Pseudo-Differential Operators on Manifolds with a Singular Boundary

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Abstract The aim of this work is to describe new interesting examples of non-smooth manifolds and elliptic pseudo-differential operators acting in functional spaces on such manifolds. Fredholm properties for these operators are studied by factorization methods, and these are based on several complex variables.

Keywords Pseudo-differential operator • Local representative • Bochner operator • Wave factorization

1 Introduction

In 1990s the author has started to develop a new approach to pseudo-differential equations and boundary value problems on smooth manifolds with a non-smooth boundary. This approach takes its origin from Vishik–Eskin's theory of boundary value problems for manifolds with a smooth boundary [3]. The author has extended a factorization idea to non-smooth situation to apply it for describing Fredholm conditions for elliptic pseudo-differential equations and boundary value problems on manifolds with a non-smooth boundary. This studying was completed in general for a two-dimensional case [11] but last years the author has found new interesting constructions for a multidimensional case also [12–15]. A special case is the author's paper [10] in which it was done for Calderon–Zygmund operators only.

A basic problem is to describe Fredholm conditions for the equation

$$(Au)(x) = v(x), \quad x \in M,$$

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where M is a compact manifold with a boundary (non-smooth as a rule), A is a pseudo-differential operator acting in certain functional spaces on M. We use Sobolev–Slobodetskii spaces $H^s(M)$, $s \in \mathbb{R}$, for studying these properties [11]. Such spaces are introduced locally using a partition of identity [3].

Some words on other approaches to the problem and related topics [2, 4–8]. Richard Melrose's programme was declared in ICM-90 [5], and it is devoted to algebraical and topological aspects of the problem, this way is reserved by papers [6, 7]. Other kind of papers was written by analysts. So, B.-W. Schulze and his group (see for example [2]) deals with general pseudo-differential operators on manifolds with cones and wedges.

"In all papers the conical domain (see Fig. 1 below) is treated as the direct product of a circle and a half-axis, then they apply the Mellin transform on half-axis, and the initial problem is reduced to a problem in a domain with a smooth boundary with operator-valued symbol. That follows further it is like the generalization of well-known results on operator symbol case. Of course, the my approach is generalization also, but it is a generalization on dimension space, and the principal difference is that I don't divide the cone, and it's treated as an emergent thing" [14]. The last is related also to papers V. Mazya, B. Plamenevskii and others mentioned in [14].

Another kind of "analytical" papers deals with very "simple" operators and boundary value problems on manifolds with very "bad" boundary (see for example [4]). The main aim for them is developing the classical Fredholm theory as far as it is possible using potential theory.

Fig. 1 Simplest manifold with a non-smooth boundary
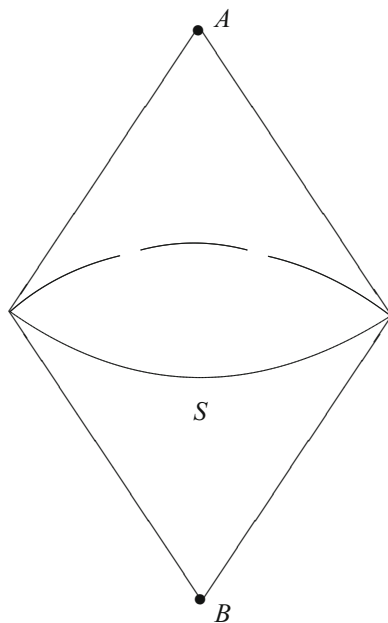
Finally recent paper [8] contains a certain new generalizations for statements of boundary value problems without smoothness requirement on boundary of a domain.

> ...we develop the global symbolic calculus of pseudo-differential operators generated by a boundary value problem for a given (not necessarily self-adjoint or elliptic) differential operator. For this, we also establish elements of a non-self-adjoint distribution theory and the corresponding biorthogonal Fourier analysis. There are no assumptions on the regularity of the boundary which is allowed to have arbitrary singularities. We give applications of the developed analysis to obtain a priori estimates for solutions of boundary value problems that are elliptic within the constructed calculus. [8]

2   Non-Smooth Manifolds and Local Representatives

Let it will not be a strange thing but in this section we'll define the declared manifold by operators which live on this manifold.

