaluated using a reference group. In relation to this, the individual reference norm is particularly appropriate for primary school age. For mathematically gifted students, however, it can be assumed that some of them create extraordinary mathematical products, find procedures etc. in relation to the peer group or the group of similarly mathematically experienced students. In these cases, one can assume a high (relative) creativity.

(d) Concerning "real-life relevance" or "usefulness", we agree with Sriraman's statement (2009, p. 15) that the results of creative mathematical processes do not always have to be applicable, because mathematics is also a world with its own value. Consequently, it seems "[…] sufficient to define creativity as the ability to produce novel or original work." This is especially valid for primary school age.

So, from a psychological perspective only the criteria flexibility—concerning mathematical products and in particular also processes of doing mathematics—and originality on an individual level seem to be adaptable for describing mathematical creativity at primary school age.

To continue with the common didactic perspective on creativity already mentioned above, we initially have to go a little further.

According to many researchers, creative mathematical processes particularly occur at *problem solving* (Chamberlin and Moon 2005; Leikin and Lev 2013; Pehkonen 1997). In recent years, the importance of *problem posing* has also been emphasised. The proximity of creativity and *problem solving* is, according to Guilford (1977), already implied by the similar understanding of the two terms: "Creative thinking produces novel outcomes and problem solving involves

[1]Changing aspects 1 and 4 are described as creative abilities of mathematically promising students e.g. by Sheffield (2003).

producing a new response to a new situation, which is a novel outcome" (Guilford 1977, p. 161). He therefore concludes that problem solving involves creative processes. Since problem solving constitutes an essential part of mathematical activities, creativity necessarily plays an important role in mathematics. However, the problem solving process is not always considered as creative. Kießwetter (1977) and Haylock (1984) only judge solution processes as creative which involve divergent thinking, like e.g. the association of distant things, the creation of new means or the novel usage of known/existing means. By contrast, a process which solely consists of applying solution schemes, logical reasoning or systematic sorting is not seen as creative.

Guilford already cited problem sensitivity as a special characteristic of a creative person. What he understands by problem sensitivity is the ability to approach the material and social environment with an open, critical attitude and to discover problems and opportunities for improvement, contradictions, inconsistencies and novelties (Preiser 1976), which is linked to *problem posing*. In mathematics, as a constantly broadening science of self-created abstract structures, the identification, extension, narrowing or widening and transferring of (new) scientific questions plays a vital role. "Problem finding" or "problem posing" per se is seen as creative act by some authors (Leung 1997; Silver 1994). It is partly also viewed as impulse for especially creative performances (Sheffield 2009). In school situations however, problem posing is often not a consequence of a genuine impulse to discover, but specifically initiated by the teacher. The specifications here may vary: based on an already solved problem, related problems can be identified and follow-up questions can be raised. Independent of a concrete problem, questions may be developed, for example, concerning a specific mathematical content or context, specific terms or numbers, specific solution strategies, but also entirely without predefined specifications. In any case, problem posing is closely related to problem solving. For a detailed overview of problem posing in mathematics learning see the report by Singer et al. (2011).

Problem solving is already an important part of mathematics classes in primary school, where mainly problems are dealt with that can be solved without a broad mathematical knowledge base. Since it is particularly challenging for young students to formulate their own questions, creativity in connection with problem posing might mainly manifest itself in the design of diverse variations of given, perhaps even partly solved problems.

With mathematically interested and experienced older students and even more with adult researchers in mathematics, problem solving and problem posing are usually embedded in more comprehensive *theory building processes*. Here, the handling of an initial problem becomes part of a circular process of problem solving and problem posing through variation and expansion and the subsequent analysis of this circle. The results and methods as well as the newly developed terms and logical relations and, respectively, the novel strategies and tools emerging from this process form a "theoretical fabric", which is then optimised, preserved and integrated into the existing knowledge base (Fritzlar 2008).

In such theory building processes, creative acts, the invention of (subjectively) new mathematical objects and structures as well as new mathematical methods play an important role. They also specifically highlight the interplay between divergent thinking, i.e. the ability to develop and elaborate diverse and original ideas with fluency, and convergent, i.e. logical and evaluative, thinking.

It can hardly be expected that young students with little mathematical experience are already capable of such theory building processes. However, concerning mathematical creativity, it could be possible that primary school children are already able to create subjectively new mathematical objects and relations and gain mathematical experiences in investigating these (Fig. 3.6). Thereby the student's invention can either be rather target-oriented, especially when they are dealing with a superordinate problem, or relatively free.

Overall, mathematical creativity in primary school age appears when students work on low-knowledge problems, vary given problems and create mathematical objects. This creativity can be especially high when students work flexibly and/or invent original products.

The close connection between creativity and problem solving or problem posing has been proposed many times (Chamberlin and Moon 2005; Haylock 1987; Leikin and Lev 2013; Leung 1997; Pehkonen 1997; Silver 1994; Sriraman 2009; Yuan and Sriraman 2011). In a case study we therefore want to explore in how far students of the fourth, fifth and sixth grade are already capable of creating *subjectively new mathematical objects*. Focusing on arithmetic, which is of particular significance for mathematical education at primary school level, possible novel mathematical objects are, for example, numbers, sequences of numbers, relations, operations and algorithms.

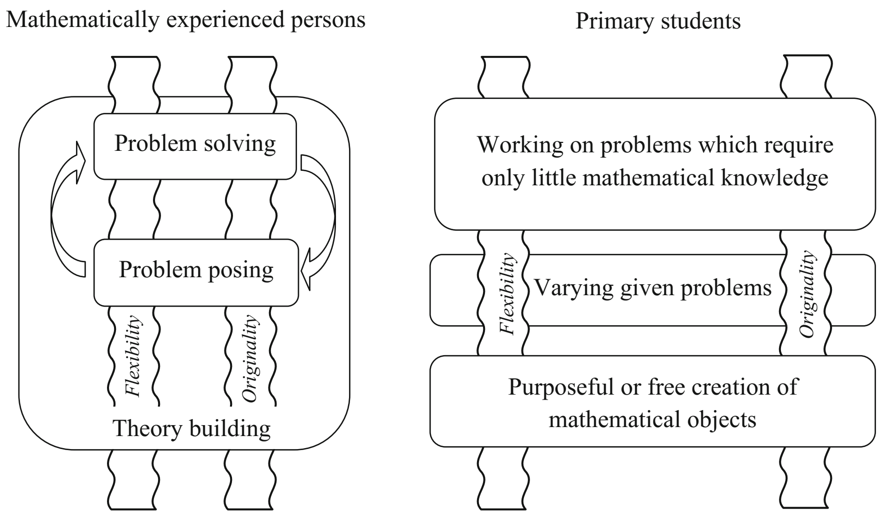


Fig. 3.6 Initiating mathematical creativity

3.6 Creativity as Inventing Mathematical Objects in Primary School

In recent literature, most researchers suggest that creativity is the result of confluence of different individual and environmental factors. According to Sternberg and Lubart (1996, 1999), for example, creativity results from the interplay between six different, but not independent resources: intellectual abilities, knowledge, styles of thinking, personality, motivation and environment. At this point, we cannot deal with all mentioned resources in depth, but for the implementation of our case study this "investment theory" approach had the following consequences: in a first "round" we worked with children participating in talent programs or special classes for mathematically gifted students because we could assume that these children would be intrinsically motivated to deal with mathematical problems and possess a broader mathematical knowledge base (relative to their age group). In addition, we have developed a collection of problems which inspire the creation of mathematical objects and relations. Finally, it was important for us to create an environment in which students can propose their ideas without the pressure of being assessed.

3.6.1 Inventing Mathematical Operations

The aim of the investigations described below was to encourage the children to invent subjectively new computing operations. The investigations were carried out with 127 fourth-graders and 33 fifth-graders. 35 of the fourth-graders participated at a fostering project for mathematically gifted and interested children. They were proposed for participation by their teachers. In addition, they had to pass an entrance test including tasks that test essential characteristics of mathematical giftedness (cf. Assmus, this volume; Käpnick 1998). The other fourth-graders took part at a mathematical correspondence circle for mathematically interested children and were also chosen by teachers. The fifth-graders attended a special school for mathematics. An entrance test had to be successfully completed for admission to this school.

To become familiar with this type of task, the students should first decrypt predefined operations. The next problem was to invent new arithmetic operations (to design appropriate tasks for their classmates to solve) and to explain how the calculation works. For example, the following is a concrete task used in the correspondence circle (original German):

"Marc finds the calculation modes addition, subtraction, multiplication and division boring. He has designed a new type of arithmetic operation and for this purpose invented this operation sign: $\diamondsuit$

He calculates: $1 \diamondsuit 3 = 5$ $6 \diamondsuit 4 = 16$ $3 \diamondsuit 1 = 7$ $0 \diamondsuit 5 = 5$

(a) Explain how Marc's type of calculation works.
(b) Find another new type of calculation. Invent an operation sign, give 4 examples and explain how your calculation works. Think of a name for your type of calculation."

For the most part, the students worked alone, partly in pairs. They received no assistance by the researchers. The students found different ways of creating new arithmetic operations. Similarly to place-value systems, they for example combined numbers into new numbers (Fig. 3.7).

In all student groups, familiar arithmetic operations were combined into new operations in a wide variety of ways. The Figs. 3.8 and 3.9 show some examples. For a better understanding, we provide the mathematical formula for describing the operation.

We could also observe combinations of known operations and transformations of numbers, for instance by rounding or summing their digits. Figure 3.10 shows an example developed by a fourth-grader. Another member of the correspondence circle had the idea to change calculation rules (Fig. 3.11).

With fifth-graders, combinations of the already presented invention strategies could be observed as well (Fig. 3.12).

Very interesting seems the operation on (Fig. 3.13) known operations invented by a fifth-grader.

We also asked the fifth-graders to investigate their new operations and look for interesting characteristics. Figure 3.14 shows some examples.

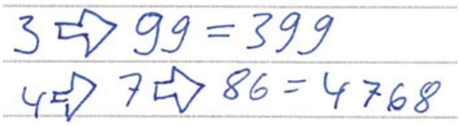
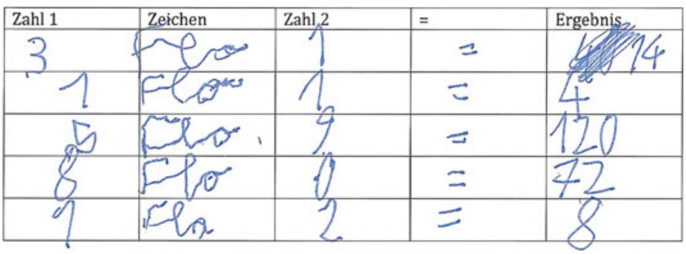


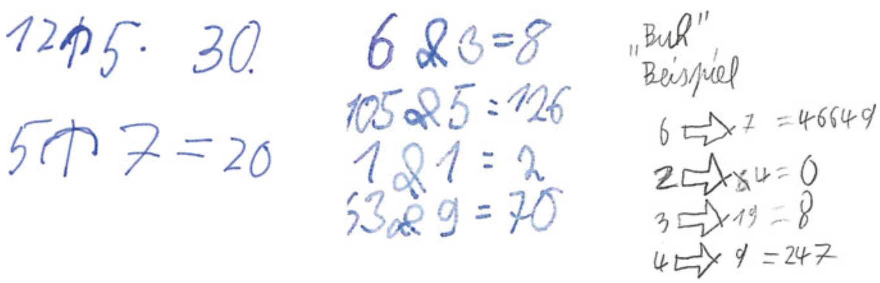
Fig. 3.7 Composing numbers (5th-grader)



Zahl 1	Zeichen	Zahl 2	=	Ergebnis
3	Flo	1	=	14
1	Flo	1	=	4
5	Flo	9	=	120
8	Flo	0	=	72
9	Flo	2	=	8

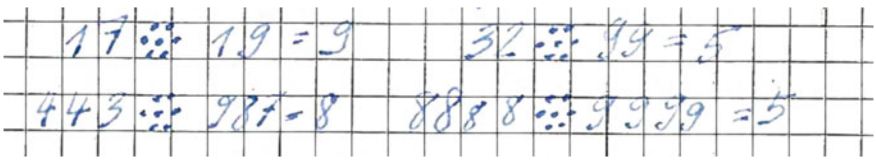
$a^2 + b^2 + a + b$

Fig. 3.8 Creating a new operation (4th-grader)



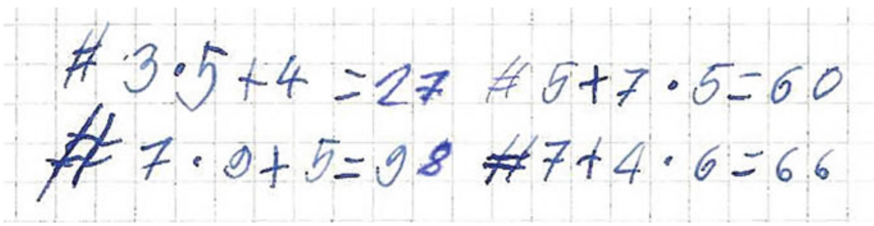
$\frac{(a-1)+(b-1)}{2}\cdot 4$ (4th-grader) | $\frac{a}{b}+a$ (5th-grader) | $a^a - b$ (5th-grader)

Fig. 3.9 Creating new operations



Repeatedly compute the sum of digits of both numbers until they are single figures; then add these figures.

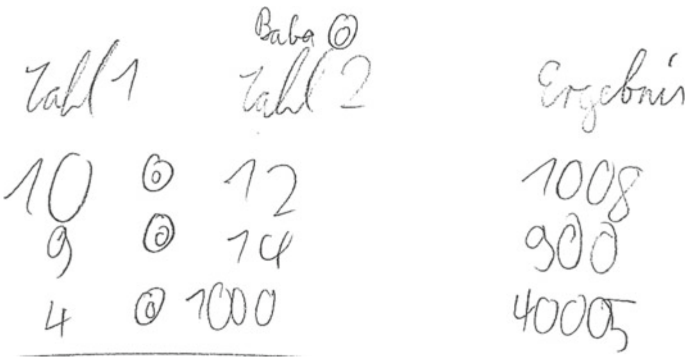
Fig. 3.10 Combining known operation and transformation of numbers (4th-grader)



If # is placed before the arithmetical task, the order of operation rule changes.

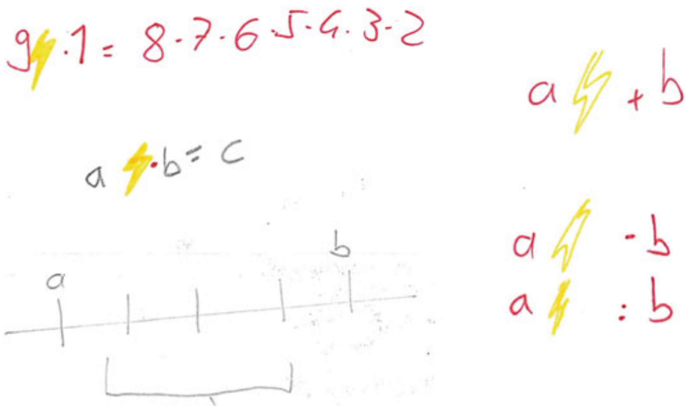
Fig. 3.11 Changing calculation rules (4th-grader)

At least in the last examples, the students' willingness and commitment for investigating their new operations demonstrate that they do not only use new names or signs but actually invent (subjectively) new mathematical *objects*.



Compose both numbers and subtract the sum of digits.

Fig. 3.12 Combining invention strategies (5th-grader)

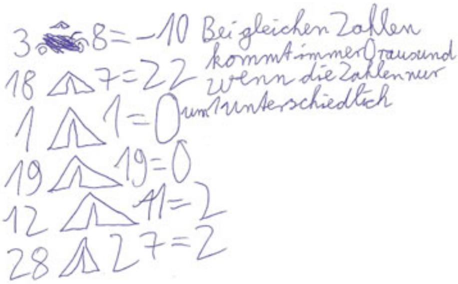


All natural numbers between two numbers *a* and *b* will be linked by the specified operation.

Fig. 3.13 Operation on known operations (5th-grader)

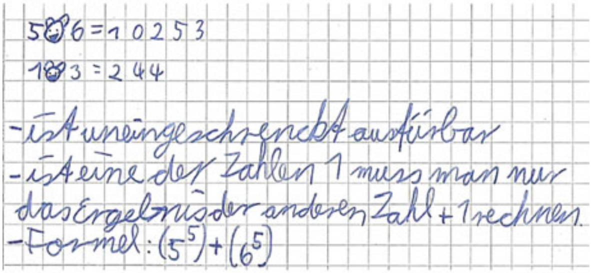
3.6.2 A Purposeful Invention During Problem Solving

The inventions of arithmetic operations described above can be seen as free creations; the following example is a purposeful creation during problem solving. In a math circle for sixth grade students, we used the following problem (cf. Kuzman 2016) for the solution of which the students invented different encryption methods.



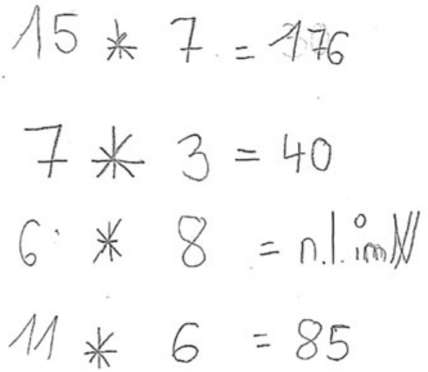
If the numbers are the same, the result is always 0.

$$2 \cdot (a - b)$$



- Can be executed without restrictions
- If one of the numbers is 1, you only have to calculate the result of the other number plus 1

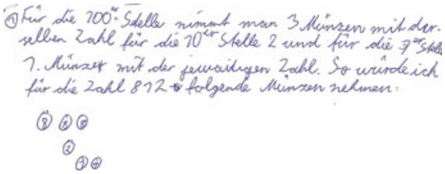
$$a^5 + b^5$$



- If the first number is less than the second number, it is not solvable in ℕ.
- If there are two equal numbers, the result is 0.
- If the first number is 1 greater than the second, the result is exactly the same as for the addition task.

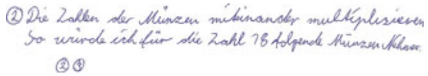
$$(a + b) \cdot (a - b)$$

Fig. 3.14 Invented operations with identified characteristics (5th-grader)



For the hundreds digit you take 3 coins with the same number, for the tens digit 2 and for the units digit 1 coin with the respective number. Thus, I would take the following coins for the number 812:

Fig. 3.15 Tim’s first idea



Multiply the numbers of the coins by each other. Thus, for the number 18, I would use the following coins: (2) (9)

Fig. 3.16 Tim's second idea

Take as many coins as is the numerical value.

Fig. 3.17 Tim's third idea

People have always tried to exchange secret messages in a form that cannot be read by outsiders. Today, it is up to you to invent an encryption method which could already have been used in ancient times when messages were still sent by courier. Only those who know the encryption method should be able to encode and decode the message.

The message can consist of a natural number between 1 and 999.

It should be encrypted by coins, each of which is labelled with a number between 0 and 9. The courier should be able to carry the coins in a small bag, which of course is thoroughly shaken during transportation. There are many coins of each kind available for encoding.

Your task is to invent a really good encryption method!

The six participating students attend the special school for mathematics mentioned earlier.

First of all, each student developed their own suggestions for the encryption, which they then presented to each other. Tim presented three ideas (Figs. 3.15, 3.16 and 3.17).

Here, the hundreds, tens and units digits are coded by three, two and respectively one coin of the corresponding sort. This code for coins, however, cannot always be definitely decoded, because for instance 811 and 188 would result in the same coin representation. (All in all, the algorithm fails at 90 numbers.) In a later discussion, there was the idea to "repair" this method, namely by using the respective coin four times in the case of identical units and tens digit number.[2]

Numbers should be encoded as a product of the coin values. However, this method fails with multi-digit prime factors (e.g. 23, 51, 91).

Here, numbers are coded by the number of coins. With this method, Tim has invented a simple procedure which however is easy to see through and not very practical: on average, 500 coins are needed for encoding one number.

[2]Having the same units and hundreds digit is also unproblematic with this improved method.

Tim's first coding idea can to a certain extent also be reversed. Jason, for example, suggested to first choose three kinds of coins and code the units digit with the corresponding number of coins with the lowest par value, the tens digit with the appropriate number of coins with medium par value and to proceed analogically for the hundreds digit. During the discussion, the students themselves realised that it is not possible to decode numbers with the digit 0 and, respectively, one- and two-digit numbers. This method, too, can be "repaired" if sender and receiver agree upon three kinds of coins. Thus, a maximum of 27 coins or, respectively, an average of 13.5 coins are needed.

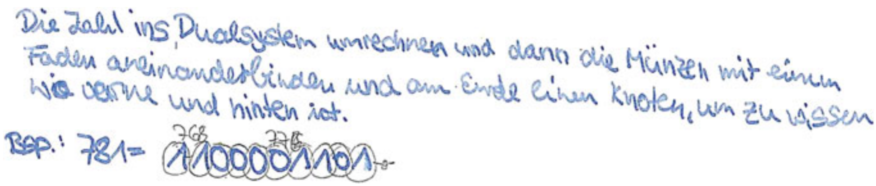
Anna presented two options (Figs. 3.18 and 3.19).

Through division, the number to be encoded can first be represented by smaller numbers, which can then be coded by coins. In the example, 781 would be represented by 86 coins with a value of 9 and one with a value of 7.

The second suggestion uses the conversion into the dual system. The coins should be tied up on a string with a knot, which makes it possible to keep the order of zeros and ones. In the follow-up discussion, the students themselves came up with the idea of associating the 2^n-digit of the binary number with the n-coin type and of putting a corresponding coin into the bag for each digit 1. Thus, the number 781 would be coded by the coins (9)(8)(3)(2)(0). Since $2^{10} - 1 > 999$ applies, all numbers required by the problem can be unambiguously encrypted with a maximum of nine coins.

…, i.e. 86 coins with a value of 9 and one with a value of 7.

Fig. 3.18 Anna's first idea



Convert the number into the dual system; tie the coins together with a string that has a knot marking its beginning.

Fig. 3.19 Anna's second idea

3.7 Conclusion and Outlook

Both examples firstly show that mathematically gifted primary school children are already capable of being mathematically creative and of developing and investigating subjectively new mathematical objects. With these results, we have achieved a main objective of our study. It also indicates the wide variety of creative products even among the small group of students participating in this experiment. A comparative observation furthermore shows that some products, e.g. in Figs. 3.11, 3.12, 3.13 or 3.19, are very rare, so in comparison to the others these products can be judged as very creative. Moreover, the students' ideas for maximally accurate and efficient encryption algorithms can be seen as seeds of theory building processes. It is, of course, also possible to follow up on the idea presented in the first example, e.g. regarding algebraic properties. Therefore, the described results and procedures also indicate further mathematical potentials beyond creativity.

As further investigations have shown, tasks like the invention of new mathematical operations may also encourage primary school students in regular classes to be creative with mathematical objects. For instance, we asked fourth-graders in two regular classes to invent new arithmetic operations like the ones presented above. Almost all of them were able to meet these requirements. Mostly they combined familiar arithmetic operations into new operations. Although the majority of these inventions were not as complex and diverse as many inventions in the gifted group, creative approaches were apparent. Thus, we think the tasks are suitable for initiating mathematical creativity (on different levels) in almost all primary school children. While in the free creation of mathematical objects, differences between gifted and non-gifted students appear in the different mathematical complexity of the invented objects, it could be possible that the purposeful creation of objects during problem solving is only achievable for gifted students. This hypothesis should be scrutinised in further studies.

The creation of mathematical objects at primary school age is not reducible to the creation of mathematical operations and algorithms. Furthermore, children can invent their own mathematical terms (e.g. special numbers like MUM- and DAD-numbers; special geometrical shapes) and formulate their characteristics. The area of geometrical patterns also offers numerous opportunities to create new figure patterns, geometrical ornaments or tilings. This means that also in primary grades and for almost all primary school students, there are many chances to be mathematically creative in the sense of creating (subjectively) new mathematical objects. Further studies with gifted students should investigate their abilities to invent different purposeful mathematical objects to review the embedded model of giftedness and creativity (left part of Fig. 3.5).

The type of tasks described in this paper does not only allow for creative mathematical action. It also creates the possibility of detailed investigations of mathematical characteristics of the created objects. For instance, the computing operations can be examined with regard to group axioms, commutativity or other algebraic features. It is also possible to investigate the relations between details of the definition of a new operation and its algebraic characteristics. In this way, students will be encouraged to take a stronger algebraic perspective, also on known operations and other mathematical objects. Our studies indicate that this is possible already for younger students.

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Chapter 4
Cognitive Variety in Rich-Challenging Tasks

Cristian Voica and Florence Mihaela Singer

Abstract Cognitive flexibility—a parameter that characterizes creativity—results from the interaction of various factors, among which is cognitive variety. Based on an empirical study, we analyze students' and experts' cognitive variety in a problem-posing context. Groups of students of different ages and studies (from primary to university) were asked to start from an image rich in mathematical properties, and generate as many problem statements related to the given input as possible. The students' products were compared in-between, and to the problems posed by a group of experts (teachers of mathematics and researchers) who received the same images as input. The study revolves around the question: "In what ways does cognitive variety depend on age or training in mathematically promising individuals?" We found that cognitive variety seems randomly distributed among the groups we tested, contradicting the intuitive idea that this is age (and training) related, except at the expert level. In addition, when talking about mathematical creativity, more sophisticated parameters, such as validity, complexity and topic variety, as well as the potential of respondents' products to break a well-internalized frame have to be taken into account. All those are to be balanced against the person's level of expertise in the specified domain.

Keywords Mathematical creativity · Problem posing · Cognitive flexibility
Cognitive variety · Expertise · Novice · Mathematically promising individuals

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4.1 Introduction

Traditionally, students' mathematical creativity has been studied based on quantitative evaluation of the following parameters: originality, fluency, and flexibility (Torrance 1974; Leikin 2009). In a problem posing context, these evaluations refer mostly to the number of posed problems that meet certain criteria (such as, for example, those described in Leikin et al. 2009). However, quantitative approach when studying mathematical creativity in a problem-posing context does not seem to be accurate enough when describing sub-components pertaining to creative students' behaviors (e.g. Kontorovich et al. 2011), and when "cognitive heterogeneity" (Abramovich 2003) is actually the norm; therefore, other parameters and/or frameworks for analyzing mathematical creativity might be needed.

In this paper, we study mathematical creativity through the lens of the subjects' cognitive flexibility, which is described by: cognitive variety, cognitive novelty, and changes in cognitive framing (e.g. Singer and Voica 2017).

We attempt to better identify and evaluate cognitive variety by trying to answer the following question: "In what ways does cognitive variety depend on age or training in mathematically promising individuals?" We tackled this by asking groups that differentiate through training and age to pose as many problems as possible, starting from a given configuration rich in mathematical properties. To gain better insights into the qualifiers best describing cognitive variety, we developed some tools for analysis, which focus mostly on qualitative aspects of the products built by the respondents.

4.2 Background

4.2.1 Problem Posing

The present study is focused on analyzing the cognitive behavior of students and experts in a problem posing context. We stand by the definition stating that problem posing is "the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems" (Stoyanova and Ellerton 1996, p. 518). As Silver et al. (1996) postulated, a problem-posing task requires the participants to formulate problem statements that are new to the poser, based on a given set of conditions.

Stoyanova and Ellerton (1996) classified problem-posing situations into three categories: free (when the participant poses a problem with no constraints on the content), structured (when the participant poses a problem starting from his/her own solution to a different problem), or semi-structured (when "an open situation is given and ask to explore the structure and to complete it by applying knowledge, skills, concepts and relationships from their previous mathematical experiences") (Stoyanova and Ellerton 1996, pp. 519–520).

4.2.2 Creativity and Problem Posing

Creativity had long been viewed as a domain-general phenomenon. However, recent evidence shows that creativity is rather domain specific (e.g. Baer 2010; Singer and Voica 2015). Particularly, mathematical creativity has specific manifestations, which take into account the deductive nature of the mathematical field (Piirto 1999), but also its inductive ways of solving problems. Moreover, creativity seems to be not only domain specific, but even task specific within content areas (Baer 2012).

Throughout the past two decades, mathematical creativity received much attention from researchers who focused on defining or characterizing it, as well as on establishing criteria for its assessment (see, for example, Sriraman 2004). Since it is so hard to precisely define what creativity is, as with general creativity, precise and generally accepted definition of mathematical creativity is very difficult to give (e.g. Leikin and Pitta-Pantazi 2013). However, while there are many competing definitions of the term mathematical creativity and, consequently, various frameworks for its quantification, literature makes a clear distinction between creativity of students who still learn mathematics in school, and creativity of professional mathematicians (e.g. Sriraman 2005).

In the present paper, we are interested in the way mathematical creativity manifests in school students. We will refer to this as “everyday innovation”, in the sense used by Kaufman and Beghetto (2009). This type of creativity can be assessed only in relation to the behaviors within a reference group, any definition of creativity being, in fact, relativistic (e.g. Liljedahl and Sriraman 2006).

As researchers consider that problem posing is one of the main tools for mathematical creativity development in all students (e.g. Sheffield 2009), we exposed students to problem-posing sessions, in order to identify existing or potentially-developing creative manifestations.

For our analysis, we use a framework based on cognitive flexibility. We have previously used this framework in a variety of situations (e.g. Pelczer et al. 2013; Singer and Voica 2015, 2017; Voica and Singer 2012, 2013), inspired by its use in studying organizational settings, where it is used to measure a person’s ability to adjust his or her working strategies in variable contexts (e.g. Krems 1995). When applying the aforementioned framework to a problem-posing situation, we can say a student displays cognitive flexibility when (s)he understands the context and the constraints of the task and uses them meaningfully. In other words, we observe cognitive flexibility when:

1. S(he) operates within an appropriate cognitive frame, which is flexible enough to change when solving problems or identifying/discovering/creating new ones (i.e. change in cognitive framing, or reframing).
2. S(he) generates new problem statements that are far from the given situation (i.e. display of cognitive novelty).
3. S(he) can come up with a wide range of problem statements, starting from the given input (i.e. display of cognitive variety).

The reason we chose this framework is that, in problem-posing contexts, a person faces the analysis of an ill-defined situation, with multiple possible solutions and personal interpretations, which leads to a type of situation similar to the way problems are addressed in organizations. Moreover, the analogy is functional in the context of unpredictable developments in contemporary information society, in which virtual interactions mediate (and often replace) direct contacts.

Our study compares elements related to mathematical creativity in relation to age and training differences. This issue is present in the literature. Some studies (e.g. Roskos-Ewoldsen et al. 2008) do not find statistical differences between younger and older participants, concerning originality. Other studies (e.g. Sak and Maker 2006) found that age is more related to originality, flexibility, and elaboration than to fluency in mathematics, but no correlation with students' grade was found.

4.2.3 Problem Validity in Problem Posing

There can be examples of new and surprising products, but which are not useful, either from a pragmatic view or an esthetic view. We would not include these into creative manifestations because, among the criteria of creativity listed in the more recent literature, there are some related to the social value of those products. These might refer to the impact at the level of a community (e.g. Gardner et al. 2001), or to their usefulness (e.g. Amabile 1996; Sternberg and Lubart 1996), or to adaptiveness with the task's constraints (Sternberg and Lubart 1996).

In the problem-posing process, new problems—sometimes unexpected problems if taking into account posers' experience—are generated. Previous research (e.g. Singer et al. 2015) highlights that there are some wordings generated by students, which look interesting and surprising, but which do not meet minimal conditions of validity. Therefore, some criteria are needed for accepting a posed problem as being valid. Abramovich and Cho (2015) used the criterion of didactical coherence in assessing preservice teachers' skills for developing curriculum materials in a technology-supported learning environment. We started the analysis of the students' posed problems by discussing their syntactic and semantic validity (Singer and Voica 2015; Voica and Singer 2013).

As referring to the syntax, *the coherence* of a mathematical problem deals with the rules and principles that govern the structure of a problem, more precisely:

- The following text components: givens, operations, constraints are present;
- The following text components: givens, operations, constraints are recognizable or identifiable;
- The givens are not redundant, or missing.

Concerning the semantics, *the consistency* of a mathematical problem refers to the existence of meaningful links among the elements of the problem, i.e.:

- The problem data are not contradictory;
- The following text components: givens, operations, constraints are correlated;
- The components of the problem statement satisfy a certain assumed mathematical model;
- The problem is solvable (Voica and Singer 2013).

A new criterion: "task consistency" was introduced in the analysis of the posed problems by Singer et al. (2017) for the case when problem-posing is conditioned in a structured or semi-structured way.

4.3 Context

4.3.1 Sample and Method

Students at pre-university level (Group 1), tertiary mathematics students (Group 2), and experts (Group 3) were exposed to a similar problem posing task, with some small variations, which will be further explained.

Group 1 consisted of students who participated in a summer camp, as winners of a national mathematics contest. The competition consisted of two rounds—a general round "for all" (approximately 25,000 students participating each year from grades 2 to 12), and a second very selective round, during which the winners were chosen. The students ranking on the top 0.5% from each level were invited to the aforementioned summer camp. Two weeks before the camp started, a call for posing problems was addressed to the future participants in the camp. The call was answered by 18 volunteers. To better understand students' proposals and cognitive mechanisms activated in solving the task, the respondents were interviewed the next day after their submission. During the interviews, the students were asked to explain their proposals and to come up with new problem statements.

Group 2 consisted of university-level students—prospective teachers of mathematics, enrolled in a course of Mathematical Education. They came in this study from two different departments of the Faculty of Mathematics while attending a course taught by the same instructor in two successive years. Their problem-posing task was formulated as a non-mandatory project, which could count for a part of their grade. The students worked individually for two weeks, and 20 of them provided the required materials.

Group 3 consisted of experienced high-school teachers involved in international mathematics competitions and researchers participating in a Mathematics Education conference. The participants agreed to take part in a problem-posing competition, with tasks of a similar type with the ones given to the previous two groups. Participants had 3 days to submit their problem statements. In total, 6 participants completed their task and posed problems. Given their advanced training, we consider members of this Group 3 as experts.

As it stems from the above paragraphs, our sample for the analysis consists of three groups of respondents, amounting to a total of 44 people.

4.3.2 The Task

All the three groups were exposed to a similar problem-posing task, based on similar images. The inputs for their tasks are presented in the following paragraphs.

(A) **For students at pre-university level (group 1)**

Martin bought two kinds of stone pieces to pave his yard. He began the work putting the pieces as indicated in the next image. Fortunately, Martin did not have to pay much attention to his work because any face of a piece he would use fits perfectly into the pavement.

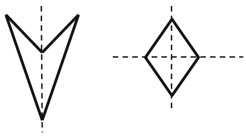
1. Observe the image and pose as many problems as you can on this pavement. For each of the posed problem, write the solution.

2. Make another drawing containing a pavement with two different kinds of pieces chosen by you. Write a list of similarities and differences you observe between the original pavement and your proposal.

(B) **For university students, and experts (groups 2 and 3)**

We consider the following geometrical context.

A



B

The tessellation in the image (A) is made using two types of congruent pieces: the first one is a dart, and the second one is a rhombus. These two pieces have symmetry axes, as one can see in image (B).
Please pose as many problems as you can, referring to this context. The problems can be from any mathematical domain and can have any level of difficulty.
For every posed problem, include also the answer and a proof, eventually a short one.

4.3.3 Some Comments on the Tasks

As one can easily see, there are slight differences in the wording of the task in the two cases. We opted for this variation for several reasons.

For Group 1, we placed the problem in an everyday-life context, to make it more attractive for students (some of them aged just 9 years old). There is no talk about symmetry (because some of the students in our sample did not know this concept), but we did say that the tiles of the pavement can be placed on any face and still match perfectly. For Groups 2 and 3, we used the mathematical name of the pieces and we referred to their symmetry properties.

On the other hand, in formulating tasks for Group 1, we preferred to remove a part of the pattern, which makes the statement more plausible, but also gives the students a reason for focusing on the possibility of completing the pavement. We also formulated an assignment to draw another pavement, just to make the Group 1's participants, who had less knowledge of geometry, to focus on the properties of the given figure and on the relationships that can be obtained. We considered that this would guide the students from our sample into analyzing the given configuration more carefully. When we did the data analysis, we equated the proposal of a new pavement with a new posed problem and we have thus included the pavements drawn by students in their lists of problems.

4.3.4 A Brief Analysis of the Task

The proposed task represents a semi-structured problem-posing situation (in the terminology used by Stoyanova and Ellerton 1996). We have chosen this task for two reasons.

On the one hand, the suggested image, rich in geometric and algebraic relationships, offers a high degree of freedom for formulating new problems. This open-ended frame leaves more room for personal initiatives, which may allow emphasizing cognitive variety.

On the other hand, we used a semi-structured problem-posing situation to get lists of problems which have some connections between them. This is useful in relation to group's behavior, precisely because we can find terms of comparison between the proposed problems within personal proposals and within groups' proposals as well.

For solving the task, participants were supposed to analyze the given image, identify the relationships that can be highlighted by the tessellation, and then formulate new problems. There are many relationships of algebraic and geometric nature between the elements of the given geometric model, which are suggested by the possibility of covering the plane with the two types of pieces. For example: analyzing the positioning of the five rhombuses in the middle, as well as the way the pieces match, we can determine the measures of the angles of these

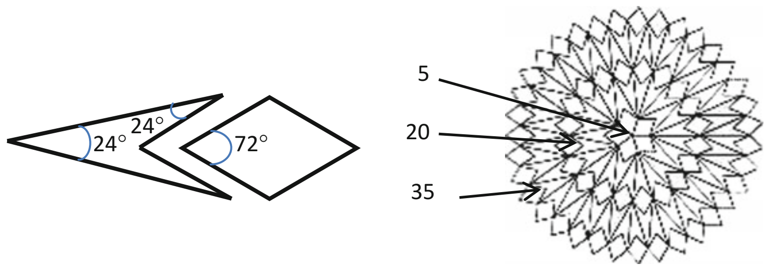


Fig. 4.1 Geometric and algebraic relations in the given pattern

quadrilaterals; analyzing the number of rhombuses successively added in the tessellation, and how the model develops, we can identify the increase in an arithmetic progression (see Fig. 4.1).

Some of the relationships suggested by the given figure are proven to be true, but this context rich in properties can also induce false perceptions. For example, observing the figure, we can say that some sides of the constituent pieces make a straight angle (which is true), or that the external vertices of the rhombi are on the same circle (which is false).

4.4 Tools for Data Analysis

To solve the task, the respondents must understand the various constraints of the given figure, build understanding around the relationships between these constraints, and formulate mathematical statements. We focused on understanding the cognitive mechanisms activated by this task, as follows:

- The link algebra-geometry. The given context is rich in information of geometric nature easily transferable in counting with possibilities of expansion in generalizations of algebraic nature. To what extent do participants detect these links and use them effectively?
- The relationship between the possibilities of expanding the model and the geometrical properties of the figure. The concentric development of the tessellation suggests that it may be continued following the same rule as for the first "circles". Does common perception act as a barrier on the explicit formulation of the likely continuation of the model also when trying to formulate a proof? This question is meaningful especially because we are dealing with both mathematically promising students of various ages, and experts.

To answer these questions, we developed several tools to help with the analysis of the participants' proposals. These tools refer to: the validity of the posed problems, the degree of complexity of the problems, and the topic variety of the lists of generated problem statements.

4.4.1 Problem Validity

We used the coherence and the consistency tools (as presented in Sect. 4.2.3) to analyze all the problems posed by the participants in the study.

We also used the criterion "task consistency", because it proved useful (when referring to posing problems based on a given image) by indicating the extent to which the proposer shows cognitive framing. In fact, understanding the frame (e.g. the fact that the given "mosaic" coverage supposes joining pieces without spaces, or that only pieces of two types are used, or that the pieces follow a pattern, etc.) is a necessary condition for a valid proposal.

We consider that a problem statement can be deemed consistent (relative to the given task) if: it capitalizes on consequences derived from the constraints imposed by the given tessellation, it highlights properties of pieces of the given tessellation, and it uses pieces of two types that admit symmetry axes in developing new tessellations. We consider that a problem statement is not task-consistent when it does not meet any of the above, even if the other validity conditions were met. For example, the following problem was proposed by one of the students in our sample:

> Let's consider the rhombus ABCD with m(A) = 60° and the diagonal AC of 12 cm. Calculate the perimeter of the rhombus.

The problem refers to one of the elements that form the tessellation (a rhombus), but the numerical data, more precisely: the measures of the angles of the rhombus do not take into account the constraints derived from the given figure (the rhombuses that form the tessellation actually have angles of 72° and 108°). We would expect that the proposer has developed a new tessellation based on this type of rhombus. If this would have been the case, the problem would have been task consistent. However, no reference was included towards a new pattern; therefore, it is rather a simple recourse to the use of the 60° angle, very frequent in canonical problem solving. Consequently, this problem was classified as being coherent and mathematically consistent, but not task consistent.

4.4.2 Problem Complexity

In the data analysis we did for this study, we associated to each problem the main concepts and techniques that were circulated by the wording and the solving of the problem statement posed by a participant. To simplify the discourse, we'll use the term *procept* meaning "a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either or both" (Tall 1993, p. 7).

We further introduce two parameters that are meant to describe the complexity of a proposal made by the participants in our study: *conceptual complexity*, which is related to the list of procepts associated with a posed problem, and *procedural complexity*, which is related to the essential steps needed to solve a posed problem. To characterize the problems based on these parameters, we endowed them with numerical measures.

To get a numerical measure of conceptual complexity, we roughly associated to every procept an integer between 1 and 4, taking into account the key-stage of education when that respective concept/procedure is studied in school: the later the concept is studied, the bigger the associated number. The association was made based on Romania's national curriculum. Generally, this curriculum has a progression of concepts and procedures relatively similar to the one at international level, which additionally allows international comparative studies and benchmarking (e.g. Martin et al. 2008; Singer 2007).

The conceptual complexity of a problem was expressed by the weighted arithmetical mean of the numerical values associated with the procepts used in that problem. In this way, to each list of posed problems a numeric sequence was associated, representing the conceptual complexity of the proposal made by a respondent.

On the other hand, we identified the procedural complexity of solving a problem by determining the number of essential steps needed to solve it. In this way, we positioned each posed problem on a scale from 1 to 5 (where 1 indicates a low procedural complexity and 5 a high procedural complexity). This second numerical sequence represents procedural complexity of the proposal made by a respondent.

4.4.3 Topic Variety

The present study is focused on cognitive variety in a sample of problem posers. To better capture this dimension, we have characterized each list of posed problems through *thematic variety* and *mathematical variety*. Thematic variety and mathematical variety are the two components of the topic variety of a posed list of problems.

Thematic variety refers to the number of distinct themes addressed in the set of problems; a topic might be a specific domain of mathematics (e.g. qualitative geometry, arithmetic, probability, etc.), or an extra-mathematical domain (e.g. rules of games, commercial offers, rules of coloring, etc.). Therefore, this parameter captures meta-mathematical aspects involved in the list of problems.

Mathematical variety refers to the number of distinct procepts circulated in the list of posed problems.

4.5 Data and Results

As mentioned above, we analyzed the data collected from the participants using the following parameters associated to each individual's list of posed problems: validity, complexity, and topic variety. In processing the data, we excluded only the problems critically invalid; we accepted mistakes or awkward wording if the mathematical meaning of the problem could be deduced. Moreover, we included in the analysis those problems that were task-inconsistent but were referring to recognizable elements of the initial geometric context, even if the relationships and constraints of the posed problem text did not take into account this context.

As a result, we had to completely remove the responses received from three of the respondents: we took this decision because the three (one from each group) only posed a single problem, which did not meet the minimal conditions of validity. For example, one of these respondents made a drawing of a quadrilateral, wrote several general mathematical relationships—such as the sum of the angles of a quadrilateral, but did not give an intelligible wording of the problem.

Consequently, the analysis we made is based on 406 proposals received from 41 respondents.

4.5.1 Group 1 Proposals—Brief Description

The 17 students participating in Group 1 generated 144 problems (average number of posed problems 8.5, median 7). The extremes: 2 posed problems (minimum), versus 25 posed problems (maximum). The next problem is found in most of the students' proposals:

> How many pieces are needed to complete the pavement?

To answer, some students carefully completed the drawing and numbered the pieces, so that they could be easily counted. (Two examples of this type are shown in Fig. 4.2.)

We have also encountered other approaches: Radu (5th grade) split the figure into congruent sectors, counted how many pieces a complete sector has, and then expanded this number to the incomplete sectors. Vlad (also 5th grade) identified various correspondences between the positions of darts (arranged with the vertices inward or outward) and the rhombuses of the median "circle" and external "circle". This reasoning helped him find the correct number of pieces.

We present below four cases from this sample, which are relevant for characterizing this group's cognitive behavior.

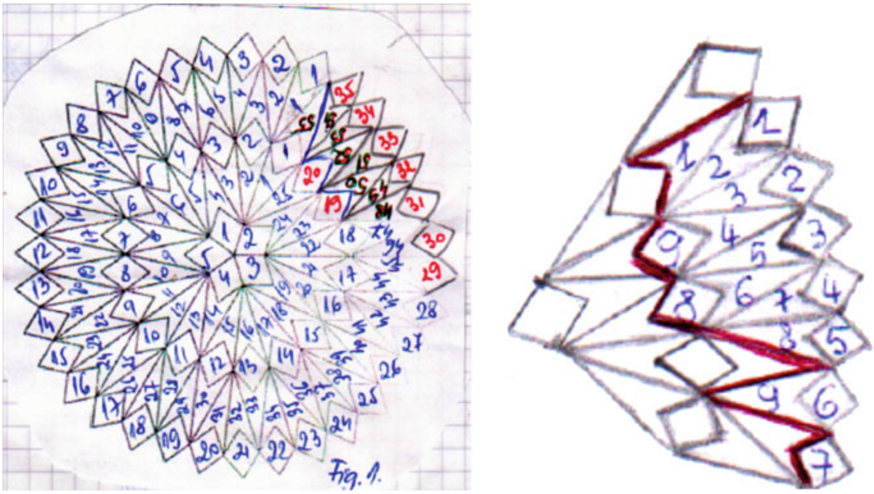


Fig. 4.2 The drawings made by two students to compute the number of missing pieces

4.5.1.1 The Case of Diana (4th Grade)

Diana proposed 25 problems. The first problems were very simple (such as: How many rhombuses are in the given picture?), but her next proposals extended in various mathematical domains and the formulations became more elaborate. Although a 4th grader, Diana displayed a certain mastery of problem-solving techniques. For example, to determine how many pieces are on the next “row”, after completing the figure, she notified one-to-one associations and geometric regularities, such as:

> From the given pavement, we understand that between each two rhombuses there are two darts, but once at two intervals there is just one dart. So, every three intervals there are 5 darts. (…) If we know this, it will be very easy to calculate (…)

Her problems cover a variety of topics. Judging by her statements, Diana seems to “take the problem to the street”—she imagines herself or characters from her close environment playing hopscotch, coloring, adding pieces, cutting pieces, buying new pieces, putting them in a variety of combinations, etc. All these show that Diana has concrete thinking, which she exploited in proposing a large gamut of problems.

4.5.1.2 The Case of Andrei (4th Grade)

Andrei posed a total of 9 problems. For 5 of his posed problems the solver needs to imagine and make different calculations about the circular “row” made of rhombuses.

As 4th grader, Andrei did not have the knowledge and tools to investigate the properties of the given geometric configuration. He filled the gap of lacking these tools through an arithmetic approach, in which recursion played an important role. Thus, he noted that the number of rhombuses in the figure, for every "row" is 5, 20, 35, and concluded that these numbers correspond to an increase by 15 (i.e. make an arithmetic progression). Starting from here, he could predict, by calculation, how many rhombuses appear on the "row" 100, or how many complete rows can be formed if he would use 2016 "arrow" pieces.

Andrei decomposed the given geometric configuration and transposed it into a union of sequences whose elements enter into different correspondences. For example, in solving one of the suggested problems, he stated that "the number of arrows in a row is formed by summing up the rows of rhombuses that are intertwined between" and expressed the assertion in the following way:

$$R1\ arrows = R1\ rombus + R2\ rombus\ (\mathrm{R} = \mathrm{row})$$

We notice that Andrei duplicated his arithmetic intuition with elaborate reasoning, quite complex for his age, when calculating the number of figures from various rows based on 1-to-1 correspondences.

4.5.1.3 The Case of Alexandru (6th Grade)

Alexandru posed a total of 7 problems. At first, Alexandru said that the given tessellation can be continued. He proved a geometric intuition—he noticed regularities in the original drawing that he emphasized graphically, by drawing rays highlighting possible extensions of the configuration (Fig. 4.3). On this basis, his next step was sketching (with dotted lines) a rhombus on an arc—as a representative of the rhombi "on the 4th zone". His geometric intuition led him to assimilate the distribution of rhombi in circular shapes and to identify new rhombuses formed by joining three pieces of the pavement (he marked a representative of this category of rhombi by hatching).

However, Alexandru's geometric intuition operated within certain limits. Thus, although the rays extending the drawing were generated as an intuition of invariance to rotation, he did not check a series of conjectures of geometrical nature that he utilized. For example, he wrongly assumed that the common vertices of the external rhombuses are situated on a circle.

Although strong, Alexandru did not actually use his geometric intuition in the problem statements formulation. When he posed problems based on the given configuration, Alexandru made a shift from geometry to algebra. None of his posed problems has a geometric nature—all of them are counting-type problems. For example, to find out how many rhombus-type pieces are in the 4th zone (representing the first level that is not shown in the figure), he identified the terms of an arithmetic progression and got the result based on the appropriate formula.

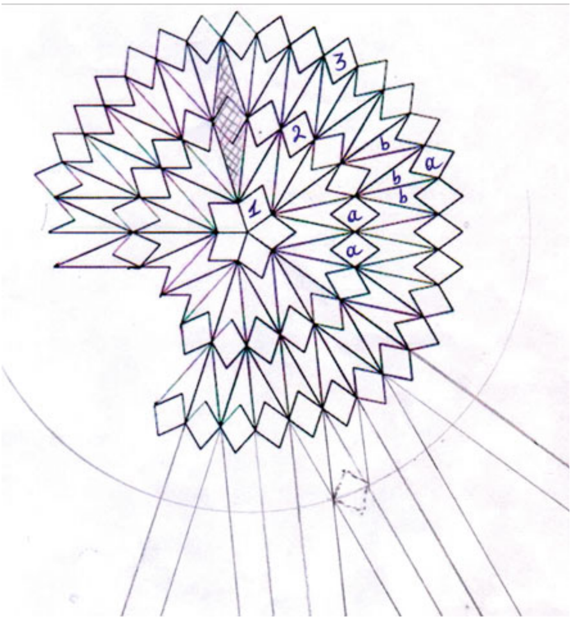


Fig. 4.3 The drawing made by Alexandru to highlight geometric regularities in the tessellation

4.5.1.4 The Case of Laura (9th Grade)

Laura posed eight problems: all of her problems are valid (mathematically coherent and consistent). In the first two problems, Laura addressed simple questions about the number of pieces of the pavement (such as: *How many tiles of each kind are missing? What is the total number of paving blocks of every kind?*). Later, she became interested in the geometric properties of the tessellation and formulated the following geometry problems:

> Determine, using the given drawing, the measures of the angles of a convex-quadrilateral-shape tile.
>
> Check if the angles of the concave quadrilateral (a.n. of the tessellation) are equal.
>
> What are the measures of the angles of the concave quadrilateral?

Laura carefully analyzed the relationships between the compound pieces, she identified details of the whole configuration (which she represented separately as in Fig. 4.4) and calculated such measures of angles.

Laura proved that she had a geometric intuition, backed by judgments she could do, including due to more advanced knowledge of geometry she possessed, compared to the younger students in her group.

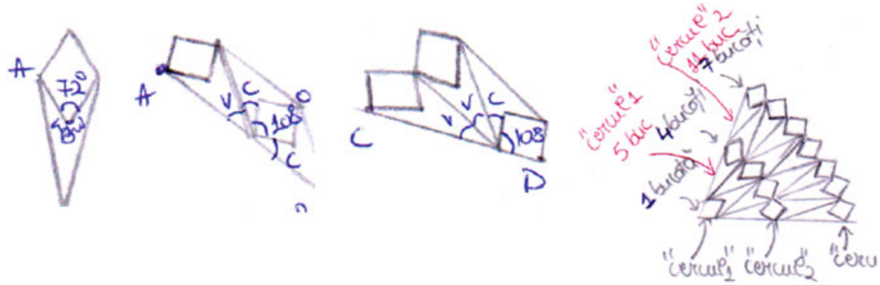


Fig. 4.4 The details identified by Laura to solve her posed problems

4.5.2 Group 2 Proposals—Brief Description

The 19 students from Group 2 generated a total of 245 problems (with an average of 12.9 posed problems, median 12); extremes: 4 posed problems (minimum) versus 25 posed problems (maximum).

Relatively many of the posed problems by students from Group 2 focused on the properties of the constituent pieces of the given tessellation and have only a weak connection with the task. For example, the next problem lies in several lists of proposals:

> Show that joining the midpoints of the sides of a rhombus, ones get a rectangle.

This problem is mathematically coherent and consistent. We considered it, however, task-inconsistent because the midpoints of the sides of the rhombus neither appear as such in the given configuration nor involve further developments in a new pattern; they are brought into discussion just because the students learned very well this theorem as part of their previous training.

We further include in our analysis three relevant cases from this group.

4.5.2.1 The Case of Simona (Undergraduate University Student)

Simona posed 23 problems. She processed the original figure in various ways, emphasizing various regularities: symmetry axes, properties of the polygons of the configuration, conditions that some points are con-circled, etc.

Simona tried an exhaustive treatment of the given configuration, structuring her posed problems in 5 content areas mentioned from the beginning: probability, geometry, algebra, mathematical analysis, and "others". Most of her problems show an explicit link with the given figure: for example, she required the calculation of the angles of the two types of tiles.

Simona transferred the initial configuration in other contexts and explored its properties in relevant ways to pose problems with meaning and substance. For example, she formulated the following problems framed in the domains probability, respectively algebra:

> Calculate the probability that, randomly selecting three external vertices of the figure, the obtained triangle is acute-angled.
>
> Describe the isometry group of the figure obtained by joining the vertices of the rhombi from the center.

Simona made also transfers within a domain of mathematics. For example, she noted that the numbers of rhombuses on each row form an arithmetic progression; hence, she formulated a problem with this topic, which she completed with the following problem:

> Prove that an infinite arithmetic progression of integers cannot have all the different terms prime numbers.

Given the correlation with the previous problem, we still accepted this problem as task consistent.

4.5.2.2 The Case of Cristina (Graduate Student in Mathematics)

Cristina posed a total of 13 problems. Initially, she identified various regularities of the figure (e.g. division into congruent sectors, regular pentagons, etc.) and marked these regularities on a drawing (Fig. 4.5). Even if the figures are not entirely correct (the points added by her not being concyclic in reality), her problem statements show that she has, at an intuitive level, an understanding of the properties of the given configuration.

The identified regularities allow Cristina to decompose the figure and, thus, to simplify the counting process. Those regularities were also used to make the transfer towards arithmetic problems because she could use geometric reasoning to prove that the numbers of rhombuses from the successive “rows” make an arithmetic progression.

Cristina posed elementary geometry problems based on the given figure (such as, for example: *How many axes of symmetry does the figure have?*), but also problems that require the calculation of measures (of angles or areas). Interestingly, she asked, (formulating the question as a posed problem) if there is another way of arranging the dart pieces between “rows” of rhombuses and she made a transfer from the given configuration to probability problems, interpreting the figure as a spinner. We noticed that Cristina consistently exploited the configuration—geometrically and algebraically. Starting from the identified properties, she succeeded to pose very different problems, which refer to counting, coloring, calculations of areas and measures of angles, or probabilities.

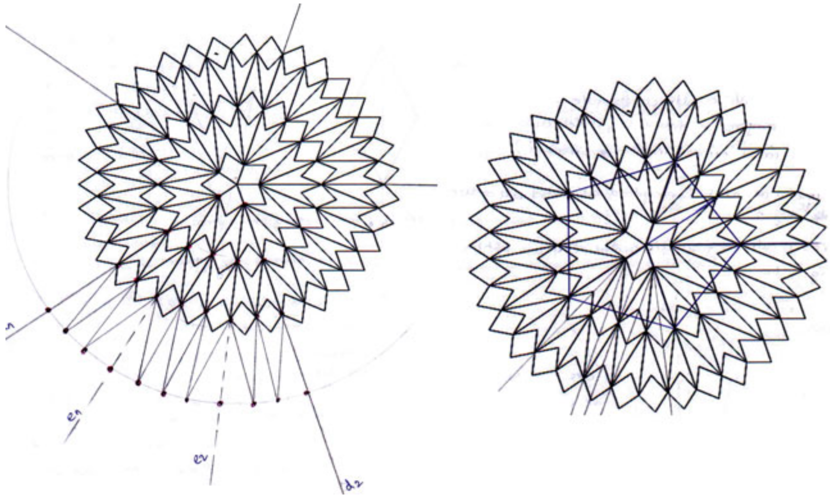


Fig. 4.5 The drawings made by Cristina to highlight geometric regularities of the tessellation (she printed the given configuration and completed it by hand)

4.5.2.3 The Case of Daniela (Undergraduate University Student in Mathematics)

Daniela posed several problems, each with several requirements. In total, there were 25 posed problems.

She used elementary configurations: in each of her posed problems a rhombus and a dart were used. Some of the Daniela's proposed problems are obvious; for example:

> In rhombus ABCD, find a symmetry-point of A about the diagonal BD.

In all her posed problems, Daniela only retains from the given configuration the shape of component pieces and their properties of symmetry; none of her posed problems refers to the geometrical configuration per se. This shows that Daniela has seen the details but not the big picture. This claim is supported by the fact that in some problems, Daniela indicated arbitrary measures of angles, with no connection to the given geometric context. Because of this, we classified all her posed problems as being task-inconsistent. However, if we look at those posed problems independently from the task, some of them are challenging (or at least less frequent among her colleagues' usual behaviors), such as for example the following:

> We make a cardboard dart ABCD and fold the figure along the line AC. Prove that after folding, the lines AC and BD remain perpendicular.

Her case is interesting because the large number of posed problems could, at first sight, indicate cognitive variety. It happened yet that her problems were not really consistent, and many of them were very similar. Consequently, the number of posed problems cannot offer alone a good indication of cognitive variety.

4.5.3 Group 3 Proposals—Brief Description

The 5 participants from Group 3 generated a total of 17 problems (average of 3.4). With one exception (a problem in which the wording is not too well defined—possibly because of its formulation in a language that is not native for the person), we classified all the posed problems as valid.

As a general feature, the problems posed by this group have a high degree of complexity and difficulty. Most problems require many essential steps to arrive to a solution, as well as strategic thinking and proof of conjectures. For comparison, in many of their posed problems, Group 2 formulated *explicitly* several sub-problems of the original problem, which decreased the level of complexity and difficulty, because through the requirements they made, they directed the path to a solution. This guidance led to a larger number of problems compared to the number of proposals made by Group 3.

We further present two relevant cases for this group.

4.5.3.1 The Case of Mihai (Ph.D. Mathematics Student)

Mihai proposed a total of 8 problems. He preferred to call the pieces of the tessellation "1-piece" and "2-piece", instead of "rhombus" and "dart". The problems posed by Mihai are from different areas of mathematics: geometry, algebra, calculus, number theory, and combinatorics. For his first problems, Mihai used geometric arguments to prove that the given model can be continued in the same way, and found recurrence relations between the terms of two sequences (representing the number of 1-pieces, denoted a_n, and 2-pieces, noted b_n, in layer n). His arguments refer to how the pieces are arranged in the given configuration. Once the general term of these sequences determined, Mihai can formulate new problems, such as, for example:

> We expand the configuration in the same way by adding successive layers of 1-pieces and 2-pieces. Prove that there is a layer of 1-piece with just 2015 pieces.
>
> Calculate the total number of pieces (denoted A_n), from which a pavement can be created with n layers of 1-pieces and $n-1$ layers of 2-pieces.
>
> Determine the solutions in whole numbers of the equation $A_n = k^2$.

To calculate the number A_n, Mihai explained the terms of the sequences $(a_n)_n$ and $(b_n)_n$ defined above. More precisely, Mihai used arguments of a geometric nature (in which the way of joining the pieces is actually used in the reasoning he made) to deduce the recurrence relationships:

$$a_{n+1} = b_n - a_n;\ b_{n+1} = 3b_n - 4a_n.$$

These relationships allowed him to explain the general terms of the sequences: $a_n = 5(3n - 2)$, $b_n = 5(6n - 1)$. Mihai used these algebraic expressions, as well as the equality

$$A_n = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n-1} b_i,$$

to deduce the equality:

$$A_n = \frac{5}{2}(3n - 1)(3n - 2).$$

Mihai utilized the geometrical configuration only once in the input stage, when information of geometrical/positional nature were used to argue on the possibility of extending the pattern and to determine recurrence relationships. After the numerical information was specified, the geometric/visual support was abandoned and the focus was in directions that are no more intuitive—like equations in the set of integers, or convergent series.

Mihai suggested problems with a high degree of difficulty and abstraction. He used different notations having in view emerging generalizations, which again indicates a capacity for high degree of abstraction. All these show that in solving this task, Mihai acted as an expert, trying to mathematically enrich the given context as much as possible. He was not interested in formulating elementary problems of a metric nature (such as, calculating measures of angles or areas), which would enable him to generate immediately many problems, but he kept a challenging degree of difficulty and complexity of the posed problems.

4.5.3.2 The Case of Adriana (Mathematics Teacher)

Adriana proposed a single problem, namely:

> To pave a dance floor, 50 tiles are used, arranged as shown above. The rhombus-shaped tiles are painted with white paint, while for the darts blue paint is used. We know that for 5 rhombuses, 20 g of paint are used; find the quantities of white paint and blue paint required.

Adriana proposed a contextualized problem: she imagined an everyday-life situation in which a geometric pattern of the given type could be used. Her problem statement starts from the assumption (which is not explicitly formulated, but can be accepted) that painting is uniformly distributed, the amount of paint (white or blue) being proportional with the painted surface.

To solve Adriana's problem, many steps are needed: determining the number of pieces of each kind; calculating measures of the angles of the two quadrilaterals; calculating the ratio between rhombus and dart areas (including, among others, finding tg12°); expressing proportions and finding unknown terms.

This shows that Adriana's problem is complex, and contains several sub-problems that a solver would need to formulate and then solve.

4.5.3.3 The Case of Ina (Mathematics Education Researcher)

Ina posed the next 4 problems:

> Prove that all three acute angles of the dart are equal.
>
> Is it possible to expand the picture in a similar way (by constructing additional arc of darts and rhombuses around the given flower)?
>
> Find the minimal number of colors used such that any two neighbor figures (with common edge) are in different colors.
>
> Find the longest way (and/or length of this way) to visit as many figures as possible – any figure no more than once.

The last problem was considered by Ina as being open-ended. This problem statement is not really well formulated—it does not specify what does "way" mean (is this about crossing the edges?) or what does it mean to "visit a figure". We considered that the wording ambiguities stem from the fact that the proposal has been made in English—which is not Ina's mother tongue. Perhaps the statement refers to the existence of an Euler path in the graph of the sides.

We acknowledged that the four problems are from different content areas (geometry, optimization, graph theory) and require different solving methods (calculate, prove or disprove, find a minimal number, and an open problem without known solution).

4.5.4 A Global Analysis on the Three Groups' Proposals

In this study we analyze the responses to a call for posing problems from three groups: students in pre-university education (Group 1), university students from the Faculty of Mathematics (Group 2) and experts (Group 3). The products realized by these groups have been organized in three clusters that are further discussed in view of the criteria presented in Sect. 4.4.

4.5.4.1 Validity

Table 4.1 shows synthetically the scores on the *validity* criterion, more specifically the number of problems posed by each group of the sample, and a report on the coherent, mathematically consistent and task-consistent problems, which is presented numerically and by percentage.

Table 4.1 reveals quite different cognitive behaviors in the three samples.

On average, the largest number of problems is recorded for the university students, and the lowest from experts. However, the higher number of posed problems by the university students is not aligned with a quality criterion—many problems in this group are not mathematically coherent or consistent.

Group 2 (university students) put forth a relatively small percentage of task-consistent problem statements. Some of the students in this group tend to retain only disparate elements of the given context, and many of the proposed problems (35%) have little connection to the initial context.

Unlike students in groups 1 and 2, the experts propose far fewer problems, but the results of their work are (with one exception) both coherent and consistent.

4.5.4.2 Complexity

As explained in Sect. 4.4.2, we associated to each list of posed problems two numerical sequences representing *conceptual complexity* and respectively *procedural complexity*. To have a global view on the complexity of the proposed lists of problems, we plotted the numerical values obtained for *conceptual complexity* and *procedural complexity* in an orthogonal system, for each group from our sample.

For the graphical representation, each number sequence was ordered increasingly, the data scaled against a reference of 25 problems—which represents the maximum number of proposed problems in our sample—and an exponential trend line was used to better describe the amplitude of problemistic spectrum of each list of problems. We used this type of representation because the exponential function highlights variation, and thus trends become more visible. In this way, for each group of the sample and for each of the utilized parameters: conceptual complexity and procedural complexity, we obtained a family of curves.

Table 4.1 The number of problems posed by the three groups, and statistical data on the validity criterion

Group	No. of participants	Total no. of posed problems	Mean of numbers of posed problems	# of coherent posed problems	# of consistent posed problems	# of task-consistent posed problems
1 (pre-university)	17	144	8.5	140 (97%)	139 (96%)	141 (98%)
2 (university)	19	245	12.9	210 (86%)	208 (85%)	159 (65%)
3 (expert)	5	17	3.4	16 (94%)	17 (100%)	17 (100%)

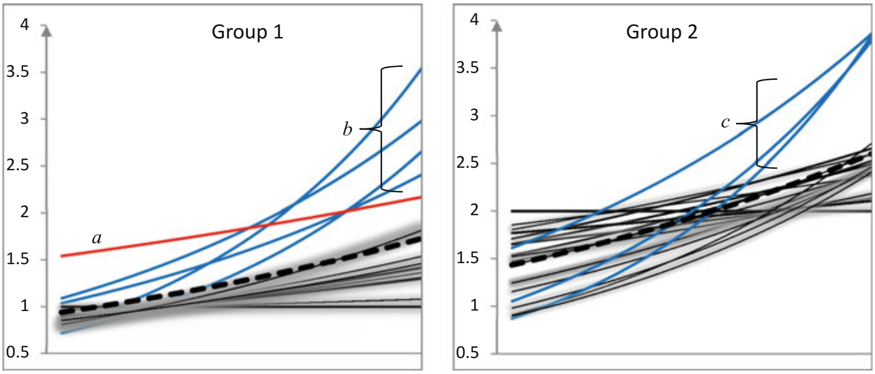


Fig. 4.6 Trends in the conceptual complexity for Cluster 1 and Cluster 2. The dotted curve represents the arithmetic mean of the data

In these representations, we were interested in the general trend of each sample, but also in the deviations. Each chart contains, as a dotted curve, the arithmetic mean of the data.

In Fig. 4.6 we graphed the conceptual complexity of the problem lists proposed by students of Group 1 and Group 2. (We have not included a similar representation for Group 3 because, being too small numerically, it is not significant in this type of analysis.)

In the next paragraphs, we discuss the interpretation of the data represented in Fig. 4.6, for the two clusters of task results (lists of posed problems) of the two groups. For Cluster 1, one can observe as a majority trend (marked on the chart with a shadow) the location of the posed problems in an area where the circulated mathematical content is at a basic level. According to the grading scale of procepts used in the study, we encounter in this area problems classified at a level of conceptual complexity between 1 and 2. This trend is natural, considering that most students in this group are at the primary-education level (10–12 years old).

In the graphical representation, we can see two types of deviations:

- The line graph denoted by *a*, corresponds to a list of problems that have constantly a degree of conceptual complexity higher than average.
- The line graphs denoted by *b*, indicate a large variation in the conceptual complexity of the posed problems. In these cases, the graph amplitude (visible in the variation on the vertical axis) is the highest as compared to the other curves that accumulate in the area of the general trend.

For Group 2—university students, the graph in Fig. 4.6 indicates a congestion (marked by glow) of the curves representing the conceptual complexity in the zone of values 1 and 2, thus clarifying a trend of Group 2 for posing problems that vehiculate procepts situated in the primary or intermediate educational stages (very rarely—high school), in which procepts practiced at the university level are usually avoided (with very few exceptions).

Deviations from the major trend—i.e. the line graphs denoted by *c* (and depicting Simona's, Stefania's, and Ioana's proposals) indicate high amplitude in conceptual complexity, covering the full scale of measurement.

We note that the dotted curves (representing the arithmetic mean of the data for each cluster) are almost "parallel" and very closely situated—that means that the two groups are behaving quite similar with respect to the conceptual complexity of their proposals. One would expect the "average" curve for Group 2 be positioned at higher values on the chart, given the higher level of education of these students, but as we can see, this did not happen.

In Fig. 4.7 we graphed the *procedural complexity* of the problem lists proposed by students in Group 1 and Group 2, following the same set of conventions: the general trend is marked with glow, and the dotted curves represent the arithmetic mean of the data.

In general, *procedural complexity* covers a range (variation between the minimum and maximum number of steps to solving) that is smaller for cluster 1 as compared to cluster 2. This is expected given the mathematical knowledge (correlated with age) of respondents from the two groups. However, the exceptions are significant.

In Cluster 1, the deviations from the general trend are of two types:

1. Situations where the posed problems are numerous and of relatively high-procedural complexity (marked by *d*)
2. Situations where most problems have procedural complexity below average (marked by *e*).

We found that deviations of type 1 have the amplitude much greater than the average in Cluster 1, but even more, the amplitude is surprisingly bigger compared to the variations in Cluster 2.

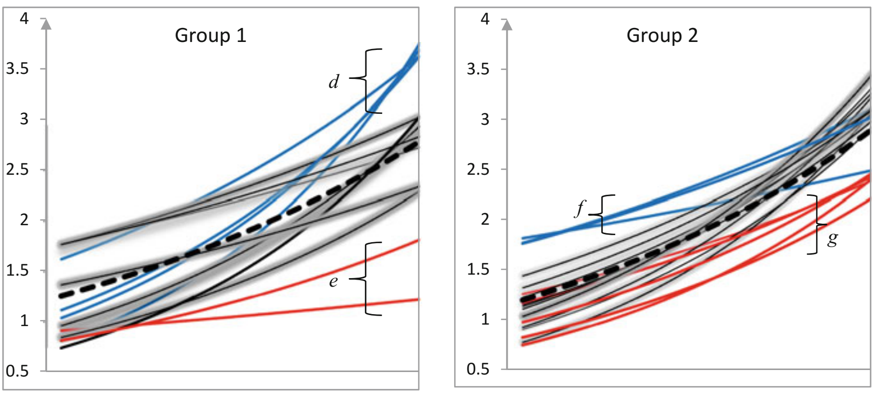


Fig. 4.7 Trends in the procedural complexity for Cluster 1 and Cluster 2. The dotted curve represents the arithmetic mean of the data

In Cluster 2, deviations from the general trend are also of two types, but different from those in Cluster 1:

1. Situations where procedural complexity is maintained at a relatively constant level and simple procedural problems are kept at a higher level than the average (marked by *f*)
2. Situations where most problems have procedural complexity below average (marked by *g*).

In Cluster 2, the family of curves illustrating the major trend has the amplitude higher than the curves representing deviations, which does not happen on Cluster 1.

In Group 3, all five participants posed problems procedurally complicated (i.e. needing at least 4 steps to solve), which seems to be a characteristic of experts.

Deviations from the general trend are more obvious in Group 1, the variety of topics being greater here. Group 2 acts somewhat uniform, making it difficult to identify obvious deviations from the major tendency behavior. Concluding, we found that the exceptions in Group 1 are farther from the general trend compared to the exceptions in Group 2. In addition, the curves indicating relative high-procedural complexity (marked by *d*, respectively *f*) have greater amplitude in Group 1, compared to those from Group 2. At the same time, however, the curves representing the average are virtually identical in the two groups.

4.5.4.3 Topic Variety

To obtain a global picture on topic variety of problems, we graphed the values obtained for each problem list through a point in the plane, in which the horizontal axis shows thematic variety, and the vertical axis—the mathematical variety of problem lists. We thus obtained the diagrams in Fig. 4.8, one for each group.

In these representations, a marker represents the characteristics of a list of problems for each cluster; we increased the marker size proportionally to the

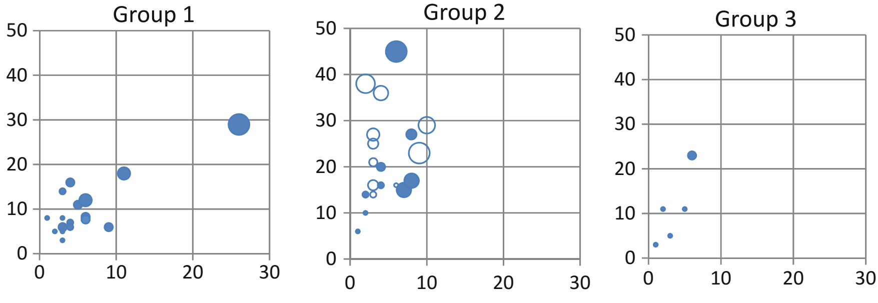


Fig. 4.8 Schematic representation of problem statements topic variety in the three clusters. Thematic variety is represented horizontally and mathematical variety is represented vertically

number of problems of each list. Those lists where a significant number of posed problems were not valid had been represented by using a no fill marker.

In these representations, we were interested in studying the accumulation-point areas per cluster, as well as occurring deviations (isolated points).

The graph for Cluster 1 shows a concentration of the markers in the 0–10 zone of thematic variety and in the 0–20 zone of conceptual variety. For Cluster 2, the diagram shows a greater conceptual scattering (crowding zone being 10–30), while the thematic variety is similar to that for Cluster 1. Submissions from students in Group 2, who posed relatively more problems that were not valid, display relatively low thematic variety, but a greater level of mathematical variety compared to those of Group 1.

4.6 Discussion

As we emphasized at the beginning, we analyze mathematical creativity of participants in our study based on a cognitive-flexibility framework, highlighting students' behaviors on three components: *cognitive variety*, *cognitive novelty*, and *changes in cognitive framing* (Singer and Voica 2015; Voica and Singer 2013).

There is a cultural component of creativity: creativity cannot manifest in the absence of an environment in which it can be nurtured and valued (Csikszentmihalyi 1996; Gardner 1993, 2006). Therefore, the elements based on which we try to determine the presence or absence of a creative behavior need to be applicable to the considered context. This aspect was included in the analysis made in the present study, by considering the validity of the students' posed problems. Psychologically, the validity of the posed problems is relevant for cognitive framing, which represents a person's ability to build an adequate representation of the situation expected to be transformed or improved. The construction of an adequate representation (which acts as a witness for cognitive framing) is a prerequisite for creative manifestations, at least in the studied context (i.e. when posing mathematical problems). In the present paper, we focused on specific behaviors that are markers of cognitive variety.

As previously defined in an organizational context, cognitive variety refers to the diversity of mental templates for problem solving that exist in an organization (Eisenhardt et al. 2010), or to the diversity of cognitive pathways or perspectives that can be mobilized in an organizational setting (Furr 2009). In a problem-posing context, an indicator of cognitive variety might be the number of distinct problem statements. We concluded that this indicator is not relevant enough, as the work of some of the participants in our study, who proposed more problems than their peers, did not satisfy the validity criterion, a criterion considered necessary to demonstrate cognitive framing. Therefore, to capture the finer aspects of cognitive variety, we defined other two possible indicators: the *complexity* of the posed problems and the *topic variety* of the lists of problems.

Briefly expressed, the complexity of a problem refers to the number of independent factors involved in the problem. The graphic representations of conceptual complexity, and procedural complexity respectively (Figs. 4.6 and 4.7) showed that there is a similar average behavior in the Groups 1 and 2. Using those graphical representations, we further analyze "positive" deviations from the average trend in the two groups: those deviations describe situations in which there are relatively many posed problems with large variation of *conceptual complexity* and relatively many problems with high *procedural complexity*.

For Cluster 1, the curves furthest from the average behavior, on both graphs, correspond to Andrei (4th grade), Alexandru (6th grade) and Laura (9th grade). If the deviation can be considered normal in Laura's case (given that, at the moment of the study, she was starting secondary school and, thus she had access to the knowledge of both primary and middle school), this deviation is exceptional for Andrei and Alexandru. The two students have come up with problem statements using a wide range of procepts—from some studied in elementary school to some usually studied in high school—and the number of steps needed to solve the problems that they proposed is higher than their group's average.

For Cluster 2, the curves furthest from the average behavior, on both graphs, correspond to Simona and Cristina. Between their proposals, there is a big difference in terms of the number of proposed problems (23 by Simona, 12 by Cristina) but, while all of Cristina's problems can be deemed valid, 4 of the problems posed by Simona do not meet this condition.

It is noteworthy here that, except Simona, the problems posed by students from the two groups who have proposed the most problems [Diana (4th grade)—25 problems, and Ana (5th grade)—15 problems in Group 1; Daniela—25 problems, and Madalina—19 problems in Group 2] have a relatively low complexity compared to the average.

We found that this trend is dominant in the samples: faced with a context of problem posing, atypical for what they are usually asked to do, many students in Groups 1 and 2 have relied on problems of low complexity, without relating the task to their current knowledge. This trend is prominent in Group 2 (university students) who frequently posed canonical problems that seem taken from a textbook for middle school. For comparison, the members of Group 3 proposed complex problems that cover a lot of procepts and require many steps in solving.

Concluding the discussion about the complexity of proposals, we found that:

1. Complexity can act as an inhibiting factor in problem posing, especially for high achieving respondents (such as those in group 3), who avoid too simple problems, thus reducing the total number of proposals;
2. The number of proposed problems seems to inversely correlate with their degree of complexity (without emphasizing a proportionality factor here).

The *topic variety* of the problems characterizes the variety of mathematical or meta-mathematical aspects targeted in a proposal. Comparing the graphic representations for the problems proposed by the three groups (Fig. 4.8), we found that, in all cases the congestion of markers occurs in the area of 0–10 for thematic variety, but there are differences regarding the accumulation zone relatively to the mathematical variety (0–20 for Groups 1 and 3 and 10–30 for Group 2). We further analyze isolated points of these graphs, considered as deviations from the major trend.

For Group 1, deviations correspond to Diana (4th grade), Andrei (4th grade), Sandu (4th grade) and Laura (9th grade). Diana, Andrei, and Sandu (4th grade) were shown thematic variety above the mean, while Diana and Laura proved a mathematical variety above the mean. These results are not surprising: in our sample, primary-school students tent to "populate" their problems with different characters and actions. By comparison, students from higher grades displayed a lower level of thematic variety, but some of them compensated that through mathematical variety.

In Group 2, there are deviations corresponding to lists of problems that do not qualify based on the validity criterion. In these cases, we found a wide mathematical variety: probably just the "freedom" to propose problems beyond the task constraints (freedom which is manifested most likely at an unconscious level) allowed these students to digress, posing long list of problems with a high variation in topic variety. We have seen that, in those cases, students do not demonstrate cognitive framing, which is a necessary condition for considering some creative manifestations. Therefore, we focus the discussion on those deviations describing lists of valid problems: they correspond to Simona and Cristina. In these cases, we found a lower thematic variety compared to deviations in Group 1, but a greater mathematical variety. Analyzing the lists of problems proposed by the two students, we noticed the existence of a true "program" for posing problems, whose design seems to have been made prior to the time of solving the task. Thus, Simona grouped her posed problems in "folders" related to different areas of mathematics; her strategy in generating problems seems to be "what probability (for example) problems could I propose?", then she moves to another mathematical domain asking the same question. Cristina exploited the given image quite systematically, and performed transfers between geometry and algebra in a natural way.

As referring to Group 3, the deviation concerning *topic variety* corresponds to Mihai, who has a training profile closer to Group 2, as Ph.D. student. The major tendency in Group 3 seems to be that of posing more targeted problems: the respondents' quality of teachers can be guessed in the background, the wordings being more elaborated, as if targeting a concrete/real solver.

From the analysis of the *topic variety* parameter, we can formulate the following conclusions:

1. A large number of posed problems have a low level of thematic or mathematical variety;
2. As advancing in school level, the thematic variety seems to diminish, but mathematical variety seems to widen.

Although the focus of this paper is on cognitive variety, we briefly take into account below the other factors of cognitive flexibility: cognitive novelty and changes in cognitive framing for our groups of respondents.

To think "out of the box", it is necessary for a student to understand the essential invariants of the given configuration, which allow a surprising problem. We further compare the following two problems generated by two students from our sample:

> Problem 1 (Posed by Madalina - university student): A tourist visiting Egypt during the holiday went to the pyramids. At first pyramid, the guide gave information about the pyramid shape and dimensions: the base is a rhombus with diagonals of 250 m and 200 m, and the length of lateral edges is 300 m. How tall is this pyramid?

> Problem 2 (Posed by Mihai, Ph.D. student): We denote by A_n the total number of pieces that can create a pavement with n layers of 1-pieces and n-1 layers of 2-pieces. Show that the series $\sum_{n=1}^{\infty} \frac{1}{A_n}$ is convergent and find the sum of the series.

Problem 1 appears to be generated by interpreting, in another mental registry a detail of the tessellation. Madalina drew the figure formed by joining a dart and a rhombus, interpreted this figure as the plane drawing of a geometric solid and posed a problem about a quadrilateral pyramid (see Fig. 4.9).

In Problem 2, Mihai defines the terms of a series, and makes a problem about its convergence.

Viewed from outside, both problems are far from the given input (the initial configuration). However, there is a significant difference between them: while problem 2 relates significantly to the essential elements of the given context, problem 1 makes a leap in which the invariants (filling with no gaps, matching pieces of two kinds, etc.) are ignored. Therefore, in the context studied in this paper, although problem 1 seems far from the given situation, and therefore a creative product, we rather consider it as a misinterpretation of the requirements, witnessing lack of cognitive framing. The same happened with other problems posed by Group 2, which have been presented to show the topic variety of proposals, but have to be excluded when discussing cognitive flexibility.

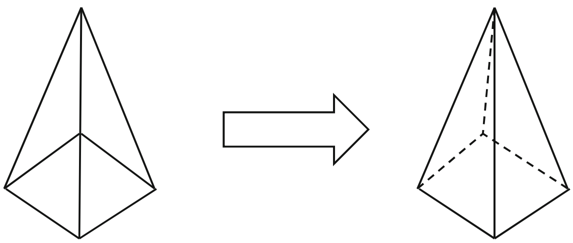


Fig. 4.9 Interpretation of a detail of tessellation as the plane representation of a 3D object

4.7 Conclusions

This study is focused on the cognitive variety of the responses given by different groups to a call for problems. The sample consisted of three distinct groups whose responses have been analyzed as three distinct clusters. Subsequently, specific cases have been discussed in each group and comparisons have been made between the clusters.

Common to all these groups is the members' selection based on their focus on mathematical problem solving and problem posing. Group 1 consists of winners of a two-round national contest in mathematics and, taking into account the selection process they followed, they can be considered mathematically promising students. The participants in Group 2, consisting of university mathematics students, have been selected by their option for (future) professional careers in mathematics and/or mathematics education. Group 3 represents experts in mathematics. Therefore, all three groups have a major interest in mathematics learning or mathematics practice.

