Chapter 15
Simulation of Stochastic Volatility Variance Swap

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Abstract This paper aims to propose efficient mathematical model of variance swap to study the effect of stochastic volatility in different time-scales on the option pricing. Two types of stochastic volatility, including Ornstein-Uhlenbeck (OU) process and Cox-Ingersoll-Ross (CIR) process are considered. Analytical solution of CIR model is presented. For the OU process, a numerical algorithm based on the finite element approach is established for solution of the model.

Keywords Variance swaps · Time-scale · Stochastic volatility · Finite element method

15.1 Introduction

A variance swap is a financial instrument which allows investors to speculate on the spread between future volatility and implied volatility. Variance swap provides us a straightforward method to cover the exposure risk of the volatility of the underlying asset. Recently, many researchers have investigated the variance pricing based on the classical Greek option with constant volatility which lead to the underlying process of a fat-tailed distribution. The stochastic volatility model is one of the

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approaches to overcome the shortcomings of the constant volatility models. Carr and Madan [1] briefly reviewed three different methods for trading realized volatility, including static replication, delta hedge, and volatility contract, and connected their work with the stochastic volatility. However, little work has been done to study the variance swap pricing problem under stochastic volatility.

The concept of time scales in Finance was first introduced by Fouque et al. in 1998, in Ref. [2] in which the option pricing model with fast-scale stochastic volatility is proposed. In practice, variations of data, include high frequency data always appears only in the short period, while low frequency data appears in the long period.

Many numerical algorithms have been proposed to study the time-scale option pricing problem. Little and Pant [3] applied the finite difference method (FDM) the variance swaps problem based on constant volatility [4], in which a two-dimensional (2D) problem was reduced to a one-dimensional one, and the price of variance swap was obtained as an average of the 2D solutions. As well known, the stochastic volatility emerges as a solution to the constant volatility, which has been studied for years. Zhu and Lian [5] applied the Fourier transformation to price variance swaps with discrete sampling times and found a closed-form solution of the Heston's two-factor stochastic volatility model [6].

In this paper, we extend Zhu and Lian's work to study the variance swap based on the fast-scale model. A little attempt has been done on using the OU process for the variance swap problem. Numerical approximation is carried out using Finite Element Method. By introducing the technique implanted by Little and Pant [3], the 3D model reduces to a 2D model. The model is then split into two stages at $t_{i-1} \leq t \leq t_i$, and $t_i \leq t \leq T$ respectively. The solution of the second stage at $t_i \leq t \leq T$ is first carried out and is then implemented to the initial solution of the first state. The solution in Zhu and Lian [5] is used as a benchmark to show the validity of our algorithm. The effect of maturity time and different time-scale rates on strike price is investigated, also the long level convergence of the strike is showed in our numerical results.

The rest of this paper is as follows. In Sect. 15.2, we set up the model. Section 15.3 concerns the numerical study of the problem. Section 15.4 is the conclusion of this paper.

15.2 Model Setup

This section concerns dynamics of the underlying asset, which can be described by the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t^{(0)}, \tag{15.1}$$

where S_t denotes the stock price, $dW_t^{(0)}$ is a Geometric Brownian motion, σ_t represents volatility which is driven by a factor diffusion process and is determined by $\sigma_t = f\left(Y_t^{(k)}\right)$, with the factor $Y_t^{(k)}$ obtained from a Gaussian OU process [7]:

$$dY_t^{(k)} = k\left(m - Y_t^{(k)}\right)dt + v\sqrt{2k}dW_t^{(1)}, \tag{15.2}$$

for the drift scale $k > 0$ and the diffusion coefficient scale $\sqrt{k}$. From (15.1) and (15.2), $W_t^{(1)}$ and $W_t^{(0)}$ is correlated with the covariation $\mathrm{cov}\left(W_t^{(t)}, W_t^{(0)}\right) = \rho_1$.

