Chapter 20
Reconciling the Realist/Anti Realist Dichotomy in the Philosophy of Mathematics

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Introduction

Mathematical philosophy typically occurs in the background of mathematics. In the vast territory that characterizes modern mathematics, positions in the philosophy of mathematics can be viewed as a map or a guide through which one can understand some of its terrain. In classical mathematical philosophy there are four positions, namely Platonism, formalism,[1] logicism, and Intuitionism (or Constructionism). Each of these positions has been expounded on at length in the literature by philosophers like Reuben Hersh, Michele Friend, Penelope Maddy, among others. Platonism is also referred to as Realism and Intuitionism (or Constructionism) is referred to as Anti-Realism.[2] These two positions as their labels suggest are dichotomous with Realism conferring ontological status to mathematical objects whereas anti-Realism emphasizes epistemology in the sense that methods of construction are necessary to construct mathematical objects. More specifically there are different conceptions for the establishment of truth in these two positions.

[1]We deliberately rule out formalism for the primary reason that in keeping with Heyting's (1974) observation: "There is no conflict between intuitionism and formalism when each keeps to its own subject, intuitionism to mental constructions, formalism to the construction of a formal system, motivated by its internal beauty or by its utility for science and industry. They clash when formalists contend that their systems express mathematical thought. Intuitionists make two objections against this contention. In the first place, …[m]ental constructions cannot be rendered exactly by means of language; secondly the usual interpretation of the formal system is untenable as a mental construction." (p. 89).

[2]In this chapter we use the terms Realism and Constructionism for these two positions.

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S. Wuppuluri and F. A. Doria (eds.), *The Map and the Territory*,
The Frontiers Collection, https://doi.org/10.1007/978-3-319-72478-2_20

For a realist, a proof by contradiction is sufficient to confer an irrational status to say $\sqrt{2}$, but for an anti-realist it is more important to know how to construct $\sqrt{2}$ or any other number for that matter! To paraphrase L.E.J. Brouwer, the founder and proponent of Constructionism, one does not ask a statement is true unless they know what it means (Bishop 1973). And further the methods used to construct an object or prove a theorem should not rely on "logical tricks" such as the law of the excluded middle. Richman (1999) illustrates this in the in direct proof of "There is a digit that appears infinitely often in the decimal expansion of π". The proof explained by Richman does not give any method for constructing these digits but merely confers an "existence status" to objects. Similarly there are other interesting and even absurd things that can proved using the Realist's criteria of an existence proof, without really knowing how to go about constructing these objects. This is the crux of the Realism-conferring status to objects without knowing what they are in the sense of being able to construct them without using the rule of the excluded middle. In other words, if a Realist proves "$\exists$O", the Constructionist would answer you have established "$\neg\forall \times \neg$O" or if the Realist proves "A $\vee$ B", the Constructionist would answer you have proved "$\neg[\neg$A $\wedge$ $\neg$B]"

The territory of mathematics particularly that found in textbooks relies on such proofs to establish results for undergraduate students. The question then is what (if any) are the benefits of using constructive methods. Further from a pluralist standpoint as expounded by Friend (2014), can one possibly hold both a realist and an anti-realist stance for particular objects or results? Better yet, in the exercise of "constructing the real numbers" (pun intended), an exercise which terminates in a real analysis course for some students, and an advanced geometry or abstract algebra course for others, can one highlight issues that arise in the philosophy of mathematics, particularly the realist and anti-realist stance to developing this mathematical object. In doing so, the territory of what constitutes a real number is illuminated by the map of developing particular constructions, especially notions of rationals and irrationals, and the subtleties of these objects. Can the seemingly dichotomous position of the realists and anti-realists find "points of convergence" (no pun intended), or can different ways to construct a particular number shed more insights for a student, and a pluralist view is thus possible? Another necessity to examine this approach is the fact that mathematical theories are constantly in a state of flux as evident in the development of non-Euclidean geometry, the paradoxes of set theory, and the development of special relativity with Minkowski's space-time metric as opposed to the older theory of Lorentz that used Newton's notions of space-time. Arguably bringing in examples from physics or examples from the physical world may be challenged by both realists and anti-realists as not being real mathematics. In the remainder of this chapter we will focus exclusively on mathematics.