In these circumstances, we looked into how cognitive variety (seen as a component of creativity) manifests in these groups.

In the present study, we discussed creativity in relation to the norm group. This option is natural because it is not about absolute creativity of geniuses who revolutionize a field, but about "small c" creativity (e.g. Bateson 1999; Kaufman and Beghetto 2009). Therefore, all comments we made are in relation to the normative behavior of the groups' members of our sample.

To better capture aspects related to cognitive variety, we have developed some tools used to quantitatively model qualitative aspects of the respondents' proposals. We thus introduced two parameters that are meant to describe the *complexity of a proposal* made by the participants: *conceptual complexity*, which is related to the list of procepts associated to a posed problem, and *procedural complexity*, which is related to the essential steps needed to solve a posed problem. Other two parameters were used to characterize the problems topic variety: *thematic variety* and *mathematical variety* of the lists of posed problems.

To study cognitive variety, we plotted the measures for the *complexity* and, respectively *topic variety of a proposal* in coordinate systems and we identified on these graphs the dominant behavior of the sample (identified by accumulation of curves or points) and isolated cases/deviations from that behavior.

The following conclusions emerge from the study:

- The existence of a cognitive frame generated by the given input for problem posing is a necessary condition to discuss an individual's mathematical creativity, and for the study of cognitive variety.
- The number of posed problems is not a sufficient indicator of cognitive variety for some reasons. For example, the concern for proposing only problems with a high degree of complexity acts as an inhibiting factor for the number of posed problems. Conversely, in many cases, students who pose a large number of problems either do not show cognitive framing, or pose problems of low level of complexity. In determining cognitive variety, it is necessary to consider several

parameters; besides the number of posed problems, we need to pay attention to the validity of the posed problems, to their complexity, and to their breadth of topics.

- The data analyzed in this paper show that, in terms of cognitive variety, there are no significant differences between Group 1 (consisting of students aged 9–16) and Group 2 (consisting of university students at the Faculty of Mathematics), but there are significant differences between these groups and Group 3 (experts). It seems that, beyond the knowledge increase and age-related cognitive maturation, cognitive variety is not age or training related up to the expert level. Expertise also may happen that it is not age-related to a certain extent (but it is very much training-related).

On the other hand, the present study confirms findings from other studies (e.g. Singer and Voica 2015) according to which, in a semi-structured problem-posing context, cognitive novelty is limited: students who are apparently more creative do not have a built-in cognitive frame and the problems they generate, although seem to be far from the given context—therefore more creative, actually are not task-consistent; in these cases, proposers did not fulfill a basic condition to be considered mathematically creative. Consequently, when dealing with concepts situated at the interaction between human knowledge and human psychology, many precautions and careful analysis are needed to formulate generalizable conclusions.

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Chapter 5
Mathematical Creativity and Its Subdomain-Specificity. Investigating the Appropriateness of Solutions in Multiple Solution Tasks

Maike Schindler, Julia Joklitschke and Benjamin Rott

Abstract In mathematics education, researchers often talk about mathematical creativity. However, we see a lack of research on the question of whether such an ability exists for mathematics in general; or whether mathematical creativity should rather be viewed subdomain-specifically; for instance, in the contexts of geometry, algebra, or arithmetic separately. In this paper, we present results of an empirical study investigating upper secondary school students' performances in Multiple Solution Tasks (MSTs). First, we elaborate on the notion of appropriateness and its influence on the investigation of creativity; and illustrate implications based on the given data. Second, we give an insight into students' performances along three different MSTs from different mathematical domains and point out correlations between students' performances in two domains: geometry and algebra. Our results do not support the construct of domain-specific or subdomain-specific creativity, but indicate that mathematical creativity should be considered task-specifically.

Keywords Mathematical creativity · Multiple Solution Tasks (MSTs)
Appropriateness · Domain-specificity

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5.1 Introduction

Creativity is an ability that is mostly associated with arts or music. Famous artists such as Mozart or van Gogh composed or painted numerous and outstanding pieces of art and are therefore regarded as creative people. Moreover, creativity can be seen as an important aspect in problem solving: Activities such as generating new solutions or elaborating on extraordinary and rare ideas may involve creativity to some extent. With the growing demand of our society for innovation and creative solutions to complex problems in domains such as technology, engineering, or natural sciences—domains in which mathematics plays a crucial role—mathematical creativity is gaining increasing significance. It is important to foster students' creativity (Sheffield 2009, 2013; Silver 1997; Hershkovitz et al. 2009) to both prepare students for their current and future lives in modern societies and to face the needs that our society encounters now and in the future. Whereas research has focused on creativity of outstanding mathematicians and exceptional people for a long time (Hadamard 1954), recent research has increasingly addressed creativity of everyone, especially students' creativity (e.g., Mann 2005).

In mathematics education, research aiming to understand and grasp students' mathematical creativity is so far quite rare (Leikin and Pitta-Pantazi 2013). Approaches to investigate mathematical creativity include problem solving as well as problem posing (Silver 1997; Leikin 2009). Many of the existing studies draw on students' products (e.g., written solutions or drawings) to assess mathematical creativity (Leikin and Lev 2013; Kattou et al. 2015). Performances are quantified along the dimensions fluency, flexibility, and originality, which originally arose from research on intelligence (Guilford 1967). Moreover, Kattou et al. (2015) found that creativity needs to be regarded domain-specific (within mathematics) and not domain-general. However, previous research has not yet sufficiently clarified whether students' performances in different creativity problems correlate and, thus, can be attributed to a single construct called students' mathematical creativity. The question of whether creativity is to be regarded a domain-specific or *sub*domain-specific ability is not yet clarified.

The purpose of this study is to investigate students' creative performances along different fields of mathematics—namely geometry and algebra. For this purpose, we conducted an empirical qualitative study with 21 upper secondary school students. For conducting a thorough data analysis, we first perused the question of which student solutions are to be taken into consideration when investigating creativity: Following Leikin (2013; Levav-Waynberg and Leikin 2012), we discuss the term *appropriateness* of solutions and investigate whether and to what extent solutions that are not correct, have flaws, or lead to a wrong solution may be regarded appropriate nevertheless. Based on these findings, we focus on students' performances along three different MSTs. In this first approximation to investigate the specificity of mathematical creativity, we found that the scores for mathematical creativity seem not to be consistent along different MSTs.

5.2 Theoretical Background

The growing demand in our society and economy for creativity is undeniable. People need skills to solve complex non-routine problems in extraordinary ways. Extraordinary and original ideas are not only important for managers and employees, but also for students in schools and high schools. Especially the fast development of science and technology requires more and more experts who are able to cope with these challenges. Special skills to solve problems creatively become more important for the coming generation, as Kattou et al. (2013) point out: “Given that students, as future citizens, will face problems that are unknown at present, it is especially crucial for them to be creative in order to efficiently tackle the challenges they will meet” (p. 180). Assessing and fostering mathematical creativity has accordingly become an important research field (e.g., Haylock 1987; Silver 1997). The recent development shows that this field is growing; in 2017 at ICME13, there was a distinct topic study group on “mathematics and creativity”. The aim to learn more about creativity and the question of how to measure and foster creativity have become important for research in mathematics education. In this line of thought, Leikin (2009) points out: “I consider developing mathematical creativity in school students to be one of the important objectives of school mathematics education. This implies that tools for the evaluation of students’ mathematical creativity are needed to realize the students’ creative potential and to assess the effectiveness of various mathematical curricula” (p. 129).

In the following, we will outline theoretical aspects regarding mathematical creativity. First, we give an insight into concepts on creativity in general. Second, we focus on creativity in the field of mathematics. Third, we give an overview on methods to quantify products in order to assess mathematical creativity.

5.2.1 What Is Creativity?

So far, researchers are discordant about a coherent definition of creativity (Sriraman 2009). There are various concepts, which cannot all be addressed in this article. However, we give an insight into the beginnings of research on creativity and introduce some prominent concepts.

The current tradition of research on creativity started in the 1940s and 50s when Guildford (1950) conceptualized creativity as one component of intelligence (Guilford 1950). Guilford’s psychological model of intelligence was the first one that comprised different forms of creativity. He differentiated, among others, between *convergent production* and *divergent production*: “Convergent production is in the area of logical deductions or at least the area of compelling inferences. Convergent production rather than divergent production is the prevailing function when the input information is sufficient to determine a unique answer. […] For example, if we ask, ‘What is *the* opposite of HARD?’” (Guilford 1967, p. 171). In comparison to convergent production, he describes divergent production as "a concept defined in accordance with a set of factors of intellectual ability that pertain primarily to information retrieval and with their tests, which call for a number of varied responses to each test item. […] [These] tests require examinees to produce their own answers, not to choose them from alternatives given to them" (Guilford 1967, p. 138). Divergent abilities are "most relevant to creative performance. [For these abilities a] […] factor of *fluency* […], a factor of *flexibility* […] and [a] […] factor of *originality* materialized. Later, in a study of planning abilities, a factor of *elaboration* was expected and was demonstrated" (Guilford 1967, p. 169; emphasis by M.S./J.J./B.R.). The dimensions fluency, flexibility, originality, and—partly—elaboration are nowadays used in many creativity tests. In this context, *fluency* stands for the ability to come up with a multitude of produced answers. *Flexibility* is to be understood as the capability to generate answers in various ways. *Originality* means the uniqueness of answers, and *elaboration* the level of details of the solutions.

In addition to the question of *what creativity is*, the question of *who or what can be creative* is discussed (Leuders 2010; Liljedahl 2013; Rhodes 1961). Rhodes (1961) describes different strands of research on creativity as *four Ps of creativity* which are *product*, *process*, *person*, and *press*. Concerning *products*, Bailin (1988) states that creativity is reflected in certain achievements or rather products. Liljedahl (2013, p. 255), for instance, focuses on *processes* arguing "that such a use of assessment of end product pays very little attention to the actual process that brings this product forth". The focus on processes goes along with problem solving processes, mostly along the lines of processes as described by Wallas (2014) in his seminal work *art of thought*. Wallas first published his book in 1926 drawing on ideas by the French mathematician Henry Poincaré, describing stages of conscious and unconscious cognitive processes with a moment of illumination. According to Rhodes (1961, p. 308) "the term process applies to motivation, creativity to *persons* —mostly to persons who were considered to be a genius". For example, Kneller (1965) or Ghiselin (1985) investigated various geniuses. Rhodes completes his remarks with the influence of the *press*: "The term *press refers to the relationship between human beings and their environment*" (Rhodes 1961, p. 308). He concludes that these four aspects are strongly interwoven.

In our investigation, we focus on Guilford's components *fluency*, *flexibility*, and *originality* by analyzing students' *products*. We do not focus on the component *elaboration* as this aspect has rarely been considered in mathematics education research as we will point out below.

5.2.2 *What Is Mathematical Creativity?*

Referring to a product-based conception of creativity (see below) and using a confirmatory factor analysis, Kattou et al. (2015) showed that creativity is not domain-general but domain-specific: "Therefore, psychologists and educators should no longer characterize individuals as creative, but instead, as creative in specific domains" (Kattou et al. 2015, p. 1022). In this respect, Kattou et al. focus on students' products as well as on creativity as a personal trait. Whether one speaks of "students' creativity" depends on the underlying conception and definition of creativity. In mathematics, the term creativity is often considered along the lines of Poincaré's (1948) results regarding the Fuchsian functions. The four-step process described by Poincaré and later expatiated by Hadamard (1954) has become synonymous to mathematical creativity for some researchers (e.g., Liljedahl 2013). Accordingly, Sriraman (2009, p. 15) defines mathematical creativity "as the publishing of original results in prominent mathematics research journals." However, researchers such as Mann (2005) state that not only famous mathematicians can be creative but also everyone else, especially students (see also Sheffield 2009; Hershkovitz et al. 2009). This apparent contradiction has been addressed in the discussion labeled "big C", referring to extraordinary creativity, and "little c", referring to everyday creativity (cf. Sriraman et al. 2014). Our understanding of creativity is not limited to big C or the so-called genius approach on creativity (Hadamard 1954) but includes subjectively new results by students (e.g., Leikin 2009).

5.2.3 Methods to Evaluate Creativity

Guilford proposed ideas to measure components of creativity, for instance, in the form of his well-known Alternative Uses Test (1967). This test is closely related to the Brick Uses Test: Here, the participants are asked to name as many uses for a brick (or another common object) as they can think of in a certain amount of time. With a view to all of the four dimensions fluency, flexibility, originality, and elaboration, the creativity score gets higher, the higher each component is rated. The fluency score depends on the number of solutions. To rate flexibility, the number of different categories of uses is taken into account. If a participant names "building a house", "building a wall", and "building a floor" the flexibility is low, whereas "throw at a cat", "make bookends", and "make a filter" show a high level of flexibility (Guilford 1967, p. 143). The dimension originality is rated based on the relative frequency of the given answer in the focused group. Elaboration refers to the level of detail: Answers such as "use the brick to filtrate tainted water" are more elaborative than "make a filter" and would be rated with a higher score. This task can be managed without a specific content knowledge due to the ordinariness of a brick.

Based on Guilford's theory, Torrance (1974) developed the Torrance Test of Creative Thinking (TTCT). This test contains slightly altered versions of Guilford's test, called "Unusual Uses Activities", as well as additional subtests. Examples of these subtests include verbal items such as the "Ask and Guess"-test where the participants are supposed to ask questions to given drawings. Other subtests are constructed to be non-verbal such as the "Picture Completion"-test, which consists of incomplete figures that have to be completed. This test has earned a widespread acceptance for the analysis of creativity in different components.

5.2.4 Methods to Evaluate Mathematical Creativity

A domain-specific adaptation of Guilford's conception of creativity is suitable to get a more detailed insight into mathematical creativity, especially with a focus on problem solving (Leikin and Lev 2013). Therefore, domain-specific research in mathematics education has focused on these seminal ideas; and has adapted them to mathematics education (see Leikin and Pitta-Pantazi 2013). Leikin (2009; Levav-Waynberg and Leikin 2012) has introduced the concept of Multiple Solution Tasks (MSTs) within the domain of mathematics education: Mathematical tasks that are supposed to be solved in different ways.

Furthermore, Guilford's and Torrance's ideas on measuring creativity have been used: Based on this, researchers such as Leikin and Lev (2013) as well as Kattou et al. (2013) developed tests, which draw on *Multiple Solutions Tasks* (*MSTs*), where the students' products' fluency, flexibility and originality are evaluated. *Fluency* is scored by the number of given answers. For the *flexibility,* the students' solutions are classified depending on their diversity. *Originality* addresses the relative frequency of a given solution in comparison to the reference group of participants. The component *elaboration* is mostly not evaluated "due to the difficulty of determining levels of elaboration in mathematical tasks" (Kattou et al. 2013, p. 174). This approach is linked to the "little c" concept of mathematical creativity: solutions are analyzed with respect to the reference group with comparable prior experience.

In contrast to the brick task, some of the MSTs deal with more or less complex problem solving tasks from different fields of mathematics. As such, they may require a certain mathematical background, which—as pointed out by Leikin and Sriraman (2017)—may lead to a correlation of mathematical creativity and mathematical ability. Especially for evaluating mathematical creativity in cases of complex problem-solving tasks, it appears reasonable to not only take into account solutions that are entirely correct but also those that are *appropriate* (Leikin 2013). Levav-Waynberg and Leikin (2012) point out that also imprecise solutions are scored when assessing creativity; Leikin (2013) explains that the term appropriateness is used instead of the term correctness "to allow evaluation of reasonable ways of solving a problem that potentially lead to a correct solution outcome regardless of the minor mistakes made by the solver" (p. 391). The scores for each appropriate solution (fluency, flexibility, originality) result in a score for mathematical creativity across a set of MSTs. In this chapter, we want to connect to this idea and focus on the question of what makes appropriate solutions appropriate. Following Leikin's work, we elaborate on what kinds of flaws solutions may entail to still being regarded appropriate. Accordingly, our first research question is as follows:

> What kinds of flaws may appropriate solutions entail to still be regarded appropriate? What does appropriateness mean in terms of mathematical MSTs?

Furthermore, there are other approaches to assess mathematical creativity. According to Silver (1997) and Bruder (2001), creativity is not only assessable through problem solving but also through problem posing. The capability to generate subsequent questions to a mathematical phenomenon shows a high level of creativity (Silver 1997). Singer et al. (2017) address problem posing as well: Here, the concept of cognitive flexibility is crucial, which comprises the dimensions cognitive variety, cognitive novelty, and changes in cognitive framing. Singer et al.'s results indicate that a student's cognitive style might be a predictor for their mathematical creativity. In both cases (focusing on problem solving or posing), mathematical creativity is seen as a relative construct because researchers focus on students' performances.

Another—non psychometric—approach is, for instance, the psychodynamic approach (Sternberg 1999), which focuses on the shifts between conscious and unconscious mind. Liljedahl (2013) uses this approach when analyzing students' essays; and draws on a social-personality approach (Sternberg 1999), since he focuses on affective components. Here, mathematical creativity is also seen as a relative construct because students' reports are in the focus and students reflect on their creative moments.

Along these concepts of assessing mathematical creativity, we see that students' performances in certain tasks are taken into regard in order to assess students' mathematical creativity. However, it is not clear whether students' performances along different tasks are consistent. In this chapter, we therefore ask the question of whether one can speak of "students' mathematical creativity" in general—or whether mathematical creativity is rather to be regarded subdomain-specifically. This leads to our second research question:

> To what extent do students' performances vary across different MSTs? Does students' mathematical creativity differ between the sub-domains of geometry and algebra?

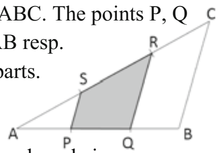
5.3 Methods Part I: Investigating Appropriateness

The investigation took place in a project at the University of Duisburg-Essen in Germany called MBF_2, which is a German acronym for *Focusing on mathematical giftedness—for upper secondary school level* (in German: Mathematische Begabung im Fokus—in der Sekundarstufe II). About 21 students (with slightly varying number from lesson to lesson) from different local schools participated in this project. They met every second week for a total of ten times. The students worked on challenging mathematical problems from different mathematical fields such as graph theory, cryptography, and spherical geometry. Furthermore, they worked on special problems that allowed us to investigate mathematically gifted behavior in upper secondary school children (Joklitschke et al. 2017). We focused on mathematical giftedness from different points of view: students' problem solving processes, student's products, and students' behavior. As some researchers see creativity as one component of gifted behavior (Renzulli 2002) we focused on mathematical creativity as well (see, e.g., Rott and Schindler 2017). There were no criteria for a previous selection of the students such as intelligence tests or certain grades in school. All participating students were highly interested in mathematics and attended this extracurricular course in their free time—independently from their regular schooling. In order to estimate students' cognitive abilities, we conducted the Culture Fair Test (CFT20-R) during the project span, which resulted in a measured IQ of 121 on average. Thus, the participating students may be considered above-average intelligent.

To assess mathematical creativity, we used MSTs and evaluated students' products. We used two geometrical problems (Figs. 5.1 and 5.2) and one algebraic problem (Fig. 5.3). For all problems, the wording of the general instructions was the same: "Solve the following problem. Can you find different ways to solve the problem? Find as many ways as possible." The three problems were given in different sessions and the students had a processing time of 30 min for each problem.

Solve the following problem. Can you find different ways to solve the problem? Find as many ways as possible.

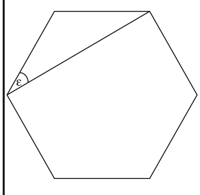
This figure is a triangle ABC. The points P, Q resp. R, S divide sides AB resp. AC each in three equal parts.



What is the area of the quadrangle in comparison to the area of the triangle?

Fig. 5.1 MST "triangle" (see Novotná 2017)

Solve the following problem. Can you find different ways to solve the problem? Find as many ways as possible.



This figure is an equilateral hexagon. How big is the angle ε? Remember, in an equilateral hexagon, all sides have the same length and all angles have the same size, which is 120°.

Fig. 5.2 MST "hexagon" (see, e.g., Schindler et al. 2016)

Fig. 5.3 MST “punch” (see Leikin and Lev 2013)

Lena produces 80 liters of punch for the Christmas market. She fills the punch in equal shares into barrels. When putting the barrels into the car, she notices that four barrels do not fit into the car and have to be left at home. Nevertheless, she wants to take all of the punch to the Christmas market. Therefore, she distributes the content of the four not-fitting barrels in equal shares to the other barrels. After this, she realizes that she had added exactly ¼ of the previous amount to each of the barrels. How many barrels did she have in the beginning?

When evaluating students’ products, we used different schemes: First, we only took into consideration *correct* solutions. In a second analysis, we took into account solutions that we considered *appropriate*. Through this approach, we analyzed differences between appropriate and correct solutions in order to investigate appropriateness of solutions thoroughly. Our analyses were based on the rating scheme as used by the research group around Leikin (e.g., Leikin and Lev 2013); once based on correct solutions, once based on appropriate ones (similar to Leikin and Lev 2013). In the first version, which we call “strict” in the following, wrong answers were sorted out and not used in the scoring. In the *second analysis*, we lowered this level to evaluate the products even if they entailed certain flaws, mistakes, or imprecisions.

For considering whether the approaches were considered appropriate, we consensually validated whether the reasoning was sufficient. Based on this, we evaluated whether the approach would be included in the first way of analysis; or only in the second one. In both ways of evaluation, we calculated a total creativity score for each problem and each student. Following Leikin and Lev (2013), we used the following formula:

$$Cr = \sum_{i=1}^{n} Flx_i \cdot Or_i$$

Here, the flexibility score and the originality score are multiplied. Then, all (here n) solutions, respectively approaches (depending on the first or second analysis), presented by each student in one task are summed up to a total score.

5.4 Methods Part II: Students' Performances Across Three MSTs

The results of the first part of our study regarding appropriateness of solutions build the basis for the second part of our investigation, where we study students' performances in the presented problems across the disciplines geometry and algebra.

To analyze to what extent the students' performances vary across the different problems, we need a non-parametric test, which measures the ordinal association between the three MSTs pairwise. Therefore, we use Kendall's Tau Test, which is designed for small sample sizes. This test quantifies the similarity of orderings of students' performances.

5.5 Results Part I: Comparison of Two Evaluation Schemes

A first overview (see Table 5.1) displays how many students worked on the three different MSTs (14–21) and on how many approaches the students worked on average (1.5–3.5 on average).

In this section, we elaborate on the students' performances by illustrating meaningful examples for each of the three MSTs. We furthermore explain which differences occur when choosing the first or the second kind of analysis. In a second part, we compare students' performances by giving an insight into examples from students in all three MSTs.

In the first analysis based on only correct solutions, all non-correct approaches (in the above-mentioned sense) were sorted out in the beginning before calculating a creativity score. In the second analysis, we widened the spectrum of included approaches and included all approaches that were considered appropriate (see above). Table 5.2 displays the number of approaches we took into account in the first and in the second kind of analysis for each task. It displays the total creativity score; not the particular components such as flexibility or originality.

Furthermore, Table 5.2 displays the differences in creativity scores resulting from the first and second way of analysis. The number of students that worked on

Table 5.1 Overview of the number of students and approaches in the three MSTs

	Triangle-problem	Hexagon-problem	Punch-problem
Number of students	21	17	14
Number of approaches	32	59	19
Average number of approaches per student	1.5	3.5	1.4

Table 5.2 Students' performances in the three MSTs (considering correct vs. appropriate solutions)

	Triangle-problem		Hexagon-problem		Punch-problem	
	Correct	Appropriate	Correct	Appropriate	Correct	Appropriate
Number of approaches included	4	21	39	59	13	19
Number of students who have a creativity-score of 0	19	7	1	0	3	0
Number of students with an improvement (2nd vs. 1st analysis)	0	20	0	11	0	9

the triangle problem was 21, whereas 17 and 14 students worked on the hexagon and punch problem respectively. For the hexagon problem, the students found by far the most approaches (i.e. 59). We see that there are major differences in the number of approaches, which were included into the particular analysis. Especially in the triangle MST (4 correct vs. 21 appropriate approaches) and in the hexagon MST (39 correct vs. 59 appropriate approaches), the differences are remarkable. This has, of course, consequences for the students' creativity scores. In the triangle problem, 19 out of 21 students have a creativity score of 0 when the first—strict—analysis is applied (because these students do not have any entirely correct solution), but only 7 students have a score of 0 when the second analysis is used (because other, partly wrong approaches are furthermore taken into account). Moreover, in all three given MSTs, the second analysis leads to an improvement of students' creativity scores.

In the following, we use examples of students' approaches for each of the three problems in order to illustrate what makes students' approaches appropriate even though they may be wrong. We furthermore point out why we think it may be important to include also those approaches that have weaknesses but are comprehensible.

5.5.1 The Triangle Problem

Steven's approach (see Fig. 5.4) shows an algebraic calculation. He uses vectors to represent the corners P, W, S, and R of the given triangle, as well as the subdivisions of the line segment AB, resp. AC in relation to A which he chooses as the origin. In the next step, he tries to calculate the surface area of the gray quadrilateral. Therefore, he sees the area as a composition of a parallelogram and a triangle. After substituting the unknown lengths with the known ratios of $\frac{1}{3} \cdot AB$ and $\frac{1}{3} \cdot BC$, respectively, and after having set the lengths of the sides AB and AC to 1, he is able

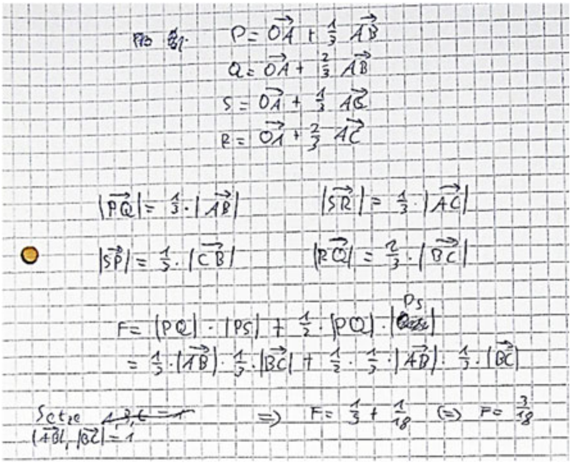


Fig. 5.4 Steven's approach (triangle problem)

to get to the result of 3/18. This outcome, however, does not refer to the triangle *ABC*, but to the parallelogram which can be constructed by adding a second triangle rotated by 180°. The last step, namely a division by 1/2, is missing. Therefore, this approach must be considered wrong. He misses the last step and, moreover, uses a special case to handle the problem. In consequence, if only correct solutions are to be considered, Steven gets 0 points for this approach. However, focusing on appropriateness, we can see a clear and comprehensible strategy. Indeed, the last step is missing and therefore, his conclusive answer is wrong, but his idea is well founded. Interestingly, no one else in our group of participants worked on this problem by using vectors, so that his approach shows a high level of originality. In the second analysis, this approach is included into the calculation scheme and raises Steven's creativity score.

The next example shows *Lilly's approaches* (see Fig. 5.5).

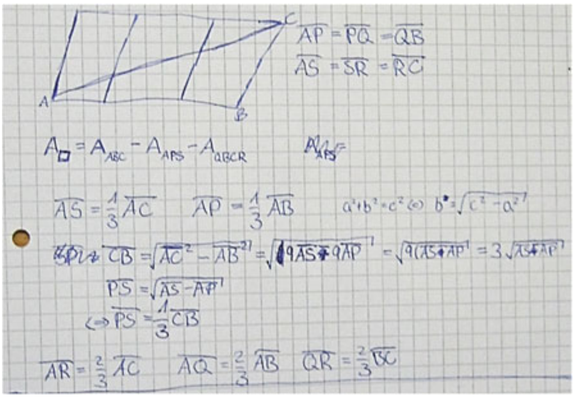


Fig. 5.5 Lilly's approach (triangle problem)

Lilly works on two ideas which are discussed in the following: On the top of her sheet, there is the triangle *ABC*, which she completes to a parallelogram by rotating the triangle at the middle of the line segment *BC*. Furthermore, she extends the lines *PS* and *QR*. To the right of Lilly's sketch, there are two lines of equations, which illustrate the same lengths for the different sections of *AB* and *AC* respectively. In the rest of her notes, she does not refer to this sketch. Of course, she refers to the triangle, but she does not elaborate on the parallelogram anymore. In her notes, she tries to express the gray surface area by building differences (see the line underneath the sketch in Fig. 5.5). Following this line of thought, to determine the particular surface area, she uses the Pythagorean Theorem to determine the lengths of some line segments but does not have any success. Finally, she writes down the proportion of the line segments and the two sides of the triangle. We see two very different approaches with which Lilly tries to solve the triangle problem. In the first one, she has the important idea of building a parallelogram but she does not proceed with this idea, even though she extends the line segments. When only correct solutions are to be considered, this solution is not included into the evaluation scheme, because there are no notes referring to the sketch or a conclusive answer. When appropriate solutions are in focus, we can see that she completes the triangle to a parallelogram. This move constitutes an important and creative part to come to a right solution. Especially the line extensions can be a hint at a reasonable idea. For this reason, we decided to include this approach in the second way of analysis. Lilly's second attempt is about the Pythagorean Theorem. Lilly tries to use this, even though the triangle is not right-angled. Furthermore it is not clear why she tries to calculate lengths of the line segments. We were not able to understand her strategy in this case. This example shows that there were cases in which an approach cannot be taken into account for a further evaluation with neither the strictly dichotomous first nor the broader-viewed second kind of analysis.

5.5.2 The Hexagon Problem

The hexagon problem was the second MST the students worked on. Most students integrated many ideas and approaches within one picture. Therefore, it is difficult to identify the particular approaches. In the following, two examples are presented, which have some similarities.

Figure 5.6 displays many approaches as carried out by Lilly. In the paper at hand, we have a closer look at one particular approach.

In this approach, Lilly draws a line from *A* to *D*. Together with the given line *AE* she writes that her auxiliary line halves the interior angle *BAF* in two angles of equal size, namely 60°. Then, she argues that the given line *AE* leads to a second halving, so that she comes to the conclusion that the angle ε is 30°. However, there is no reasoning provided about *why* the mentioned lines exactly halve the focused angle. Probably she had reasons of symmetry in mind, but we can neither be sure about this nor is symmetry a trivial argument in this problem. The non-trivial

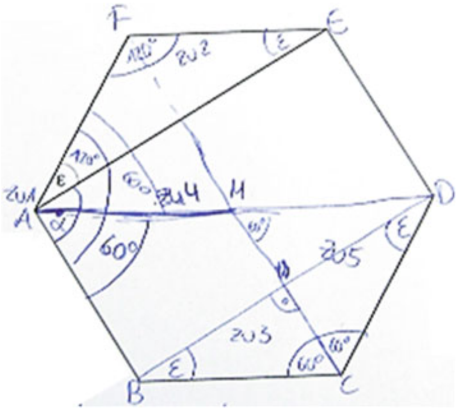


Fig. 5.6 Lilly's solution (hexagon problem). (Students' note "AD divides the angle in 2 equal 60° angles. AE divides one of this in two angles of equal size 30°")

argumentation why *AD* and *AE* are halving the angle ε twice and the fact that she misses to note some explanations may indicate that she is acting intuitively or is not aware of the correct mathematical argumentation. However, she comes to the right solution that *FAE* is 30°. We see that Lilly's approach has the right outcome but her reasoning is insufficient. For this reason, this approach is excluded under the perspective of the first analysis which—because of the high flexibility—leads to a reduction of her creativity score. In the second analysis, we decided to consider this approach appropriate and include it because of the right solution and an understandable way of solving the problem. Even though it may lack proper mathematical reasoning, it is a character of mathematical creativity.

Tina (Fig. 5.7) works on three approaches.

In the first one, she calculates the size of ε from the angle sum of a right triangle. Her next approach, which will be analyzed in the following, is similar to Lilly's solution presented above. Tina also divides an angle: She focuses on the angle *BAE* and divides it into thirds. She then comes to the right solution of 30°. Here, two aspects in the argumentation are missing. First, it is not pointed out why the angle

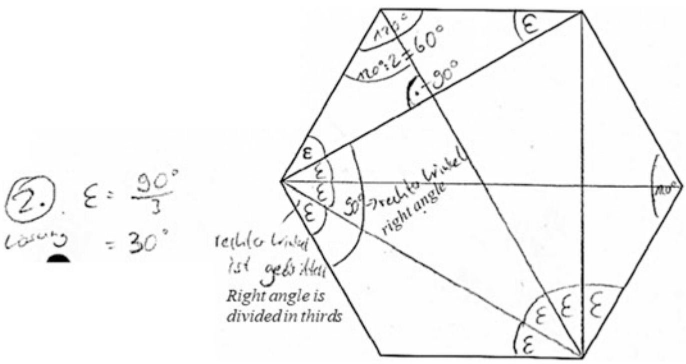


Fig. 5.7 Tina's solution (hexagon problem)

BAE is right-angled. She could argue drawing on the alternate angle, which she used in her first approach. However, she does not mention this. Another explanation could be that Tina does not feel the necessity to motivate this. Second, it is not obvious why the inserted line segments *AC* and *AD* divide the right angle into thirds. This approach is—compared to Lilly's approach—more sophisticated. Tina is probably not aware of the reasons and therefore this approach might be excluded in the first analysis, although her solution is right. However, we consider this approach appropriate, because it shows a clear idea that can be completed. Tina's third approach is very similar to her previous one. She divides the interior angle into four parts like Lilly. As in the approach before, Tina does not give any reasons for why her division is correct. Here, the evaluation follows the same line of argumentation as in Lilly's case: In the first analysis, the approach is excluded, whereas in the second analysis this approach is included. Again, we consider approaches appropriate for assessing students' mathematical creativity that lack certain explication of steps in the student's proof.

5.5.3 *Christmas Punch Problem*

The third problem was an algebraic one. And it seems to be the most complex one of the problems we used: The maximum number of solutions provided by the students was two, but most of the students only worked out one approach. In the following, one example is given.

In Olive's approach (see Fig. 5.8, left), we see some notes at the beginning, which build the central information for solving the problem.

The most interesting lines to understand her procedure are "$3/4 + 1/4 = 1$" and "4 barrels = 3/4 filled". Here, it becomes clear that Olive's idea refers to the wrong underlying set. She assumes that the barrels are filled to 3/4 in the beginning (Fig. 5.8, right). She therefore comes to the wrong solution of 16 barrels in total. In the first analysis, because of this mistake, her approach cannot be integrated into

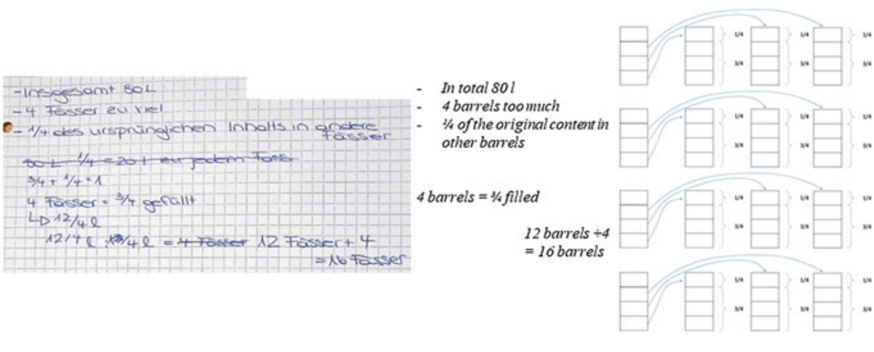


Fig. 5.8 Left: Olive's approach (Christmas punch); right: illustration to Olive's approach

the calculation scheme. However, when focusing on appropriateness, we see an approach which, due to a wrong text comprehension, entails a reasonable strategy to solve the problem. Of course, the mathematical notation is not elaborated but her thoughts are structured and clear. Therefore, this approach is considered appropriate and integrated into the calculation scheme. Because Olive did not work on a further approach, she gets a score greater than 0 only if this approach is taken into account. Nevertheless, many of the students had a very similar solution so that the originality score and thus the whole creativity score is rather low, but higher than 0, in general.

These examples illustrate that under the "strict" scheme of the first analysis, several approaches have to be excluded, which hint at creative thoughts and original ideas. Given that only entirely correct approaches are taken into consideration, we feel that the students' scores are heavily influenced by their ability to produce correct solutions. Therefore, we follow Leikin's (2013) suggestion to replace the notion of correctness with the notion of appropriateness.

In sum, considering our first research question of w*hat kinds of flaws appropriate solutions may entail to still be regarded appropriate and what appropriateness means in terms of mathematical MSTs*, we found that in most cases, the appropriate solutions

- lack certain steps in the reasoning of the proof but still are reasonable and understandable,
- provide an appropriate strategy and are understandable, but include other mistakes (arithmetical, algebraic, or geometrical) leading to a wrong solution, or
- are incomplete but may lead to a correct solution and could be complemented to a correct solution.

5.6 Results Part II: Students' Performances Across Three MSTs

In the next part of our study, we focus on students' performances across different MSTs.

Based on our previous finding that a scoring based on appropriateness gathers a broader spectrum of students' creative approaches, we ranked students' performance in each MST in order to compare students' ranks. Thereby, we aimed at analyzing the consistency of students' performances across three MSTs. By doing so, we wanted to investigate whether students' performances in different MSTs correlate and, thus, can be attributed to a single construct called students' mathematical creativity or not.

Table 5.3 Correlations between each two MSTs

	Problem		
	Triangle	Hexagon	Punch
Triangle	–	r = .233 (p = .280)	r = −.019 (p = −.939)
Hexagon	–	–	r = −.281 (p = .257)
Punch	–	–	–

Calculated with Kendall's tau

Table 5.3 shows the pairwise rank correlation coefficients between each two MSTs which were calculated with Kendall's tau. There is a weak rank correlation between the triangle MST and the hexagon MST (r = .233). This positive rank correlation is not significant (p = .283). If we look at the other two comparisons (triangle MST and hexagon MST; hexagon MST and punch MST), we see no positive correlation (r = −.019, resp. r = −.281). Actually, there is a weak negative rank correlation between the triangle MST and the punch MST, but with p = .939 this correlation coefficient is not convincing. We see that in each comparison there is no informative rank correlation. That means that there is no statistical evidence for a successful or unsuccessful performance along the three MSTs. However, it needs to be taken into consideration that due to the small sample size and the above-average IQ, the group under investigation is not representative; with according consequences for the results of the statistical analysis.

To give a qualitative insight, we focus on three students (Kirsten, Claire, and Phil) in the following and present their performance across the three tasks. We choose these students because their diverse performances are generic for the students in the project.

5.6.1 Kirsten's Case

In her approach to solve the triangle problem (see Fig. 5.9), Kirsten divides the grey area into a parallelogram and a triangle by drawing an auxiliary line through S that is parallel to *AB*. Then, she uses the intercept theorem to reason on proportions between different line segments. In comparison to the reference group, her approach is elaborated, as she is the only student who motivates the equality of the lengths of the line segments. Because of the originality of her approach, she receives a high creativity score (Cr_1). Although she does not write down another solution, she gets one of the highest total creativity scores (Cr) in this MST.

In the second geometrical MST (the hexagon problem), her performance is also high (see Fig. 5.10). She works out four approaches. In her first approach, she makes use of the angle sum. Therefore, she focuses on an isosceles triangle, which consists of one interior angle of the hexagon which is given with 120° and two angles ε. Then, she calculates the correct size of the angle. This approach was

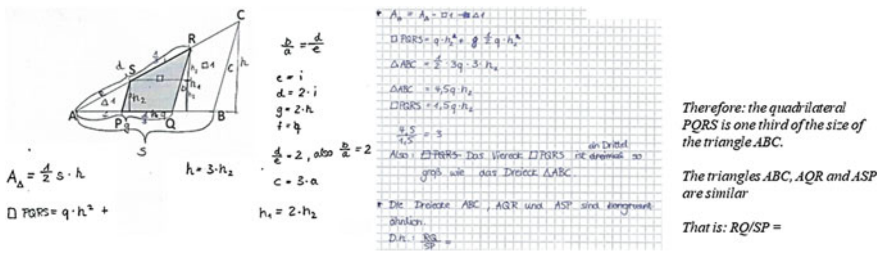


Fig. 5.9 Kirsten's solution for the triangle problem

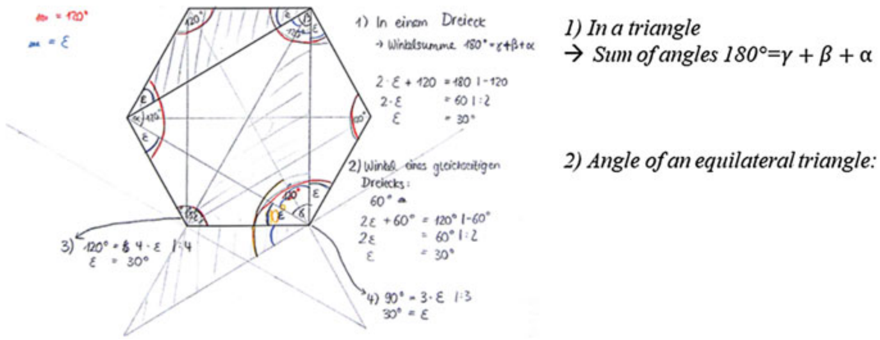


Fig. 5.10 Kirsten's approaches for the hexagon problem

worked out by several students. Therefore Kirsten's originality score is only 1 for this approach. The flexibility is 10, because this approach was her first one. Her second approach is from another category, so that the flexibility is scored with 10 for that approach as well.

The originality is also scored with 10: she inscribes an equilateral triangle and thereby divides the interior angle into 2 angles ε and one angle of the equilateral triangle which is 60°. In her next approach, she divides the interior angle into four angles of equal size. In the analysis, we take this solution into account, although there are missing steps regarding the argumentation (appropriateness). Because she uses the strategy again to divide an angle, the flexibility is 1, but the originality is rated with 10, because this solution does not appear often. In her last approach, Kirsten divides a right angle into thirds. Therefore, flexibility is rated with 1 again. The originality is also rated with 1, because this approach is common. With this performance, Kirsten's creativity score in this task is one of the highest.

In the third MST (the punch problem), Kirsten's performance (Fig. 5.11) is weaker. She works out one insight-based solution. Indeed, her solution is correct, but most students have a solution from the same category so that the originality score is 0.1. Therefore, her creativity score for this MST is lowest among all students participating in this study.

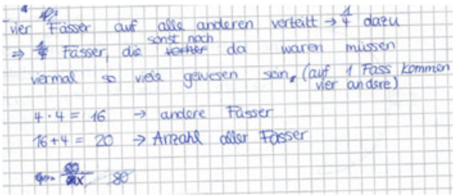


Fig. 5.11 Kirsten's approach for the punch problem

Kirsten's case illustrates that there are students who perform highly in some MSTs but low in other ones. In this particular case, the creativity scores were high in both geometry MSTs, whereas they were low in the algebra MST. Her creative performance appears to be content- or subdomain-dependent. This might indicate, for instance, that her abilities in the domain of geometry are stronger than in algebra; or it might reflect different norms that are established in her regular mathematics teaching in these different domains. It might also indicate that she has more practice with geometrical proofs than with problem solving in algebra. Finally, the differences might rely on difficulties to understand the situation in the last MST or on other influencing factors (concentration, motivation, etc.). However, what we see is that her creative ability does not come to light equally in the three MSTs. Her case indicates that mathematical creativity should possibly be considered subdomain-specifically (here: in geometry vs. algebra) rather than in general.

5.6.2 *Claire's Case*

Claire is one of the mathematically strongest students in the course. She performs very well in almost all problems given in the course.

In the first MST, her performance is not outstanding, but good (see Fig. 5.12). She works out two approaches, both rated with an originality of 1. In the first one, she completes the triangle to a parallelogram (although she writes "rectangle") and reasons why the grey area is 1/3. Remarkable is her note at the end in which she states that this relation only works with the area in the middle. In her second approach, she divides the given triangle into nine small similar triangles of equal size and determines the ratio.

In the hexagon problem, Claire works out four approaches (see Fig. 5.13).

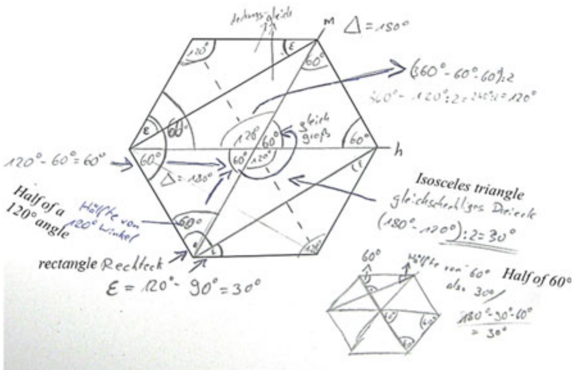
In one approach, she inscribes a rectangle und then subtracts 90° from the interior angle of 120°. Because of the low originality (rated with 1) she gets 10 points (10 for flexibility). Additionally, she provides two approaches in which she uses the angles sum of triangles—once in an isosceles triangle (similar to Kirsten, see above) and the other time in a right triangle. Both are rated 1 for originality and 10, respectively 1 for flexibility. In her last approach, she halves one interior angle twice. Here, she gets 1 point for both flexibility and originality. Although Claire

If you complement the triangle to a rectangle [she means parallelogram], you can recognize easily the share is one third. The area for the rectangle [not correct but this is what she said] has been multiplied by two. Therefore, the share of the area is also true for the triangle.
But this works only for the piece in the middle.

Parallelogram is just the same as a rectangle [she means the calculation of the area]
because of< ...
With this, the big triangle can be divided into several smaller ones of equal size.
In total: 9 small triangles
Selected area: 3 small triangles
Therefore 1/3 of the whole

Fig. 5.12 Claire's approaches for the triangle problem

Fig. 5.13 Claire's approaches for the hexagon problem



provides many ideas, her creativity score is below average, because her presented ideas show only a low level of flexibility and originality.

In the punch problem, Claire is one of the few students who works out two different approaches (see Fig. 5.14). Her first approach draws on a diagram, which illustrates the distribution of the barrels. Compared to the solutions of the peers, this solution is original, because only one other student used this idea as well. Therefore, she gets a score of 100. In her second approach, she uses two variables in a system of equation. This is also rated with a score of 100. As in the triangle and hexagon problem, she describes her ideas by giving additional information or writing down exact intermediate steps. In this problem, Claire has the highest score of all participants.

As in Kirsten's case, we see that Claire works on MSTs on different levels. Claire performs outstanding in the triangle MST and in the punch MST. In the hexagon MST, she performs below average as compared to her peers. We can

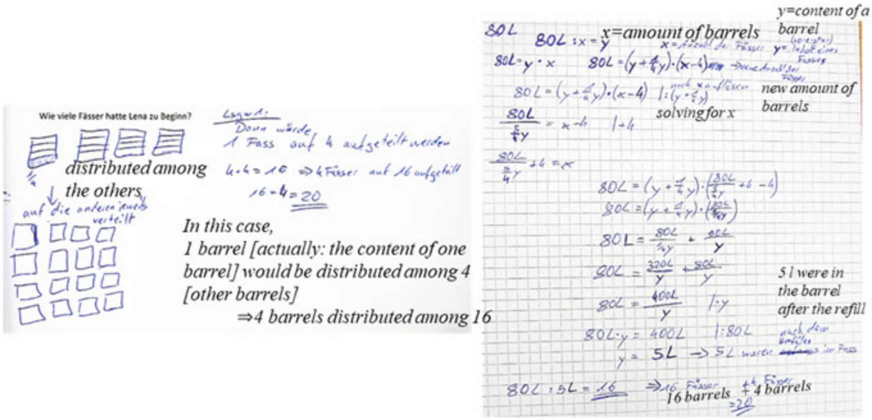


Fig. 5.14 Claire's approaches for the punch problem

assume that she might have a certain affinity to algebraic problems, because here, she shows a very sophisticated solution. An alternative reason could be motivational or affective influences or issues. Claire's case suggests that creativity might not even be conceptualized subdomain-specifically, but only task-specifically, depending on the particular MSTs.

5.6.3 *Phil's Case*

In the following, we give an insight into Phil's performances, which are lower than in the above-mentioned cases. When working on the triangle problem (see Fig. 5.15), Phil did not come to a conclusive result. We see that he sketched a right triangle as a special case and completed it in another sketch to a square. On his sheet of paper there are also some notes, in which we see rudiments of how to calculate the surface area of the grey area as the sum of a rectangle and a triangle. In all of his notes, we cannot identify a clear strategy that would give insight into his ideas. Even when including appropriate solutions, he gets a creativity score of 0 in this task.

In the hexagon MST, Phil's performance (see Fig. 5.16) is strong. He gets a high score for fluency, because he has figured out four approaches. In the first approach, he divides one of the interior angles into a right angle and ε. This approach is rated with 10 for flexibility and 10 for originality. The second approach is from another category (10 for flexibility), but not very original (rated with 1): He calculates the angle ε by the differences of interior angles of a right triangle. The next approach is similar to the first one: He uses a right angle. This results in 0.1 points for flexibility, but 10 points for originality. In his last approach, he refers to the sum of interior angles of a kite. Even though this approach gets 10 points for originality,

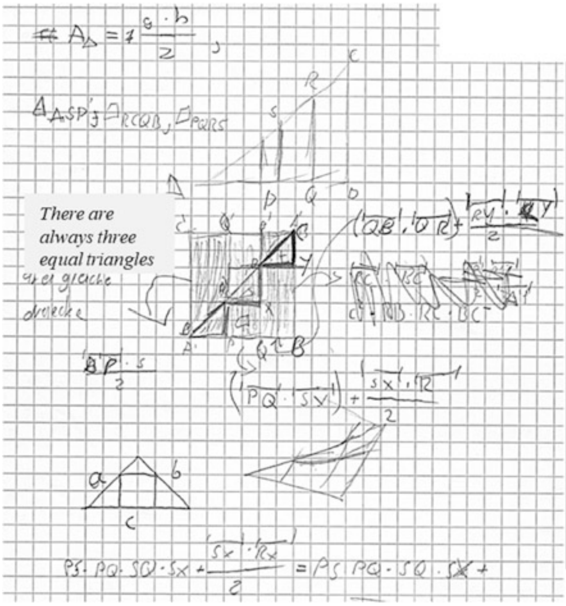


Fig. 5.15 Phil's approach to the triangle problem

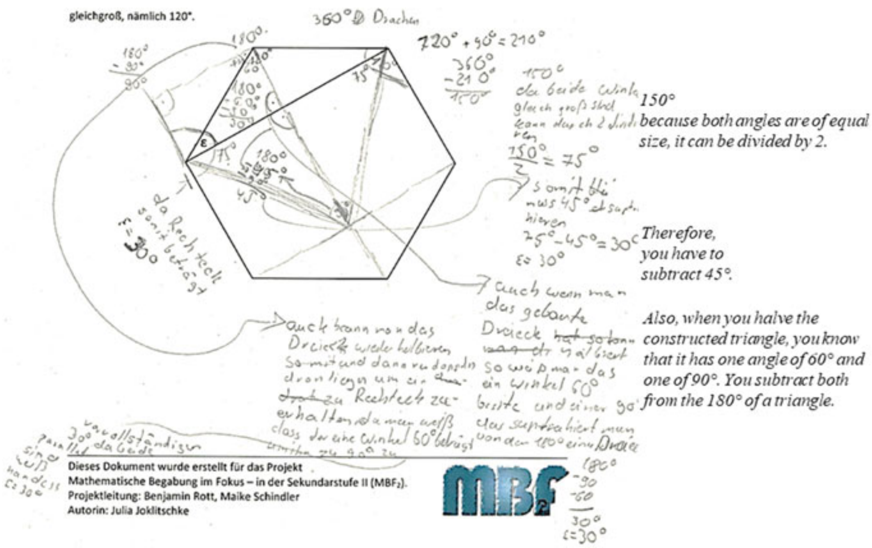
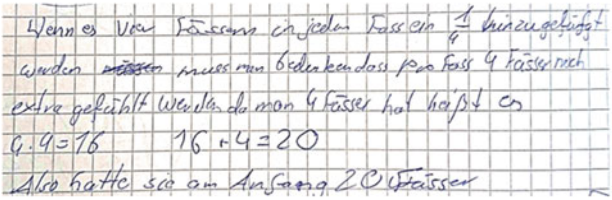


Fig. 5.16 Phil's approaches for the hexagon problem



If ¼ is added to each barrel, you have to consider that for each barrel 4 [other] barrels will be filled [spelling mistake in German]. Because you have 4 barrels, this leads to:
4·4=16

Therefore, she started with 20 barrels.

Fig. 5.17 Phil's approach for the punch problem

the flexibility is rated with 1, because Phil has worked in this category before. A difficulty for rating Phil's work lies in the fact that it does not become clear in which order he noted his ideas. However, a calculation that is based on another order would lead to the same total score.

Phil's approach for the punch problem (see Fig. 5.17) shows an insight-based solution (similar to Kirsten's) which is rated with 0.1 for originality. He does not work on another approach, so that his creativity score for this problem is 1.

Phil only performs on a high level in the hexagon MST. It seems as if he does not find a proper access to the triangle MST. In the punch MST, he works out a common solution. This case is complex because of two reasons: First, Phil does not perform better in one of the disciplines algebra or geometry. Second, he does not improve over the three problems. This might indicate that performances in MSTs depend on the problems themselves and that mathematical creativity is not a question of domain-specificity or even subdomain-specificity.

5.7 Discussion and Outlook

The purpose of our study was to investigate students' creative performances along different fields of mathematics. For inquiring into this question, we first intended to peruse the question of what *appropriateness* of solutions—a concept as offered by Leikin (2013)—may entail and imply in contrast to *correctness* of solutions.

We investigated these questions based on an empirical study with upper secondary school students, who were mathematically interested, showed strong cognitive abilities, and who worked on three MSTs—two geometrical tasks and one algebraic task—that were evaluated quantitatively as well as qualitatively.

A review of the literature in the field of creativity reveals that there is no consistent definition of the term creativity. This goes along with a wide field of ideas about how to conceptualize and operationalize creativity (e.g., Haylock 1987; Sriraman 2005). There are, for example, different assumptions about which persons can be mathematically creative—only professional mathematicians or everyone. This aspect goes along with the distinction between relative creativity and absolute creativity—called "little c" (for relative) and "big C" (for absolute creativity).

We assume that everyone can be creative—at least in a relative way. In mathematics, research focuses on assessing creativity by using problem posing and in problem solving (Silver 1997). Therefore, different approaches are used to assess mathematical creativity. In our study, we focused on a product-based evaluation method where students' products from problem solving tasks were analyzed and rated in the dimensions of fluency, flexibility, and originality (see, e.g., Leikin and Lev 2013; Kattou et al. 2013).

We used the existing and broadly accepted evaluation scheme developed by Leikin and Lev (2013) to quantify mathematical creativity in students' products. As suggested by Leikin (2013), we took into account appropriate solutions. In the first part of this chapter, we investigated the characteristics that appropriate solutions may have. We found that appropriate solutions may either be correct, may lack certain steps in the reasoning of the proof (while still being reasonable and understandable), or include other mistakes (arithmetical, algebraic, or geometrical) leading to a wrong solution while still providing an appropriate strategy.

We found that non-correct appropriate approaches provided us with insights into students' creative potential even though they contained flaws such as lacks in ways of reasoning, mistakes in calculations, or a missing answer. In our paper, we used cases of students to illustrate the value of focusing on appropriate solutions (as suggested by Leikin 2013) for thoroughly evaluating mathematical creativity of students' products. In qualitative analyses, we illustrated that despite their partial incorrectness, these products include creative efforts that would have been disregarded by excluding these approaches.

However, we also experienced that in some cases it is difficult to decide whether a student's approach is to be considered appropriate or not. In these cases, we drew on a consensual validation, in which we discussed whether student approaches impart a comprehensible strategy and can be included into the second analysis or not. Even though this procedure results in high efforts, taking into account appropriate solutions uncovers a broad spectrum of creative approaches which otherwise were excluded, if only correct solutions were regarded. Of course, when appropriate solutions are considered, this has the potential effect that the originality and flexibility scores that are assigned to some approaches are lowered due to a higher number of graded approaches and therefore a higher frequency of solutions.

Furthermore, our study emphasizes that students' prior knowledge should be considered when assessing mathematical creativity. The used MSTs are from the fields of geometry and algebra. To solve these problems in multiple ways, a certain spectrum of mathematical background is necessary. In this respect, Singer and Voica (2017) found that "that creativity manifestation is conditioned by a certain level of expertise" (p. 75). It is therefore difficult to distinguish clearly between mathematical abilities and mathematical creativity. Research results rather indicate that creativity and expertise mutually influence and support one another (Singer and Voica 2017). However, with our understanding of appropriateness of solutions, we see a potential path to face this issue (at least partly). Through including not only correct solutions but also partially correct approaches, the view on mathematical creativity is broadened.

Naturally, the idea of taking into account appropriate solutions (and not only correct ones) is especially useful when the given tasks are complex problems; such as geometrical proofs. If the students solve problems which do not require complex reasoning or proofs, it is eventually more appropriate to focus only on correct solutions. For example, when using a problem in which the students are asked to fill in a number pyramid (see Kattou et al. 2013) it is not necessary and probably not even adequate to lower the criterion of correctness. A modification of the evaluation scheme has to be carefully considered against the backdrop of the given problems, their complexity, and the required reasoning.

From the perspective of test theory, there is an important discussion about the visible *performance* of students and their rather invisible *potential* (Foth and van der Meer 2013). In our study, we were able to have a look at a broader spectrum of students' potential by also taking into account incomplete approaches. Through focusing on appropriate solutions (similar to Leikin 2013) and elaborating on the notion of appropriateness, our study contributes to efforts to bridge this gap between potential and performance. But even when regarding appropriate solutions, it is challenging to grasp students' potential regarding mathematical creativity. To face this issue, we think that a shift in the perspective from products to processes is significant. There is more empirical research needed that inquires into students' mathematical creativity from a process-view. First studies (e.g., Schindler et al. 2016; Schindler and Lilienthal 2017a) hint at the potential that empirical studies focusing on students' processes may have for extending the body of knowledge regarding mathematical creativity. Especially, eye-tracking appears to be a research method with certain potential for investigating mathematical creativity from a process-view in the future (Schindler and Lilienthal 2017b).

We studied students' performances across three different MSTs; two geometric and one algebraic problem. Focusing on rank correlations and using Kendall's Tau Test, we did not find a statistical significant correlation between students' performances in different MSTs. This indicates that MSTs in different subdomains but even within a single mathematical subdomain (geometry) do not require exactly the same competencies. In particular, we hypothesize that it is rather content-knowledge than a creative ability that affects students' performances in these MSTs. A qualitative analysis of the students' products shows a broad spectrum of performances across the three MSTs. There are cases such as Kirsten's who seems to prefer geometric problems, or Claire who might show outstanding results in all domains, or Phil who only performs well in one geometrical MST, but not in the other one.

In the sense of mathematical creativity as a domain-specific construct (Kattou et al. 2015), we would have expected a stronger consistency within the ranks regarding the three MSTs. There are various possibilities to explain our results. First, we investigated a selected group of students. These students were mathematically interested and had—as the results from the intelligence test showed—an above average IQ of 121 on average. Given that high intelligence might correlate with mathematical giftedness (Foth and van der Meer 2013), we assume that a ceiling effect might affect our analysis and disguise possible rank correlations. Second, the students did not have prior experiences working with MSTs. We assume that this might influence the results as well. Accordingly, we recommend to conduct creativity tests with students who are familiar with MSTs. Third, because of the ordinal scaled data in our analysis, a direct comparison was not possible. Therefore, we had to work with ranks and rank correlation, but we have to have in mind that this method does not represent their performances adequately.

Referring to mathematical creativity as a domain-specific concept (Kattou et al. 2015), our results gave hints that the construct of mathematical creativity is not as consistent and homogeneous as it might appear eventually. Possibly, creativity in the field of mathematics should be rather viewed as a subdomain-specific construct (which means that there could be a construct like *geometric creativity* or *algebraic creativity)*; or even as task-specific. This relates to Singer et al.'s (2017) research on cognitive styles. In particular, they found "that cognitive flexibility—a basic indicator of creativity—inversely correlates with a style that has dominance in metric GN [Geometric Nature] and structured CD [Conceptual Dispersion], showing that the detected cognitive style may be a good predicator of students' mathematical creativity" (p. 37). We think that this interesting phenomenon requires more research in the future.

In comparison to studies such as Kattou et al.'s (2015), we cannot present quantitative values such as correlation coefficients or model calculations that are statistically reliable. This relies on the fact that our group of participants was small and not representative. Our intention was rather to share and discuss thorough considerations about mathematical creativity and its evaluation. Finally, we hope that our contribution can lift future scientific discussion on the evaluation of mathematical creativity, the notion of appropriateness, and the question of whether mathematical creativity is to be considered domain-specifically, subdomain-specifically, or rather task-specifically.

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Part II
Characteristics of Students with Exceptional Mathematical Promise

Chapter 6
Characteristics of Mathematical Giftedness in Early Primary School Age

Daniela Assmus

Abstract This study examines characteristics of mathematical giftedness in second graders. First, a possible system of characteristics was designed in theory. Then, this system was verified by giving a paper-and-pencil test with tasks developed for this purpose to 182 mathematically gifted children as well as 69 children of a reference group. In addition to the written tests, semi-structured interviews were conducted with all the participants regarding their results and strategies. The outcomes of the study show that all of the analyzed characteristics of mathematical giftedness can be confirmed. These include the ability to memorize mathematical issues by drawing on identified structures, the ability to construct and use mathematical structures, the ability to switch between modes of representation, the ability to reverse lines of thought, the ability to capture complex structures and work with them, the understanding of relational concepts and the ability to use relational concepts and connections.

Keywords Mathematical giftedness · Characteristics · Primary grades
Second graders · Comparative study

6.1 Mathematical Giftedness as Domain-Specific Giftedness

Before conducting theoretical and empirical investigation of mathematical giftedness, it is of essential importance to answer the question of whether giftedness should be regarded as domain-general or rather domain-specific. The answer to this question has consequences in terms of the modelling of the construct as well as in terms of choosing adequate instruments of diagnosis. Currently, giftedness is predominantly understood as domain-specific. This can be seen, for example, in

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terms of multidimensional models, such as the *Munich Model of Giftedness* (Heller 2004) or the *Differentiated Model of Giftedness and Talent* of Gagné (2003), which distinguish between various areas of ability, e.g. intellectual, creative and socio-affective abilities. In these models, however, the domains are quite broad, so that no more precise differentiation with regard to intellectual abilities occurs. It remains unclear whether domain-specific characteristics might exist in this area as well. However, it is of great interest for research on mathematical giftedness to understand its relatedness to intellectual abilities.

Basically, three models can be considered for investigating the relationship of mathematical and intellectual abilities (Heilmann 1999):

Model 1: Mathematical giftedness as a component of general intellectual giftedness,
Model 2: Mathematical giftedness as general intellectual giftedness in combination with specific mathematical abilities,
Model 3: Mathematical giftedness as a specific giftedness which can also occur independently of general intellectual giftedness.

Model 2 as well as model 3 consider a specific giftedness, i.e. a high display of specific mathematical abilities which occur additionally to or independently of higher intellectual ability. In model 1 no specific giftedness is assumed. Instead, general intellectual giftedness is seen as a necessary prerequisite for mathematical giftedness. This way, mathematically gifted people must always display high intellectual abilities. The fact that in reverse, not all intellectually gifted must display high mathematical abilities can be explained by looking at other factors that influence the development of giftedness, as model 1 shows. Especially, factors that refer to interest and motivation carry particular importance. It is possible that mathematical giftedness occurs when very early on, specialized engagement with mathematical content is spurred as specific giftedness (while simultaneously neglecting the development of competences in other disciplines), even if—according to this model—general intellectual giftedness exists, primarily developed in one domain (Heilmann 1999).

For all three models, indicators and arguments can be found in academic discourse. Indicators in terms of the validity of model 1 are derived from psychological correlation studies in which medial to high correlations within the total population were proven between IQ-results and mathematical achievement (e.g. Primi et al. 2010; Taub et al. 2008). Individual studies with mathematically gifted children and teenagers which reported on this extreme group spoke of similar correlations (Birx 1988).

Model 2 is supported among others by one study conducted by Gawlick and Lange (2010) in which 684 fifth graders took part in an intelligence test (CFT-20R) as well as in an indicator task test by Käpnick (1998). Statistical analyses showed that a model explained the results of the Käpnick test best in which the general factor of intelligence as well as mathematically specific factors had an impact on mathematical performance. The extensive analyses by Lubinski and Humphreys

(1990), in which data of various tests (i.e. mathematically specific tests and tests on cognitive abilities) from over 900 American highs schools were evaluated, provided indicators for validity of model 1 as well as of model 2: “For the most part, mathematical talent appears intimately related to general intelligence as indexed by conventional measures, however mathematical giftedness also appears to be somewhat specific” (p. 340).

Indicators for the domain specificity of mathematical giftedness (model 3) can be found in a study by Benbow and Minor (1990). Together with 106 mathematically and 20 linguistically gifted teenagers (by measure of SAT), as well as 18 persons who belonged to both groups, different intelligence and ability tests were carried out. The groups of mathematically and linguistically gifted persons distinguished themselves in nearly all areas of ability. “The verbally precocious scored higher on verbal and general knowledge types of tests, the strengths of the mathematically precocious were in the nonverbal abilities” (p. 25). However, this can be explained by the fact that both constructs of mathematical and linguistic giftedness show great differences and can therefore not be part of a general intelligence.

Moreover also, in favour of model 3 speaks the fact that in various case studies, children and teenagers have proven their mathematical giftedness in challenging mathematical situations despite any signs of general intellectual giftedness (e.g. Assouline and Lupkowski-Shoplik 2005; Käpnick 1998; Kontoyianni et al. 2013; Nolte 2004). Hence, many but not all children and teenagers showed that mathematical giftedness comes along with a high IQ. This implies a sort of specific giftedness which may exist together with a general intellectual giftedness but not necessarily.

As the studies at hand show, the question of relationship between mathematical and intellectual giftedness cannot be unequivocally answered. Considering also studies and theoretical concepts which exclusively look at the phenomenon of mathematical giftedness and ignore explicitly examining the relationship with general intellectual abilities, mathematical giftedness is often times considered a domain-specific giftedness (e.g. Benölken 2015; Fritzlar 2013; House 1999; Käpnick 1998; Krutetskii 1976; Singer et al. 2016).

This view is manifested in the efforts to describe the construct of mathematical giftedness as detailed as possible while also deriving mathematically specific abilities and patterns of action which when exhibited on a large scale could be characteristic for mathematical giftedness. As a consequence, over the course of the previous three decades, various lists of characteristics were developed (e.g. Käpnick 1998; Kießwetter 1985; Krutetskii 1976; Miller 1990; Sheffield 2003). Identifying mathematical components of giftedness allows for the conclusion that also the entire construct is domain-specific.

However, it is also problematic that mathematics specificity of some listed components is not always immediately evident. When for example characteristics of giftedness, such as a high ability of abstraction and generalization or a particular flexibility in thinking are mentioned, these terms themselves do not always display their reference to mathematics directly. “These investigations […] hardly make explicit what it is that is ‘specifically mathematical’ about these abilities. Rather, it seems to be general ability that is needed for mathematics albeit to a larger extend but necessary also for all other intellectual challenges" (Heilmann 1999, p. 39; translation by the author).

In contrast to this, other authors are of the opinion that ability does not exist in isolation, but always tied to specific content. Therefore, initially general abilities may occur as specific abilities during specific tasks (e.g. Gullasch 1973). Similarly, Krutetskii remarks:

> Certain features of a pupil's mental activity can characterize his mathematical activity alone – can appear only in the realm of the spatial and numerical relationships expressed in number and letter symbols, without characterizing other forms of this activity and without correlating with corresponding manifestations in other areas. Thus, *mental abilities that are general by nature* (such as the ability to generalize) *in a number of cases can appear as specific abilities* (the ability to generalize mathematical objects, relations, and operations). There appears to be every basis for speaking of *special, specific* abilities, and not of general abilities that are only refracted in a unique way in mathematical activity. (Krutetskii 1976, p. 360; emphasis in original)

Summing up the depictions discussed above, mathematical giftedness is regarded as a domain-specific giftedness in this article. It can occur coupled with general intellectual giftedness but also independent of it. The following working definition forms the basis for all further elaborations:

> Mathematical giftedness is an extraordinarily high potential, compared to others of the same age, to successfully solve mathematically challenging questions and problems. (cf. Nolte 2013)

It is assumed that mathematical giftedness can be described with so-called characteristics which refer to a high display of topic-specific abilities or patterns of action.

6.2 Characteristics of Mathematical Giftedness

One emphasis of researching mathematical giftedness lies on developing mathematics-specific characteristics of giftedness. This is expected to be of great use especially for identifying mathematically gifted children and youths. The comprehensive studies of the Russian psychologist Krutetskii (1976) were groundbreaking in this field: Using a variety of research methods, he assembled an overview of different components of mathematical giftedness; the "structure of mathematical abilities" (1976, p. 350). Similar lists were established by other authors (e.g. Greenes 1981; House 1987; Miller 1990) in the following years. In many cases it was, however, not specified, which age groups the respective compilations referred to. It is possible that some characteristics only manifest themselves strongly in certain learning and development steps that cannot occur until a certain age or stage of development. It is therefore questionable, whether lists of characteristics without a specified target group can be applied to primary school children without any restrictions. It might be necessary to define a selection or even other characteristics of giftedness.

The German mathematics educationalist Käpnick (1998) explicitly investigated primary school children's characteristics of giftedness and tested them empirically with third and fourth graders. During his research, he identified the following mathematics-specific abilities as characteristics: (1) remembering mathematical facts, (2) structuring mathematical facts, (3) mathematical sensitivity and mathematical fantasy, (4) transferring mathematical structures, (5) intermodal transfer and (6) reversing lines of thoughts (Käpnick 1998, translated by the author; cf. Benölken 2015). Like other authors he points out that abilities are not always necessarily developed equally, which means that individual profiles of giftedness can vary quite a lot.

Regarding the early primary school age, no studies have been published to my knowledge which deal with characteristics of mathematical giftedness that have been validated by comparative studies involving mathematically gifted children and those who are not. The study at hand attempts a contribution to reducing this lack of research. The study builds on Käpnick's research and its goal is to develop mathematics-specific characteristics of giftedness that can already be observed in second graders.

6.3 Design of a Combined Test and Interview Study

The basis of the conducted research was a theoretical analysis of already existing systems of mathematical giftedness characteristics regarding their transferability to younger children. In addition, studies on developing mathematical competencies during the first school years were examined to include further possible distinctive features of younger mathematically gifted children. These theoretical analyses led to the conception of a preliminary system of mathematical giftedness characteristics in second graders. In order to verify the characteristics system, special indicator tasks were developed among other measures. Solving these tasks successfully suggests a high level of the individual characteristics. For various reasons (difficult operationalization of some characteristics, restricted test length due to the young age of the participants), not all theoretically established characteristics resulted in the development of indicator tasks. The following characteristics were included in a total of 10 subtasks: (1) Ability to memorize mathematical issues by drawing on identified structures, (2) Ability to construct and use mathematical structures, (3) Ability to switch between modes of representation, (4) Ability to reverse lines of thought, (5) Ability to capture complex structures and work with them. A selection of tasks is presented in the next chapter. In order to avoid repetition, the individual characteristics of giftedness will be explained content-wise in the sections: description of the task and results.

6.3.1 Research Methods

These tasks were given to mathematically gifted as well as “ordinary” second graders under similar conditions in a paper-and-pencil test. After the paper-and-pencil test, all participants were asked by specially trained pre-service teacher students about their solutions and strategies by means of semi-structured, recorded interviews. On the one hand, these interviews were supposed to help understand the thoughts and notations of the children, and to recognize practical approaches even if the solutions were not correct. Thus, it was possible, to apply a rather differentiated evaluation system that displayed differences in levels directly in the allocation of points. On the other hand, during these interviews, the used methods and strategies could be defined retrospectively and analyzed in addition to the problem-solving success.

6.3.2 Choice of Test Persons

The participants of the test group were 182 (149 males, 33 females) children that were chosen by teachers to participate in an extracurricular fostering project for mathematically gifted second graders. The reference group (N = 69; 34 males, 35 females) was composed of children from two urban as well as two rural school classes.

The teacher’s judgment was chosen as selection instrument for the following reasons: on the one hand, there was no empirically approved test for the age group under scrutiny that would reflect the represented understanding of mathematical giftedness. On the other hand, we wanted to avoid that the test group consisted of pupils involved in longer-standing projects fostering mathematically gifted children, as this would have implied an advantage in experience with unfamiliar and challenging tasks in comparison to the children from the reference group. Therefore, the only remaining possibility was to hand the choice over to persons who regularly experience the children in relation to others of their age group.

Because of the subjectivity of individual experiences and possible stereotypical views, however, the teacher’s judgment is not an optimal selection instrument. The following measures have been taken to increase the reliability of the selection:

- The schools were informed in written form about the realization of an extracurricular fostering project for mathematically gifted second graders. In order to provide support to teachers regarding questions such as how to identify a case of mathematical giftedness or which children are eligible for such a project, the information flyer also provided a number of indicative questions. These questions (Assmus 2017) targeted changed behaviour regarding mathematical contents or the case of students being underchallenged by regular classes. The characteristics of giftedness examined in this particular study were not adopted.

- In addition, after registration of the children teachers were asked to write an open assessment. This should explain what makes the child stand out regarding their handling of mathematical contents, how their mathematical giftedness can be evaluated and why the teacher considers additional fostering reasonable. For 91% of the 226 students, teachers complied with this request.
- Before the empirical study, all children participated in a trial lesson during which they could gain an insight into the project and independently decide whether or not they wanted to participate. In addition, a pre-test was conducted with all children (also the reference group) intended to help the participants familiarize themselves with the test procedure.

202 children registered for the fostering project eventually participated in the actual empirical study. Prior to the study, based on the teachers' assessments and first observations, 20 children were classified as probably not mathematically gifted, which is why their results were not considered in the statistical analysis.

Hence, 182 children remained in the test group. It is possible that a few more children among the test group had been suggested for the project based on teachers' misconceptions, but this could not safely be predicted prior to the study. Teachers of the reference classes were also asked to assess the mathematically giftedness of the students. Those children who were described as mathematically gifted by the teachers or candidates for the project anyway were not eligible for the reference group.

To enable comparative evaluations of the two participating groups, this study classifies all remaining children in the test group as mathematically gifted and all children in the reference group as not mathematically gifted, even though this might relate to misconceptions in individual cases.

The two groups differed significantly in gender proportions (test group: 149 males, 33 females; reference group: 34 males, 35 females). Since the test group consisted of potential candidates for the fostering project, and since their registration for the project depended on external assessments by teachers and parents as well as the children's own interest, it was not possible to influence the gender relations in this group. This aspect, however, must be considered in the analysis.

6.3.3 Data Analysis

For data analysis, qualitative and quantitative methods were combined. An overview is given in Fig. 6.1.

First, the semi-structured interviews[1] and the notes from the paper-pencil-test were analyzed in the sense of a qualitative content analysis (Mayring 2015), employing inductive category formation (number 1 in Fig. 6.1). As the majority of children's task-solving procedures could reasonably be accounted for, i.e. the

[1]For this, the relevant parts of the interviews were transcribed.

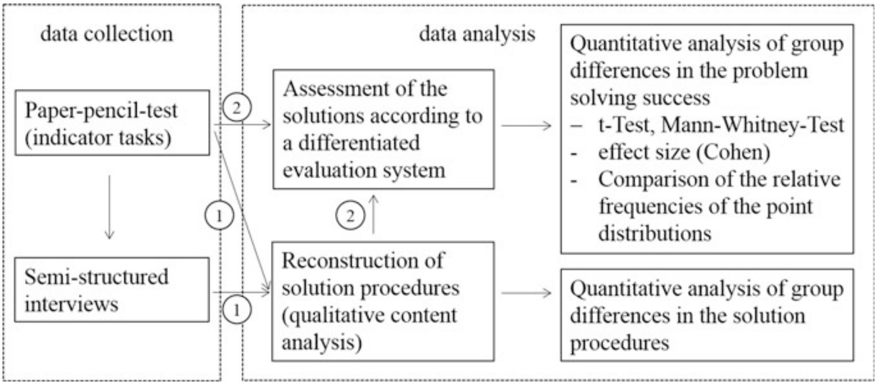

Fig. 6.1 Data analysis

meaning of the notes was mostly clear,[2] a differentiated evaluation system was used to assess the results for each task in terms of their (partial) accuracy (number 2 in Fig. 6.1).

The data thus obtained was subjected to various statistical analyses: The data was first analyzed using tests of significance regarding the group differences in the problem-solving success. The t-test was used to make sure that possible differences in the results between the two groups were not due to the varying gender ratio. Afterwards, the results of the test and the reference group participants were compared using the Mann-Whitney test. Additionally, the effect size d (according to Cohen 1969) was calculated. Finally, the group-specific allocations of points were examined in detail for each task.

Additionally, the results from the analysis of the interviews were, for each task, quantitatively examined regarding differences between the groups. The group comparisons referred to the task-specific requirements and used procedures, identified and used mathematical structures as well as possibly occurring mistake patterns.

6.4 Tasks Used in this Study

Among others, the following tasks were used (for reasons of space, the instructions are not always represented in their original form).

[2]It should be mentioned that the described procedures after the test cannot be claimed to definitely correspond to the thinking processes during the study. This, however, applies to all attempts to reconstruct thinking processes and thus does not only concern this study.

Fig. 6.2 Number square

8	2	1	9
7	3	4	6
6	4	3	7
9	1	2	8

Fig. 6.3 Solution template

Task 1: Number square
The children were given 60 s to look at the number square (Fig. 6.2) and memorize the numbers and their placement. The task was then to correctly complete the solution template (Fig. 6.3) with the numbers from memory. Based on the assumption that short-term memory is limited to seven to eight unstructured individual pieces of information (Anderson 2007), the correct memorization of numbers should require the usage of mathematical structures. If these structures are optimally employed, the numbers to be memorized can be reduced to 4, possibly even to one. If the child e.g. recognizes that always the outer neighboring numbers form the sum of 10 and all numbers are arranged point-symmetrically to the centre, it is sufficient, e.g. to learn the numbers 1–4 in the upper two rows. All other numbers can be filled in by reconstructing them with the help of the structures mentioned above.

With this task the "ability to memorize mathematical issues by drawing on identified structures" is examined.

Task 2: Tim's figure
In Fig. 6.4a, b, the empty circles should be completed with numbers so that the following rules apply:

1. When two numbers are directly above and below one another, the lower number is always greater by 1 than the upper number.
2. When two numbers are adjacent, the right-hand number is always greater by 4 than the left-hand number.
3. In diagonals, the sum of the two numbers in the corners is always the number in the middle circle.

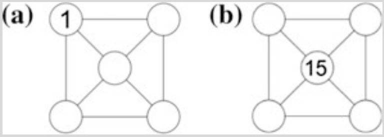

Fig. 6.4 Solution template

Firstly in this task it is important to understand the mathematical structures described in the text. ("Ability to construct and use mathematical structures".) In order for this to happen, mathematical content and relational terms (above/below, right/left, greater) as well as terms, such as "sum" must be grasped and connected with the given numbers. In task a based on the 1 in the upper left circle, the rules could be applied one after another. In the follow-up task b, not the first number of the sequence is given, but the last one, which can be considered as reversed situation. Thus, for succeeding at this task, this reversal must be recognized ("Ability to reverse lines of thought"). For solving this task, different approaches are possible. For example an arbitrary number can be placed in the left upper corner which forms the basis from which all other given rules are realized. In the end, it will be checked whether the results in the middle show the number 15. If this is not the case then the number in the left upper corner will be changed until the middle number is correct. The reversed line of thought becomes apparent here in recognizing the reversed situation (Assmus 2016; Fritzlar 2010). Otherwise, if one follows the third rule and looks for decompositions of 15, two-way associations are used (the 15 is connected with its decompositions as a number triple. These can be turned into addition as well as subtraction tasks). The establishing of two-way associations is considered another aspect of the ability to reverse lines of thought (Assmus 2016; Krutetskii 1976).

Furthermore, for correctly solving this task it is necessary to simultaneously follow all the given rules ("Ability to capture complex structures and work with them"). In task b this is not possible by following through with the results forwards or backwards. When expectably approaches to solving the tasks are carried out, the change of individual numbers always impact the realization of all rules likewise. In order to solve the task successfully, one has to handle the complex situation and has to also simultaneously pay attention to the impact on the validity of rules.

Task 3: Number triangle ("Ability to construct and use mathematical structures")

(a) Find a rule in this number triangle, draw lines and add appropriate calculations.
(b) Invent your own number triangle in which two rules can be discovered.

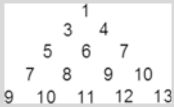

Fig. 6.5 Number triangle

The number triangle is a mathematical pattern whose constitution is based on several mathematical structures. In order to name in (a) a general rule that applies to the number triangle, a mathematical structure must be recognized. Recognizing can be made visible by giving suitable tasks and by describing the structure.

Completing the task successfully requires a grounded understanding of the term rule which second graders may not have acquired yet. In order to reduce misunderstandings and insecurities because of unfamiliar terms used in the task, the respective term is jointly worked out in the introductory task. Therefore, in the first part always two diagonally neighboring numbers are connected from the upper left to the lower right by drawing a line. Additionally, there are matching supplement tasks given to the number triangle (1 + _ = 4, 4 + _ = 7, 3 + _ = 6, 8 + _ = 11). Their supplementary summands are equal. This way a rule is developed which is worked out with the test leader by calculating the tasks, matching them with the lines, and recognizing a regularity ("always plus 3") which is formulated and written down by the children.

In (a) another rule must be given. The following rules could be used, for example:

- From the upper left to the lower right the numbers increase by 2.
- Vertically down, the numbers increase by 5.
- Within one horizontal row, numbers increase from left to right by 1.

Both subtasks refer to the "Ability to construct mathematical structures". While this applies in task (a) to recognizing already existing structures, task (b) asks for the construction of a new pattern. Additionally, this may expose mathematical creativity.

Task 4: Figurative numbers

Here grows a magic forest, which always takes the shape of a rectangle. Every year, the forest changes according to a certain rule.

(a) How many trees does the forest comprise in year 5?
(b) How many trees does the forest comprise in year 9?
(c) One year, the forest counts 110 trees. Which year is that?

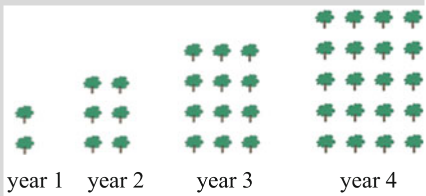

Fig. 6.6 Figurative numbers

The first two subtasks require the identification and usage of a valid rule of formation of mathematical sequence ("Ability to construct and use mathematical structures"). In task (a) the elementary number of the immediately subsequent figure is searched. Hence, a near generalization (Stacey 1989) of the mathematical structure is necessary. In contrast to this, in task (b) a further distanced figure is inquired which demands a further generalization (Stacey 1989). The problem to be solved in task (c) is the opposite of the problems presented in tasks (a) and (b), because the elements that are given and the elements to be identified are reversed ("Ability to reverse lines of thought"). For second graders, however, it is not yet possible to solve this task directly by reversal operations, but through continuation of the previously chosen solution strategy. This task reveals the recognition of reversed situations, which is considered one aspect of the ability to reverse lines of thought (Assmus 2016; Fritzlar 2010).