The scale of k affects the volatility process significantly. When k is large, the process is referred to the fast scale process; otherwise, is known as the slow scale process. The value of a variance swap at the expiration date can be written as $V_T = L^*(\sigma^2 - K)$, where σ^2 is the final realized volatility, K is the strike price and L is the variance amount. In the risk-neutral world, the value of variance swap at time t is denoted by $V_T = E\left[e^{-r(T-t)}\left(\sigma_R^2 - K\right)L\right]$. We let $V_0 = 0$, because there is no cost to enter a swap at the right beginning. Based on the martingale property, $K = E[\sigma^2]$ is obtained. The problem thus becomes to calculate the realized volatility σ^2. According to Little and Pant [3], the final realized volatility is defined as

$$\sigma_R^2 = \frac{AF}{N}\sum_{i=0}^{N-1}((S_{i+1} - S_i)/S_i)^2, \tag{15.3}$$

where AF is an animalization factor and N is number of the expected scheduled trading days in the observation period. The AF value of 252 is used when the sampling frequency is every trading day, 52 for everyweek and 12 for everymonth. As it is shown in (15.3), there are two underlying processes in the final payoff function, which makes the problem difficult to deal with. In this work, the method used by Zhu and Lian [5], and Little and Pant [3] are implemented. Firstly, a new variable I_t is introduced as

$$I_t = \int_0^t \delta(t_{i-1} - \tau)S_\tau d\tau, \tag{15.4}$$

where δ is the Dirac-delta function, which means $I_t = 0$ if $t < t_{i-1}$, and $I_t = S_{i-1}$ if $t \geq t_{i-1}$. By the usual no-arbitrage argument, we rewrite (15.1), and (15.2) into the forms:

$$\begin{cases} dS_t = rS_t dt + \sqrt{Y_t}S_t dW_t^{(0)}, \\ dY_t = \left(k(m - Y_t) - \lambda v\sqrt{2k}\right)dt + v\sqrt{2k}dW_t^{(1)}, \end{cases} \tag{15.5}$$

Where λ denotes the Risk price which is the same as Heston [6]. Letting $U(t, S, Y, I)$ be the price of a derivative whose payoff at time point t_{i+1} from t_i is $((S_{i+1} - S_i)/S_i)^2$, and according to Fouque and Sircar [8] and Feyman-Kac Theorem [5], we obtain:

$$U_t + \frac{1}{2} Y S^2 U_{SS} + \rho v S \sqrt{2kY_t} U_{SY} + v^2 k U_{YY} + r(U_S - U) + \left(k(m - Y) - \lambda v \sqrt{2k}\right) U_Y + \delta(t_{i-1} - t) U_I = 0 \tag{15.6}$$

with the terminal condition $U(t, S, Y, I) = (S/I - 1)^2$. According to the definition of $U(t, S, Y, I)$, we have $e^{r(t_i - t)} U(t, S_t, Y_t, I_t) = E_0^Q\left[(S_i/I_i - 1)^2\right]$. Based on the definition (15.4), I_t is deterministic and is only related to the previous state of S_{i-1}, which means that even though S is a stochastic process, I_t can be determined if we fix the previous stage of S_{i-1}. The variation of a deterministic process equals to zero, and the proof of this argument can be found in Klebaner et al. [9]. Because the discounted price of U is a martingale, by $V(t, S, Y, I) = e^{-rt} U(t, S, Y, I)$, we obtain.

$$dV = e^{-rt}(-rUdt + dU) \tag{15.7}$$

From Ito's formula,

$$\begin{aligned} dU &= U_t dt + U_S dS + U_Y dY + U_I dI + \frac{1}{2} U_{SS}[dS, dS] + \frac{1}{2} U_{YY}[dY, dY] + U_{XY}[dX, dY] \\ &= \Big(U_t + \frac{1}{2} Y S^2 U_{SS} + \rho v S \sqrt{2kY} U_{SY} + v^2 k U_{YY} + r(U_S - U) \\ &\quad + \left(k(m - Y) - \lambda v \sqrt{2k}\right) U_Y + \delta(t_{i-1} - t) U_I \Big) dt + \phi(X, Y, I, t) dW \end{aligned} \tag{15.8}$$

where ϕ is a complicated function of X, Y, I and t. By substituting (15.8) into (15.7), and using the Martingale Representation Theorem, (15.6) is obtained by taking expectation of (15.7). Let $x = \ln S$, $y = \ln Y$ and $\gamma = \ln I$, lEq. (15.6) becomes

$$U_t + \frac{1}{2} y U_{xx} + \rho v \sqrt{2ky} U_{xy} + v^2 k U_{yy} + r(U_x - U) + \left(k(m - y) - \lambda v \sqrt{2k}\right) U_y + \delta(t_{i-1} - t) U_\gamma = 0 \tag{15.9}$$

with the terminal condition $U(T, x, y, \gamma) = (e^{x - \gamma} - 1)^2$. Based on the property of Dirac-delta function, Eq. (15.9) can be represented by two-stages PDEs.