There are different views of constructive mathematics (Bridges and Richman 1987; Raatikainen 2004) which suggest that old mathematical concepts need to be relearned and this is a non-trivial task, hence the recommendation to begin with younger students of mathematics. Schechter (2001) points to seemingly trivial notions that many take for granted such as inequality and apartness of real numbers also need to be carefully distinguished keeping with Brouwer's suggestion to constructionists that meaningful distinctions need to be maintained. One of the classical notions in analysis is that of an infimum of a set S of real numbers. Schechter writes:

> Suppose S is a set of real numbers, and r is a real number. To show constructively that $r = \inf(S)$, we must prove that $r \leq s$ for every $s \in S$, and we must also construct numbers $s_1, s_2, s_3, \ldots \in S$ satisfying $r > s_k - 1/k$. It is not enough merely to show the classical "existence" of some s_k's with that property.

The constructionist aspect suggests that merely having an algorithm is sufficient to meet the demands of constructionist mathematics. But Bishop (1967) never really explained what constitutes an algorithm for it to meet the burden of being constructionist. This leaves a very large grey area where algorithmic mathematics can be argued as being constructionist mathematics, a view which is corroborated by Richman (1999). However there is some clarification for what these grey areas might be. According to Mandelkern (1989), Errett Bishop said the following to explain what constructive mathematics is:

> How do you know whether a proof is constructive? Try to write a computer program. If you can program a computer to do it, it should be constructive. Notice I said write the program. Don't necessarily run it on the computer and wait around for the result.

In the 21st century, we have the advantage of retrospective on these words because of the huge program of experimental mathematics established by the Borweins, which not only involved writing a computer program but actually running it to ends never thought possible by Bishop.

Exploring the Grey Areas: Constructing the Real Numbers

The real numbers can be constructed in numerous ways. Typically one begins with the construction of Q, the set of rational numbers, which is an ordered field but not complete. For completeness considerations one has to venture into constructions that are too technical to discuss in this chapter. However the idea of infinity has to be developed since the types of sets one encounters now are infinite sets. Just like the natural numbers are countably infinite, the set of rationals are also countably infinite because it can be put in one-to-one correspondence with the natural numbers. For the realist there is no issue with lining up two infinite sets since the idea of an actual infinity is accepted, however for the constructionist there is a major issue here because the notion of actual infinity is rejected for "potential infinity". Actual infinity to the Constructionist suggests infinity is a closed realm that can be manipulated like an object as opposed to having different existential possibilities. Even though the arithmetic of infinity, called transfinite arithmetic is not viewed favorably by Constructionists (e.g., Kronecker who was an adherent of finitism), strangely enough the development of this theory by Cantor involved many constructionist proofs which are explored in the next section.

Constructing Objects in R

If one started with two numbers "a" and "b" and thought of them as lengths with $b < a$, then one can show the constructability of Q simply through Euclidean constructions, i.e., arithmetic with x and + gives it the properties of a field. In other words the four operations of arithmetic work and result in constructible lengths. In this process numbers such as $\sqrt{2}$, $\sqrt{3}$, … and well as nested radicals like $\sqrt{\sqrt{2}}$ etc. also arise which do not belong to Q.

There are three ways to deal with these new objects, either formally by extending the field of rational numbers to $Q\sqrt{a}$ for every new number $\sqrt{a}$ and showing arithmetic still works, leading to the construction of a tower of quadratic field extensions which in essence show that Euclidean numbers could be given the structure of a finite field. Another alternative for constructing Euclidean numbers like $\sqrt{2}$ is showing that an algorithm exists for constructing these numbers as multi-decked fractions called continuous fractions. The third alternative is viewing these numbers as being algebraic, i.e., as numbers that are solutions to polynomial equations in one variable with integer coefficients. $\sqrt{2}$ is the solution of $x^2 = 2$. Expressing these numbers as continued fractions allows for a constructive proof of establishing their irrationality. For example,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}$$

And this representation establishes irrationality because of another constuctive result that confers irrational status by producing an infinite continued fraction, as opposed to the traditional proof by contradiction that does not help us to construct the number.

By looking at the set of all the algebraic numbers, we produce not only all the rational numbers as solutions to these equations but all the numbers that are not rational like $\sqrt{2}$.