In addition, all three tasks require students to switch between modes of representation ("Ability to switch between modes of representation"). I distinguish between external and self-initiated modes of representation change (Assmus 2017). Externally initiated representation change occurs when a task is not solvable without this change. This is the case in the current task. Because an amount is inquired, a change from a graphical to a symbolic representation is necessary to solve the task successfully. Additionally, a self-initiated change can be applied, if the test persons already work with symbolic representations i.e. when they have chosen an arithmetical approach. Referring to the findings by Käpnick (1998) it is assumed that mathematically gifted second graders distinguish themselves from non-gifted children by applying self-initiated representation changes.

6.5 Results and Discussion

The children of the test group clearly solve the indicator tasks better than the children of the reference group. For all tasks, the differences between mathematically gifted and non-gifted second graders were very significant ($p < .01$). The differences that were calculated using the effect sizes d mainly varied between a half and a total standard deviation (sometimes even more). According to Cohen (1969) this equals a medium to high range. The results therefore support the allocation of the tested mathematics-specific abilities in the characteristics system of mathematical giftedness in second graders. Slightly lower effects (d = .3 to d = .4) were only observed in tasks regarding memorizing mathematical issues by drawing on identified structures.

Moreover, differentiated evaluations are noted regarding differences between the children of the test and the reference group for all studied characteristics of mathematical giftedness.

Ability to memorize mathematical issues by drawing on identified structures

The mathematically gifted second graders were better able than non-gifted second graders to mentally store structured number arrangements. In doing so, they used structures more often and more successfully than children who were not mathematically gifted. Structures which optimally reduce the information to be memorized were almost exclusively used by mathematically gifted children. This became particularly clear in task 1: what proved to be particularly successful was the combination of the point reflection of the number arrangement towards the figure centre and the addition to 10 with the outer adjacent numbers. All children who used this strategy (11 in the test group, 1 in the reference group) accurately reconstructed the numbers.

In the test group the high share of structure-based memory strategies corresponds with the expectations. It was shown in previous studies with older test persons (van der Meer 1985; Käpnick 1998) that mathematically gifted children structure sets of facts, which they are asked to memorize, after mathematical aspects already in the phase of acquiring information. Also in Dubrowina's follow-up studies to Krutetskiis's inquiries similar observations have been made. The mathematically gifted children, in contrast to the non-gifted children, did not only remember individual facts but were also able to reproduce their mathematical relations. Especially relevant for memorizing mathematical sets of facts was apparently their mathematical relation because if children forgot something, it was usually just a piece of information (Krutetskii 1976).

The tests presented in this article, however, also showed that not all and not only mathematically gifted second graders store the given information using structures. Especially simpler mathematical structures like the point reflection of the number arrangement in task 1 was also used by not mathematically gifted children (test group: 50.8%, reference group: 39.1%). Hence, significant differences between both groups existed above all in the type of recognized structures as well as when it came to the success of applying them.

Ability to construct and use mathematical structures

In various mathematic situations, the mathematically gifted second graders demonstrated that they were able to recognize, construct and use mathematic structures better than not mathematically gifted children of the same age group. Most of them identified arrangement structures in number patterns, such as in task 3 (e.g. the numbers increase by 2 diagonally down to the left), and proved this by stating the regularities in writing (or verbally). In tasks regarding the continuation of patterns (e.g. task 4), many gifted children recognized the underlying structure rules and they were usually able to use these rules for forming the subsequent component, i.e. for a near generalization (Stacey 1989). Especially with non-linear structures, however, not all of them succeeded in using the same rules for defining a component that is further away (far generalization, Stacey 1989), e.g. identifying the number of elements in the 9th figure. However, in contrast to most not mathematically gifted second graders, many mathematically gifted second graders were able to find suitable solution approaches.

Thus, 52.7% of the test group children as compared to 28.9% of the children in the reference group provided adequate solution strategies, 42.3% as compared to 14.5% implemented them appropriately and 28.6% as compared to 9% obtained the correct result for the 9th figure. The groups thus did not only differ concerning the proportion of those who identified an appropriate approach to solving the task, but also in the further implementation of the solution strategy. Not mathematically gifted children seemed to have more difficulties to adequately implement an identified solution strategy than mathematically gifted children. Further differences were found concerning the handling of inadequate solution strategies. A high proportion of both groups used linear solution strategies (Test group: 29.7%, reference group: 24.6%). However, while the non-gifted children mostly overgeneralized partial structures and repeatedly added a constant number (e.g. 20 + 8 + 8 + 8 + 8 + 8, because the difference between the 4th and 5th figure is 8), many mathematically gifted children identified correct structures, but implicitly assumed a proportional growth (e.g. “year 4 + year 5 = year 9”) for simplifying the calculation. In mathematics classes in Germany, students are rarely confronted with non-proportional situations until the end of the second school year. Thus, they are used to obtaining correct solutions by applying this strategy in similar tasks, which is why this “trick” might appear practicable. Hence, it is possible that students in the test group recognized useful structures, but did not test the compatibility of the proportionality assumption and the structure of the task.

A tendency of differences might be observed in the types of recognized and used structures. In the present study, this has been indicated particularly through applying graphic strategies for continuing figure patterns. Both groups were prone to mistakes in their use of graphic strategies. However, while the mistakes of mathematically gifted children mostly resulted from imprecisions in their drawings, many not mathematically gifted children drew quadratic or other arrangements which appear to result from a failure to accurately recognize or implement the figure sequence.

In this case it is possible that non-gifted second graders focus more on superficial structures, whereas gifted children of the same age group mainly tend to recognize and use deeper structures.

Furthermore, most mathematically gifted children considered, in contrast to most non-gifted children, mathematical structures as rules when constructing their own number arrangements, like for example in a self-invented number triangle (see Fig. 6.5).

To sum up, it can be said that the differences were very prominent between both groups when it came to recognizing and using mathematical structures in all tasks given. This is in line with the results of the previously mentioned comparative studies by Krutetskii (1976) and Käpnick (1998) with older test persons. The matching of “Abilities to construct and use mathematical structures”, as a characteristic of mathematical giftedness is further supported by this or a similar characteristic is mentioned in nearly all lists of characteristics. Also studies by Mulligan and Mitchelmore (2005, 2009) which deal with awareness of mathematical pattern and structure, though not explicitly with the topic of mathematical giftedness but also with mathematically gifted children as test persons, provide indicators that mathematically gifted children show a greater ability to structure and working with mathematical objects than non-gifted ones.

Ability to switch between modes of representation

The mathematically gifted second graders could easily relate different forms of representation in a task and used the differently represented information for solving a task. They seldom had difficulties with changes in representation/mode, which were necessarily required for naming the solution. Concerning tasks which were iconically represented, but allowed for both a graphic and computational solution (e.g. task 4), mathematically gifted second graders, in contrast to not mathematically gifted children of the same age group, sooner switched to solving the problem in the symbolic mode.

Thus, 28% of the children in the test group used adequate mathematical strategies to determine the number of elements in the 5th figure, even though the result can mostly be obtained with minimal effort by graphic continuation of the figure sequence so that switching modes is not necessarily required. In the reference group, corresponding computational solutions practically did not occur. Instead, the 4th figure was cognitively or graphically expanded, or a completely new drawing was created. This difference became even more pronounced in subtask b. The comparison of the participants with viable solution strategies (test group: 52.7%; reference group: 28.9%) showed that the majority (61.9%) of the test group children worked with viable solution strategies in the symbolic mode. By contrast, the majority (11 of 20 children) of the reference group children with appropriate solution strategies kept the iconic mode given in the task.

The studies by Käpnicks (1998) provided corresponding results of third and fourth graders. In a similar task on figurative numbers, the share of mathematical procedures in a task on far generalization of the test group was also significantly greater than in the reference group. This could be explained with special abilities that enable children to switch between modes of representation and a preference for symbolic representations that mathematically gifted children exhibit, which also Fuchs (2006) observed in her studies on problem-solving behavior. However, connections can also be assumed when it comes to recognizing and applying structures because mathematical structures make it much easier to solve tasks of this nature than a strictly graphic approach.

Ability to reverse lines of thought

The ability to reverse lines of thought was better developed with mathematically gifted second graders than with not mathematically gifted children of the same age group. This particularly manifested itself in tasks which require the recognition of a reverse situation and allow for the usage of two-way associations (like, e.g., in task 2).

In the present study, significant differences between the two groups could be observed: 60% of the test group versus 17% of the reference group could accurately implement two or three rules. A larger proportion of the test group (44%) than of the reference group (23.1%) proceeded by firstly orienting themselves towards the third rule, i.e. they used two-way associations. In addition, the results obtained by the test group children using this procedure were by 20% better than those achieved by the reference group children (average result in the test group: 57.1%, reference group: 37.5%).

Regarding the continuation of patterns, mathematically gifted children displayed a significantly better developed ability to reverse lines of thought with tasks posing reversed problems (cf., e.g., task 4c).

Thus, the study showed that most test group children applying successful solution strategies in the previous subtask managed to continue with these strategies in the task posing the reversed problem. In the reference group, this could only be observed to a lesser extent. As for faulty solution strategies, mathematically gifted children were also better able to continue these strategies than non-gifted ones. It might be possible that not mathematically gifted children often do not succeed at solving the tasks posing reversed problems because they cannot identify the parallels to the previous subtasks and thus consider the reversed tasks in isolation. The subtask posing reversed problems is thus perceived as independent task for which a new solution not relating to the other subtasks should be found. This also corresponds to the observations made by Krutetskii (1976).

The ability to reverse lines of thought at working backwards (no exemplary task is provided here) was better developed with mathematically gifted as compared to not mathematically gifted second graders, and many gifted children displayed initial approaches to working backwards. However, a correct reversal of operations and sequences of operations at a task with four steps was only possible for a few mathematically gifted second graders.

Even if this study clearly demonstrated that there were significant differences between the mathematically gifted and non-gifted children when it comes to reversing lines of thoughts, it must be said that also mathematically gifted second graders had difficulties with correctly reversing their line of thought. Quite a few mathematically gifted children found it difficult to find an approach to solving a task which includes a reversed question. This is in line with the results of Käpnick (1998), who observed third and fourth graders exhibiting this characteristic of giftedness to varying degrees. Whether it is appropriate to speak of different types of giftedness, of a quality feature that the reversion of line of thought illustrates or this characteristic may be developed at certain points in time can neither be answered conclusively on the basis of the study by Käpnick nor on the basis of the study at hand.

Ability to capture complex structures and work with them

Mathematically gifted second graders were more successful at simultaneously considering several relevant details in mathematical situations than non-gifted children of the same age group. In task (2b), the children of the test group demonstrated that they were able to complete Tim’s figure in a way which corresponded to considerably more rules than the children of the reference group did. Thus, 60% of the test group, as opposed to 17% of the reference group, managed to complete the figure so that two or three rules were correctly implemented.

This potential characteristic of giftedness has not been observed systematically by previous studies. However, indicators in terms of its validity can be found in, e.g. Nolte and Kießwetter (1996) who concluded from their investigations in the problem field of “Folding Task” that mathematical giftedness can be detected by “equal consideration of the details with regard to complex problems” (p. 156, Translation by the author).

Altogether, the study presented in this paper implies that giftedness characteristics which have been developed for third and fourth graders can already be observed in second graders. It could therefore be confirmed that Käpnick's results also apply to younger children. Moreover, "Ability to capture complex structures and work with them" can be added as one further talent characteristic. However, it can be assumed that this is not an age-specific capability, but rather a characteristic which is also found in older pupils. Furthermore, this study confirms that abilities like reversing lines of thought are not always developed in an equally strong way.

6.6 Further Investigations

The study at hand was extended by further investigations. This was done to ensure, enrich, and specify previous results. For this purpose extra classes were given to selected mathematically gifted second graders and also in an ordinary primary school class (second grade).

60 participants (52 males, 8 females) of the test group of the comparative study described above were selected to take part in a fostering project at university bi-weekly. In this project children engaged in challenging mathematical problems mostly independent of further instructions and partially on their own. Additionally, the learning material from the extra classes was used under similar conditions in the second grade of a primary school, including 22 children (8 males, 14 females) that have participated as a comparative class in the study presented above. In both groups, progress protocols of the participating children were compiled for every lesson which listed essential observations regarding the child's solution procedures and solutions as well as other aspects, such as motivation, cooperation with other children, etc. The interpretation of the protocols was enriched by the written notes of the children.

The investigations brought forward many interesting results which cannot all be depicted here. The following results are of greatest significance:

The assumption gained from the main study that mathematically gifted second graders focus more on deep structures while those deemed not mathematically gifted pay rather attention to superficial structures has been concretized. This can be seen in the task below:

Additionally the children worked on tasks, such as these:

- Write the numbers of series 5 above.
- How many numbers are in series 7 (15)?
- Which number are in series 8 (9) in 13th place?

In this task, different structures can be recognized. On the one hand, there are structures which describe the order of numbers. This includes, e.g. that the numbers rise starting from 1 to the middle number (which corresponds with the number of

Series 1				1			
Series 2			1	2	1		
Series 3		1	2	3	2	1	
Series 4	1	2	3	4	3	2	1
Series 5	___	___	___	___	___	___	___

Fig. 6.7 Number arrangement

the series) from left to right and then fall again. When these structures are recognized the tasks above can be answered correctly by completing the following series mentally or in written form. It is not possible to find a way to solve this task apart from the order of numbers. These structures are described as superficial structures in this example. On the other hand, there are deep structures which when applied open the opportunity of different mathematical approaches. This way, for example, the amount of elements in each series can be described as an order of odd numbers whose following elements can be either calculated recursively by the continuous addition of 2 or explicitly, e.g. via (number of series) + (number of series − 1).

The investigations showed that the children of the ordinary school class exclusively worked on the tasks with the help of writing down and counting the numbers. In contrast to this, in the gifted group nearly half of the children made use of structural characteristics which allowed for a mathematical solution of the task. Conclusively it can be said that the type of structures that were used differed considerably between the two groups.

Furthermore, the investigations provide clues concerning the correlation between abilities to construct and use mathematical structures and abilities to reverse lines of thought. Already the main study showed that tasks with reversed questions can be better solved when the correct solving of the problem or the recognition of sensible structures in preceding parts of the task was accomplished. Further investigations enrich these results as above all the use of relations and structures which allow for a shortening of the solution process have proven beneficial. This can be seen, e.g. when looking at the reversed question of the task in Fig. 6.7. (In which series are there 39 numbers?) Here primarily children were successful who solved the preceding tasks mathematically by applying deep structures. Even more clearly this relation can be seen in the following task (Fuchs and Käpnick 2004):

A dog chases a rabbit. The rabbit has a head start of 22 ft. But he can only do 6-ft-long jumps, the dog's jumps, however, are 8-ft-long.

(a) After how many jumps has the dog caught up with the rabbit?
(b) How many feet of a head start would the rabbit have had at the beginning, if the dog had caught up with the rabbit after 15 jumps?

Task (a) can be solved notably more efficiently if the difference between the jump length of the rabbit and that of the dog is used. Since the dog catches up by 2 ft every time he jumps, the result can be determined by calculating how often the jump difference of 2 ft appears in the head start of the rabbit. Additionally, there are also (here not described) other approaches to the solution which generate the correct answer but don't make use of the jump difference.

In the investigations, only mathematically gifted children managed to solve this task. It was salient that the correct results of task (b) were exclusively determined by approaches based on the abridging structuring via the jump difference of 2. The recognition that the jump difference is the length that the dog catches up every time he jumps, seems, therefore, a significant parameter for the solution of task (b) and this way, also for reversing the line of thought in this task. Possibly, reversing the line of thought is achievable in reversed questions especially when mathematical situations and solving processes are reduced in their complexity by building structures so that reversed processes do not demand high cognitive efforts and are easily accessible.

In addition to the theoretically developed characteristics of giftedness, already the main study exposed that there could be a particular age-specific difference between gifted and non-gifted second graders when it comes to understanding and using relational concepts and connections. What was shown inductively by the data was more closely investigated in the follow-up investigations. They exhibited strong differences:

- when it comes to using the term half: In contrast to non-gifted children, the gifted ones showed that they knew what the relativity of the term half entails. Great differences between both groups occurred in that gifted children as opposed to the non-gifted knew about the term half. Significant difference between both groups appeared when the children had to work with 'half' repeatedly. Non-gifted children often gave independently from the initial number the identical number.
- when it comes to understanding and applying relational terms which describe differences between numbers/amounts: Understanding and dealing with information, such as increase by 5, 1 year older did not pose a problem to the mathematically gifted children. Tasks that included several common relational information, the gifted children, in contrast to the non-gifted ones, were able to order the information in accordance with their relations without further problems.

This component is closely connected to dealing with structures. However, I see here essential new aspects which could be eligible for describing the difference between mathematically gifted and non-gifted children at primary school level. The understanding of relational terms and their successful application to mathematical situations was for every child an intended learning objective. But it is assumed that building an understanding for relational numbers is only completed towards the end of the full development of the number concept (Fritz and Ricken 2008; Krajewski 2008) so that some children are possibly missing the prerequisite for understanding relational terms at an early school age. Also the application of relational concepts in context is regarded as especially difficult (Stern 1998). It is possible that insight is especially gained in the above mentioned differences between mathematically gifted and non-gifted children with regard to their developmental stages. The differences might also be explained and traced back to higher-developed cognitive abilities coupled with a faster understanding of mathematical concepts and also a head start of mathematically gifted children as they are usually more experienced when it comes to engagement in mathematical content. It could be assumed that the observed differences decrease with age as children develop a growing understanding of relational concepts the older they get. This way this potential indicator of giftedness could be especially dependent on development stages and might, therefore, be age- or class-specific which results in an exceptional status of this indicator among the characteristics of giftedness.

Furthermore, these investigations showed that several mathematically gifted second graders display creativity in dealing with mathematics. Moreover, some children showed special competences in constructing and using mathematical analogies (cf. Assmus and Förster 2015).

All in all, it can be concluded that the results of the investigations did not contradict in any case the results of the main study. All investigated characteristics displayed great differences between the two groups during the extra classes. However, it needs to be said that the number of the participants, especially in the comparative group, was rather small so that the results are not generalizable. Also, during those times that children worked independently, entirely controlled conditions were not realizable as participants sometimes worked together or communicated about their tasks. The latter can be simultaneously regarded as an advantage and a disadvantage. It is disadvantageous because it was not always possibly to evaluate whether the children could have achieved the same results on their own. This way, quantitative statements can only be made when tasks were solved individually (as for example in the dog-rabbit-task). Of advantage is, however, that the extra classes seem to resemble quite closely the pedagogical work done in school so that other facets can be explored than in a standardized testing situation. This way, the further investigations provide beneficial insights and additions to the main study.

6.7 Summary

The results of the conducted study suggest that the cognitive abilities of mathematically gifted and non-gifted second graders differ in the examined areas. Therefore, it is reasonable to assume that the following abilities represent characteristics of mathematical giftedness in early primary school children.

- Ability to memorize mathematical issues by drawing on identified structures,
- Ability to construct and use mathematical structures,

- Ability to switch between modes of representation,
- Ability to reverse lines of thought,
- Ability to capture complex structures and work with them,
- Understanding of relational concepts and ability to use relational concepts and connections,
- Ability to construct and use mathematical analogies,
- Mathematical creativity.

However, the final two characteristics have not been verified in comparative studies under controlled conditions.

Finally, it is important to mention that many of these abilities are not restricted exclusively to the mathematically gifted, but that they are, up to a certain level, developed by all pupils. Thus, the differences between gifted and non-gifted children occur only when solving comprehensive and challenging mathematical tasks (Nolte 2004). However, merely displaying a high level of the mentioned cognitive capabilities is usually not enough for outstanding mathematical achievements. Generally, this requires further favorable personality characteristics, such as an extraordinary interest for mathematics as well as concentration and endurance. A supportive environment can also have a favorable effect here. With this in mind, the construct of mathematical giftedness is not reducible to cognitive factors.

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Chapter 7
The Cognitive Demand of a Gifted Student's Answers to Geometric Pattern Problems

Analysis of Key Moments in a Pre-algebra Teaching Sequence

Angel Gutierrez, Clara Benedicto, Adela Jaime and Eva Arbona

Abstract Mathematically gifted students require specific teaching methodologies to foster their interest in mathematics and their engagement in solving problems. Geometric pattern problems are an interesting context in which to introduce algebra to those students. We present the case of a nine-year-old student engaged in a teaching unit based on geometric pattern problems that was aimed at helping him start learning algebra, equations, and algebra word problems. To analyze and assess the cognitive effort the student made to solve the problems, we used a particularization to this context of the cognitive demand model. We analyzed answers typical of the different kinds of problems posed throughout the teaching unit, showing the student's learning trajectory and related characteristics of mathematical giftedness.

Keywords Geometric pattern problems · Levels of cognitive demand
Linear equations · Mathematical giftedness · Pre-algebra · Primary school

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7.1 Introduction

Mathematically gifted students (hereafter referred to as "gifted students") tend to show unusual paths of reasoning and methods of solving problems. Authors such as Freiman (2006), Greenes (1981), Krutetskii (1976), and Miller (1990) have suggested a number of characteristics of gifted students related to aspects of their mathematical or social activities. Some characteristics are quite general, such as good memory or the enthusiasm for mathematics, while others are more specific, such as the abilities to identify patterns and relationships among different elements, generalize and transfer mathematical ideas or knowledge from one context to another, or invert mental procedures of mathematical reasoning. These abilities are especially useful in particular contexts, such as the one we are dealing with in this chapter: the use of geometric pattern problems to introduce gifted students to algebraic language and equations. Other traits of gifted students are that they need much less time than average students to solve problems (Budak 2012) and require challenging problems to maintain their interest during the mathematics classes (Kennard 2001). This raises, for teachers and researchers, the issue of finding criteria to determine problems' appropriateness for certain specific gifted students.

Research also shows that gifted students understand and learn mathematical concepts quite quickly, so it may be useful to teach them some advanced topics that open a door for them into new kinds of challenging problems (Cai and Knuth 2011; Diezmann and Watters 2002; Kennard 2001). For gifted students in upper primary grades, one such topic is elementary algebra, since it provides them with new tools to solve problems when arithmetic is not sufficient. Different researchers have designed curricular variations and teaching units to introduce elementary algebra to ordinary groups, including gifted students (Gavin et al. 2009).

A successful methodology to initiate students into elementary algebra is posing *geometric pattern problems* (Cai and Knuth 2011; Rivera 2013). What we are here calling geometric patterns have also been called visual patterns, pictorial patterns, growing patterns, or just patterns by other authors. Typical geometric pattern problems include questions asking students for the value of the term in a position of the sequence (direct questions). They may also pose questions where the value of a term in the sequence is given and students are asked to calculate the position of that term in the sequence (inverse questions).

The literature has reported many teaching experiments involving direct questions given to students of different ages, from kindergarten (Papic et al. 2011) and early primary (Radford 2011; Rivera 2010; Warren 2005) to lower secondary (Warren et al. 2016). Research has shown a variety of strategies used by students (Rivera 2010) and different focuses of attention to analyze this context (Cai and Knuth 2011), with some research focusing on gifted students (Amit and Neria 2008; Benedicto et al. 2015; Fritzlar and Karpinski-Siebold 2012).

Rivera and Becker (2005) identified two methods to analyze these patterns: numerical and figural. García-Reche et al. (2015) described several strategies to calculate the value of a term of the sequence, labeled: counting, recursive, functional, and proportional. Radford (2006) described several types of generalization, ranging from naïve trial and error to the sophisticated symbolic algebraic generalization.

There are very few publications about inverse questions in geometric pattern problems. Rivera (2013) reported an experiment where Grade 2 students solved pattern problems represented by means of manipulatives and drawings. The students found the inverse tasks very difficult, with very few students solving them meaningfully. According to Rivera, a source of difficulty was the language, since children confused data and result values. Warren (2005) conducted an experiment where Grade 4 students solved several geometric pattern problems including *reversing the thinking* questions. This author reported that this type of questions was very difficult for most children, although she included neither examples nor descriptions of students' answering strategies in the paper.

The context of geometric pattern problems seems especially interesting in promoting access to pre-algebraic concepts in mixed-ability classrooms, since all students, regardless of their mathematical ability and previous knowledge, may explore geometric patterns and obtain some answers (Smith et al. 2007). This context has been shown to be particularly useful for gifted students, since they may advance faster and further than average students. However, there have only been a few publications reporting gifted students' behavior when solving these problems. Amit and Neria (2008) confirmed that generalization via pattern problems is an appropriate gateway to developing the algebraic skills of gifted students. Fritzlar and Karpinski-Siebold (2012) explored the algebraic abilities of primary school students aged 9–10 of varying performance levels, including gifted students. As expected, the more able students got the better results, although none were able to adequately answer questions about the nth term of sequences. Benedicto et al. (2015) asked gifted children in Grades 5, 6 (primary school), and 7 (secondary school) to calculate the values of the 5th, 20th, 100th and nth terms in a triangular numbers pattern. They found that some students in the sample were not able to obtain any kind of expression for the nth term, others verbalized a recursive expression (the nth term is obtained by adding the numbers from 1 to n), and the most talented were able to write an algebraic expression to calculate the nth term (the value of the nth term is $n + (n + 1)/2$). These results prove that some questions commonly asked in geometric pattern problems may be appropriate for some students but not for others, even among gifted students, so teachers need to evaluate the suitability of questions to students. Therefore, again, a reliable criterion to decide on which questions are appropriate for specific gifted students is needed.

Giving a meaning to equations and learning to solve linear equations is one of the possible objectives of posing students geometric pattern problems. When they start learning to solve linear equations, students are often asked to solve simple equations such as $3x + 4 = 19$. However, in the context of geometric pattern problems, students may also be faced with solving complex equations such as $2(x + 1) + 2(x + 2) = 86$. Filloy et al. (2008) suggested the existence of a didactical obstacle in between simple and complex linear equations that explains the difficulties students find in solving complex linear equations even when they solve simple equations easily.

There is increasing agreement among mathematics education researchers and teachers that a successful methodology to promote meaningful learning in all students, particularly in gifted students, is to pose them challenging tasks that promote high-level thinking (Silver and Mesa 2011). An issue in this context is to have at our disposal a theoretical tool to discriminate tasks that promote high-level thinking from those that do not promote it. The *cognitive demand* model may evaluate the intellectual effort required when students solve mathematics problems, so it helps decide on which problems are more appropriate to promote high-level thinking in different kinds of students. To assess the power of tasks to help develop students' mathematical thinking, Stein et al. (1996) analyzed a diversity of types of tasks, varying from ones requiring only recall from memory to others requiring complex and original use of mathematical knowledge. To allow teachers to select tasks with an appropriate level of challenge or demand for their pupils, Smith and Stein (1998) stated a set of criteria to classify mathematical tasks or problems into four *levels of cognitive demand* corresponding to different grades of cognitive effort required to solve them.

Most researchers determine the level of cognitive demand of a problem by analyzing the statement of the problem (Boston and Smith 2009; Wijaya et al. 2015), but this procedure does not acknowledge that a problem may be solved correctly in several ways that require from students different degrees of cognitive effort. Instead, we have adopted an original approach that uses the levels of cognitive demand to also make an analysis of students' answers to those problems. In this way, we can better understand their processes of reasoning and decide on the appropriateness of tasks (Benedicto et al. 2015). This way of using the cognitive demand model has proved in our research experiments to be a framework that reliably identifies problems appropriate to students with diverse mathematical capabilities, in particular to gifted students. It has also allowed us to analyze individual students' answers to different problems, providing information about the students' learning trajectories.

To use the levels of cognitive demand to analyze students' answers to geometric pattern problems, we have rephrased the generic characteristics of the levels to make them appropriate to the particularities of the geometric pattern problems posed and the answers to their questions.

The issues that emerged in this introduction are related to researchers' interest in knowing how gifted students solve problems and progress in learning more abstract and complex strategies and how to determine which questions are appropriate to promoting gifted students' high-level thinking while learning and understanding mathematics (pre-algebra in particular). The objective of this chapter is to offer some answers to these questions in the context of geometric pattern problems and pre-algebra, to gain knowledge about gifted students' behavior, and to evaluate the cognitive effort required to solve the problems posed to them. This information could help teachers prepare sets of problems tailored to the particular needs and expertise of their gifted pupils. The specific objectives of the research we present here are:

(i) To identify and analyze the solution strategies gifted students use to solve geometric pattern problems and the evolution of these strategies throughout the course of a teaching unit.
(ii) To analyze the relationships between types of geometric pattern problems and the cognitive demand required by gifted students' solution methods.
(iii) To analyze the relationships between the complexity of the generalizations made by gifted students and the cognitive demand required by their methods to answer the inverse relationship tasks.

To provide information on these objectives, we carried out a teaching experiment aimed at guiding a nine-year-old gifted student to the ultimate learning objective of being able to solve verbal algebraic problems based on linear equations. The intermediate objectives were to help the student: understand and learn the process of mathematical generalization, contextualized in geometric patterns; start managing the basic components of algebraic reasoning in order to learn to translate verbal descriptions of the general terms of sequences into algebraic expressions; and learn and understand a meaning of linear equations and the procedures for solving them.

7.2 Theoretical Framework

The description and analysis of the teaching experiment presented in the next sections is based on three elements that integrate our theoretical framework: the geometric pattern problems, which are the environment where the teaching experiment took place and the student's algebraic thinking arose; the cognitive demand model, which is the analytic tool used to interpret and categorize the different instances of cognitive effort made by the student participating in our experiment when solving the problems we posed him; and the characteristics of mathematical giftedness, since we have observed a gifted student's behavior to identify traits of giftedness present in his mathematical activity that explain the student's success in learning algebraic language and solving linear equations and algebraic word problems.

7.2.1 The Solution of Geometric Pattern Problems

Radford (2000) differentiated algebraic thinking from algebra; the former refers to the use, possibly intuitive, of basic algebraic concepts such as unknown, variable, and generalization (i.e., expressions of the general term of a sequence) without employing the algebraic symbolic system of signs, and the latter refers to the explicit use of the analytic ways of representing and managing the above mentioned concepts in contexts such as solution of equations. According to Radford (2010), algebraic thinking may adopt different forms depending on the kinds of tasks posed. Our research is situated in the context of algebraic thinking, so it is necessary to characterize the particularities of our teaching experiment.

Geometric pattern problems typically show, as data, a pictorial representation of the first terms of an increasing sequence of natural numbers (see some examples below), although some authors may present as data non-consecutive terms. The problems pose students *direct questions* (Amit and Neria 2008) about some terms of the sequence, usually asking them: to calculate the values V_n of *immediate*, *near*, and *far* terms (Stacey 1989); to *verbalize a general rule* valid for calculating any specific term; and to *write an algebraic expression* for such a rule [i.e., to write an algebraic expression in mathematical terms for the function $V_n = f(n)$]. Students may also be asked *inverse questions*, consisting of calculating specific cases of the *inverse relationship* (Rivera 2013), that is, to get the place n of a term given its value V_n (i.e., mathematically speaking, to solve the equation $f(n) = V_n$).

The pictorial representation of the sequence provides students with objects carrying numerical information and graphically exhibiting the algebraic relationship between the terms of the sequence, that students can identify and induce in different ways (Amit and Neria 2008; Rivera and Becker 2005). Rivera and Becker (2005) differentiated between *figural* and *numerical* procedures of using the information provided by the geometric patterns, depending on whether students use the graphical representation of the terms to answer the questions or only pay attention to the numerical values of those terms (i.e., the number of elements in the pictures of the terms), respectively.

According to García-Reche et al. (2015), students use several strategies to get the numeric answers to the direct questions:

Counting: Students reproduce the graphical pattern by drawing the requested term and count the number of elements in the new drawing to get the numeric answer. Students do not use any mathematical property of the sequence to get the answer.
Recursive: Students identify the graphical or numerical pattern of growth in the sequence, relating each term to the preceding one(s). They then calculate the terms one by one until they get the requested term. This strategy is helpful in calculating immediate and near terms, but it is too time consuming when calculating far terms and does not work to get the general term of the sequence.
Functional: Students identify a mathematical expression that allows them calculate the value of any specific term of the sequence. This expression can also be used as the general term of the sequence.
Proportional: Students use the ratio between the values of two specific terms to calculate the value of any other term of the sequence. For instance, if V_3 is 5 times V_1, then V_9 is 15 times V_1. This strategy usually produces wrong answers.

7.2.2 The Cognitive Demand Model

The cognitive demand model identifies four levels of cognitive effort required from students to solve mathematical problems based on the complexity of the reasoning the students used to produce the answers. The basic characteristics of the tasks typically associated with each level are (Smith and Stein 1998):

- *Memorization*: tasks asking students to reproduce facts, rules, formulas or definitions previously learned or information explicitly presented in the statement of the task.
- *Procedures without connections* to concepts or meaning: tasks focused on getting correct answers but not on connecting to the underlying contents, requiring students to perform in a routine manner an algorithmic process already learned.
- *Procedures with connections* to concepts and meaning: tasks focused on discovering the underlying contents and gaining mathematical understanding of them, requiring students to perform an algorithmic process that is not routine, since it presents some ambiguity on how to carry it out.
- *Doing mathematics*: tasks requiring complex and non-algorithmic thinking from students. Students have to understand the underlying mathematical contents and explore their relationships.

A more detailed and operational description of the levels of cognitive demand can be found in Smith and Stein (1998). As in the cognitive demand model, in this text the terms *algorithm* and *procedure* are equivalent, and they should be understood in a broad way, including the well-known algorithms for arithmetic calculations, solving equations, getting the derivative of a polynomial function, etc., and also any procedure to get a result by means of a purposeful sequence of steps, such as calculating the angles of a convex polygon with a protractor or drawing the element of a geometric pattern following the terms given.

7.2.3 Particularization of the Cognitive Demand Model to the Geometric Pattern Problems

The characterization of the levels of cognitive demand by Smith and Stein (1998) is good to give a general idea of their meaning and to show the main differences between them, but it is not sufficiently precise to evaluate geometric pattern problems or students' answers to these problems. To make operational and helpful use of the levels of cognitive demand to analyze specific students' answers, we needed to particularize the general descriptions of the levels to the specific context of the geometric pattern problems. To do this, we matched up the core characteristics of each level of cognitive demand to the specific characteristics of the questions posed in these problems and rephrased the characteristics to include aspects of geometric pattern problems. We present below a synthetic analysis of each type of task included in the geometric pattern problems used in our teaching unit and then a table with the detailed characteristics of each level of cognitive demand in this context. A thorough description and validation of the process of transformation of the initial characteristics of the levels into the specific characteristics can be read in Benedicto et al. (2017).

The ordinary procedures of calculating immediate and near terms of geometric patterns require only continuing the numeric or geometric structure shown by the terms given in the statement of the problem, either by drawing the requested term and counting its elements or by recursively determining its value (for instance, by adding 3 again and again). To do this, students do not need to be aware of the algebraic relationship underlying the pattern (i.e., the general term of the sequence) and only need to make a limited cognitive effort to do such calculations, which corresponds to the procedures without connections level of cognitive demand.

Far terms cannot be calculated without finding an algebraic relationship underlying the sequence and using it. To solve these tasks, typical students analyze previous information (terms in the statement of the problem and immediate and near terms already calculated) to connect relevant data from them and get a relationship or rule that can be used to calculate the value of any other specific term. This is not a routine procedure, since it requires using underlying algebraic relationships and has to be carefully applied in different ways to different patterns. Therefore, the cognitive effort necessary to calculate far terms corresponds to the procedure with a connections level of cognitive demand.

To verbally or algebraically express a general rule for calculation of terms, students need to find an expression for the algebraic relationship underlying the sequence (i.e., its general term) and be able to abstract that relationship in order to express it without the support of specific terms. There is not an algorithmic guide that can help primary school students to solve this kind of task, so they need complex non-algorithmic thinking to analyze the task, extract useful information, and make appropriate use of it. All this activity requires originality and a considerable cognitive effort from students, which corresponds to the doing mathematics level of cognitive demand.

To calculate inverse relationships, typical students' answers are based on the rules of generalization obtained in direct questions. The rules may have a simple algebraic structure (for instance, $V_n = 3n + 2$) or a complex one (for instance, $V_n = 3n + 2(n + 1) + 1$), so the levels of cognitive demand required to solve the inverse tasks vary depending on the complexity of the generalization made. When the rule of generalization has a simple structure, students simply need to make arithmetic calculations in the appropriate order, determined by an easy algorithm, so students have to make limited cognitive effort, which corresponds to the procedures without connections level.

When the rule of generalization has a complex structure, students need to explicitly state and solve an equation to get the answer. When these tasks are posed before students know how to solve equations, they can only find the solution by making a (sometimes carefully organized) trial-and-error checking of possible values. This process of solution is quite straightforward and it does not require understanding of the algebraic structure of the sequence, so it only requires limited cognitive effort, which corresponds to the procedures without connections level.

When students have learned to solve equations, they may state and solve an appropriate equation derived from the rule of generalization. Stating a correct equation cannot be made without understanding the algebraic relationship underlying the pattern, so it requires a quite high cognitive effort, which corresponds to the procedures with connections level.

Table 7.1 presents the characterization of the levels of cognitive demand particularized to geometric pattern problems (Benedicto et al. 2017) that we have used to analyze the student's outcomes presented in next sections. Table 7.1 does not include the level of memorization because it is not used in the analysis made in this chapter.

Table 7.1 Characterization of the cognitive demand of answers to geometric pattern problems

Levels of Cogn. Dem.	Categories	Characteristics of the task
Procedures without connections	Process of solution	• Is algorithmic. The procedure consists of following the pattern shown in the statement to calculate (either recursively or functionally) immediate or near terms, either graphically by drawing the terms and counting their elements, or arithmetically by calculating the number of elements of the terms. However, the students do not understand the underlying algebraic structure of the sequence. The calculation of the inverse relationship is based on a learned sequence of basic arithmetic operations or on checking possible answers by trial and error
	Objective	• Focus students' attention on producing a correct answer (the number of elements in an immediate or near term), but not on developing understanding of the algebraic structure of the sequence
	Cognitive effort	• Solving it correctly requires limited cognitive effort. Little ambiguity exists about what has to be done and how to do it, because the statement clearly shows how to continue the sequence
	Implicit content	• There is implicit connection between the underlying structure of the sequence and the procedure used. However, students do not need to be aware of such connection since they may answer the question by drawing terms and counting their items
	Explanations	• Requires explanations that focus only on describing the procedure used. It is not necessary to identify the relationship between the answer and the term
	Representation of solution	• A geometric representation is used to get the number of elements and an arithmetic one to write the result. Students use the representations without establishing connections between them or with the algebraic structure of the sequence
Procedures with connections	Process of solution	• The data or the answers to previous tasks suggest general functional procedures that are connected to the underlying algebraic structure. The students understand the algebraic structure of the sequence and they use it to solve the task, but

(continued)

Table 7.1 (continued)

Levels of Cogn. Dem.	Categories	Characteristics of the task
		they are not able to obtain a general algebraic expression. The calculation of the inverse relationship is based on solving the equation of the general procedure previously obtained
	Objective	• It directs students' attention to the use of general procedures aiming to deepen their understanding of the underlying algebraic structure of the sequence
	Cognitive effort	• Solving it correctly requires rather considerable cognitive effort. Students may use a general procedure, but they need to have some understanding of the algebraic structure of the pattern
	Implicit content	• To solve the task, students need to explicitly consider the algebraic relationship between any term and its value underlying the correct procedures of solution
	Explanations	• Requires explanations referring to the general algebraic relationship between the terms and their values, based on using specific cases (particular terms of the sequence)
	Representation of solution	• The solution connects several representations. Geometric, arithmetic, and algebraic representations may be used, and students use those which help them to make an abstract reasoning
Doing mathematics	Process of solution	• Requires complex and non-algorithmic thinking. The statement does not suggest any way to get the general term of the sequence. Students have to understand and analyze the algebraic structure of the sequence to get a general algebraic expression that lets them obtain any term of the sequence
	Objective	• Requires students to analyze the solutions to previous tasks and possible limitations to get an algebraic expression of the general term of the sequence
	Cognitive effort	• Requires considerable cognitive effort, since it is necessary to use abstract reasoning to determine how to algebraically represent the general term
	Implicit content	• Requires that students access relevant knowledge and previous experiences (immediate, near, and far terms) and make appropriate use of them in working through the task to get an algebraic expression of the general term
	Explanations	• Explanations consist of the proof of the algebraic expression of the general term
	Representation of solution	• The solution is based on an algebraic representation, which may be connected to geometric and/or arithmetic representations

7.2.4 Characteristics of Mathematically Gifted Students

As mentioned in Sect. 7.1, researchers have determined quite many of the characteristics of gifted students' behavior. A few of these characteristics are pertinent to the differential particularities of geometric pattern problems (Greenes 1981; Krutetskii 1976; Miller 1990):

- An unusual quickness in learning, understanding, and applying mathematical ideas: Gifted students understand and learn very quickly. They usually need only a few explanations; they even grasp new ideas before the teacher has finished explaining them.
- Ability to see mathematical patterns and relationships, sometimes in original ways: Gifted students have a high capability for identifying regularities and complex structures in patterns, extracting them from empirical contexts, and characterizing them in general terms.
- Ability to generalize and transfer mathematical ideas to another context: Gifted students are able to detect general relationships when observing specific cases and are able to extract the relationships they have identified in specific contexts and formulate them in general terms.
- Ability to invert mental procedures of mathematical reasoning: Gifted students are able to manage unidirectional relationships in ways that allow them to see whether they can be inverted and, when possible, the direct or the inverse relationship can be used. They are then able to create new procedures by inverting the steps in known procedures.
- Flexibility to change: Gifted students are able to quickly move from one problem-solving strategy to another if they believe that the new one will be more useful or easier.
- Development of efficient strategies and abbreviation of problem-solving processes: Because gifted students tend to see better procedures of solving a type of problem, they are able to more efficiently solve other problems of the same type.

We present in this chapter a research experiment where a nine-year-old gifted student solved a teaching sequence based on geometric pattern problems aimed to teach him to generalize and to solve verbal algebraic problems based on linear equations. The analysis of the student's behavior presented in next sections will show that he had the above mentioned traits of mathematical giftedness and he put them to work when solving the different kinds of problems.

7.3 The Research Methodology

This research is based on an experimental case study, analyzed qualitatively to provide answers to the specific objectives stated in the first section. We present and analyze data from a teaching experiment with a gifted student who solved a sequence of problems aimed at guiding him in understanding and learning the basic algebraic concepts necessary to solve verbal problems based on linear equations. Most problems in the sequence were geometric pattern problems.

7.3.1 Sample and Experimental Setting

The subject participating in this study was Juan (a pseudonym), a 9 year old who had been identified as gifted student after having been administered the standard identification procedure used by the educational authority. Furthermore, Juan had proved to have a very high mathematical talent during his participation in several mathematics workshops conducted by the authors over several years. In the Spanish educational system, 9-year-old children are typically in primary school Grade 3, but Juan had been accelerated one grade. We invited Juan to participate in our study because he had showed a high interest for mathematical problem solving and he was willing to learn more complex mathematics.

The teaching experiment was an out-of-school workshop having the format of clinical interviews, which were conducted by the fourth author. It started in August during the summer holidays after Juan had finished Grade 4 and ended in December of the same year, when Juan was studying Grade 5. It was not possible for the researcher-teacher and the student to meet each other, so the sessions were conducted by means of videoconferences using Skype, which were audio- and video-recorded using screen-capture software.

As the first action to start working on a problem, the teacher posted a document for Juan with the statement of the problem. Juan could draw or write in a notebook, but he had to answer and give explanations verbally, except when he had to write algebraic expressions. The teacher asked Juan to explain his answers whenever he did not do it spontaneously. When Juan started writing algebraic expressions, he used a word processor and shared his screen with the teacher so she could read what Juan was writing. The information written in the notebook was not relevant for our analysis because it mostly consisted of calculations that Juan described verbally to the teacher when necessary.

7.3.2 The Teaching Unit

The experimental teaching unit consisted of a sequence of problems divided into three parts. To easily identify the problems in each part of the teaching unit in this text, we have labeled them as Problems 1.*n*, 2.*n*, and 3.*n*. Some problems were taken from the literature and the others were created by the authors to fit specific requirements for methods of solution or structure of the general relationships of their sequences.

The first part had the objectives of (i) teaching Juan to identify and verbally express generalizations of the relationships underlying the sequences represented by the patterns and handle those generalizations, and (ii) helping Juan start implicitly using variables and unknowns while solving the problems. Over nine sessions, Juan solved 20 geometric pattern problems ordered according to their difficulty. All problems had the same structure and included the same tasks (Fig. 7.1):

Marc and his friend want to make sets of houses with sticks as follows:

1 house 2 houses 3 houses

a) How many sticks will they need to make 6 houses? How did you know?

b) How many sticks will they need to make 11 houses? How did you know?

c) Can you tell me a way to calculate how many sticks will they need to make 44 houses? How did you know?

d) If they have 51 sticks, how many houses can they make? Explain how you got the answer.

Fig. 7.1 A typical geometric pattern problem (1.8) from the first part of the teaching unit

Three direct questions (a–c) asking for calculation of the values of an immediate, a near, and a far term in the sequence, and an inverse question (d) asking for the term in the sequence having a given value.

From the very beginning, Juan correctly solved most problems, showing a high ability to generalize from the data presented in the problems, which is one of the previously mentioned characteristics of mathematical giftedness.

The second part of the teaching unit aimed at (i) introducing Juan to the use of algebraic symbols (letters, equal sign, parentheses, etc.) to algebraically represent his generalizations and (ii) introducing Juan to the solution of linear equations contextualized by an applet representing a balance. Over three sessions, Juan solved six geometric pattern problems different from the ones in the first part. All problems had the same structure and included the same tasks (Fig. 7.2): A direct question (a) asking for calculation of a near term of the sequence, a question (b) asking a written algebraic representation of the calculations made in question a, and an inverse question (c) aimed to be solved using an equation. The aim of these problems was no longer to teach Juan to get generalizations, so we removed the

At the school we have learned how to build wooden cabinets having as many shelves as you like. We used pieces of wood to make the cabinets in this way:

1 shelf 2 shelves 3 shelves

a) How many pieces of wood do we need to make a cabinet with 13 shelves? How did you know?

b) Write down the formula you used in the previous question.

c) If we have 98 pieces of wood, how many shelves can the cabinet have? How did you know?

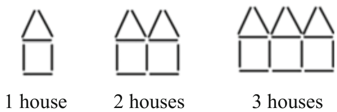
Fig. 7.2 A typical geometric pattern problem (2.5) from the second part of the teaching unit unnecessary questions and included new questions focusing on the algebraic representation of the general relationship.

To answer question b, Juan learned to write algebraic expressions representing the calculations made in question a. Next, to answer question c, Juan learned to write an equation by using his answer to question b and the data in question c. In the first problem of this part, Juan was introduced to the meaning of equation as equilibrium using an applet (NLVM 2016) showing a balance that allows representation and solution of linear equations by adding pieces to or removing them from the balance beams. When there is not equilibrium, the balance swings down. Juan quickly learned to write algebraic expressions and solve linear equations with the help of the balance model, showing one of the traits of mathematical giftedness.

The third part of the teaching unit aimed at (i) teaching Juan to transform algebraic expressions and simplify complex linear equations, (ii) teaching Juan solve algebraic word problems and gain practice in solving linear equations, and (iii) showing Juan the usefulness of equations in solving a diversity of mathematical problems. Over two sessions, Juan solved seven geometric pattern problems, including shortened versions of six problems from the first part of the teaching unit in which Juan was not able to correctly solve the inverse question or he solved it by trial and error. He also solved four linear equations to gain practice and six algebraic word problems. All geometric pattern problems had the same structure and included the same tasks (Fig. 7.3): A question (a) asking for a written algebraic representation of the general term of the pattern, a question (b) asking about the possibility of shortening the algebraic expression produced in a, and an inverse question (c) asking for a transformation of the expression in b into an equation and its solution.

In objective (iii) of this part of the teaching unit, the geometric pattern problems used were the ones from the first part of the teaching unit whose inverse questions Juan had found very difficult to solve and on which he had had to use trial and error (Table 7.3). This showed Juan that having learned to state and solve equations allowed him to solve these problems easily.

Marc and his friend want to make sets of houses with sticks as follows:



a) Write down an algebraic formula to calculate the number of sticks necessary to make any number of houses.

b) Do you believe that it is possible to get a simpler formula? If so, write it down.

c) If they have 96 sticks, how many houses can they make? Explain how you got the answer.

Fig. 7.3 A typical geometric pattern problem (3.5) from the third part of the teaching unit, related to problem 1.8 (Fig. 7.1)

7.3.3 Source and Analysis of Data

In this chapter we analyze Juan's answers to the geometric pattern problems he solved in the three parts of the teaching unit, but we do not take into consideration the word problems or the linear equations he solved in the third part. We make a multi-faceted analysis of those answers based on the three theoretical constructs described in the second section of the chapter. The main analysis is based on matching the answers to the characteristics of the levels of cognitive demand in Table 7.1, in order to get information about the cognitive effort required to solve the problems and look for a learning trajectory throughout the course of the teaching unit. However, we have also analyzed Juan's answers in relation to his procedures of using the graphic data of the problems (numerical and figural; Rivera and Becker 2005), the procedures he used to calculate specific terms of the sequence (counting, recursive, functional, and proportional; García-Reche et al. 2015), and the appearance of traits of mathematical giftedness. This multi-faceted analysis provides a more complete picture of this student's behavior and helps in understanding why he made more or less cognitive effort in the solution of different problems.

7.4 Analysis of the Cognitive Demand of Student's Solutions

In this section, we present examples of the various types of strategies Juan used to solve the geometric pattern problems and, based on Table 7.1, analyze the levels of cognitive demand necessary to produce those answers. Due to the differences between strategies of solution of direct and inverse questions, we present them in two different sub-sections.

7.4.1 Analysis of the Cognitive Demand of Solutions to Direct Questions

Two relevant aspects of the strategies of solution of direct questions in geometric pattern problems are the ways of using the graphical information and the procedures used to calculate the values of specific terms and obtain the general relationships needed to calculate any term. We are considering Rivera and Becker's (2005) figural and numerical procedures of use of the pictorial information. Juan used one or the other depending on the complexity of the geometric pattern of each problem. We are also considering the counting, recursive, functional, and proportional types of calculations described by García-Reche et al. (2015). Examples 1–4 present a diversity of answers corresponding to the different types, showing that Juan made calculations of the recursive and functional types, but he never used the counting and proportional types.

My mother has bought a strange plant. It grows during the night and, when we get up in the morning, we see new leaves:

a) How many leaves will the plant have on day 5? How did you know?

b) How many leaves will the plant have on day 13? How did you know?

c) How many leaves will the plant have on day 65? How did you know?

Fig. 7.4 Problem 1.7

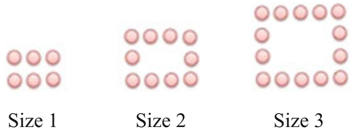
Example 1: *Recursive numerical counting* from the pattern and a cognitive demand in the *procedures without connections level*. In a few problems, Juan identified the constant difference between the total number of objects in two consecutive given terms and recursively used this difference to get the answers for the immediate or even the near terms. In problem 1.7 (Fig. 7.4), in order to calculate the number of leaves the plant has after 5 and 13 days, Juan made recursive calculations:

Juan: On the fifth day it has 13.
Teacher: Very good. How do you know?
Juan: I was adding 3.
Teacher: And what about day 13?
Juan: 38 [the correct answer is 37, but he made a mistake in the calculations].
Teacher: It is 37. How did you calculate this?
Juan: Plus 3, plus 3, plus 3.

The geometric pattern clearly suggests that each day the plant has three new leaves, so the geometric pattern induced Juan to use a recursive strategy to solve the immediate and near terms. However, when he had to answer question c, Juan moved to a functional strategy. The recursive strategy is algorithmic, since it consists of adding 3 a number of times and requires a very low cognitive effort since following it does not require awareness of the algebraic relationship underlying the sequence. The objective of this solution method was producing a correct result but not gaining understanding of the structure of the sequence. Juan's answers to a and b were typical of the procedures without connections level.

Example 2: *Functional numerical counting* from the pattern and cognitive demand in the *procedures without connections level*. In some problems, Juan counted the total number of objects in each given term and worked with the numeric sequence without paying attention to the graphical information of the pattern. In problem 1.5, in order to calculate the number of tiles around a pool of size 5 (Fig. 7.5), Juan made a guess about a general functional relationship:

We want to build a swimming pool with tiles around it. We want to know how many tiles will be needed for pools with different sizes:



a) How many tiles do we need for the pool of size 5? How did you know?

Fig. 7.5 Problem 1.5

Juan: We need 22 tiles.
Teacher: Why?
Juan: I discovered that it [the number of tiles] is 4 times the size of the pool plus 2.
Teacher: How did you discover this?
Juan: Looking at the examples. The first pool has 6 [tiles], and $4 \times 1 + 2 = 6$. The second has 10, and $4 \times 2 + 2 = 10$. And the third has 14, and $4 \times 3 + 2 = 14$.

Juan either was not able to find an adequate decomposition of the geometric pattern or directly got the numeric values of the given terms and used trial and error to look for an arithmetic way to relate the value of each term to its position. Despite the result he obtained, consisting of finding a general formula, he did not show an awareness of the underlying algebraic structure of the sequence, since he described the procedure used to solve the task but was not able to give a reason for it. The strategy of the solution chosen may have required rather cognitive effort to produce the general relationship, but it did not require understanding the structure of the sequence. The objective of this solution was focused only on producing a correct result. This solution then corresponds to the procedures without connections level.

Example 3: *Functional figural decomposition* of the pattern with a cognitive demand in the *procedures with connections level*. Most geometric patterns show procedures to split the figures into parts that can be considered like independent patterns, making it easy to find a general procedure to calculate the values of the terms in the sequence. This is Juan's explanation of his method of calculating the number of chairs around the tables in question a of problem 1.12 (Fig. 7.6):

Juan: I think this is correct. I take 3, or whatever number, times 3 and add 2.
Teacher: Fine. How did you get it [this procedure]?
Juan: For one table, [I added] the numbers [of chairs] above and below, 1 and 2, and $1 \times 3 = 3$. For two tables, 3 above and 3 below, $2 \times 3 = 6$ and $3 + 3 = 6$. And we still have to add those two [chairs on the sides]. For three tables, $5 + 4 = 9$ and $3 \times 3 = 9$.

My parents are organizing a family party. There are many people in my family, and my parents do not know how many will come. They have to decide how many tables and chairs they will need:

a) How many chairs will there be around 6 tables? How did you know?

d) If there will be 50 guests, how many tables will they need? How did you know?

Fig. 7.6 Problem 1.12

To answer question a, Juan divided the chairs into three sets (top, bottom, and sides), and this helped him elaborate a general procedure to calculate the number of chairs for any number of tables. This answer offers a functional relationship to calculate the number of chairs based on the algebraic structure of the sequence represented by the geometric pattern. The geometric pattern can be easily split, which helps in understanding the algebraic structure of the sequence. This allowed Juan to understand the algebraic structure of the pattern and produce a generalized functional relationship. The strategy of the solution chosen required rather considerable cognitive effort to produce the general relationship. Therefore, this answer is in the procedures with connections level. This transcript is also an example of the functional strategy of solution of geometric pattern problems that Juan used in most problems.

Example 4: *Functional figural decomposition* of the pattern and cognitive demand in the *doing mathematics level*. Question b in the problems in the second and third parts of the teaching unit (Fig. 7.2) asked explicitly for an algebraic expression of the general rule of calculation of terms that should have been derived from question a. This new question meant an increase in Juan's cognitive effort while solving the problems, particularly for those problems whose pattern was more complex. We present Juan's answer to questions a and b of problem 2.5 (Fig. 7.2). After having read question a, Juan spent about 2:50 min thinking about it. Then he said:

Juan: I have a way. In the [cabinet] 1, $1 \times 4 + 2$ [pieces of wood]. In the 2, $2 \times 4 + 2$. In the 3, $3 \times 4 + 2$.
Teacher: Where do the 4 and the 2 come from?
Juan: The 2 is because … in the first … there are two extra pieces on the top of the cabinets … like the roof. And the 4 is because they [the shelves] are 4 and 4 [each shelf].
Juan: [He wrote] $13 \times 4 + 2 = 54$.

Next, to answer question b, Juan started writing:

Juan: $N =$ [then he deleted the text and wrote again] $S =$
Juan: Will you understand if I write shelves?

Teacher: Yes.
Juan: If I write *S* instead of *N*?
Teacher: Yes. You can use any letter you like. *S* for shelves is fine.
Juan: [He wrote] $S \times 4 + 2$.

To answer question a, Juan divided the cabinets into shelves (made of four pieces) and the two extra pieces on the top. This was a quite complex graphical structure that did not suggest a procedure of decomposition to get the general term. The abstract explanation he offered for his answer to question a correctly justified the meaning of the coefficients in the algebraic expression. To answer question b, Juan had to use the information produced in question a and transform it into a functional algebraic expression using the initial letter of *shelf* to give meaning to the algebraic expression. The cognitive effort required to produce the answers was high, since Juan had to decompose the graphical pattern and obtain an algebraic expression for such decomposition, following a non-algorithmic complex way. This solution was in the doing mathematics level. The teaching method of suggesting that students write the initial letter of the unknown instead of its whole name proved to be successful in helping Juan make the transition from verbal expressions to algebraic expressions and understand the use of letters in algebraic expressions. This is clearly seen when he changed the letter *N* to *S* because the unknown were shelves.

The examples of solutions presented here demonstrate that Juan was able to interpret correctly most patterns presented to him, showing an ability to interpret them and convert them into meaningful arithmetic or algebraic information after a process of generalization from the particular cases worked out. This behavior is quite different from what is observed in average students in Grade 5, therefore showing traits of mathematical giftedness related to identification of patterns and generalization of mathematical ideas.

7.4.2 Analysis of the Cognitive Demand of Answers to Inverse Questions

As explained in Sect. 7.2.3, strategies for calculating inverse relationships are conditioned by students' awareness of algebra (or lack of it) and ability to solve equations (or lack of it). When the teaching experiment started, Juan had not received any previous instruction on algebra, so he neither knew how to write algebraic expressions nor solve equations. He also had not studied square roots. Examples 5–7 demonstrate that, during the first part of the teaching experiment, he showed three strategies to calculate inverse relationships (trial and error, wrong inversion, and correct inversion) that we consider typical of students without knowledge of algebra. Examples 8 and 9 then show the great change that occurred in Juan's ways of approaching and solving the problems when he started to learn algebraic concepts, use its system of signs, and solve equations.

Example 5: *Trial and error direct calculations*, with a cognitive demand in the *procedures without connections level*. When the general procedures Juan had found for the direct questions were of the linear type $y = ax + b(x \pm c) \pm d$ or the quadratic types $y = x^2$ or $y = (x \pm a)(x \pm b)$, Juan was unable to find a procedure to invert such complex expressions and he resorted to trial and error. We present below Juan's answer to the inverse question in the problem of the walls (problem 1.11, Fig. 7.7), where the generalization he got in the direct questions was the functional relationship $V_n = (n + 1) \times 2 + n$.

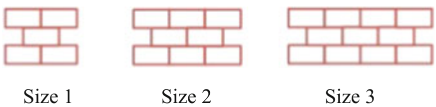
Juan: I had a procedure, but it doesn't work. … I did 38 divided by 2 minus 2.
Juan: [after 1:35 min of making calculations] I believe it is 13.
Teacher: Why?
Juan: I have tried numbers [for the size of the wall].
Teacher: What did you do while you were trying numbers?
Juan: I did $13 + 1 = 14$ … No, sorry, it is 12. Because $12 + 1 = 13$, $13 \times 2 = 26$ and $26 + 12 = 38$.

Trial and error is an algorithmic process that does not connect to the algebraic structure of the pattern and is only aimed at getting the correct answer. Juan did not understand the algebraic structure of the generalization he had obtained in previous questions of the problem, so he could not use it. He made a limited cognitive effort to get the answer, since he only had to make direct arithmetic calculations by checking different values for n until he found the correct one. He used an arithmetic representation that did not help him to connect to the algebraic properties of the sequence. He only correctly answered the question when the teacher helped him. Therefore, this strategy required a cognitive effort in the procedures without connections level.

Even though this trial and error strategy was far from our objectives of learning, Juan showed an ability to look for a different strategy of solution when he was not able to use a more correct one. Furthermore, nobody taught him this trial and error strategy; rather, it was Juan who developed it. Therefore, Juan demonstrated traits of mathematical giftedness related to flexibility in changing his focus and ability to develop efficient strategies to solve problems.

Example 6: *Correct inversion* of the order of operations, with a cognitive demand in the *procedures without connections level*. When the direct calculations were of types $y = ax$ or $y = ax \pm b$, most of the time Juan correctly applied the inverse

A group of masons has to build walls of different sizes, as shown below:



d) If they have used 38 bricks, what size is wall they have built? How did you know?

Fig. 7.7 Problem 1.11

arithmetic operations to get the position of the term, such as in question d of problem 1.12 (see Example 3, Fig. 7.6). To answer the direct questions, Juan used the general relationship $V_n = 3n + 2$. His answer to the inverse question d was:

Juan: It is 16 [tables]. … I did 50 minus 2, which is 48, and then divided by 3.

Inverting arithmetic operations is not just a matter of memory: it is a simple algorithm that can be applied in a straightforward way, requiring very limited cognitive effort because the student only needed to make basic arithmetic calculations. The aim of this procedure is to get a correct solution: the explanation was just a description of the calculations made, and it is not necessary to be aware of the algebraic structure of the sequence to get the correct answer. Therefore, this solution required a cognitive effort in the procedures without connections level.

Example 7: *Wrong inversion* of the order of operations, with a cognitive demand in the *procedures without connections level*. Although Juan correctly solved most problems like the previous example, sometimes he was not aware of the relevance of the order of calculations. In the problem of the friezes (problem 1.10, Fig. 7.8), the generalization he got in the direct questions was the functional relationship $V_n = 2n + 1$.

Juan was first asked to calculate the number of triangles made with 20 sticks (question d). After his wrong answer, the teacher helped him by making him aware of the need to consider the order of calculations.

Juan: I did 20 divided by 2 minus 1.
Teacher: Well, … look, before [in questions a to c] you first multiplied by 2 and then added 1. Now, what do you have to do first, subtraction or division?
Juan: Subtraction.

Juan was then asked to calculate the number of triangles made with 31 sticks (question e).

Juan: It is 15. … I subtracted 1 from 31 and got 30, and 30 divided by 2 is 15.

Juan did not understand the algebraic structure of the pattern, which induced him to decide on an incorrect order for the calculations. He made a limited cognitive effort to give a solution because he did not try to analyze the way he had made the calculations in the direct questions. Furthermore, the teacher helped Juan to

John wants to make friezes with sticks to decorate his bedroom. The friezes are made of triangles, as follows:

d) If John has 20 sticks, how many triangles can he make? How did you know?

e) If John has 31 sticks, how many triangles can he make? How did you know?

Fig. 7.8 Problem 1.10

understand the order in which the inverse calculations had to be made, so he repeated the teacher's instructions (first subtraction, then division). Besides, as he did not know how to represent the arithmetic expression algebraically, he had to use the arithmetic representation of the data. So, this solution required a cognitive effort in the procedures without connections level.

At the beginning of the second part of the teaching experiment, Juan learned to state and solve linear equations. From that moment, he was able to answer any inverse question by writing and solving an equation, as shown in Example 8.

Example 8: *Statement and solution* of an equation, with a cognitive demand in the *procedures with connections level*. We present now Juan's answer to the inverse question (c) in problem 2.5 of the second part (Fig. 7.2), which follows the answer we presented in Example 4, where Juan wrote $S \times 4 + 2$ to answer question b. When the teacher asked him to answer question c, he immediately gave the correct answer (24 shelves) by doing the inverse calculations. However, the teacher asked him to solve the question by using algebra.

In previous sessions, Juan had solved problems 2.1–2.4. When solving problem 2.1, the teacher introduced him to the use of the virtual balance (NLVM 2016) to represent and solve equations. The teacher guided Juan to understand the objective of solving equations by maintaining the balance in equilibrium while reducing the number of pieces in order to isolate the unknown. He practiced by solving a few equations ($ax + b = c$) with the virtual balance.

After having solved problem 2.1 with the virtual balance, Juan did not need to use it anymore; instead, he preferred to write a simulation of the virtual balance using the word processor (Fig. 7.9):

Teacher: First, you have to write the balance, like the equation. OK?

Juan: Do I write *S* times 4 plus 2?

Teacher: *S* times 4 plus 2 is a side of the balance. Now we know the [number of] pieces of wood. In question a, we knew the number of shelves but not the number of pieces. Now we do not know the number of shelves, so we write *S* instead of 13. So, what do we need now?

Juan: The balance?

Teacher: We already have a part. We need the other part, OK? … What should we write in the other part [of the balance]?

Juan: *X*.

Teacher: Look at question a. We had the first part of the balance [she meant $S \times 4 + 2$] and, as we knew how many shelves we had, we used 13 instead of *S*. Right? And we calculated the number of pieces we needed. Do we now know how many pieces of wood we have?

Juan: Yes.

Teacher: So we write it down, OK?

Juan: [wrote] $S \times 4 + 2 = 98$.

Juan now continued representing the equation on the word processor screen as if it were in the balance (Fig. 7.9a) and solving it by reproducing the compensations

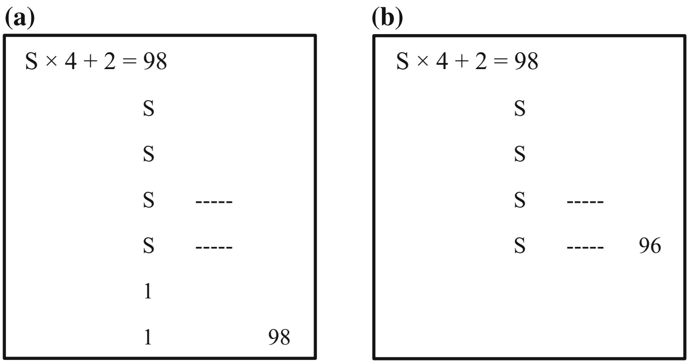


Fig. 7.9 Steps in the solution of problem 2.5, in the second part of the teaching unit

that he would make in the balance (Fig. 7.9b). Finally, he calculated 96 ÷ 4 = 24 shelves.

Juan's first answer to this question was based on arithmetical inversion, like in Example 6, because that procedure was easy for him. However, unlike in Examples 5–7 above, Juan had now learned to state and solve linear equations, so, when asked by the teacher, he was now able to correctly connect the pattern and the generalization he had obtained to the algebraic expression representing it and state (with some help by the teacher) and solve the equation.

Once Juan learned the procedure to write equations for the geometric pattern problems, this procedure became algorithmic for him, since he learned to combine the answers to questions a and b and the data in question c in a meaningful way related to the algebraic structure of the sequence represented, and from this he was able to write the corresponding equation. This algorithmic procedure cannot be followed automatically, but it is necessary to understand the specific algebraic structure of the sequence to decide on the way to write the equation. We see that, in this problem, the solution required Juan to put forth a quite high cognitive effort in order to decide which parts of the information available were useful and how to combine them. Juan used algebraic and diagrammatic (a balance-like diagram) representations to state and solve the equation. Therefore, this solution required a cognitive effort in the procedures with connections level.

Example 9: *Algebraic solution* of a problem that Juan *could not solve arithmetically in the first part of the teaching unit*, with a cognitive demand of the new solution in the *procedures with connections level*. To close this section, we present an example of the solution of a geometric pattern problem from the third part of the teaching unit. As mentioned in the description of the teaching unit, these problems were aimed at showing Juan the power that equations have in solving the problems that he found too difficult during the first sessions.

Problem 3.6 was an advanced version of problem 1.11 (Example 5, Fig. 7.7). As shown in Fig. 7.3, the new version of the problem explicitly asked for an algebraic expression of the general term of the sequence, then it asked for a simplification of that expression if possible, and, finally, it asked an inverse question, to be answered by solving an equation. The algebraic expression Juan wrote in question a was $(S + 1) \times 2 + S$, the same expression he used in problem 1.11. After question b, Juan simplified this formula to $3S + 2$. Question c asked for the size of the wall having 50 bricks, so Juan wrote, without any help:

Juan: $3S + 2 = 50$
$3S = 48$
$S = 48 \div 3$
$S =$ size 16

This answer required a cognitive effort in the procedures with connections level, for the same reasons stated for Example 8. The cognitive effort of a long answer depends on its most complex parts. In this case, for the problems in the third part of the teaching unit, Juan was able to solve question c (state and solve equations) more efficiently, since he had gained practice in operating with parentheses, simplifying the equations, and solving them. However, he had to make the same mental reasoning and the same cognitive effort as in previous problems in order to understand the algebraic structure of the graphical pattern and adequately combine the data to translate them into an equation.

The examples of solutions presented in this section demonstrate that Juan was able to interpret relationships and invert them arithmetically or algebraically. This behavior corresponds to the ability of gifted students related to identification and inversion of mental procedures.

7.5 Discussion on the Analysis of the Student's Answers

In the previous section we have presented examples of the different types of answers to the geometric pattern problems solved by Juan, and we have analyzed them from the viewpoints of the cognitive demand model and algebraic thinking. To discuss this student's behavior and his learning trajectory in the teaching experiment, we present information about the types of answers and the cognitive demand required to answer the direct questions (Table 7.2) and the inversion questions (Table 7.3) in all the problems solved in the first part of the teaching unit. After that, we also analyze the answers to the geometric pattern problems solved in the second and third parts of the teaching unit.

The first striking result from Table 7.2 is that, since the very first problem, Juan was able to solve correctly all direct questions without significant help (although he sometimes made errors in mental arithmetical calculations) by formulating and using appropriate verbalizations of the general term of the sequences. Only in problem 1.17, he was not able to calculate by himself the near and far terms, and the teacher helped him to get a generalization so he could try to answer the inverse question.

Table 7.2 Cognitive demand of the answers to the direct questions in the first part of the teaching unit

Problems	a) Immediate terms	b) Near terms	c) Far terms	Cognitive demand*	
1.1	Numer. - Recurs.	Figural - Funct.	Figural - Funct.	P. wout c.	P. with c.
1.2	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.3	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.4	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.5	Numer. - Funct.	Numer. - Funct.	Numer. - Funct.	P. wout c.	
1.6	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.7	Numer. - Recurs.	Numer. - Recurs.	Numer. - Funct.	P. wout c.	
1.8	Figural - Recurs.	Figural - Funct.	Figural - Funct.	P. wout c.	P. with c.
1.9	Numer. - Recurs.	Figural - Funct.	Figural - Funct.	P. wout c.	P. with c.
1.10	Numer. - Funct.	Numer. - Funct.	Numer. - Funct.	P. wout c.	
1.11	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.12	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.13	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.14	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.15	Numer. - Funct.	Numer. - Funct.	Numer. - Funct.	P. wout c.	
1.16	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.17	Numer. - Recurs.	No answer	No answer	P. wout c.	- -
1.18	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	
1.19	Numer. - Funct.	Numer. - Funct.	Numer. - Funct.	P. wout c.	
1.20	Figural - Funct.	Figural - Funct.	Figural - Funct.	P. with c.	

* Levels of cognitive demand: *P. wout c.* = procedures without connections; *P. with c.* = procedures with connections.

Furthermore, Juan made calculations of recursive and functional types, but he never used the counting and proportional types. He only used valid mathematical procedures to solve the problems, trait related to the unusual quickness in learning, understanding, and applying mathematical concepts typical of gifted students.

The facts that Juan had never had prior contact with mathematical generalization or algebra and that he had never solved geometric pattern problems before this teaching experiment clearly point to the presence of some traits characteristic of mathematically gifted students. Some of these are the ability to recognize mathematical patterns and structures, the ability to generalize mathematical ideas, and the ability to invert mental processes. It is clear from Table 7.2 that these abilities were very well developed in Juan.

Table 7.2 also shows that the student did use functional procedures to solve most of the 20 geometric pattern problems. He used the recursive strategy to find the immediate term in five problems and to find the near term in one of those problems. This is a clear sign of high mathematical talent, since this flexibility in changing the basic strategy of solution to a more efficient one is not found in average Grade 5 students. However, it is most interesting to note that he used this method of solution even to calculate the immediate terms, for which it is not necessary. He soon noticed that the last questions in the problems required a generalization, so when he started solving a problem, he worked to find a general rule and he applied it to calculate all the terms, exhibiting his ability to develop efficient and shorter problem-solving strategies.

Table 7.3 presents the types of solutions to the inversion questions in the first part of the teaching unit. As showed by Examples 5–7, all the solutions are in the procedure without connections level, since both arithmetic inversion and arithmetic

Table 7.3 Types of answers to the inversion questions in the first part of the teaching unit (two types in the same problem mean two attempts of solution)

Problems	**d) Inverse questions**	
1.1	Correct inversion	
1.2	Trial and error	
1.3	Wrong inversion	
1.4	Correct inversion	
1.5	Correct inversion	
1.6	Wrong inversion	Trial and error
1.7	Trial and error	
1.8	Trial and error	
1.9	Trial and error	
1.10	Wrong inversion	Correct inversion
1.11	Wrong inversion	Trial and error
1.12	Correct inversion	
1.13	Trial and error	
1.14	Trial and error	
1.15	Trial and error	
1.16	Trial and error	
1.17	- -	
1.18	Trial and error	
1.19	Trial and error	
1.20	Trial and error	

trial and error procedures were applied without needing understanding of the algebraic structure underlying the geometric patterns representing the sequence.

Table 7.3 shows other traits of giftedness, as the abilities to invert mathematical processes and develop efficient strategies. In this case, the efficient use of such abilities by Juan was limited by his ignorance of algebraic language and linear equations. However, Juan managed to find an alternative procedure to solve the inversion questions, since he consistently used arithmetical inversion, and, when this strategy was not useful due to the complexity of the general expression he had found (Table 7.4), he resorted to a careful trial and error. The very few publications analyzing students' answers to inverse questions in geometric pattern problems have not yet described a new behavior that we found in our experiment: a student systematically using, from the very beginning, an *organized trial and error* procedure to solve inversion questions. He tried a possible solution, and, if it was too big or small, he tried a smaller or bigger number, continuing this process until he found the correct solution.

Cai and Knuth (2011) described several learning trajectories of students solving geometric pattern problems, but none of them fit Juan's behavior. Tables 7.2 and 7.3 show that Juan was quite consistent throughout the course of the teaching unit in his methods of solving problems with shared characteristics (for instance, problems with the same grade of algebraic complexity), which is another contribution of this research to the knowledge about gifted students.

The problems in the second part of the teaching unit (Fig. 7.2) were focused on introducing our student to the basic concepts of algebra, to algebraic symbolization, and to the statement and solution of linear equations. On the direct questions of problems 2.1–2.3, Juan needed some help from the teacher to correctly write the algebraic expressions—in particular the use of parentheses—but he no longer needed help in the three last problems. To represent the unknown, he even wrote the initial letter of the objects presented in the pattern (days, shelves, etc.). In the direct questions, as shown by Example 4, Juan worked at the doing mathematics level, since he was discovering new algebraic ideas. On the inverse questions, all answers consisted of stating and solving an equation based on the data in the question, so the cognitive demand in all the student's answers was in the procedures with connections level.

The problems in the third part of the teaching unit (Fig. 7.3) were all solved by stating an algebraic expression for the generalized relationship in the sequence and

Table 7.4 Solutions to the inverse questions in the first and third parts of the teaching unit

Problems	Generalization	Inverse questions	Problems	Inverse questions
1.1	$y = ax$	Correct inversion	3.1	Equation
1.2	$y = ax + b$	Trial and error	3.2	Equation
1.6	$y = ax + b(x + c)$	Wrong inv.—Tr. & err.	3.3	Equation
1.7	$y = ax + b(x + c)$	Trial and error	3.4	Equation
1.8	$y = ax + b(x + c)$	Trial and error	3.5	Equation
1.11	$y = ax + b(x + c)$	Wrong inv.—Tr. & err.	3.6	Equation
1.16	$y = ax - b(x - c) + d$	Trial and error	3.7	Equation

then stating an appropriate equation and solving it, so the student worked at the procedures with connections level. Table 7.4 presents a comparison of the strategies of solution of the inverse question in the problems posed both in the first and third parts of the experiment. After the student had all the necessary knowledge and abilities to understand and use the algebraic approach to the geometric pattern problems, he used them confidently to solve those problems without difficulty.

In line with gifted students' quickness in learning and understanding mathematical concepts, Juan only needed the help of the virtual balance in very few problems, since he began to solve question c by imagining the balance and writing the transformations of equations paralleling the manipulations made in the balance. This internalization of the balance and the way he represented it on the word processor screen (Fig. 7.9) is a consequence of the giftedness trait of development of efficient strategies.

7.6 Conclusions

We have presented the case of a nine-year-old student in primary school Grade 5 who worked on an experimental pre-algebra teaching unit. As an answer to the first research question, we have shown that, as the experiment advanced, the student also progressed in his learning of the different concepts, structures, and procedures necessary to meaningfully learn basic algebra and linear equations and appropriately modified his strategies to solve the problems. This experiment shows that mathematically gifted students are much faster than average students in understanding and learning mathematical contents, but they also need to be taught mathematics.

We have used the cognitive demand model to evaluate a student's cognitive behavior in the different questions in the problems and also in the consecutive parts of the teaching unit. Related to the second research question, we have shown that, in trying to solve the problems during the first and second parts of the teaching unit, the student made all necessary cognitive effort, as much as was possible due to his limited knowledge of algebra. The model has proved to be useful to differentiate the cognitive effort required from the student by the different types of questions posed.

As for the third research question, the student exhibited ability to adapt his solving strategies and his cognitive effort to the mathematical complexity of the generalizations he had obtained and the algebraic tools he had learned at each moment. By the end of the teaching unit, the students showed the ability to work confidently on solving linear equations and algebraic word problems. This behavior is also typical of mathematical giftedness.

This research shows only the case of one student, so we do not suggest that this behavior may be generalized; however, it has some similarities to—and also differences from—other cases reported in the mathematics education literature. This research experiment was done in a laboratory context, but we believe that the teaching unit may be modified to adapt it to the context of ordinary schools, where it could be useful for teachers of the upper primary grades and lower secondary grades when starting to teach pre-algebra to all their pupils, but with the possibility of paying special attention to the gifted students.

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Chapter 8
Twice-Exceptional Students: Students with Special Needs and a High Mathematical Potential

Marianne Nolte

Abstract Twice-exceptional students are students with both a high potential for mathematical abilities and educational special needs. They are particularly at risk for neither their potential nor their disabilities being recognized. Students who work below their potential are called underachievers. This chapter discusses whether the special learning conditions of twice-exceptional students need a differentiated approach than what is usually applied for underachievement. Furthermore, by means of examples of affected students, the implications for learning processes are illustrated. The focus is put on mathematical giftedness occurring together with learning disabilities (LD), attention deficit disorders (ADD), attention deficit disorders with hyperactivity (ADHD), and autism spectrum disorders (ASD).

Keywords Twice exceptional · Special needs · Underachiever
Mathematical giftedness · Learning disabilities · Attention deficit disorders
Autism

8.1 Introduction

Twice-exceptional students are students who have both mathematical potential and handicaps or educational special needs. Twice exceptionality needs theoretical discussion as well as empirical research because there is still a research gap in the subject in the field of mathematics education. Research on this topic has been impeded because there are two different theoretical domains involved in discussions about affected children: mathematically promising students and students with handicaps or educational special needs, such as (learning) disabilities and disorders. In addition, these two fields deal with different paradigms, which poses the question

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of whether clear definitions of giftedness and learning disabilities are possible (for an overview see Scherer et al. 2016; Singer et al. 2016). One reason for the research gap may be the low prevalence. Although information about the prevalence of twice exceptionality has to be used very carefully (Nielsen 2002), one of my studies showed that about 15% of identified mathematically gifted students were affected (Nolte 2013). Due to its low incidence, discussions about twice exceptionality in mathematics education are mostly based on case studies (e.g., Montgomery 2003b).

One essential problem in working with twice-exceptional students is the "masking effect": twice-exceptional students are in particular at risk that neither their potential nor their disabilities are recognized. There are different reasons for low performance in mathematics, such as inadequate methods, a low level of potential, and barriers in learning processes that may hinder the development of mathematical achievement. To underline that an achievement is not a firm trait in a student, the term *promising student* often is used (Sheffield 1999). Mathematics teachers have to address all students with all their different needs. Nevertheless, noticing students' needs is especially difficult when it comes to twice exceptionality. Twice-exceptional students require both a challenging learning environment and support to overcome barriers in their learning processes. For a student with a high potential in mathematics, performance at a high level can be expected. Nevertheless, this is not always the case. Various factors may lead to an underachievement. However, can we call twice-exceptional students underachievers if difficulties, handicaps, and disorders have an impact on their learning processes?

The complexity of the learning situation of twice-exceptional students deserves closer attention. First of all, the general aspects of the development of achievement in both fields have been described and summarized in a model developed by the author that takes into account barriers in learning processes. If promising students do not achieve at the level they may be capable of raises the question of whether they are underachievers. In the context of twice exceptionality especially this is not an easy question. Therefore, the section about underachievement proposes to make distinctions between different reasons for levels of achievement.

This chapter examines the examples of four students to illustrate different aspects of the situations of twice-exceptional students. Because they are often used as examples for twice exceptionality, four different disorders and their impacts on problem-solving processes are outlined: learning disabilities (LD), attention-deficit disorders (ADD), attention-deficit disorders with hyperactivity (ADHD) and autism-spectrum disorders (ASD).

Cases of affected students are used to exemplify difficulties of achievement with problem-solving and communication processes. Furthermore, they give an impression about why it is so hard for their teachers to notice the potential as well as the difficulties that come with the disorders. In the section about the masking effect, considerations about reasons for not noticing either high potential or special needs are summed up. The chapter closes with considerations on how to work with affected students based on a distinction between situational and long-time interventions.

8.2 Systemic Approaches Towards Giftedness

Within the field of giftedness, the development of high competences are currently described in models based on influencing factors (e.g., Gagné 2004; Heller and Perleth 2008; Renzulli 2012). Some authors make a distinction between potential to perform and developed performances; for example, Gagné (2004) and Wieczerkowski and Wagner (1985) phrase potential as an inherited gift and systematically developed performance as talent. This differentiation underlines the importance of educational influences on the developmental process. Furthermore, because activities of the students moderated by individual factors such as interest and motivation play an important role in modelling the development of giftedness, actual models have described the developmental process as being guided by intrapersonal and environmental catalysts.

> Giftedness designates the possession and use of untrained and spontaneously expressed outstanding natural abilities or aptitudes (called gifts), in at least one ability domain, to a degree that places an individual at least among the top 10% of age peers. Talent designates the outstanding mastery of systematically developed competencies (knowledge and skills) in at least one field of human activity to a degree that places an individual at least among the top 10% of 'learning peers' (those who have accumulated a similar amount of learning time from either current or past training). (Gagné 2013, p. 5)

In addition to these models, Ziegler et al. (2013) points out that the situation of a child may change over time depending on various conditions. Focusing on the activities of students with his actiotope model of giftedness, Ziegler et al. (2013) refers also to the interplay between environmental aspects and individual activities. However, because the situation of a student may vary with the teacher, the situation in the classroom, and biographic aspects, it is reciprocal: these factors influence the student's activities. Theoretical positions on giftedness argue that performance cannot be regarded as synonymous with potential to perform: High potential measured by intelligence tests does not guarantee high performance. Crucial is the interplay between several factors that lead to the activities of a child in a certain situation and over a certain period of time. These positions lead to an approximation between positions about giftedness and research on expertise.

8.3 Systemic Approach Towards Learning Disabilities and Disorders

In the field of special educational needs there was a similar shift from an individual perspective focusing on the needs of a child that come with weaknesses or disorders[1] towards a more systemic approach (Hassanein 2014; Nolte 2000).