Stage one for $0 \le t < t_{i-1}$,

$$U_t + \frac{1}{2} y U_{xx} + \rho v \sqrt{2ky} U_{xy} + v^2 k U_{yy} + r(U_x - U) + \left(k(m - y) - \lambda v \sqrt{2k}\right) U_y = 0, \lim_{t \uparrow t_{i-1}} U_i = \lim_{t \downarrow t_{i-1}} U_i. \tag{15.10}$$

Stage two for $t_{i-1} \le t \le Y$,

$$U_t + \frac{1}{2} y U_{xx} + \rho v \sqrt{2ky} U_{xy} + v^2 k U_{yy} + r(U_x - U) + \left(k(m - y) - \lambda v \sqrt{2k}\right) U_y = 0, U_i(x, y, \gamma, T) = (e^{x - \gamma} - 1)^2. \tag{15.11}$$

Let $t_{i-1} = T - \Delta t$, and $\Delta t = T/N$, $N = 1, 2, \ldots, \tilde{N}$ stage one and stage two should be solved by backward algorithm.

15.3 Numerical Analysis

In Subsect. 15.3.1, the finite element method (FEM) is applied to solve the model with the CIR process, and our approach will be benchmarked by making a comparison between the approximate solution and the semi-analytical solution at each point. Also, for the reason that it is no analytic solution for the model with the OU process, we investigate the relationship between the time scale rate k, the maturity time T and the expected value of strike price through numerical solution based on the FEM method in Subsect. 15.3.2.

15.3.1 Validity Study

In order to show that the proposed model is applicable, we apply the model (15.12) and assumption as used by Zhu and Lian [5], and compare the approximated solution with the closed form solution. The Heston Model:

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t d\tilde{W}_t^S, \\ dY_t = k^*(\theta^* - v_t)dt + \sigma_V \sqrt{v_t} d\tilde{W}_t^V, \end{cases} \tag{15.12}$$

where $k^* = k + \lambda$ and $\theta^* = k\theta/(k + \lambda)$ are the risk-neutral parameters, λ is the premium of volatility risk [6]. The parameters used here is the same as those in Zhu and Lian [5], namely $k^* = 11.35$, $\theta^* = 0.022$, $\sigma_V = 0.618$ while we choose $v_0 = 0.5$ in this paper. We also apply the same assumption, as in Zhu and Lian [5] that the strike price is defined by

$$K_{\text{var}} = \kappa^* 10^4 = e^{r\Delta t} E[(S_{t_i} - S_{t_{i-1}})/S_{t_{i-1}}] * 10^4/T \tag{15.13}$$

which is only related with the time step size and value of v_t. Figure 15.1 shows the comparison of the FEM approximation and the exact solution. Clearly, in Fig. 15.1, the strike price falls as maturity time increases. Even though the difference between the two methods becomes larger and larger as the maturity time increases, we can apply this method for the reason that exact solution does not always exist, and even in Zhu and Lian's paper, they derived the semi-analytic solution with the integral form instead of the exact solution. The FEM method is accurate to describe the tendency and in most cases the maturity time cannot be that long. In order to increase the absolute accuracy and make our method more persuasive, we calculate the κ in Eq. (15.13) instead of K_{var} with the mesh size of 100, and then obtain Fig. 15.2.