An interesting question now is that of countability—if Q is countable, are the Algebraic numbers also countable? At first glance this seems like a preposterous question because of the abstract nature of such a set. But Cantor's proof for the countability of these numbers is a good example of a constructive proof because it relies on the tabulation of polynomials each given a particular index. Thus, for a general polynomial $a_0 + a_1x + a_2x^2 + \ldots a_nx^n$, the index used is $n + |a_0| + |a_1| + |a_2| + \ldots |a_{n-1}| + |a_{n,}|$ which neatly generates every polynomial and every algebraic number orders according to the index of the polynomial that generates it. This interesting object is called the height function and results in a systematic enumeration of the algebraic numbers! (Fig. 20.1).

The question now is why this approach is better. Before jumping to any conclusions about a preference for either approach, we critique each of these philosophies.

Table of Cantor's Height Function

Index	Polynomials($\cdots = 0$)
0	—
1	—
2	x
3	$x^2, 3x, 2x+1, 2x-1$
4	$x^3, 2x^2, x^2+x, x^2-x, x^2+1, x^2-1, 3x, 2x+1, 2x-1, x+2, x-2$
5	$x^4, 2x^3, x^3+x^2, x^3-x^2, x^3+x, x^3-x, x^3+1, x^3-1, 3x^2, 2x^2+x, 2x^2-x, 2x^2+1, 2x^2-1$ $x^2+2x, x^2-2x, x^2+2, x^2-2, x^2+x+1, x^2+x-1, x^2-x+1, x^2-x-1, 4x, 3x+1,$ $3x-1, 2x+2, 2x-2, x+4, x-4$
6	$x^5,$ $2x^4, x^4+x^3, x^4-x^3, x^4+x^2, x^4-x^2, x^4+x, x^4-x, x^4+1, x^4-1,$ $3x^3, 2x^3+x^2, 2x^3-x^2, 2x^3+x, 2x^3-x, 2x^3+1, x^3+2x^2, x^3-2x^2, x^3+2x, x^3-2x, x^3+2, x^3-2,$ $x^3+x^2+x, x^3+x^2-x, x^3-x^2+x, x^3-x^2-x, x^3+x^2+1, x^3+x^2-1, x^3-x^2+1,$ $x^3-x^2-1, x^3+x+1, x^3+x-1, x^3-x+1, x^3-x-1,$ $4x^2, 3x^2+x, 3x^2-x, 3x^2+1, 3x^2-1, 2x^2+x+1, 2x^2+x-1, 2x^2-x+1, 2$ $x^2-x-1, 2x^2+2x, 2x^2-2x, 2x^2+2, 2x^2-2, x^2+3x, x^2-3x, x^2+3, x^2-3$ $5x, 4x+1, 4x-1, 3x+2, 3x-2, 2x+3, 2x-3, x+4, x-4$
7	$x^6,$ $2x^5, x^5 \pm x^4, x^5 \pm x^3, x^5 \pm x^2, x^5 \pm x, x^5 \pm 1,$ $3x^4, 2x^4 \pm x^3, 2x^4 \pm x^2, 2x^4 \pm x, 2x^4 \pm 1, x^4 \pm 2x^3 x^4 \pm 2x^2, x^4 \pm 2x, x^4 \pm 2, x^4 \pm x^3 \pm x^2,$ $x^4 \pm x^3 \pm x, x^4 \pm x^3 \pm 1, x^4 \pm x^2 \pm x, x^4 \pm x^2 \pm 1, x^4 \pm x \pm 1$ $4x^3, 3x^3 \pm x^2, 3x^3 \pm x, 3x^3 \pm 1, 2x^3 \pm 2x^2, 2x^3 \pm x, x^3 \pm 1, x^3 \pm 3x^2, x^3 \pm 3x, x^3 \pm 3,$ $2x^3 \pm x^2 \pm x, 2x^3 \pm x^2 \pm 1, 2x^3 \pm x \pm 1, x^3 \pm 2x^2 \pm x, x^3 \pm 2x^2 \pm 1, x^3 \pm 2x \pm 1, x^3 \pm x^2 \pm 2x,$ $x^3 \pm x^2 \pm 2, x^3 \pm x \pm 2, x^3 \pm x^2 \pm x \pm 1$ $5x^2, 4x^2 \pm x, 4x^2 \pm 1, 3x^2 \pm 2x, 2x^2 \pm 3x, 2x^2 \pm 3, x^2 \pm 4x, x^2 \pm 4, 3x^2 \pm x \pm 1, 2x^2 \pm 2x \pm 2$ $2x^2 \pm 2x \pm 1, 2x^2 \pm x \pm 2, x^2 \pm 3x \pm 1, x^2 \pm x \pm 3,$ $6x, 5x \pm 1, 4x \pm 2, 3x \pm 3, 2x \pm 4, x \pm 5$
8	$Ax^q \pm Bx^{6,5,4,3,2,1,0} \pm Cx^{5,4,3,2,1,0} \pm Dx^{4,3,2,1,0} \pm Ex^{3,2,1,0} \pm Fx^{2,1,0} \pm Gx^{1,0} \pm Hx^0$ Where $q + A + \lvert B\rvert + \lvert C\rvert + \lvert D\rvert + \lvert E\rvert + \lvert F\rvert + \lvert G\rvert + \lvert H\rvert = 8$, where $q \leq 7$, and all other exponents are less than the previous exponent.
9	$\vdots$
$\vdots$	$\vdots$