[1]The terms *disabilities* and *disorders* are not always used as equivalent; however, in this article, they are. As a more general term to describe different kinds of deficits, the term *weakness* is used in this article.

An individual perspective offers a child a special (e.g., medical) treatment. From a systemic perspective, performance does not exclusively depend on inborn potentials and restrictions. Further, equally important influencing factors are the child's learning environment at home and in school, learning biography, and, with this, what a child does and can do under certain circumstances. In discussions about dyscalculia, these aspects are called questions about the matches between children and their possibilities and the conditions of their situations (e.g., Lorenz 1998). Difficult situations at home and in school cause a risk that a student will not perform at an appropriate level.

Furthermore, Betz and Breuninger (1982) underline the interplay between beliefs and actions of persons who are involved in learning processes. For instance, repeated experience of failure, and with this low performance, can lead to low levels of self-efficacy and self-esteem in students. Students may show adaptive behavior that can, for example, result in avoidance of obstacles and aversive reactions. Teachers and parents also interpret the achievement as the behavior.

8.4 Model for Acquiring Mathematical Competencies Including Barriers

Approaches towards both the development of giftedness and developmental disorders explain the dependency of the achievements of a student on various variables. Thus, the accent on the activities of students and the interplay between involved persons in both approaches can be generalized for the development of achievement for all students.

In both fields, there are discussions about labelling and identifying a student as gifted or showing a special disorder. In the field of special educational needs, a broad and ongoing discussion points out that there is a dilemma between the identification of a disorder as a prerequisite for interventions and the problems that come with labelling a person as disabled (Hassanein 2014). Both fields point out the interplay between the activities of a student in learning processes and, with this more or less explicit, the importance of intrapersonal and environmental variables. Both fields, however, do not explicitly combine a high potential with a certain barrier. When it comes to twice-exceptional students, both a high potential and a learning disability or a disorder must be assumed. A disability or disorder may cause a barrier that restricts the activities of students and thus high achievement in the field of their potential. This barrier may be one cause for underachievement in mathematics in the sense that the word has commonly been used. Figure 8.1 uses aspects of the development of mathematical competencies that have both fields in common. The barrier underlines the difficulties twice-exceptional students have to overcome because of the disorder.

Combining aspects from both fields, the model describes the development of mathematical competencies in general as the result of a developmental/learning

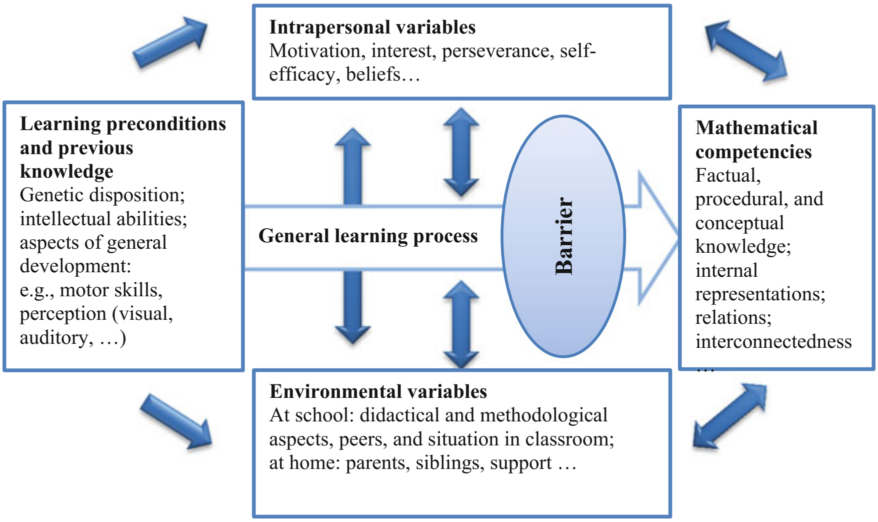

Fig. 8.1 Barriers as an influencing variable in mathematical learning processes (adapted from Nolte 2013)

process, which is based on learning preconditions including inherited dispositions, perceptual preconditions, and further developmental preconditions. These preconditions have an impact not only on intrapersonal variables such as interest and perseverance but also on the students' inner dialogues and thus on their self-esteem. Mathematical competencies such as knowledge about facts and concepts depend on the activities of a child, which are further influenced by attribution and beliefs about self-efficacy. They also have an impact on how teachers interpret students' achievement. While students may have good learning preconditions depending on their activities, in the case of disorders, learning preconditions can also cause restrictions in the developmental or learning process.

The arrows in both directions refer to the interplay between competencies and intrapersonal variables as well as the interplay between competencies and environmental variables. Environmental aspects, such as teachers' pedagogical interventions or parents' support, show a reciprocal effect on intrapersonal variables as well as on beliefs about competencies.

Twice-exceptional students have additional barriers in the learning process that may hinder the full development of potential. Such barriers in the context of mathematical giftedness could be, for instance, a learning difficulty in reading, writing, and spelling.

This model can be adapted to the conditions of a special student. Preconditions may vary depending on the learning environment. A high mathematical potential can be combined with, for instance, learning disabilities or kinds of disorders. Disabilities or disorders may build barriers that make it harder for students to enhance their capabilities. Depending both on the level of performance and on a child's activities, the reaction of teachers, peers, and parents will differ. This situation is not stable and may change if the activities and/or the behavior of the interconnected persons change. Thus, environmental variables influence achievement and at the same time achievement has an impact on the environment. This is similar for intrapersonal variables. Recurring experiences of success, disappointment, or obstacles have an impact on attributions and beliefs for both students and teachers.

Barriers during the developmental process do not necessarily lead to learning disorders. Taking into account the interplay between individual and environmental factors, high competencies can be gained if in a particular situation there is a match between the conditions of the learning environment and a child's possibilities (von Aster 2000). Neither low nor high performances are regarded as stable; rather, they are regarded as dynamic due to the underlying interplay between several influencing aspects.

8.5 Approach to a Conceptualization: Twice Exceptionality Within the Field of Mathematics Education

Gifted students who have to overcome barriers or who show weaknesses in their learning process are labeled twice exceptional. Twice-exceptional students have also been referred to as dual exceptional (Brody and Mills 1997; Silverman 1997) or double exceptional (Montgomery 2003b) or with combinations of giftedness and disorders, such as gifted with ADHD (Fugate and Gentry 2015).

Giftedness does not imply protection against developmental or other disorders (Bachmann 2008; Silverman 2006). Twice exceptionality in mathematics education refers to a high potential in mathematics. The barriers and weaknesses mathematically gifted students have to face can be very different. This includes physical limitations and handicaps such as hearing or visual disorders (Brandl and Nordheimer 2016), developmental disorders, learning disabilities, attention deficit disorders, or autism spectrum disorders. The three last appear most frequently in the discussion about twice exceptionality (Pereira et al. 2015).

> However, when giftedness is seen as developmental advancement or as advanced abstract reasoning ability, it becomes apparent that a bright student may have difficulty reading, writing, spelling, calculating, or organizing. Giftedness can be combined with blindness, deafness, cerebral palsy, other physical handicaps, and psychological dysfunctions. It provides no immunity against physical diseases and accidents that impair functioning. (Silverman 2006, p. 28)

Often twice-exceptional students do not achieve at the level of their potential. Can we call them underachievers?

8.5.1 Underachievers

Underachievement seems easily to be defined: It occurs when achievement is lower than potential. With this approach, underachievement can be defined as (a) a discrepancy between performance and IQ, i.e., the potential measured by an intelligence test is higher than the achievement that has occurred. Another approach is to define underachievement as (b) a discrepancy between the expected performance and the shown performance. Combined with definitions about underachievement, there are conceptualizations of the terms *achievement*, *potential*, and *intelligence* used in the particular definitions. Reis and McCoach (2000) discuss the problems involved with this variety of definitions and also question the underlying concepts of achievement and intelligence. Nevertheless, a discrepancy between potential and achievement is commonly the basis for the definition of underachievement.

Underachievement is observed on different levels of potential and is relatively stable (Sparfeldt et al. 2006). Holling et al. (2004) distinguish between general and partial underachievement. Many underachievers show emotional and motivational problems (Hanses and Rost 1998). Commonly discussed traits of gifted underachievers are, for instance, low academic self-efficacy, low academic self-perception, low academic self-esteem, and further motivational components combined with a perfectionistic approach to tasks (see McCoach and Siegle 2003; Sparfeldt et al. 2006). The reasons behind this may be multifaceted. Referring to the model (Fig. 8.1), because the described traits influence the activities of a child, environmental variables should also be taken into account. If teachers or parents do not realize the high potential of a child, the learning environment may not be demanding enough. But motivation depends also on interesting challenges. These are even important for the regulation of attention (Durstewitz et al. 1999; Güntürkün and Westphal 2009). With this, boredom or the motive to adapt to an expected (average) level often lead students to hide their capabilities (Nolte 2016).

8.5.1.1 Underachievement as a Collective Term for Different Disorders

Underachievement is also used as a collective term for different disorders. Underachievement, described as a "discrepancy between expected achievement (as measured by standardized achievement test scores or cognitive or intellectual ability assessments) and actual achievement (as measured by class grades and teacher evaluations)" (Reis and McCoach 2000, p. 157), allows the inclusion of both students with learning disabilities and those with disorders. This brings up the question of whether students with disorders are covered under the concept of underachievement. Reis and McCoach (2000) answer this question by excluding students with a diagnosed learning disability: "To be classified as an underachiever, the discrepancy between expected and actual achievement must not be the direct result of a diagnosed learning disability" (p. 157). This detailed definition requires a differentiated identification of both high potential and learning difficulties and disorders. It is not clear whether affected students may also fall under the general description of underachievement. The discrepancy between IQ and performance or expected versus shown performance does not give differentiated information about reasons for the level of achievement. Furthermore, an identification based on IQ may not be correct, because developmental disorders or disabilities often build a barrier for successfully working on tests (Harder 2009).

8.5.1.2 Implications

Often, students are first identified as having emotional or behavioral problems, but these problems may be the result of repeated failure of success or a mismatch between potential and given tasks. It may be helpful for teachers and parents to be conscious of the fact that emotional and behavioral problems can be observed on the surface while the needs of a twice-exceptional student may be hidden.

If students have a learning disability and high potential at the same time, and if they show general low achievement it is only partial underachievement (see Holling et al. 2004). This may be the case if the high potential is unidentified or if students cannot develop their potential because they are unable to compensate for their barriers.

If students have a disorder such as ADD/ADHD or ASD, the characteristics of the disorder may be a barrier in learning processes, and this may lead to a lower level of performance than student could have shown without the disorder. In these cases, the low level of achievement can be described as *subordinate*.

Based on these considerations, the term *underachiever* should be used with care. Several conditions may cause students not to perform at the level of their potential. Figure 8.2 shows different possibilities for underachievement in twice-exceptional students. Here, the word *disorders* is used as an umbrella term for a superordinate concept for different disabilities and disorders. The figure exemplifies an analytical differentiation between different possibilities, knowing that levels of achievement and disorders fall on a continuum.

Some of the students can compensate for disorders while others cannot (von Aster 2000). The four fields show the different possibilities. If students show certain disorders and at the same time perform at the expected level they can be called twice exceptional. If achievement does not meet the level appropriate to the potential, the twice-exceptional student is also an underachiever. Furthermore, students with no disorders who show a high potential are gifted students. Students who do not show disorders and do not achieve at the expected level are called underachievers.

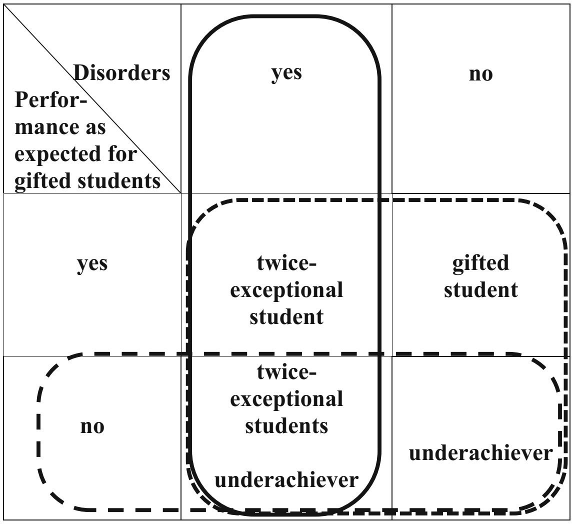

Fig. 8.2 Underachievement and twice exceptionality: a given student may or may not have a disorder. A student who performs as a gifted student (in spite of the disorder) is considered a twice-exceptional student. A student who does not perform as expected of a gifted student is considered twice exceptional and an underachiever

8.5.2 Mathematically Promising Students, Learning Disabilities, and Disorders

A closer look at four disorders that are often discussed in the literature about twice exceptionality illustrates the barriers affected students have to overcome. Other than Timo's case, the examples are based on case studies done by the author within the framework of a project called PriMa.[2] Within this project at the University of Hamburg, mathematically talented primary school students in Grades 3 and 4[3] can participate in an enrichment program. After an identification process that is comprised by several steps (Nolte 2012), we work with about 50 students every second week using complex fields of problems (Nolte 2012; Nolte and Pamperien 2017). Since 1999 more than 6000 students have participated in the talent search and about 950 students in Grades 3 and 4 have been accepted into this fostering program. Students with disorders or weaknesses participate frequently in this project (about 15% of the students in the program; see Nolte 2013).

The described examples will give an impression of the complexity and interconnectedness of the various aspects of every single case and underline the prototypical aspects of each disorder.

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[2]PriMa is an abbreviation of "*Primary* grade students on different ways towards *mathematics*." It is a cooperation project started in 1999 by the Hamburger Behörde für Schule und Berufsbildung, the William Stern Society (Hamburg) and the University of Hamburg.

[3]Since 2010 we have extended the program up to Grade 7.

8.5.2.1 Learning Disabilities

"Students who are gifted and also have learning disabilities are those who possess an outstanding gift or talent and are capable of high performance, but who also have a learning disability that makes some aspect of academic achievement difficult" (Brody and Mills 1997, p. 282).

The term *learning disabilities* is used in different ways in different countries [for a survey, see Scherer et al. (2016)]. One difference lies in the use of this term with general learning disabilities as opposed to special learning disabilities in a particular subject. Furthermore, sometimes there is a distinction made between difficulties, disabilities, and disorders, for which difficulties are regarded as a weaker form. A commonly used term (in the context of medical research) is *special learning disorder* (American Psychiatric Association 2016b). In general, the term *learning disabilities* describes a level of performance well below the average level described in curricula. Unlike the focus on discrepancies between performance and curricula, in the context of giftedness the term *learning disability* is also used to describe a discrepancy between the abilities of a child in certain domains (intrapersonal perspective). Consequently, learning disability can be used to describe average performance compared to high performance. This raises questions of how to determine how great the level of differences needs to be to be described as having a disability. Preckel and Baudson (2013) point out that individual weaknesses and strengths (distinct profiles of potentials in intelligence tests), such as high mathematical abilities and lower verbal abilities, occur more frequently in gifted students. However, at what point should a discrepancy between abilities be called a barrier? This depends on the situation. If there are possibilities of compensation, and the applied methods and materials match the child's learning precondition, then disabilities will not build barriers in learning processes.

8.5.2.2 Reading, Writing, and Spelling

Amongst the twice-exceptional students, mathematically gifted children very often show difficulties in the domains of reading, spelling, and writing. The difficulty can be a learning disability compared to the curricula or a weakness—and in this sense a learning disability—compared to a high level of potential in mathematics. Difficulty in reading mathematical problems may especially hinder a student working on problems, causing the teacher to observe mathematical achievement below the potential of the student.

Justin

Justin is an example of a student with a discrepancy between his mathematical and his reading capabilities. At the time we worked with him, he was an 8-year-old boy who participated in the PriMa project during the third and fourth grade. After a short time in the fostering project, he seemed to lose motivation. His mother was

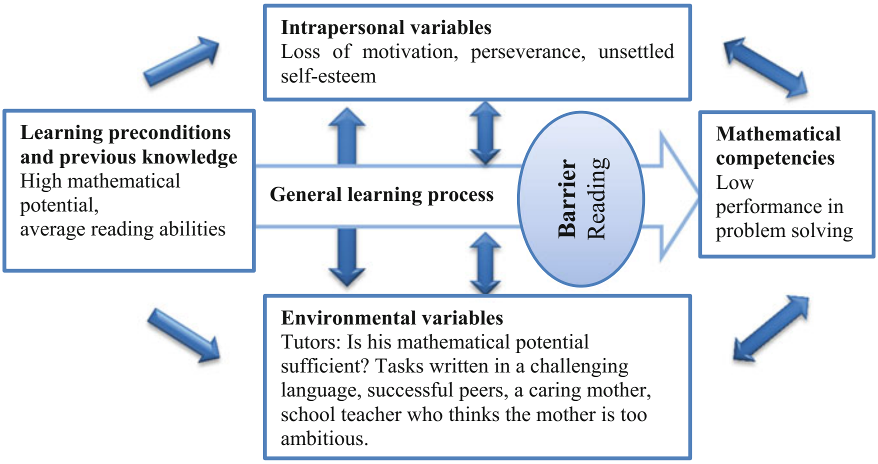

Fig. 8.3 Influencing variables of Justin's learning process

concerned because she did not understand why he lost motivation. The tutors told us that Justin often seemed to misunderstand the tasks. Both we and Justin were uncertain whether he was identified correctly as mathematically gifted. Therefore, we tested him again. When we gave him a problem, we observed that he had difficulty reading it, so we tested his reading abilities. The test showed that his reading abilities were at an average level (PR 35[4]). We assumed that the discrepancy between his potential in mathematics and his reading abilities caused his irritation and his loss of motivation. As the level of mathematical problems given in the fostering project increases (compared with the level at school), the demands on reading mathematical texts increase as well (Nolte 2013).

Going back to the model, Justin's achievement was not as high as expected. His motivation to work on the problems and his self-esteem decreased. The tutors recognized his low motivation and his low performance and questioned the quality of the test results we got during the talent search process. Because of the discrepancy between his potential and his performance, Justin can be described as underachieving. However, it was only after identifying the reason for his low level of performance that corresponding methodological interventions could be developed (Fig. 8.3).

In terms of his environment, we as his teachers tested his reading abilities and found that this crucial learning precondition was insufficient for reading challenging mathematical problems. We discussed the following measures with Justin and his mother: As compensation for his average reading capabilities during the fostering lessons, the tutors or his peers supported him to understand the texts.

[4]PR is an abbreviation of percentile rank. It indicates the percent of cases that are at or below a score in tests.

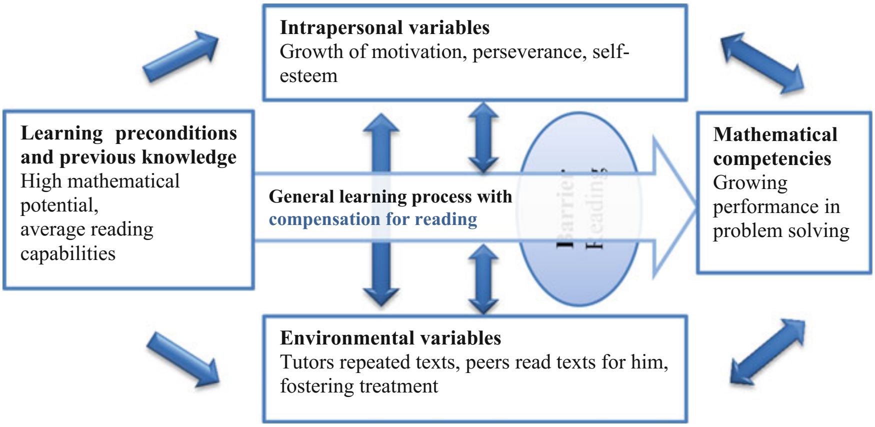

Fig. 8.4 Influencing variables of Justin's learning process with compensation for reading

This immediately had an impact on his performance, his motivation, and his self-esteem. We explained the test results to Justin and encouraged him to find his own way of compensation. So, as a first approach we tried to make the conditions of the learning environment more suitable for him. At home, his mother took care of an intervention (therapy) to improve his reading abilities. In the long term, this also influenced the development of his mathematical potential (Fig. 8.4).

8.5.2.3 Why May Average Capacity in Reading Cause Problems?

The problems we developed for mathematically gifted students are written at a very low level of redundancy and the given information is complex. The used language can be described as academic discourse or academic language (Gogolin 2012; Gogolin and Lange 2011). Although the problems are phrased mathematically correctly and adapted for children, understanding the phrasing of these problems is far more challenging than understanding the phrasing of problems normally given in school. Comprehensive reading depends on reading fluency (reading speed and understanding). The speed in the reading process is crucial, because if the reading process is too slow, the information to be stored in working memory is too high and due to this, internalizing the meaning is not possible. In well-known contexts, weak reading capabilities or slow reading may be compensated by interpretation of the context, but the low level of redundancy in our tasks requires comprehension of every word. Besides understanding the mathematical content of the problems, reading is also needed for interaction and therefore important for the communication processes, e.g., reading what is written on the blackboard or on other student's papers. Furthermore, the chance to reread a problem reduces the load of information that must be stored in a student's working memory. Due to the high-level mathematical terminology, weaknesses in reading also effect verbal communication capabilities. A child who has difficulty reading also has difficulties internalizing mathematical terminology into his spoken vocabulary, which thus creates a barrier in problem-solving processes. So, mathematical problem-solving processes are also mediated by different learning preconditions, in this case a weakness in comprehensive reading.

8.5.2.4 Weaknesses in Perception as a Special Learning Disability

Besides focusing on learning content such as mathematics or language, learning disabilities are also described as focusing on perception or sensory disorders. Silverman (1997) lists, for instance, "auditory processing weaknesses, … visual perception difficulties, spatial disorientation" (p. 4). Discussions about developmental disorders describe sensory disorders such as weaknesses in auditory or visual perception as risk factors, but not as causes for learning disabilities. Due to the fact that these factors may occur alongside learning disabilities but do not occur in all cases, individual weaknesses may be overcome by one student but not by all students (von Aster 2000; Nolte 2001). In mathematical learning processes, visualizations of mathematical content are an important tool for exemplifying this content. Therefore, weaknesses in visual perception should rarely appear together with mathematical giftedness.[5] This assumption is confirmed by the results of a questionnaire we gave to about 400 parents of students who participated in the talent search process in the PriMa project. We asked about learning disabilities and developmental disorders they observed during the developmental process of their child. About one third of the parents answered and among them about one third mentioned learning disabilities or (developmental) disorders. No child was mentioned showing weaknesses in visual perception. However, seven students had auditory processing disorders, partly in combination with disabilities in reading and writing.

Lars

Lars's (9 years old) case gives an example for barriers caused by auditory perception disorders. He also participated in our fostering program in class three and four and showed shifting levels of participation. When the lessons began, he was highly motivated, but over time his engagement went down. We tested his intelligence (PR 99.9), his mathematical school achievement (PR 97), and his reading capabilities. The results he obtained in tests in vocabulary (PR 90), stumbling words[6] (PR 58), and reading (41 out of 50 points) were different. During trial lessons at the university, he placed 95 out of 370. These trial lessons are the first

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[5]Assouline et al. (2012) refer to studies, which underline the importance of visual spatial skills for mathematical abilities.

[6]The stumbling words test measures understanding while reading. Students must identify words that are useless to various sentences. The test requires automatically reading at a high speed.

step of the talent search process in the PriMa project (see Nolte 2012). The second step is a mathematics test for mathematically gifted students. In this test, he was nearly in the highest third (76 out of 226). Taken together, the results of the different observations and, taking into account Lars's discrepancies between different test results, we observed a very high intelligence, high mathematical potential, and average verbal capabilities. Comparing his IQ with his capabilities in the stumbling words test, there was a difference of about five standard deviations. But what are the reasons for this discrepancy? His parents told us about a diagnosis of auditory perception disorder. In his school report states that he is well integrated into the class community and that he likes to support his classmates, but does not always seem to be able to control anger (Nolte 2013).

8.5.2.5 Why May Disorders in Auditory Perception Cause Problems?

Students with auditory perception disorders must make an effort to understand what is going on in the classroom. During lessons, there are always noises in the background. For most of the students, it is easy to focus on the relevant noises, such as the voice of the teacher or a partner. This is not the case for students with auditory perception disorders. This may explain symptoms of fatigue, such as levels of participation that are different at the beginning of fostering lessons than they are later on in those lessons. In particular, communication and interaction can be hard for children with auditory perception disorders because they may have difficulty in understanding others say (Nolte 2000, 2004), thereby making it difficult to figure out what is really meant in communication processes. Disorders in auditory perception can therefore be a reason for misunderstandings that may lead to emotional reactions. In general, even if students try to compensate for sensory disorders in perception, it is far more exhausting for them to participate in communication processes and to concentrate on working on a problem. These aspects could also explain uncontrolled anger (see Lars).

Communication processes normally depend on interpretations, so that in well-known contexts it is not necessary to understand every single word. This is different in a learning environment for mathematically gifted students, where the level of necessary concentration is higher. Going back to the model (Fig. 8.1), Lars's disorder in auditory perception builds a barrier. It was exhausting to overcome this barrier and his decreasing participation can be seen as missing perseverance. Lars did not work with as much concentration as other children. While his performance was still high, it did not reflect his potential.

Disorders in auditory perception are often not recognized, and children's efforts to compensate for auditory problems, such as repeatedly asking questions, may be misinterpreted as inattention. If the teachers know about this weakness, it is easy to support the student. Clear language, a low level of background noise, and frequent verification as to whether the student has internalized the context or not are all helpful.

8.5.3 Autism Spectrum Disorders (ASD)

Autism spectrum disorders (ASD) are one of the best known pervasive developmental disorders in the discussion about giftedness (For an overview see, e.g., Cash 1999; Foley Nicpon et al. 2011; Montgomery 2003a). Recently, the name of these disorders has been changed from different subtypes of autism (e.g., Asperger) to autism spectrum disorders in order to underline “that these subtypes are most likely a variation of the same underlying condition or etiology” (Young and Rodi 2014, p. 759). Characteristics of autism spectrum disorders (ASD) vary in a broad range: They fall on a continuum, but the focus lies on delays in the development of socialization and communication skills. “People with ASD tend to have communication deficits, such as responding inappropriately in conversations, misreading nonverbal interactions,… In addition, people with ASD may be overly dependent on routines, highly sensitive to changes in their environment, or intensely focused on inappropriate items” (American Psychiatric Association 2016a). This intensity of focus occurs also with mathematics. Focusing is a strength that is very useful in the context of learning mathematics. It is therefore no surprise that studies confirm the coexistence of mathematical giftedness and ASD (Chiang and Lin 2007, p. 552).

Knowledge about ASD and high mathematical capabilities are often based on case studies.

> To date, only one empirical study (Foley Nicpon et al. 2012) has reviewed the cognitive and academic profiles of individuals with very high cognitive ability (an ability index score of 120 or higher) and ASD. This study yielded information that gives a fuller understanding of the very broad cognitive range of individuals with ASD, specifically addressing cognitive strengths and weaknesses relative to the different ASD diagnoses. (Assouline et al. 2012, p. 1782f)

Regularly, students who participate in our fostering project are diagnosed with an autism spectrum disorder. In the above-mentioned questionnaire, 5 out of 132 parents told us about an ASD diagnosis. In most of the cases, we recognized that the students needed attention, but in a different way compared to those with a severe disorder. This was different with Leon.

Leon

At the beginning of our fostering program,[7] Leon’s parents informed us that he, a 9-years-old, was autistic. He showed characteristics ascribed to ASD such as “a rigid fascination with an interest; a need for precision; intellectual rigidity; a lack of social skills; the need to monopolize conversations and situations; … difficulties in adapting to the way of thinking of others; and a tendency to introversion” (Cash 1999, p. 23). We tested his intelligence (PR 99.9), his mathematical school achievement (PR 97), and also his reading capabilities. Similar to Lars’s case,

[7]Leon participated in the project PriMa during Grades 3–6.

the tests we performed, vocabulary (PR 90), stumbling words (PR 58), and reading (42 out of 50 points) gave different results. As result of our observations during trial lessons during the talent search process, he placed 9 out of 370. His results in the mathematics test for gifted students were also very high, placing 25 out of 226. His results in the stumbling words test were about five standard deviations from his IQ. Nevertheless, his reading capabilities were at an average to a higher level.

Leon felt completely comfortable when working with us at the university. He was fascinated by doing mathematics. His achievements were impressive. Most of his thoughts and computations he did mentally, using an almost unlimited number space. He preferred to work alone and was never distracted by anything while working on mathematics problems. In plenary discussions or in discussions with his tutors he precisely explained his ideas at a very high level of language. However, due to his need for precision, in general, participation in discussions with others was difficult for him. Likewise, plenary phases were a challenge for him, especially if other contributions were not as exactly and completely as his own thoughts. It was hard for him to endure others not being as fast and as capable in deeply understanding mathematical content as he was (Nolte 2013).

8.5.3.1 Why Does Weakness in Social Interaction and Communication Processes Cause Problems?

Working with children with characteristics of ASD in a group is a challenge for all participants. The rigid fascination with an interest is very helpful when it comes to problem solving with its need for volition and perseverance. Working precisely is also an important aspect in mathematical thinking processes. However, intellectual rigidity may be a hindrance for broadening the mind for different approaches to content. In communication and interaction processes with a whole group or with a partner, an exchange of ideas is an important tool to learn more about the content. Furthermore, discussions with others are based on the capabilities of the interacting persons to grasp their ideas, to understand their ideas, and to compare them with own ideas. Thus, difficulties in adapting to others' ways of thinking are a hindrance to participating in these processes. However, thinking the ideas of others through offers a chance to learn new approaches to content. This way, communication about mathematics broadens knowledge. Participating in plenary discussions enables the participants to link different ideas at a higher intellectual level. Therefore, the rigidity that comes with ASD is a barrier for flexibility and so perhaps for creativity.

8.5.3.2 Approaches to Support Students

There are therefore some traits of ASD that support a high level of mathematical thinking and other traits that make it hard to show fluency and flexibility. In this case, normative interactions and hints can be helpful, such as "mathematicians always look for patterns" or "now it is your task to understand the pattern behind what others are saying."

In terms of interaction with peers, the need to monopolize conversations and situations combined with rigidity may make it hard to bear such a child. In Leon's case, his brilliance in mathematics required emotional support for other students. His rigidity in conversations was a challenge to cope with for peers and teachers. Peers were confronted with the fact that their contributions were often of a lower level of mathematical insight. This experience caused some of the other children to feel that they were less capable than they actually were.

Knowledge about characteristics in communication processes is helpful. In Leon's case, teachers had to take into account that his insistence on considerations or facts was as stressing for him as they were for the teachers. Another characteristic often found with ASD is taking everything said literally: "Can you solve this task?" may lead to a simple answer of "Yes," while "Please solve this task!" may lead to actually working on the task.

8.5.4 Attention Deficit Hyperactive Disorder (ADD/ADHD)

Attention deficit disorders can be observed with (ADHD), without hyperactivity (ADD), or as a mixed type. The core symptoms that teachers and parents often describe are that children with attention deficit disorders do not listen, have difficulties organizing themselves, lose things, seem to be very forgetful, and/or show a high level of distractibility. We see impulsive behavior in communication processes when children cannot wait and interrupt others (see Schulte-Markwort 2004, p. 14ff) or have excessive motoric restlessness (Asherson 2013, p. 2). Another difficulty lies in the shift of attention. "However, recent studies suggest that ADHD is best perceived as having three main components consisting of an inattention factor, a hyperactivity-impulsivity factor, and a general factor that combines symptoms from both symptom domains" (Asherson 2013, p. 3). Diamond (2005, 2011) underlines that students with ADD and ADHD should be distinguished because of "dissociable cognitive and behavioral profiles" (2005, p. 808). Although the American Psychiatric Association (2016c) did not follow her proposal to discuss the subtypes as different, her observations show the necessity for treating the groups differently. The inattentive subtype is "not so much distractible as easily bored and under-aroused" (Diamond 2005, p. 808). Looking for distraction means looking for events and things that may be interesting. "Where hyperactivity is prominent, children with ADHD tend to be frenetic" (Diamond 2011, p. 322). Both types show difficulty in concentrating as long as other students. Nevertheless, both types need different interventions. As an example, Diamond refers to test situations. Hyperactive children often get more time to work on a test, but "children with the inattentive subtype often perform better when challenged by presenting test items at a quick rate" (Diamond 2011, p. 322).

Timo

At the time we worked with him, Timo[8] was an 8-year-old boy whose mother asked me for advice because she thought her son might be underachieving. She described him as oblivious and absent minded in everyday life situations. This, however, was not the case with content he was fascinated by. At home he engaged himself with a great amount of concentration in technical questions or in mathematics. In first grade, he started to lose his motivation and seemed to be unhappy. His performance in mathematics quickly dropped to an average level. In achievement tests, his performance was on an average level because he tended to overlook some of the given tasks; therefore, his performance in school gave no hint of underachievement. Despite what his performance in school suggested, intelligence test results showed high potential in mathematics combined with an average potential in language. His high potential in mathematics was confirmed by observations on how he solved complex mathematical problems. In order to identify causes for these inconsistent observations, a child and youth psychiatrist was consulted. Eventually, he was diagnosed with attention deficit disorder (ADD) and hyperacusis (Nolte 2016).

8.5.4.1 Why Do Attention Deficit Disorders Cause Problems?

Learning processes require the potential to focus on information, to analyze ideas, and to store knowledge in memory. Even in classrooms at the primary level, the information that should be learned is embedded in longer processes of activities. Often even solving easy tasks demands following several steps. Students with ADHD frequently show difficulties in focusing long enough, working on tasks step by step and pursuing a goal. Problem-solving processes are far more demanding. Problem-solving requires selecting the most relevant information, planning different steps, organizing material, and proving results. Parts of this process may be repeated so that the amount of information that must be organized increases. During the whole process, it is important to focus simultaneously on both the different steps and the goal in order to get the solution, which requires a frequent shift of attention. The problem solver has to evaluate the quality of previous steps on the path towards a solution. Because the path from the question to the solution can be long, for some students it is hard to hold their attention at an appropriate level. The difficulty rises with the amount of steps and with the complexity of the problem. Lucangeli and Cabrele (2006) point out that students with ADHD or ADD may "fail in a variety of cognitive tasks because of their inability to focus on the most relevant information" (p. 56).

Observations of mathematically gifted students give a contradictory impression. Because the traits of mathematical giftedness include the ability to handle complex information, skip steps and abbreviate analytic-synthetic activities (Krutetskii 1976, p. 107f), mathematically gifted children can often focus on the essential content in

[8]Timo did not participate in the PriMa project. Here the results of a case study are described (Nolte 2016).

problem solving. Furthermore, interest and motivation play an essential role when it comes to perseverance. A child with ADD/ADHD does not show difficulty in focusing on a topic in every situation. That is why Timo's attention to mathematics and technical questions does not contradict the ADD diagnosis.

Nevertheless, some of the students get lost in repeatedly, coming up with a number of ideas and approaches to reach a solution but not taking the next steps to actually prove the applicability of their ideas. In such cases, a high level of productivity and creativity may not be efficient and may even become a hindrance in solving the problem. This experience of lack of success is disappointing and may lead to emotional problems.

In group processes, there are always background noises. Students who work together obviously talk to each other. Teachers may communicate with students. Similar to students with auditory weaknesses, this noise level is hard to bear for a child with ADD/ADHD. They cannot shield themselves against irrelevant stimuli, which are hard to avoid when dealing with other children.

Taken together, "maintaining an appropriate problem-solving set to achieve a future goal, inhibiting an inappropriate response or deferring a response to a more appropriate time representing a task mentally (i.e., in working memory), cognitive flexibility, and deduction based on limited information" (Barry et al. 2002, p. 260) are necessary tools in problem solving processes. However, to maintain a focus, inhibit an impulse, and to defer a reaction are difficult for students with ADHD/ADD..

8.5.4.2 Approaches to Support Students with ADD/ADHD

We did not observe that most of the students in our fostering project paid more attention to the problems than other students did. The following conditions, derived from the literature and our field observations, are supportive in fostering programs and in school (Fielker 1997; Schulte-Markwort et al. 2004; Nolte 2016):

- The problems are challenging at an appropriate level.
- The students are well treated by a therapist.
- The lessons follow a structured and a familiar plan.
- The lessons are based on acceptance and a good relationship between affected students and teacher and other participants or classmates.
- The teachers are aware of the importance of offering possibilities for students to get the necessary information when they lose their train of thought.
- The teachers offer space for moving, especially for students with hyperactivity.

8.6 The Masking Effect

Twice-exceptional students need support to develop their high potential and at the same time support to overcome their barriers or to compensate for disorders. However, often neither the high potential nor the disorders are recognized.

This *masking effect* is well known in discussions about twice exceptionality. "When gifts and handicaps exist in one individual, they often mask each other so that the child may appear 'average' or even as an 'underachiever'" (Silverman 1989, p. 37). According to Baum (1990), students may be identified as gifted without the learning disability being identified or vice versa. A third category consists of students "whose giftedness and disability mask each other" (Besnoy et al. 2015, p. 108).

Due to the masking effect, it is difficult to get clear information about the prevalence of twice exceptionality. Nielsen (2002) points out that estimates about its prevalence should be handled with caution. About learning disabilities, Nielsen (2002) writes that "empirical data regarding the actual incidence of gifted children with LDs, however, are virtually nonexistent" (p. 94). In order to learn more about the masking effect, in our case studies we compared the school reports of the students (Justin, Lars, and Leon) to their test results, to observations during the fostering program, and with information given by the parents (Nolte 2013).

Justin's teacher did not recognize the high discrepancy between his reading capabilities and his mathematical potential. This is understandable because in school Justin participated successfully in reading competitions. In addition, at school he learned Italian. In these lessons, his teacher recognized problems in reading and informed the mother. When we tested Justin, in addition to a standardized test with average results, we used a text about science written for primary grade students. With these conditions, it was hard for him to grasp the information. Because the text was more difficult than the usual texts in school, it became obvious that his speed of reading was rather slow.

Lars's teacher did not recognize his auditory perception weaknesses. She observed behavioral problems such as an inappropriate control of anger. She and a psychologist proposed an intervention and suggested therapeutic treatment for his behavior. His school reports described his achievement as very high. A small hint of his difficulties was the report that stated that his spelling was graded as good instead of very good in the other school subjects. Although his teacher was not informed about his high IQ (PR 99.9), she could have recognized his weakness in grammar and spelling. Here, his performance was not in line with a child in third grade, even with abilities at an average level. There may be different reasons for a teacher not to identify a weakness in a child. One may be the level of potential and achievement of other students in the class: Perhaps compared with other children, Lars's achievement was high enough.

Timo is an example of a child who shows various difficulties. Timo's teacher did recognize neither his attention deficit disorders, his hyperacusis, nor his impressive mathematical potential. Because he overlooked some of the tasks given in tests and adapted his performance to the level of most of his classmates, his performance in school was at an average level (Nolte 2016).

In general, knowledge about the existence of the masking effect enables teachers to detect inconsistency in the learning behavior of a student. But matching the learning environment to students' needs requires knowledge of their learning preconditions.

8.6.1 Further Considerations About the Masking Effect

The masking effect is based on the ability of students to compensate for their learning disabilities or disorders. At the same time, these can be a hindrance to showing high potential. Like the above examples show, average achievement can be at a level lower than potential. Students who do not attract attention sometimes adapt their achievement to what they think is expected from them. Similarly, they do not attract attention if they hide their barriers or if the achievement of others is also at a low level. In such cases, parents who ask for support for the development of the high potential often are misunderstood as being too ambitious.

Twice-exceptional students need both support and challenges. A way to successfully recognize strengths and weaknesses in a twice-exceptional child may be to offer challenging tasks (as we observed with our students). Confronted with rather difficult, thought-provoking mathematical problems, perhaps the students can show their high potential. Furthermore, difficult tasks may be a hindrance to compensating for weaknesses, which can help uncover disorders.

Besides the cognitive aspects, the emotional situation of the child should also be taken into account. Compensation strategies are not always successful. Efforts to overcome barriers may be exhausting. Due to their high cognitive potential, twice-exceptional students are normally conscious about the fact that "there is something wrong." That is why they are at risk for emotional problems (Bachmann 2008). Justin and Timo lost their motivation, Lars got angry in such a way that it was mentioned in his school report and, furthermore, some of the students we worked with seem to work under high pressure to overcome a barrier. Therefore, on the behavioral side, intrapersonal variables (see Fig. 8.1) that support the development of high mathematical potential are not visible.

These examples also show that the emotional reactions of the students can mask a problem that is not visible at first.

Emotional and behavioral problems may occur

1. With learning disabilities
   - when children are aware of the discrepancy between their potential in different areas
     e.g., weakness in reading competencies compared with high mathematical potential
   - when children experience repeated situations of misunderstanding
     e.g., weakness in auditory perception combined with high mathematical potential
2. With ADHD/ADD
   - when children are excluded from activities with others due to their behavior
   - when students cannot control their impulsiveness and/or forget objects or appointments

3. With ASD
   – when children are isolated due to their social problems.

8.7 What Should Be Done?

Going back to Fig. 8.1, on the side of student learning disabilities, special educational needs such as disorders or learning disabilities are risk factors for underachievement and for emotional problems. There are also several risk factors on the side of the environment. At the behavioral level, a teacher may notice the activities of a student or a withdrawal from activities. How to interpret the behavior depends on the knowledge about possible reasons for that behavior. Due to the interconnectedness of environmental and intrapersonal variables, analyzing the challenges of the learning environment and, in some cases, a differential diagnosis are preconditions for interventions.

For Justin, it was helpful for him to be aware of the discrepancy between his reading and his mathematical abilities. He learned to compensate for his disability by asking his tutors or peers to reread the texts if necessary. His tutors also changed their behavior. Lars needed a confirmation in communication processes to avoid misunderstandings. He also learned to ask questions. His tutors no longer interpreted repeated questions as inattention. Because we knew about Leon being autistic, we did not interpret his need to dominate conversations as impolite and developed methods to work with him in a way that the other students also got the necessary attention. As Timo was not challenged in school he needed problems of a higher difficulty.

Although these are only four cases, they allow some general remarks. If we go back to Fig. 8.1, it is possible to describe every child far more extensively. In everyday work with students, their situations in the classroom and relations with teachers, therapists, parents, and siblings influence developmental and learning processes, as does the type of disorder they have. Nevertheless, students have in common a barrier they must overcome, and teachers should support their students to develop compensation strategies.

Fostering a twice-exceptional student cannot wait until therapeutic interventions are successful. Therefore, in all cases a distinction between situational and long-term interventions is helpful (Nolte 2000, 2004). In learning processes, a situational intervention matching the students' needs with their learning setting allows students to participate in lessons in school and in fostering programs. A situational intervention is based upon the use of materials and methods that support a student in a way that the disability or the disorder does not hinder a student's work on a problem. To develop such interventions is the responsibility of the teacher, perhaps together with the student. These interventions are orientated on the phenomena: Students who cannot read texts get help by having the text read to them. Students who have weaknesses in motor skills need perhaps larger material that they can handle. Students who cannot focus in communication processes need different support. In many cases, situational interventions help to immediately overcome a barrier. However, long-term interventions and/or therapeutic interventions or treatments are often needed to support the development of the student (Nolte 2016).

For therapeutic interventions, specialists for different kinds of disorder are needed. Even though there are many forms of treatment known for disorders, how to overcome disorders in the field of giftedness remains an unanswered question. Traits of students with high abilities allow affected students to better develop self-regulating methods and to grasp new ideas more easily than other students. Some training is based on this (e.g., Fischer-Ontrup and Fischer 2016). Nevertheless, it may be hard to train students' capabilities "step by step" if they are used to grasping ideas (in mathematics) immediately.

On one hand, recognizing strengths and weaknesses is a precondition for teachers to foster a promising student and to support a student to overcome barriers or to develop compensation strategies. On the other hand, disabilities and disorders as well as potentials are located on a continuum. It is a challenge for teachers to keep the balance between noticing a barrier and carefully observing a student's development: Do the developed methods match the student's needs or is it necessary make a different psychological or medical diagnosis? Working with students requires balanced and individual support. Besnoy et al. (2015) point out that, "unfortunately, the multifaceted characteristics and varying needs of each twice-exceptional child prevents implementing uniform instructional approaches across the entire population" (p. 108).

Nevertheless, exposing children to challenging problems that can be solved on different levels may be helpful for a variety of reasons:

1. Mathematically gifted students normally enjoy working on harder problems. Offering problems that can be solved in different ways and on several levels gives them a chance to demonstrate high potential (Nolte and Pamperien 2017).
2. Working on challenging tasks may uncover masked disabilities and disorders.
3. For students with ASD, Foley Nicpon et al. (2011) point out that "the social skills issues of some intellectually gifted children dissipate when they are challenged and placed in appropriate classes" (p. 12). In general, for gifted students, challenging problems help avoid boredom and, thereby, a variety of emotional and behavioral problems.

> These results suggest that educational programs that are designed specifically to address the academic and social needs of gifted students can be successful in reversing many underachievement behaviors, particularly those that are due to a mismatch between students' needs and the school setting. (Matthews and McBee 2007, p. 167)

For teachers with high mathematical and pedagogical content knowledge, challenging problems are an adequate tool to identify high mathematical potential: This is part of the professional knowledge of a mathematics teacher. However, this addresses only one side of twice exceptionality. The diagnoses of disabilities and disorders must be in the hands of specialists. “School personnel almost always determine if a student is gifted and talented, but it takes a psychologist, psychiatrist, or another trained mental health professional to complete the twice-exceptional classification by appropriately diagnosing an ASD” (Foley Nicpon et al. 2011, p. 11). This can be generalized for ADD, ADHD and other developmental disorders.

The masking effect is not only observed in relation to achievement. Phenomena on the behavioral level such as inattention may be caused by ADD, ADHD, diabetes, lack of sleep, lack of interest and sorrow because of the death of a pet, and so on. In order to identify twice exceptionality and the different reasons for students’ attributions, behaviors, and levels of achievement, a team of various specialists is needed for children’s development of mathematical giftedness. Although, today the existence of twice exceptionality has been confirmed by many researchers, most of the approaches to support students focus only on one of the two parts: giftedness or disorder. Therefore, identification and support is still a challenge for students as well as parents, teachers, and specialists.

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Part III
Teaching Strategies to Foster Creative Learning

Chapter 9
Flexibility of Pre-services Teachers in Problem Posing in Different Environments

Wajeeh Daher and Ahlam Anabousy

Abstract Previous studies showed that problem posing tasks encourage students' creativity that includes three components: fluency, flexibility and originality. The present research intends to examine mathematics pre-service teachers' flexibility in problem posing, when they pose problems on a specific mathematical situation called the Paper Pool. The pre-service teachers worked in four environments: with technology and with what-if-not strategy, with technology but without what-if-not strategy, without technology but with what-if-not strategy, and without technology and without what-if-not strategy. Seventy-nine pre-service teachers participated in the research, where 19–21 of them worked in each environment. Qualitative methods (deductive and inductive constant comparison methods), as well as quantitative methods (Means, standard deviations, ANOVA, effect size and Scheffe's post hoc test) were used to verify the participants' flexibility in the four environments. The research findings showed that the participants used three posing approaches that included six strategies. At the same time, the participants posed ten problem types. The group, who used technology and what-if-not strategy, used all the six strategies and raised all the types of problems. It can be concluded that technology, as well as what-if-not strategy, affected positively students' problem posing. At the same time, the combination between technology and what-if-not strategy affected positively the participants' flexibility in problem posing more than any one of the two tools alone. These results were affirmed qualitatively as well as quantitatively.

Keywords Flexibility · Problem posing · Tools · Pre-service teachers

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9.1 Introduction and Literature Review

9.1.1 Introduction

The present chapter examines pre-service teachers' flexibility in problem posing, when they pose problems in four different environments about a specific mathematical situation called the Paper Pool. According to the NCTM Illuminations site,[1] students are asked, through the Paper Pool investigation, to play a game called Paper Pool. It is played on rectangular grids made of congruent squares. Furthermore, this game is played by hitting a ball from the left bottom corner of a table (a rectangle), so that it travels at a 45° diagonal across the grid. If the ball hits a side of the table, it bounces off at a 45° angle and continues its travel until it hits a pocket. Students continue the investigation of the Paper Pool situation by exploring different and numerous tables and organizing the results. Using the collected data, they attempt to find a relationship between the size of the table, the number of hits, and the pocket in which the ball lands. The game provides an opportunity for students to develop their understanding of different mathematical concepts as ratio, proportion, greatest common factor and least common multiple.

In the current research, seventy nine pre-service teachers from two education colleges were asked to pose problems concerning the Paper Pool situation. The pre-service teachers worked in four environments that differed in their use of tools; specifically a technological tool—An applet, and a problem posing tool—the what-if-not strategy. In the first environment, the pre-service teachers worked without technology and without what-if-not strategy. In the second environment, the pre-service teachers worked with technology, but without what-if-not strategy. In the third environment, the pre-service teachers worked without technology, but with what-if-not strategy. In the fourth environment, the pre-service teachers worked with technology and with what-if-not strategy. The participating pre-service teachers were encouraged to pose as many problems as they wanted regarding the mathematical situation. This activity had many potentialities for the pre-service teachers, as introducing them to mathematical creativity (where they are involved with an activity in which they ask different and new questions), mathematical problem posing (posing problems instead of solving problems) and tool use in problem solving (in our case problem posing). These three aspects are major aspects in mathematics education, which makes it necessary that mathematics pre-service teachers are prepared to work with them, so that they take care of them in their future teaching of mathematics.

Thus, the present chapter is considering three important aspects of mathematics learning and teaching, namely problem posing, creativity and tool use.

[1]At http://illuminations.nctm.org/unit.aspx?id=6526.

9.1.2 Literature Review

9.1.2.1 Problem Posing

Problem solving is one of the cornerstones of mathematical activity (e.g., NCTM 2000). Problem posing and problem solving are connected and natural companions (Bonotto 2013; Kilpatrick 1987; Koichu and Kontorovich 2013; Silver et al. 1996; Singer and Voica 2012). Stoyanova and Ellerton (1996) describe mathematical problem posing in detail as "the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems" (p. 518). Problem posing involves processes of generation of new problems, as well as processes of reformulation of given problems (Duncker 1945; English 2003; Silver 1994; Whitin 2006). Moreover, problem posing, as Cai et al. (2015) argue, is a critical aspect of the work of teachers, both in posing problems for their students and in helping their students pose interesting problems in the mathematics classroom. Furthermore, Cunningham (2004) showed that providing students with opportunities to pose problems enhanced their reasoning and reflection. Silver (1995) identified three types of students' experiences in problem-posing that provide opportunities for them to engage in mathematical investigation: problem posing prior to problem solving when problems arise from a particular situation, problem posing during problem solving when the student intentionally changes a problem's conditions, or problem posing after solving a problem when the student modifies his/her experience in a problem-solving context or applies it to new situations. At the same time, there are three categories of problem-posing experiences: free, semi-structured, and structured. In the free situations students pose problems without any restrictions. Semi-structured problem posing occur in situations where students are asked to write problems, which are similar to given problems or to write problems related to specific pictures and diagrams. Structured problem posing occurs in situations where students pose problems by reformulating already solved problems or by varying the conditions of a given problem.

Researchers were interested to investigate students' problem posing in the Paper Pool situation. Two studies that investigated students' problem posing in the Paper Pool situation, are those of Silver et al. (1996) and of Kontorovich et al. (2011). The results of the first study showed that the participating pre-service and in-service teachers posed problems in two manners: keeping problem givens fixed and varying the givens. Moreover, most of the posed problems dealt with the relationships between and among the table dimensions (length and width), the number of times the ball hits the sides on its path to its final destination, and the final pocket the ball enters. The second study results related to flexibility showed that the participating high-achieving secondary school students, working in groups, used three problem posing strategies: (1) Accepting the givens; (2) varying the givens; and (3) introducing new types of givens. At the same time, the participants raised three types of problems: (1) Analyzing the ball path; (2) finding the ball path(s) under given constraints; and (3) “static” geometrical problems. In the present study, we intended to study pre-service teachers’ flexibility when posing problems in the Paper Pool activity. What distinguishes the present research is the four environments in which the participants worked: with technology and with what-if-not strategy, with technology but without what-if-not strategy, without technology but with what-if-not strategy, and without technology and without what-if-not strategy. Here flexibility is associated with the number of questions’ types on a problem, or with the number of implemented posing strategies (Leikin et al. 2009).

Problem posing has the potentiality to affect positively the problem solving capacities of students, as well as other aspects of students’ learning; for example their critical and creative thinking (e.g., Bonotto 2013; Stoyanova and Ellerton 1996). Several researchers studied the relationship of students’ problem posing and their creativity (e.g., Silver 1997; Silver and Cai 2005; Singer et al. 2015; Voica and Singer 2013; Van Harpen and Sriraman 2013; Yuan and Sriraman 2010), where creativity is one aspect of mathematics learning that mathematics education researchers have been paying attention to in the recent two decades.

Previous research on the relationship of creativity and problem posing has pointed at the relationship between these two constructs. Hadamard (1945), as reported by Silver (1994), identified the ability to find key research questions as an indicator of exceptional mathematical talent. Silver (1994) adds that the apparent link between problem posing and creativity is represented in the fact that posing tasks have been included in tests designed to identify creative behavior. Moreover, Yuan and Sriraman (2010) argue that any relationship between creativity and problem posing is probably the product of previous instructional patterns. Furthermore, Voica and Singer (2013) found evidence of links between problem posing and cognitive flexibility, where cognitive flexibility is represented by variety, novelty, and changes in framing.

In addition to the above, researchers pointed to the difficulty of students to pose creatively mathematical problems. Van Harpen and Sriraman (2013) found that even mathematically advanced high school students had trouble posing good quality and/or novel mathematical problems. Moreover, Singer et al. (2015) found that fourth to sixth graders with above-average mathematical abilities, when they modified a given problem, had difficulty in posing problems that showed understanding the deep structure of mathematical concepts and strategies.

The relationship of creativity and problem posing in mathematics education is but one of the essential issues studied in the context of students’ creativity in mathematics.

9.1.2.2 Students’ Creativity in Mathematics

Mathematics in general (e.g., Matsko and Thomas 2015; Sriraman 2009) or some aspects of it (Tabach and Friedlander 2013) have been associated with creativity, so it is only natural to encourage students’ creativity in mathematics learning, problem solving and problem posing. Keeping this in mind, mathematical creativity has recently come to be considered a major component of education (Van Harpen and Sriraman 2013) and an essential skill that teachers should enhance in all students (Kattou et al. 2013; Mann 2005). Researchers in mathematics education have attempted to define the characteristics of and the meanings associated with mathematical creativity (Voica and Singer 2013), where researchers' views on creativity have changed over time (Leikin and Pitta-Pantazi 2013). No accepted definition of creativity has been achieved (Mann 2006; Sriraman et al. 2013), where some of the existing definitions of creativity are vague or elusive (Sriraman 2005), which indicates the complexity of the creativity construct. Creativity definitions given by the various researchers could be categorized into four types according to the object of their focus (Leikin and Pitta-Pantazi 2013; Runco 2004). The focus is either on the creative person, the creative processes, the creative product or the creative environment. Some researchers are concerned with just two types: the final product and the process (James et al. 2010). Defining creativity as process, researchers describe it as the ability to think conceptually. One of the definitions of creativity as a process includes describing its three components: fluency, flexibility and originality (Guilford 1975; Leikin et al. 2009; Torrance 1974). Flexibility, which is examined in the present chapter, is associated with the number of answers'/questions' types on a problem, or with the number of implemented problem solving or posing strategies (Leikin et al. 2009; Torrance 1974).

Researchers in mathematics education have studied different issues of mathematical creativity, as its manifestation in learning and teaching, including using different theoretical frameworks to study creativity and the relationship of creativity with other educational constructs. One such issue is the characteristics of students' creativity. Kiymaz et al. (2011) found that pre-service teachers developed different problem-solving behaviors in different problem situations, encountering various difficulties due to their algorithmic strategies. They also found that the pre-service creative thinking skills mainly depended on personal and extra cognitive factors. Another such issue is how to nurture students' creativity (Leikin et al. 2011; Mihajlović and Dejić 2015; Prusak 2015). For example, Prusak (2015) found that telling mathematical stories and sharing these stories in the classroom nurtures eleventh and twelfth-grade students' creativity. Furthermore, Mihajlović and Dejić (2015) found that open-ended problems promote the creativity of elementary school students. An additional creativity issue that researchers considered is the collectivity of the mathematical creativity. For example, Levenson (2011) argued that collective flexibility could be used to describe a process as well as a product. Moreover, she claimed, based on her findings, that it seems farfetched to talk about collective originality.

Regarding the relationship of creativity with other educational constructs, researchers especially studied its relationship with problem posing (see above), with giftedness (e.g., Sriraman et al. 2013; Sriraman and Haavold, in press), with ability (Kattou et al. 2013; Lave and Leikin 2013), with expertize (Leikin and Elgrabli 2015) and with students' previous knowledge represented in different grades (e.g., Leikin and Kloss 2011; Tabach and Friedlander 2013). For example, Leikin and Elgrabli (2015) argue that students' creativity represented in their discovery skills can be developed along different levels of problem solving expertise, but the range of this development depends on the expertise. Moreover, students' creativity represented in the discovery process is rooted in the problem solving expertise of a student. In addition, Kattou et al. (2013) found a positive correlation between mathematical creativity and mathematical ability. Furthermore, they found that mathematical creativity is a subcomponent of mathematical ability.

Researchers also studied students' creativity in different contexts and when using different tools (e.g., Lavigne and Mouza 2012).

9.1.2.3 Using Tools to Promote Creativity

Tools have been present in mathematics learning (Kidwell et al. 2008), teaching (Goos and Soury-Lavergne 2010) and teacher education (Mousley et al. 2003) for a long time. Moreover, tools can be used to encourage students' creativity (Kynigos and Moustaki 2014). Solomon and Schrum (2010) says that web tools support students in learning different skills; some of which are communication, collaboration, and creativity. Moreover, these tools help overcome environmental and cultural barriers to creativity (Victor and Vidal 2009).

The what-if-not strategy, as a tool that supports learners in their problem posing, was suggested by Brown and Walter (1990). This strategy is based on the assumption that modifying the components of a given problem can yield new and intriguing problems. To apply this strategy, students are encouraged to traverse three levels, beginning with the re-examination of the given problem, in order to derive closely related, new ones. At the first level, students are asked to produce a list of the attributes of the problem. At the second level, for each attribute, they must ask the what-if-not question and suggest alternatives to the given problem. At the third level they formulate new problems and questions, inspired by the alternatives. Shriki (2013) says that what-if-not strategy, as a tool for problem posing, can yield new and stimulating problems that ultimately may result in some interesting investigations. Lavy and Bershadsky (2003) say that the what-if-not strategy could make the learners rethink the geometrical concepts they use while creating new problems, as well as helping them make connections between the given and the new concepts and as a result deepen their understanding of these new concepts. Moreover, Lavy and Shriki (2007) say that the usage of the problem posing strategy assists students in the activity of discussing a wide range of ideas, and in considering the meanings associated with a mathematical problem rather than merely focusing on finding its solution. The previous findings indicate the need for strategies as what-if-not strategy, which encourage the deep learning of students and which are not used enough by them (Mishra and Iyer 2015). This deep learning could result in more creativity of students when posing mathematical problems. The present chapter examines whether the what-if-not strategy indeed contributes to pre-service teachers' creativity in problem posing.

As to the use of technology, as a tool, in problem posing, researchers noted that problem-posing and conjecturing activities can become richer and more profound when technology is involved, because the software can rapidly and efficiently take care of the technical work, such as computing and graphing (Lavy and Shriki 2010). In particular, problem posing using dynamic geometry software involves various effects of the interface of the tool on the procedures used by the students and on their understanding (Christou et al. 2005). Moreover, technology provides dragging functionality that gives students the opportunity to use visual reasoning and to generalize problems and relationships (Sinclair 2004). In other cases, it helps them check the validity of new mathematical situations (Lavy and Shriki 2010). Examining the use of digital technology for mathematical problem posing, Abramovich and Cho (2015) found that the use of graphing software as a medium for problem posing is conducive for developing sophisticated questions about algebraic equations with parameters.

In addition to studying the use of technology in problem posing, researchers in mathematics education studied the use of technology to encourage students' creativity (Lavigne and Mouza 2012). Lavigne and Mouza (2012) pointed at the virtual open environments that have little structure as providing opportunities for autonomy and creative expression. Moreover, Hong and Ditzler (2013) argue that the space for creative activities is virtually unlimited and creative processes and activities are enriched from working in virtual space and in collaboration with others, especially peers.

Furthermore, Leikin (2011) concludes that it is necessary to involve technological tools that promote mathematical creativity in students and support teachers' attempts to scaffold students' mathematical inquiry. Specifically, researchers suggest that technology-based activities open opportunities for promoting students' mathematical creativity (e.g., Hoyles 2001; Yerushalmy 2009). Yerushalmy (2009) argues that technology supports students' questioning, conjecturing and discovery, which promotes their mathematical creativity.

Researchers also studied teachers' views regarding using technology in activities that promote students' creativity. They found that in-service and pre-service teachers vary in their views of using technology to promote students' creativity (Bolden et al. 2010; Panaoura and Panaoura 2014). Panaoura and Panaoura (2014) found that few pre-service teachers associated creativity with technology, while Bolden et al. (2010) found that most of the participants associated creativity with the use of resources and technology. These findings further emphasize the need for tools to support students' creativity in problem solving and problem posing.

9.2 Research Rationale, Goals and Questions

9.2.1 Research Rationale and Goals

Pre-service teachers should be exposed to problem posing (Chapman 2012) and creativity (Bolden et al. 2010) to encourage them to integrate problem posing and creativity tasks in their future teaching of mathematics. This exposition needs to be accompanied by research that examines the product of the participating pre-service teachers' creativity, in our case their flexibility in problem posing in a specific situation called the Paper Pool. On the other hand, researchers in mathematics education tried to examine the contribution of tools to students' problem posing (Abramovich and Cho 2015; Christou et al. 2005; Lavy and Shriki 2010) and mathematical creativity (Kynigos and Moustaki 2014; Solomon and Schrum 2010; Victor and Vidal 2009). These attempts are still limited, which points at the need of further research that verifies the effect of tools on students' creativity, problem posing and creativity in problem posing.

To elaborate more, the present chapter attempts to contribute to the research related to the use of tools to promote learners' creativity in problem posing, by examining the effect of using two tools by pre-service teachers to pose problems about a specific mathematical situation called the Paper Pool. These tools are similar in some properties, while they are different in other properties. Their similarity lies in their being cognitive tools for exploration in mathematics learning (Pea 1987), while their difference lies in that what-if-not strategy is more structural than the applet. Furthermore, the applet is more representational than what-if-not strategy, where we can change the dimensions of the table in the applet and watch the change in the number of hits of the ball, as well as the pocket in which the ball lands. Comparing the contributions of the two different-structuring and different-functioning tools is one issue that needs further studying. We attempt, in the present chapter, to contribute to this issue.

Moreover, the present chapter verifies the effect of tools on pre-service teachers' flexibility combining between qualitative and quantitative methods. This combination adds to the validity of the results. It also takes care of different aspects of the pre-service teachers' flexibility, as categories of this flexibility (taken care of by the qualitative methodology), as well as the significance of the difference between the flexibility scores of the four research groups (taken care of by the quantitative methodology).

9.2.2 Research Questions

The present study intends to study pre-service teachers' flexibility in problem posing related to the Paper Pool activity. This flexibility is associated with the pre-service teachers' strategies in posing problems, as well the types of problems they posed. The research context includes four environments that differ in using technology, on one hand and what-if-not strategy, on the other hand. The research questions are:

1. What are the pre-service teachers' strategies in posing problems about the Paper Pool activity with/without technology and with/without what-if-not strategy?

2. Do the pre-service teachers' scores in strategies for problem posing differ significantly according to the use of what-if-not strategy or technology?
3. What are the types of pre-service teachers' posed problems about the Paper Pool activity with/without technology and with/without what-if-not strategy?
4. Do the pre-service teachers' scores in the types of posed problems differ significantly according to the use of what-if-not strategy or technology?

9.3 Method

It is the goal of the present chapter to study the effect of tools on pre-service teachers' flexibility in problem posing, where they pose problems on a specific mathematics situation called the Paper Pool. This makes the context of pre-service teachers' work of special importance for its influence on their learning, in our case problem posing.

9.3.1 Research Context and Participants

The participants in this research were pre-service teachers who specialized in mathematics teaching. The research was conducted in the academic year 2013–2014. The participants were majoring as mathematics teachers in the middle school. The participants were in their second year of preparation as mathematics teachers, where this preparation takes four years. The participants had previous knowledge of technology through a course called "Technology for education", in their first year and another course called "technology for the discipline" in their second year. In the course "Technology for education", the pre-service teachers were introduced to the use of open software, as Microsoft Office, in education, while in the course "technology for the discipline", the pre-service teachers were introduced to the use of specific software, as Excel, applets (not the Paper Pool applet) and GeoGebra, in mathematics education. This introduction of the mathematical software was done through the instructor's presentation, as well as through the pre-service teachers' work with them to practice them and to solve mathematical problems with them. All the participants were introduced to what-if-not strategy in their second year of study through the course "Didactics of teaching mathematics". At the same time, all the participants were introduced to creativity in the same previous course, where this introduction was limited to working with three problem-solving activities. Before the experiment reported in the present chapter, the participants had not been introduced to problem posing. In the frame of the current experiment, a part of the participants were introduced to the Paper Pool applet through letting them work with it to solve the mathematical problem. Another part of the participants were introduced to problem posing and what-if-not strategy in one lesson lasting for one and a half hour. During this lesson, the participants posed questions on two mathematical situations which the Paper Pool was not one of them. The average age of the participants was 21.23 years.

The participants were divided into four groups randomly. The groups differed in their use of technology and their use of “what-if-not?” strategy. Each group included 19–21 participants. In more detail, two groups were introduced to the what-if-not strategy and were requested to work with the Paper Pool activity using this strategy. One of these two groups was also introduced to technology, so the group members could work on the task using both what-if-not strategy and technology. One of the other two groups was introduced only to technology, in our case a Java applet, and group members worked on the task using the applet. The fourth group was not introduced to either technology or what-if-not strategy. Table 9.1 shows the number of the participants in each group.

The participants in each group were given one hour to carry out the task individually. Moreover, the participants in the two groups who worked with technology used an applet called “the Paper Pool Applet”.

9.3.2 Data Collecting Tools

The data we used were the problems that the pre-service teachers posed for the Paper Pool activity. We worked out these data in order to arrive at categories of the problems that the participating pre-service teachers posed. This was done in order to find the actual problem types’ space of the problem situation. It was also done to find the actual problem strategies’ space of the problem situation. Afterwards, the two types of flexibility for each participant were computed. Thus the data of the present research could be described as textual, which we analyzed for qualitative analysis, and numeric, which we analyzed for quantitative analysis.

9.3.3 Data Analysis Tools

To analyze the data, we first excluded problems that were not mathematical or were unsolvable. For example, the problem “what happens if we change the table’s color?” was not considered mathematical.

We considered two types of flexibility for a participant: the number of different problem types posed by a participant, and the number of different posing strategies that she/he used. This analysis method follows Kontorovich et al. (2011) who analyzed secondary school students’ creativity in problem posing. Moreover, it follows studies that examined students’ flexibility in using strategies (e.g., Elia et al. 2009) and, at the same time, those who examined students’ flexibility by looking at solution types rather than strategies (e.g., Leikin and Lev 2007). In addition, this analysis of two types of flexibility follows Silver (1997) who described students’

Table 9.1 Participants' distribution in the four research groups

		What-if-not strategy		Total
		With what-if-not	Without what-if-not	
Technology	With technology	19	19	38
	Without technology	20	21	41
Total		39	40	79

strategies and problem types when posing problems in the frame of the Paper Pool activity. We are aware of the different analysis methods of students' creative activity, whether this activity was individual (Leikin and Lev 2007; Shriki 2013; Singer and Voica 2015) or collective (Levenson 2011, 2014), but examining the same mathematical situation, as well as the same educational construct (specifically students' flexibility), we followed Kontorovich et al. (2011) in examining pre-service teachers' flexibility.

It should be noted that our consideration of the solution types differs from researchers' consideration of fluency, because here we looked at types rather than answers. For example, the pre-service teacher who gave the two problems (In which pocket does a ball fall in a table whose length is twice its width?; In which pocket does a ball fall in a table whose dimensions are 8 × 16) got just one point because both of the problems belong to the same type (Asking about the pocket in which the ball falls). In the case of fluency we would have given the pre-service teacher two points.

Analyzing the collected data qualitatively, we used primarily deductive content analysis, keeping in mind the inductive content analysis to arrive at new categories of problem types and problem strategies. In more detail, we performed constant comparisons between the units of the gathered data (the posed problem) in order to categorize them into problem types or strategies. This deductive content analysis explains partially the categories and situations of problem types and problem posing strategies at which we arrived, as part of them was reported in previous literature (Silver et al. 1996; Kontorovich et al. 2011). The inductive content analysis gave us new categories as specific/general and object/relation. Each one of the authors coded the participants' posed problems. Cohen's kappa (Cohen 1960) was used as a measure of agreement between the two raters. The inter-rater reliability for flexibility of questions' types was 89.1%, and for flexibility of posing strategies was 88.8%. These high percentages suggest a good agreement between the two raters and therefore a satisfactory reliability for each component of creativity.

To analyze the data quantitatively, we used the SPSS 21 package. Means, standard deviations, ANOVA and effect size were performed to examine whether there are significant differences between pre-service teachers' flexibility scores in problem types and posing strategy types of the participating pre-service teachers in the four environments. When ANOVA gave significant results, Scheffe's post hoc test was performed to decide which environment contributes more to the flexibility of the participating pre-service teachers.

We combined between qualitative and quantitative methods of analysis, so that we can advance our understanding about the phenomenon under the investigation (Niglas 2000), in our case pre-service flexibility in problem posing, in tools-environments. In addition, this combination would constitute triangulation of methodologies (Denzin 1978). The qualitative methodology provides us with categories of flexibility types and strategies, as well as the content of each of the types and strategies, while the quantitative data enables us to verify the significance of using tools in problem posing.

9.3.4 The Task

The task was adapted from Silver et al.'s (1996). It was chosen for it is rich enough to stimulate the generation of interesting problems and conjectures and, at the same time, it is accessible for learners as it requires only knowledge of rather basic mathematical concepts (Silver et al. 1996).

In addition, this task fits working with, using traditional means, as well as technological means. Following is the task text. Figure 9.1 shows the text of this task.

9.3.5 The Software

Two of the groups used an applet called the Paper Pool Applet[2] (see Fig. 9.2). The Paper Pool applet enables the user to change the dimensions of the table (its length and width), and to change the speed of the ball. It also enables the user to show or hide the track of the ball path, the grid of the table, the table itself, and the count of hits.

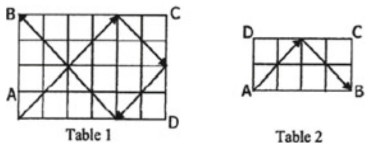
9.4 Findings

9.4.1 Strategies of Problem Posing

One goal of the research was to categorize the strategies that the pre-service teachers used to pose problems. This was the goal behind the first research question. To answer this question, we examined the different strategies that the participants used, keeping in mind the categories of strategies reported by previous research as Silver et al. (1996) and as Kontorovich et al. (2011). We also carried out inductive content analysis to arrive at additional categories not reported in the literature. The analysis resulted in three approaches. The first relates to the object of the problem: a

[2]At http://illuminations.nctm.org/Activity.aspx?id=4219.

Imagine billiard tables like the one shown below. Suppose a ball is shot at 45° angle from the lower left corner A of the table. When the ball hits a side of the table, it becomes off at a 45° angle. In Table 1, the ball travels on a 4x6 table and ends up in pocket B, after 3 hits on the sides. In Table 2, the ball travels on a 2x4 table and ends up in pocket B, after 1 hit on the side. In each of the figures shown below, the ball hits the sides several times and then eventually lands on a corner pocket.



Based on this situation, pose and write down as many interesting mathematical problems as you can.

Fig. 9.1 The Paper Pool task

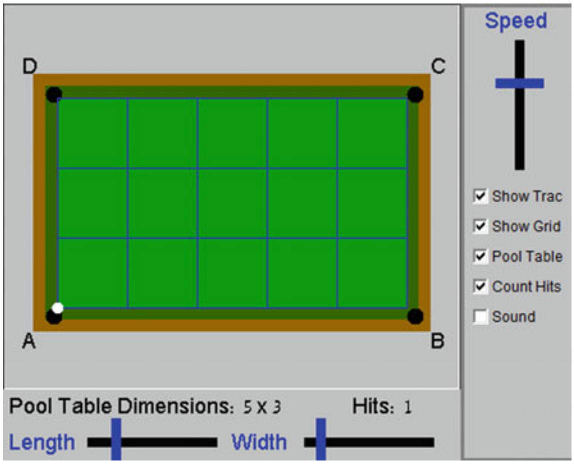


Fig. 9.2 The interface of the Paper Pool Applet

mathematical object or a mathematical relation. The second relates to the givens of the problem: keeping the givens, or varying the givens. The third approach relates to generality of the problem: asking about a specific object, or asking about a general object. We present examples on the different types of problems in the three approaches in Tables 9.2 and 9.3. Table 9.2 presents examples from problems posed on specific object/relation, while Table 9.3 presents problems posed on general objects/relation. Table 9.2 differentiates between 'keeping the givens' and 'varying the givens', while Table 9.3 deals only with 'varying the givens', for generality, by its nature, needs varying the givens of a problem.

The four groups of pre-service teachers differed in their use of strategies in problem posing. Both of the groups who did not use technology used mostly two strategies: keeping the givens or varying the givens, while asking about a specific mathematical object. The group who used technology but not what-if-not strategy

Table 9.2 Problems posed on a specific mathematical object/relation

Topic	Givens	
	Keeping the givens	Varying the givens
Mathematical object	What is the length of the path resulting from a ball hitting the sides of the 2 × 4 table?	What is the angle that we should use in throwing the ball in order to get equal number of hits in the two given tables?
Mathematical relation	How do the dimensions of the Table 2 × 4 determine the number of ball hits?	What happens to the relation between the number of ball hits and table dimensions on a table with 6 pockets?

Table 9.3 Problems posed on a general mathematical object/relation

Topic	Givens
	Varying the givens
Mathematical object	What are the table's dimensions that guarantee no intersections of the ball path until it hits a pocket?
Mathematical relation	What is the relationship between the table's dimensions and the number of ball hits?

used mostly four strategies: varying the givens and asking about a general mathematical object or relation, or varying the givens and asking about specific mathematical object or relation. The fourth group, who used technology and what-if-not strategy, used all the six strategies.

9.4.2 *Differences in Scores of Flexibility of Strategies in the Four Environment*

A second goal of the research was to examine whether the differences between the scores of the flexibility of strategies in the four environments were significant. This goal was behind the second research question. To answer this question, we computed the means and standard deviations of pre-service teachers' flexibility related to the different strategies, in the four groups (see Table 9.4).

The results of the ANOVA test showed that the mean score of flexibility related to problems' strategies is significantly different among the four research groups, with F(3.75) = 16.36, $p < 0.001$, $\eta^2 = 0.396$. Based on Cohen's standard, we have large effect size here. Moreover, the results of Scheffe's test showed that the mean score of flexibility related to strategies in the presence of technology and what-if-not strategy was significantly higher than in the case of the other three groups ($p < 0.05$). At the same time, no significant difference was found between the mean scores of flexibility related to problems' strategies among the three groups who did not use technology or/and did not use what-if-not strategy.

Table 9.4 Mean and standard deviation of students' flexibility related to strategies in the four groups

Group	N	Mean	SD	Range
Without technology and without what-if-not	21	1.95	0.59	1–3
Without technology and with what-if-not	20	2.00	0.46	1–3
With technology and without what-if-not	19	2.32	0.48	2–3
With technology and with what-if-not	19	2.89	0.32	2–3

9.4.3 Types of Posed Problems

A third goal of the research was to categorize the types of problems that the pre-service teachers posed. This was the goal behind the third research question Here, we also used deductive content analysis, keeping in mind the types of problems reported by previous research as Silver et al. (1996) and as Kontorovich et al. (2011). We also carried out inductive content analysis to arrive at additional problem types not reported in the literature. Doing so, we arrived at ten problem types: problems about the pocket in which the ball falls, problems about table's dimensions, problems about the ball path, problems about the number of ball hits, problems about the number of thrown balls, problem about the pockets' number, problems about the angle of throwing the ball, problems about the ball speed, geometric problems related to the ball path, and problems engaging two or more of the previous issues. Table 9.5 shows an example on each of the problem types posed by students in the fourth group. We chose the fourth group because all the problem types were posed by its members.

Only participants who used technology and what-if-not strategy raised all types of problems. Moreover, only participants who worked with technology posed problems about the speed of the ball, while only participants who worked with what-if-not strategy posed problems about the number of balls.

9.4.4 Differences in Scores of Flexibility of Strategies in the Four Environment

A fourth goal of the research was to examine whether the differences between the scores of the flexibility of problem types in the four environments were significant. This goal was behind the fourth research question. To answer this question, we computed the means and standard deviations of pre-service teachers' flexibility related to the different types, in the four groups. Table 9.6 shows the results.

The results of the ANOVA test showed that the mean score of flexibility related to problems' types is significantly different among the four research groups, with F (3.75) = 78.5, $p < 0.001$, $\eta^2 = 0.758$. Based on Cohen's standard, we have large effect size here. Moreover, the results of the Scheffe's test showed that the mean

Table 9.5 Examples on each of the problem types that the fourth group posed

Problem type	Examples
Asking about the pocket in which the ball falls	In which pocket does a ball fall in a table whose length is twice its width?
Asking about the table's dimensions	What are the table dimensions when the number of hits is two?
Asking about the ball path	What is the path of the ball in a Table 6 × 4, when it starts from pocket C?
Asking about the number of ball hits	What is the greatest number of hits in a table m × n that the ball makes before falling into a pocket?
Asking about a situation when more than one ball are thrown	What happens when a ball is thrown from A and a second one from C? Will they collide?
Asking about a situation when the number of pockets changes	What happens if the number of pockets is 6? 2?
Asking about the angle of throwing the ball	What is the angle that we should use to throw the ball in order to fall into pocket A?
Asking about the ball speed	What happens when the speed of the ball increases?
Asking a geometry question related to the path	Would the ball path in a table m × n make a parallelogram?
Asking about two or more of the previous issues	What is the relationship between the angle of the table and the pocket in which it falls?

Table 9.6 Mean and standard deviation of students' flexibility related to types of problems in the four groups

Group	N	Mean	SD	Range
Without technology and without what-if-not	21	2.33	0.48	2–3
Without technology and with what-if-not	20	4.05	1.19	2–6
With technology and without what-if-not	19	4.16	1.30	2–7
With technology and with what-if-not	19	7.11	0.81	5–8

score of flexibility related to problems' types in the presence of technology and what-if-not strategy was significantly higher than the other three groups, while the mean score of flexibility related to problems' types without technology and without what-if-not strategy was significantly lower than mean score of each of the other mean groups ($p < 0.05$). At the same time, the mean score of flexibility related to problems' types with technology and without what-if-not strategy was not significantly different from that without technology and with what-if-not strategy ($p = 0.990$).

9.5 Discussion

It was the goal of the present chapter to examine mathematics pre-service teachers' flexibility in problem posing in four environments differing in the use of technology and what-if-not strategy. The research findings showed that each of the tools used by the participating pre-service teachers (technology and what-if-not strategy) affected positively their flexibility. Here the what-if-not strategy supported the mathematics learners in their varying of task givens (Brown and Walter 1990). Moreover, asking what-if-not leads students to ask deep questions (Mishra and Iyer 2015) through organizing their problem posing activity. As to technology, it supported the participating students in their generalizing activity (Tabach 2011). This support is due to the technology potentialities and its varied modes of use, as data collection and analysis, visualization, and checking (Doerr and Zangor 2000). These potentialities helped students that used technology to understand the mathematical situation, and therefore have more means to vary their posed problems. This facilitation of the learning process is described by researchers who pointed at the increasing opportunities for students that technologies provide, so that they become more able to explore the mathematical situation (Crespo and Sinclair 2008). This enables them to modify given problems into higher quality problems (Cai et al. 2015).

Furthermore, it could be argued that the Paper Pool activity is a rich context that encourages students' mathematical activity, including their creativity. This is so because the Paper Pool as a mathematics situation is accessible to learners as it requires only knowledge of basic mathematical concepts and, at the same time, can stimulate the generation of interesting problems (Koichu and Kontorovich 2013; Silver et al. 1996). This rich context was essential in realizing the potential of the technology used (Clements 1995).

The research findings also show that the group, who combined between technology and what-if-not strategy, used three approaches in problem posing, where these approaches included six strategies, and raised, at the same time, ten problem types. The other three groups of pre-service teachers' used only part of the strategies and types. This implies that combining between these two tools makes mathematical problem posers aware of more posing strategies and problem types. The findings also showed that the numbers of strategies and problem types, used by the group that combined between technology and what-if-not strategy, are greater than those reported in Kontorovich et al. (2011) and Silver et al. (1996). This could be due to the potentialities of the two tools that directed the participating mathematics learners in their problem posing activity. Bonotto (2013) argues that suitable cultural artifacts can become a meaningful source for problem-posing activities. In our research, the combination of two cultural artifacts contributed to the flexibility of pre-service teachers in a cultural authentic activity, namely the Paper Pool activity.

The influence of each of the tools and their combination could be explained as contributing to the problem solving/posing processes described by Polya (1945). In more detail, the tools contributed to understanding the mathematical situation through emphasizing the different components of this situation, as well as the relation between these components. The applet interface shows the different components of the situation, while the first step of what-if-not strategy shows these components too. Manipulating the applet interface, as well as posing problems according to the second step of what-if-not strategy, facilitates seeing relations between the situation components. Moreover, the both tools support the learning in the second process described by Polya; i.e. devising a plan. Technology does this through helping the learning in actions as checking, looking for a pattern, and drawing a picture. On the other hand, what-if-not strategy does that through helping the learner in actions as making an orderly list, eliminating possibilities and working backwards. Devising the plan, generally it is not difficult to carry it out. It could be argued that technology supports the learner more than what-if-not strategy in the fourth process of problem posing. This is so because technology, due to its potential to facilitate reworking with the same process or example, helps looking back at what we have done, what worked for us or did not work. This will enable the learner to decide what strategy to use to pose future problems.

In addition to the argument above, the difference between the flexibility of the participants from the four groups occurred as a result of the different environments' characteristics; i.e. the different contexts. This is present in the findings that only students who used technology asked about the speed of the ball, which could be due to the presence of this factor in the applet interface. On the other hand, the number of balls is not present in the applet interface, so students who worked with technology and without what-if-not strategy did not pose problems about this variable, while students who worked with what-if-not strategy did. So, task contexts are of great implications regarding the type of activity and learning with which mathematics learners are engaged (Joubert 2013). In addition, the task contexts are part of the teaching contexts which impact students' learning (Sarrazy 2002). Sarrazy (2002) asks what affects students' responsiveness to a problem, aside from inter-individual differences. He answers this question by saying that the inter-class variations could explain the variability of students' responsiveness to a problem. In our case this interclass variability was represented in the pre-service teachers' use of tools.