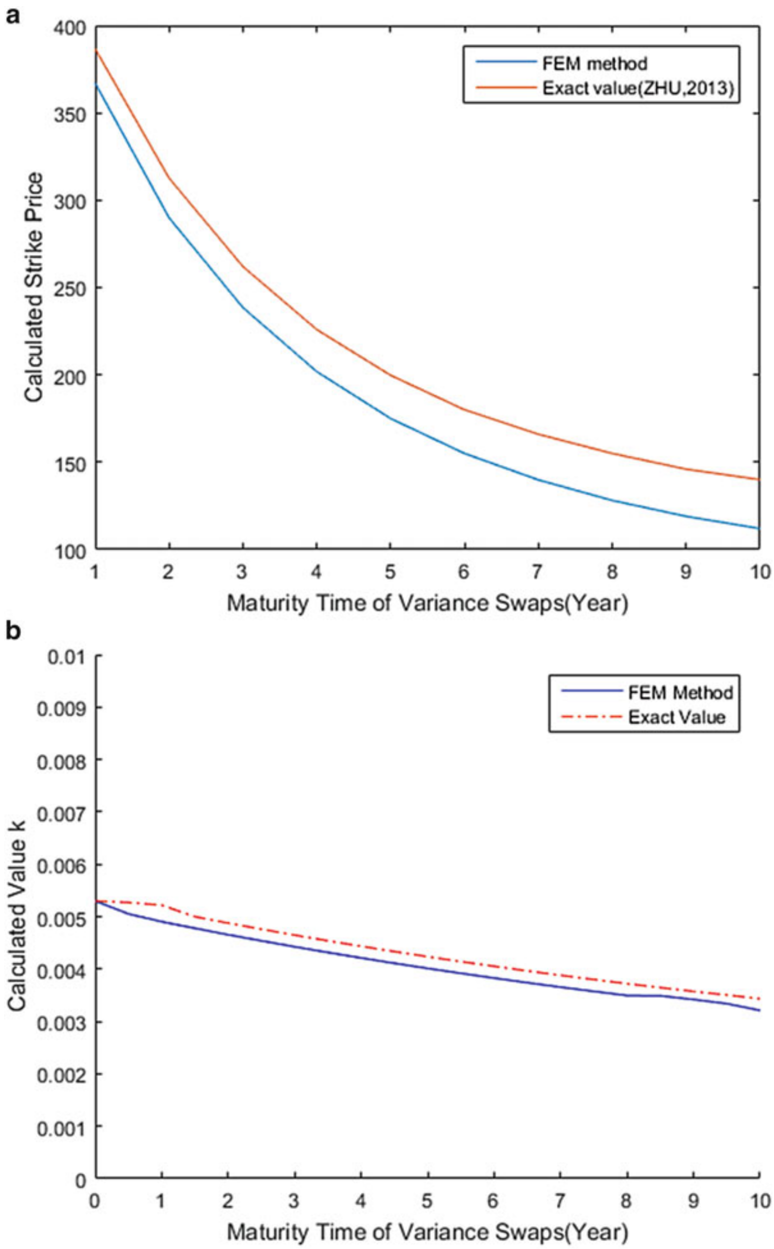

Fig. 15.1 Calculated strike values as a function of maturity time

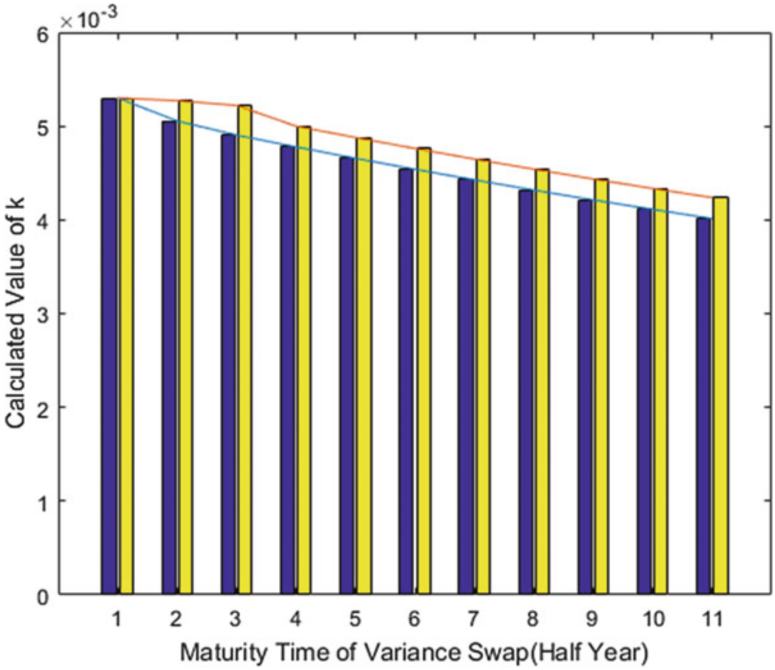

Fig. 15.2 Relationship between maturity time and strike price

15.3.2 Numerical Results of Our Model

Zhu and Lian's paper is based on Heston's two-factor stochastic volatility model. In Heston's model, the stochastic volatility process is a CIR process, from which it is easy to construct a closed form solution by using Heston's Scheme [6]. However, if the stochastic volatility process is the OU process instead of the CIR process, it is not easy to obtain analytical solution by simply constructing a specified form. The approximate solution is thus obtained instead.

As it is shown in Fig. 15.3a, the strike price is inverted anti-correlated with the time scale rate. There is a mechanism behind the phenomenon: Lager k brings more risk exposure, which contributes more to the strike price. But this effect will not go to infinity, when k is larger than one, the strike price experiences a slightly decrease and converges to zero.