2.1   Pseudo-differential Operators

The main object of the paper is a linear bounded operator $A : H^{s_1}(M) \to H^{s_2}(M)$ which is called a pseudo-differential operator under following assumptions. We'll suppose that operator A is composed by a certain operator-function $A(x_0)$, $x_0 \in \overline{M}$, so that for arbitrary smooth functions φ, ψ on M with supports concentrated in small neighborhoods U, V ($U \subset V$) of x the following representation for the operator A

$$\varphi \cdot A \cdot \psi = \hat{\varphi} \cdot (A_{x_0} + T_{x_0})\hat{\psi},$$

holds, where A_{x_0} is an operator defined by formula

$$(A_{x_0}u)(x) = \int\limits_{D} \int\limits_{\mathbb{R}^m} e^{i\xi \cdot (x-y)} \tilde{A}(\eta^{-1}(x_0), \xi)\tilde{u}(\xi)d\xi dy, \quad x \in D,$$

$\eta : V \to D \subset \mathbb{R}^m$ is a local diffeomorphism, $\tilde{A}(x, \xi)$ is a certain function defined on $V \times \mathbb{R}^m$, $\hat{\varphi} = \varphi \circ \eta^{-1}$, $\hat{\psi} = \psi \circ \eta^{-1}$, $T_{x_0} : H^{s_1}(\mathbb{R}^m) \to H^{s_2}(\mathbb{R}^m)$ is a compact operator, $\tilde{u}(\xi)$ denotes the Fourier transform in m-dimensional space

$$\tilde{u}(\xi) = \int\limits_{\mathbb{R}^m} e^{ix \cdot \xi} u(x)dx,$$

$\forall u \in S(\mathbb{R}^m)$ (Schwartz class of infinitely differentiable rapidly decreasing functions at infinity).

Remark 2.1 Generally speaking the domain D depends on the point x_0 but we'll see below in a lot of cases it isn't essential.

Definition 2.2 The operator $A_{x_0} : H^{s_1}(D) \to H^{s_2}(D)$ is called a **local representative** of the operator $A : H^{s_1}(M) \to H^{s_2}(M)$ at the point $x_0 \in \overline{M}$, and domain D is called a **canonical** domain.

A structure of the set D can be different in dependence on a placement of the point x_0. For inner points $\overset{\circ}{M}$ we have $D = \mathbb{R}^m$, for points of smoothness on ∂M we have $D = \mathbb{R}^m_+$, for conical points $D = C^a_+$ and so on. One has painted a simple example of a non-smooth manifold M on the Fig. 1. There are distinct types of local representatives in dependence on a point kind. These local representatives are defined by different formulas for inner points $\overset{\circ}{M}$, for points of smoothness on ∂M, for points on S which is a smooth edge of a wedge, and separately for conical points A and B. Below we'll describe these local representatives and methods for their studying.

Definition 2.3 We say that a manifold M has a non-smooth boundary if there is at least one local representative of a pseudo-differential operator for which $D \neq \mathbb{R}^m, \mathbb{R}^m_+$, and the function $\sigma(x_0, \xi) = \tilde{A}(\eta^{-1}(x_0), \xi)$ defined on $\overline{M} \times \mathbb{R}^m$ is called a **local symbol** of a pseudo-differential operator A at the point x_0.

2.2 Canonical Domains: A Half-Space, a Cone, a Wedge

Since the definition of a pseudo-differential operator is a local and we use the "freezing coefficients principle" or, in other words, "local principle" then we'll omit a pole x_0 in a symbol of a pseudo-differential operator. Thus we have following types of local operators related to an initial pseudo-differential operator A. First it is the operator

$$u(x) \longmapsto \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^m} \tilde{A}(\cdot, \xi) u(y) e^{i(x-y)\cdot\xi} d\xi dy, \quad x \in \mathbb{R}^m. \tag{2.1}$$

for a point $x_0 \in \overset{\circ}{M}$.

If $x_0 \in \partial M$ and x_0 is a smoothness point then we need another formula

$$u(x) \longmapsto \int\limits_{\mathbb{R}^m_+} \int\limits_{\mathbb{R}^m} \tilde{A}(\cdot, \xi) u(y) e^{i(x-y)\cdot\xi} d\xi dy, \quad x \in \mathbb{R}^m_+. \tag{2.2}$$

For invertibility of such an operator with symbol $\tilde{A}(\cdot, \xi)$ not depending on a spatial variable x_0 one can apply the theory of the classical Riemann boundary

value problem for upper and lower complex half-planes with a parameter $\xi' = (\xi_1, \ldots, \xi_{m-1})$. This step was systematically studied in the book [3]. But if the boundary ∂M has at least one conical point, this approach is not effective.