As to the quantitative results, using Scheffe's test showed that the mean score of flexibility related to strategies, as well as problem types, in the presence of technology and what-if-not strategy was significantly higher than in the case of the other three groups. At the same time, no significant difference was found between the mean scores of flexibility related to problems' strategies among the three groups who did not use technology or/and did not use what-if-not strategy. These results show the effectiveness of combining between tools to encourage students' problem posing and their flexibility in the activity of problem posing. Researchers in mathematics education pointed to the effect of tools on students' creativity (Lavigne and Mouza 2012; Solomon and Schrum 2010; Victor and Vidal 2009). Our experiment reported in the present chapter emphasizes the potentials for students' creativity when providing them with tools.

Moreover, the mean score of flexibility related to problems' types without technology and without what-if-not strategy was significantly lower than mean score of each of the other mean groups. Furthermore, the two groups who used one tool only did not differ significantly in problem types. These results once again show the effectiveness of combining between tools in the mathematics classroom. Previous attempts have been done to combine tools and resources in the mathematics classroom in order to influence positively students' learning of mathematics. Cross (2009) reports the combining between argumentation and writing in the mathematics classroom to enhance students' achievement, where the results revealed significant differences between the groups; specifically the students who engaged in both argumentation and writing had greater knowledge gains than students who engaged in argumentation alone or neither activity.

9.6 Conclusions and Implications

Ellerton (2013) described his research as "a first step in understanding pre-service students' responses to being confronted with problem-posing activities directly linked to the curriculum" (p. 88). The research reported in the present chapter could be described as trying to verify and to understand the effect of tools, specifically technology and what-if-not strategy in pre-service teachers' flexibility in problem posing. Moreover, the present chapter shows that combining between tools is effective for students' learning of mathematics, especially their flexibility.

Examining the use of technology and what-if-not strategy by pre-service teachers, it was found that each tool affected positively the pre-service teachers' problem posing activity. Moreover, combining the two tools affected positively the pre-service teachers' activity more than any one of the two tools alone. These findings indicate the need for different tools in the mathematics classroom (Bonotto 2013; Brown and Walter 1990; Maschietto and Trouche 2010; Yerushalmy 2009), as well as their combination. It is claimed that combining between tools and strategies could be more effective on students' learning of mathematics than one tool/strategy (Cross 2009). This is especially true when each tool/strategy serves different function. In our case, technology serves visualization of mathematical relations (Doerr and Zangor 2000) and generalization (Leung et al. 2006), while what-if-not strategy serves organization of students' thinking and behavior (Brown and Walter 1993).

The present chapter examines how the use of tools affects students' creativity in problem posing. More studies are needed to verify the effect of this use on students' creativity in mathematical problem posing, as well as problem solving, especially when these tools are combined. Moreover, the chapter examines this issue of tools' use in a specific mathematical situation that could be described as a game (Math Forum at NCTM 2016). Studies are needed that verify the issue in regular mathematical lessons.

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Chapter 10
Ornaments and Tessellations: Encouraging Creativity in the Mathematics Classroom

Hana Moraová, Jarmila Novotná and Franco Favilli

Abstract The presented paper focuses on encouraging creativity in mathematics lessons in heterogeneous mathematics classrooms. It is an extended version of a paper presented at ICME 2016 in Hamburg. It describes teaching experiments conducted within the frame of the project M^{3}EaL: Multiculturalism, Migration, Mathematics Education and Language (526333-LLP-1-2012-1-IT-COMENIUS-CMP), a multilateral project whose aim was development of teaching units to support immigrant pupils and pupils from different socio-cultural backgrounds (m3eal.dm.unipi.it). The paper shows that if pre-service and in-service teachers face a situation in which they cannot rely on traditional textbooks and ways of doing mathematics—in this case a culturally heterogeneous classroom—they tend to be very creative in planning their lesson and at the same time encourage their pupils' creativity. Thus cultural heterogeneity may be perceived as advantage as it may result in breaking down stereotypes in mathematics classrooms.

Keywords Creativity · Substantial learning environment · Socio-cultural background · Teaching in culturally and linguistically heterogeneous classrooms

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10.1 Introduction

The world is becoming a global village, and it no longer holds that a teacher can expect to be teaching a more or less homogeneous class with pupils with more or less similar socio-cultural and linguistic backgrounds. Migration to EU countries results in the need to introduce practices into mathematics classrooms that will allow pupils from various socio-linguistic and cultural backgrounds to be successful, follow the instruction and curriculum, and build on their identity. This corresponds to the principles of inclusive education, which means that all pupils and students are welcome to attend their neighbourhood schools in age-appropriate, regular classes and are supported to learn, contribute and participate in all aspects of the life of the school (UNICEF). The premise is that a more inclusive education will promote a more inclusive society that respects diversity. Inclusive schools should include everybody, celebrate differences, support learning and respond to individual needs, including the needs of learners from various cultural and linguistic backgrounds.

This concept has caused educators and researchers to look for teaching units and learning environments that are able to support the needs of all the students at their schools and that will help teachers who deal with complex situations involving teaching a heterogeneous group of pupils with very diverse social, cultural and educational backgrounds and very diverse needs.

A questionnaire survey conducted within the frame of the project M^{3}EaL: Multiculturalism, Migration, Mathematics Education and Language (co-funded by the European Commission under its Longlife Learning Programme, project number: 526333-LLP-1-2012-1-IT-COMENIUS-CMP) clearly showed that teachers feel that they feel the need to have materials developed especially for classrooms with cultural heterogeneity. The situation becomes even more complicated if teachers are expected not only to present the required mathematical content to their pupils but also to develop their creativity regardless of their students' cultural and linguistic backgrounds. The goal of the presented research was to show that a teacher does not need to look for new teaching units or new topics for each area of school mathematics and each age group. Instead, they can creatively adapt and modify one selected environment to suit the particular situation of the class they are teaching. Another area of interest is the study of the impact of these changes on the creative potential of the basic teaching unit.

The project research team developed and organized pilot implementation of seven teaching units by in-service teachers from six European countries: Austria, the Czech Republic, France, Greece, Italy and Norway. The teaching units were designed to meet the needs of culturally and linguistically heterogeneous classrooms (Favilli 2015). The goal was to show that teachers can adapt basic teaching units creatively in such a way that they help learners from different linguistic and cultural backgrounds become involved in their classes without actually needing detailed lesson plans and materials. In this chapter, we illustrate this on a teaching unit developed by the Czech project team as a teaching unit for heterogeneous lower-secondary pupils. When piloted by in-service teachers from three countries, it was adapted to meet the needs of each particular educational system and student age group. The Czech teaching unit was based on ornaments from different cultures as a source of mathematics to be taught. The developed unit combined mathematics, art and creativity to boost pupils' motivation and to allow pupils to excel regardless of their cultural or linguistic backgrounds. The pilot implementation in three countries (the Czech Republic, Italy and Austria) showed that creativity was needed not only on the pupils' but also on the teachers' part. The participating teachers adapted the original teaching unit with a lot of creativity and thus adapted it to the specific situation of their own classroom and curricula.

The chapter comes out of the ressearch study presented at ICME 2016 (Moraová and Novotná 2016).

10.2 Theoretical Background

10.2.1 Teaching Heterogeneous Classrooms

The results of the project entitled 'Context problems as a key to the application and understanding of mathematical concepts' sponsored by the Grant Agency of the Czech Republic (16-06134S) showed that most primary and lower-secondary classes in the Czech Republic are heterogeneous. The situation in many other countries is similar. However, not enough attention has been paid so far to teaching mathematics in culturally heterogeneous classrooms. If researchers pay attention to teaching mathematics in culturally heterogeneous classrooms, they usually focus on linguistic issues (e.g., McDermott and Varenne 1995). But what we really have to ask is how mathematics is specific and what the dangers of teaching it in culturally and linguistically heterogeneous classrooms are. Mathematics educators (e.g., Barton et al. 2007; Bishop 1988; César and Favilli 2005) often speak of in-service teachers' call for teacher training that would provide them with the tools essential for work in linguistically and culturally heterogeneous classrooms. Mathematics educators are fully aware of the fact that a pupil who must simultaneously master both a new language and mathematics content faces a very difficult situation (Moschkovich 2007, 2012; Barwell et al. 2007; Norén 2008; Barwell 2015; Sibanda 2017).

Work in culturally and linguistically heterogeneous classrooms requires materials that do not block understanding of mathematics for any of pupils in the classrooms. Arslan and Altun (2007) are aware of the importance of learning environments. They use the term 'socio-cultural learning environments' that comes out of the concept of 'socio-cultural norms' as it is used by Sullivan et al. (2003). Arslan and Altun (2007) point out that the usually recommended learning environments may be alien to certain groups of pupils, and teachers must take steps that will decrease this alienation. These modifications affect several norms: mathematical norms among which Arslan and Altun place 'principles, generalization, procedures and results that are the outcome of mathematics education as well as its tool' (p. 109). The teacher's task is to work with such learning environments that are comprehensible for and accessible to all pupils in the particular group. This is the only way to achieving equity in education and to preventing exclusion of some groups of pupils from education.

At the same time, we can observe in mathematics education an increasing interest in and stress on creativity, communication and interaction in mathematics lessons, on careful and unequivocal formulation of ideas, on multimodality of representations (e.g., Yackel and Cobb 1996; Kynigos and Theodosopoulou 2001). This development in mathematics education will be beneficial for all pupils, including pupils from different cultural and linguistic backgrounds. We are convinced that creativity is also the way to teaching mathematics in culturally and linguistically heterogeneous classrooms. The below presented research study looks for confirmation of this conjecture.

10.2.2 Creativity in Mathematics Lessons

The paper presented here shows how pupils' creativity and flexibility can be awakened by pre- and in-service teachers' efforts to develop teaching materials that can be used in culturally heterogeneous classrooms by making use of the pupils' different cultural backgrounds and experience and adapting an existing teaching unit to the needs of particular groups of pupils. At the same time, the developed teaching unit and all its modifications for different classrooms and age groups works with pupils' creativity, integrating elements from art and history, providing space for inquiry and independent discovery of various concepts and properties, and increasing pupils' activity and motivation.

The teaching unit discussed is called 'Ornaments'. The following section describes the development of the teaching unit, the potential of this environment and modifications to the teaching unit by in-service teachers in their pilot implementations.

10.2.3 Definition of Mathematical Creativity

Mathematical creativity is a concept that raises much interest in mathematics education research. There have been a variety of views of creativity and mathematical creativity (e.g., Leikin 2009). There are a number of definitions of mathematical creativity in literature. Krutetskii (1976) characterized mathematical creativity as independence and originality. Sriraman (2005) describes mathematical creativity 'as an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population' (p. 75). Ervynck (1991) claims that mathematical understanding, intuition and insight form the basis of mathematical creation. Leikin (2009) states that

> creative products, therefore, help in the understanding of mathematical relationships because they reference former results and often respond to current needs; they are original because they can lead to unpredicted outcomes. . . . Mathematical creativity in school students is evaluated with reference to their previous experiences and to the performance of other students who have similar educational history. (pp. 130–131)

10.2.4 Ornaments

The original teaching unit developed by the Czech team within the frame of the aforementioned project was based on the concept of substantial learning environments (SLE) developed by Wittmann (1995, p. 366), namely, on the concept that 'a good teaching material for teachers and pupils should be the one which has a simple starting point and a lot of possible investigations or extensions'. It was also based on Duval's theory of registers of semiotic representations, i.e., on how students perceive and apply different usages of representations for the same concept and transformations between these representations. Since mathematical concepts and relations are abstract, they cannot be seen and felt in daily life (Duval 1993, 2000). It is important for students to recognize and use different representations of a concept and to shift from one presentation to another. This can be managed by using different representations for the same concept at the same time and in the same context (Winslow 2003).

The simple starting point in this case was a number of ornaments whose origin was in different cultures; they are used with the intention of allowing minority pupils to be heard, to present ornaments typical for their culture or home and to break the wall between home and school culture between the mathematics naturally used at home and the mathematics used at school (Meany and Lange 2013).

An ornament represents 'an artifact that can be exploited by the teacher as a tool of semiotic mediation to develop genuine mathematical signs that are detached from the use of the artifact but that nevertheless maintain with it a deep semiotic link' (Mariotti 2009, p. 427). Ornaments can be considered also from Radford et al.'s (2005, p. 120) perspective:

> The apprehension of the object in its cultural dimension—i.e., the apprehension of the cultural conceptual content and meaning of the object—requires students to engage in an interpretative and imaginative process whose outcome is an alignment of subjective and cultural meanings.

In a paper focusing on creativity, Csikszentmihalyi (1996) discusses the characteristics of the creative process and the significance of environment on the development of creativity. The creative process is divided into five stages: preparation (becoming immersed in a set of problematic issues that are interesting and arouse curiosity), incubation (during this period, unusual connections are likely to be made), insight (pieces of ideas fall together), evaluation (deciding whether the insight is valuable and worth pursuing) and elaboration (validating the insight).

Csikszentmihalyi (1996) lists major elements in the social milieu that can encourage creativity: training, expectations, resources, recognition, hope, opportunity and reward. Singer and Moscovici (2008) propose a frame for organizing classroom interactions within a constructivist approach. Their model consists of three general phases they refer to as immersion, structuring and applying, each with two sub-phases that highlight specific roles for the teacher and the pupils.

One of the ways of helping learners that are not from the mainstream population become involved in mathematics classes is to use various artefacts coming from different cultures. Bishop (1988) analysed educational situations involving cultural issues. He argues that up to early 1980s, mathematics knowledge was regarded as 'culture-free'. This is currently not the case. Studies from mathematical history demonstrate clearly that mathematics has a cultural history. Bishop (1988, p. 180) supports it by citing several anthropological and cross-cultural studies focusing on different cultures. He argues that

> mathematics must now be understood as a kind of cultural knowledge, which all cultures generate but which need not necessarily 'look' the same from one cultural group to another. Just as all human cultures generate language, religious beliefs, rituals, food-producing techniques, etc., so it seems do all human cultures generate mathematics.

The Czech project team used their own experience with methods supporting creativity in lessons in addition to the experience of their colleagues abroad. Based on their experience and expertise, they decided to develop a teaching unit entitled Ornaments. Ornaments undoubtedly meet the criteria of an SLE. They offer a rich source of mathematics (they can be used when teaching many different topics and areas) and at the same time allow introduction of culturally heterogeneous contexts, show that very different cultures have very different but equally elaborate and intriguing ornaments and provide space for creativity—drawing ornaments, tiling, making tessellations (tiling of a plane using one or more geometric shapes) and bringing photographs and images from home and using them as a background to mathematics and art activity.

10.3 Methodology

10.3.1 Ornaments in Pre-service and In-service Mathematics Teacher Education

The background research questions of the reported research study were: Do teachers need detailed lesson plans in order to be able to teach in culturally and linguistically diverse classes? Or can inclusive mathematics education be successfully conducted by creative teachers if they are given a learning environment with an outline of its possible uses in mathematics education that they then adapt and develop according to the needs of the curriculum and their specific group of learners?

To answer this question, several teaching units were developed by project partners and piloted at schools by in-service teachers. The researchers studied how the teachers coped with adapting and using the developed teaching unit.

In the process of developing the unit, the Czech project team decided to use the Ornaments teaching unit as part of pre- and in-service teacher training targeted at developing teachers' skills in the area of work in culturally heterogeneous classrooms. The aim of the workshop was to show pre- and in-service teachers that development of culturally heterogeneous materials and materials that enable pupils bring their own culture into the classroom and thus make mathematics more meaningful to them is not very difficult and that any teacher can do it with not that much effort. Another goal was to find which activities pre- and in-service teachers regarded as suitable for developing the creativity of different age groups of pupils and which activities they would choose if they were asked to plan a teaching unit, based on the environment of the Ornaments teaching unit, for a culturally heterogeneous classroom.

The workshop started with the introduction of issues of cultural heterogeneity in contemporary Europe and proceeded to a problem-posing activity. The pre- and in-service teachers were shown ornaments from different cultures and were invited to pose as many problems using the content as possible. '*In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi*' [In mathematics the art of proposing a question must be held of higher value than solving it] (Cantor 1867 in Henrard (2006, p. 2). The irony in the derivatives discounting).

> The formulation of the problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill. To raise new questions, new possibilities, to regard old problems from a new angle requires creative imagination and marks real advance in science. (Einstein and Infeld 1938 in https://www.colinvandenbroek.com/design-problem-solving/, accessed 27 September 2017)

Problem posing is an important component of the mathematics curriculum and is considered to be an essential part of doing mathematics (NCTM 2000; Tichá and Hošpesová 2010). It is an activity mathematics teachers do almost every day when they need to supplement problems from the textbook. It is also a very creative activity in which much is revealed about understanding of mathematics and different concepts and topics. Problem posing, moreover, can be used both as a part of teacher training and in regular mathematics lessons to see how well pupils have grasped a particular concept, procedure or topic.

The participants were then shown ornaments from different cultures and asked to think of as many mathematical areas, concepts and topics as possible that could be developed in this environment. The ornaments at this stage were selected at random. In this teacher training seminar, the objective was to show to pre- and in-service teachers the potential of the environment and the scope of their own creativity and resourcefulness. The participants themselves were not culturally heterogeneous. But all of them were likely to be teaching in culturally heterogeneous classrooms in the future and thus would have to be able to plan lessons that come from cultural settings familiar to all pupils in the class. Thus ornaments from all parts of the world were selected (Native American, Indian, Polynesian, Arabic, Celtic, Aboriginal, Roma, Scottish, Moravian etc.). It was up to the participants which ornament they would choose and how they would use it in their mathematics classroom.

The participants brainstormed the following topics:

- The Pythagorean theorem: Measure and calculate with the triangles in the presented ornaments.
- Compare line symmetry, rotation and translation. What is typical for which ornaments?
- Find all the different geometrical shapes you can find in one ornament; name them and describe them.
- Study the concept of tessellation; find which ornament can make tessellation.
- Copy the ornaments on a square grid, look at their area. Use square grids of different scale to study proportionality.
- Calculate the proportion of area of one colour.
- How much fabric (tartan) with this ornament would you need to make, for instance, one kilt?
- Find the generating element.
- How many lines of symmetry are there in a specific ornament?
- The least common multiple (in case of Indian line ornaments).
- How many beads are needed to make one segment of Native American ornament?
- How much tape is needed to decorate a wall of certain dimensions?
- Patchwork and ornaments: What geometrical shapes are possible for production of patchwork?
- How many threads of each colour do we need to make one segment of tartan?
- Draw symmetrical ornaments, copy them from the original or have pupils create their own ornaments.

The outcomes of the workshop were used to construct a lesson plan in which ornaments were used to teach line symmetry. The lesson plan focused on the development of pupils' creativity. The proposed teaching unit was piloted in three different settings and with pupils of different ages. The findings from the pilot implementation were analysed with respect to modifications done in the different settings as well as with respect to the results achieved in the implementation.

The methodology adopted was a workshop. The workshop is intended not just as a physical place, but as a classroom activity where doing and thinking are closely related and a teaching situation where the meaning of mathematical objects are constructed through experiences that are rich and motivating for pupils.

Based on observations and materials collected in the workshop, a primary teaching unit for Ornaments was developed. This primary teaching unit was then piloted in three different environments (in different countries and with different ages of pupils).

10.3.2 The Teaching Unit Developed

Lesson 1

- Title of the lesson: Ornaments
- Revision of symmetry: look for lines of symmetry in different types of letters (http://m3eal.dm.unipi.it/images/doc/07.teaching_materials/02/CZ_TeachingUnit_EN_web.pdf; 10 min)
- Lead-in: Presentation on types of ornaments in different cultures (10 min)
- Main activity: Linking school knowledge to real objects pupils come across in their everyday lives, active search for mathematical relations (also outside the context of school mathematics) and development of flexibility (beyond application of learned algorithms)
  - Show ornaments from different cultures
  - For one or two show ornaments, show the different types of symmetry and transformations
  - Give each student one ornament and ask them to find all lines of symmetry
  - Ask students to name and copy all symmetrical geometrical figures in the ornament
  - Ask students to formulate a conclusion about the typical ornaments of a particular culture
- Homework: Bring an ornamental decorated object from your home and bring pictures of various ornaments from your holidays. (Active search for information from other sources.)

Lesson 2

- Lead in: Present your ornaments; what types of ornaments are they and what line symmetries did you find? (Presentation of individual discoveries)
- Main activity: Creative use of prior knowledge in new situations
  - Give each student one of the three ornaments (Celtic, Native American, Arabic rosette) and a square grid with different scales
  - Ask students to find all lines of symmetry in their ornament
  - Ask students to copy the ornament onto a square grid
  - Ask students to count the number of at least partially coloured squares
  - Ask students to calculate the area of the ornament (taking partially coloured squares as covered squares)
- Follow-up: Copy the following chart on the whiteboard

Scale	0.5 cm	0.75 cm	1 cm	1.25 cm	1.5 cm	2 cm
Area						

What proportionality is there between the scale and area? (Generalization based on discovered partial results, institutionalization.)

Creativity here was expected in the first lesson of this plan of a teaching unit where pupils were expected to create ornaments with line symmetry and especially in the assignment for homework where pupils would copy ornaments from their everyday life and from their homes and cultures (or other cultures should they wish to).

The teaching unit was then sent to the project partners (in Vienna, Austria and Pisa, Italy) for piloting to see its potential for use in a culturally heterogeneous environment. In all the three pilot implementations, attention was paid to keeping all aspects of the teaching unit that encourage the development of creativity, especially allowing the search for individual approaches to solving the problems and making conclusions from the findings.

10.4 Modifications in Different Countries

The developed teaching unit was then distributed to in-service teachers in three countries (Austria, Italy, the Czech Republic) who had agreed on piloting it in their own classroom. Although the initial plan for the first and second pilot implementations was to implement the teaching unit as the project partners developed it and leave modifications for the third pilot implementation, the reality of different classrooms, age levels and conditions caused the pilot teachers to adapt the lesson plans in creative ways in order to meet the needs of their particular classrooms. The conditions in the three countries varied not only in the age of pupils but also the curricula, the number of lessons available, the number of migrants in the classroom and the motivation and attitude of the pupils, thus providing wider and deeper feedback as to the educational consistency and the didactical effectiveness of the proposed teaching unit.

10.4.1 Implementation in the Czech Republic

In the Czech Republic, the activity was piloted in the fifth grade at ZŠ Fr. Plamínkové s RVJ in Prague by the teacher with 24 years of practice. The group was culturally homogeneous, which made the selection of ornaments arbitrary in the sense that any ornaments could be chosen and only with respect to the mathematical content. The cultural background of pupils did not have to be taken into account. The research team started by closely analysing the *Framework Education Programme for Primary School Education in the Czech Republic* (MŠMT 2013) and *School Education Programmes for Primary School Education in the Czech Republic* (http://www.plaminkova.cz/skolni-vzdelavaci-program) to see which of the topics listed in the above described proposal were suitable for this age group. Czech fifth graders do not yet have knowledge of symmetries and do not work with them explicitly, but are likely to have intuitive knowledge of them. In the fifth grade they learn to work in a square grid and can build their pre-concepts of plane geometry (area and perimeter). They have not been introduced to the concept of proportionality yet.

10.4.1.1 Classroom Piloting

Thus the decision was made by the team to adapt the teaching unit and to pilot the teaching unit in two lessons in the following way: Lesson one would be based on introduction to ornaments, discussion of ornaments, their types, shapes, differences in cultures, basic elements of elements and namely discussion of Native American ornaments (made of beads). Children were given a square grid (0.5 cm) and a Native American ornament (Fig. 10.1) and were asked to copy it accurately square by square. They were then asked to calculate the number of squares in one colour, which was an introduction to area because of the scale.

Children were then given 1 cm square grid and another example of a Native American ornament that had been embroidered rather than made of beads, i.e., it was made of rectangular, not square elements (Fig. 10.2). Two parts of the ornament had been copied on a square grid (one rectangle made of three times three

Fig. 10.1 Native American ornament

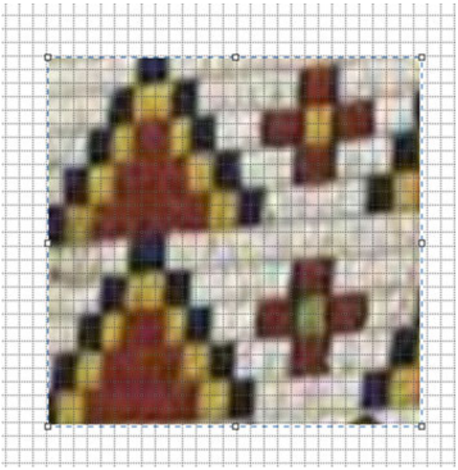

Fig. 10.2 Rectangular elements

squares). The children were asked to copy these figures into a 1 cm square grid. As each square was 1 cm square, the pupils could then easily state the area of different geometrical figures they had drawn (rectangle, two rectangles, cross, pyramid etc.). The same was done with perimeter.

Lesson 2 gave children more space for creativity. The lesson combined elements of both maths and an art lesson. Children were shown two original Native American ornaments (Figs. 10.3 and 10.4). They spoke about the figures they could see in them (revision of mathematical language).

Fig. 10.3 Native American ornaments

Fig. 10.4 Native American ornaments

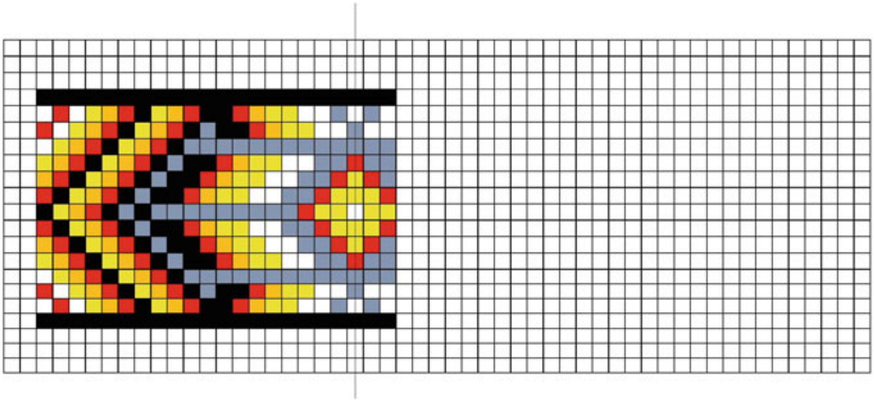

Fig. 10.5 Model of an ornament with lines of symmetry

Then they were given a model of an ornament with lines of symmetry indicated and asked to finish the ornament (Fig. 10.5).

Having finished this, they were offered beads and square grids and asked to model their own ornaments that would resemble Native American ornaments (e.g., would be based on similar geometrical figures).

For homework they were asked to look for ornaments they have at home and copy them onto a grid to be brought to school.

Having concluded the teaching unit, the research team interviewed the teacher and two of the children. The teacher in general evaluated the teaching unit as motivating for her pupils. Pupils were occupied and working hard for most of the two lessons. The materials allowed differentiation in the lesson (selection of figures that were more or less difficult, number of figures assigned to pupils in which they were to state area and perimeter). The teacher suggested that with this age group, only a 1 cm square grid be used in all activities as it makes counting different squares more meaningful from the very beginning. The pupils spoke in the interview about enjoying the possibility of having art and mathematics combined and having the chance to work with the new structures from a mathematics lesson to create ornaments of their own.

10.4.1.2 Conclusions from Piloting in the Czech Republic

The piloting of the lesson showed that a teaching unit based on Native American ornaments was very interesting and motivating for the pupils. The parts where the pupils actively worked with the square grid and coloured pens and pencil, drew and completed the symmetrical figure and created a figure of their own made them very much involved; through these activities they learned a great deal of mathematics without being fully aware they were doing more than just moving in the world of Native American ornaments. At the same time, they were introduced into the world of the concepts of area and perimeter as well as given a chance to activate their knowledge of shapes and geometrical figures. There are many possible ways to introduce area and perimeter, but this one seemed to be very motivating. It combined creativity, fun, culture and mathematics. The teacher doing the pilot implementation, although she recommended some changes, e.g., use of 1 cm square grids as a more direct way to the concept of perimeter and area, found the unit very interesting and easily usable in her teaching. The whole lesson supported pupils' individual creative discovery. The creative potential of the teaching unit was fully used. The activity meets the criteria for SLE.

10.4.2 Implementation in Italy

The activity in Italy was designed for and developed by a group of students from two different classes at the Istituto Comprensivo 1 of Poggibonsi (Province of Siena) during the weeks of flexible teaching. In these weeks, the classes are open to carry out various disciplinary or interdisciplinary activities and develop projects outside the school. The group consisted of 15–18 students in the second classes of lower secondary school who voluntarily worked on this project. The length of practice of the two piloting female teachers was 30 and 37 years.

The topics dealt with in the teaching unit allowed retrieval of previous pieces of knowledge but also allowed them to be seen in a different, more creative way. Significant added value was represented by the affectivity, since the activity urged reference to cultural aspects characteristic of the country of birth of the students. The teaching unit topics allowed the introduction of teaching methods such as problem posing that had not yet been used for the development of certain mathematical skills.

There were four basic reasons for choosing to develop the teaching unit designed by the Czech team of the M^{3}EaL project, even though some relevant changes were made, due mainly to the students' higher age and mathematical knowledge:

- To look at reality with mathematical eyes
- To develop intercultural education, accompanied by the desire to let students know about other cultural roots (in accordance to the educational model described by the Italian Ministry of Public Instruction, MPI 2007)
- To develop a positive attitude to mathematics through meaningful experiences (as suggested by the Italian National Guidelines, MIUR 2012)
- To describe, name and classify geometric figures, identifying their relevant elements and symmetries and in order to make all students able to reproduce these figures (as prescribed by the Italian National Guidelines, MIUR 2012).

All activities of the teaching unit were designed with reference to isometric transformations, a topic only partly introduced in the previous school year. The topic was addressed using a mirror: By putting a drawing or an object in front of the

Fig. 10.6 Discovering features of line symmetry

mirror and observing the reflected image, the students were able to discover the key features of the axial symmetry (Figs. 10.6 and 10.7).

The next step was to build, as compositions of symmetries with parallel or perpendicular axes, the other isometric transformations—translations and rotations—and identify their basic features (Fig. 10.8 and 10.9). Afterwards all students built some dynamic models that allowed them to represent the abovementioned geometric transformations.

The object-action connection makes the student free to design and to interpret, and it is for this reason that it becomes important to see, observe and interact with a dynamic, non-static object.

Static environments in the classroom limit the students, forcing them to consider one aspect only of the didactical scenario and do not help them to analyse it from different points of view. Furthermore, such static environments do not stimulate and foster students' curiosity and creativity. Above all, neither allows them to speculate or even less to argue, thus excluding a substantial part of those relevant educational processes that are fundamental in the start and development of mathematical

Fig. 10.7 Discovering features of line symmetry

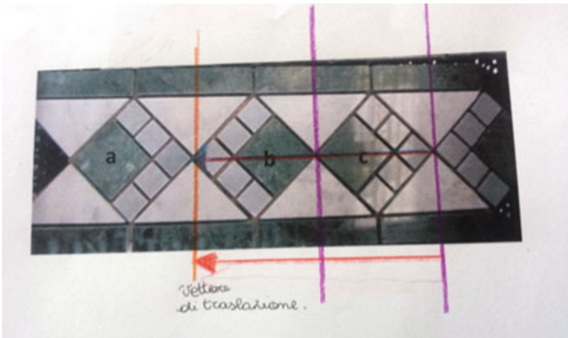

Fig. 10.8 Identification of basic elements

thinking. Therefore, in the design and the implementation of the teaching unit, besides the use of the dynamic models already known by students, the plan was to develop the activity with the use of a 'room of mirrors', a new experience for the students. The pilot implementation was planned to cover four lessons. The implementation is described below.

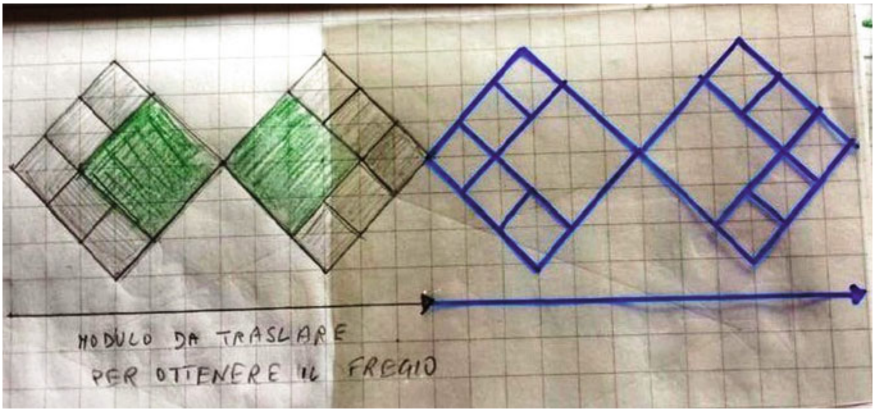

Fig. 10.9 Identification of basic elements

10.4.2.1 Classroom Piloting

Lesson 1

In the first lesson, two ornaments, a decoration and a rosette, were delivered to the students, who had been divided into four groups of four or five students each. Students were then asked to analyse the ornaments using a flat mirror in order to identify what kind of isometric transformations had allowed the creation of the ornaments. The didactical purpose of this activity was to rethink students' prior knowledge and to compare the results of the activities carried out in the different groups, with the main objective being to investigate students' conjecturing and their ability to support their own arguments and confront possible arguments from other classmates.

Decoration group

Students held a mirror vertically on the drawing (Figs. 10.6 and 10.7) and observed that there is an axial symmetry with Square a as its module. When considering Squares a, b and c (Fig. 10.8), double axial symmetry with parallel axes can clearly be seen. Then, looking more closely, if we take the Figures a and b as a whole, a translation to the right can also be detected.

At this point students asked themselves the question, 'How long is the translation vector?' Using the dynamic model, they verified that the length of the translation vector is exactly twice the distance between the two parallel axes that give rise to the movement.

Rosette group

This is the description by the group working on the rosette of its own activity: 'In this pattern we saw immediately that there is axial symmetry with incident axes

Fig. 10.10 Rosette

(Fig. 10.10). Then A pointed out to all of us that there is also a rotation, because it is produced by the composition of two symmetries with incident axes. We then decided to draw the axes of symmetry and we were able to find the centre of the rotation. To check the rotation we took the transparency and had our dynamic model (Fig. 10.11)'.

It is very interesting to realize that the students showed the module on the decoration and the axes of symmetry in order to explain their reasoning and referred again to the dynamic model in order to remove any possible doubts.

In this first lesson, to get confirmation of what had been observed, it was very helpful to use a mirror and the dynamic models that the students had built.

Once reviewing the different isometric transformations was finished, students were given individual homework assignments to write a summary of the work done in the class (log book) and search for and bring, on the day of the next lesson, objects and/or fabric available in the house that contained decorations and were from their country of origin or obtained in countries they had visited.

Lesson 2

Both the objects and the fabrics were made available to students. Each group selected an item of their own choice: a piece of fabric from Senegal and two other pieces of fabric that a student uses at home to cover sofas. The three pieces of fabric had different patterns, thus allowing each group to make different choices for different reasons.

The following task was assigned to each group:

1. Give a motivation for the choice of the object/fabric
2. Identify the isometric transformations with the help of a mirror plane
3. Reproduce the chosen ornament on the two sheets with squared grids that have been given
4. Identify the pattern generated on the fabric
5. Present other groups the chosen decoration, providing each of them with the pattern generated and the instructions to create it.

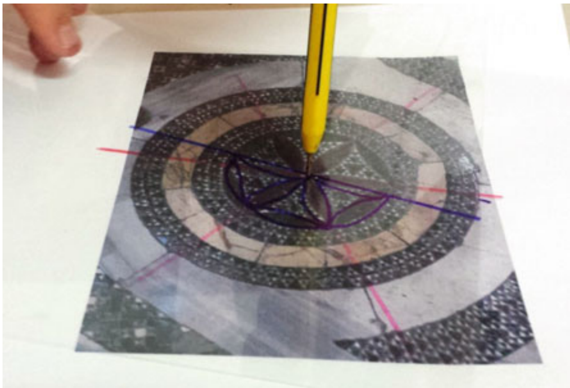

Fig. 10.11 Checking rotation

Clover group

The members of the group that had chosen the ornament with a clover pattern (Fig. 10.12) gave the motivation their choice as the fact that it was 'easy and pretty'. In fact, as they wrote in their report, they were initially mistaken: 'We thought only of simple symmetries in both the petals and the pattern from square to square, but then looking more closely with the mirror, we realized that there was a small flower stem and that it was not an axial symmetry, then, but a central symmetry…. For this we used the transparency with a snap fastener with the popper, to better understand it'.

The work done (Fig. 10.13) provides clear evidence of their research. From the base pattern, with successive rotations of 90°, they came to represent the decoration of the fabric. The discussion in the group clearly showed how the different colours (black and red) of the flowers helped students' development of their thought.

Small frame group

The group that chose the decoration with small frames (Fig. 10.14) stated that they did it because 'it looked like one of the small frames that were made during the elementary school years'. Students easily reproduced the pattern, immediately

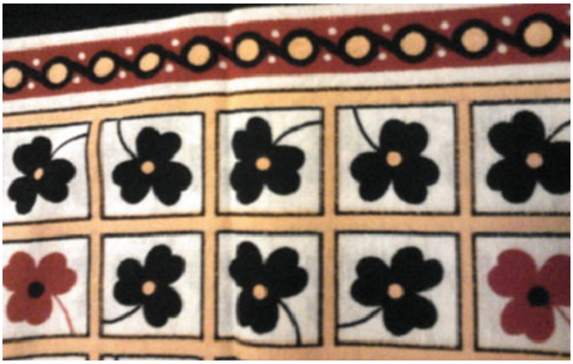

Fig. 10.12 Ornament with clovers

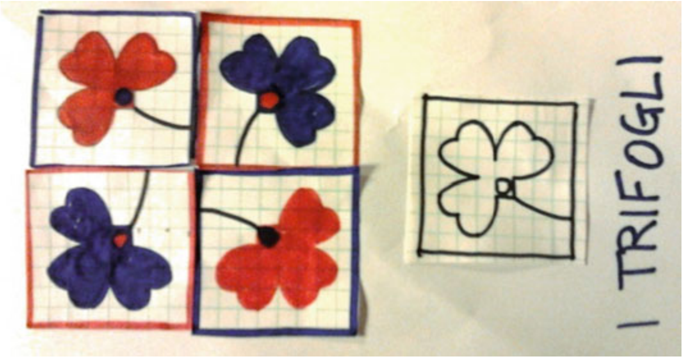

Fig. 10.13 Representation of the decoration

Fig. 10.14 Decoration with small frames

identifying and highlighting the translation (Fig. 10.15) which allowed them to reconstruct the entire decoration. Also on this occasion, after the moment of euphoria they felt about the speed of their execution, students paid attention to the central decoration that was 'made of small rectangles'. The students realized that there were other symmetries: in particular, a pair of central symmetries that consisted of a series of four 90° rotations (Fig. 10.16).

Lozenge group

This group chose a very colourful decoration that had curves and straight lines and two different ways of decorating (Fig. 10.17). The rather complicated structure of the ornament made it difficult for the group to represent the fabric design on paper, and a little help from the teacher was necessary. The students then realized that only four rotations of 90° were needed to recreate the ornament and then easily identified and composed the basic pattern.

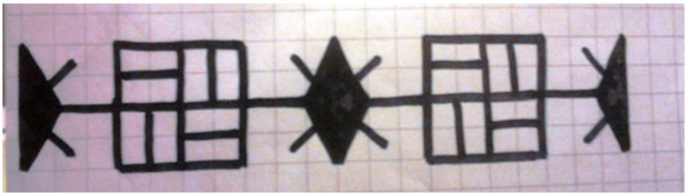

Fig. 10.15 Highlighting the translation

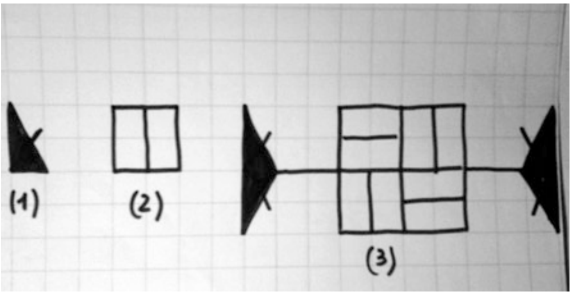

Fig. 10.16 Central symmetries

Fig. 10.17 Example of decoration chosen by Lozenge group

Rosette group

The rosette (Fig. 10.18) undermined the students: They failed to identify the starting pattern, could see some symmetries, but only in two pairs of the ornament elements and recognized the rotations but could not explain how to make use of them. Therefore, the group decided to give up for the moment and agreed with the two teachers to work more on this ornament in the next lesson.

Lesson 3

From the 'room of mirrors' to proportionality

The lesson started with the teachers taking a few mirrors into the classroom and assigning the students the following task.

Consider the patterns you made in the previous lesson that originated from the rosette ornament (Fig. 10.19). Place the patterns one at a time in the 'mirror room'

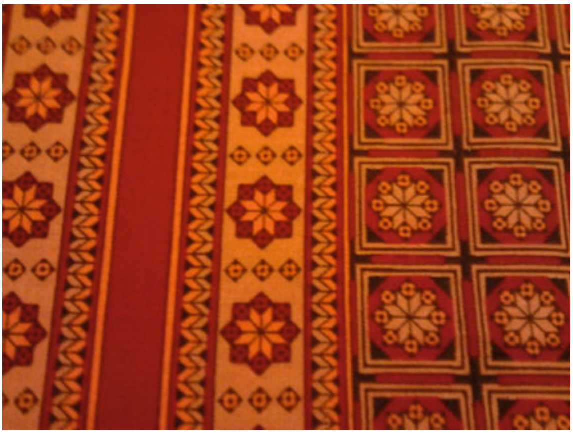

Fig. 10.18 Rosette

Fig. 10.19 Pattern originating from the rosette ornament

(the new setting obtained through the introduction of the mirrors) and tell how many of them are needed to complete the ornament (Fig. 10.20a–c).

The teachers' intention was to promote and encourage students' creativity and elicit their ability to abstract and conjecture.

Students began to put the mirrors over the pattern at an angle of 90°. When they did this, they saw four images (three are reflected and one is the real one) forming a flooring, that is, an entire decoration. Having then put the mirrors to another pattern, they realised that 'with the 45° angle, however, we get 8 images, so they doubled. Yet the angle is decreased; more precisely, it's halved!' This remark stimulated the curiosity of the students, who began to try other pieces of the ornament. Then, without any hint from the teachers, they came to say that if the angle of the mirror

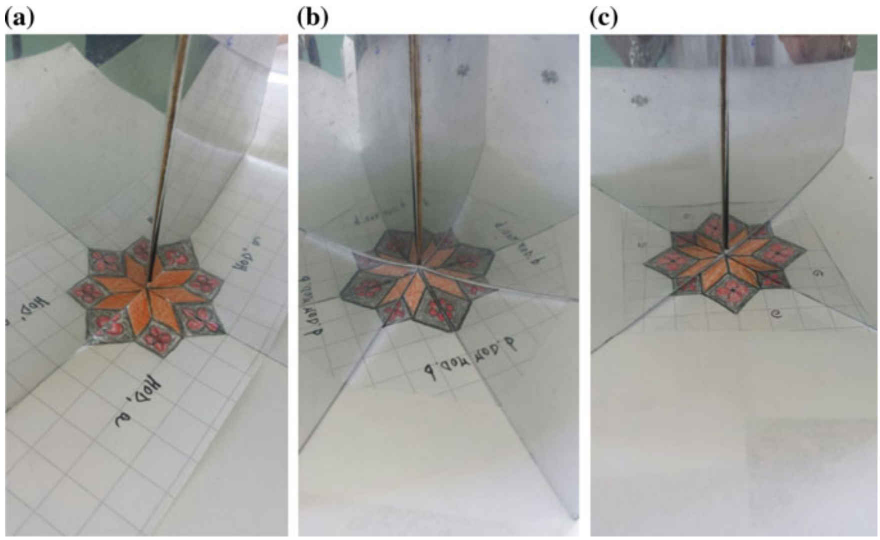

Fig. 10.20 Completing the rosette

room decreases, the number of images that are necessary to complete a rosette increases.

The teachers wanted to get deeper into the topic and suggested that students use the protractor to find the angle between the mirrors and place a thin object such as a pencil in the 'room of mirrors'. They also suggested building a table with the angle between the mirrors and the corresponding number of images obtained. The angle will have a measure equal to a sub-multiple of the full angle. In the table, then, there are going to be not only the pairs (90, 4), (180, 2) and (45, 8) but also, for example, (30, 12), (40, 9). The table will clearly show the relationship *angle width ←→number of images* because it is easy to see that if the angle halves, becomes a third and so on, the number of images doubles, triples and so on. Students could, therefore, observe that the product of the angle width by the number of images is constant and equal to 360°, the full angle. In this way the students discovered intuitively, but at the same time rigorously, the law of inverse proportionality!

Afterwards, the teachers decided to ask students to represent the data in the table by points on a Cartesian plane and connect them. In this way, students were able to realise that the points can be seen as elements of a curve that they did not know yet: a branch of hyperbole.

It is here that the student B asked: 'Why does this graph start at 10°? If I close the room, that is if the angle is zero, what happens? I do not see anything so I don't get any figure; the images are zero… but then it does not work… there's something wrong.' This student's doubt became a resource for everyone! The teachers suggested then that students put a piece of string in the room of mirrors and look attentively what happens when closing mirrors slowly. The closure action enabled students to understand that the images are not zero but infinite: ‘In fact, we do not see them because they are inside!’ Once again the dynamism of an object led to examination of an important limit case that would not be easy to deal with and understand using only arithmetic, since division by zero is impossible. In this way, the students, through an operation that they verified to be impossible, were able to grasp the idea of infinity.

Lesson 4

Art tessellations

The students had already worked on the tessellation of the plan and knew what the regular polygons (equilateral triangles, squares, hexagons) were that make it possible and the reason why. A slightly modified version of this activity—which had already been carried out—was proposed to the students, with the aim of unleashing their ‘artistic creativity’.

Students were asked to cut out a part of a square and place it on the opposite side. In this way they produced a pattern that, by subsequent translations, created a tessellation. The same activity could be proposed using other regular polygons, such as an equilateral triangle. The creativity of the students transformed the patterns they created into subjects that became the ‘heroes’ of these new and very personal tessellations (Fig. 10.21).

The activity was very much enjoyed by the students who, after some temporary confusion due to the actual construction of the pattern, had fun creating beautiful floors while showing imagination and artistic sense.

Unlike what happened with the square and the parallelogram, the use of the triangle as the polygon to start from proved to be difficult. Where should the cut part be placed in order to obtain a tessellation? Is it OK to put it on any of the other two sides? Or is it necessary to place it on the same side that it was cut from? This question arose spontaneously and led to a good discussion that was developed with good arguments. Once again, the hand, in a context that was emotional and

Fig. 10.21 Example of a personal tessellation

meaningful, encouraged the spontaneous birth of interesting questions that, properly managed by the teacher, gave the opportunity to break new ground or retrace paths already experienced, but with a different perspective, developing a continuous reconstruction of knowledge.

10.4.2.2 Conclusions from Piloting in Italy

The Italian research team and the teachers doing the pilot implementation were firmly convinced that more than just having students accumulate knowledge by merely passing on to them notions and information that are often not interlinked or interrelated, teachers should try to stimulate students' aptitude to pose problems that can increase their motivation and foster discoveries. The teaching unit described above falls within this framework, making use of workshop activities in such a way that learning is really centred on the student's needs and characteristics. The student is the investigator and, as such, acquires the ability to identify, accept, confront and solve new problems, both individually and in groups.

The development of the teaching unit is rooted on three methodological cornerstones:

1. Setting context problems
2. Fostering the asking of questions
3. Working in groups so that the heterogeneity of the students is a resource for the entire class, with a view to getting more and more inclusive learning.

With respect to development of creativity, all stages of the teaching unit were based on students' creative approaches to problem solving, discovery of patterns, rules and regularities, generalization and active search for needed information. In this case, the activity also meets the criteria for SLE.

10.4.3 Implementation in Austria

The teaching unit was piloted by a mathematics teacher with five years teaching experience working in an upper secondary school near Vienna. The teacher chose to conduct the pilot implementation of Lesson 1 during a regular mathematics class (50 min) in the sixth grade and of Lesson 2 during a 50-min class using fieldwork (i.e., collecting information outside the classroom, laboratory etc.) as a teaching method. Eight students (aged 17–18), three of whom were migrant students, attended the class.

The original teaching unit, designed by the M^{3}EaL project Czech team, was implemented with a completely different age group with very different skills and motivation. While primary school children tend to be very eager to be creative and draw and work with beads, upper secondary school students need not only different content but also different methods and motivation.

10.4.3.1 Classroom Piloting

The teacher introduced the topic for Lesson 1 by bringing to the classroom several objects with Japanese, South African and US motives (Fig. 10.22) from her private possession. The students formed groups of two and were asked to look for symmetries and different geometrical figures and then compare the different kinds of figures and symmetries in each of the different cultures that they found in the objects. Each group then briefly presented their findings in front of the whole class, and the other groups wrote them down in their notebooks. At the end of the lesson, the teacher asked the students to bring actual examples of ornaments or pictures of ornaments from different cultures to the next lesson, as suggested in the original proposal by the Czech research team. The students argued, however, that only very few of them (or their families) actually had suitable ornaments or pictures at home. Repeating the lesson with more objects from the teachers' collection was seen as not very interesting by both the teacher and the students. The students then came up with the idea to go out into nature and bring pictures of symmetries or geometric figures that are found in flowers or plants instead. The teacher argued that if symmetry in nature would be interesting for the students, it would be better to actually make a field work session out of Lesson 2 instead of just looking at the pictures. It therefore was decided that Lesson 2 would be modified, and students would go out together with the teacher, look for symmetries in nature and take photos for later discussion of symmetry and scale in class (the last part that occurred back in classroom was not part of the pilot implementation).

Lesson 2 started with the teacher reminding the students of the different kinds of symmetries and figures as well as special angles (e.g., from Fibonacci numbers). Then the teacher and the students went out into a field near the school to look for the occurrence of symmetries and geometric figures in both natural and artificial

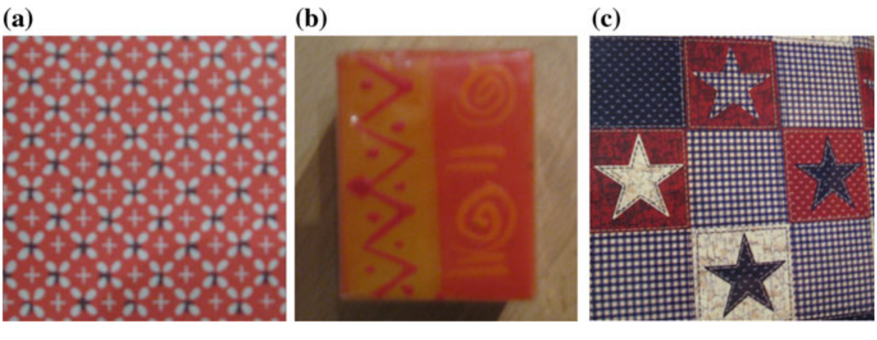

Fig. 10.22 Japanese, South African and US motives

Fig. 10.23 Students looking for symmetries

objects. Students first looked for the occurrence of certain angles on plants. Very soon they realized that 137.5° was a very frequent angle on a number of species of plants, a fact that impressed the students very much. Students took pictures of the objects to use in the next lesson.

The unit continued with the students looking for symmetries, particularly for mirror symmetry (Fig. 10.23). The students were mostly able to state that the object did actually show some kind of symmetry but were not always able to name the kind of symmetry concerned. As a result, the students fairly often pointed out symmetries and the teacher explained the particular symmetry of the object.

Students then started a discussion about how exact these symmetries actually were. The teacher used this opportunity to point out that real objects (regardless of whether they were artificial objects such as the ones she brought into the classroom in Lesson 1 or natural objects such as grass) are never exactly symmetric in a mathematical way and that this is where modelling comes into play.

At the end of the lesson, artificial objects (e.g., advertising pillars and patterns on t-shirts) were also checked out, and the students and the teacher discussed whether the patterns and/or the form of the pillar have cultural and/or practical reasons. Several of the shirt patterns were photographed; the patterns came from different cultural backgrounds that the students (according to their own statements) did not know when they bought the shirts. The teacher's homework was to determine the cultural backgrounds and cultural relevance of the photographed patterns.

The lesson ended back in the school building, where the homework assignment was repeated.

10.4.3.2 Conclusions from Piloting in Austria

This pilot implementation showed that even if the unit is modified and—at least superficially—moves away from the intercultural aspects, these aspects can easily be brought back into the minds of the students by referring to everyday objects and their cultural connections.

In the following interview with the research team, the teacher spoke very highly of the students' involvement in the field work and their creativity when looking for patterns and describing patterns on everyday objects. The possibility of being active and creative in this implementation again helped the students discover enthusiastically. Although the cultural focus disappeared in the second lesson and the attention was shifted to natural objects, the unit retained its active and creative drive and was appreciated both by the teacher and the students. Even if the teaching unit is designed in this way, the creative potential of the primary teaching unit is not lost. Furthermore, the criteria for SLE are fully met.

10.4.4 Conclusions from the Three Pilot Implementations

It can thus be concluded that all three adaptations of the original teaching unit were creative and triggered pupils' creativity. Teachers used elements of fieldwork, arts and crafts to make the lessons more creative and motivate their pupils. While in Austria the lesson was changed to going out of the classroom to look for 'ornaments' in wildlife, Italian and Czech pupils used drawing to discover properties such as line symmetries, rotations and translations, and area and perimeter.

Creativity was an essential element both at the stages of lesson planning and conducting the lessons. Only creative teachers are able to adapt a teaching unit designed as an SLE and use the potential of the suggested setting to teach the mathematics they need in the particular group and at the particular level of their pupils. And pupils, if engaged in creative activities, are more likely to have positive attitudes to mathematics and learn it without actually being aware of doing 'difficult' mathematics. In line with the principles of inquiry-based education, where pupils discover mathematical concepts and procedures by making conjectures by testing and through independent discovery, creative activities give pupils the space needed for discovery of many important concepts and procedures. Creativity brings fun and is challenging but also supports discovery. Its place in mathematics lessons is crucial.

This teaching unit allows teachers to meet today's students' needs without sacrificing the teaching of the basic concepts of the discipline. Even the realization of ornaments, an activity that makes the student free to experiment and indulge in fantasy, offers an emotional dimension that is important, because learning is difficult if the sphere of emotions is not positively affected. On the other hand, teamwork allows students to learn how to defend their conjectures and at the same time accept change when others' arguments are clear and justified.

The whole activity, therefore, is based on fundamental aspects of the learning; in fact, it requires students to be active, constructive, collaborative, contextual and thoughtful. This way it provides excellent opportunities to build skills.

10.5 Conclusions

There are many problems mathematics teachers face nowadays: They work with unmotivated students and in socially heterogeneous cultures. This makes it crucial for a teacher to be able to design teaching units that allow students to see reality from different perspectives and also to develop greater self-knowledge. The three pilot implementations of the teaching unit described above show that it has the potential of meeting today's students' needs without sacrificing the teaching of the basic concepts of the discipline. Even though mathematics is often seen as an abstract matter, it can instead become somewhat closer to them and to their reality. The use of everyday objects such as ornaments gives the subject an affective aspect that is not to be neglected. Creation of ornaments helps the students become free to experiment and indulge their imagination and offers an emotional dimension that is important.

The paper presented here shows that SLEs, as defined and developed by Wittman, support creativity in mathematics lessons—both on the teachers' and the pupils' part. If given the cultural background and the environment, teachers are naturally motivated to use their creative potential to look for the mathematics that can be discovered and taught in that particular environment.

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Chapter 11
Instances of Promoting Creativity with Procedural Tasks

Michal Tabach and Alex Friedlander

Abstract The learning of algebraic procedures in middle-school algebra is usually perceived as an algorithmic activity, achieved by performing sequences of short drill-and-practice tasks, which have little to do with conceptual learning or with creative mathematical thinking. The goal of this chapter is to explore possible ways by which all middle-grade students can be encouraged to apply higher-order thinking in the context of tasks that integrate procedural work, conceptual understanding and creative thinking. Each of the five instances presented in this chapter was intended to promote creative thinking in the context of procedural tasks. An a-priori task analysis and data collected in some of our previous studies indicate the presence of many learning competencies and high levels of mathematical creativity in the participating students' work. Thus, we conclude that certain procedural tasks have a strong potential to promote higher-order, and creative thinking.

Keywords Procedural tasks · Creative thinking · Drill-and-practice tasks

11.1 Introduction

The learning of algebraic procedures in middle-school algebra is usually perceived as an algorithmic activity, achieved by performing sequences of short drill-and-practice tasks, which have little to do with conceptual learning or with creative mathematical thinking. Traditionally, both research and classroom teaching distinguished between procedural and conceptual tasks (e.g., Rittle-Johnson and Alibali 1999), and as a result, a considerable part of teaching and learning algebra has focused on routine practice and the application of rules, procedures, and

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techniques. More recently, an integrated approach between learning of rules and procedures on one side, and a deeper understanding of their meaning and a flexible choice of solution methods on the other side were recommended (Kieran 2004; Star 2007; NCTM 2000; Friedlander and Arcavi 2012; 2017).

The aim of this chapter is to explore the potential of integrated procedural tasks—i.e., tasks that add a conceptual dimension to the learning of algebraic procedures, to promote mathematical creativity for ***all*** middle-grade students. In the following sections, we present and analyze five exemplary tasks. In this chapter, we review our current research on the potential of some middle-school algebra tasks that address both domains of procedural and conceptual learning, with a focus on aspects related to students' creativity. Our analysis will focus on aspects of students' creative thinking, and will be based on (a) a task analysis according to a set of cognitive skills (sometimes referred to as students' learning competencies) required by each activity—as proposed in Friedlander and Arcavi (2012; 2017), and (b) an analysis of findings on students' work on each of the five exemplary tasks—as reported in our previous research or in our unpublished experimentations.

11.1.1 Students' Learning Competencies

Student competencies are referred in literature in a wide variety of contexts—for example, as structured sets of knowledge, skills or attitudes. Some of them are also listed in the literature as being traits of gifted students. In our context, we refer to students' learning competencies as defined by Friedlander and Arcavi (2017). In their book, student competencies are viewed as a set of categories of generic skills required in students' attempt to solve a task. Friedlander and Arcavi analyze their collection of about fifty algebraic tasks according to eight student competencies—the following six of which are relevant to the goals of this chapter:

- *Representing, modeling, and interpreting*: Algebraic expressions, equations, and functions represent quantities, patterns of, and relationships between quantities. As such, they can be used as mathematical models of real or mathematical situations, relationships, or properties. Students should be required to analyze and operate on these models, interpret the results, and on that basis, derive new knowledge about the situations they represent.
- *Reverse thinking*: Some tasks require students to reverse the 'direction' of an activity by resorting to backward thinking, or reconstructing a procedure already performed but missing. Thus, students should be able to reconstruct expressions or equations based on partial information or on the given final result.
- *Generating examples*: Students should be required to explore, try, create, and review their understandings by generating examples or counterexamples to a given assertion. Providing a variety of examples or counterexamples, and comparing them with those provided by others also stimulates divergent thinking.

- *Generalizing:* Students should experience a main activity in algebra consisting of identifying patterns, relations, properties, and processes of variation, and expressing them in symbolical, graphical, numerical, or verbal representations.
- *Justifying and proving*: Students should also experience algebra (and not only geometry) as a domain that allows them to engage in proofs of statements/properties either provided by the teacher or conjectured by themselves.
- *Divergent thinking*: students should encounter tasks that involve and promote multiple-solution methods, a wide variety of answers, meaningful mathematical discussions, and opportunities for creative solutions. Divergent thinking involves the creative generation of multiple answers to a particular problem (in contrast to convergent thinking, which aims for a single, correct solution).

In the following sections, we will attempt to establish which of these learning competencies are required by our integrative procedural tasks.

11.1.2 Students' Mathematical Creativity

There is no one acceptable definition of creative mathematical thinking in general, or of creative mathematical thinking in the context of school mathematics, in particular (Leikin and Sriraman 2017). Nevertheless, there are some accepted characteristics for creative mathematical thinking. For example, Guilford (1967) linked creative thinking with divergent thinking (or production). Divergent thinking in mathematics involves the creative generation of multiple solution paths and/or multiple answers to a particular problem. Torrance (1974) identified creativity in general by specifying three components: *fluency* refers to the number of ideas generated in response to a prompt, *flexibility* is assessed by the number of apparent shifts in approaches taken when generating responses to a prompt, and *novelty* considers the originality of the ideas generated in response to a prompt.

In our view, creative thinking in mathematics involves thinking on a problem from different perspectives and points of view, and is characterized by fluency, flexibility and originality. In the following sections, we will attempt to analyze data on students' work on our integrative procedural tasks according to these three characteristics of creative thinking. Also, we would like to note that manifestations of creativity and higher-order thinking in procedural tasks will be considered in the context of two different kinds of tasks:

- procedural tasks that allow or require students or groups of students to produce *multiple solutions* according to their mathematical knowledge, ability and creativity (i.e., fluency/flexibility/originality)
- procedural tasks that allow or require students or groups of students to employ *multiple solution methods*, and to choose a method or to solve the task at hand in a variety of ways according to their mathematical knowledge, ability, cognitive preference and creativity.

As mentioned above, in the following sections we examine the potential of five different instances to provide students opportunities to employ creative thinking related to understanding and practicing algebraic procedures—such as symbolic manipulations of algebraic expressions and solving equations. In two of the presented instances we also examine the potential of using technological tools in achieving this purpose.

11.2 Five Instances of Procedural Tasks

11.2.1 Employing Creative Thinking to Understand Distribution

The *Identical Columns* activity

Rationale. Spreadsheets can serve as a bridge between arithmetic and algebra (Haspekian 2005; Wilson et al. 2005). Moreover, they have the potential to provide a natural need for using a rich variety of symbolic expressions ("formulas") to create a large number of numerical tables.

We describe here briefly a study (Tabach and Friedlander 2008) that involved a spreadsheet-based activity called *Identical Columns* (Fig. 11.1). The purpose of the activity was to allow students to consider the conceptual aspects of learning the transformational skill of using the distributive law to produce equivalent algebraic expressions.

In the first task, the students were required to perform the following operations:

- fill in each of the two speadsheet Columns A and B a given set of numbers, that form an arithmetical sequence. [Note: The choice of arithmetical sequences, rather than discrete numbers, was not inherent to the task, but it allowed students to fill in these columns by using formulas.]
- construct in Column C, the sum of the corresponding numbers given in Columns A and B (A + B)
- construct in Column D the sum 2A + 2B
- use Columns A, B, or C in order to create two other columns that are identical to Column D.

The purpose of this task was to stress the symbolic equivalence between the expressions 2A + 2B and 2(A + B)—i.e., the symbolic representation of the distributive law. The expected spreadsheet formulas were 2(A + B),

A + A + B + B, A + B + C, or 2C. The request to create two (rather than just one) corresponding columns was intended to encourage students to think creatively about general relationships, rather than look for idiosyncratic numerical connections.

Copy the following pairs of numbers to your spreadsheet:

	A	B	C	D	E	F	G
1	4	16					
2	2	13					
3	0	10					
4	-2	7					
5	-4	4					
6	-6	1					
7	-8	-2					
8	-10	-5					

1. a) Write in Column C the sum of the numbers from Columns A and B (=A+B).

   b) Write in Column D the sum =2·A + 2·B

   c) Use Columns A, B, or C in different ways, to create additional columns that are identical to Column D.

   Write down the *Excel* formulas that you used to obtain the identical columns:

   Your first formula ____________________

   Your second formula ____________________

   d) Use symbolic language to write the relations between the formula 2·A + 2·B and your formulas for identical columns.

   2·A + 2·B = ________________

   2·A + 2·B = ________________

2. a) Write in Column G the product =10·(A + B).

   b) Use only Columns A and B in different ways to create other columns that are identical to Column G.

   c) Use symbolic language to write the relations between the formula 10·(A + B) and your formulas for identical columns.

   10·(A + B) = ______________

   10·(A + B) = ______________

Fig. 11.1 *Identical Columns*—a spreadsheet-based activity (Tabach and Friedlander 2008)

In the second part of this activity, students were asked to write in Column G the formula 10(A + B) and again, to use Columns A and B in order to create two columns of their own that are identical to G. The expected spreadsheet formulas for this task were 10A + 10B, or (A + B)10.

Note that the two tasks required the consideration of the distributive law, in two different directions and meanings—by factoring out and by expanding a symbolic expression.

The use of spreadsheets in this activity was intended to provide students with numerical support for symbolic transformations, to enable them to produce their own expressions, and to test their hypotheses using the resulting numbers. In these tasks, obtaining identical number columns indicates the equivalence of symbolic expressions, whereas a mismatch indicates the need for additional adaptations of the employed formula.

Learning Competencies. Our task analysis led us to link this activity to the following competencies listed in the introductory section as having a potential to elicit mathematical creativity:

- *Representing, modeling, and interpreting*: Students are required to look for relationships between symbolicaly represented numbers or quantities, to analyze and operate on these representations, interpret the results, and on that basis, derive new knowledge about the relationships they represent.
- *Reverse thinking*: The activity requires students to employ the distributive law in both directions—expanding expressions of the form k(A + B) and factoring out expressions of the form kA + kB.
- *Generating examples*: Students are expected to analyze the given numbers, and to generate examples of algebraic expressions (formulas in spreadsheet terminology) that create a required result.
- *Generalizing*: The activity requires students to identify relations and properties, and express them in numerical and symbolic representations.
- *Divergent thinking*: Students encounter an activity that involves both multiple solution methods (for example, examining the given numbers and generalizing, or identifying and applying a general rule), and a wide variety of possible solutions (i.e., a wide variety of equivalent expressions).

Method. The activity was presented to three classes of seventh grade students, around the middle of the first year of a beginning algebra course, conducted in a learning environment composed of both spreadsheets and paper-and-pencil activities. The work on this activity (Fig. 11.1) was conducted with 41 pairs of students in a computer laboratory. The data of this study was based on 41 *Excel* files produced as part of their regular work, and as such, they reflect final results, rather than solution processes. In the analysis of student responses to the two spreadsheet-based parts, we considered both the final results and the strategies used.

Findings. Table 11.1 presents the expressions produced by the participating students, categorized by the employed strategies deduced from the data. We would like to note that all the expressions are indeed equivalent, a fact that can be attributed to the use of spreadsheet.

The students' work contained 22 and 31 different expressions for the first and the second tasks, respectively. This richness of expressions was not expected, and we see it as an indication of students' ability to employ original thinking.

In the first part of the activity, we categorized 58% of the expressions as based on symbolic reasoning (i.e., applying symbolic procedures), whereas 42% of the expressions were considered the result of a generalization activity (i.e., applying numerical considerations). In the second task, a total of 92% of the employed solution methods suggested applying symbolic procedures, and only 8% of the

Table 11.1 Expressions and strategies in the *Identical Columns* activity (N = 41 *Excel* files, Tabach and Friedlander 2008)

Strategy	Sample of expressions	
	Task 1 N = 87 Expressions (frequency %)	Task 2 N = 71 Expressions (frequency %)
Numerical considerations	C + C (42%) C · 4/2 2C A + B + C	10C (8%)
Distributive law	2(A + B) (24%) (A + B) · 2	10A + 10B (57%) 10B + 10A (A · 5 + B · 5)2 5((A · 2) + (B · 2)) (A · 2 + B · 2) · 5 (A + B) · 5 + (A + B) · 5 (A + A + B + B) · 5 (A + B + A + B) · 5 5 · (A + B) + 5 · (A + B) (A · 6 + B · 6) + (A + B) · 4
Additive	A + A + B + B (20%) (A + B) + (A + B) B + B + A + A	A + A + ⋯ + B + B⋯ (10%) A + B + A + B⋯ A + A + 8A + B + B + 8B
Commutative	A · 2 + B · 2 (5%) 2B + 2A B · 2 + A · 2	(A + B) · 10 (6%) 10(B + A)
Other symbolic strategies	(A + 2) · 2 + (B − 2) · 2 (9%) A/0.5 + B/0.5 ((A · 4) + (B · 4))/2 C − A + B + A · 2 A·4·4/8 + B·10−4/20 2A + 2B + C − C	(A + B)2 · 5 (19%) 5 · 2 · (A + B) (A + B) · 5 · 2 (A + B) · 2.5 · 4 (A + B)/0.1 10(A + B + A − A) (A + B) · 20/2 5 · 2 · (A + B) (5 + 5)(A + B) (15 − 5)(A + B) −10(−A + −B) (A + B)4 · 4 · 10/16 · 75/300

expressions were considered to be based on numerical considerations. This difference between the two tasks can be attributed to the numbers involved (a factor of 10 as compared to a factor of 2), to the students' realizing the advantages of using symbolic considerations for this particular task, or to the fact that the operation of expanding is intrinsically easier than that of factoring out.

The Creativity Perspective. The main purpose of this study was to investigate possibilities to promote the understanding and learning of algebraic procedures. The variety of strategies employed and the resulting equivalent symbolic expressions indicate yet another way of promoting creative thinking with a procedural task.

The students as a group exhibited fluency in terms of the number of solutions produced for each task, and in terms of the number of solution strategies employed. As mentioned in Table 11.1, 22 and 31 different expressions were created for the first and the second task respectively—as compared to our expectation of at most four expressions for each task. As shown in the last row of Table 11.1, the students also demonstrated a high degree of fluency of solution methods—besides the four main strategies of employing numerical considerations, the distributive law, additive and commutative reasoning.

The participating students displayed flexibility by producing in most cases two solutions (i.e., two equivalent expressions) that were considerably different one from the other, and were obtained by employing different solution strategies.

A low percentage of students' employing a certain strategy was considered an indicator of a high degree of group originality. Thus, in the first task less than 10% of the students employed a commutative, or another symbolic strategy, and as a result they were considered original solution methods. Likewise, in the second task, less than 10% of the students applied numerical considerations, additive, or commutative strategies, and as a result they were considered original solution methods.

11.2.2 Students as Designers of Procedural Tasks

The *Make-a-Quiz* activity

Rationale. Algebraic expressions produced in a process of algebraic simplification are equivalent. Consequently, the concept of equivalent expressions is traditionally linked with the algebraic procedures related to simplifying expressions. In one of our studies (Tabach and Friedlander 2017) we explored the potential of some problem *posing* tasks related to the topic of equivalent expressions, to promote conceptual understanding, procedural fluency and creative mathematical thinking.

Method. The *Make-a-Quiz* activity presented in this instance required 56 ninth-grade students learning in three heterogeneous mathematics classes of one urban middle-grade school, to design a multiple-choice questionnaire on the topic of equivalent algebraic expressions. In each of the six test items, students were given an algebraic expression (the stem), and were asked to provide several correct answers (equivalent to the given expression), and several incorrect distractors (non-equivalent to the given expression).

The instructions of this activity were: "*Write four distractors for each quiz item. Try to give more than one correct answer and some good distractors*". The following expressions were given to the participating students as the stems of the six test items: (a) $7 - 2 \cdot (\mathrm{x} - 3) =$; (b) $5\mathrm{x} - 2\mathrm{x} \cdot (\mathrm{x} - 3) =$; (c) $\frac{x}{2} + 2x =$; (d) $10 - \frac{1}{4}x + \frac{x}{2} =$; (e) $1 - \frac{\mathrm{X}-7}{2} =$; and (f) $(5 - \mathrm{x}) \cdot (6 + \mathrm{x}) =$.

In order to provide examples of correct answers, students were expected to be familiar and creative with regard to accepted ways of simplifying expressions, whereas the construction of "good distractors", required creativity based on awareness of misconceptions and errors in performing operations on algebraic expressions.

Student work on the *Make-a-Quiz* activity requires both procedural skills (proficiency in creating equivalent expressions), and a deeper understanding of procedures of simplifying algebraic expressions.

Learning Competencies. We assumed that work on this type of activities requires several learning competencies that we would like to relate to creative thinking—such as

- *Representing and interpreting*: Students are expected to realize that the set of expressions that are equivalent to a given one is diverse and includes a wide varity besides the simplified form of the given expression. Students are also expected to be aware of common misconceptions and errors related to the mathematical concept of equivalent expressions by creating "good" distractors for non-equivalent expressions.
- *Reverse thinking*: Rather than simplifying expressions in an algorithmic process, students construct complex expressions that are equivalent to a given relatively simpler one.
- *Generating examples*: Students are encouraged to explore, try, create, and review their understandings by generating examples and counterexamples of equivalent expressions.
- *Divergent thinking:* Students are expected to provide a wide variety of equivalent expression and "good" distractors for a given expression.

Method. A total of 56 ninth-grade students learning in three heterogeneous mathematics classes of one urban middle-grade school participated in the study. Their background in algebra consisted of a two-year beginning algebra course that included the basic procedures related to simplifying algebraic expressions and solving linear equations. Before the *Make-a-Quiz* activity, the students were given a regular multiple choice questionnaire on the same topic, in order to get acquainted with the structure of multiple-choice items, with thinking processes involved in answering this type of items, and with principles involved in their design.

Findings. Table 11.2 presents some of the more frequent types of responses provided as equivalent expressions (i.e., correct answers) analyzed with regard to their level of originality, whereas Table 11.3 presents the more frequently found responses of non-equivalent expressions (i.e., distractors) categorized by levels of awareness of errors, for four of the six quiz items.

The Creativity Perspective. As stated in our task analysis above, we assumed that work on this kind of activities requires and promotes creative mathematical thinking. The findings indicate that the students produced for each stem: between 11 and 32 different equivalent expressions, and between 66 and 102 different non-equivalent expressions. Thus, as a group, the students exhibited a high degree of fluency in terms of the number of solutions.

Table 11.2 Frequent responses for equivalent expressions by level of originality (Tabach and Friedlander 2017)

Level	Item			
	$7-2(x-3)=$	$5x-2x(x-3)=$	$\frac{x}{2}+2x=$	$(5-x)(6+x)=$
Low[a]	$7-2x+6$; $13-2x$	$5x-2x^2+6x$; $11x-2x^2$	$2.5x$; $\frac{5x}{2}$	$30-x-x^2$; $30+5x-6x-x^2$
Medium[b]	$-2x+13$; $\frac{7}{1}-2\cdot\left(\frac{x}{1}-3\right)$	$(20x-9x)-2x^2$; $5x+6x-2x^2$	$\frac{5x}{10}+\frac{30x}{15}$ $\frac{2x}{4}+\frac{2x}{1}$	$-x^2-x+30$; $30-x^2-x$
High[c]	$7+2(3-x)$; $2(-x+3)+7$	$5x+2x(3-x)$; $5x+2x(-x+3)$	$\frac{10x}{4}$; $\frac{3x}{2}+x$	$(5\cdot 6+5x)-(x\cdot 6+x\cdot x)$; $(5-x)6+(5-x)x$

[a]Minor change by direct derivation of the given expression
[b]Slight change in addition to direct derivation
[c]Non-routine change

Table 11.3 Frequent responses for non-equivalent expressions, by level of awareness of errors (Tabach and Friedlander 2017)

Level	Item			
	$7-2(x-3)=$	$5x-2x(x-3)=$	$\frac{x}{2}+2x=$	$(5-x)(6+x)=$
Low[a]	$2x+6+7$; $15x$	$3x(x-3)$; $5x-2x^2-6x$	$\frac{x}{2}\cdot\frac{2x}{1}$ $\frac{4x}{2}$	$(5-x)(6x)$; $(5–6)(x+x)$
Medium[b]	$2x+1$; $5x+15$	$5x+2x(x-3)$; $5x-2x(3-x)$	$\frac{x+2x}{2+1}$ $x+4x$	$(6-x)(5+x)$; $30+x-x^2$
High[c]	$1–2x$; $5(x-3)$	$-3x(x-3)$; $5x-2x^2-6$	$\frac{3x}{2}$ $\frac{x+2x}{2}$	$(5+x)(6-x)$; $30-x^2$

[a]provided by one or two students
[b]provided by 3–7 students
[c]provided by eight or more students

As for originality, our findings indicate that (1) about a third of the 56 participating students were able to display a medium or high level of originality in the construction of equivalent expressions, and (2) about 80% of the students demonstrated medium to high level of awareness of errors—i.e., provided distractors based on commonly encountered errors.

11.2.3 Solving Equations in Multiple Ways

The *Solve It in Many Ways* activity

Rationale. Equations can be solved by different methods, having thus the potential to promote mathematical creativity in the context of procedural learning. Research on students' ability to choose, employ and integrate various tools in the solution of algebraic tasks in general and of algebraic procedural tasks in particular, is scant. The limited knowledge on this issue is also related to the fact that regular classroom tasks in algebra frequently recommend explicitly the representations and tools that should be employed.

The goal of a study conducted by one of the authors and his colleague (Friedlander and Stein 2001) was to investigate (a) students' ability to solve algebraic equations in an environment that provides a variety of tools, (b) students' ability to choose, employ and integrate various representations and tools in their solution process, and (c) students' view of their solution tools.

In the present chapter the focus will be on the perspective of promoting creative mathematical thinking.

The *Solve It in Many Ways* activity provides students a linear equation, a quadratic equation, a system of two linear equations, and a system of a linear and a quadratic equation. Two out of the four equations are presented in the leftmost column of Table 11.4. The students are required to solve the given equations in *as many different ways* as they can, and express and to justify a personally preferred solution method. At the beginning, the activity also provides four tools for solving equations: paper and pencil, and three computerized tools—graph plotter, algebraic symbol manipulator and spreadsheets.

Learning Competencies. We would like to express here the assumed potential of such an activity in terms of the required learning competencies as stated by Friedlander and Arcavi (2017).

- *Representing and interpreting*: For each equation, students make frequent transitions between its representation as an equality between two expressions that can undergo the same operation on both its sides (when using an algorithmic paper-and-pencil solution method), its representation as an equality between two graphically represented functions (when using a graph-plotter), its representation as a sequence of (some true and some false) numerical equalities (when using spreadsheets), and its representation as a "black box" that produces answers without an interpretation (when using an algebraic symbol manipulator).
- *Reverse thinking*: In some cases (for example, when employing trial-and-error substitutions for solving quadratic equations, or verifying numerical sequences with spreadsheets) students are expected to solve equations by starting from assumed solutions, rather than proceeding from the given equation towards a desired solution.
- *Justifying and proving*: In this activity, solution methods become objects that are compared, their properties are discussed, and personal preferences of some of them are expressed and justifyed.

Table 11.4 Solution methods employed by the interviewed students (number of pairs using each method[a], Friedlander and Stein 2001)

Eq.	Tool			
	Paper and pencil	Graph plotter	Alg. symbol manipulator	Spreadsheet
$1.2(x-0.5)=8.4$	• Expanding the expression and solving for x. (6) • Dividing first by 1.2 and then solving for x. (2)	• First, an unsuccessful attempt to graph the equation by entering it as a whole, and then (a) give up. (1) or (b) graph each side separately and trace the intersection point (eventually changing step size or scaling). (2) or (c) graph the left side, trace and monitor the y-coordinate. (1) • Direct performance of stage (b) or (c). (2)	• Entering the equation and pressing the "Solve" key. (6)	• Entering a sequence of numbers in Column A (e.g., from 1 to 10 in steps of 0.1), possibly changing the step size. Then, entering in Column B the left side or the whole equation, as a formula, copying it downwards and looking for the appearance of the right-side value (8.4), or the True value. (4) • Unsuccessful attempts (confusing the role of the independent and dependent variable, omitting the sequence of independent variable). (2)
$x^2-5x+6=0$[b]	• Dividing by x (correctly) and unable to continue. (2) • Dividing by x (incorrectly) and receiving an incorrect solution. (3) • No attempt (awareness of an unknown method). (1)	• Graphing each side separately and tracing the intersection point (possibly changing step size or scaling). (2) • Graphing the left side, tracing and monitoring the y-coordinate. (3) • Mentioning graphing, but no attempt. (1)	• Entering the equation and pressing the "Solve" key. (6)	• Entering a sequence of numbers in Column A (e.g., from 1 to 10 in steps of 0.1), possibly changing the step size. Then, entering in Column B the left side or the whole equation as a formula, copying it downwards and looking for the appearance of the right-side value, or the True value. (3) • No attempt. (3)

[a]Occasionally, the same pair produced more than one written (paper and pencil) solution for the same equation
[b]During their course work, the students did not encounter the algebraic algorithm for solving quadratic equations

- *Divergent thinking:* The activity provides four tools for solving equations. However within these constraints there are many degrees of freedom in choosing a sequence among the given tools (possibly omitting some of them), and in following a different solution path for each chosen tool.

Method. The study was conducted within the learning environment of a junior-high school mathematics curriculum, integrating an interactive computerized learning environment. However, the students had no previous experience with parallel work with several computer tools, and with the need to choose a tool according to their considerations. Task-based interviews were conducted with six pairs of 13–14 year-old higher, and average ability students.

At this stage of learning, the interviewed students were not acquainted with the algorithmic solution of a quadratic equation. During their work, the students received no instructions or hints with regard to their choice of tools or solution methods.

Findings. Although in their actual work, all six pairs chose to start each of the four solutions on paper, the students were less committed to this tool in their comments made at the end of each task. In the case of the linear equation, most students expressed a preference to a solution on paper. In the other three tasks, however, most students chose one of the two computerized tools as their first explicitly expressed preference.

With regard to their choice of a preferred tool, the students displayed in their actual work a preference for manual, algebraic algorithms. In their comments, however, they frequently expressed a preference for a technological tool, and could provide an explanation for their opinion. The main criteria that influenced the students' choice of tools were its potential to display the solution process, its potential to allow a higher extent of student involvement in the solution process, and its compliance with accepted norms of work.

The Creativity Perspective. In this task creativity can be expressed by the variety of solution methods—rather than by a variety of solutions.

With regard to the group's fluency, the students employed 26 solution methods for the first equation and 24 methods for the second—an average, of four different solution methods for each equation for each pair (see Table 11.4).

Flexibility in this task expressed itself both by the students' ability to make transitions between four different tools and by their ability to produce as a group various solution strategies for each tool.

Although the students were not acquainted with the formal algorithmic solution method of quadratic equations, some of them were able to solve the quadratic equations given in this task in a non-algorithmic way. We considered these cases as instances of original thinking.

Thus, with regard to the issue of creative thinking, the study indicated that the interviewed students were able to employ creative thinking. They were able to employ a variety of solution methods, representations and tools, and to make connections between various meanings of the equation concept.

11.2.4 Creative Visualization of Equivalent Expressions

The *Crossed Squares* activity

Rationale. A context-based task should be experientially real for the student, and should serve as a basis upon which a mathematical concept can be built. Among its many cognitive and affective advantages, a context-based approach emphasizes the potential of using algebraic models and skills in other fields, and allows students to be flexible in their thinking, as the learning is not focused around mastering algorithms or procedures. Mason et al. (1985) state that "In order to have clear, confident and automatic mastery of any skill, it is necessary to practice, but the wish to practice will arise naturally from stimulating contexts" (p. 36).