Also, Fig. 15.3b shows the relationship between the maturity time T and the strike price. Obviously, the strike price is anti-correlated with the maturity time. It decreases sharply when the maturity time T is less than 1.5 years and approaches a steady level when time goes by. This agrees with the result proved by Zhu and Lian [5]. The result verifies that volatility provides a measure of risk exposure. The longer the investors hold the contract, the higher risk they have to take.

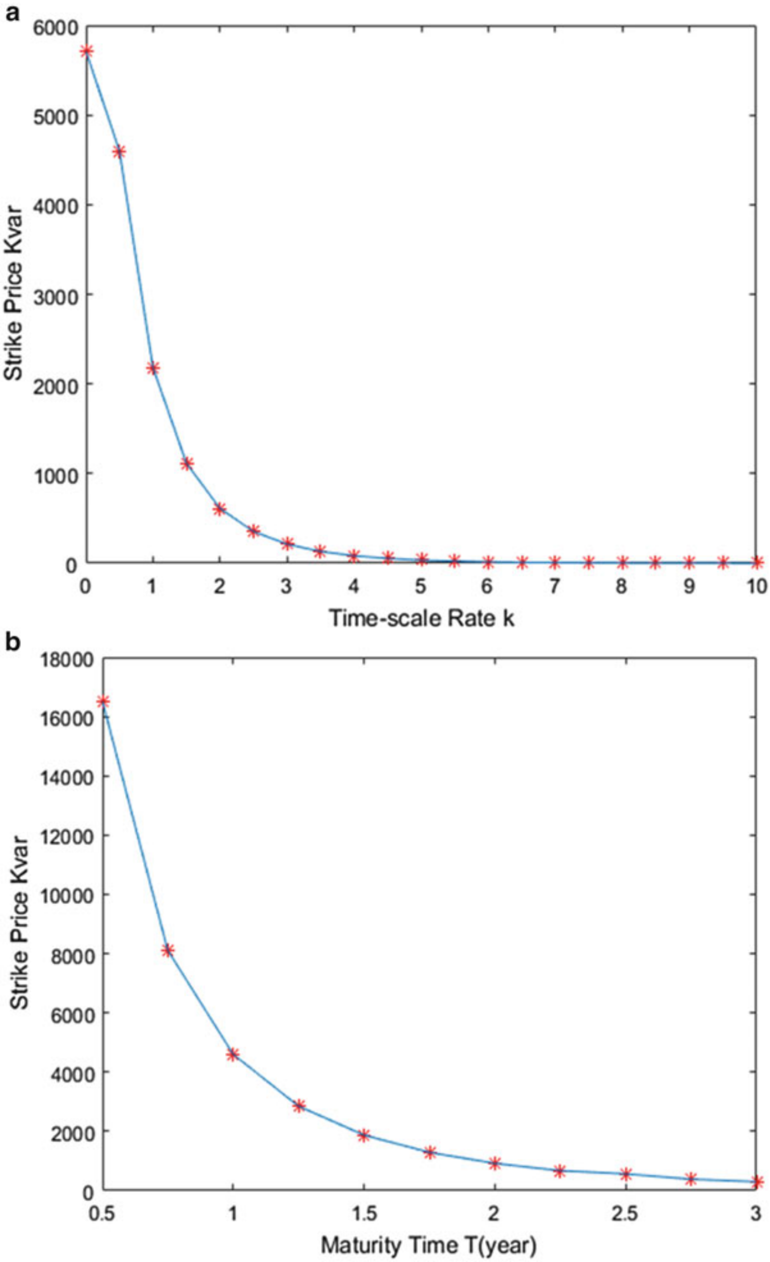

Fig. 15.3 Relationship between the maturity time and strike price

15.4 Conclusion

In this paper, we apply the finite element method to obtain the approximate solution of variance swaps under stochastic volatility. The time scale rate of stochastic volatility is considered to describe the long term and short term perturbation and draw the conclusion that the strike price of variance swap is anti-correlated with the time scale rate, especially when k is less than one. Also, for the reason that the volatility is a measure of risk, the strike price falls when the maturity time increases. We have also compared the results produced by the FEM method with the model with the CIR process for describing the volatility and found that our approximate solution agrees with the exact solution. The significance of this work can be illustrated in two aspects. First of all, the exact solution can only be obtained for specified models. For most PDE, we cannot derive the closed form solution, which makes the numerical approach necessary. Besides, even though most work has considered the stochastic volatility, they do not study the property of the stochastic volatility, we apply the time scale rate to describe our model and show how it works on the variance swaps pricing.

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