A conical point x_0 at the boundary is such a point for which its neighborhood is diffeomorphic to the cone $C_+^a = \{x \in \mathbb{R}^m : x_m > a|x'|, x' = (x_1, \ldots, x_{m-1}), a > 0\}$, hence the local definition for pseudo-differential operator near the conical point is the following

$$u(x) \longmapsto \int\limits_{C_+^a} \int\limits_{\mathbb{R}^m} \tilde{A}(\cdot, \xi) u(y) e^{i(x-y)\cdot\xi} d\xi dy, \quad x \in C_+^a. \tag{2.3}$$

A k-wedge point x_0 at the boundary is such a point for which its neighborhood is diffeomorphic to the wedge $W_+^{a_k,k} = \{x \in \mathbb{R}^m : x = (x'', x', x_m), x'' = (x_1, \cdots, x_k), x' = (x_{k+1}, \cdots, x_{m-1}), x_m > a_k|x'|, a_k > 0\}$. In other words $W_+^{a_k,k} = \mathbb{R}^k \times C_+^{a_k}$, where $C_+^{a_k}$ is a cone in $\mathbb{R}^{m-k}$. Hence the local definition for pseudo-differential operator near the k-wedge point is the following

$$u(x) \longmapsto \int\limits_{W_+^{a_k,k}} \int\limits_{\mathbb{R}^m} \tilde{A}(\cdot, \xi) u(y) e^{i(x-y)\cdot\xi} d\xi dy, \quad x \in W_+^a. \tag{2.4}$$

To study an invertibility property for the operator (2.3), (2.4) the author has introduced the concept of wave factorization for an elliptic symbol near a singular boundary point [10, 11] and using this property has described Fredholm properties for an equation with the operator (2.3), (2.4).

2.3   Class of Symbols

To describe invertibility conditions for operators (2.1)–(2.3) we need to fix a class of local symbols under consideration.

Definition 2.4 A local symbol belongs to the class S_α if it satisfies the following condition

$$|\sigma(x_0, \xi)| \sim (1 + |\xi|)^\alpha, \quad \forall x_0 \in \overline{M}, \ \xi \in \mathbb{R}^m.$$

The number α is called an order of a pseudo-differential operator.

Such symbols and corresponding operators we call **elliptic** ones.

According to [3] such operators with local symbols from S_α are linear bounded operators acting from $H^s(D)$ to $H^s(D)$, and everywhere below we consider only symbols from the class S_α.

2.4 Main Theorem

Theorem 2.5 *An elliptic pseudo-differential operator* $A : H^s(M) \to H^{s-\alpha}(M)$ *with continuous local symbol has a Fredholm property iff all local representatives* $A_{x_0} :$ $H^s(D) \to H^{s-\alpha}(D)$ *are invertible.*

Proof Since the definition of a pseudo-differential operator given above assume that an operator is defined locally then a local principle [9] implies this assertion.   □

2.5 Wave Factorization and Invertibility of Local Operators

We'll give the definition of a wave factorization with respect to a k-dimensional wedge because a cone is a particular case of a wedge $W_+^{a_0,0} = C_+^{a_0}$.

Let $\overset{*}{C_+^a} = \{x \in \mathbb{R}^m : ax_m > |x'|\}$ be a conjugate cone, and $T(\overset{*}{C_+^a})$ be a radial tube domain over the cone $\overset{*}{C_+^a}$ [1, 11, 16], it is a subset of $\mathbb{C}^m$ of the following type $T(\overset{*}{C_+^a}) = \mathbb{R}^m + i\,\overset{*}{C_+^a}$.

Definition 2.6 Wave factorization of a local elliptic symbol $\sigma(x_0, \xi)$ with respect to the wedge $W_+^{a_k,k}$ is called its representation in the form

$$\sigma(x_0, \xi) = \sigma_{\neq}(x_0, \xi) \cdot \sigma_{=}(x_0, \xi),$$

where factors admit analytic continuation into radial tube domains $T(\pm \overset{*}{C_+^{a_k}})$ over cones $\pm \overset{*}{C_+^{a_k}}$ for almost all $\xi'' = (\xi_1, \cdots, \xi_k)$ with estimates

$$|\sigma_{\neq}^{\pm 1}(x_0, \xi + i\tau)| \leq c_1(1 + |\xi| + |\tau|)^{\pm \ae_k},$$

$$|\sigma_{=}^{\pm 1}(x_0, \xi - i\tau)| \leq c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - \ae_k)},$$

where c_1, c_2 are constants.