Fig. 20.1 Enumeration of algebraic numbers

A Critique of Realism (Platonism)

According to Davis and Hersh (1981) your typical mathematician is a Platonist on weekdays and a formalist on Sundays. In other words, when the mathematician is actually doing mathematics he is convinced, at least implicitly and subconsciously, that he is dealing with an objective reality whose properties he is attempting to determine. However, when the mathematician is challenged to give a philosophical account of this reality, most of them would prefer to pretend that he does not believe in it after all. For instance, when the French mathematician, and Bourbakian, Jean Dieudonne was asked about his thoughts on the nature of mathematics, he answered that: "when philosophers attack us with their paradoxes we rush to hide behind formalism and say, mathematics is just a combination of meaningless symbols, and then we bring out Chapters 1 and 2 on set theory. Finally we are left in peace to go back to our mathematics and do it as we have always done, with the feeling each mathematician has that he is working with something real. This sensation is probably an illusion, but is very convenient." (1970, p. 145). So from this apparent contradiction between doing mathematics and thinking about mathematics, we can pose the following question: if the existence or non-existence has no impact on how we do mathematics, are mathematical objects even relevant?

Mathematical realism posits that mathematical objects exists independently of the human mind, language, and practices. However, these mathematical objects are not causally efficacious, or even observable. That means that mathematicians can work on mathematical problems, prove theorems and make computations, without ever encountering these abstract mathematical objects. In other words, human mathematical activity is possible regardless of the ontology of mathematics, unless there is some unknown link between human intuition and this abstract world of mathematical objects—which leads us to a second line of criticism raised against Platonism. Benacerraf (1973) formulated what is perhaps considered the most influential objection to Platonism and mathematical realism. The short version of the argument goes something like this: according to Platonism, mathematical objects are abstract objects that exist outside the spatio-temporal world of physical things like stars, cars and human beings. It is generally agreed upon that abstract entities cannot interact with concrete entities. So how can humans, who are very much concrete entities, acquire knowledge of abstract entities like mathematical objects? According to Davis and Hersh (1981), Platonists believe that human intuition must be the link between human awareness and mathematical reality. Take for instance the continuum hypothesis.[3] Its validity depends the version of set theory that is being used, and it is therefore undecidable (Gödel 1940; Cohen 1963). The Platonists, according to Davis and Hersh (1981), would say that this situation is just an example of human ignorance, and that human intuition must be developed until this situation can be resolved and truth established. The problem is of course that Platonists have yet to describe and explain human intuition, and how it could perceive an ideal and abstract reality, similarly to how our senses perceive a physical reality. Platonism in mathematics now has two problems that make it a difficult philosophy of mathematics for the rational and scientifically oriented person.

A third issue that has also been raised against Platonism, although not as influential as the previous two, is the identification problem first developed by Benacerraf (1965). The identification problem contends that since there are an infinite number of ways of identifying the natural numbers with sets, no particular set-theoretic method can be determined to be true. For instance, we could identify the natural numbers with sets in the following two ways: A: $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\{\emptyset\}\}$, $3 = \{\{\{\emptyset\}\}\}$ and so on, while set B: $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$,

[3]The proposal originally made by Georg Cantor that there is no infinite set with a cardinal number between that of the infinite set of integers x_0 and the infinite set of real numbers (the "continuum").