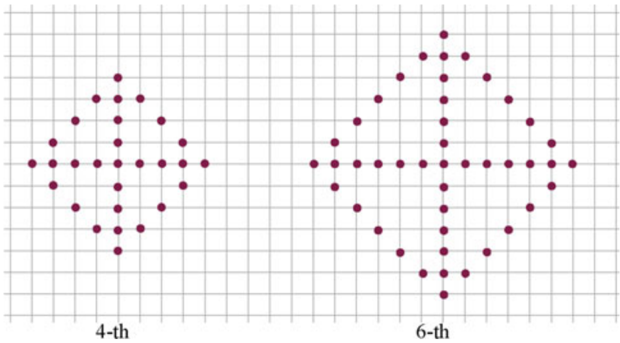
However, the finding of context-based activities that promote both creative mathematical thinking and understanding of algebraic procedures is not an easy task. Mason et al. (1985) also indicate that sequences of figures may be used as an appropriate context to represent the concepts of variables and equivalence of algebraic expressions. By working in a contextual situation, the concept of equivalence is raised by an authentic need to compare different ways of counting that are designed by students to describe the same sequence—and not by an arbitrary requirement to simplify abstract expressions presented by the teacher. Such a comparison requires flexible thinking by the students. We claim here that in addition to promoting conceptual learning and creative mathematical thinking, work on this kind of activities facilitates processes related to understanding and performing the algebraic procedure of simplifying algebraic expressions.

We present and discuss here the *Crossed Squares* activity (Fig. 11.2) as an illustrative example of such a context-based task.

Learning Competencies. As in the case of the previous instances, we analyze here the assumed potential of the *Crossed Squares* activity in terms of the learning competencies that we would like to relate to creative thinking.

- *Representing, modeling, and interpreting*: In figure sequences, algebraic expressions represent patterns and relationships between the place of a figure in the given sequence and the number of components contained in that figure. As a result, the expression is a mathematical model of a concrete situation, based on the properties of the figures. Students should be able to analyze and compare alternative models that correspond to the same sequence, interpret the results, and on that basis, derive new knowledge about the meaning of algebraic generalizations, equivalent expressions, and processes of simplifying expressions.
- *Reverse thinking*: The connection between an expression and a counting method is a two-way relationship A counting method determines a generalized expression and on the other side, a given expression allows us to identify the corresponding counting method. We also note that most activities based on figure sequences (even though not this particular one) require "backward thinking" in terms of finding the place value in the sequence of a figure made of a given number of elements.

The two crossed squares below are the fourth and the sixth in a sequence of crossed squares that increase at a fixed rate.



a) How many dots make up these two shapes?

b) Check your answer to a) by counting in a different way.

c) Use your two ways of counting to determine how many dots you will have in the eighth crossed square.

d) Find other methods of counting these dots. Discuss your methods of counting and write the corresponding expressions

Fig. 11.2 The *crossed squares* activity

- *Generating examples*: Students are encouraged to provide examples of counting methods and their corresponding algebraic expressions and compare them with those provided by others.
- *Generalizing:* Students identify patterns, relations, properties, and processes of variation, and express them in numerical, symbolical or verbal representations.
- *Justifying and proving*: If working in groups, students explain and justify the relation between their counting methods and the resulting algebraic expressions.
- *Divergent thinking:* The activity requires multiple solution methods, a wide variety of answers, meaningful mathematical discussions, and opportunities for creative solutions.

Method. The *Crossed Squares* activity was not employed as a tool in a formal research study. However, the activity was presented as a paper-and pencil task towards the end of the first year of a beginning algebra course, conducted in a learning environment composed of both spreadsheets and paper-and-pencil activities. The activity was a part of the students' regular classwork, and recommended to be conducted in groups. The findings are based on teacher observations conducted in several experimental classes.

Findings. The participating students used a variety of ways to count the number of dots in the given figures, and as a result produced a variety of corresponding algebraic expressions for the n-th shape in the sequence—each of them being equivalent to $8n$—3 (Fig. 11.3).

The Creativity Perspective. We claim that the task of producing and discussing various expressions by employing a variety of counting methods is a manifestation of flexible thinking. With regard to the issue of creativity, in this context-based approach, the expressions produced by students during the activity represent different and creative ways of modeling the same phenomenon. Students were working in pairs, and in many cases each of them employed initially a different counting method. As a result, they had to switch roles between presenting and justifying their counting method and corresponding generalization, and listening and understanding the solution presented by their peer. In some cases, the pair worked together to "decipher" the link between an expression and a counting method.

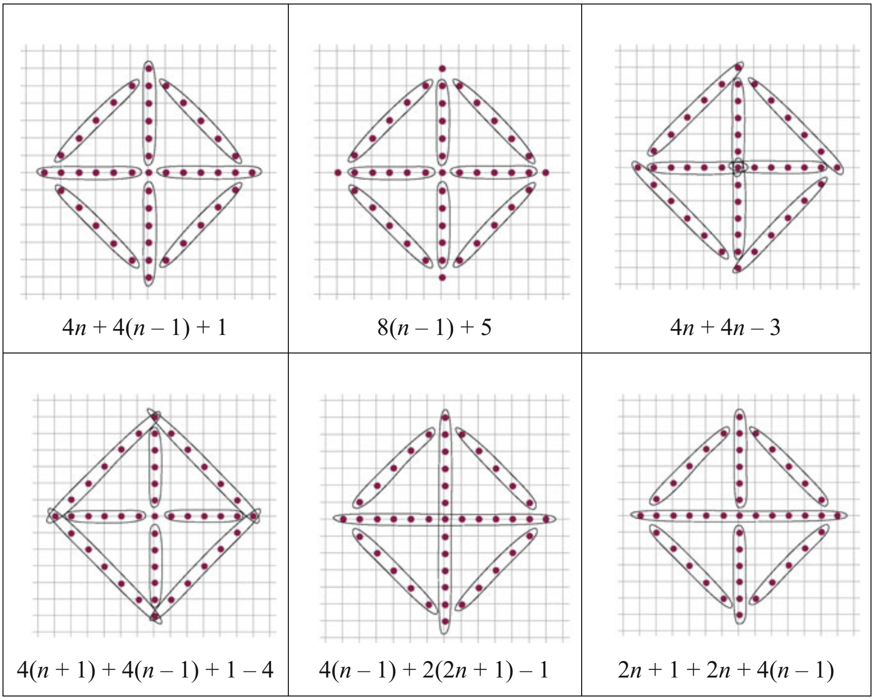


Fig. 11.3 Counting methods and their symbolic representation

11.2.5 Thinking Divergently in Reversed Procedural Tasks

The *Express Yourself* activity

Rationale. Divergent thinking is frequently involved in mathematical tasks aimed to promote multiple solution methods, a wide variety of answers, meaningful mathematical discussions, and opportunities for creative solutions. However, these characteristics are less frequently encountered in procedural algebraic tasks.

The search for an expression that is equivalent to a given one involves several procedures: expansion, simplification, rearrangement of terms and sometimes, replacing one symbol by another with related meanings (for example, replacing the division sign by a fraction bar). These procedures are needed for solving many mathematical exercises and problems in school algebra and calculus. Classroom experience and research (for example, Matz 1982) show that learning these procedures as an arbitrary set of rules frequently leads to short-term retention and misconceptions, and as a result, to an erroneous performance.

One of the pedagogical strategies with potential to support such meaningful learning is to require students to create expressions under given constraints.

In contrast with "direct" algorithmic procedural tasks, reversed procedural tasks require reversed thinking—i.e., reversing the "direction" of an activity, and reconstructing expressions or equations according to given parts, or the final result of an exercise.

The activity presented and discussed in this section (Fig. 11.4) is taken from a study by Friedlander and Arcavi (2012) and from a book by the same authors (Friedlander

1. Complete the blanks to obtain expressions that are equivalent to **15*x***

   a) $30x - \square$ b) $30x + \square$ c) $\square - 30x$

   d) $30x \cdot \square$ e) $30x \div \square$ f) $\square \div 30x$

   Find another solution for each case.

2. Write two expressions in each slot.

Expression	as a sum	as a difference	as a product	as a quotient
x				
$-x$				
x^2				
$-x^2$				

Fig. 11.4 Two reversed procedural tasks (Friedlander and Arcavi 2017)

and Arcavi 2017) on the potential of integrating procedural and conceptual knowledge. We claim that this kind of tasks has the potential to promote both procedural skills and divergent thinking.

The main activity in these tasks is to construct examples of more complex expressions that are equivalent to the given simple results. This activity requires experimentation, trial and error, and monitoring of results. In the process, students are expected to solve many exercises posed by themselves as intermediate steps towards pursuing the goal.

We assume that certain tasks set in a procedural context have the potential of promoting students' symbol sense (as defined by Arcavi 2005) and mathematical creativity—in addition to promoting their procedural knowledge.

Besides the requirement of "going backwards" from simple to more complex equivalent expressions (which is in a sense the reverse of simplifying), reversed procedural tasks require elevating the level of ingenuity and inventiveness. The two reversed procedural tasks presented here require students to produce multiple solutions, and as a result, allow them to suggest (and later discuss) solutions according to their level of mathematical knowledge and creativity.

Learning Competencies. A task analysis of the assumed potential of this activity led us to the following list of required learning competencies.

- *Representing and interpreting*: Students are required to construct equivalent expressions in a non-algorithmic way and apply numerical and operational properties. Thus, they are encouraged to consider the similarities and differences between arithmetic and algebra, to understand the meaning of the equivalence of algebraic expressions, and to apply properties of operations (associativity, commutativity, and distributivity). These tasks aim to confront students with the need to reflect, compare, check, and discuss procedures as well as concepts (e.g., equivalence).
- *Reverse thinking*: The main activity in this task is to construct examples of more complex expressions that are equivalent to the given simple results—i.e., "going backwards" from simple to more complex equivalent expressions (which is in a sense the reverse of simplifying).
- *Generating examples*: Students are expected to construct two examples (in the first task) and eight examples (in the second task) of expressions that are equivalent to each of the given results.
- *Divergent thinking*: The activity requires students to construct a wide variety of answers, it encourages meaningful mathematical discussions, and provides opportunities for creative solutions.

Method. The activity was administered as a questionnaire to 56 students from three ninth-grade classes, and their work was collected and analyzed. The classes were a part of the experimental design process of the *Integrated Mathematics* (2015) middle-grades mathematics program.

Findings. The responses presented in Table 11.5 below show that students were able to employ divergent thinking, and to provide a wide variability of answers.

Table 11.5 Examples of student answers to Task 1 in the *Express Yourself* activity

$30x - \_\_\_ = 15x$		$30x + \_\_\_\_ = 15x$		$30x \cdot \_\_\_ = 15x$	
Correct	Incorrect	Correct	Incorrect	Correct	Incorrect
(N = 68)	(N = 43)	(N = 62)	(N = 52)	(N = 55)	(N = 50)
$15x$	$(3.872x)^2$	$-15x$	$-3 \cdot - 5x$	$\frac{1}{2}$	$\sqrt{\frac{1}{4}x}$
$5x \cdot 3$	$(\frac{40x}{2} + \frac{10x}{2})$	$-14x - x$	(-15)	0.5	$0.5x$
$\sqrt{225}x$	$5x + (25x)^2 - 635x$	$50x - 65x$	$\frac{30}{-2}$	$\left(\frac{1}{4} + \frac{1}{4}\right)$	$-2 + 75x$
$5 \cdot 3x$	$[-(-15x)] + 30x$	$\frac{\Pi \cdot x^2 \cdot (-3^2-6)}{\Pi \cdot x}$	$(\sqrt{225})^2 \cdot -1$	$\sqrt{0.25}$	$15:100 + 10 \cdot 5x$
$30x + 15x$	$-5x \cdot -3x$	$70x - 85x$	$\frac{-3}{-21} \cdot 7 - 1$	$2 \cdot 0.25$	$30x \cdot 3:6$
$30x:2$	$15x \cdot 2 + 5x$	$(\log 100000) \cdot -3x$	$(30x + 30x):4$	$0.125 \cdot 4$	$(10^2:200x)$
$3 \cdot 5x$	$(-5x) -5 \cdot 5x + 5x$	$-(3 \cdot 5x)$	$(-3.14159265358 \cdot \pi)$	$\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10}$	$2-45x$
$\frac{45}{3}x$	$(\frac{-60}{2})x - 45x$	$-(2 \cdot 7.5x)$	$((5x + 2x)^0 0 - 17x)$	$(\frac{370x}{370x}):2$	$\frac{25x-40x-10x+x}{4} \cdot 2$
$10x + 5x$	$+(-15x)$	$687x - 702x$	$(\frac{-7x}{-7x}:2 - 30)$	$\frac{1}{4} \cdot 2$	$\left(3x\frac{1}{2} - 3x\right)$
$30x \cdot 0.5$	$52x + 37x$	$18x - 33x$	$4(5x + 1.25x)$	$0.05 \cdot 10$	$(\frac{1}{3} + \frac{1}{3} + \frac{1}{3})$

The Creativity Perspective. In this case, we had to consider the students' final answers, rather than their solution methods, as evidence of creative thinking. The first notable fact in Table 11.5 is the high number of expressions produced by the participating students—a total of about 110 symbolic expressions for each exercise. In each case, the distribution of correct and incorrect responses is slightly different—but in all cases, more than half of the expressions were correct. However, in spite of the large number of incorrect answers, we can claim that the task elicited a high level of fluency.

We claim that the participating students displayed a high level of flexibility—even though they seemed to focus on varying the arithmetical operations involved in the numerical coefficients.

11.3 Summary

The need to foster creative mathematical thinking in school mathematics nowadays is acknowledged by the mathematical research community (e.g., Leikin and Pitta-Pantazi 2013) and to some extent, by some educational policy documents. As noted by Tabach and Friedlander (2013), there is an inherent tension between two seemingly opposing curricular goals: learning procedures and applying them in routine tasks, on the one hand, and learning concepts and employing more advanced thinking strategies in solving non-routine problems, on the other hand.

In this chapter we presented five examples of tasks that address both domains of procedural and conceptual learning, and examined methods of assessing their potential to induce creative thinking in all students.

Table 11.6 summarizes the learning competencies and components of creative thinking found in each of the five tasks presented here.

The check-list presented in Table 11.6 shows (a) the presence of many learning competencies required by each of our five instances, as found by an a-priori task analysis based on Friedlander and Arcavi (2017), and (b) considerably high levels of creativity in the solutions of the participating students, as defined by Torrance (1974), and indicated by data found in some of our previous studies.

There are some obvious connections between the set of learning competencies employed in our task analyses, and that of the three components of creative thinking deduced from our empirical data. Thus for example, a task that requires students to generate examples, is likely to promote fluency, a task that requires divergent thinking is likely to induce flexible thinking.

Table 11.6 also indicates that tasks that address a wide range of learning competencies have the potential of promoting students' mathematical creativity as well. The learning competencies required by a task can be deduced from an a-priory analysis, whereas its potential to elicit creativity can be established by conducting a carefully structured empirical study. Our findings indicate that task analysis may be a reliable first indicator of its potential to promote mathematical creativity as well.

Table 11.6 Distribution of learning competencies and components of creative thinking by instances

		1. Identical columns	2. Make a quiz	3. Solve in many ways	4. Crossed squares	5. Express yourself
Learning competencies	Representing, modeling, interpreting	√	√	√	√	√
	Reversed thinking	√	√	√	√	√
	Generating examples	√	√		√	√
	Generalizing	√			√	
	Justifying, proving			√	√	
	Divergent thinking	√	√	√	√	√
Creativity components	Fluency	√	√	√	[a]	√
	Flexibility	√		√	[a]	√
	Originality	√	√	√	[a]	

[a]No empirical data at this stage

The occurrence of original thinking seems to be a function of individual students or of class norms, rather than of certain task characteristics. However, we assume that tasks that do ***not*** require some of the learning competencies described and discussed above, are most likely closed tasks that will not allow for creative thinking.

To conclude, we claim that open-ended procedural tasks that require students to employ reversed thinking, to provide multiple examples, and to choose their solution tools and solution methods, have the potential to promote a higher degree of conceptual understanding and mathematical creativity.

We note that in order to establish the nature of the learning processes and the quality of the learning outcomes, more classroom implementation and research is needed to accompany the teaching and learning of this type of tasks.

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Part IV
Tasks and Techniques to Enhance Creative Capacities

Chapter 12
Gifted Students Approaches When Solving Challenging Mathematical Problems

Andreas Poulos and Joanna Mamona-Downs

Abstract This study explores the different solving approaches of three mathematically gifted students attempting to solve a primarily Euclidean geometry problem, specifically a geometric characterization problem. We are comparing the various methods, steps and tactical maneuvers of the individuals as they transition between using geometric and algebraic tools. The posed problem proved to be very challenging without a computational approach. We present and analyze all the efforts and attempts with special interest in the combination of analytical thinking and experimentation using software. The analysis reveals the students' individual ability towards mathematical problem solving in general.

Keywords Gifted · Problem solving · Analytical/experimental approaches

12.1 Introduction

This study focuses on the steps that gifted solvers of mathematical problems follow. We are particularly interested in examining the solvers' lines of thought when they are confronted with problems of characterization, i.e. when they are asked to determine which mathematical objects satisfy given properties (instead of finding the properties of an object). This theme is rarely taken into account in the problem-solving literature. We regard it analogous to the framework of 'example generation', (e.g. Watson and Mason 2005) and 'heuristic refutation' (De Villiers 2000) where the appearance of counterexamples leads to the reframing of the task environment. The problems under consideration require a level of knowledge of mathematical concepts and theoretical constructs up to the last year of High School. We present the fieldwork material built on a problem that was constructed for this

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particular level of mathematical attainment; as a result of the data processing, we obtained interesting solution approaches as well as other relative problems posed. We compare our findings with the findings of other related studies on the skills and behaviours of gifted students. The following problem is the one used in the fieldwork:

> Count the number of equilateral pentagons of length 1 with integer size angles in degrees.

This problem is a partial generalization of the simpler problem "Count the number of equilateral quadrilaterals (rhombuses) of length 1 with integer size angles in degrees". It is easy to calculate that there are 90 solutions to this problem. One of these solutions is the square, and the remaining solutions are rhombuses whose two different angles vary from 1° to 179°. The solutions are reduced to 89 since we get pairs of congruent solutions. The given problem is also a specific case of the following general problem "Count the number of equilateral polygons of length 1 with integer size angles in degrees".

It is a general consensus that the main sources of mathematical problems are:

(a) Mathematical research and research in other domains of science,
(b) Problem-posing for examination purposes, from high school exams up to the level of mathematical competitions,
(c) Problem-posing targeted to specific cases, such as research on the detection of special knowledge and skills (Mamona-Downs and Downs 2005).

The specific problem used in this study does not belong to the first category, since it is not an open problem in mathematical research. One could possibly think that this problem could be set as a problem at mathematical competitions. This, however, will prove not to be the case, since the time required to solve this problem, especially without the use of a software, greatly exceeds the time limits of a competition. This problem was posed as an instrument in a research program in Mathematics Education, in order to observe the problem-solving approaches that gifted solvers employ, and the specific mathematical domains, which they draw in their work for the resolution of the problem (Poulos 2016).

According to Singer and Voica (2013, p. 13) "we use the term "expert" in problem-solving in a broad sense, referring to three types of persons:

(1) the mathematician who runs research activity,
(2) the teacher who trains students for math competitions, and
(3) the high achiever student validated by the results obtained in math competitions".

We carefully selected the participants of this study to fall into the third category. Research on this type of solvers is rather limited, although in the past decade, there has been an increase of interest in this subject, Cai et al. (2005), Koichu (2010).

12.2 The Theoretical Background of the Research

12.2.1 The Tradition of Polya

It is widely accepted that the main core of issues in advancing the solving of mathematical problems lie in Polya's tradition. The fundamental ideas appearing in his book "Mathematical Discovery" inspired us to conduct the part of our research presented here. In this book he mentions: "In fact, what is presented here are not merely solutions but case histories of solutions. Such a case history describes the sequence of essential steps by which the solution has been eventually discovered, and tries to disclose the motives and attitudes prompting these steps. The aim of such a careful description of a particular case is to suggest some general advice, or pattern, which may guide the reader in similar situations", Polya (1981, p. x).

From this passage, we can see that Polya prompts us to observe carefully the sequence of steps that solvers follow when they approach a mathematical problem. In fact, the close examination of the unfolding of the solution offered by a solver constitutes not only a crucial aspect of research in mathematical problem-solving, but even more generally, of research in Cognitive Science.

Schoenfeld (1985) suggests the following four factors that affect problem-solving performance:

(i) the problem solver's mathematical knowledge;
(ii) knowledge of heuristics;
(iii) affective factors which influence the way the individual views problem-solving and
(iv) the managerial skills associated with selecting and implementing appropriate strategies.

As far as the process of problem-solving as a personal endeavor is concerned, we believe that the following extract from Polya's writings is of significant importance: "The reader who has spent serious effort on a problem may benefit from the effort even if he does not succeed in solving the problem. For example, he may look at some part of the solution, try to extract some helpful information, and then put the book aside and try to work out the rest of the solution by himself. The best time to think about methods may be when the reader has finished solving a problem, or reading its solution, or reading a case history. With his task accomplished and his experience still fresh in mind, the reader, in looking back at his effort, can profitably explore the nature of the difficulty he has just overcome" (Polya 1981, pp. xvii–xviii). The above describes an ideal behaviour of individuals working in solving a problem, which was either set in a mathematics class, or chosen by them out of personal interest, or given to them as a research item.

12.2.2 Recent Developments on the Multiple Facets of Problem Solving

Lester (2009) analysed the 'state of art' in early 90s by conducting a large survey on the studies on problem solving. His first conclusion was that between 1990 and 1994, the research on problem solving was not the most popular subject in the field of mathematical education. The accuracy of this statement can be verified easily by searching the published studies in journals of Didactics of Mathematics, and the proceedings of International Conferences such as I.C.M.E., (International Congress on Mathematical Education). Therefore, the following question arises "which are the reasons for this change of interest of the researchers"? Lester suggested the following three reasons:

(a) The first and obvious reason is that the research community was interested in other issues. Even though problem solving was one of the four pillars of the Curriculum that the National Council of Teachers of Mathematics (N.C.T.M.) in U.S.A. had suggested, new issues had attracted the researchers' attention internationally, many of which had not immediate relation with problem solving. Such topics include students' and teachers' perceptions of the nature of Mathematics, the study of social and cultural factors that affect the ways of learning Mathematics, the applications of Mathematics, etc.
(b) Many researchers and teachers believe that the questions concerning problem solving have been answered, or as Lester mentioned, "We believe that we know everything around mathematical problem-solving". In fact, a vast majority of the researchers on the Didactics of Mathematics and mathematics teachers in the United States had the opinion that the positions described by the "Standards" constitute a complete set of guidelines on problem solving; consequently, there was no need to either alter or improve them. Lester challenged these views directly and emphatically, since he believed that even the "Standards" had not been written under a complete research program targeted directly on efficient problem solving.
(c) Finally, he emphasized the fact that problem solving is a highly complex and multifaceted form of human activity that includes procedures far more complicated than simple recollection of methods. In addition, psychological, social, and other factors complicate problem solving even more.

With regard to the nature of problem-solving Mamona-Downs and Downs (2005) pointed out special aspects that constitute its 'identity'. They analysed how problem-solving impinges on other aspects of the mathematical work, such as the conceptualization of new concepts and assimilation of new theories, the handling of different representations of mathematical constructs, the realization of the mathematical structure embedded in a given task and the subtle differences and overlays between problem-solving and proof.

With respect to the means developed in order to identify the mental processes of the solvers during solving mathematical tasks, English et al. (2008, p. 6) emphasize that "the researchers have developed very few new tools that allow us to observe, substantiate and measure the skills which are believed to contribute in problem-solving".

Berzsenyi (1999, p. 186) notes that "to solve a problem, one usually needs at least one bright idea or a non-routine application of some method". Therefore our research goal is to detect possible bright ideas appearing in solvers' efforts and to analyze them.

12.2.3 Krutetskii's Cognitive Approach

Krutetskii (1976), in his quest for realizing the subjects' cognition when solving problems, underlined that one should distinguish between the processes of "solving a problem for myself" and that of "solving a problem for others". How did he overcome this dichotomy? During his experiments, he explained to the students involved exactly what he required of them. He did not ask them to explain their steps, only to talk out loud during the process of solving a problem. He stressed the fact that he was not interested in the actual solution of the problem, but rather in the procedure of getting there (or up to whichever point they reached). He encouraged them to think out loud, as they would have done if they had solved the problem on their own. Also, he did not interrupt and did not make any comments or remarks during the experiments. Krutetskii was especially interested in the time required to solve a problem, as a criterion of skill and talent. This criterion is not encountered in most of the studies of similar context. In some problems, the students sketched figures. These figures not only played an essential role towards the solution of the problem, but also were of significance for the interpretation by the researcher of the students' way of thinking. In addition, erroneous paths of solution and unhelpful actions employed by students were extremely informative and provided additional insight to him.

Krutetskii emphasized that "we know that during the process of problem solving many complex factors are in play, such as prior experience, assimilated knowledge, skills already acquired, etc. Therefore, the big question is to choose problems whose solutions do not depend directly only on one of these factors". For this reason, he avoided problems that required very narrow and specific knowledge or skills. He believed that some individuals' mind could be classified under what he refers to as a "mathematical cast of mind" i.e. a tendency to conceive the world in a mathematical way, something that is apparent in the cases of a gifted student in Mathematics. He assumed that such a tendency might have a biological basis.

He identified three such types of "mathematical casts of mind".

1. The analytic type, who tends to think with lectical-logical terms.
2. The geometric type, who tends to think with optical-figurative terms.
3. The harmonic type, who combines characteristics of both of the two previous types.

We were interested in determining which kind of 'mathematical cast of mind' the subjects of our research largely adhered to, and in fact, if this broad categorization is helpful to our understanding of the argumentation paths that they follow during problem solving.

12.2.4 Studies Targeted at Gifted Students

In relation to the mathematically gifted performance in problem solving, Saul (1999, p. 83) points out that "the thoughts of the mathematically talented do not differ greatly from those of the other students, except that they are more efficient". However, as there is no adequate research evidence in order to verify, or refute Saul's conclusions, we relate his statement to the question.

"In what exactly do "strong" solvers of mathematical problems differ from the "weak" ones?".

Research on this question is focused on the study of individual cases of solvers, who had either great ease or unusual difficulties in solving the same problem. The tendency of conducting individual studies was dominant during the 1970s–1980s, and only during the '90s research started systematically examining larger groups of people with regard to issues arising from the differences between "strong" solvers and "novice" ones. The development of Cognitive Science and of Artificial Intelligence affected not only the research questions that the community was interested in putting forward, but also the terminology used to describe differences, such as the ones mentioned above. Therefore, we come across less on the duality "strong" and "weak", and more on the pairing "successful" and "unsuccessful", "experienced" and "beginner" solver, etc.

We believe that the research on how gifted students solve problems has a long way yet to go before can be deduced concrete profiles. This subject resembles the tip of the iceberg, whose greatest part is hidden under water.

It is also interesting to investigate how experts in general, in addition to the gifted students, solve mathematical problems. Expert mathematicians for example are, by profession, strong problem solvers, without necessarily being gifted, and they are usually extremely competent in problem solving. This issue is raised in the research work of Schoenfeld and Herrmann (1982), and Silver and Marsall (1989).

Levav's and Leikin's research (2010) which is based on Polya's, Schoenfeld's and Kruteskii's ideas is also relevant to our theme. They conducted a wide scale research on the role of *multiple solutions on mathematical problems* as a criterion of the quality and the level of mathematical thinking. Levav and Leikin categorize the solver's multiple approaches to a problem with respect to the following criteria:

(a) Different representations of a mathematical concept;
(b) Different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic; or
(c) Different mathematical tools and theorems from different branches of mathematics (Levav-Waynberg and Leikin 2010, p. 766).

Mamona-Downs (2008) goes further on the significance of multiple solutions to a task provided by a solver. She claims that this is indicative of a successful 'structural appraisement' of the task environment; something that leads the solver to change the focus of his efforts and encourages him to produce different solutions.

12.2.5 The Use of Technology in Problem Solving

The question to what degree the use of software contributes to problem solving is of particular interest to the present part of our research. Sanchez and Sacristan (2003, p. 116), point out that "There is a fundamental difference in the construction of the geometrical figure between doing it with paper-and-pencil and doing it in a dynamic geometry environment: whereas in the first one it is the construction of a particular case, in the latter one it is actually the construction of a general case".

In Yevdokimov's paper on the relation between problem-solving and problem-posing, it is stated that "…, a very important factor [for succeeding in problem-solving/posing] is the development of students' skills for multiple flexible transitions from visual thinking to analytic one and vice versa" (2005, pp. 263–264).

Concerning the use of technology in problem solving Santos-Trigo's statement is interesting: "The coordinated use of digital technologies allows for diverse ways to identify, formulate, represent, explore, and solve problems situated in different fields or contexts. Consequently, new routes can emerge for learners to construct and comprehend disciplinary knowledge" (Santos-Trigo and Moreno-Armella 2016, p. 191).

Regarding the use of software at problem-solving, Dick (2007, p. 338) introduced the term "mathematical fidelity" and identified three reasons why we have limited fidelity.

(1) The non-correct syntax of code,
(2) Under-specifications of the mathematical structures, and
(3) Limitations on the representation of continuous phenomena based on discrete structures and finite arithmetical calculations.

Last but not least, reference needs to be made to Terence Tao's views on problem-solving, which are apparently very similar to Polya's views. Tao, himself an outstanding problem solver, underlines the significance of modifying the task environment, following in fact Polya's tradition. He considers it to be an 'aggressive type of strategy' that "… helps in getting an instructive feel of what strategies are likely to work, and which ones are likely to fail" (Tao 2006, p. 5).

12.3 The Framework of the Research

This paper is a case study, "a case history" in Polya's words, which portrays the solution paths of competent young mathematicians when solving a given problem. The participants provided extensive scripts that expounded their thought processes during the solution. Also, we asked them to loudly expose their mental arguments as much as possible; their utterings were taped. The scripts, the thinking aloud protocols, together with the first researcher's observations during the initial phase of tackling the problem and detailed questioning a posteriori on the lines of argumentation written down by the subjects constituted the core material for the analysis. The method of analysis was purely qualitative. The main research questions were:

1. *Which factors determine the various approaches of young mathematicians when they attempt challenging problems? What matters mostly for the resolution of the problems of characterization?*
2. *Do the solvers agree with communicating and exemplifying their results with the researchers in detail?*
3. *Could we identify the 'turning points' in their solution path as "Eureka" or "Aha!" moments?*
4. *What are the relative merits that our subjects ascribed to an analytical and an experimental solution of the same problem in a computer environment?*

12.4 Method

We conducted our research following the ethnographic method, see Eisenhart (1988).

12.4.1 Profile of Participants

The participants in our research program were three strong problem solvers aged 18–22 year old. We were very well informed about their mathematical background, we knew their scores in difficult mathematical competitions, their projects relevant to problem-solving, their mathematical 'hang-ups' and overall achievements.

At the beginning of the program the first author explained to the three participants the purpose of the study and the fact that we were interested in the different approaches they would employ to tackle the problem. He informed them that he was going to record their arguments in each stage of the problem-solving process, the way they expressed themselves, the explanations they gave to questions after the solving phase.

The profile of the three solvers is the following: The first solver was 18 years old at the time and he currently studies mathematics; he was a member of the Greek, Balkan and Olympic teams and excelled in all these competitions. The second solver was 22 years old, a magna cum laude graduate of one of the Departments of Mathematics of a Greek University; he also had a Master's degree. The third solver was 19 years old, an undergraduate student of an Engineering School, and had participated in mathematical competitions. All three solvers had experience in tackling complex problems and they were also comfortable in using various kinds of computer software.

The fieldwork took place during the 8th Summer School of Mathematics, organized by the Greek Mathematical Society in July 2014. The solvers did not meet with each other, and none of them knew that two other students were attempting the same problem. The first author met with each student on different days. He informed them that he posted a problem online (about 20 days prior to the meetings), on a 'Forum of mathematical discussions' and nobody had offered a solution so far. All three solvers seemed enthusiastic to tackle the proposed problem, and agreed readily to participate in this research program. The subjects gave their permission to the researchers to publish the main parts of their scripts and transcription documents of their thinking aloud protocols during the problem solving process. All sessions with every individual student lasted over 2 h divided in two periods, including the time spent on explanatory questions on their approaches. It was agreed that they would continue to work on the problem on their own and keep informing the researchers of possible different methods and ideas that they might exploit. It was made clear to them that, while continuing working on the problem at their own homes, they were free to consult books and, in general, any source of information they thought would lead to the solution.

In the following subsections, we outline the attempts of the first and the second solver, and we present the entire archive of the sequence of steps of the third one, who was especially expressive.

12.4.2 The Participants' Interviews

In this subsection we provide a detailed account of what the three students wrote down, together with parts of their thinking aloud protocols. We also provide the explanations they gave to the first researcher (who was present at the time), on their specific solution paths.

12.4.2.1 What the First Solver Tried

The first approach was purely geometrical. He realized that certain basic figures would constitute part of the solution, but from his actions in this step, he didn't obtain a substantial result.

His second approach was geometrical at the beginning, and shifted gradually to elementary Number Theory using formulae of Combinatorial Enumeration. His insistence, for a considerable amount of time, in using Combinatorial Enumeration results did not prove fruitful.

In his third attempt, the student eliminated some of the conditions of the problem, in order to facilitate his solving. In this attempt, as it was recorded, he was initially inclined to believe that the number of possible solutions was very large. Actually, this was the reason that he tried to reduce the problem to a problem of Combinatorics. In particular, he recalled that in the book by Herman et al. (2003), which he had studied while preparing for the International Mathematical Olympiad (I.M.O.), there was a theoretical topic that could prove useful to him in this problem. He mainly recollected Polya's theory on Enumeration, and a proposition called Burnside's Lemma, that is a result in group theory that gives a formula to count objects, where two objects that are related by symmetry are not to be counted as distinct. This Lemma refers to the number of partitions of a set. First, the student focused on a quadrilateral instead of working on a pentagon. He represented by Φ the set of permutations of the tuple with components the four angles of the quadrilateral, denoted by $a = (a_1, a_2, a_3, a_4)$. His main idea was to apply a permutation $\varphi \in \Phi$ on a. This particular permutation φ applied to the tuple 'a' gave:

$$\varphi(a_1, a_2, a_3, a_4) = (a_2, a_3, a_4, a_1)$$

and so $a_1 = a_2, a_2 = a_3, a_3 = a_4, a_4 = a_1$.

The iterations of the application of φ gave the following results:

1. $\varphi(a) = a$, this results in the quadrilateral being a square, as shown above.
2. $\varphi o \varphi(a) = a$, this results in the quadrilateral being a parallelogram and not a square, i.e. $\varphi o \varphi(a_1, a_2, a_3, a_4) = (a_3, a_4, a_1, a_2)$.
3. $\varphi o \varphi o \varphi(a) = a$, this results in the quadrilateral being a square again, i.e. $\varphi o \varphi o \varphi(a_1, a_2, a_3, a_4) = (a_4, a_1, a_2, a_3)$ and finally,
4. $\varphi o \varphi o \varphi o \varphi(a) = \varphi^{(4)}(a) = a$.

The student was puzzled by the result of this iteration, which he employed in the hope that Burnside's Lemma would give the known answer for the case of quadrilaterals and so he would use it in the case of the pentagon. As this was not the case, the student abandoned this approach.

On the next day, during his fourth attempt, the student was interested in generalizing the problem in the case of a polygon with n sides, and integer size angles in degrees; he found it very difficult to proceed and in the end dropped also this line of argumentation. Finally, he thought of constructing equilateral triangles situated on top of rhombuses. From his gesticulations we perceived it as an "Aha!" moment. He consequently drew the Fig. 12.1b, c. Figure 12.1b consists of an equilateral triangle situated on top of a rhombus, and this provides 89 different solutions, with angles ranging from 1° to 89°. We note that the solution (90°, 150°, 60°, 150°, 90°) corresponds to Fig. 12.1a.

In the end, the first solver found 90 different solutions as a result of the above-described approach and indeed without the aid of any computer software.

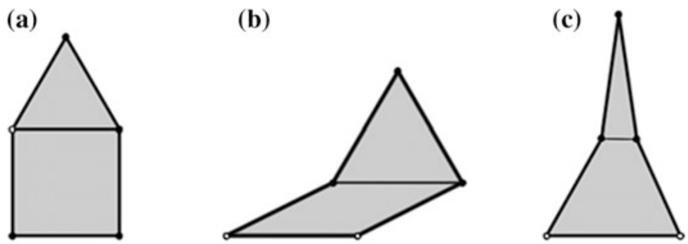


Fig. 12.1 .

12.4.2.2 What the Second Solver Tried

His first approach was also purely geometrical; he described cases of symmetrical solutions, and he highlighted this fact by saying "I notice that the symmetry of a vertex with respect to the corresponding diagonal forms a new non-convex shape". This idea led him to come up with "pairs" of solutions as in the group of Fig. 12.2.

The first author, who was present, observed that later on, the student realized that this kind of symmetry was not "reliable", in the sense that in some cases, either the convex or the non-convex shape of the pair did not have the desired properties; for example, the size of the angles for the non-convex pentagon of the pair were not integers in degrees. It is equally interesting that in this case, the student himself realized his mistake regarding the "symmetry" and reported it. Towards the end of this first attempt, he decided that an algebraic approach would be preferable to a geometric one.

His algebraic approach was essentially a further exploitation of the previous geometric one. He used trigonometric equations. Soon he realized that solving these equations by hand was an extremely tedious and time-consuming task, even more so since the number of cases was large. At this point, he started entertaining the idea of writing a computer program to solve the problem. However, during this attempt, he was using only pen and paper.

The next day he exploited the geometrical properties and relations of the task environment, being well acquainted with the management of geometrical figures. He used coordinates to describe the vertices of the pentagons, resorting in Analytic and elementary Vector Geometry. He initiated, as the first solver did, the notion of "hats", i.e. name a pentagon ABCDE, and without loss of generality, choose the diagonal AC. The triangle ABC is a "hat" and has two orientations "upwards" and "downwards" with respect to the diagonal AC. The orientation of the "hat" determines whether the pentagon is convex or non-convex. He realized via the use of "hats" the existence of $2 \cdot 89 + 2 = 180$ solutions of convex and non-convex pentagons, though a lot of these solutions turned out to be symmetrical. A concise description of the sequence of steps he followed in order to solve the problem follows in the next paragraph.

The following idea occurred to him: "each time, choose the measure of two consecutive angles, e.g. θ_1, θ_2, in integer degrees, then the position of the four out of the five vertices of the pentagon are controlled". What remains to be described, is

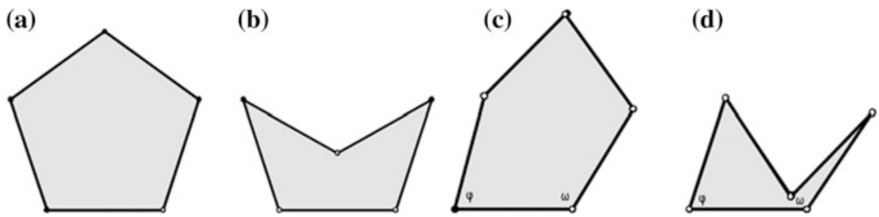


Fig. 12.2 .

the compatible hat, namely, the fifth vertex that will be determined by two sides. The corresponding angle to this vertex is defined by the trigonometric equation:

$$\theta = 2\arcsin\sqrt{\frac{3 + 2[\cos(\theta_1 + \theta_2) - \cos\theta_1 - \cos\theta_2]}{4}} \tag{12.1}$$

Therefore, Eq. (12.1) must be solved for all possible pairs θ_1, θ_2, as in Fig. 12.3, where $\theta_1 = \hat{B}$ and $\theta_2 = \hat{C}$. To begin with, he noticed that he must have $\theta_1 \le \theta_2$, otherwise he would have to check the pairs of angles twice. Also, the two sides AB and DC will intersect if $\theta_2 \le 90° - \theta_1$.

Finally, he concluded that it was sufficient to check for $1° \le \theta_1 \le \theta_2 < 180°$, since two consecutive angles of every pentagon have to be convex, otherwise the pentagon would have at least three non-convex angles, that is, it would have at least $3 \cdot 180° = 540°$ which is a contradiction. Therefore, (12.1) should be solved $179 + 178 + \cdots + 1 = 16{,}110$ times and one should also check that $\hat{E}$ is an-angle of integer degrees, see Fig. 12.3.

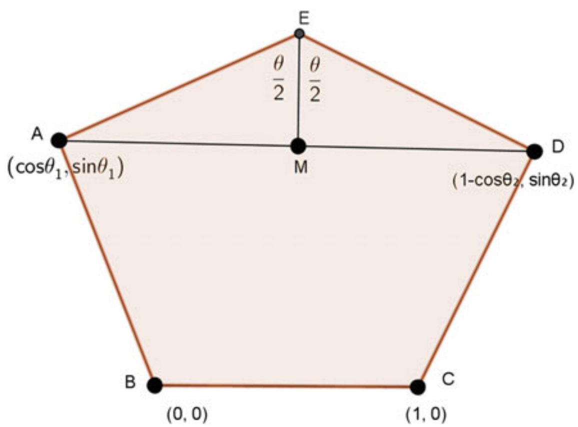
Even in this case, it is possible that the angles $\widehat{EAD}$ and $\widehat{EDA}$ are not of integer degrees. Also, as we mentioned earlier, the hat, in this case AED has two orientations, "upwards" and "downwards".

What remains is to calculate the angles $\hat{A}$, $\hat{D}$ and to check that the corresponding figure is an acceptable solution.

He first calculated the coordinates of the vertex E. The coordinates of the midpoint M of the diagonal AD are

$$\mathrm{M} = \left(\frac{\cos\theta_1 + 1 - \cos\theta_2}{2}, \frac{\sin\theta_1 + \sin\theta_2}{2}\right).$$

Fig. 12.3 .



Since AED is an isosceles triangle, ME will be also the corresponding height from the vertex E of the triangle. The line passing through the points A and D has slope $\frac{\sin\theta_2 - \sin\theta_1}{1-\cos\theta_2-\cos\theta_1}$. The equation of the line passing through ME will be

$$y - \frac{\sin\theta_1 + \sin\theta_2}{2} = \left(\frac{1-\cos\theta_2-\cos\theta_1}{\sin\theta_1 - \sin\theta_2}\right)\left(x - \frac{\cos\theta_1 + 1 - \cos\theta_2}{2}\right).$$

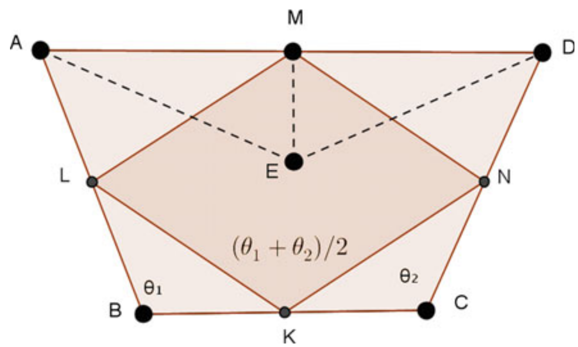
The line segment ME has length $\|\mathrm{ME}\| = \cos\left(\frac{\theta}{2}\right)$. The resulting two E points will correspond to the two orientations of the hat. If we adjust the "hat" to Fig. 12.4, this will define a pentagon.

Let L, M, N and K be the midpoints of the sides of the quadrilateral ABCD. Then LBK is an isosceles triangle, so we can deduce for the $\widehat{BKL} = \frac{180^\circ - \theta_1}{2}$. The triangle KCN is also an isosceles triangle, so again we can deduce that $\widehat{CKN} = \frac{180^\circ-\theta_2}{2} = \widehat{CNK}$. The outcome is that $\widehat{LNK} = \frac{\theta_1+\theta_2}{2}$. The quadrilateral LKNM is a parallelogram, since its vertices are the midpoints of a quadrilateral, hence $\widehat{KNM} = 180^\circ - \widehat{LKN}$. Therefore, we will have that $\widehat{MND} = 180^\circ - \widehat{MNC} = \frac{\theta_2}{2} - 90^\circ$. The coordinates of the point N are $\left(\frac{2-\cos\theta_2}{2}, \frac{\sin\theta_2}{2}\right)$. He also calculated the lengths of the sides MD and MN of the triangle MND.

Using the sine rule, he obtained $\frac{\sin MND}{MD} = \frac{\sin MDN}{MN} \Rightarrow D = \arcsin\left(\frac{\sin MND}{MD} \cdot MN\right)$, and $\widehat{MAL} = 360^\circ - \theta_1 - \theta_2 - \widehat{MDN}$, where $\widehat{EDM} = \frac{180^\circ - \theta}{2}$ for the hat pointing upwards. For the hat pointing downwards, he obtained that $\widehat{EDN} = \widehat{MDN} - \frac{180^\circ-\theta}{2}$ and $\widehat{EAL} = \widehat{MAL} - \frac{180^\circ-\theta}{2}$.

He considered 'translating' all of the above steps into a program using Mathematica 7.0, software with which he was very familiar. He wrote down the necessary algebraic relations and equations, and carried out all of the calculations

Fig. 12.4 .



meticulously, in order to simplify algebraic relations as much as possible. He expected that the program would help him crosscheck the solutions found before, as it outputs shapes of the possible solutions.

After all these calculations executed by the program, the final output were the figures of the pentagons with the string of the five angles printed underneath each figure. He verified the resulting arguments from Eq. (12.1) in order to exclude the ones with non-integer size and he additionally rejected the self-intersecting pentagons. Overall he came up with a total of 121 different solutions.

Figure 12.5 shows some of the output results of the program.

12.4.2.3 What the Third Solver Tried

We include the complete record of the attempts of the third solver, written and transcribed, in order to portray as accurately as possible the way he worked throughout the entire process.

"The problem asks us to count the different equilateral pentagons, with integer angles in degrees. When I first thought of this problem I had the impression that it was a geometrical problem, however, I thought later it is probably a Number Theory problem, since it asks for integer arguments. The requirement about the integer arguments created many difficulties in the trigonometrical equations I had to solve. The equations involved in this problem are defined on the reals, yet I am restricted to natural numbers. Therefore, I didn't manage to solve the problem analytically, whatever "analytically" is supposed to mean in a Number Theory problem, where integer values are sometimes required as solutions. What I tried to do was to check what values had to be excluded, handling the problem with pen and paper. Finally, when I realized that I had run out of ideas for an analytical solution, I ended up writing a program to calculate all the solutions. The program was based on a plan devised while trying to solve the problem geometrically. While analysing the problem, I will write in the first and the second person, as this is the way I think and explain things to myself".

Let me analyze the problem:

– This problem looks interesting. What exactly does it asks for, pentagons with integer angles in degrees? I just noticed the fact that the measure of the angles must be integers.
– A! I have the regular pentagon. What other pentagons could satisfy these conditions? Is the regular pentagon the only such pentagon?
– This isn't necessarily true. I imagine that the vertices of the pentagons have "joints", that is, I could move these joints and preserve the length of the sides. Then the angles would change, and we would obtain a lot of such figures.
– Maybe this problem isn't so simple after all. These pentagons cannot be random. We must have some constraints imposed on the angles, by the condition of equality of all the sides.

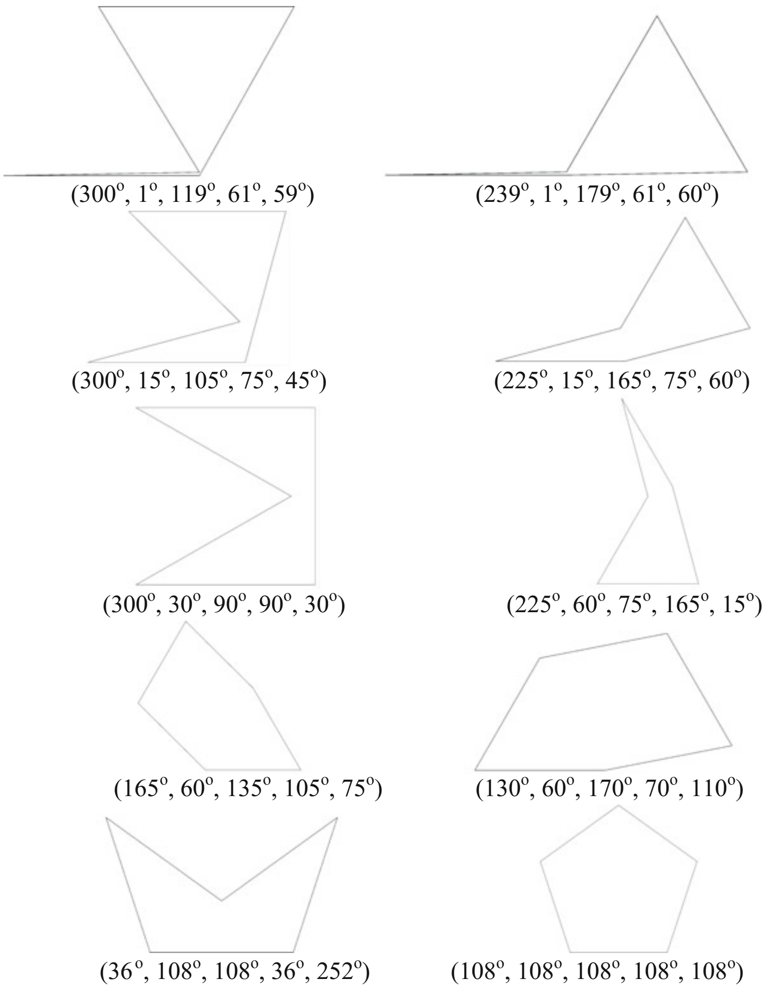
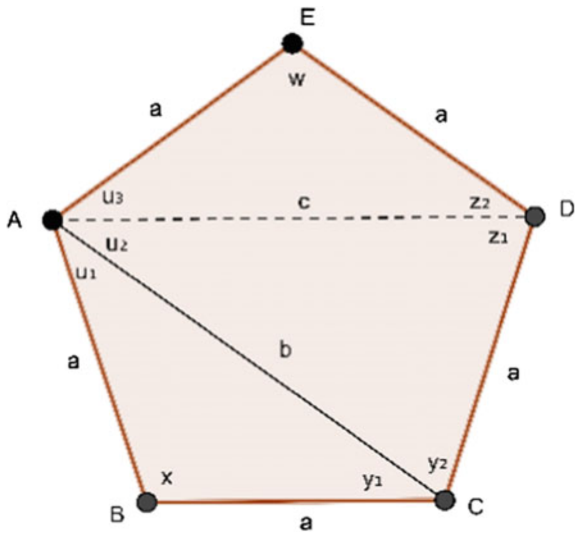


Fig. 12.5 .

– Yes, that is probably the case. I rather not focus on the pentagon as yet. Maybe there is a faster way.
– Why is the problem asking only about pentagons? Would it be easier about n-agons, with less than five sides? Let's start from there.
– It is trivial in the case of triangles. There exists only the unique solution of the equilateral triangle, with angles (60°, 60°, 60°). No doubt about that.
– What about quadrilaterals? The sides of the quadrilateral could move just as in the case of the pentagon. I suspect that in this case there isn't a unique solution, although I'm thinking of the square. Of course, the equilateral quadrilaterals are special, and are referred to as rhombuses.

- What are the properties of the rhombus that make it so special?
- A rhombus must be a parallelogram.
- This means that its opposite angles are equal. Let x and y be the measure of the non-equal angles, then I will have $2 \cdot x + 2 \cdot y = 360°$, that is: $x + y = 180°$.
- And now?
- Now we have the pairs of integers (x, y) and both x and y can vary from 1° to 179°, therefore, I will try possible pairs. Obviously I do not have to check all pairs, for example the pair (1, 1) is not such a pair.
- Good, then I will fix x and check if $y = 180° - x$ is a positive integer.
- Good! So I have 90 different solutions, while x varies from 1° to 90° and while y varies from 179° to 90° respectively. There is no need to check for x varying from 90° to 179°, this will give me congruent solutions. This is a good method. Can I apply this method to a pentagon as well?
- Probably yes, but it won't be enough to fix only one angle in order to determine the equilateral pentagon completely. I didn't have to fix an angle in the triangle case. In the quadrilateral case I had to fix one angle. I think that in the pentagon case I will have to fix two angles.
- It is not smart to fix all five angles. Fixing four of the five angles, determines the fifth angle, since the sum of the measure of all the five angles has to be 540°. This isn't very useful, I have not so far used the fact that all sides are equal in any of the solutions. If I fix three of the five angles, the fact that all sides must be equal will determine uniquely the other two angles. Therefore, I conclude that if I fix two consecutive angles, the equality of the sides will somehow affect the other three angles.
- Good. So what shall I do? Without loss of generality I will fix two consecutive angles x, y, whose arguments are integer numbers in degrees, and I will calculate the three remaining angles, call them z, w, u, and I will check if their measures are also integer numbers in degrees.
- Yes, this sounds good! Now let's see, why is this in fact true? If I fix two consecutive angles x, y, I fix the placement/position of three consecutive sides of the pentagon, namely, AB, BC and CD and what remains to find is the position of the other two sides AE and ED. The other two sides will be unique (for convex pentagons), since the sides have to be equal, and the vertex E that connects them will be on the perpendicular bisector of the line segment AD (see Fig. 12.6).
- This means that I can determine the angles trigonometrically, and I can find constraints in my choices. The sum of the measure of the two angles x and y_1 have to be less than 180°. The triangular inequality on the triangle ABC must hold true, even after I fix the two consecutive angles.
- At this point I note that I do not want to deal with non-convex pentagons, I will solve this later. Also, the constraints that follow from the triangular inequality aren't a problem. In the final program I wrote, I only excluded the choices of the angles x and y that did not result to a pentagon.
- So what should I do now? Find the relation that describes z, w and u, in terms of x and y?

Fig. 12.6 .



– Exactly! Denote by 'a' the length of the sides. Using the cosine rule, I can find the length of the diagonal b that divides the angle y into two angles: $b^2 = 2a^2 - 2a^2\cos x$.
– Now, I will try to find the diagonal c.
– In order to do this, I need to calculate y_2. Since $y_2 = y - y_1$, I need first to calculate the angle y_1. This is easy, since ABC is an isosceles triangle, so I can write: $y_1 = (180° - x)/2 = 90° - x/2$ so it follows that $y_2 = y + x/2 - 90°$. Now using the cosine rule again on the triangle ACD, I get the following result: $c^2 = b^2 + a^2 - 2ab\cos(y_2) = b^2 + a^2 - 2ab\sin(y + \ x/2)$.
– If I rearrange this, I will obtain an expression of c in terms of a, b, x and y. However, c has another expression as well. The one that follows if we apply the rule of cosines in the triangle AED: $c^2 = 2a^2 - 2a^2\cos(w)$. Therefore, I have: $2a^2 - 2a^2\cos(w) = b^2 + a^2 - 2ab\cos(y + x/2)$. Using $b^2 = 2a^2 - 2a^2\cos x$, simplifying by the common factor a^2 from all the terms, then using the trigonometric identities $\cos(2x) = 1 - 2\sin^2(x)$ and $\sin^2(x/2) = (1 - \cos(x))/2$ and finally, using the addition formulae for the cosines, I obtain the "beautiful" relation: $\cos(w) + 1/2 = \cos(x) + \cos(y) - \cos(x + y)$.
– Good, but I still haven't calculated u and z, how can I do this? It seems to me it would be difficult to calculate them directly, because these angles are divided into by the diagonals b and c (see Fig. 12.6). Is there a smarter way to do this?
– Yes, without loss of generality the same line of thought applies if instead of x and y, I start with y and z. It doesn't matter which pair of consecutive angles within the pentagon I start with. In a sense, I will apply cyclic permutation to my "beautiful" relation, and I will get to the following five relations. These five

relations, together with the total sum of the angles $x + y + z + w + u = 540°$, turn the problem into the following system of equations:

$$
\begin{aligned}
\cos(w) + 1/2 &= \cos(x) + \cos(y) - \cos(x + y)\\
\cos(u) + 1/2 &= \cos(y) + \cos(z) - \cos(y + z)\\
\cos(x) + 1/2 &= \cos(z) + \cos(w) - \cos(z + w)\\
\cos(y) + 1/2 &= \cos(w) + \cos(u) - \cos(w + u)\\
\cos(z) + 1/2 &= \cos(u) + \cos(x) - \cos(u + x)\\
x + y + z + w + u &= 540°.
\end{aligned}
$$

- Now things are getting harder. How can I solve these equations only for integer values? Can I exclude any pairs (x, y) by further exploring these equations?
- I think of using complex numbers to express cosines, so that the operations are easier to handle, and to get simpler expressions for z, w and u. For example, I will denote by $x' = e^{ix} - 1$, so $x = \mathrm{Re}(x' + 1)$. This gives me the "beautiful" relation $x' \cdot y' = w' - 1/2$. This idea didn't help at all in the end, because it leads to wrong conclusions, due to the fact that the real part of the product of complex numbers is not equal to the product of the real parts of the two complex numbers. So I can't write that $w = \mathrm{Re}(w' + 1) = \mathrm{Re}(x'y' + 1/2)$.
- What else? Perhaps I could expand the sine as a Taylor series, yet I don't see how this could help.
- I'll use the computer program I wrote to find solutions for now, and we'll see.

A final comment on the program I wrote. I used Python, and asked the program to find all convex and non-convex solutions. For the non-convex solutions, it suffices to notice that a non-convex solution has a unique corresponding convex. In every convex pentagon, I tested each angle, (trying to find its symmetric vertex with respect to the corresponding diagonal), to get a possible non-convex solution. Every such possible solution of course needs to be verified to see if it is acceptable. I have attached my solutions."

The third solver later sent us an additional clarifying note on the role that his program played on the overall solution of the problem.

"The program I wrote on Python uses the formulae I discovered. I have been careful to check if the choice of two consecutive angles in degrees results in an acceptable solution. In other words, every ordered pair (x, y) varying from 1° to 179° produces the rest of the angles, starting from the angle w and checking the constraints. The next step is the calculation of the other two angles z and u, and it checks if z, u and w are integers, and if they correspond to a convex pentagon. If these are true, we obtain a string of five numbers (x, y, z, w, u) that corresponds to a convex pentagon. A final verification is whether the sum of these five numbers is 540°, and it examines cyclic permutations of these angles, clockwise and counter-clockwise, in order to discard congruent pentagons. Finally, it asks of each convex pentagon to find its corresponding non-convex one. Also, we can ask the program to check if a corresponding solution exists for each specific choice of angles x and y".

As we can see from our analysis above, the third subject gave a very detailed description of his attempts to find the number of the equilateral pentagons with integer angles in degrees. This is indicative of the degree of his involvement with the problem, his mature exercise of metacognitive skills, and also of the fact that he was willing to unfold the path of his reasoning for the sake of the research.

The list of the solutions used is set out below in order to enable the reader to follow the number and the type of solutions (Table 12.1).

12.5 Analysis of the Students' Efforts Vis-à-Vis Our Research Questions

First question: *Which factors determine the various approaches of young mathematicians when they attempt challenging problems? What matters mostly for the resolution of characterization problems?*

If we analyse the "history" of the solutions of a problem, as proposed by Polya (1981), we can be led to interesting deductions about the sequence of the essential steps the gifted students follow and the strategies they use in order to find a solution to a problem.

The effect of the solver's formally acquired mathematical knowledge cannot be easily assessed. The imprint of that knowledge can be observed, the similarities and the differences between different trajectories of the background knowledge of solvers can be compared, yet the 'engineering' of how this knowledge is evoked in the process of solving is difficult to gauge. First, there are certain important factors towards building up the background knowledge: for example there are factors such as conceiving and comprehending important mathematical concepts, theoretical constructs and more generally theories from different mathematical fields. For problem-solving purposes it is important to succeed in bringing to the surface this knowledge, which otherwise remains 'inert'. Besides, on top of the acquired knowledge, another important aspect for working effectively in problem solving is exercising the managerial aspects, i.e. the executive control. Last but not least, one has to realize the structure of the task environment at hand (Mamona-Downs and Downs 2005). On a more practical level, preparing for and participating in mathematical competitions does boost strengthening of solving abilities. However, it is not easy to estimate the relative merit of any of these factors in every individual solver, and how each of these factors reinforces the others.

Krutetskii (1976) proposed a number of mathematical casts of mind for the mathematically gifted that influence the solving approaches. We cannot determine accurately which of these casts the problem solvers in this study belong to. They

Table 12.1 The list of solutions

Number	Angle	Angle	Angle	Angle	Angle
Solution	A	B	C	D	E
1	300	1	119	61	59
2	300	2	118	62	58
3	300	3	117	63	57
4	300	4	116	64	56
5	300	5	115	65	55
6	300	6	114	66	54
7	300	7	113	67	53
8	300	8	112	68	52
9	300	9	111	69	51
10	300	10	110	70	50
11	300	11	109	71	49
12	300	12	108	72	48
13	300	13	107	73	47
14	300	14	106	74	46
15	300	15	105	75	45
16	300	16	104	76	44
17	300	17	103	77	43
18	300	18	102	78	42
19	300	19	101	79	41
20	300	20	100	80	40
21	300	21	99	81	39
22	300	22	98	82	38
23	300	23	97	83	37
24	300	24	96	84	36
25	300	25	95	85	35
26	300	26	94	86	34
27	300	27	93	87	33
28	300	28	92	88	32
29	300	29	91	89	31
30	300	30	90	90	30
31	252	36	108	108	36
32	239	1	179	61	60
33	238	2	178	62	60
34	237	3	177	63	60
35	236	4	176	64	60
36	235	5	175	65	60
37	234	6	174	66	60
38	233	7	173	67	60
39	232	8	172	68	60
40	231	9	171	69	60

(continued)

Table 12.1 (continued)

Number Solution	Angle A	Angle B	Angle C	Angle D	Angle E
41	230	10	170	70	60
42	229	11	169	71	60
43	228	12	168	72	60
44	227	13	167	73	60
45	226	14	166	74	60
46	225	15	165	75	60
47	224	16	164	76	60
48	223	17	163	77	60
49	222	18	162	78	60
50	221	19	161	79	60
51	220	20	160	80	60
52	219	21	159	81	60
53	218	22	158	82	60
54	217	23	157	83	60
55	216	24	156	84	60
56	215	25	155	85	60
57	214	26	154	86	60
58	213	27	153	87	60
59	212	28	152	88	60
60	211	29	151	89	60
61	210	30	150	90	60
62	209	31	149	91	60
63	208	32	148	92	60
64	207	33	147	93	60
65	206	34	146	94	60
66	205	35	145	95	60
67	204	36	144	96	60
68	203	37	143	97	60
69	202	38	142	98	60
70	201	39	141	99	60
71	200	40	140	100	60
72	199	41	139	101	60
73	198	42	138	102	60
74	197	43	137	103	60
75	196	44	136	104	60
76	195	45	135	105	60
77	194	46	134	106	60
78	193	47	133	107	60
79	192	48	132	108	60
80	191	49	131	109	60

(continued)

Table 12.1 (continued)

Number Solution	Angle A	Angle B	Angle C	Angle D	Angle E
81	190	50	130	110	60
82	189	51	129	111	60
83	188	52	128	112	60
84	187	53	127	113	60
85	186	54	126	114	60
86	185	55	125	115	60
87	184	56	124	116	60
88	183	57	123	117	60
89	182	58	122	118	60
90	181	59	121	119	60
91	179	60	121	119	61
92	178	60	122	118	62
93	177	60	123	117	63
94	176	60	124	116	64
95	175	60	125	115	65
96	174	60	126	114	66
97	173	60	127	113	67
98	172	60	128	112	68
99	171	60	129	111	69
100	170	60	130	110	70
101	169	60	131	109	71
102	168	60	132	108	72
103	167	60	133	107	73
104	166	60	134	106	74
105	165	60	135	105	75
106	164	60	136	104	76
107	163	60	137	103	77
108	162	60	138	102	78
109	161	60	139	101	79
110	160	60	140	100	80
111	159	60	141	99	81
112	158	60	142	98	82
113	157	60	143	97	83
114	156	60	144	96	84
115	155	60	145	95	85
116	154	60	146	94	86
117	153	60	147	93	87
118	152	60	148	92	88
119	151	60	149	91	89
120	150	60	150	90	90
121	108	108	108	108	108

appeared to think in lectical-logical and optical-figurative terms; however, only on the bases of the results of the present study, we refrain from explicitly identifying them as being of the harmonic, the analytic or the geometric type.

The students also posed the more generalized case of the equilateral polygon. We think that they did it, firstly, out of pure curiosity about the general case and secondly, because they tacitly hoped that the solution of the generalized polygon would lead to the solution of the specific pentagon problem in a more simplified way, in the spirit of Tao (2006). For characterization problems, in particular, the demarcation of the general case reveals the core of the structure of the mathematical objects in question. This, in turn, facilitates the building up of the list of exemplars for the object, i.e. when we are dealing with classification issues. On the topic of generalization, Krutetskii (1976, p. 335) wrote: “A generalized solution of problems (a tendency to solve each specific problem in a general form) is typical of capable adolescents generally”.

Overall, all solvers displayed a progressive heuristic behaviour. Even though they gave the impression that they abandoned a solving path that did not advance the process towards the solution, this path emerged in many instances in a subsequent approach. (Koichu 2010).

Second question: *Did the solvers comply with communicating and exemplifying their solution paths with the researchers in detail?*

The experiment was conducted in a relaxed atmosphere. We were well acquainted with the participants, who understood the purpose of this research well, and showed a great interest in the problem. All these factors played an essential role to the execution of the experiment in a proper way.

Our experience in this research project was that the personal relationship between the researcher and the talented students plays a very important role in enabling them to articulate their thought processes. Also, it must be clear from the beginning what is expected of the student. If a student has to solve a problem in a very limited time frame, then obviously, he/she won’t be willing to spend considerable time in explaining the steps he has taken in order to solve the problem. However, if the student understands that it is important for the sake of the research requirements to explain each and every step taken, then the student complies with the research protocol.

Krutetskii (1976) observed a qualitative difference between the approach of “solving a problem for myself” and of “solving a problem for others”. This is a key issue for the interpretation of our subjects’ behaviour. In addition, we conclude that they were focused on what the problem asked for, ascertaining that they were fully conscious of the characterization aspect of the problem; in this way they did not feel the need to pose questions on possible other properties of these kind of pentagons.

Third question: *Could we identify the ‘turning points’ in their solution path as Eureka” or “Aha!” moments?*

We were interested in seeing whether our subjects were aware of the ‘turning points’ in their solution process, and to what extent they expressed these vividly at the time, or whether they comprehended these as such after revising their solution. In fact, we did not detect instances where the students expressed explicitly any real excitement while working on the problem, i.e. a kind of “Eureka /Aha!” moments. However, from their expressions and gestures it was evident that a crucial idea had occurred to them during the progress of the solution, such as for the first solver the realization of the significance of constructing a ‘hat’, or for the second and the third solver the fact that the sizes of the first two consecutive angles determine the location of the vertices of the next two angles. What seems to be true, and we say this from our own experience, is that usually, after solving the problem completely, the solver is in a position to point at these moments. This indicates the exercise of metacognitive acts in problem solving and therefore requires a degree of mathematical maturity.

Fourth question: *What are the relative merits that our subjects ascribed to an analytical and an experimental solution of the same problem in a computer environment?*

There are many difficulties regarding to the analytical approach to this problem, and there are some conjectures that have to be tested; for example, are there solutions of the type (a, b, c, b, a) or of the type (a, b, a, c, c) in which there is symmetry (beyond the regular pentagon). In fact there are only three solutions of this type i.e. the (108°, 36°, 252°, 36°, 108°), (300°, 30°, 90°, 90°, 30°), and (60°, 150°, 90°, 90°, 150°).

Testing a conjecture with computer programs offers the advantage of visual apprehension and of a very rapid experimentation, which helps the location of counterexamples. As Butler et al (2010, p. 425) put it: “With new computer based tools, geometrical thinking can become again a central source of insights when exploring new domains of knowledge and modeling”. We conclude that this particular problem highlights the solvers’ capabilities both in analytical and experimental solving, by using software. For example, when experimenting with software the students found out that a small change to the measure of an angle makes a big difference.

We would like to note that, along with the list of the solutions they obtained from their software programs, our subjects desired to actually see the geometric shapes corresponding to these solutions. This way, they confirmed by themselves the correctness of the solutions they obtained.

It was clear from the analysis of their written solution documents and their answers in the consequent interviews, that the solvers had developed the ability of frequent and successful transitions between visual apprehension and analytical thinking. In addition, the problem solvers had increased mathematical fidelity during problem solving with the use of software, according to the terms and the conditions described by Dick (2007).

As far as the question of this research is concerned, the solvers agreed that the use of a software program helped them in the monitoring of the environment of the problem, in creating simulations, and in checking the results. More importantly, software use provided a substantial verification of their acts and reasoning. Overall, whether the use of software is ‘legitimate’ in solving mathematical problems constitutes a topic of concern not only for the researchers in Mathematics, and in Mathematics education, but also for the teachers of Mathematics.

12.6 Open Questions and Conjectures Regarding the List of Solutions

Solving this problem traditionally with pen and paper, without the use of software, is not easy at all. At first, we were especially interested to see which part of their mathematical knowledge the subjects accessed. They brought into play different areas of Mathematics that they had at their disposal, such as theorems of Combinatorial Enumeration (Burnside Lemma), Complex Numbers, Elementary Vector Calculus, Analytic Geometry, theorems of Trigonometry, and basic geometric tactics. The efforts that depended solely on analytical treatments, involving their mathematical background from the above-mentioned areas, did not prove effective in providing the solutions.

We must underline that all three subjects implemented executive control in the process of solving, as we observed during their individual accounts, spoken or written. They posed very pointed questions, for example whether axial symmetry possibly played a role in the solution of the problem. This conjecture was brought up by their observation of cases like the ones shown in Figs. 12.7 and 12.8, which seem to be possible solutions of the problem, albeit that they were proved wrong.

In Fig. 12.7 ABC is an isosceles triangle and BDEC is an isosceles trapezoid with AB = BD = DE = EC = CA. However, these figures soon proved to be pseudo-solutions. Actually, the only figures with an axial symmetry that constitute solutions are shown in Fig. 12.9 (108°, 108°, 108°, 108°, 108°), in Fig. 12.10 (60°, 150°, 90°, 90°, 150°), in Fig. 12.11 (30°, 90°, 90°, 30°, 300°), and in Fig. 12.12 (252°, 36°, 108°, 108°, 36°)

Of a similar nature was their following question: "By looking at the complete list of the solutions to the problem, that is, at all the tuples (a, b, c, d, e), how could we detect solutions that have axial symmetry?" Moreover, "do tuples of the form (a, b, c, c, b) indicate axial symmetry?"

All problem solvers applied the method of triangulating the pentagon, resulting in two isosceles triangles ABC and AED, and a third triangle ADC, whose two sides were equal to the bases of the isosceles triangles, as shown in Fig. 12.13.

Applying the method of triangulation led them to the following concern: "How difficult it is to create and solve a system of parametric equations using the cosine rule, in which the angles of the pentagon have integer size angles in degrees" The

Fig. 12.7 .

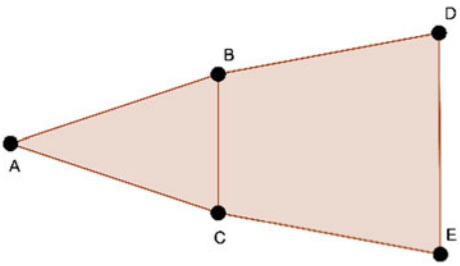


Fig. 12.8 .

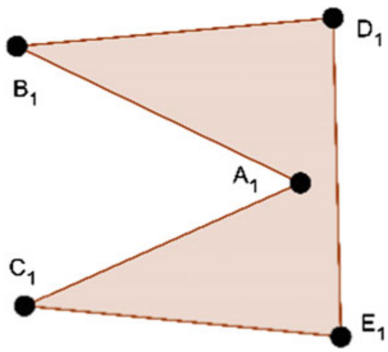
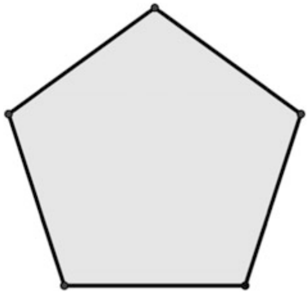


Fig. 12.9 .



first observation, which gives a partial answer to this question, is that the angles of the triangles could have non-integer measures, whereas the angles of the corresponding pentagon could have integer measures. This observation definitely complicates this kind of approach. In addition, in practice the subjects found that such a system of equations was almost impossible to solve only "manually".

It is apparent from Figs. 12.9, 12.10, 12.11 and 12.12 that both a convex and the corresponding non-convex pentagon constitute solutions to the problem. The following question emerged: "Does every convex solution correspond to a non-convex solution?". They pinpointed, as an example, the two non-convex solutions of the problem shown in Fig. 12.14 (300°, 15°, 105°, 75°, 45°) and Fig. 12.15 (225°, 15°, 165°, 75°, 60°). They also observed that some vertex of the non-convex angle of the pentagon has a symmetric vertex with respect to a corresponding diagonal as axis of symmetry. The corresponding convex pentagons of Figs. 12.14 and 12.15 also constitute solutions to the problem, namely (60°, 135°, 105°, 75°, 165°) and (135°, 60°, 165°, 75°, 105°).

Another concern was put forward: "*Each pentagon has five different diagonals. Does each convex solution correspond to five non-convex solutions?*" The students did not elaborate on this issue.

The first solver also suggested that "*it might be useful to work on specific cases of pentagons made by compounding geometrical figures having sides of equal length, which could convince us that there exist "enough" solutions to the problem,*

Fig. 12.10 .

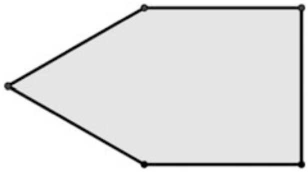


Fig. 12.11 .

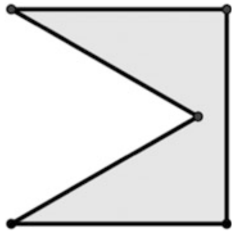


Fig. 12.12 .

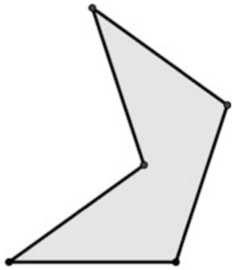


Fig. 12.13 .

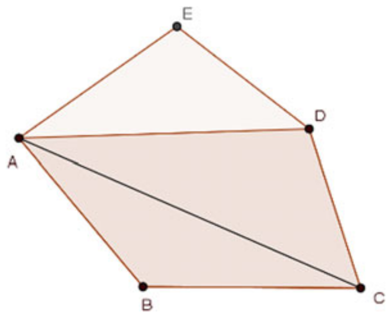


Fig. 12.14 .

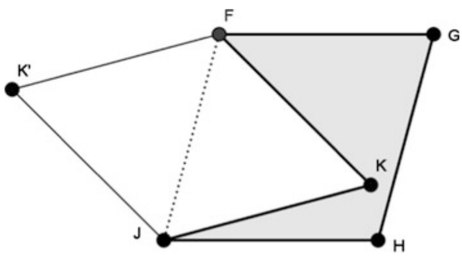


Fig. 12.15 .

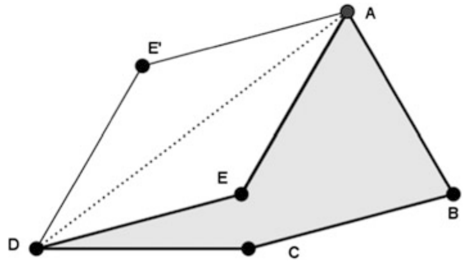


Fig. 12.16 .

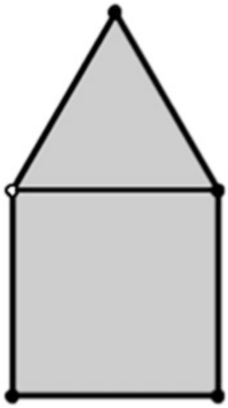


Fig. 12.17 .

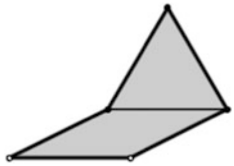


Fig. 12.18 .

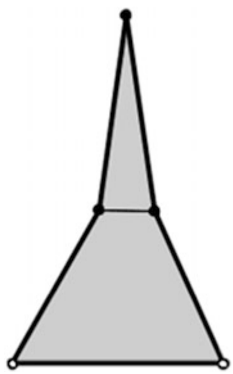


Fig. 12.19 .

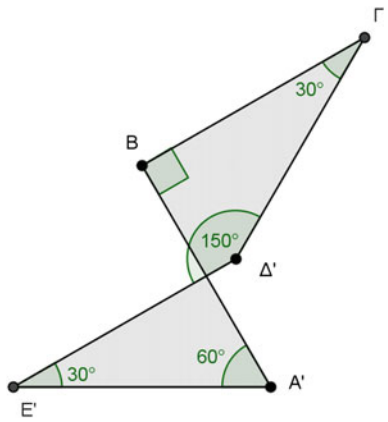
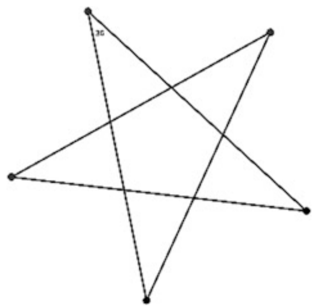


Fig. 12.20 .



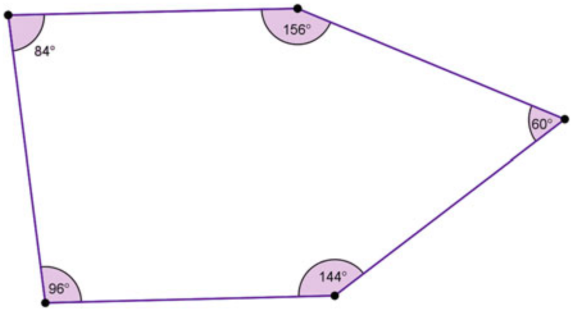
and from these solutions some ideas might arise, which can help us confront the problem in a more global way". For this reason he constructed the group of pentagons shown in Figs. 12.16, 12.17 and 12.18.

Figure 12.15 corresponds to a compound shape, that of an equilateral triangle on top of a square, and has equal sides and angles (60°, 150°, 90°, 90°, 150°). It is a special case of the pentagon shown in Fig. 12.17. Figure 12.18 shows a pentagon formed by putting together an isosceles triangle to an isosceles trapezoid and it does not belong to the list of solutions of our problem, no matter how plausible it may seem.

It is interesting that the solvers were not concerned with self-intersecting pentagons such as these shown in Figs. 12.19 and 12.20 [Fig. 12.20 shows the so-called pentalpha (36°, 36°, 36°, 36°, 36°)]. The self-intersecting polygons are not part of the school curriculum of Mathematics in Greece. It is also noteworthy that while using software programs, this kind of polygons appeared, yet the students rejected these solutions, as if they did not accept them as 'legitimate' ones.

After all these meticulous attempts to solve this problem an unavoidable question comes up: "*Does the list of solutions contain any hidden information?*" The answer to this question is positive and nothing short of stunning and surprising. **The**

Fig. 12.21 .



solutions to the problem are the regular pentagon and 120 other pentagons, that correspond to compound shapes, namely, to rhombuses with an equilateral triangle attached to one of their sides! Only two of our problem solvers observed the appearance of the angles of 60° or 300° in every solution, but none of them expressed it as an undoubted outcome.

On the whole, any mathematician, who resorts to a computer program for finding the solutions to a problem similar in character to the one we encounter in this paper wonders whether the list of solutions given by the program is indeed correct and complete. Actually our problem solvers were concerned by this same question “are the solutions in the list correct, do they correspond to actual solutions?” as it was evident from the interviews with them. And for that reason, they made sure that they checked all 121 solutions either “by hand”, i.e. printing the results of the program and rejecting the self-intersecting pentagons, or by inserting a subroutine in the program in order to do this automatically. An example of accepted pentagon with angles (156°, 60°, 144°, 96°, 84°) is shown in Fig. 12.21.

It is apparent that the whole procedure is time-consuming and beyond our “agreement” with the participants about the time-span of the fieldwork. Every solver’s goal was to tackle the problem within certain time limits. However, it was proved that for this kind of problems recourse to software aid is unavoidable and consequently brings up the question to what extent we can rely on its outcome. This general issue we could say is to some degree a philosophical one, and depends very much on the general culture of Mathematics of our era.

In our time, the use of software programs has “infiltrated” the process of problem-solving and their use in mathematics classrooms at school should be seriously considered, as Santos-Trigo and Moreno-Armella (2016) suggests.

12.7 Final Remarks

The analysis of the way in which these achieved solvers approached the problem, proved very rewarding. Naturally, our starting point was the assumption that we can study the abilities of gifted students in Mathematics using "unusual" and challenging mathematical problems.

How can we be certain that the information we gathered on the solution paths followed by the three gifted students, represent the maximum of their strength as solvers? We tried to achieve this by meticulously recording their writings "in vivo", and by additional questioning following a successful or an unsuccessful try. We believe that this study provides a faithful representation of their work. Furthermore, their understanding of the purpose of the research assures us that we 'extracted the most' they could do at the time of the fieldwork.

The transition from one form of approach to a different one is a characteristic of the mathematically gifted, which is a fact well known in bibliography, "they are flexible in thought, and can move easily from one line of thought to another, even if these are completely different" (Assouline and Lupkovich 2011, pp. 162–163).

Another very important issue that concerns our research is to what extent gifted students conceptualize the subtle differences and interconnections between problem-solving and proving (Mamona-Downs and Downs 2013). This touches both the analytical and the experimental approaches. Our subjects explicitly said that they did not feel that they 'proved' the existence of all possible solutions. One of the reasons for this is that they were dependent on the program they wrote. The list of solutions proves that certain anticipated solutions exist, while other anticipated solutions do not! The program also printed shapes that were unexpected.

Our final comment relates to the issue whether the problem we used in this study is 'appropriate' to evaluate the abilities of the gifted students, given that (to the best of our knowledge) it requires the use of software for its solution. The answer to this question is affirmative, taking under consideration the fact that the students themselves developed the program that gave the list of the solutions. Moreover, the use of such type of problems has the following advantage: it poses the dilemma of whether a solution or a list of solutions achieved in this way is complete. Therefore, observing the solvers' efforts and recording their conjectures while trying to remove all their uncertainty on this, led to a number of useful accounts of their solving behaviour; accounts which not only reveal the solving strategies of these gifted students, but enrich our understanding of their attitudes towards 'doing Mathematics' in general.

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Chapter 13
Repeated Participation at the Mathematical Olympiads: A Comparative Study of the Solutions of Selected Problems

Ingrida Veilande, Liga Ramana and Sandra Krauze

Abstract The paper analyzes the works of students who have participated in at least three *Open Mathematical Olympiads of Latvia* (LOMO) in the 6th, 8th and 9th grade. The question of the research discussed in the paper is: *What knowledge and problem-solving skills do students demonstrate in the solutions of algebra and number theory problems?* Six algebra and number theory problems were selected for the research, as solving them requires sufficiently high levels of abstract thinking, algebraic reasoning, and an accurate use of the mathematical language. The authors developed a special coding system for students' works and elaborated an assessment tool for assessing each individual student's levels of problem-solving competence. The implementation of this tool enables comparing the student's problem-solving success in different grades, and it enables comparing the specific properties of the solutions presented by the group of students as well. While showing that a few students were good problem solvers, the data collected revealed deficiencies of algebra knowledge in a significant part of students' works.