The number $\ae_k$ is called an index of the wave factorization.

The following theorem in general was proved in [11]. There are some examples of elliptic symbols admitting the wave factorization.

Let us denote by S_k a smooth sub-manifold of M consisting of k-wedge points.

Theorem 2.7 *If the elliptic local symbol* $\sigma(x_0, \xi)$ *admits wave factorization with respect to the wedge* $W_+^{a,k}$ *for all* $x_0 \in S_k$ *and* $|\ae - s| < 1/2$ *then all local representatives* $A_{x_0}, x_0 \in S_k,$ *are invertible.*

2.6 The Bochner Operator

Let $S(\mathbb{R}^m)$ be the Schwartz space of infinitely differentiable rapidly decreasing at infinity functions. Let us define the following function

$$B(z) = \int\limits_{C^a_+} e^{iy \cdot z} dy, \quad z = x + i\tau \in T(\overset{*}{C}{}^a_+)$$

and introduce the following

Definition 2.8 A Bochner operator is called the following linear operator

$$(Bu)(x) = \lim_{\tau \to 0} \int\limits_{\mathbb{R}^m} B(z - y)u(y)dy,$$

where $\tau \to 0$ along arbitrary non-tangential way, $\tau \in \overset{*}{C}{}^a_+$ [1, 16].

Remark 2.9 For this case C^a_+ the Bochner kernel can be calculated exactly [10, 16]

$$B(z) = \frac{a\Gamma(m/2)}{2\pi^{\frac{m+2}{2}}} \frac{1}{\left(z'^2 - a^2 z_m^2\right)^{m/2}}, \quad z' = (z_1, \cdots, z_{m-1}),$$

where Γ is Euler Γ-function, but all above is valid for arbitrary **sharp convex cone** in $\mathbb{R}^m$ although we don't know an explicit form of the kernel $B(z)$.

Proposition 2.10 *The operator* $B : L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$ *is a linear bounded operator.*

Proof It follows from the fact that the operator B is Fourier image of a multiplication operator on an indicator of the cone C^a_+. $\qquad\square$

Remark 2.11 It is easy to prove that $B : H^s(\mathbb{R}^m) \to H^s(\mathbb{R}^m)$ is also linear bounded operator for $|s| < 1/2$.

Let us note that the Bochner operator plays an important role and permits to construct an inverse operator for local operators (2.3), (2.4) [10, 11].

3 Hidden Parameters

Everywhere above we have assumed that following parameters are constants.

3.1 Order of an Operator

First an order of an pseudo-differential operator can vary from a point to a point. Simple example is an elliptic local symbol of following kind

$$\sigma(x_0, \xi) = (1 + |\xi|^2)^{\alpha(x_0)}.$$

So there is the following

Problem 3.1 What one can say on boundedness and invertibility of such a pseudo-differential operator in Sobolev–Slobodetskii spaces $H^s(M)$?

3.2 Index of Wave factorization

Since index of factorization (according to Vishik–Eskin theory) determines a quantity of boundary conditions and index of wave factorization also it is very interesting situation when such indices vary from a point to a point. Thus the following question arises.

Problem 3.2 Is it possible the situation when one needs different quantity of boundary conditions on distinct parts of a boundary?

3.3 A Variable Size of a Cone

Here we consider a case when size of a cone varies from a point to a point. It means that size of a canonical cone C_+^a can vary, in other words we need to consider a cone of a variable size $C_+^{a(x_0)}$. According to the definition of the operator B one can construct the following operator

$$(B_{var}u)(x) = \frac{a\Gamma(m/2)}{2\pi^{\frac{m+2}{2}}} \lim_{\tau \to 0+} \int_{\mathbb{R}^m} \frac{u(y)dy}{((x'-y')^2 - a^2(x)(x_m - y_m + i\tau)^2)^{m/2}}.$$

Problem 3.3 What one can say on boundedness of the operator B_{var} in Sobolev–Slobodetskii spaces $H^s(\mathbb{R}^m)$?