3 = {Ø, {Ø}, {Ø, {Ø}}}, … Benacerraf then simply asks which of these two consists of true identity statements? A or B? Both procedures could be used to define the natural numbers, and the two sets are isomorphic in their structure, but the definitions and arithmetical statements are not identical in the two sets. For instance, the two sets differ as to whether $0 \in 2$, insofar as $\varnothing$ is not an element of $\{\{\varnothing\}\}$ (Benacerraf 1965).

A Critique of Constructionism

Constructionism then seemingly offers the mathematicians a foundation for mathematics that avoids many of the paradoxes of Platonism. Yet only a few mathematicians have embraced constructionism, even though mathematicians often value constructive results with algorithmic meaning (Davis and Hersh 1981). Why is that? Perhaps the most basic and foundational consequence of constructionism, as opposed to Platonism, is the rejection of mathematical truth independent of the human mind. To the Platonists, mathematics can and must provide truth and certainty or "where else are we to find it?" (Davis and Hersh 1981); the purity of mathematics itself would be threatened. The constructionist denies mathematical truth as independent of human intuition and human mental constructions. To them, mathematics is a (inter-)subjective enterprise, in which understanding, intuition and human mental constructions are the foundations. This view of mathematics as a human, fallible and flawed enterprise becomes intolerable to the Platonists, who sees mathematics as infallible, perfect and eternally true, waiting to be discovered.

Now, the nature of truth is more of an esoteric critique, as most working mathematicians do not concern themselves with the philosophical mysteries of the foundations of mathematics—they just do mathematics. However, there are other, more mundane and practical reasons for why the mathematical community has rejected mathematical constructionism. One reason is that mathematicians do not want to give up many of the results that are valid within Platonism, or classical mathematics, but that would be rejected within mathematical constructionism, or as David Hilbert reportedly said in 1924: "the goal (of mathematics) is to obtain more, not less theorems." (Hesseling 2003, p. 74). To the constructionists, the many extra theorems of classical mathematics add no value, as they are not proved according to the principles of constructionism (as outlined earlier in this paper). One consequence of this, is that constructionism is probably less useful to the physical sciences than classic and Platonist mathematics, as the physical sciences are not directly dependent, or even concerned, with the ontological foundations of mathematics. Fewer valid mathematical results would produce a smaller toolbox for the physical sciences.

Other reasons, which are also less philosophical in nature, comes from how results are obtained in Platonist mathematics and constructionist mathematics respectively. Proofs that use classical techniques that are allowed in Platonist mathematics, but not constructionist mathematics, are often short, elegant and clever—ideas that are closely related to the concept of mathematical beauty—while the corresponding constructive proof is longer and far more convoluted.[4] The constructivist proof has lost all of its elegance (Snapper 1979). There are also theorems that are proved in constructionist mathematics, but that are considered meaningless and invalid in Platonist mathematics due to different definitions of concepts. One such example is the theorem that states that every real-valued function which is defined for all numbers is continuous. This sounds like a strange statement outside constructivist mathematics, but within constructionist mathematics a real-valued function is defined for all real numbers if and only if for each real number r, which has been constructed, the real number $f(r)$ can be constructed. Therefore, any discontinuous function that a Platonist mathematician might mention, would not satisfy this constructive criterion (Snapper 1979). Results like this seem so bizarre to many mathematicians, that they reject constructionist mathematics in its entirety.

Constructionism and Pedagogy

Brouwer's First Act of Intuitionism is the foundation for his intuitionist beliefs. In it, he separates mathematics from mathematical language and logic, and defines mathematics as a mental exercise. Mathematics is constructed by the mind by performing changes on its own thought in time, then abstracting away from the particulars of these constructions (Brouwer 1907). Brouwer's rejection of mathematics as pure logic was a reaction to the strong relationship between semantic and ontological realism in Platonism. The Platonist would argue that our mathematical theories should be taken at face value and that they are true, and that they could not be true in the absence of mathematical objects. Or, as Davis and Hersh puts it: "To show that all of mathematics is just an elaboration of the laws of logic would have been to justify Platonism, by passing on to the rest of mathematics the indubitability of logic itself." (1981, p. 332). Brouwer, on the other hand, meant that the truth of a mathematical proposition can only be determined by a mental construction that proves it to be true. He therefore, for instance, rejected the principle of the excluded middle, and contended that our usual logical principles were abstracted from our dealing with finite sets, and these principles could not be applied to infinite sets (Ferreiros 2008).