Keywords Algebraic reasoning · Assessment · Mathematical creativity Mathematical Olympiads · Problem solving

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13.1 Introduction

13.1.1 Open Mathematical Olympiad

Students' mathematical competitions have become popular globally. They are of different types and forms: there are both in-person and remote competitions; the problems must be solved either in the classical way—by writing down a full solution for each task, or as multiple-choice tasks. The competitions also differ in their content: some Olympiad problems are entertaining by their nature, while some other mathematical competitions motivate school students for in-depth studies of mathematics and serious preparatory work (Singer et al. 2016). The competitions are organized for students of different age and are based on different principles of participant selection. They can be characterized as the inclusive-type, appropriate for students with average knowledge, and the exclusive-type, where only the most capable students are invited. For instance, the most prestigious competition—the International Mathematical Olympiad, is attended by students with outstanding skills from many countries (Kenderov 2009). The role of the Olympiads is highly valued, as they enrich the mathematical curricula of schools, promote students' interest and activity, help identify the gifted students and reinforce their education (Thrasher 2008).

In 1974, as an alternative to the State Mathematical Olympiad, the *Latvian Open Mathematical Olympiad* (LOMO) was started as an "inclusive" event, giving the opportunity to every Latvian school student to test their mathematical skills, regardless of their school grades or of the teachers' opinion on the preparedness of the given student. The problem sets for each grade include five tasks from algebra, geometry, combinatorics, and number theory, which every contestant has to solve within 5 hours. Both parts of a student's written work—the fair copy and the draft —must be submitted to the jury.

While LOMO is the most popular students' Olympiad, which in the past years has gathered a record number of participants—over 3000, the statistics shows a trend that the scores of the younger contestants are higher than those of senior grade students. One of the reasons is that the tasks given to the younger grades are more entertaining and can be solved with calculations, logical reasoning, and simple combinatorial methods. Apart from the skills defined in the school curriculum, solving high-school level problems requires a greater problem-solving experience. This could be one of the reasons for the lower number of participants in senior grades.

13.1.2 Assessment of Olympic Works

The assessment of the Olympiad works is not an easy task. There are different systems of assessment, such as the one used at the Moscow Mathematical Olympiad, consisting of plus-minus signs specifying the correctness of the solution (Galperin and Tolpigo 1986). Szetela and Nicol (1992) focused on students' use of different strategies in problem solutions and proposed to categorize the responses in the solutions. The experts of the *Trends in International Mathematics and Science Study* (TIMSS) emphasize the importance of cognitive skills—knowing, applying the knowledge, and reasoning—in assessing students' achievements and describe the main components of these skills (Mullis et al. 2009).

In Latvian Open Mathematical Olympiads, the works are assessed using a 10-point grading scale and the solution of each problem is assessed by experts, who develop individual criteria relevant to the problem, based on the general guidelines developed by the jury. Evaluators examine solutions of the problems presented in the fair copy; and they occasionally check the notes in the draft, when some doubt arises. The number of works in each class constitutes several hundred, while the time devoted their evaluation is rather limited in order to ensure a timely announcement of results. To help ensure the quality of assessment, it would be useful to provide recommendations obtained from a more profound study of the solutions submitted by students.

13.2 Possible Conclusions from Students' Works

Most Latvian mathematics textbooks include a rather low proportion of proof problems. For instance, when reviewing a grade 8 mathematics textbook (France et al. 2010), which corresponds to the newest curriculum, only 8.5% of the problems included in the algebra section require justifying statements or making proofs, while 15% of the problems are of an analytical nature aiming to assess, compare, or put forward a hypothesis. The teachers have to provide the classes with additional materials so that students can acquire relevant problem-solving skills. Such preparation for classes requires additional work of the teachers. Sweller et al. (2010) discuss the difficulties of teaching and learning problem-solving skills. They note that problem solvers have to be taught by emphasizing working examples of problem solution strategies that are domain-specific. However, Singaporean researchers recognized problem-solving as one of the fundamental goals of teaching mathematics and they elaborated a special program as a support material for teachers (Dindyal et al. 2012).

The Olympic problem set differs significantly from the tasks of school mathematics. Therefore, the students who have succeeded at the Olympiad demonstrate mathematical abilities that are considered as significant components of giftedness by several researchers (Sriraman 2008; Kontoyianni et al. 2013). For example, the ability to abstract, generalize and discern mathematical structures, the ability to master the principles of logical thinking, the flexibility of mathematical operations, and an intuitive awareness of mathematical proof are considered by Sriraman (2008, p. 94) as components of mathematical giftedness. The best LOMO students' works are a manifestation of the above-mentioned skills and original solutions.

However, a large proportion of Olympiad participants do not achieve sufficiently good results. In 2012, when correcting the works of 6th grade students, we noticed that the drafts of students' Olympic works provided useful and interesting ideas that had been left unimplemented in problem solutions presented in the fair copies. The question arises what is considered by students to be a complete problem solution? We had the opportunity to review the works of the same students as they repeatedly participated in the Olympiad in grades 8 and 9. Taking into consideration that applications of algebra and number theory results are the basis of the solutions of many Olympiad problems, we analyzed the students' works in detail and sought answers to the following question:

– What knowledge and problem-solving skills do the students demonstrate in the solutions of algebra and number theory problems?

13.3 Components of Problem-Solving Process

Problem solving is the essence of mathematics. Different models are elaborated by researchers to interpret the sequencing stages of the problem-solving process. These studies were originally addressed to teachers in order to enhance students' problem-solving skills. One of the most popular studies is Polya's four-step method (1945) that incorporates the principles: *Understand the problem*, *Devise a plan*, *Carry out the plan*, *Look back*. Through the years, researchers have analyzed the process of problem solving and developed various specialized study frameworks, according to the postulated objectives of the studies. An important aspect is the evaluation of students' knowledge, skills, abilities and creativity (Organization for Economic Cooperation and Development 2013; Szetela and Nicol 1992; Rott 2013). With the intention of describing the cooperation of teachers and students when learning inquiry-based problem-solving methods Singer and Moskovici (2008) created the IMSTRA model (*Immersion, Structuring, Applying*). Recognizing the significance of problem posing in the teaching-learning process, as a result of empirical studies, Singer and Voica (2012) have developed a conceptual problem-solving model describing the phases which are sequentially performed by the solver. They defined four operational categories—*decoding, representing, processing*, and *implementing*, where each of the categories contains a set of complex operations. Their introspective approach allowed the authors to emphasize the importance of cognitive processes so that the problem solver would create a mental model according to the given problem (*representing* phase), which they would then transform into a mathematical model (*processing* phase), sequentially using this model in solving the problem (*implementing* phase).

In our study we were using only the written Olympiad works submitted by students, where the problem-solving process was implicitly reflected in the draft copy. When applying Polya's problem-solving principles (1945) to students' drafts, it is possible to determine whether the student has understood the problem, while completed operations do not indicate whether the student has had a problem-solving plan, as oftentimes the drafts are rather chaotic.

As a result of having studied students' problem solutions, we consider the problem-solving process includes the following sequential stages:

- Conceptual understanding of the given task
- Implementation of the basic knowledge for solving the task
- Experimentation and investigation
- Formulation of regularities
- Estimation of discerned regularities
- Hypothesizing
- Justification of proposed statements.

Starting with exploration of the problem solution in draft copy and proceeding with evaluation of the problem solution in fair copy, we can observe what problem-solving stages the student has completed, what their quality is, and what are the student's problem-solving skills as a whole. Problem-solving skills include the knowledge of mathematical facts, the precision of doing mathematical operations, the mastery of reasoning methods and problem-solving strategies or heuristics, hypothesizing and justification, as well as a correct use of the mathematical language. Thus, by empirically analyzing different models expressing the relationship between mathematical abilities and mathematical creativity, Kattou et al. (2013) discovered that the components of creativity—fluency, flexibility, and originality—may be considered as sub-components of mathematical abilities.

The solution significantly depends on the student's beliefs of what a complete problem solution should be. Szetela and Nicol (1992) emphasize the importance of understanding a student's thought process in order to assess the solution; however:

> Students are prone to make calculations without explanations and calculations alone often fail to sufficiently reveal the nature of the solver's work. (p. 42)

Given that grade 6 problems are not overly complicated, along with the above-mentioned observation (Szetela and Nicol 1992), a part of the LOMO grade 6 participants decided that instead of a complete proof it is enough to give the answer "yes" or "no", or "is/not possible". All in all, the solution paths of junior-grade problems were rather uniform compared to the various problem solving strategies chosen by grade 9 students.

We chose a unified method for classifying problem solutions—we divided the solutions of each problem into general stages naming them solution steps. We classified these stages to determine what level of problem-solving competence the student has achieved. We based this classification on the original Bloom's Taxonomy, which comprises six major categories in the cognitive domain: *Knowledge*, *Comprehension, Application, Analysis, Synthesis,* and *Evaluation* [the original Bloom's Taxonomy is described in Anderson et al. (2001)]. Over time this model was transformed by proposing that the category *Knowledge* be regarded as a separate taxonomic dimension to avoid its dual nature, and by changing the sequence and naming of some categories (Anderson et al. 2001). In our opinion, the hierarchical model of the original Bloom's Taxonomy (from the simple to the complex,

Table 13.1 Description of levels of competencies used in the research on students works

Principles by Bloom's taxonomy	Explanation	Levels of competencies
Conceptual understanding pre-processing of data	Demonstration of knowledge and comprehension Application of facts, concepts and procedures to the given data	*Level 1* (L1)
Analysis	Investigation of the properties of given objects and operational processing implementing different methods	*Level 2* (L2)
Synthesis	Posing of the hypothesis, construction of algebraic formulas, generalization	*Level 3* (L3)
Evaluation	Reasoning, estimation, explanation, justification	*Level 4* (L4)

from the concrete to the abstract) is well-suited for the systematization of problem solution steps seen in students' solutions. Table 13.1 gives five general problem-solving phases which correspond to particular categories of Bloom's Taxonomy. Each of these phases fits a certain level of the student's problem-solving competence (further referred to as *level of competence*). An example of the classification and coding of a problem solution steps is presented in Appendix 3.

For describing studies of the problem-solving process, Rott reviewed a broad literature, from both psychological and pedagogical perspectives (Rott 2013). The Schoenfeld's structured stepping-stones model, which Rott refers to and which describes the solution as a path through problem space, corresponds to the framework of our study. Students' Olympic works are like a mosaic composed of various operations or activities. The type and complexity of these activities can be very different for various age groups. Thus, for instance, junior grade problems can be solved by applying rather simple heuristic strategies, such as the trial and error method or enumeration method, whereas problems for senior graders cannot be solved by such methods (see, for example, the solution of Problem 9.3 in Appendix 1). Therefore, the classification of solutions by defining the problem solution steps and the corresponding level of competence, in our opinion, is the most appropriate choice to compare students' achievements in solving problems of various degrees of complexity.

13.4 Method

13.4.1 Problem Set

The principles of assembling an Open Mathematical Olympiad problem set are traditionally based on the increasing role of discrete branches of mathematics versus continuous branches in the school mathematics curriculum (Andzans et al. 2006).

A problem set includes combinatorial, algorithmic and number theory problems (altogether constituting 50%). Especially in junior grades, considering the fact that students do not have sufficient knowledge of algebra and elementary geometry yet, the proportion of combinatorial problems is higher.

The problem topics correspond to the curriculum. Every problem can be solved by applying the knowledge of school mathematics in addition to logical thinking. Talented students with sharp logical thinking theoretically would not need to prepare for the Olympiad. However, training in problem solving significantly increases the possibility of qualitatively solving a yet-unfamiliar problem. In addition to that, the school curriculum practically does not require to justify one's statements, while for Olympic problem solutions it is the argumentation that largely counts.

For the purpose of this research project we selected problems of algebra and number theory. Such problems are traditionally included in the problem sets of Mathematical Olympiads (see, for example, the collections of Olympiad problems Djukić et al. 2011; Andreescu and Enescu 2011). Solving these problems requires quite high levels of abstract thinking and algebraic reasoning. These abilities, as well as an accurate use of the mathematical language, play a significant role in the presentation of a problem solution, as mentioned by Windsor (2010). A student's ability to describe numerical quantities and to transform their interconnection algebraically is a characteristic trait of abstract thinking in mathematics. The problems selected are rather different; nevertheless, solving them requires somewhat similar technical and also argumentation skills. Upon reviewing the students' solutions, we researched the algebraic models they had created, as well as the students' knowledge of number theory, problem solving methods, and their technical skills.

We selected two problems from the problem sets of each grade:

Problem 6.1. (LOMO 39, 2012) The numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 are on the board. Erase any two of them (e.g. a and b) and write the number $a + b + 2$ instead. The process is repeated until only one number is left. Prove that irrespective of the sequence of operations the same number will be left at the end. Determine this number!

Problem 6.4. (LOMO 39, 2012) Is it possible to write (a) six; (b) seven different natural numbers around the circle so that the sum of any two adjacent numbers were a prime number and that all prime numbers produced through summation were different?

Problem 8.2. (LOMO 41, 2014) Find all natural numbers that do not exceed 1,000,000 and that are diminished 15 times when their first digit is erased!

Problem 8.5. (LOMO 41, 2014) The cells of a 3×3 size square have to be filled with such natural numbers the sums of which in all rows, columns, and in both diagonals, are equal. Number 24 is given in the middle of the upper row. Is it possible that the lower left corner contains the number (a) 7; (b) 17?

Problem 9.1. (LOMO 42) Find two numbers whose difference is 2015 and whose product is a minimum.

Problem 9.3. (LOMO 42, 2015). Prove that expression $x^5 - 5x^3 + 4x$ is divisible by 120 for any integer x.

The jury of LOMO always later publishes the experts' solutions for every problem offered at the Olympiad (see Appendix 1). It has given us the possibility to compare students' solutions with the experts' view. In most cases students started solving problems in a similar way as experts, but often lost the path failing to solve it.

According to the mathematics curriculum for grade 6, students have to learn operations with rational numbers and are introduced to solving simple linear equations. Therefore, grade 6 LOMO problems mainly are of a combinatorial nature. Problem 6.1 is a combinatory problem with some algebra elements. Algebraically presented givens implicitly point to the implementation of algebra to discover a general solution. The problem can be solved numerically, too. Number theory Problem 6.4 is of a combinatorial nature and comprises two parts. To solve one case of the problem the contestant has to present just one example, while to solve the other part it is necessary to prove the impossibility of the requested arrangement.

The students of grades 8 and 9 could have the knowledge about the algebraic notation, transformations of expressions, and solving of equations. To qualitatively solve number theory Problem 8.2 it is necessary to implement a decimal notation for numbers and to create and solve an algebraic equation. Problem 8.5 is a combinatory one where case (a) is not possible to solve without introducing symbolic variables and solving equations. Those students who tried to solve this problem just numerically failed to reach a result. Problem 9.1 is an algebra problem that has a short solution by the analytic investigation of the created function. A relevant part of the contestants considered this as a number theory problem and found the answer in whole numbers. Some students demonstrated original solutions with well-grounded reasoning. Problem 9.3 is a number theory problem, where the shortest solution can be reached by factoring of given expression. For this reason, appropriate algebraic skills are needed, demonstrated by only a few students. The application of modular arithmetic is another way of solving the problem; however this requires some algebra knowledge, as well, to shorten the enumeration of cases.

In solving the selected problems, along with arithmetic and algebraic calculations and transformation, an important role is played by the justification and proof of the results obtained. Students must demonstrate understanding of mathematical terminology, the characteristics of whole and prime numbers; they need to be able to generalize, to build algebraic formulas and equations, as well as to justify the results and prove the statements. In the solutions proposed by students, we examined their argumentation skills, calculation and algebraic skills, as well as the application of number theory results.

13.4.2 Selection of Participants

We determined that 87 students of grade 6, 8 and 9 had repeatedly participated at Open Mathematical Olympiads in 2012, 2014, and 2015. Because of their regular participation in the Olympiad, we called them students who are *interested in mathematics.* Among them there were several contestants who scored high in all mentioned Olympiads, while others had varying degrees of success. The best solutions of Olympic problems certainly demonstrate the traits of mathematically gifted students, such as the ability to create abstract models and use different problem-solving methods and heuristics, as well as the ability of reasoning and justifying.

In this study, we analyzed the students' problem-solving skills in the fields of algebra and number theory. Out of five problems in each grade we chose two relevant ones as described above. We divided all students into three groups based on the evaluation by the jury. Subgroup A included students with the top scores—between 13 and 20 points in total for both problems. Subgroup B included students with average scores—between 7 and 12 points in total for both problems. Subgroup C included students with scores below 7 points. Each student, depending on his or her achievements, was included in one of these subgroups for each grade. The combination of these subgroups formed 18 different groups. Table 13.2 shows the number of students per group.

The data summarized in Table 13.2 demonstrate that there were no students who would have solved grade 9 problems much better than similar-themed problems in previous years; that is, the table does not have groups CBA or BBA. The 41 students who had at least once achieved the rating of group A were selected as a focus group of our research. The works of the remaining students failed to provide a sufficient amount of useful information for this study.

13.4.3 Method of Coding Students' Solutions of Selected Problems

When systematizing students' works, we noted all the activities performed in the solution of each particular problem both in the draft and the fair copy—a study of

Table 13.2 Division of students by assessment group

Assessment groups	AAA	AAB	AAC	ABB	ABC	ACB	ACC
Number of students	4	5	5	5	8	2	5
Assessment groups	BAB	BAC	BBB	BBC	BCB	BCC	
Number of students	2	4	3	11	1	14	
Assessment groups	CAC	CBB	CBC	CCB	CCC		
Number of students	1	2	2	2	11		

Table 13.3 Coding system of mathematical skills applied in students' solutions

Technical skills	Code	Modelling skills	Code	Problem-solving skills	Code	Argumentation skills	Code
Knowledge reproduction	k	Creation of algebraic notation	a	Test of examples	e	Knowledge reproduction	k
Numerical calculations	n	Visualizing	v	Numerical estimations	s	Description of operations	d
Algebraic transformations	a	Creation of combinatory structures	c	Use of algebraic methods	a	Reasoning, estimation	r
Solving of linear equations	s			Use of number theory results	n	Statement without reasoning	s
				Heuristic methods	h	Justifying	j

examples, calculations, formulas, algebraic transformations, graphic drawings, tables, and texts. These activities were classified according to their relation to the respective *solution step* implicitly showing the decision "How to solve the given task" made by the student.

To better characterize each step, we related them to some special mathematical competencies that OECD (Organization for Economic Cooperation and Development 1999) expert panels have defined as general mathematical skills. In our research we examined how the solutions of each problem reveal each student's modelling, technical, problem solving, and argumentation skills (the details are presented in Table 13.3). Written activities within one step indicated mathematical skills applied by each student. By examining the presentation of a solution in the fair copy, we saw how contestants evaluated themselves—whether they wrote down the solution of the selected task, only presented the answer, or omitted it. We evaluated each step in students' solutions by the quality of their performance: high, average, or low. The elaboration and evaluation of separate solution steps by our experts' team is presented in Fig. 13.1. Students' decisions on solving the problem as well as their self-evaluation can only be determined from the context.

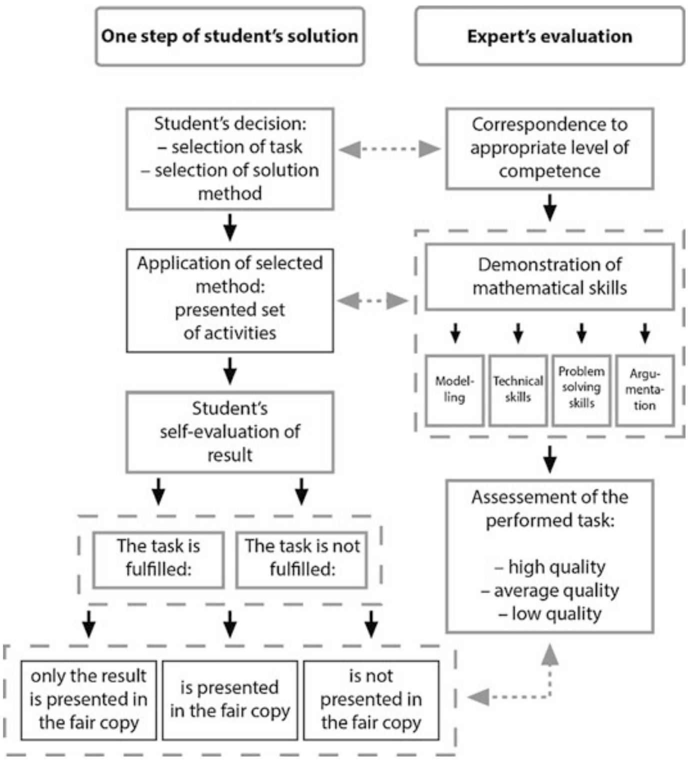

Fig. 13.1 Construction of solution step and method of description

Every step of students' solutions was measured by 3 parameters: the application of mathematical skills (see Table 13.3), an appropriate level of competencies (see Table 13.1), and the quality of performance. We determined and numbered all steps performed by students in their solutions for every separate problem. Every step was associated with a respective level of competencies and coded corresponding to applied mathematical skills.

One step can be coded with more than one code: for instance, the factorization of expression (see Problem 9.3 LOMO 42) was coded both as a way of a solving problems—the application of problem-solving skills and the application of technical algebra skills. The number of the step was added to the code. An example of the solution of problem 6.1 (see Appendix 2, Solution of Problem 6.1, presented by student Emils) and an example of coding of Emil's work is given in Appendix 3.

We created the *General Map of Competencies* (GMC) and the *Expanded Map of Competencies* (EMC) as tool for assessing the quality of solutions. The Expanded Map of Competencies is a table of codes related to respective mathematical skills and quality (as an example, see Table 13.6 in the Appendix 3). The General Map of Competencies is a sequence of 12 digits, where every three sequential digits characterize a respective level of competencies. All steps of level 1, level 2, level 3, and level 4 were counted and were distributed according to the quality of performance. We found that the decimal-basis coding system of elements, which is described by Leikin as a tool for evaluating the flexibility component of students' creativity (Leikin 2013), obviously presents the quality of a problem solution and the contestant's competence. Taking into account that the number of steps in every solution is fewer than 10 on each level of competencies, we added the number of steps at each separate level of competencies, multiplying the number of high quality steps by 100 and that of average quality steps—by 10. For example, if a contestant had made 9 steps solving one problem and got the following map of quality: 100 211 210 010, we could see that there was one high-quality step of the first level; 2 high-quality steps, an average one and an unsuccessful step of the second level of competencies; two high-quality and one average step of level 3, and an average step of level 4. We created the GMC and the EMC maps for solutions of every selected problem for every student's work.

Comparing the General Maps of Competencies, we can see whether students have managed to reach the highest level of problem solving, as well as the quality of their presentation. The Expanded Maps of Competencies helped us investigate some specific aspects—such as, the use of algebra.

13.5 Analysis of Results

The data collected in the General Maps and in the Extended Maps of Competencies let us analyze the achievements of each individual student, as well as compare the results of the entire focus group. The GMC generally characterizes student's success in problem solving, whereas the EMC shows the specific prevalence of applied

Table 13.4 Examples of general maps of competencies

Problems	Emils					Anna					Matiss					Rebeka				
	L1	L2	L3	L4	Sc.	L1	L2	L3	L4	Sc.	L1	L2	L3	L4	Sc.	L1	L2	L3	L4	Sc.
6.1	210	110	400	100	**10**	200	000	200	100	**10**	200	000	200	000	**6**	100	100	300	010	**6**
6.4	300	000	300	100	**9**	010	010	110	300	**9**	100	000	200	110	**8**	110	000	100	000	**4**
8.2	100	100	400	200	**10**	100	100	300	100	**10**	000	100	300	000	**4**	100	100	400	000	**7**
8.5	000	210	200	100	**8**	100	110	200	200	**3**	110	000	010	100	**4**	100	110	400	100	**8**
9.1	100	200	400	100	**10**	110	010	000	010	**3**	020	000	100	000	**2**	210	010	010	000	**2**
9.3	200	200	300	200	**10**	100	110	010	101	**1**	200	010	001	011	**4**	100	100	100	110	**4**

mathematical skills. Table 13.4 presents GMC maps for some students, whose results at the Olympiads were different. The right side of every GMC presents the score achieved.

The table shows that the best score was achieved by Emils. He used a variety of methods to elaborate and justify the hypotheses set, as represented by the number of solution steps of levels 3 and 4. Emils has increased his expertise in solving problems with algebra elements. He invested a considerable effort in solving grade 9 problems, which fruitfully resulted in the discovery of various regularities and defining of hypotheses, as we see on level 2 and level 3. This student persistently studied the given problems in depth. Other students did not complete as many steps of the highest levels. For example, Anna's GMC maps imply that she has to improve her algebraic skills required for solving the last three problems. We may see that Anna did well in some steps of level 3 and level 4 to solve Problem 8.5. Nevertheless, the steps made failed to lead to a complete solution of the problem, as one may conclude looking at the score obtained at the Olympiad. Different conclusions can be drawn when looking at Matiss' maps, which demonstrate a significant decline in the student's mathematical abilities. The average quality of solution has decreased. Rebeka's maps show that she produced useful statements, but failed to present an appropriate reasoning in the solutions of grade 6 and 8 problems. Additionally, looking in the Expanded Maps of Competencies, we found that several steps in Emils' solutions presented argumentation skills and the use of algebra. Anna used algebraic notations but lacked technical skills to solve equations or to make algebraic transformations. In the Olympiad works of grades 6 and 8, she provided exhaustive explanations and justified the hypotheses, but in the work of grade 9, Anna failed to do the reasoning because of insufficient algebra skills. The Expanded Maps of Competencies of Matiss and Rebeka demonstrate that they used the numerical calculations in most of the solutions. Matiss presented his solutions mainly in a descriptive form, and both Matiss and Rebeka made statements without providing any proof.

By evaluating the data obtained, we concluded that the quality assessment of the third competence level—synthesis level—characterizes such creativity components of students' solutions as fluency and flexibility (Leikin 2013). We noticed that the quality of the third level steps strongly depends on the useful outcome of solution steps of analytical and technical competencies. The elaboration of these steps demonstrated students' mathematical knowledge which is of great importance in the creative process, as noted by Leikin (2013). The number and variety of selected solution steps do not always guarantee that the student will fully solve the problem. And vice versa—a minimal number of solution steps does not always mean that the problem is not being solved. Some students considered certain problems to be so easy that they correctly described the problem solution directly in the fair copy.

When evaluating data of the entire focus group, we concluded that Olympiad achievements of the 11 students had increased. For 10 students, we determined a constantly low quality of works and the remaining 20 students demonstrated a more or less rapid decline of problem-solving skills.

We conducted a special analysis of the quality of solution steps of the third and fourth levels using codes from the General Maps of Competencies. We added numbers that represent the quality of steps at level L3 and separately at level L4 for each grade and divided them by 10. Compared to the results demonstrated by grade 9, the problem-solving skills have decreased (see Figs. 13.2 and 13.3).

When solving grade 9 problems most students failed to obtain or see the required regularities, which the proved statements had to be deduced from. They could not write the necessary explanations and reasoning. (See, for example, the solution of Problem 9.1, presented by student Arvis in Appendix 2. He found the correct answer but did not justify it.) As a result the students applied a small number of the fourth level steps in total.

We tried to determine an apparent reason for students' diminishing problem-solving skills in the Expanded Maps of Competencies. Taking into

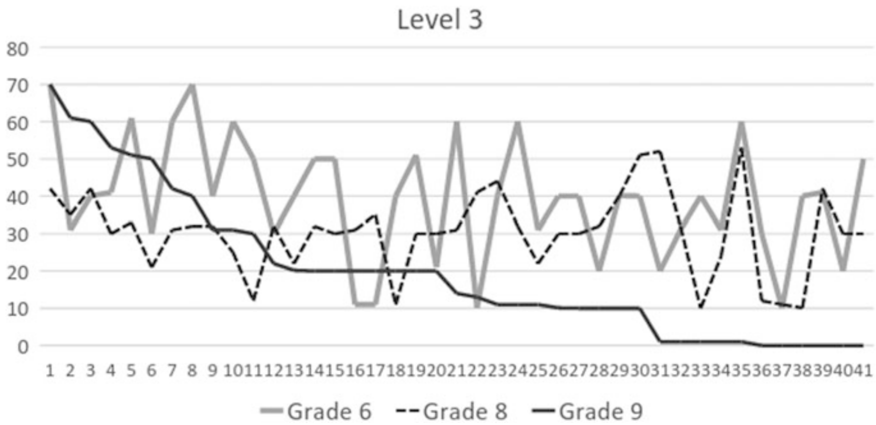

Fig. 13.2 Comparison of synthesis competence at level L3 for each student

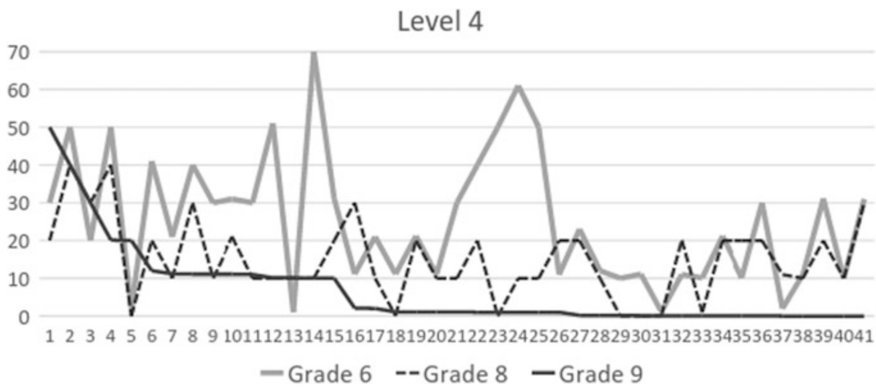

Fig. 13.3 Comparison of evaluation competence at level L4 for each student

account that solving the selected grade 9 problems requires algebra knowledge and techniques, we created sub-maps from the GMC representing only the steps requiring algebra skills. Grade 6 problems can be solved with numerical calculations and logical reasoning. Only some students used algebra in their solutions. Therefore, we created sub-maps only for grades 8 and 9. Figure 13.4 depicts the sum of both sub-maps for grade 8 and grade 9.

Figure 13.4 depicts a decrease in the number of algebraic solution steps and the quality of respective technical performance. Some students misunderstood the givens, considering Problem 9.1 as a number theory problem, while some other students tried to solve Problem 9.3 as an algebra task. For example, Diana decided that only whole numbers must be researched in the solution of Problem 9.1. She started to list the pairs of numbers (see Fig. 13.5) and then she calculated the products of these whole numbers (see Fig. 13.6).

Only five students chose similar approach as shown in the solution suggested by the experts (see Appendix 1, Solution of Problem 9.1)—they constructed the quadratic function.

Most students in the selected focus group had quite a correct conception of how to solve the Problem 9.3—they understood that the expression must be initially factorized to find the prime divisors of the given expression. Unfortunately, not all

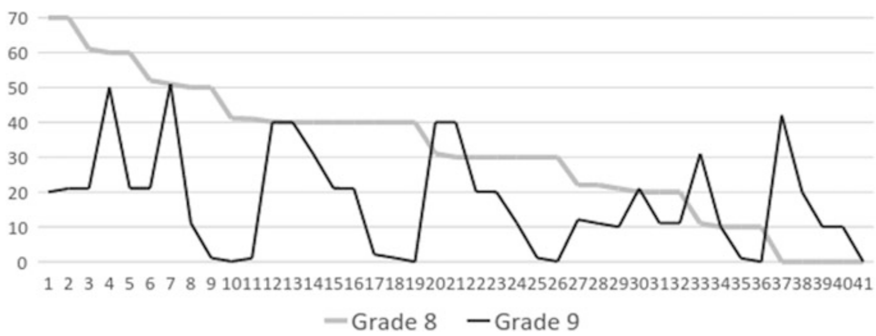

Fig. 13.4 Comparison of students' algebra skills solving 8th and 9th grade problems

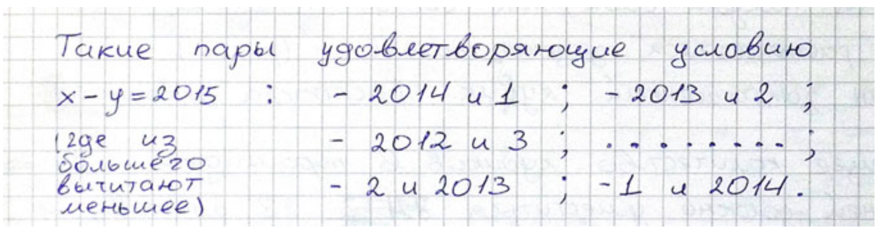

Fig. 13.5 Selection of the pairs of numbers in Diana's solution of problem 9.1

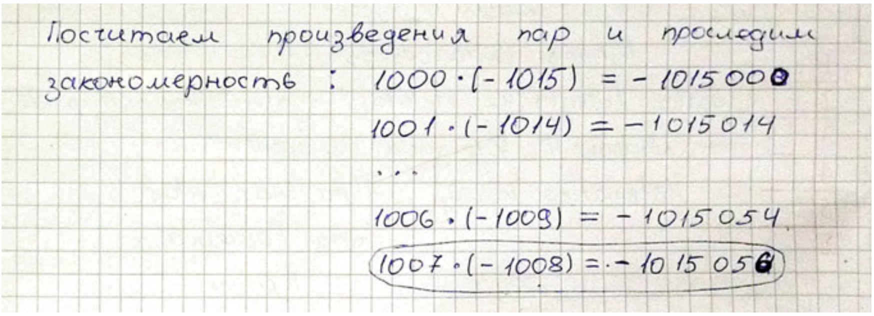

Fig. 13.6 The search of the minimal product in Diana's solution of problem 9.1

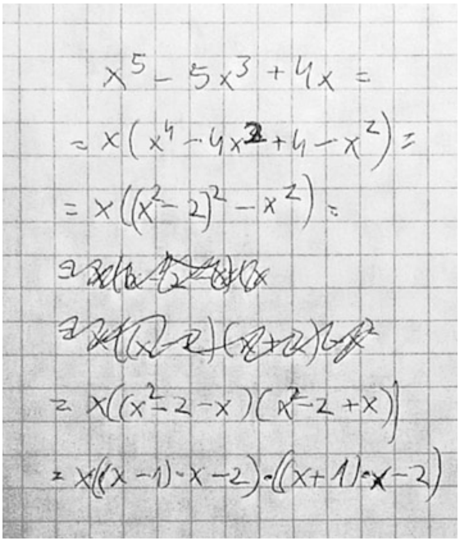

Fig. 13.7 Factoring of the expression in Martin's solution of problem 9.3

of them had a sufficient level of mathematical skills to complete the solution and they failed to reach a sufficiently high level of abstraction. Figure 13.7 shows an example of how Martins tried to factorize the given expression. However, he did not get the completely factored form. This indicates that the student had quite a good understanding of problem solving and of what was required from him. However, Martin's average level of competency was low.

13.6 Conclusion

The assessment tools presented here—the *General Map of Competencies* (GMC) and the *Expanded Map of Competencies* (EMC) for evaluation of students' solutions—can be applied for analyzing the achievements of each individual student or a group of students. The GMC is useful to produce overall statistics of the work performed by a contestant. When comparing the GMC with the score obtained in the Olympiad one may see to what extent the solution is not complete. To get more comprehensive statistics, the student's solution can be compared to an expert's solution, offered by the jury, to determine which of the selected tasks in the student's solution leads to a successful result. A more detailed overview of the activities is incorporated in the EMC, where every solution step has been categorized by the mathematical skills applied. Such EMC maps are extensive (see example Table 13.6 in the Appendix 3); however, they allow investigating sequences of similarly categorized steps.

When searching for original solutions, we found that the students' solutions of Problems 6.1, 6.4 and 8.2 were similar to the solutions offered by the experts (see Appendix 1). The students' works differed only by the quality of argumentation. The students of the focus group had chosen diverse solution methods for Problem 8.5: they either used the arithmetical approach or introduced one or more variables to create equations and even tried to solve a system of equations with 8 variables that made the process of solving very complicated (see Fig. 13.8). Some contestants were close to the perfect solution of the problem, but they were lacking experience

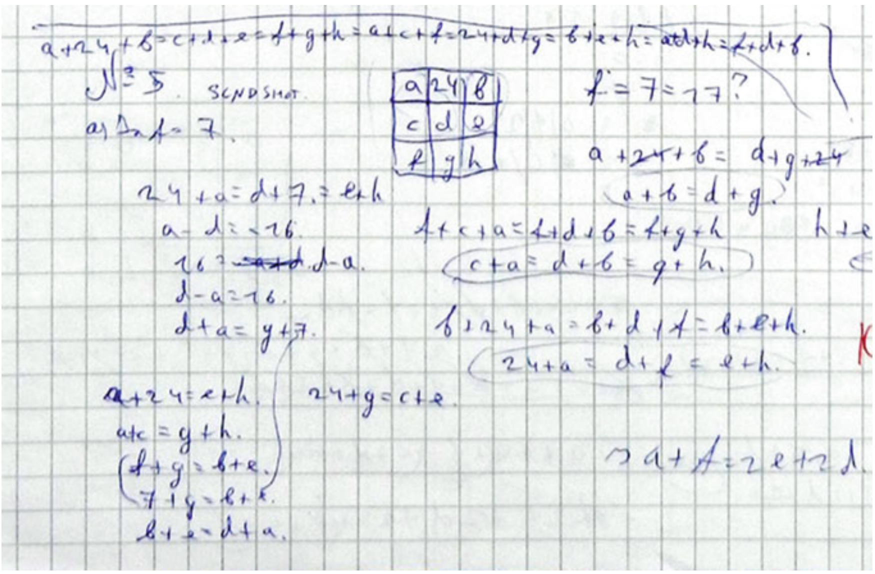

Fig. 13.8 Selection of 8 variables to solve problem 8.5

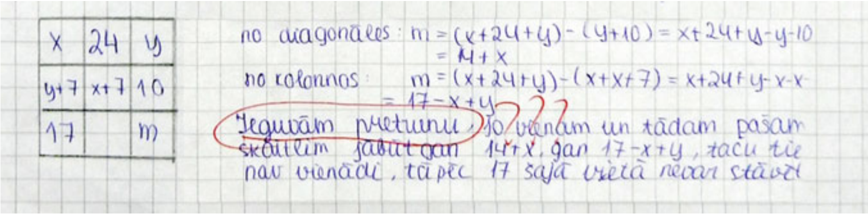

Fig. 13.9 “Contradictory result” in Sandra’s solution of problem 8.5

in the solving of Diophantine equations. Figure 13.9 shows an example of Sandra’s work where she argued that the number m cannot equal to $14 + x$ and to $17 - x + y$ at the same time, and she called this result as contradiction (see Fig. 13.9). An original approach was used by student Janis (see Solution of Problem 8.5, presented by student Janis in the Appendix 2). He researched the last digits of the numbers to find valid examples. Still case (a) was not completed because the demonstration of the example is not the correct proof of the task.

We found only a few original solutions in the Olympic works of grade 9—they differed from the problem solutions offered by the experts as well as from most solutions presented by other contestants. For example, Alex researched problem 9.1 applying the inequality between the arithmetical mean and geometrical mean. He wrote some sparse notes in the draft. He assigned the sum of two numbers a and b by $2c$ (see the Fig. 13.10). Alex drew a picture to visualize the inequality and tried to prove it using variables x and y, where x could be the half-perimeter of an

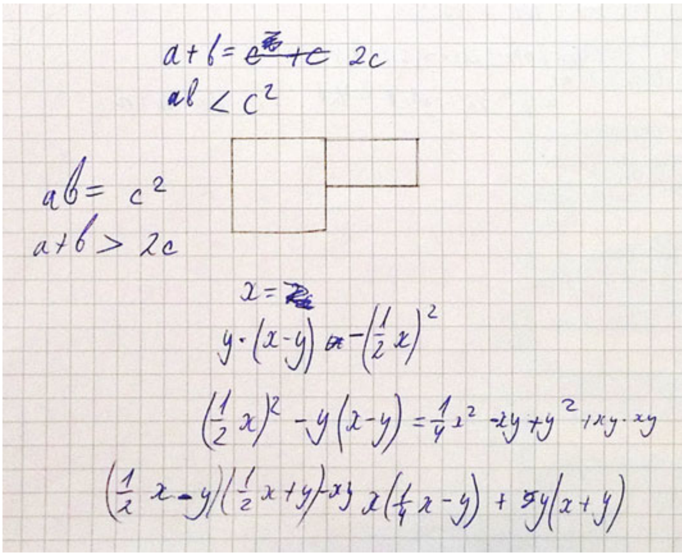

Fig. 13.10 Problem 9.1. The reconstruction of the inequality between the arithmetical mean and geometrical mean in the draft of Alex’s work

arbitrary rectangle, and y could be the length of one of its sides. Then the half of x expresses the side lengths of the appropriate square. The algebraic transformations used by him were not correct. Nevertheless the solution of the problem is well presented in his fair copy. This solution is translated and included in the Appendix 2 (see Appendix 2. Solution of Problem 9.1, presented by student Alex).

Several students had tried to solve Problem 9.3 by implementing modular arithmetic. Elena researched the properties of the separate terms. She proved the expression's divisibility by 5, noted that the last digit of the given expression is 0, and declared that the expression was divisible by 8 and by 3 without the proof (see Elena's solution of Problem 9.3 in Appendix 2). Dmitirij proved the divisibility of the expression step by step. He used an algebraic approach to show the divisibility by 2, 3, and 5 (see the example of the expression's divisibility by 3 in Fig. 13.11). Unfortunately the divisibility by 8 was not proved.

We have to remark that we cannot consider the examples of problem solutions by students Alex and Dmitrij as creative; these examples rather indicate broader experience and deeper knowledge of problem solving.

If looking at the total results obtained by the 351 participants of LOMO 42 in grade 9, we can compare the score of those 87 students who repeatedly participated in the Olympiads with the score of the other 254 contestants. Figure 13.12 compares the share of students in the respective groups of scored points (the maximum score of a student's solutions of LOMO problems is 50). Even though a significant part of the selected 87 students failed to demonstrate good mathematical skills in the solutions of Problems 9.1 and 9.3, they had better scores in the solution of other grade 9 problems and achieved better results than the other 254 contestants. Such a comparison implies that the contestants who repeatedly participate in the Olympiad have better problem-solving skills on the average.

The data collected on the reasons explaining why students demonstrated a low level of algebra competencies in their works does not provide an answer. However, it raises some open questions:

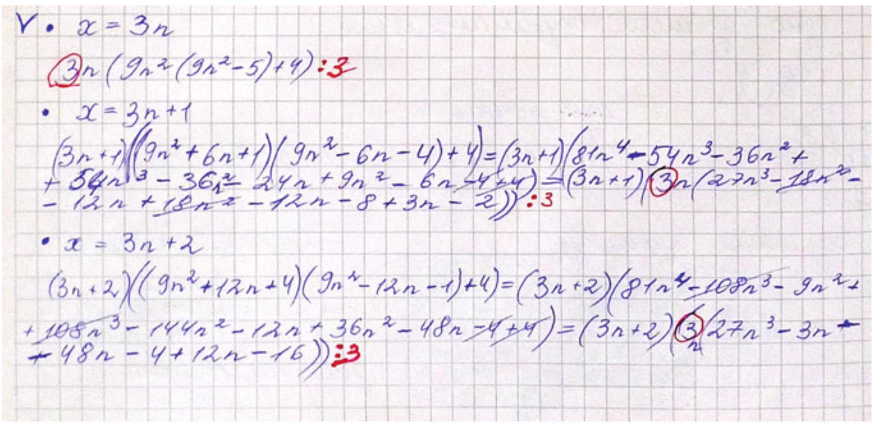

Fig. 13.11 Problem 9.3. Dmitrij's proof of the expression's divisibility by 3

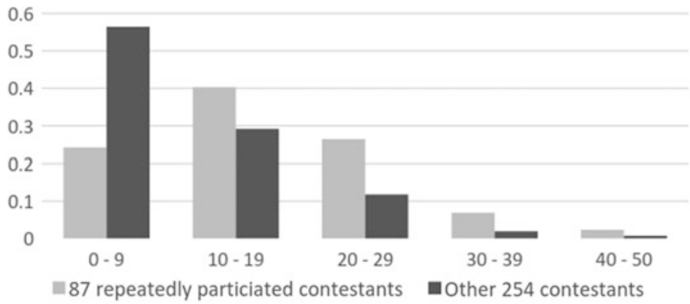

Fig. 13.12 Comparison of 9-graders' results in LOMO 42 (2015)

– How do students acquire school mathematics, do they have an experienced teacher?

Students need mentors who would help them acquire various algebra skills, gain new knowledge, and broaden their problem-solving competencies. Teachers' professionalism is a key prerequisite for developing students' argumentation and justification skills, as indicated by Kattou et al. (2013).

– Do students have the opportunity to prepare for the mathematical contests? Are they motivated by their teachers, peers, parents, or themselves? Can they attend the extracurricular activities in mathematics?

Research of participation in International Mathematical Olympiads Subotnik et al. (1996) highlights the significance of special programs for gifted youth, student attendance in interest groups and in clubs, parental support for talent development and collaboration between parents and school, which are crucial to develop the mathematical abilities of gifted students. At present, a variety of programs for training gifted students are being implemented, however, their efficiency requires a more profound examination (Singer et al. 2016). Therefore, more systematic empirical studies are required to answer the open questions concerning the opportunities for gifted students and for the students who are interested in mathematics.

Appendix 1

Solutions of LOMO Problems Offered by Experts

Solution of Problem 6.1. (LOMO 39, 2012)

Two numbers are erased and their sum plus 2 is written instead. As the other numbers do not change, their sum stays the same. The common sum of the given numbers increases by 2 after every operation. Finally, the last number equals $S + 2 \cdot n$

where S is the total sum of all given numbers, but n is the number of the executed operations. There were 10 numbers on the board at the beginning, but after each operation the amount of the numbers diminishes by one. Therefore $n = 9$. The number left on the board is:

$$S + 2 \cdot n = (1+2+3+4+5+6+7+8+9+10) + 2 \cdot 9 = \frac{(1+10) \cdot 10}{2} + 18$$
$$= 55 + 18 = 73$$

Solution of Problem 6.4. (LOMO 39, 2012)

Case (a) It is possible to write six numbers around the circle as required. See the example in Figure 13.13.
Case (b) It is not possible. The sum of two different natural numbers is at least 1 + 2 = 3. All prime numbers except the number 2 are odd. Accordingly, the sum of two adjacent numbers must be an odd number, that is, the parities of any two adjacent numbers must differ. Whatever is the layout of the seven numbers around the circle, there is going to be one spot where two numbers of the same parity with a total sum of an even number bigger than 2 (it is not a prime number) will be next to each other.

Solution of Problem 8.2. (LOMO 41, 2014)

Express the number in expanded form $x \cdot 10^k + Y$ where x is the first digit, but Y is the k–digit number ($1 \le k \le 5$). Then $x \cdot 10^k + Y = 15 \cdot Y \Rightarrow x \cdot 10^k = 14 \cdot Y \Rightarrow x \cdot 2^k \cdot 5^k = 2 \cdot 7 \cdot Y$.

We conclude that x is divisible by 7. Taking into account that x is a one-digit number $x = 7$ and $Y = 2^{k-1} \cdot 5^k = 5 \cdot 10^{k-1}$, $1 \le k \le 5$. There are five natural numbers that satisfy the solution—75, 750, 7500, 75,000, 750,000.

Solution of Problem 8.5 (LOMO 41, 2014)

The variable x represents the number in the central cell of the given square. The variable y represents the number in the middle cell of the lower row. Then the sum of numbers in every row, column or diagonal is $24 + x + y$. The cells can be filled step by step (see Figs. 13.14 and 13.15).

Case (a)

We determine that the middle number in the last column must be -10, which is not a natural number. Consequently, the lower left corner cannot contain the number 7.

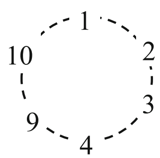

Fig. 13.13 Arrangement of six numbers around the circle

Case (a)

	24	
	x	
7	y	

→

	24	17+y
	x	
7	y	

→

	24	17+y
	x	
7	y	17+x

→

	24	17+y
	x	–10
7	y	17+x

Fig. 13.14 Filling of the square in case (a)

Case (b) Similarly the cells can be filled in the case (b):

	24	
	x	
17	y	

→

	24	7+y
	x	
17	y	

→

	24	7+y
	x	
17	y	7+x

→

	24	7+y
	x	10
17	y	7+x

→

	24	7+y
14+y	x	10
17	y	7+x

→

x–7	24	7+y
14+y	x	10
17	y	7+x

Fig. 13.15 Filling of the square in the case (b)

Fig. 13.16 One of the possible answers in the case (b)

6	24	9
16	13	10
17	2	20

Case (b) Similarly the cells can be filled in the case (b):

The sum of numbers in the diagonal is $3x$. Then $y = 2x - 24$. Choosing $x = 13$ we determine one of the answers (see Fig. 13.16.).

Solution of Problem 9.1. (LOMO 42, 2015)

The letter x and the expression $x + 2015$ are used to represent the given numbers. The function representing their product is $f(x) = x \cdot (x + 2015) = x^2 + 2015x$. The graph of this function is a parabola that opens upward. The x-coordinate of their vertex is $x_0 = \frac{-2015}{2} = -1007.5$. The function has the minimal value at this point. Consequently, the two numbers are 1007.5 and -1007.5.

Solution of Problem 9.3 (LOMO 42, 2015)

The factoring of the expression is

$$\begin{aligned} x^5 - 5x^3 + 4x &= x \cdot (x^4 - 5x^2 + 4) \\ &= x \cdot (x^4 - x^2 - 4x^2 + 4) = x \cdot (x^2(x^2 - 1) - 4(x^2 - 1)) \\ &= x \cdot (x^2 - 1) \cdot (x^2 - 4) = x \cdot (x - 1) \cdot (x + 1) \cdot (x - 2) \cdot (x + 2) \\ &= (x - 2) \cdot (x - 1) \cdot x \cdot (x + 1) \cdot (x + 2). \end{aligned}$$

We determined that the given expression is a product of five consecutive whole numbers. At least two of the numbers are divisible by 2, while one of them is also divisible by 4. At least one number is divisible by 3, and at least one number is divisible by 5. Consequently, the product is divisible by $2 \cdot 3 \cdot 4 \cdot 5 = 120$.

Appendix 2

Solutions of Problems, Presented by Students

Comment. We tried to keep in translation the writing style presented by students. Usually students did not use correct mathematical language and failed to construct grammatically correct sentences. Sometimes they omitted the subject or the verb.

Solution of Problem 6.1, presented by student Emils

Taking into account that we reach every next number in the equation $a + b + 2 = c$, we can predict:

d = the number that left the last,

there are 10 numbers, therefore 5 of them will be *A*, but others 5—will be *b*.

In one's turn $d = a_1 + a_2 + a_3 + a_4 + a_5 + b_1 + b_2 + b_3 + b_4 + b_5 + (n \cdot 2)$

n = the number of attempts of the operation $a + b + 2 = c$

n = 9 because after every attempt left per 1 number less on the blackboard, so only one number left after 9 attempts.

Not important, which number is a_1, which number is a_2, etc., because all given numbers will be used sooner or later. The common sum of given numbers is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$, therefore $d = 55 + (9 \cdot 2) = 55 + 18 = 73$.

Solution of Problem 8.5, presented by student Janis

Problem 5, version (a). If we insert number 7 in the left lower corner, the square is impossible to complete. The square in the picture is almost completed (see Fig. 13.17). However, number 7 does not allow the highlighted diagonal (all sums are 35). If number 7 could be turned into number 17 at least, the square would be completed as I am going to demonstrate in version (b).

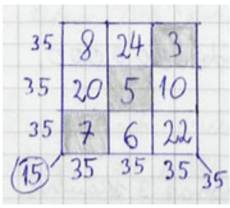

Fig. 13.17 Filling of the square in case (a)

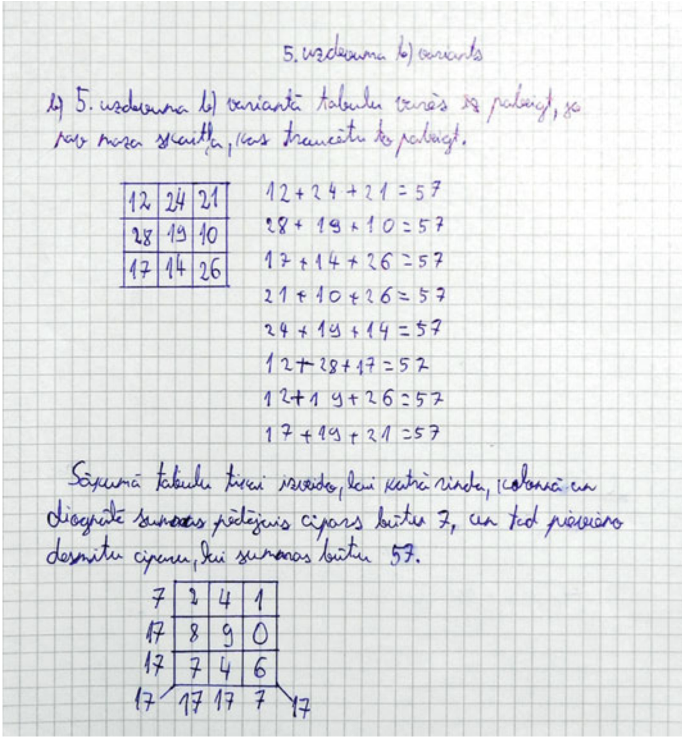

Fig. 13.18. Solution of case (b), presented by Janis

The square in Problem 5 version (b) will be possible to complete because of the lack of a small number that disturbs completion. At the beginning the square is drawn up so that the last digit of the sum in each row, column and diagonal is 7. Then a number of tens is added so that the sums would be 57 (see Fig. 13.18).

Answer: it is impossible to complete the square in version (a), but it is possible to complete it in version (b).

Solution of Problem 9.1, presented by student Arvis

Numbers where the difference is 2015 will have the smallest product when the second number is negative, because if a negative number is subtracted, it is added. For example, $2-(-4)=2+4=6$. Therefore the smallest product for whole numbers is 1008 and -1007. But it can be slightly increased if we take numbers 1007.5 and -1007.5.

Solution of Problem 9.1, presented by student Alex

We assume that 2015 and 0 are such numbers. If it is not true, then one of the numbers is less than 0 (for the product of two numbers to be less than 0, one number is <0, but the other is bigger). We assume that the positive (P_1) is subtracted from the negative ($-P_2$):

$-P_2 - P_1 < 0 < 2015$ is not good.

Second version:

$P_1 - (-P_2) = P_1 + P_2$—can be completely equal to 2015.

As we have to determine the smallest $P_1 \cdot (-P_2)$, let's determine the biggest $P_1 \cdot P_2$, so that $P_1 + P_2 = 2015$.

According to the rule of the square where the perimeter is equal to the perimeter of other rectangles (not squares), a square will always have the biggest area.

That is, $P_1 \cdot P_2$ is going to be the biggest if $P_1 = P_2$.

As $P_1 + P_2 = 2015$, $P_1 = P_2 = 1007.5$.

Consequently, the smallest product is going to be $-(1007.5^2)$. Truly:

$1007.5 - (-1007.5) = 2015$, so these are the numbers we need. Answer 1007.5 and -1007.5.

Solution of Problem 9.3, presented by student Elena

If this number is divisible by 120, it contains prime factors 2, 2, 3, 2, 5. This number is divisible by 5, because subtrahend $(-5x^3)$ is divisible by 5, and the transformed expression $x(x^4 + 4)$ is also divisible by 5. In order to prove it, let's determine how many different numbers can be obtained in the case of each digit (see Fig. 13.19).

If the last digit of number x^4 is 1, then, if we add 4, it is divisible by 5. Accordingly, every product is divisible by 5. It is likewise with the digit 6 at the end of a number. If the last digit of number x^4 is 5, then x contains 5. But all addends contain 5, therefore $\vdots$ 5.

We see that the sum of the last digits is 0 (see Fig. 13.20). 0 is divisible by $2 \cdot 2 \cdot 2 = 8$ therefore the given expression is divisible by 8. The expression is divisible by 3 too if x is a whole number. The expression $x^5 - 5x^3 + 4x$ is divisible by 120 if x is a whole number.

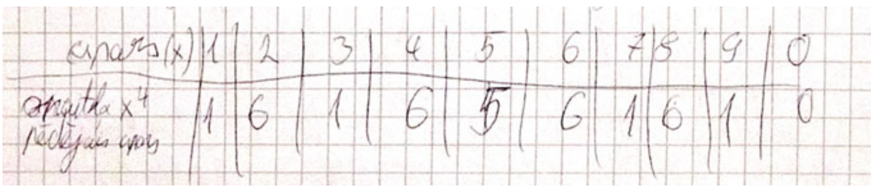

Fig. 13.19 Last digits of the terms x and x^4

Fig. 13.20 Research on the last digits of given terms

Appendix 3

Example of Coding System for Problem 6.1 (LOMO 39)

We selected all solution steps contained in the works of the students of the selected focus group. They are as follows:

1. Try one or some examples to see how the given algorithm works.
2. Describe one step of the process algebraically.
3. Systematically investigate several examples to detect common properties of the given numbers during the process.
4. Visualize the algorithm—construct the tree, draw the arcs, or use colored pencils—to visualize the calculation of an example.
5. Calculate the sum of the given numbers (S = 55).
6. Express the sum of the given numbers S algebraically.

Consider and explain what happens at every step of the process:

7. the sum of the given numbers does not change,
8. the total sum of the numbers on the black-board increases by 2,
9. the amount of the numbers on the black-board diminishes by 1.

Calculate the number of steps in the process:

10. the use of an algebraic formula (10 – n = 1),
11. calculate (10–1 = 9).

Calculate the increase of the sum S:

12. the use of an algebraic formula (2n),

13. numerically (2·9).

Write the calculation formula:

14. algebraically ($S + 2n$),
15. numerically ((1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) + 2·9).
16. Calculate the result (73).
17. Justify the invariance of the result obtained.

It is not necessary to execute all of these steps to complete the solution of the given problem. The best students' solutions were similar to the expert solving path, which contains the following steps: 5, 7, 8, 9, 10, 12, 14, 16, and 17.

All the steps mentioned were coded corresponding to the competencies needed for problem solving. Some of the steps are attributed to different groups of competencies. For example, the creation of an algebraic formula to describe the whole process corresponds to modelling skills, technical skills, and problem-solving skills. Problem-solving skills applied in the solutions represent such heuristic strategies as the method of trial and error (step 1), the visualization of the process (step 4), systematic investigation (step 3), the detection of relevant properties of the process (steps 7–13)—here explanations and reasoning are coded as argumentation skills; the creation of algebraic formula (step 14), justifying the invariance of the process (step 17). Students' decisions to complete calculations or to apply some algebraic terms are coded as technical skills. If a student additionally formulated the established definitions and described the applied operations, we coded these argumentation skills as knowledge reproduction and description (see Table 13.3).

Table 13.5 contains codes for each solution step of Problem 6.1. We added the number of the step to the code. In the case of *Problem solving* some steps are coded by the same code because the meaning of these steps is the same. For example, step

Table 13.5. Coding system of solution steps for Problem 6.1 LOMO 39

Level of competence	Modelling skills	Technical skills	Problem-solving skills	Argumentation skills
	Step: code	*Step: code*	*Step: code*	*Step: code*
Level 1	Step 2: a2	Step 1: c1 Step 5: c5	Step 1: e1	Step 8: r8
Level 2	Step 6: a6	Step 3: c3 Step 11: c11	Step 3: e3 Step 10 or 11: ap10	Step 10: r10 Step 11: r11
Level 3	Step 4: vm4 Step 10: am10 Step 12: am12 Step 14: am14	Step 12: a12 Step 13: c13 Step 14: at14 Step 15: a15 Step 16: c16	Step 4: vp4 Step 12 or 13: ap12 Step 14 or 15: ap14	Step 7: r7 Step 9: r9
Level 4			Step 17: h17	Step 12: r12 Step 13: r13 Step 14: r14 Step 15: r15 Step 17: j17

Table 13.6. Expanded map of Emils' mathematical skills

Level/skills	Modelling	Technic	Problem solving	Argumentation
Level 1	a2	c1; c5	e1	k1; k2
Level 2		c3; c11	e3; ap10	r11
Level 3	a6	c13; at14; a15; c16	ap12; ap14	d14
Level 4			h17	r13; r14; j17

12 and step 13 are coded by *ap*12, where the second letter is used to differentiate it from the code *am*12 used for coding the same step as a case of modelling.

Student Emils used 10 steps in the draft and fair copy for solving this problem. He tried to calculate the example (step 1) and made one mistake (this step was evaluated with average quality). Then he systematically investigated the examples (step 3) in the draft (he scored average quality because he called an example "proof"). He calculated the sum S (step 5) and the increase of the sum (step 13). In the fair copy, Emils used an algebraic description (step 2), created and explained the formula (step 14), transferred this into a numerical form (step 15) and calculated the result (step 16). He calculated and reasoned the number of steps in the process (step 11) and justified the invariance of the process (step 17). All the steps completed were of a high quality, except two. The quality map GMC of the steps completed is as follows: 210 110 400 100.

Emils made 3 steps of the first level of competencies and two of the second level, where two of these steps were considered to be of an average quality. Other five steps were completed very well. Emils presented exhaustive explanations and a correct justification, as demonstrated in the argumentation section in Table 13.6 that characterized Emils work in details. His solution obtained the highest score.

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Chapter 14
Complex and Open-Ended Tasks to Enrich Mathematical Experiences of Kindergarten Students

Viktor Freiman

Abstract At a very young age, some children already manifest unusually strong mathematical abilities that need to be fully developed and nurtured in school. The chapter investigates in what way a kindergarten curriculum can offer all students a richer mathematical experience by means of open-ended and complex tasks. Hence, I developed and implemented challenging activities for kindergarten students. The data collected during the experiment helped us examine learning opportunities within challenging situations in terms of the mathematics structures the kindergarten students create during such activities and the strategies they use. The chapter analyses how kindergarten students approach three challenging situations, showing a great variety in students' authentic strategies and mathematical approaches. While some students struggle with increasing complexity of tasks but still remain engaged and try to overcome obstacles, others seem to exhibit more structured (in terms of mathematical relationships), systematic (in terms of problem-solving strategies), and abstract (in terms of mathematical symbolism) approaches. In addition, all students, even at a very young age, can benefit from a classroom culture of questioning, investigating, communicating, and reflecting on more advanced and meaningful mathematics that can help develop their mathematical mind.

Keywords Mathematical giftedness · Precocious abilities
Kindergarten curriculum · Open-ended and complex tasks

14.1 Introduction: Context and Background

The benefits of complex and open-ended tasks in enriching the mathematical experiences of all learners in a regular classroom, including those with potentially high abilities, have been well documented in the literature (Leikin 2006; Rotigel and Fello 2004; Singer et al. 2016; Sheffield 1999; Taylor 2008). These benefits can

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be observed as early as in pre-school and kindergarten. For example, Leeson (1995) investigated kindergarten students' spatial constructions by describing students' advances in problem solving under adult guidance. Leeson shows that a child presented with task of counting "hidden" cubes in a 2-D representation of a 3-D construction can be able to "see" hidden parts; a discussion with the interviewer allowed the child to reflect upon her reasoning and in so doing, she was able to re-organise her understanding of the situation (Leeson 1995). The results of this study, according to the author, support the view that many students in early school years are under-challenged (Leeson 1995). Wadlington and Burns (1993) studied teachers' views of activities for gifted pre-schoolers and kindergartners; some teachers mentioned a "need for greater challenges and additional enrichment in the regular classroom" (p. 50). Several teachers from this study used activities that were open-ended and could be extended to match children's personal interests and skills while keeping the experiential and exploratory character (Wadlington and Burns 1993).

At the same time, researchers expressed concerns that kindergarten curriculum focuses too often on "readiness" exercises while overlooking the potential of nurturing children's curiosity and ability to reason mathematically about their world (Pletan et al. 1995). More recent studies have shown that some teachers still keep beliefs that solid "basic skills" must be given a priority to ensure academic success, especially for students coming from economically and ethnically disadvantaged regions, thus overlooking add-on value of problem-solving, inquiry-oriented learning activities (Stipek 2004).

By addressing the need to support teachers in changing their practices, Tirosh and Graeber (2003) emphasise three key elements of changing teaching practices by "(1) providing opportunities for children to solve mathematical problems in their own ways, (2) listening to children's mathematical thinking, and (3) using children's mathematical thinking to make instructional decisions" (p. 648). In the study mentioned above (Leeson 1995), revealing a student's ability to deal with spatial construction became possible in a one-to-one conversation with the child. How can teachers do so in a whole-group teaching situation? By investigating this question, it would be possible to get more insight into the interplay of abilities, commitment, affect, and opportunities that constitute, according to Leikin (2009), key components of mathematical potential.

Imagine a kindergarten classroom with 36 5- to 6-year-old children and two classroom teachers presenting a fashion show. In fact, six pairs of winter clothes, each pair consisting of a coat and a hat, were brought out to let children try them out in different combinations. Three pairs were for boys and three others for girls. At each round of the show, one child was called out by the teacher and was asked to choose a set of clothes and try them on in the front of their peers. While making their choice, the children had to respect one more condition: Each time the chosen pair should be different from ones shown before. Initially perceived as a game, the activity drew a lot of enthusiasm in students: All children were excited with the task and everyone wanted to have a turn to participate in the show. Not surprisingly, since many choices are available at the beginning, children can easily find an appropriate solution according to their preferences: Boys preferred to choose clothes for boys, and girls chose clothes for girls.

At some point, one boy appeared to be puzzled: There were no more boys' clothes left to make up a pair that would be different. The teacher discussed the situation with the class to remind them of the condition: The pair should be different and has to include one hat (for boys or for girls) and one coat (for boys or for girls). However, this does not necessary imply that boys must wear only clothes for boys and girls only clothes for girls. Eventually, the children conclude from this definition of a pair that a pair can also be mixed—one item for boys and another for girls. The activity becomes even more entertaining and more laughs can be heard in the classroom. More children raised their hands to be called on. Now was the time to ask a "serious" question: If the show continues, will we have enough choices for everyone to participate? Children started making suggestions and trying to give some arguments for yes or no answers. The search for all possible combinations continued in paper-and-pencil mode, an important part of the activity, in which some children, while reflecting on their experience, demonstrated a more systematic way to look for different combinations, noticing patterns, and eventually coming to some generalizations, thus entering into higher levels of mathematical thinking (Clements and Sarama 2009).

The described activity was a part of a set of challenging mathematics activities in the form of complex and open-ended tasks I developed in parallel with the regular kindergarten curriculum and experimented with during a school year on a weekly basis (one hour per week). The idea was to prepare students for the enriched curriculum all of them had to follow in Grades 1–6. The detailed report about experimentation in this curriculum and its use for identification and fostering of mathematical giftedness in early grades was described in our previous publications (Freiman 2006, 2010). In this chapter, I present theoretical foundations and data from an exploratory small-scale study of mathematical enrichment for young children (5 years old) whose goal was to (1) design challenging situations using open-ended and complex mathematical tasks and (2) experiment with these situations in a classroom to reveal patterns of students' mathematical thinking. In terms of research questions, I aimed to investigate:

- What are the learning opportunities within challenging situations in terms of mathematics structures created by the kindergarten students?
- What are the strategies students use within the open-ended task and what are mathematical abilities they demonstrate while solving challenging problems?

The topic of the study is particularly relevant for the current context of early school years in Canadian schools, which have recently been implementing more systematic early childhood and kindergarten curricula. For example the New Brunswick Provincial Department of Education and Early Child Development (DEECD 2012) requires teachers to introduce mathematical concepts through specially designed activities that foster reflection, reasoning, and problem-solving using manipulations, exploration, and experimentation with concrete material.

In this respect, without specifically targeting mathematically gifted students, the study I describe in this chapter seeks to provide with some insight into young children's capacities to work with more complex mathematical structures and develop a taste for deeper questioning and investigation of advanced mathematical topics. In following sections, I will outline the theoretical background of our activities and analyse three examples of open-ended and complex tasks that help to reveal different types of mathematical representations that may point out higher abilities in task commitment, self-regulation, and self-efficacy and be interpreted as precursors of mathematical giftedness in some of our participants.

14.2 Some Initial Observations from History and Modern Pre-school Practice

The story of 9- or 10-year-old Gauss solving the routine problem of calculating the sum of the first hundred natural numbers is one of the well-known examples of this kind. While all other children of his class were desperately trying to add the terms one by one, Gauss impressed the teacher by finding a quick and easy way to do it by regrouping the terms in a special way (see, for example, Dunham 1990, pp. 236–237). This unusual insight into mathematical structure at such a young age is described in the literature as an example of a precocious mathematical mind that needs to be further developed and nurtured.

Another example of a very early insight into mathematical structures that comes from a more recent time was given by the Russian mathematician Kolmogorov (1988), who recalled that at the age of 5 or 6 he was pleased with his discovery of the regularity of the sum of consecutive odd numbers resulting in a full square $1 = 1^2$, $1 + 3 = 2^2$, $1 + 3 + 5 = 3^2$, etc.

I became interested in the study of mathematically precocious children when reflecting on our classroom experience with 4- to 5-year-old children using educational software with some mathematical tasks while attending computer classes at a private pre-school and kindergarten in the Montréal area in Québec, Canada.

I noticed that some students always chose more challenging activities; went through all the levels up to the highest ones; understood each activity almost without any explanation from the teacher; demonstrated very systematic approaches to the problem; had very sharp selective memory of important facts, details, and methods; were very creative in their work with open-ended problems (such as creating puzzles and patterns); and often proudly shared their discoveries with their peers.

For example, working with counting tasks such as finding a domino piece with a number of dots corresponding to a number from 6 to 9 that they have been shown, some children counted all the dots on almost every card using their fingers; others first chose one that contained more than 5 dots; and most counted the dots, and if they did not get the right result, they jumped randomly to another with a similar number of dots. There was also a small group of children who tried to spot a card with less than 8 dots on it. Finally, one child clicked immediately on a card with 7 dots, saying “I know it’s this one because 5 and 2 make 7.”

Analysing children’s strategies, I could notice their different approaches to numbers. Some children see cards as pictures with objects to count and they use the same strategies they would with manipulative objects (such as toys). Other children try to use a different, more complex approach: thinking more globally (“I see it’s 5 here; I know that 7 is less than 8”) and abstractly (number as an abstract characteristic of a set of dots) along with using a number of shortcuts that help them to increase the efficiency of their mathematical thinking based on grasping patterns and relationships.

Our next example was a comparison task with two cards shown to the child: one with a certain number of dots arranged within a 3 × 4 array (12 dots maximum) and another one with a number from 1 to 12 written on it. The child had to decide whether two cards present the same number or not. For most 5-year-old children, this is a relatively simple task, but within a time limit (introduced in the game) it becomes an extremely challenging activity for children whose strategy of counting is limited to “finger pointing.” The best winning strategy was found by children who used estimation (I know that I have many more dots here than the number 3 on the other side) and counting with eyes (i.e., without fingers). Some children gave surprisingly deep comments such as “I know this number of dots is 12 because I see 4 rows of 3 dots, which make 12,” which demonstrates precocious insight into numbers and number relationships.

Some other tasks gave children opportunities to create patterns, for example, by constructing a character for which they had to choose a particular feature (like a blue hut) or of they could build their own character following some pattern they would define by themselves. This second option was seen by many children as an art activity, although our observation shows that some 4-year-old children created characters based on more complex patterns of a mathematical nature (such as colours, backgrounds, and pieces of clothing). One activity presented a 6 × 6 grid with a set of different puzzles to reproduce (pictures were given as a model), and another involved creating your own puzzle, where many young children (4–5 years old) tried to “draw another picture.” Again, there were few children who spontaneously built more mathematically abstract tessellations using complex, sometimes symmetrical configurations of shapes, which is something that is more likely to occur with older children already familiar with geometric transformations such as reflection or translation. Another activity presented a factory for making chocolate chip cookies. One mode of this activity asked children to put a number of chips in a cookie corresponding to a randomly given number (1–10). Another mode prompted them to create a cookie with an arbitrary chosen number of chips. However, this free choice gave us a chance to observe certain children making cookies with consecutively chosen numbers from 1 to 10 repeated in two rows. And even more, they were so fascinated with their result that they started to repeat the same pattern more and more without any visible fatigue, although it was a routine repetition of the same procedure. This appears to be an example of mathematical creativity of a particular kind: seeing the beauty of mathematics in repeating a pattern.

Equalizing tasks are complex for very young children. For example on activity involving feeding rabbits with carrots shows rabbits “waiting for food”; another shows an empty field in which a child has to put out carrots, keeping in mind that each rabbit should get one carrot. In fact, the child has to control two conditions at the same time to ensure that the number of rabbits is equal to the number of carrots. Our observation shows that some children decide to arrange carrots in a certain geometric pattern (row, stair, or array), which helps them keep control of conditions, this showing a more complex way of thinking.

Finally, working on ordering tasks such as ordering seven *matryoshka* dolls by increasing or decreasing size, some children proceeded by trial and error, while others did it more systematically (looking at neighbours and changing the order if necessary). A few did it in a very systematic way, by starting with putting the smallest or biggest one first, then going to the next smallest or biggest and so on. This strategy allowed them to simplify the process of problem solving, and at the same time, showed their ability to apply more complex thinking.

In these examples taken from the childhood of historical mathematical prodigies such as Gauss and Kolmogorov or from current explorations of pre-schoolers in computer-based environments, the exceptional abilities of precocious children become apparent during activities that are rather spontaneous. The question then arises whether such abilities could be developed in a more systematic way. Hence, the study is based on the belief that all young children deserve a richer learning environment, and it is driven by the quest to develop children’s highest intellectual potential in mathematical abilities by implementing a more challenging curriculum for four- to five-year-old students enrolled in kindergarten classes. It is worth mentioning that in Québec, Canada, as in other provinces and territories, kindergarten attendance is mandatory for all children from the age of five. In the next section, I will review the literature on mathematical abilities in general, their development during early years, and teaching strategies that can foster such development.

14.3 Literature Review

According to Krutetskii (1962), the abilities involved in doing mathematics are composed of general abilities (diligence, persistence, productivity, active memory, concentration, and motivation), general mathematical abilities (flexibility and dynamic thinking), and specific mathematics abilities. Many researchers relate mathematical ability to intelligence. Young and Tyre (1992) give some practical characteristics of intelligence as the ability to deal with new situations; see relationships, including complex and abstract ones; learn and apply what has been learnt to new situations; inhibit instinctive behaviour; handle complex stimuli; and respond quickly to information. It also includes a group of mental processes involving perception, association, memory, reasoning, and imagination (Young and Tyre 1992).

Other researchers consider giftedness as an intersection of different factors. For instance, Mingus and Grassl (1999) focused their study on students who displayed a combination of willingness to work hard, natural mathematical ability, and creativity. In their model, besides natural mathematical ability, they look into some non-mathematical abilities such as willingness to work hard (i.e., being focused, committed, energetic, persistent, confident, and able to withstand stress and distraction) or high creativity (i.e., capacity for divergent thinking and for combining the experience and skills from seemingly disparate domains to synthesise new products or ideas). As results of the identification process, the authors labelled students possessing a high degree of mathematical ability, creativity, and willingness to work hard as “truly gifted.” (Mingus and Grassl 1999).

Ridge and Renzulli (1981) define giftedness as an interaction among three basic clusters of human traits: above average general abilities, high levels of task commitment, and high levels of creativity. By their definition, gifted and talented children are those possessing or capable of developing this composite set of traits and applying them to any potentially valuable area of human performance. Greenes (1981) analyses various strategies used by children working on mathematical problems. She points out seven attributes that characterise the gifted student in mathematics: spontaneous formulation of problems, flexibility in handling data, data organisation ability, mental agility or fluency of ideas, originality of interpretation, ability to transfer ideas, and ability to generalise. Many of these characteristics can already be detected during the early years (pre-school and primary grades). Krutetskii (1976) noticed in his study that although precocity is not a necessary feature of giftedness, some highly precocious children did demonstrate a clear interest in mathematics, worked with mathematics with pleasure and without compulsion, mastered different mathematical skills and habits faster, and attained a comparatively high level of mathematical development compared to other children of the same age.

From the developmental point of view, cognitive and meta-cognitive patterns of young children’s mathematical development were studied in depth within different psychological theories in the 70s and 80s, for example, Piagetian studies (Rosskopf 1975), Gelman’s study of children’s understanding of numbers (Gelman and Gallistel 1978), and Resnick’s developmental theory of number understanding (Resnick 1983). The last two decades have brought additional dimensions to teaching and learning mathematics, such as creativity.

Hershkovitz et al. (2009) analysed characteristics of tasks that are effective in helping all students to become more creative. By focussing on giving opportunities to all, the authors argue for putting a great deal of care and delicateness into choosing such tasks in terms of level of challenge or difficulty (Hershkovitz et al. 2009). According to them, “on the one hand, the task should be challenging enough to encourage interesting solutions, and on the other hand, it should also enable weaker students to reach some solutions, perhaps less elegant” (p. 258); thus, open tasks can be appropriate to create such opportunities.

Kulm (1990) remarked that since so much of school mathematics in the past has been focused on practising skills, the completion of a large number of exercises in a limited time period has been accepted not only as a measure of mastery but as an indication of giftedness and potential for doing advanced work. On the other hand, higher order thinking in mathematics is by its very nature complex and multi-faceted, requiring reflection, planning, and consideration of alternative strategies. Only the broadest limits on time for completion make sense on a test purposing to assess this type of thinking.

Similar observations were made by Greenes (1981), who argued that the bulk of our mathematics program has been devoted to the development of computational skills, and we tend to assess students' ability or capability based on successful performance of these computational algorithms (so-called good exercise doers) and have little opportunity to observe students' high order reasoning skills. According to the author, sometimes even a very banal math problem might deliver a clear message about distinguishing the gifted student from the good student. Greenes (1981) analyses a very simple word problem (given to fifth grade children): Mrs. Johnson travelled 360 km in 6 h. How many kilometres did she travel each hour? One bright student surprised the teacher by having difficulty in solving this easy problem. Finally, the teacher realised that the student had discovered that nothing was said about whether the same number of kilometres had been travelled each hour. This example demonstrates the child's ability to detect ambiguities in the problem, which may be a sign of mathematical giftedness.

Summarizing these observations, I could say that using routine drill tasks involving numerous standard algorithms do not, in general, offer a good opportunity to identify and nurture mathematical talents. Sheffield (1999) calls such routine tasks "one dimensional." As an example, she cites a class of third and fourth graders reviewing addition of two-digit numbers with regrouping. Children were asked to complete a page of exercises such as 57 + 45, 48 + 68, and 59 + 37. As it usually happens with brighter and faster students, they finished all the exercises before their classmates. So the teacher "challenged" them with three- or four-digit addition. Although the calculations become longer and more time consuming, the tasks themselves were not more complex or more mathematically interesting. As a better didactical solution for these children, Sheffield suggests the use of meaningful and more challenging tasks such as the following:

Find three consecutive integers with a sum of 162.

Working on this task, students can continue to get practice in adding two-digit numbers with regrouping, but they also have the opportunity to make interesting discoveries. Students who are challenged to find the answer in as many ways as possible; pose related questions; investigate interesting patterns; make and evaluate hypotheses about their observations; and communicate their findings to their peers, teachers, and others will get plenty of practice adding two-digit numbers, but they will also have the chance to do some real mathematics (Sheffield 1999, p. 47). She claims that by using an open-ended heuristic model that connects five elements of problem-solving process, namely, 'relating', 'investigating', 'creating', 'communicating', and 'evaluating', with each other forming "a star inside of the pentagon," we can contribute to a student's creative development of mathematical abilities (Sheffield 1999).

I also find useful Burjan's (1991) recommendations to use open-ended investigations and open-response problems rather than multiple-choice short questions, problems allowing several different approaches, non-standard tasks rather than standard ones, tasks focusing on high-order abilities rather than lower-level skills, complex tasks requiring the use of several "pieces of mathematical knowledge" from different topics rather than specific ones based on one particular fact or technique, and knowledge-independent rather than knowledge-based tasks.

Another aspect of choosing appropriate problems for the identification of mathematically bright children has been analysed by Greenes (1997), who has underlined the importance of the use of rich problems and projects in which students can demonstrate their talents. The author mentions that such problems

- allow integration of the disciplines (application of concepts, skills, and strategies from the various sub-disciplines of mathematics or from other content areas, including non-academic ones);
- are open to interpretation or solution (open-beginning and open-ended problems); require the formation of generalisations (recognition of common structures as basic to analogue reasoning);
- demand the use of multiple reasoning methods (inductive, deductive, spatial, proportional, probabilistic, and analogue);
- stimulate the formulation of extension questions; offer opportunities for first-hand inquiry (explore real-word problems, perform experiments, and conduct investigations and surveys);
- have a social impact (well-being or safety of members of the community); and necessitate interaction with others.

14.4 Theories that Guided the Design of Our Activities

Based on the available literature, I started to look at didactical tools that could help teachers create a meaningful learning environment in which richer (in my study, this means complex and open-ended) tasks give young students a chance to enhance and deepen their mathematical experiences as early as kindergarten.

Although some authors point to the fact that solving mathematically demanding problems requires rich knowledge about numbers and number relationships that is not normally available to elementary school students (see, for example, Lorenz 1994), others (e.g., Krutetskii 1976) affirm that at the age of 7 or 8, gifted children already begin to "mathematize" their environment, giving particular attention to the mathematical aspects of the phenomena they perceive. They realise spatial and quantitative relationships and functional dependencies in a variety of situations: They see the world through mathematical eyes.

These children are eager to learn mathematics, they enjoy it, and teachers should use every opportunity to nurture their fresh young minds. Thus, a special environment has to be created in order to maintain their genuine interest. I shall call this environment challenging, as it is composed of a variety of situations that provoke mathematical questioning, investigations, and use of different strategies; reasoning about problems; and reasoning about reasoning. These situations also require special abilities to organise and re-organise mathematical knowledge to solve new problems by developing and using different strategies.

The theoretical framework draws on the work of Krutetskii (1976), who postulated that a mathematically able child can, at a young age:

- formalize a problem situation by linking logically related data,
- generalize particular cases by combining separate data into more general structures,
- curtail mathematical operations keeping in mind all intermediate steps,
- demonstrate a flexible way of thinking switching easily from one idea to another, and
- rationalize their thinking by critically evaluating different ways to solve a problem.

Krutetskii claimed that these components of mathematical activity are integrated into a specific mental structure called the "mathematical cast of mind." Moreover, drawing on these principles, I assume that mathematically able children, even at a very young age, are inclined to more abstract (theoretical) thinking in their approach to reality and mathematical problem solving; that is, they are likely to engage in thinking for the sake of thinking and not only for the sake of getting things done (reflective thinking), to be concerned with the structure of relations between concepts and not only with concrete objects or actions on them; to be critical with respect to the validity of their own and others' claims; to view mathematical statements as conditional and hypothetical (systemic thinking); and to be aware of the arbitrary and conventional character of representations of concepts (analytic thinking).

I was aware that very young learners may not exhibit all these features of theoretical thinking. However, in a specially constructed learning and teaching situation, children's behaviour can hint at the potential for the development of these features in further advanced mathematical learning.

In his theory of didactical situations, Brousseau (2002) describes the so-called paradox of devolution of situations. He states that in a situation where the teacher

> is induced to tell the student how to solve the given problem or what answer to give, the student, having had neither to make a choice nor to try out any methods nor to modify her own knowledge or beliefs, will not give the expected evidence of the desired acquisition. (p. 41)

But at the same time, the teacher has a social obligation to "teach everything that is necessary about the knowledge. The student—especially when she has failed—asks her for it" (p. 41). This situation is obviously paradoxical:

> The more the teacher gives into her demands and reveals whatever the student wants, and the more she tells her precisely what she must do, the more she risks losing her chance of obtaining the learning which she is in fact aiming for. (p. 41)

Brousseau (2002) thus claims that everything the teacher undertakes in order to make the student produce the behaviours that she expects tends to deprive this student of the necessary conditions for the understanding and the learning of the target notion. Questioning further the nature of the ability to reflect on a mathematical task, I find interesting links between Brousseau's theory and the works of the Russian methodologist Shchedrovitskii (1993) and his colleagues from the Moscow Methodological Circle, which are probably less known to Western educators.