4 A Wedge with a Variable Size

Here we'll consider more complicated manifold M with so-called k-wedges with a variable size. To study such singularities one can apply the developed technique for describing sufficient invertibility conditions of local operators.

Definition 4.1 k-wedge of a variable size $W_+^{a_k(x_0),k}$ is a smooth sub-manifold $S_k \subset M$ consisting of points x_0 in which a local representative of an operator A has the form

$$u(x) \longmapsto \int\limits_{W_+^{a_k(x_0),k}} \int\limits_{\mathbb{R}^m} \tilde{A}(\cdot,\xi)u(y)e^{i(x-y)\cdot\xi}d\xi dy, \quad x \in W_+^{a_k(x_0),k}, \tag{4.1}$$

where the function $a(x_0)$ is defined on S_k, is continuous, takes its values on an interval (b_k, c_k) and has finite limits in points b_k, c_k.

4.1 Refined Theorem and Sufficient Conditions

Let a manifold M be such that its boundary ∂M includes a smooth part and smooth sub-manifolds S_k which are k-wedges, $k = 0, \cdots, m - 2$. A sub-manifold S_{m-1} is a closure of a smooth part of a boundary ∂M. For this piece of a boundary one can use Vishik–Eskin theory [3].

In this section we'll add to local representatives the operator (4.1) and formulate the following

Theorem 4.2 *Let A be a pseudo-differential operator with continuous elliptic local symbol $\sigma(x_0, \xi)$. The operator $A : H^s(M) \to H^{s-\alpha}(M)$ has a Fredholm property iff all local representatives $A_{x_0} : H^s(D) \to H^{s-\alpha}(D)$ are invertible. If the local symbol admits the wave factorization with respect to k-wedge points $x_0 \in S_k$, $|\ae_k - s| < 1/2, k = 0, \cdots, m - 2$, then all such local representatives $A_{x_0} : H^s(D) \to H^{s-\alpha}(D)$ are invertible.*

4.2 From a Half-Space to a Half-Line: Degenerating Wedge

If $b_1 = 0, c_1 = +\infty$ we have the wedge on Fig. 2. For $b_1 = 0$ we obtain a plane, and for $c_1 = +\infty$ we obtain a half-line. The author has made some attempts to describe such local representatives of an operator A [13, 14] but it is not clear up to now how one can work with such singularities.

5 Exotic Singularities

There are a lot of possibilities to construct another types of singularities combining mentioned above cones, wedges and their modifications. Some variants were presented in [12].
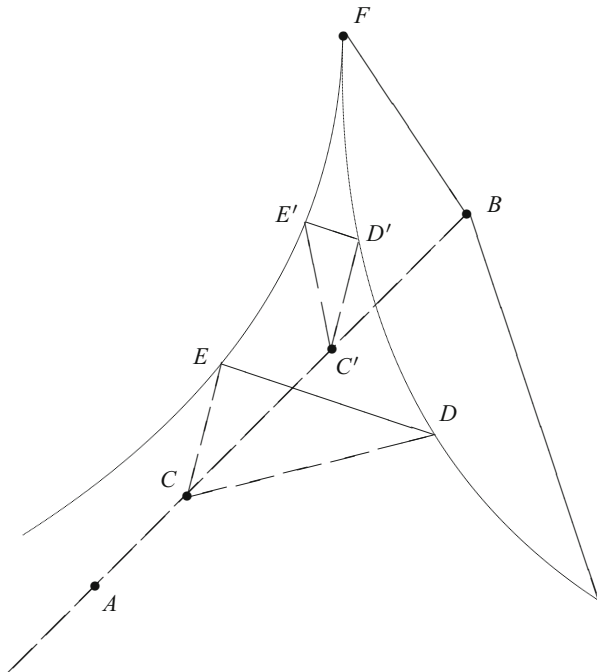
Fig. 2 A wedge of a variable size

6   Conclusion

The author hopes these considerations will help to enlarge a set of admissible manifolds with singular boundaries and to get answers to some formulated questions. It seems for all existing theories of boundary value problems for elliptic pseudo-differential equations on manifolds with singular boundaries one needs an invertibility of local representatives. Finding effective necessary and sufficient conditions for this property is a very hard problem, and any result in this direction will be a great achievement.

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