Shchedrovitskii (1993) illustrates his logical analysis of teaching and learning with a similar paradoxical phenomenon. He remarks that when we as educators want our children to master some kind of action, we often tend to teach it directly by giving children tasks which are identical with this action. But classroom practice shows that the children not only do not learn actions that go beyond the tasks, they do not even learn the actions that we teach them within the tasks. The author shows that in direct instruction, the "inputs" (what we teach) and the "outputs" (what children learn) are the same identical actions. He demonstrates that since all elements of this schema are identical, there is no need for the development of "ability," that is, of the possibility of constructing a similar action in different circumstances.

For example, in some textbooks, the teacher can find the task of constructing a square-like shaped surface out of four identical cubic blocks. This activity would present little challenge for children. It does not require any construction of a process of learning as a movement from the known to the unknown. It is, instead, a move from the known (a shape of the square as an image in a child's mind) to the known (reproduction of the shape by means of cubes), but in a different form. In this situation, many children succeed with little effort and few mistakes. In my experiment, I asked my students to produce a square with no given number of cubes, and some of them have not only produced squares of different size but also could make rectangles instead of squares. They might reveal either a lack of understanding of the notion of a square or an inability to keep under control the condition of the "square-ness". According to Shchedrovitskii (1993) this situation has more potential in terms of learning where the need to learn something new would arise from some obstacle students face when create their own solutions.

Shchedrovitskii's logical analysis of the learning of some action is based on the principle that the subjective conditions of activity, or "abilities," are just "copies" of actions mastered by the individual and then appearing in specific new situations. So, the abilities are the same actions but *in potentio* and need to be developed by engaging children in a process of construction of actions that are "new" to them, often in a situation of rupture (for example, facing an obstacle), thus making an active use of their abilities.

The third aspect of nurturing early mathematical ability is related to the idea that, in order to access a higher level of knowledge or understanding, a person has to be able to proceed at once with an integration and reorganisation of previous knowledge. Sieprinska (1994) illustrates it with references to the Piaget's theory of equilibration of cognitive structures as well as to the Bachelard's notion of epistemological obstacle (Piaget 1985; Bachelard 1967). According to this views, a passage, for example, from arithmetic to algebra via whole numbers requires a perception that natural numbers are no longer collections of objects (e.g., pizzas, cakes, or apples) but a structure with operations that can be a base for further generalisations.

Sierpinska (1994) sees the need for "reorganisations" as one of the most serious problems in education. In teaching, we do not follow the students' "natural development" but rather precede it, trying as much as possible, of course, to find ourselves within our students' "zones of proximal development." But we cannot just tell the students "how to reorganise" their previous understanding: We cannot tell them what to change and how to make shifts in focus or generality because we would have to do this in terms of a knowledge they have not acquired yet. So, we must involve students in new problem situations and expect all kind of difficulties, misunderstandings, and obstacles to emerge, and it is our main task as teachers to help the students to overcome these and become aware of differences, in the hope that then the students will be able to make the necessary reorganisation.

All three researchers suggest models of teaching that give the teacher efficient tools for dealing with before mentioned didactical paradox: meaningful didactic situations (Brousseau 2002), construction of new means by means of reflective actions (Shchedrovitskii 1993), and stimulating of "good understanding" (Sierpinska 1994). These ideas guided me in designing the teaching approach I call the *challenging situation approach* (Freiman 2006).

14.5 Challenging Situation Approach to Foster Mathematical Giftedness in Early Grades

The existing literature has shown that challenging problems are seen as suitable for identification of mathematically gifted students, as they reveal strategies and processes of thought (Heinze 2005). Many authors analyse the teacher's role in the process of identification of mathematically able children. Kennard (1998) affirms that the nature of the teacher's role is critical in terms of facilitating pupils' exploration of challenging material. Hence, the identification of very able pupils becomes inextricably linked with both the provision of challenging material and the forms of teacher-pupil interaction capable of revealing key mathematical abilities.

The author supports an interactive and continuous model for providing identification through challenge that integrates the following strands: an interpretative framework employed by classroom teachers to identify mathematically able pupils; selection of appropriately challenging mathematical material; forms of interaction between teachers and pupils that provide opportunities for mathematical characteristics to be recognised and promoted, and continuous provision of opportunities for mathematically able children to respond to challenging material. Hence, the author stresses the need to search for different ways of interacting with pupils that "maximise the opportunities for simultaneously recognising and promoting mathematical abilities" (Kennard 1998). In Kennard's case study based on this model, the identification was conducted by the so-called teacher-researcher in the classroom environment where the pupils were being taught as well as observed. The questioning approach was used in order to reveal aspects of pupils' mathematical approaches and understanding.

In our study, I used a similar methodology. Based on the work of Krutetskii (1976), Sheffield (1999), Shchedrovitskii (1993), Brousseau (2002), and Sierpinska (1994), we designed challenging situations that could stimulate mathematical questioning and investigation along with reflective thinking in young students (Freiman 2010).

We postulated that the use of teaching approaches based on challenging situations would help to engage all students in meaningful learning through early beginning of work on challenging mathematical tasks in 3- to 5-year-olds (fostering precocious minds), stimulating questioning (fostering the critical/reflective mind) encouraging searching for new original ideas by means of open-ended tasks (fostering the creative/investigative mind), promoting full and correct explanations (fostering the logical/systematic mind), introducing children to the complexity and variety of mathematical concepts and methods (fostering looking at the world with mathematical eyes), and providing children with tasks that require complex data organisation and reorganisation (fostering selective/reversible/analytical/structural mind).

Actually, any textbook problem can be turned into either a challenging learning situation or a dull exercise. Challenging situations cannot be used only on exceptional occasions in a teaching approach. Some of them must, of course, be carefully prepared, but, for the approach to work, it must become a style pervading all teaching all the time at all levels of education. The teacher must be ready to use any opportunity that presents itself in class (e.g., a puzzling question posed by a student, an interesting error, or an unusual solution) to interrupt the routine and engage in reflective and investigative activities on the spot or suggest that students think about the problem at home. Thus, in fact, what is needed is not occasional challenging situations, but a "challenging learning environment."

In the challenging situations used in my own teaching, I favour open-ended problems that are situated in conceptual domains that are familiar enough to children that they can appropriate the situations as their own and engage in an interplay of trials and conjectures, examples and counter-examples, and organisations and reorganisations (as stated by Arsac et al. 1988, p. 7). In each situation, I observe various elements of children's mathematical behaviour: how children enter into the situation (introductory stage and pre-organisation) and how different ways of presenting the problem affects the children's work (Brousseau 2002); how children construct their processes of problem solving (choice of strategy, use of manipulative, systematic search, autonomy, self-control, and mathematical components);

how children act in cases of an errors (destroy their previous work and start from scratch or try to modify/correct certain actions); how children modify their strategies when the conditions are slightly or completely changed; how children present their results (orally or in writing, clearly or not, communicating with other participants or not [children or adults], symbolism used by children, and organisation of results [on paper]).

Moreover, the context of the challenging situation allows children to move more quickly beyond the level established by a regular curriculum without losing their interest and motivation to learn more. In fact, our practice demonstrates that within a challenging environment, very young children actively demonstrate their willingness to learn more advanced and abstract topics such as big numbers, zero, infinity, negative numbers, fractions and proportional reasoning, logical inference, variables and functions, shapes and their properties (definitions and proofs), geometric transformations, and equations with missing terms, among many others, by means of the complex and open-ended tasks I describe in the next section.

14.6 Methods: Teaching Experiment in Kindergarten Using Complex and Open-Ended Tasks

The paper uses data collected for a larger study on identification and fostering of mathematically gifted elementary students (Freiman 2006, 2010). The former was conducted at a private school (1–6) in the Montréal area in Québec, Canada, with most students coming from immigrant families, often having neither French nor English as mother tongue. While the school offered an enriched mathematics curriculum for all its students starting from Grade 1, and many students it enrolled were coming for the kindergarten, we (together with the school principal and teachers) decided to enrich the kindergarten curriculum, as well. The need for enriched activities for kindergarten students had become apparent over the years, as the regular curriculum did not provide them with the material necessary for fostering their mathematical development, especially in regard to the requirements of the challenging curriculum that I used starting from Grade 1 (the *Défi mathématique* collection; Lyons and Lyons 1989, 2001–2002). This curriculum was based on the ideas of discovery-based learning, focused on developing of deeper reasoning and understanding. The use of tasks that were complex and open-ended was a regular part of the teaching and learning practice for which kindergarten was not sufficiently preparing the students. The instruction in the kindergarten was bilingual, with a half day in French and a half day in English. The experimental activities were developed and implemented by the author within the French curriculum.

In order to fill the gap, I developed an enriched course offered to all kindergarten students on a weekly basis (one hour per week). In selecting and designing the tasks, I used a challenging situations teaching approach, developing activities that stimulate mathematical questioning and investigations along with reflective thinking. Some of the activities were adapted from the Grade 1 textbook. Others were created by the author. In this paper I present data based on these tasks. During the school year our experiment was conducted, 36 students (27 girls, 9 boys) took part in the activities. My role in the study was as a teacher-researcher (Lesh and Kelly 2000).

Each class started with some general questions referring to students' previous experience such as What did we do last time? What problem did we have to solve? What was our way of dealing with the problem? What strategies did we use? This questioning aimed at provoking reflection on the problems that the children solved as well as on the methods that they used. Without this reflection, a rupture situation (in Shchedrovitskii's (1993) sense) would never arise, because a rupture is a break with previous knowledge, which needs to be brought to mind.

At the same time, I asked questions aimed at demonstrating the level of the children's understanding of the underlying mathematical concepts or methods that I was introducing (using appropriate vocabulary and/or symbolism). During this initial discussion I usually tried to bring in a new aspect that would provide children with an opportunity to ask new questions and look at the problem in a different way. Sometimes I could simply ask them what they thought we should do that day.

Thus, I was able to pass to the new situation/new problem/new aspect of the old problem. To do so, I used provoking questions, interesting stories, or introductory games. Following Shchedrovitskii (1993) and Brousseau (2002), we tried to avoid the teaching paradox by not providing children with direct descriptions of the tasks or methods of solutions while keeping them open and with increased levels of complexity. I also tried to keep their attention and motivate them to investigate these tasks with more depth and better understanding of underlying mathematical structures and relationships [in the sense of Sheffield (1999) and Sierpinska (1994)] and eventually to develop their mathematical abilities [following Krutetskii's work (1976)].

After the introductory stage of an activity, children began investigating a problem using different manipulatives: cubes, geometrical blocks, pebbles, etc. They worked alone or in small groups. During this phase of investigation, the role of the teacher became more modest: we gave children a certain amount of autonomy so that they could familiarize themselves with the problem, choose necessary materials, organise their work environment, and choose an appropriate strategy.

However, some work had to be done by the teacher to guide children through their actions. I had to make sure that the child understood the problem, the conditions that were given (rules of the game), and the goal of the activity. As the children moved ahead, I tried to verify their levels of control of the situation, i.e., what they were doing at any given time and what the purposes of their actions (activating reflective action) were. I had to keep in mind that exploration is used not only as a way to direct children towards performing certain actions, but also it is primarily an introduction to mathematical concepts or methods.

Therefore, the teacher needed to be prepared to introduce the necessary mathematical vocabulary along with its mathematical meaning as well as mathematical methods of reasoning about the concepts and about the reasoning. In this experiment, I tried to choose those mathematical aspects that are considered difficult and are not normally included in the kindergarten curriculum.

For example, when I wanted to introduce an activity with patterns, I would organise a game involving some physical activity. I would start putting students in a line with the order "boy, girl, boy, girl," etc., asking them who should be the next person in the line. Many children were actively involved in the task of building the line and were happy to discover a pattern. As the game goes on, children get used to looking for familiar patterns. This was the time to challenge them more. For example, we would ask them how many children would be in a line with the pattern "boy, girl, boy, girl," etc. Since there were only eight boys in the classroom, one child made the hypothesis that there would be 8 + 8 children in the line. After such a line had been completed, the teacher's silence was broken by a child's voice: "we can add one more child to the line—a girl in the beginning." Then I started a new "pattern": "boy, girl, boy, girl, boy, boy." Many children protested, saying that the pattern was wrong. Some of them, however, started thinking of a different pattern, such as "glasses, no glasses, glasses, no glasses," etc.

The situations were designed to give children an opportunity to take a different look at the mathematical activities that they usually do; question their knowledge about mathematics, trying to discover hidden links between different objects; discover structures and relationships between data; and learn to reason mathematically based on logical inference, while at the same leaving some space for the children's mathematical creativity. I used different didactical variables (in Brousseau's sense) in order to create obstacles that would make the children reorganise their knowledge and create new means in order to overcome the obstacle [with reference to the work of Schedrovitskii (1993) and Sierpinska (1994)].

The data of this study were collected over the period of 3 months (from March to May) and included video-recording and samples of students' work. The analysis I present in this paper is based on the samples of students' work (reports written by each student after each activity where she was asked to explain her strategy and her solution); the researcher also asked students questions about what they were doing and why and how they were doing it; these conversations were video-recorded to capture students' oral explanations (which were transcribed). Finally, field notes were used to mark observations of students' work during the activity (what they were doing and how).

I will now analyse three examples of challenging situations for which the data were collected. While focusing mainly on students' authentic work within a complex and open-ended task, the focus of the analysis was on a multitude of solutions in terms of the variety of mathematical structures they created and the diversity of their problem-solving strategies. The data analysis included classification of students' work according to the type of strategies they were using during the investigation of an open-ended task (systematic or non-systematic, see Situations 1 and 2), type of mathematical representation in students' drawing (more concrete forms, illustrating real animals, or more abstract forms, illustrating models they used; use of numbers to express solution, disposition of the objects, type of mathematical structure—like 1 to 1 correspondence, see Situation 3). According to the theoretical framework and the study design, I attribute this variety to the openness of the task as well as to its complexity, which also allowed students to demonstrate their (in some cases precocious) mathematical abilities.

14.7 Analysis of Challenging Situations and Students' Mathematical Behaviour

Situation 1. Breaking down numbers into different sums

The task given to the students arises from an activity suggested in the Challenging Mathematics Grade 1 textbook (Lyons and Lyons 1989, 2001–2002) that is formulated in the following way: Find all possibilities for dividing five counters into two groups. The textbook presents empty boxes in which right and left hands are drawn. For each pair of hands, children are asked to draw a total of five buttons in each hand with a different combination for each pair (like 2 buttons drawn in the left hand and 3 buttons drawn in the right hand).

As shown in many studies in mathematics education and psychology, children have difficulty learning numbers because of their limited conceptual understanding. For instance, children may see addition as a "counting on" with the necessity of getting a final result: the sum. Therefore, they do not understand that the expression $2 + 3 = 1 + 4$ makes sense. They think that $2 + 3$ can only be equal to 5. Thus, the operational structure of numbers might be inaccessible to young learners. This type of obstacle, which is related to making sense of the equal sign in such equations such as $8 + 4 = \cdots + 5$, has been analysed by a number of authors (e.g., Carpenter et al. 2003; Knuth et al. 2006). In our study of the topic with kindergarten, Grade 3, and Grade 6 children, we found consistent difficulties with equations $c = a + \cdots$ and $a + b = c + \cdots$ and even more surprisingly, we found children's misconceptions appeared more often in higher grades (Freiman and Lee 2004).

Based on this research, we found it useful to start incorporating structural aspects of numbers in mathematics teaching from a young age and pursue the work later on. By asking kindergarten students to find different ways of breaking 5 down into a sum of two numbers, I aimed to introduce different properties of numbers (such as "five-ness") and operations (such as the commutative law and the neutral element), thus bringing more complexity to mathematical experiences of young children.

The situation was presented in a following way: The teacher shows the children 5 counters (all are blue circles 3 cm in diameter) and asks them what one could do with them. The students make their guesses. The teacher does not make any comments while starting to play with counters, putting some of them in one hand and some in the other.

During the teacher's demonstration with hands, a new didactic variable (*variable didactique*, Brousseau 2002) is introduced to try to direct the children's search towards a pre-planned activity. The students commented on these actions in attempts to guess what this could mean in terms of mathematical tasks (the children knew that since this was a math class, the activity must be mathematical). So, they proposed counting counters, ordering them, and using them as instrument to draw circles following the border of the button. Here was an emergence of spontaneous brain-storming thinking processes: They tried to predict the possible nature of a mathematical activity yet unknown to them.

They could use their previous experience in order to build links between different activities. Thus, they had an opportunity to ask questions, make hypotheses, and learn about the rules of the "game" (instructions to follow and conditions to respect). Finally, they approached the formulation of the problem. In this very beginning stage of activity, I could already identify children who participated more actively in the discussion and manifested their interest, understanding, and insight more explicitly than others.

I observed that some children looked a bit confused as they saw that neither counting nor drawing happened. They had to adjust their guesses to the new situation. At one point, a child noted: "you are always changing the number of counters in each hand"; thus the word *partition* was used by the teacher. This made it possible to talk about different ways to divide counters. At the same time, I did not tell the children how to do this or how to validate a solution; thus, I let the children organise the process of solution, construct necessary tools, and try different strategies. Allowing children to have this openness and autonomy is very important for any learning to occur (Brousseau 2002; Shchedrovitskii 1993).

Finally, all students understood that the goal of the task was finding all the ways to do partitioning of counters into two groups. Still nothing had been said about the possibility of using the number 0 as an option, permission to commute terms such as 2 + 3 and 3 + 2, or what one can call two different partitions. The students worked on the problem in the following setting: Children sat around six tables with five to six children per table. They had to communicate their solutions by drawing them in the *Défi-1* workbook (Lyons and Lyons 2001). The researcher and two teachers moved from one table to another checking on the children's work and giving them some neutral hints (such as "look, you already found this solution; would you try to find another one?").

Our observations during the activity showed that there was variation in the way the children organised their work. The children put the counters on the table in a variety of different configurations: lines, squares, circles, and towers. I interpret this spontaneous organisation of material as an important indicator of mathematical ability (thinking in terms of structures). This could be a sign of a high level of thinking discipline. It is also plausible to suggest that the partition is being made in the child's head at this moment. It could also indicate that a child grasps the general mathematical structure of the task. Other children start immediately to move the counters from one hand to the other, imitating the teacher's demonstration. There could be two kinds of explanations of this phenomenon: They are trying to understand the problem or they have the need to simply touch the counters.

However, I could already evaluate the children's readiness to solve the problem: some of them seemed to know how to proceed, while others showed signs of confusion. While looking for solutions, some children formed two groups of counters on the table; others kept the counters in their hands. In their work organisation, I saw very different approaches: Some children were very systematic and orderly and others were chaotic and messy. While several children understood the need to verify the solutions (respecting the conditions of five counters and different partitions) and were able to do it, some children could not do it even with help of the teachers.

It is interesting to note that I did not explicitly ask children to use any ordered disposition of counters in their drawings. However, I observed some particular dispositions of circles (domino, rows, circle, and triangle); in challenging situations, some children give themselves new tasks or new interpretations of the task, thus moving beyond pre-planned activity.

Figures 14.1 and 14.2 schematically reproduce examples of the children's written work, with two schemas representing the more systematic approaches in Annie's and Fannie's work (Fig. 14.1) and one (Yvan's work) representing a non-systematic approach where the child loses control of some of conditions (e.g., one repartition appears twice with more than 5 buttons; Fig. 14.2).

One more observation: While some children could keep the entire process under control, others lost control when passing from the manipulation to the communication on paper. They considered it a different task: focusing on particular aspects of drawing instead of on the mathematical "parameters" of the task. They could even draw a completely different partition from the one found with the counters.

Our particular attention was drawn to the few students who proceeded systematically in their search of all the possibilities (e.g., 5 and 0, 4 and 1, 3 and 2), considered zero a significant element in their partition (neutral element property), made a distinction between a + b and b + a cases (commutative property), and kept the total number of counters constant (addition as operation). These students not only succeeded in this partition problem but were also able to solve the similar problem that came next (three counters) without any reference to using manipulatives; they just wrote all solutions and seemed to have grasped a general mathematical structure in a more abstract form, unlike other children, who started to solve the problem of partitioning the set of three counters in the same way that they approached the previous one: placing counters in their hands and trial and error partitions.

Situation 2. Discovering square numbers

Including complex concepts at lower levels of schooling than required in the curricula is known as potentially enriching for mathematically gifted learners. An example is Diezmann and English's (2001) study, where they describe young children fascination with multi-digit numbers and the challenges they encounter while working with them earlier than prescribed in the curriculum, starting at 5 years old.

Discovering and investigating square numbers is another example of a task that is not typical for kindergarten students; most curricula introduce them later in

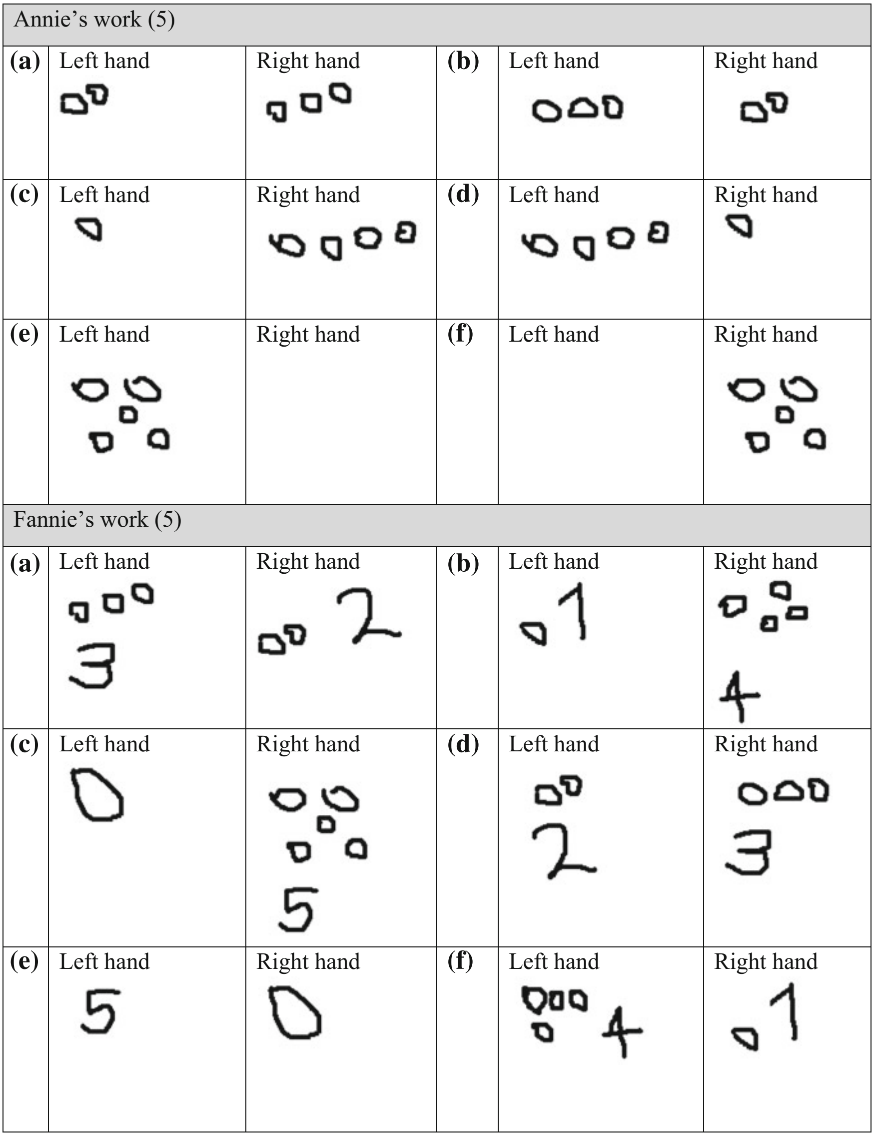

Fig. 14.1 Kindergarten children's systematic approach to breaking down the number 5

elementary school in Grades 4 or 5, along with even, odd, and prime numbers. In our experiment, I asked students to build squares with small cubes (centicubes). Each child had about 30 cubes to work with.

The whole-class discussion started with some general questions about geometric shapes children were familiar with: Children were asked to compare different shapes, explain how they recognized a particular type of shape, and describe the

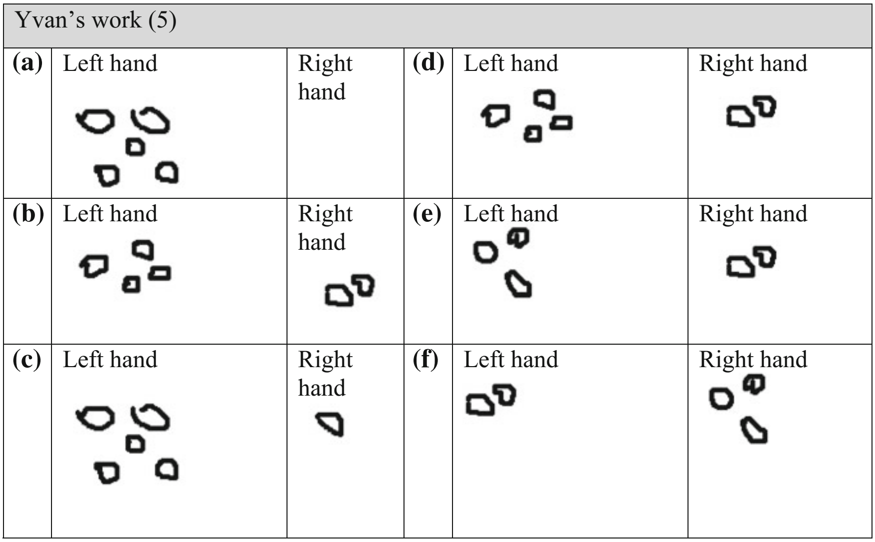

Fig. 14.2 Kindergarten children's non-systematic approach to breaking down the number 5

important characteristics of these shapes. I was trying to focus children's attention on the notion of "square" without asking how to construct one. By giving as few instructions on how to proceed in building squares as possible, I was trying to keep the task as open as possible, thus leaving children on their own. I asked only that they make a protocol of their investigation by drawing their constructions and writing down the number of cubes used in each construction. The work was done individually by each child.

In our observation of students' work, I focused on the structures they were building and the organization of their work. During the activity, the teacher's role was limited to pointing out errors in construction, e.g.: Why do you think it is a square? Are you sure it is a square? No indications were given about where exactly the error was or how to correct it. Sometimes we saw children building the contour of a square (using only the perimeter). In such cases, I asked students to continue "filling in" the whole square (leaving no "holes" inside it).

Some students started to build a "big" square using all their cubes. This strategy led them immediately to the quite complicated task of keeping "square-ness" and counting shapes. Other students started with a small number of cubes, making 2×2 or 3×3 constructions. Some of the children who had built their first square (such as 2×2) continued adding new blocks to their constructions to get new squares (such as 3×3, 4×4, and so on). Others started from scratch; using this strategy, they often lost control of the square-ness of their shapes. I observed the difficulties these children had in organizing and, if necessary, re-organizing their work. One child was able to construct a "pyramid" with base of 6×6 and each square level decreasing by 1 until 2×2. No child thought of one cube as a "square" number.

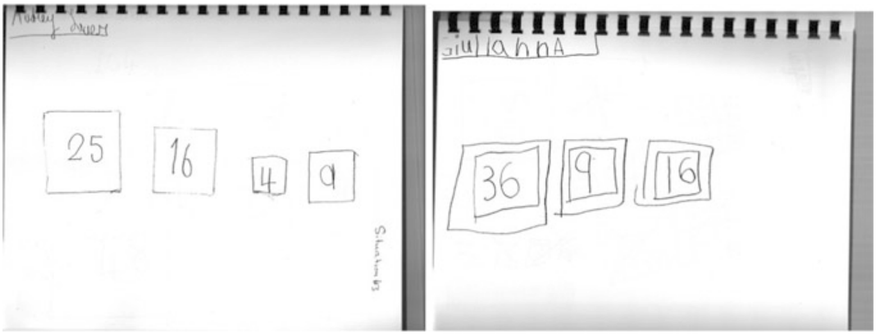

Fig. 14.3 Investigations with systematic search

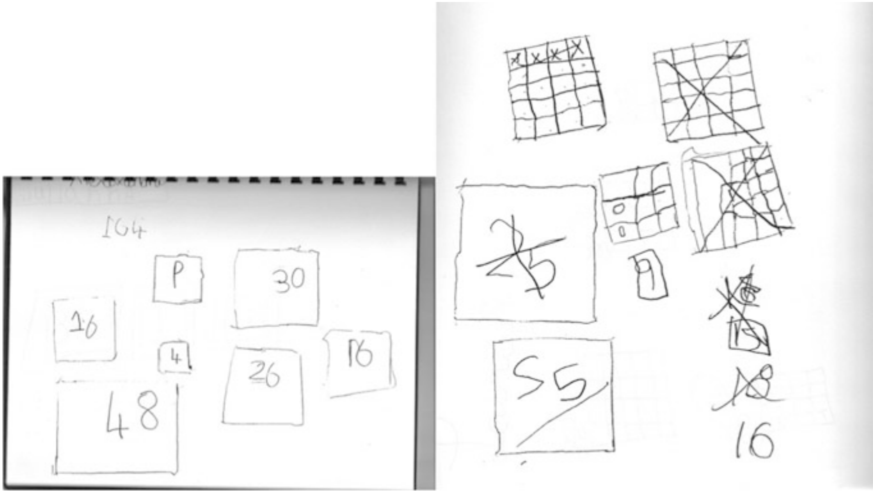

Fig. 14.4 Investigations with less systematic search

Figure 14.3, presents the students' reported representations of their constructions. Some children show very systematic work with a high level of metacognitive control of the task conditions (Fig. 14.3), while others showed rather messy work with several errors, although some presented some correct solutions (Figs. 14.4 and 14.5).

Situation 3. Rabbits and Carrots

This situation consisted of three different parts. It started in the approach to Easter, which in the local culture involves children visiting animal farms. Our initial conversation with children was therefore very open: I only asked students about their experiences visiting farms and asked for their observations. When rabbits were mentioned, I asked about the food they are given and then presented the children with counters and coffee sticks. At the same time, I asked them for ideas for a

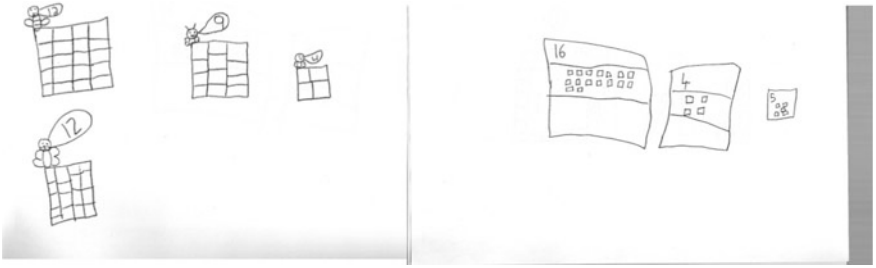

Fig. 14.5 Investigations where students lost control of counting the blocks in the square (such as 12 and 9 on the left) or building an array ($n \times n$, such as 16 and 5 on the right)

mathematical activity related to this context. Students suggested several things showing much creativity, for example, counting (adding) rabbits and building rabbits using a counter for the nose, two sticks for ears, and four counters for the legs.

Having in mind the importance of developing more efficient and advanced strategies in counting, three tasks I elaborated included (1) using manipulatives to compare two sets of physical objects in a task involving having enough carrots for each rabbit, (2) doing the same task but with pictures representing carrots and rabbits so that it would be necessary for students to develop a strategy of comparison without manipulation of physical objects (changing of the didactical variable), and (3) having rabbits of different sizes, which would add complexity in that the rule for carrot distribution would be based on the size of the rabbit: bigger rabbits get more carrots and smaller rabbits get fewer.

When presenting the first task to the students, I suggested that a counter should represent a rabbit and a stick represent a carrot. Each child had a plastic glass with a certain number of sticks (varying from 5 to 20) and a certain number of counters (varying from 5 to 20) on a plastic plate. The task was to feed each rabbit the same number of carrots. The task was formulated by the students. No explanations on how to approach the task were given. Students were able to choose their own ways to represent the problem.

Unsurprisingly, all students tried to produce a 1 to 1 correspondence on the table, but their representations were different. For instance, when placing the counter with the stick, the three most common structures were a counter at the end of a stick, a counter in the middle of a stick, and a counter beside a stick. The five most common ways the students placed these combinations (or pairs) of sticks and counters were in line, in a chain, in several rows, no particular order, and following the border of the table. The three possible cases that students had to deal with were not enough rabbits (more carrots than rabbits; in this case, students had to investigate if it was possible to give two carrots per rabbit), not enough carrots (students had to explain how to fix the issue, for example, tell how many carrots were missing), or the same number of carrots and rabbits (which means there was exactly one carrot per rabbit).

Generally, all students succeeded the task in some way. In one case, a student made the small mistake of giving two carrots to one of the rabbits, but then realized the error and succeeded in correcting it. Another student was very concerned with a lack of carrots: She was very stressed that some of her rabbits would not have food. She was happy to get additional carrots from her peer who had more carrots than rabbits, so the getting help from her peer allowing her to overcome this additional challenge and find a solution she was satisfied with.

The students' oral explanations included several interesting remarks. The importance feeding each rabbit was a theme that came up often (see some examples of students' remarks):

> *I put one carrot for each rabbit: They will all have something to eat.*
>
> *I gave one carrot to each rabbit.* [indicates this with her finger while moving over the construction]
>
> *Each rabbit has eaten one carrot.*
>
> *I placed carrots and rabbits side-by-side* [to make sure each rabbit has a carrot].

Some children showed their happiness in feeding each rabbit while making the observation that there were more carrots than rabbits, so some carrots remained:

> (remark) *I have 5 carrots too many*
>
> (explanation) *Each rabbit has one carrot, there is enough for everyone, but some other carrots remain*

One student shared a comment regarding the possibility to give two carrots per rabbit:

> *Two carrots (are given) because rabbits like to eat a lot of carrots.*

Having insufficient carrots for their rabbits was another example of students' concerns:

> [issue] One rabbit has been left [without a carrot].
>
> I did not have enough carrots; I've added some.

In their final remarks in their oral explanations, I noticed that some children used numbers in their explanations:

> I have 42 rabbits and one carrot [for each of them].
> I have 19 rabbits:
>
> Teacher: How many carrots?
> Student: *19*.
> Teacher: Do you think you have enough?
> Student: *Yes*.
> Teacher: Why do you think it is enough?
> Student: *There are no more left*.

However, it was obvious that some students were having difficulty explaining their solutions; sometimes their explanations did not follow their constructions:

I do not know.

I counted with my hands [fingers].

One child remembers correcting her mistake on the teacher's demand:

I added more carrots since I do not want to have two rabbits at a time [which means two rabbits for one carrot].

On student mentioned that she was not able to give a second carrot to the rabbits:

I cannot give a second one since there will be not enough for every rabbit.

From the videos, I also learn that some students spent a great deal of time constructing some particular configurations to represent the problem. Some also made the decision to modify configuration during the activity.

After the hands-on activity (and making sure all students' constructions were put away), I asked students to reflect on the activity by explaining it in a written report to a friend from another group who did not do the activity. This reflective activity [in Shchdrovistskii's sense (1968)] allowed the researcher to observe in greater detail the mathematical structures created by the students representing the solution.

Dismantling the students' initial constructions allowed us to see whether they would reproduce their construction exactly the way they did it or the way to represented the problem. Indeed, some of the students not only tried to exactly reproduce their configuration, some even drew the shapes of the sticks and for the counters. Regarding the mathematical structure, 24 of the 32 students produced

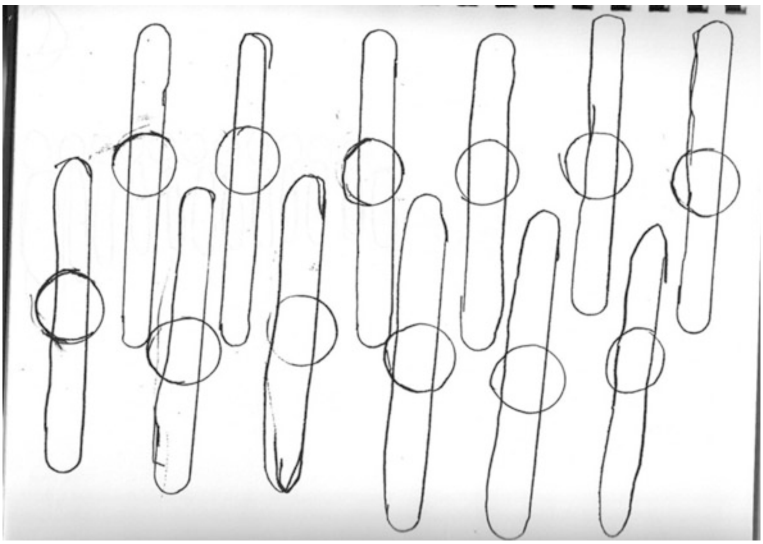

Fig. 14.6 A copy of a hands-on investigation reproduced on paper

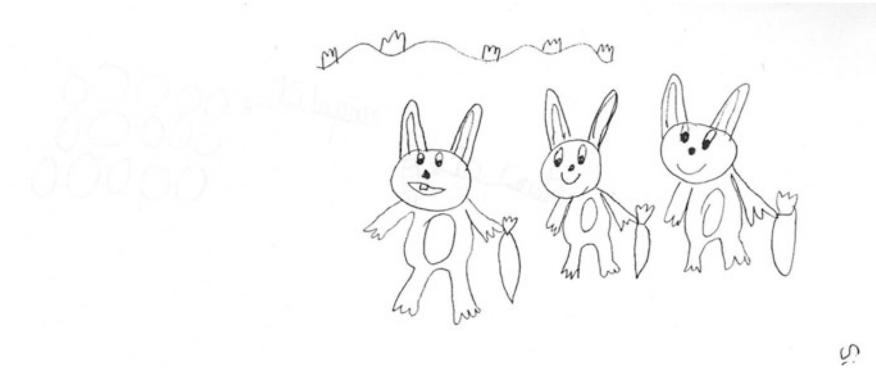

Fig. 14.7 Drawing representing "real" rabbits and carrots with five carrots missing but planted behind

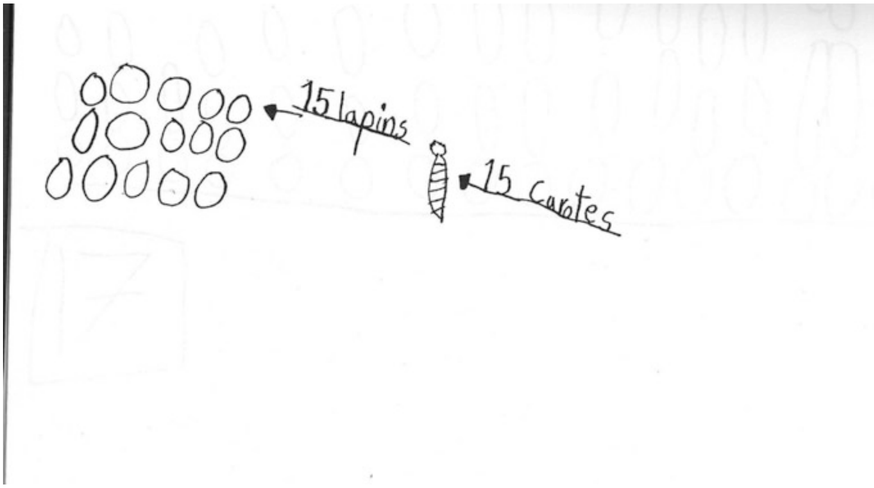

Fig. 14.8 Use of numbers in the report

a 1 to 1 correspondence and 8 made an exact copy of their initial work from memory (Fig. 14.6). One student who was very concerned with having five more carrots took care to draw them on her report (Fig. 14.7). Eleven reports did not have apparent links to their initial constructions. Thirteen students replaced the counters and sticks with drawings of "real" rabbits and carrots. Ten reports used numbers to illustrate the students' solutions (Fig. 14.8).

As a final observation from this activity, I would mention yet another possible extension. To the students who completed their task before their colleagues I gave an additional task: I told them that each counter represented one leg of a rabbit. So the task was the same but with increased complexity. Only a few students who tried it managed to solve it correctly.

The second task, one week later, allowed the introduction of a different didactical variable (in Brousseau's sense): Students were given hand-outs (on paper) with pictures of bags of carrots (represented first by a picture of the rabbit and then, in more abstract way, by a tally) and rabbits sitting around the bag (represented first by a picture of a carrot, then, in a more abstract way, by small circles). The task remained the same: Feed each rabbit with the same number of carrots. But this time objects were not given. Students had to reorganise their strategies (in Sierpinska's sense) in order to be able to verify the correspondence (carrot to rabbit) or to use some way of counting.

Our observations showed that all students first tried to draw a line connecting a rabbit to a carrot—often losing the path when the drawing became too messy. As a strategy, they tried to connect the rabbit to the carrot closest to it. The tasks with pictures that had smaller numbers of carrots and rabbits were easier for the students, and the configurations that were more organised were more easily handled by the students. In some cases, I saw students' make an effort to group the carrots (by twos). When the number of carrots did not match the number of rabbits, students tried (unlike in the situation with objects) to give more carrots to one rabbit (when there was a surplus) or divide carrots into two parts, giving a part to each rabbit (fractioning).

The third part of the activity presented a different task. There were now families of rabbits (father, mother, and baby), with each family member represented by a counter of a different size: small, baby; medium, mother; large, father. Each child had 12 carrots, each represented by a coffee stick. The task was to give all the carrots to the rabbits in a way that the fathers got more carrots than the mothers and the babies got fewer carrots than each of their parents. Having three conditions to control (total number of rabbits, size of the rabbits, and the number of carrots according to the size) many students faced important challenges and only a few (6 of the 24 students participating in the activity) could find the necessary distribution of 3 to the baby, 4 to the mother, and 5 to the father. Several students were able to follow the size and distribution according to the size conditions but got lost on the total number of carrots, while others struggled with the first two conditions.

14.8 Discussion and Conclusion

Data collected during our exploratory study about learning opportunities provided by open-ended mathematical tasks during enrichment activities in kindergarten (Question 1) and the development of students' mathematical abilities (Question 2) have provided some insights about mathematical structures students discover and explore as well as a variety of strategies they use while solving complex problems. Situation 1 provides evidence of the use of the commutative property by some students within a task involving breaking down a number (addition), which has also been reported by a number of studies National Research Council (2009) that mention young students' informal use of the commutative property of addition. The possibility of discovering square numbers using manipulations with cubes by young children in Situation 2 had already been noted at the end of the 19th century by Massey (1895) in the context of the use in kindergarten of Froebel Gifts, which allow young learners to explore complex mathematical structures involving numbers. Situation 3, which was even more complex in terms of didactical engineering (referring to Brousseau 2002), showed a variety of mathematical structures constructed by our students and methods they used to determine whether there would be enough food for the rabbits. The richness of such inquiry-based activities for deeper conceptual understanding of mathematics confirms, among others, Garcia and Ruiz-Higueras's (2013) study, which investigated quantification skills and number sense in pre-school children doing a population modelling task with silkworms, with numbers emerging naturally within a communicative situation (similar to ones I designed in my study). It also corroborates the work by Skwarchuk et al. (2016), who reviewed a number of studies revealing the importance of inquiry-based guidance of 3- to 6-year-old children using open-ended questioning (similar to method used in my approach), which seems to be helpful to facilitate problem solving, reasoning, and learning by discovery (p. 141).

Regarding the second question of our study, it is no surprise that the multitude of learning paths prompted by an open-ended task led to a multitude of strategies and solutions, some correct and others not, that may indicate significant differences in students' abilities, as also noted by Skwarchuk et al. (2016). For instance, in the first situation, I can detect some strategies that show the more systematic approach used by some students when finding different ways of breaking down the number 5. Some students already show that their grasp of number sense in terms of additive structures, the commutative property, and the concept of zero goes beyond the stages identified by Piaget (1972) for this age category.

The second situation presented particular difficulty in terms of the ability to control several conditions of the task (such as the "square-ness" of the shape and the number of cubes used). But what is more striking in this context is the fact that the situation stimulated investigation in all students in the classroom, showing their interest and engagement beyond concern with the immediate result (i.e., whether the solution was correct or not). Along with Balfanz et al. (2003), I noticed some inconsistency in students' results over a long period of time (from one task to another). A child may find an ingenious solution one day and fail on a similar task the next day. What is more important, however, again in agreement with Balfanz et al. (2003), is that students are ready for (in terms of the engagement) and excited by a challenging, comprehensive, and developmentally appropriate mathematics program (p. 267), which thus contributes to a rise in a precocious mathematical cast of mind in some children as a precursor of mathematical talent (Krutetskii 1976).

The introduction of new pre-school and kindergarten curricula aimed at increasing conceptual understanding, reasoning, problem-solving, and communication in many countries, such as the example of New Brunswick, Canada, mentioned in the Introduction, brings up the question of how open and complex tasks can enrich the mathematical experiences of young children and thus foster the development of mathematical thinking and creativity in all students (DEECD 2011).

The findings presented in this chapter (which focuses specifically on kindergarten students) provide with the theoretical foundations and practical examples of students' mathematical investigations of open-ended tasks. The show young children's capacity to investigate situations that enable multiple solutions, have different answers, have solutions ranging from simple and immediate to less immediate and more original, are challenging, and can be extended by further questions such as "Why?" and "What if…?" These situations might further enable students' generalization and abstraction, encourage their investigation of a variety of discussions and argumentations and their use of deep mathematical principles, and use and extend students' existing knowledge: These are the kinds of tasks that Hershkovitz et al. (2009) mention as suitable to promote creativity.

The examples analysed in this paper illustrate how these tasks can be implemented with kindergarten students in a cycle of whole-class discussion: individual or small-group investigation (with different manipulative), communication of results, reflection, and further questioning. By introducing these tasks, teachers can create opportunities for more mathematically promising students (referring to Sheffield 1999) to show their personal commitment, interest (affect), and higher ability (Leikin 2009), while using more systematic and efficient strategies and encouraging self-control and self-efficacy in young children can be viewed as pre-cursors of high mathematics achievement in higher grades. The study represents initial yet valuable steps in the research program which was pursued in different educational contexts in 2005–2014 [including the use of virtual problem-solving environments to enrich students' mathematical experiences, Freiman (2009)] thus helping to increase understanding of the potential of open and complex tasks to enhance the development of mathematical high achievers from an early age.

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Commentary Paper: A Reflection on Mathematical Creativity and Giftedness

Linda Jensen Sheffield

Abstract This chapter is a commentary on the earlier chapters in this monograph, which grew out of the 13th International Congress on Mathematical Education (ICME) Topic Study Groups (TSGs) on *Activities for, and Research on, Mathematically Gifted Students* (TSG 4) and *Mathematics and Creativity* (TSG 29). This chapter begins with a brief historical review of some of the research on mathematically gifted, creative, talented and promising students and then focuses on comparing and commenting on three main areas that are discussed in the earlier chapters: (1) Perspectives on Mathematical Creativity and Giftedness, (2) Connections between Creativity and Giftedness in Mathematics, and (3) The Learning Environment. The chapter concludes with a discussion of implications for educational policies and opportunities based on some of the findings in these chapters and gives suggestions for further study.

Keywords Mathematical creativity · Mathematical giftedness · Exceptional mathematical promise · Heuristics Problem solving · Problem posing

Introduction

In July 2016, the 13th International Congress on Mathematical Education (ICME) met in Hamburg, Germany. During the Congress, Topic Study Group 4 (TSG 4) focused on *Activities for, and Research on, Mathematically Gifted Students* and TSG 29 addressed *Mathematics and Creativity.* All ICMEs in the twenty-first century had hosted Topic Study Groups, Working Groups and/or Discussion Groups on mathematical creativity and giftedness, but this was the first to offer each TSG the opportunity to have both a topical survey written before the Congress and a monograph following the Congress. TSG 4 did publish a topical survey before the

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conference, and this current monograph is unique in that it combines the closely related work of two topic study groups—those on mathematical creativity and giftedness. It builds on work presented in both of those TSGs during the conference.

This combination is backed by the ICME affiliated organization, the International Group for Mathematical Creativity and Giftedness (MCG; www.igmcg.org). MCG had its start with a conference in Muenster, Germany in 1999, and officially affiliated with ICME in 2011. The purpose of the group is to bring together mathematics educators, mathematicians, researchers, and others who are interested in nurturing and supporting the development of mathematical creativity and the realization of mathematical promise and mathematical giftedness. This monograph is the latest publication in furtherance of that work.

As noted in several of the chapters in this monograph, research on mathematical giftedness and creativity has gained steam around the world in the last twenty years. As the world becomes more and more inter-connected, with a growing reliance on technology, the nurturing of creative and talented math, science and technology students becomes increasingly important. At the same time, we need to build on all the research in this area that has gone before us. Virtually all the authors in this monograph cite the work of researchers into general creativity and giftedness such as Guilford, Hadamard, Polya, Torrance, and Wallas from the 1940s through the 1960s as well as the work of Krutetskii and others on specific mathematical aptitudes and abilities from the 1960s and 1970s.

In the United States, when Sputnik was launched in the 1950s, a national outcry went up deploring the status of the country and its students in mathematics and science. In the 1960s and 70s millions of dollars were poured into the National Science Foundation and other programs to support the development of gifted and talented mathematics and science students and their teachers, but that dropped off by 1980 when the National Council of Teachers of Mathematics noted that, “The student most neglected, in terms of realizing full potential, is the gifted student of mathematics. Outstanding mathematical ability is a precious societal resource, sorely needed to maintain leadership in a technological world” (NCTM 1980, p. 18). The NCTM followed this with position statements on *Vertical Acceleration* (NCTM 1983) and *Provisions for Mathematically Talented and Gifted Students* (NCTM 1988), but by 1990, those were no longer official NCTM positions. NCTM did not have another official position statement on this topic for nearly thirty years until the NCTM Board approved a position on *Providing Opportunities for Students with Exceptional Mathematical Promise* (NCTM 2016). This built on the twenty-year-old *Report of the NCTM Task Force on Mathematically Promising Students,* students who were defined as “those who have the potential to become the leaders and problem solvers of the future” The Task Force called for a multi-pronged strategy that increases the numbers and levels of mathematically promising students by maximizing their abilities, motivation, beliefs, and experiences/opportunities, all variables that could be changed from their present state (Sheffield et al. 1995). The most recent NCTM position statement notes that they “seek to broaden the range of students identified as “students with exceptional mathematical promise” while acknowledging that each and every student has mathematical promise” (NCTM 2016).

This current monograph, as well as several recent international publications, reflects this broad concept of mathematical creativity and giftedness. Recent works on this topic include the book *Creativity in Mathematics and the Education of Gifted Students* (Leikin et al. 2009), the special issue of *ZDM, The International Journal on Mathematics Education* on Creativity and Mathematics Education (Leikin and Pitta-Pantanzi 2013), the special issue of *ZDM, The International Journal on Mathematics Education* on Mathematical Creativity and Giftedness in Mathematics Education (Singer et al. 2017), and the topical survey, *Research On and Activities for Mathematically Gifted Students* (Singer et al. 2016). Each of these, as well as the proceedings from the conferences of the International Group for Mathematical Creativity and Giftedness, and several other outstanding books and articles that are cited in these chapters have added a great deal to the knowledge base in the field. The stimulating chapters in this monograph continue that proud tradition while adding important new research-based information to this very important field of study.

Perspectives on Mathematical Creativity and Giftedness

The articles by Nolte (2018), Gutierrez et al. (2018), Veilande et al. (2018), Assmus (2018) and Assmus and Fritzler (2018) all contain a discussion of just what is meant by giftedness, and specifically by mathematical giftedness. Several of these address the issue of whether there are links between general intellectual giftedness and specific mathematical giftedness. Assmus (2018), for example, outlines three models—mathematical giftedness as one component of general intellectual giftedness, mathematical giftedness as general intellectual giftedness in combination with specific mathematical abilities, and mathematical giftedness that can occur independently of general intellectual giftedness. Assmus cites a study by Benbow and Minor (1990) of 300 mathematically gifted and 150 linguistically gifted thirteen-year-old students as identified by scores in the top 0.01% on the SAT. Since there were only 18 students in both groups, she infers that you cannot conclude that mathematical giftedness is included in general intellectual giftedness. Nolte (2018) cites Gagne’s Differentiated Model of Giftedness and Talent (2004) in distinguishing between giftedness in at least one domain as untrained and spontaneous natural abilities and talent in at least one field as outstanding mastery of systematically developed competencies. While there is no resolution in these chapters as to whether mathematical giftedness is a subset of, intersecting or separate from general intellectual giftedness, the focus of the chapters is on specific mathematical giftedness in students of different ages.

In discussing characteristics of mathematically gifted students, several authors mention those described by Krutetskii (1976) and others. Gutierrez et al. (2018) expound on the ability to see patterns and relationships, generalize and transfer ideas, quickly learn, understand and apply mathematical ideas, and move from one problem solving strategy to another while Veilande et al. (2018) note the ability to abstract, generalize and discern mathematical structures, think logically, and operate flexibly. In a similar manner, Freiman's study (2018) uses a framework that draws on Krutetskii's description of a mathematically able child who can link logically related ideas, generalize cases, and flexibly switch from one idea to another when solving problems. Note the common strand in these chapters of the ability to generalize from patterns and relationships and move flexibly from one problem solving method or solution path to another. These characteristics are similar across all ages whether authors are describing students in kindergarten such as those studied by Freiman, the nine-year-old described by Gutierrez and associates, or young teenagers such as those in the report of Veilande and colleagues. It is not unusual for mathematicians to define mathematics as the study of patterns, and therefore the ability to see and generalize from patterns is a logical trait of mathematically gifted students.

The commonly mentioned trait of flexibility in mathematically gifted students in these chapters leads naturally to a discussion of mathematical creativity. Pitta-Pantazi, Kattou and Christou have an excellent overview of the state-of-the-art in mathematical creativity in their chapter (Pitta-Pantazi et al. 2018). They use the 4Ps theory from Rhodes (1961) to organize their chapter, describing the components of creativity as:

- Product—the communication of a unique, novel and useful idea and concept
- Person—the cognitive abilities, traits and experiences of the individual
- Process—the methodology that produces a creative product, and
- Press—the relationship between the individual and the environment

Joklitschke et al. (2018) and Assmus (2018) also mention Rhodes 4Ps in the discussions of their research. Even though the authors of the other chapters on mathematical creativity don't specifically mention Rhodes description of these four components, they also discuss several of the same categories.

In analyzing creativity, in a similar manner to giftedness, most of the authors discuss both domain-general creativity, and domain-specific mathematical creativity. As with the relationship of mathematical giftedness to general intellectual giftedness, there seems to be no consensus among researchers of the relationship of domain-general creativity and domain-specific creativity. Of course, in this monograph, the emphasis is specifically on domain-specific mathematical creativity rather than general creativity.

In terms of the product, when studying creativity in the work of school students, an important distinction must be made between creative products that are unique and have a major impact in the world ("Big-C") and those that are unique and novel to the students in their particular environments ("little-c"). Rhodes (1961) acknowledged this when looking at the person and the press, noting that the individual's experiences and relationship to the environment had an effect on the creativity. The chapters in this monograph appropriately most commonly address "little-c" creativity.

In addressing “little-c” creativity, another common theme that appears in the majority of the chapters on creativity is the question of how to assess creativity. Most of these have adopted some form of Guilford’s (1967) description of divergent thinking in his Structure of the Intellect and Torrance’s (1974) Tests of Creative Thinking that define four components of creativity—fluency, originality, flexibility and elaboration (Pitta-Pantazi et al. 2018; Joklitschke et al. 2018; Tabach and Friedlander 2018; Daher and Anabousy 2018; Assmus and Fritzlar 2018). Fluency is generally measured by counting the number of responses to a given task and is distinguished from flexibility, which is defined as the number of different categories of responses to a task. Fluent thinkers generate many ideas and possibilities while flexible thinkers break a preconceived solution path, transform representations, reverse procedures, or transform the problem to find a new way of thinking. Originality is defined as a solution or idea that is unique or different, often measured against others who have been chosen as a comparison or experimental group. Elaboration is based on details that have been added to a solution or idea. This category generally is not addressed in the discussions of mathematical creativity in these chapters, but is occasionally included in discussions of mathematical creativity, sometimes in the form of elegance of solutions rather than elaboration.

Assmus and Fritzler (2018) argue that if fluency is measured as a quick creation of many answers to a mathematical stimulus, students may come up with a high number of very similar solutions that are not creative, and that actually tend to use convergent rather than divergent reasoning.

Voica and Singer (2018) use a slightly different framework based on cognitive flexibility that is inspired by its use in studying organizational settings. They analyze the validity and complexity of responses, in addition to what they term topicality, which is composed of both thematic variability and mathematical variability. Complexity of responses is also a key component of the assessment used by Gutierrez et al. (2018) as they analyze the cognitive effort a student uses in solving problems.

In transferring the analysis of general creativity to mathematical creativity, several of the authors had adopted an assessment similar to that described by Leikin and her colleagues (for example, Leikin et al. 2009; Leikin and Pitta-Pantazi 2013) to quantify mathematical creativity in what they term multiple solution tasks (MSTs) where students are asked to solve a problem in as many ways as possible. The multiple solutions are then assessed for fluency, flexibility, and originality. Elaboration is not scored when using this adaptation of assessing mathematical creativity. Joklitschke et al. (2018) analyze this method of assessing mathematical creativity using two geometry problems and one algebra problem with approximately 20 high school students in a university-based supplemental mathematics class. Solutions are scored two ways—one using a strict analysis where solutions must be completely correct following the guidelines from Leikin and a second time using a more forgiving method where incorrect solutions with relatively minor errors are scored as well. As expected, creativity scores increase using the more forgiving method, but it is unclear which method gives a more accurate picture of a student’s mathematical creativity, although the more forgiving method does appear to give more insight into students' approaches to problem solving. It is interesting to note that there is no significant correlation among the scores on the three different tasks or even between the two geometry tasks. If these MSTs are designed to measure a single domain-specific concept of mathematical creativity, you would expect them to be correlated.

In addition to MSTs, another popular method of assessing mathematical creativity is the use of problem posing where students pose meaningful mathematical problems related to a given situation. (Voica and Singer 2018; Singer et al. 2016; Daher and Anabousy 2018; Sheffield 2009; Pitta-Pantazi et al. 2018; Moraová et al. 2018). In analyzing the work of pre-service teachers, Daher and Anabousy (2018) analyze their flexibility in posing problems related to a paper pool task. Flexibility is also the focus for Voica and Singer (2018) who score problems posed based on cognitive flexibility by analyzing students' problems based on cognitive variety, cognitive novelty and changes in cognitive framing. To further refine their analysis, they also look at the variation in cognitive and procedural complexity. This further refinement is important due to the seeming drop in flexibility as problems posed become more complex. This look at the difficulty or complexity of solutions is also apparent in the scoring schemes of Veilande et al. (2018) as they analyze students' progress in problem solving with repeated participation in Mathematical Olympiads. As solutions and problems posed become more complex, by necessity, the numbers of responses often decrease. Therefore, it is not feasible to identify mathematically creative or gifted students by simply counting different responses.

The multiple definitions of both mathematical giftedness and mathematical creativity and the lack of a clear-cut correlation to general giftedness and creativity point to the difficulty in measuring these constructs. There does seem to be some consensus, however, on the inclusion of flexibility in a measure of mathematical creativity and the inclusion of difficulty or complexity in a measure of mathematical giftedness. It is also worth noting that speed of computation did not seem to be included in any of the assessments, although many of the measures did include tasks with a time limit, often simply due to the amount of time that students attended a given class or program. Different tasks also indicated varied levels of mathematical creativity or giftedness when students' responses to more than one task were analyzed, thus signifying that there is no single agreed-upon instrument to identify gifted or creative mathematics students.

Connections Between Creativity and Giftedness in Mathematics

Perhaps because of the lack of consensus on the definitions and identification instruments for mathematical creativity and mathematical giftedness, it is difficult to determine connections between them. As noted in the ICME-13 Topical Survey on mathematically gifted students, some researchers define mathematical creativity as the highest level of mathematical achievement, and therefore mathematical expertise is a prerequisite for mathematical creativity while others claim that mathematical creativity is one subcomponent of mathematical ability and still others assert that every student is capable of mathematical creativity regardless of the level of mathematical ability or achievement (Singer et al. 2016). Therefore it is important to define just what is meant by mathematical creativity as well as mathematical giftedness in any given study.

Pitta-Pantanzi et al. (2018) expound further on this in a discussion of instruments used to identify mathematically gifted students. They cite researchers who use assessment of creativity as one component in assessing giftedness. This is true of both general creativity and giftedness as well as domain-specific creativity and giftedness in mathematics. In addition, they caution that some researchers warn that emphasizing mastery of mathematical procedures can hinder students' creativity since students might have an over-reliance on memorized algorithms and procedures.

Joklitschke et al. (2018) note that the methods used to evaluate mathematical creativity in students is directly linked to the "little-c" concept of creativity. They point out that solutions are generally scored for originality based upon how uncommon ideas are in relation to the peer group under consideration and not in reference to the larger body of knowledge. Thus even very young children might be considered to be mathematically creative. Freiman (2018), therefore, describes a program that allows kindergarten students to demonstrate their mathematical creativity in response to open-ended problems.

In addition to the responses to open-ended problems from primary students in the study by Freiman (2018), responses from middle school students in Tabach and Friedlander's (2018) investigation, the nine-year-old in Gutierrez et al. (2018) research, and second grade students in the study by Assmus (2018) all demonstrate that young students are quite capable of constructing mathematical knowledge that is new to them and unique in their school settings. Assmus and Fritzler (2018) sum it up stating that there are many different perspectives on the relationship between giftedness and creativity that are not necessarily contradictory but result from different understandings of giftedness and creativity. Their research uses a collection of problems designed to inspire primary students to create mathematical objects and relations, and they include several examples of inventions of arithmetic operations and encryption methods from fourth, fifth and sixth grade students.

The Learning Environment

As noted in the recent NCTM position statement "Students with exceptional mathematical promise must be provided with differentiated instruction in an engaging mathematics learning environment that ignites and enhances their mathematical passions and challenges them to make continuing progress throughout their K–16 schooling and beyond. They must have a variety of opportunities inside and outside of school to develop and expand their mathematical talents, creativity, and passions" (NCTM 2016, p. 1).

Several of the chapters in this volume outline models, heuristics, strategies, programs and tasks designed to do just that. There does not seem to be a significant difference between the strategies and tasks recommended for developing mathematical creativity and those recommended for the development of mathematical giftedness. For example, Assmus and Fritzlar (2018) in their chapter on mathematical giftedness and creativity describe "theory building processes" where the initial problem becomes part of a circular process for problem solving and problem posing. This process is echoed in several of the chapters that describe multiple solution tasks, open-ended problems and problem posing.

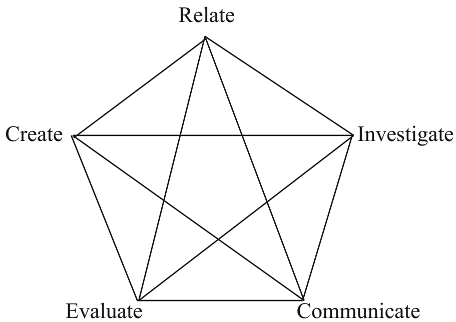
Nearly forty years ago, I described a problem solving heuristic designed for this same goal. Instead of using the straightforward, seemingly linear, four-step heuristic that is based on (sometimes erroneously) Polya's problem solving model that is common in many U.S. mathematics textbooks of (1) Understand the problem; (2) Devise a plan; (3) Carry out the plan; and (4) Check, this model shown in Fig. 14.1 describes a process that is much more open and more closely reflects problem solving and problem posing strategies of career mathematicians.

In solving and posing problems, students may start at any point on the diagram and proceed in any order that makes sense, often repeating stages as the problem is more clearly defined, new strategies and methods of solution arise, and additional questions are examined. They might do the following:

- Relate the problem to other problems that they have solved.
- Investigate the problem using a variety of strategies and models. Think deeply and ask questions.
- Evaluate their findings, not just at the end, but throughout the problem-solving process.
- Communicate their results. Work with colleagues to refine solutions.
- Create other solutions and methods of solution as well as new questions to explore (Jensen 1980).

Many of the tasks discussed in these chapters follow a similar process. A strict adherence to any heuristic or strategy can be problematic, however, and not give

Fig. 14.1 Heuristic for mathematical discovery



students the freedom necessary to be creative and dig deeply into mathematical concepts and relationships. For example, Nolte (2018) brings up several important points related to the often-overlooked issue of twice-exceptional students. She notes that students with autism spectrum disorders (ASD) often have deficits or delays in the development of socialization and communication skills. These students may become fixated on a single solution to a problem and have difficulty or lack desire to find or understand other possible methods or solutions. They may monopolize a class discussion of problem solutions and want everyone to solve problems as they do rather than attempt to understand others. Students with attention deficit disorders (ADD) may have trouble focusing on a problem long enough to work out step-by-step procedures to arrive at a solution. Reading difficulties and auditory and visual processing weaknesses also can contribute to mathematical deficiencies, but as Nolte points out, these are often masked by students' strong intellectual capabilities, and therefore students might show average or above average performance in a mathematics class, but be capable of much more without a teacher being aware of it. Several of the strategies that Nolte recommends for these students are useful strategies to incorporate for others, but are especially important for these twice-exceptional students. This includes giving support as necessary, such as helping students understand and focus on the solution that the problem is looking for, reminding students to look for patterns and understand a pattern that another student is describing, developing a supportive classroom atmosphere, and working with specialists and families on long-term interventions as necessary. She concludes by saying that it is important to expose these students to challenging problems that can be solved in different ways and on several levels, pointing out that for some of these students, especially those with ASD, that some of their social problems dissipate when they are challenged appropriately and given the opportunity to interact with others in the solution or creation of interesting problems. These recommendations are reflected in many of the other chapters as well, even though this is the only chapter specifically focused on twice-exceptional students.

Assmus and Fritzlar (2018) also assert that the tasks they used to investigate mathematical giftedness and creativity are suitable for almost all primary school children. The circular process they describe of problem solving and problem posing with variation, expansion and analysis utilizing novel strategies and tools to build new mathematical objects and structures may be used with most students, even though students' responses may vary.

The majority of chapters that describe tasks used to investigate mathematical creativity or mathematical giftedness discuss the use of open-ended/multiple solution or problem-posing tasks, with several of these authors describing the use of both of these in their programs. Some mathematics educators distinguish between open-beginning, open-middle and open-ended problems or tasks, but that distinction does not appear in these chapters although the concepts, if not the terminology, is used. In these chapters, the terms open-ended or multiple solution tasks are used to describe similar types of challenges. Open beginning tasks start with a situation that can be interpreted mathematically in different ways, and students are encouraged to define and explore according to their own interpretation. Open middle tasks are more well-defined, but a variety of methods or models might be used to solve the problem. Open-ended tasks often have more than one correct answer. For example, on a very simplistic level, instead of a closed task such as “What is the sum of 18 + 25?”, an open beginning task might ask students to list attributes of the expression 18 + 25 and explore patterns of these attributes, an open middle task might ask students to find the sum of 18 + 25 in as many different ways as possible and to compare their models and methods to those of classmates, and an open-ended task might tell the students that the sum of two whole numbers is 43, and ask them to make an organized, exhaustive list of all the possible ways to get this sum using two whole number or two two-digit addends. Each of these might be expanded to problem-posing tasks by asking students to pose and perhaps solve their own related problems after solving the original problem. Teachers can find several interesting tasks of this type designed to challenge students at all levels in these chapters as well as on websites such as www.openmiddle.com. On this website for K-12 students and their teachers, tasks often start with the directions “Use the digits 1 to 9, at most one time each, to fill in the boxes to create a true number sentence.” The equations suggested then range from those designed for kindergarteners, such as “Create the largest possible sum for …”

$$\square + \square = \square$$

to those designed for high school students such as “Make a solution as close to 100 as possible for…”.

$$\int_{\square}^{\square} x^{\square}\,dx$$

Another common theme in the chapters is programs outside of the regular school day. Another Topic Study Group (TSG 30) focused on the related subject of Mathematical Competitions during ICME 13 so that topic is not directly included in this monograph, but several chapters include a discussion of students who have taken part in competitions (for example, Poulos and Mamona-Downs 2018 and Voica and Singer 2018), and Veilande et al. (2018) followed students with repeated participation in Mathematical Olympiads. Competitions are an important means of challenging not only gifted mathematics students but may be enjoyed by a wide range of interested students. Competitions and other extra-curricular activities often give students opportunities that may not be present in the regular mathematics classroom such as the ability to abstract, generalize, discern mathematical structures, and think logically and creatively in solving challenging problems (Veilande et al. 2018).

Technology also played an important part in several of the programs described in this monograph (Pitta-Pantazi et al. 2018; Tabach and Friedlander 2018; Daher and Anabousy 2018; Gutierrez et al. 2018; Poulos and Mamona-Downs 2018). Tabach and Friedlander (2018) discuss the use of spreadsheets as a bridge from arithmetic to algebra with the potential to provide a natural need for using a variety of symbolic expressions. Students displayed a high degree of fluency, flexibility and originality as they wrote expressions to create equivalent columns in a spreadsheet. Their final responses were always correct and did not have the issues of incorrect responses that some of the tasks in studies in other chapters had, since the spreadsheet always gave immediate feedback as to whether the expressions resulted in equivalent numbers in the columns. Gutierrez et al. (2018) also used technology in their work to help a nine-year-old begin to use algebra. They found that the use of a balance applet was useful in helping the student learn to solve simple linear equations.

An applet also was an integral part of the study by Daher and Anabousy (2018). Pre-service teachers were found to be more flexible in their problem posing when using a paper pool applet, especially when this was combined with specific instruction on using a what-if-not strategy. Similarly, Poulos and Mamona-Downs (2018) found that the subjects of their study had difficulty solving the problem they posed without the use of technology (in this case, dynamic geometry software), and their three mathematically gifted students were much more successful when they used the software.

As technology capabilities grow by leaps and bounds every year, teachers and students will be increasingly challenged to keep up, choosing the best means to develop problem solving and problem posing techniques that make the best use of these aides without becoming overly dependent upon them.

Implications for Educational Policies, Opportunities, and Further Study

If we believe that all students have the right to be challenged and engaged in their mathematics programs and to learn something new everyday, then it is important that teachers, parents, students, researchers and other stakeholders learn as much as possible about best practices in supporting and developing mathematical creativity and giftedness. This is not only important to the students themselves, but also for the future of our world that desperately needs leaders who understand and can solve the challenges that we face.

As noted in the NCTM position statement, opportunities to pursue interests, develop expertise, and maintain passion for mathematics must be available to a wide range of students (NCTM 2016). We need to consider at least a dual purpose in the creation and fostering of programs for these students. Students who are currently performing at the highest levels of mathematics need to be challenged to continue to progress while maintaining their interest and passion for the subject. At the same time, we need to increase the numbers, levels, and interests of students who are capable of learning mathematics at a much deeper level.

The chapters in this monograph point to numerous ideas to accomplish these dual goals. The authors of these chapters seemed to have reached consensus on the importance of using open-ended problems, multiple solution tasks, and problem posing activities that encourage depth of knowledge, flexibility and creativity. All of these have the advantage of offering a relatively simple way to differentiate mathematics instruction, using tasks that have a low floor so that nearly all students can engage in solving the problem on some level and can extend this with new questions or solutions that are personally interesting, thus raising the ceiling so students challenge themselves to continually progress in their understanding and mastery of mathematics.

One area of research that was discussed in the TSG 4 topical survey prior to ICME 13 (Singer et al. 2016) that did not receive much attention in these chapters is whether mathematical giftedness is a creation or a discovery. That is, is mathematical giftedness something that students are born with that teachers and others need to discover or is it something that can be "created" in a much larger segment of our student population? If we accept the dual purpose of mathematics education as ensuring that students who are already demonstrating mathematical expertise, as well as those who are not there yet, have opportunities for continuous progress, perhaps this is not a critical distinction. However, far too often in many schools, students who are at the top of the class are bored as they are forced to wait for other to catch up or to act as assistant teachers in helping other students master mathematics concepts that they themselves understood years earlier. On the other hand, students who are capable of far higher levels of mathematical creativity and expertise may never be given the opportunity to express and develop these abilities.

I am reminded of a Javits grant that we received many years ago from the United States Department of Education (USDoE) to nurture mathematical talent in elementary students, Project M^3: Mentoring Mathematical Minds (www.projectm3.org). When applying for the grant, I approached a local elementary school principal about the possibility of this program for the gifted students in his school. His reply was that there were no gifted students in the school, and he asked me if I just wanted him to create some. Even though I was pretty sure he was being sarcastic, after thinking about it for a brief period, I replied, "Yes, let's try that." Somewhat surprisingly, he agreed, and the teachers were willing, so we proceeded to identify twenty of the fifty second grade students in his schools to take part in our "gifted" mathematics program the following year. After selecting students that we deemed to be in the top 40% of their cohort group, at the beginning of third grade, we gave them a well-respected standardized achievement test, the mathematics section of the Iowa Test of Basic Skills, to get a baseline on their basic mathematics skills in concepts, problem solving and computation. Their overall mathematics scores had a mean at the 23rd percentile, meaning that 77% of beginning third graders in the United States scored at a higher level on this test. You can imagine the parents' surprise when they were notified that their children had been selected for a gifted mathematics program. By the end of the first year in the program, however, this mean had increased to the 71st percentile, by far the highest growth scores of any of the schools in the program, where students in other schools had beginning means above the 95th percentile. All experimental groups in the study, including those with very high initial scores showed significant progress in their mathematical understanding, especially on open response questions adapted from released items from the National Assessment of Educational Progress (NAEP) and the Trends in International Mathematics and Science Study (TIMSS) (Gavin et al. 2007, 2009; Sheffield and Gavin 2006).

Following the success of Project M^3, we later received a grant from the National Science Foundation (NSF) to extend the curriculum to the primary grades for five- to eight-year-old students in randomly assigned heterogeneous classrooms, Project M^2: Mentoring Young Mathematicians (www.projectm2.org). One of the things we looked for with this grant was whether a curriculum that incorporated problem solving and problem posing with open-ended tasks utilizing oral and written discourse that asked students to investigate geometry and measurement concepts and relationships could result in increased numbers of students as much as two standard deviations above the mean on an open response assessment that incorporated released items from NAEP and TIMSS. We found that as many as 7% of the students in the Project M^2 program scored more than two standard deviations above the mean, while only 0.5% of students in the control group scored at this level. The Cohen's d effect size for students in the M^2 program was as high as 2.68 compared to students in the control group (Casa et al. 2017; Gavin et al. 2013a, b; Sheffield et al. 2012). Success with these programs lends credence to the claim that gifted mathematics students can be created, or at least that strong mathematics programs that expect and support students to attain significantly higher mathematics achievement can be successful.

This ability for students to progress mathematically to far higher levels supports the Report of the Task Force on the Mathematically Promising that mathematical promise is a function of ability, motivation, belief, and experiences and opportunities, all variables that can be maximized (Sheffield et al. 1995). The fact that ability can expand as the brain changes and grows depending on experiences is important to understand. Over twenty years later, NCTM reiterated this in their position statement on students with exceptional mathematical promise by asserting "mathematical promise is not a fixed trait; rather it is fluid, dynamic, and can grow and be developed" (NCTM 2016, pp. 1–2). The ability to be successful learning mathematics is not something that is fixed at birth, and adults who claim that they or their children do not have a mathematical mind can undermine students' mathematical accomplishments. As found by Dweck in her research on a growth mindset vs. a fixed mindset, when students believe that their brains change and grow, they are capable of changing their learning pathways and achieving at much higher levels (Dweck 2006). Boaler has applied this growth mindset to mathematics, noting that it is especially critical for mathematics students to understand that brains grow and change more when making mistakes, and that brain activity after making a mistake solving a math problem is greater in students with a growth mindset than in students with a fixed mindset. It can be dangerous, however, to claim as Boaler does that it is a myth that some students are mathematically gifted and therefore to infer that there should not be programs for mathematically gifted students (Boaler 2016, p. 94). Some might take that to mean that we do not need to differentiate mathematics programs or offer challenging mathematical opportunities for our highest achieving students.

We need to remember that there are students in each class who are performing at much higher levels than their classmates, and that these students need to be given experiences that allow them to engage in productive struggle and make mistakes with increasingly difficult and complex mathematical problems. They also need time to discuss findings and questions that arise with teachers, mentors, and peers who are at a similar level of understanding. It is worth noting that Boaler's first recommendation for teaching heterogeneous groups is very similar to the recommendations in these chapters for teaching mathematically gifted and creative students. That is that students need the opportunity take mathematics to different levels and not be given closed problems that are suitable for only a small subset of the class, where some will fail and others will not be challenged. She claims that it is "imperative that tasks are open-ended, with a low floor and a high ceiling" (Boaler 2016, p. 115). She makes a strong case for de-tracking mathematics programs in schools where students are placed with classmates who are perceived to have similar high, middle or low math abilities. This can be harmful to students who are not placed in the higher tracks when they perceive themselves to be incapable of and not given an opportunity to learn higher-level mathematics. It can also be harmful to students in the highest level track if they perceive themselves to be incorrectly placed and incapable of learning complex mathematical concepts the first time they are given challenging work. Unfortunately, some schools that have de-tracked have stopped offering opportunities to learn high-level, engaging, complex mathematics. For elementary students, this might mean that they no longer have an opportunity for a pull-out program where they are given time to engage with peers to learn mathematics not taught in their regular classroom. At the secondary level, the school may stop offering college-level mathematics courses such as Advanced Placement Calculus or Statistics, using the excuse that not all students are ready for this, so no one is given the opportunity.

Traditionally in the United States, the first mathematics class that students take in high school in ninth grade has been Algebra I. Geometry in tenth grade and Algebra II in eleventh grade often follow this. For most students, this is the end of their high school mathematics classes, while others take pre-calculus as seniors (grade 12). For advanced students, this sequence may start in seventh or eighth grade, allowing students to take college-level mathematics classes such as Advanced Placement Calculus before graduating from high school. Students taking high school classes early have not always been successful, however.

> What the members of the mathematical community—especially those in the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM)—have known for a long time is that the pump that is pushing more students into more advanced mathematics ever earlier is not just ineffective: It is counter-productive. Too many students are moving too fast through preliminary courses so that they can get calculus

> onto their high school transcripts. The result is that even if they are able to pass high school calculus, they have established an inadequate foundation on which to build the mathematical knowledge required for a STEM career. Nothing demonstrates this more eloquently than the fact that from the high school class of 1992, one-third of those who took calculus in high school then enrolled in precalculus when they got to college, and from the high school class of 2004, one in six of those who passed calculus in high school then took remedial mathematics in college. (Bressoud et al. 2012, p. 2)

This led to the following in the NCTM position statement on students with exceptional mathematical promise:

> When considering opportunities for acceleration in mathematics, care must be taken to ensure that opportunities are available to each and every prepared student and that no critical concepts are rushed or skipped, that students have multiple opportunities to investigate topics of interest in depth, and that students continue to take mathematics courses while still in high school and beyond. (NCTM 2016, p. 1)

In the United States, requirements and opportunities for high school mathematics differ widely between states and between districts within each state. Some districts, especially those with high populations of students from poverty or students of color, offer no mathematics classes beyond Algebra II. In other districts, especially those with high populations of wealthy, white or Asian students, students might begin taking high school mathematics classes (that would normally begin in ninth grade) as early as seventh or even sixth grade and high schools offer a variety of college-level mathematics classes. Note that in most states in the US, students are allowed to stop taking mathematics courses as soon as they have finished three years of required high school mathematics. Thus, in those states, if students have taken some of their high school mathematics courses while in middle school, they may take no mathematics for the last year or two or more of high school, and thus be at a disadvantage when entering college. Students can also get college credit for passing courses such as Advanced Placement Calculus or Statistics while in high school, and many then choose a college major that requires no additional mathematics. For these students, some of whom are the most advanced mathematics students in high school, acceleration simply means that they can stop learning mathematics as soon as possible. Opportunities for advanced mathematics classes in high school must be available and open to all interested students, but care must be taken to ensure that these students are well-prepared, eager to take advantage of these classes, and continue their engagement with high-level mathematics through college and into careers.

In addition to possibilities to explore and pose mathematical problems in innovative ways from kindergarten through college as part of the regular school curricula and to take advanced mathematics courses in high school, students of all ages need a variety of extra-curricular experiences with engaging, thought-provoking, and creative mathematics that there might not be time for during the regular school day. These might be open to a wide range of interested students. This includes such things as the engaging technology, competitions and special summer and after-school programs described in this monograph as well as work with mentors and math clubs and circles.

In order to provide students with the learning opportunities described in these chapters, teachers must have preparation in their graduate and undergraduate college and university programs as well as ongoing professional development throughout their careers. Problems such as the paper pool problem that Daher and Anabousy (2018) used with their pre-service teachers should be used throughout teacher preparation programs. Teachers need to experience creative problem solving and problem posing themselves, along with appropriate use of technology, in order to be able to support and encourage their students in similar experiences. Involving teachers in mentoring gifted students such as the nine-year-old described by Gutierrez et al. (2018) is also an excellent way to give teachers insight into students' reasoning abilities and to support them in planning appropriate experiences, whether this is in-person or online as in this chapter. Lesson study groups where teachers jointly plan and observe each other's lessons, paying special attention to the variety and uniqueness of students' questions and solutions are another means of helping teachers recognize and nurture students' talents and creativity. Practicing and pre-service teachers who analyze students' responses to multiple solution tasks or open response questions from competitions or other activities also gain greater awareness of the variety and depth of students' thinking processes and can use that to better plan problem solving and posing experiences for their students.

Teachers also need easy access to print, electronic, and human resources and the support of administrators, families, and other stakeholders. State and national laws and regulations governing public education need to acknowledge the importance of serving the needs of our most expert mathematics students, those who are already achieving at the highest levels as well as those who could achieve at much higher levels given the proper support and encouragement. Growing exceptional mathematical promise is critical for society as well as the students themselves.

To better discover and promote exemplary practices in the recognition, support and creation of these students with exceptional mathematical promise, we need to continue and increase research into mathematical creativity and giftedness. This monograph is one additional step in that direction, and hopefully is just the beginning of growing research and collaboration across the world. Building on the research reported here and in other recent publications mentioned earlier, future research should continue to guide our practices by addressing these issues, including:

- Studying the effects of various means of structuring school environments and extra-curricular activities and the role of the teacher and the curriculum for the long-term development of students' mathematical creativity and giftedness and their societal impact
- Investigating the best means of educating and supporting teachers in the advancement of their students' mathematical creativity and giftedness
- Examining effects of policies and procedures advocating for identification of and services for mathematically gifted students.

As scientific and technological innovation are increasingly important to the growth of new industries and careers, and, in general, our quality of life and the health of our planet, we need individuals with the knowledge, skills and passion for defining and tackling some of the most difficult problems ever encountered, and most of these will require mathematical expertise and creativity. There are students with exceptional mathematical promise in every country and from every demographic, and we need to work together to find the best means to find, develop and create them.

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