Take for instance the infinite series of the natural numbers: $1 + 2 + 3 + 4 + 5\ldots$ which is clearly a divergent series. However, if we treat and manipulate this series as if it was a finite series, we can see all kinds of strange effects. Srinivasa Ramanujan presented a simple heuristic example of this in chapter 8 of his first notebook:

He first assumes that the sum of the series can be expressed as $c = 1 + 2 + 3 + 4\ldots$ He then goes on to multiply this equation by 4, and subtract the second equation from the first equation:

[4]See for instance a classic and constructive proofs for the fundamental theorem of Algebra.

$$\begin{aligned} c &= 1+2+3+4+5+6\ldots \\ 4c &= 4+8+12\ldots \\ -3c &= 1-2+3-4+5-6\ldots \end{aligned}$$

Ramanujan then uses the fact that the alternating series of $1 - 2 + 3 - 4 + 5\ldots$ is the power series expansion of the function $\frac{1}{(1+x)^2}$, but with $x = 1$. He can then say that $-3c = 1-2+3-4+5\ldots = \frac{1}{1+1^2} = \frac{1}{4}$. Dividing both sides by -3, one gets: $c = -\frac{1}{12}$.

Which is clearly an absurd result, but illustrates how strange results can appear if you treat an infinite (divergent) series as a finite series. We chose to call this a *platonic leap of faith*, and it illustrates how logic and human intuition diverge (!) when we move from the finite to the infinite.[5]

Intuitionists, or constructionists, thus find non-constructive existence proofs unacceptable. Non-constructive existence proofs are proofs that claim to demonstrate the existence of a mathematical entity having a certain property without producing a method for generating such an entity. The difference between providing a method for creating a certain mathematical object and simply proving that such an object must exist, is in many ways related to the ideas of *need for certainty* and *need for causality*, which are two subcategories of what Harel (2013) calls *intellectual need*. Intellectual need is essentially defined as the knowledge an individual needs to learn, acquire or construct, to solve a particular problem. The *need for certainty* is, according to Harel (2013), based on a Piagetian theory of equilibration, a natural human desire to know whether a conjecture is true or false. Truth and certainty, however, may not be enough for an individual. The individual will often also want to know how and why something is true. The need for causality is a person's desire to explain and to determine a cause of phenomenon. Constructive proofs can be compared with a need for causality, while non-constructive proofs can be said to be more closely related to a need for certainty: "Mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain." (Steiner 1978, p. 135). A typical example of noncausal, and non-constructive, proof would be the proof by contradiction to establish the irrational status of $\sqrt{2}$.

However, the analogy between constructionism in mathematical philosophy and the need for causality in teaching and learning (didactical situations) may not be perfect. Proofs by mathematical induction are for instance not rejected a priori, as they could be seen as a sort of iterated modus ponens, which is a logical principle generally accepted by the intuitionists. Within the mathematics education community, there are those who claim that proofs by induction establish certainty, but they do not provide an explanation for why a proposition is true: "a proof that explains must provide a rationale based upon the mathematical ideas involved, the

[5]A rigorous proof $\zeta(-1) = -1/12$ can be found in: Stopple, J. (2003). A primer of analytic number theory: from Pythagoras to Riemann. Cambridge University Press.

mathematical properties that cause the asserted theorem to be true." (Hanna 1990, p. 9). Harel proposes a possible resolution to this ostensible difference between constructive proofs and proofs that explain, by drawing on the ideas of Brouwer: "Hanna (1990), who argues that proofs by mathematical induction, for example, are proofs that prove but do not explain. Our position is different. We hold that it is the individual's scheme of doubts, truths, and convictions in a given context that determines whether an argument is a proof or an explanation." (2013, p. 128). Here, Harel presupposes mathematics as a human and mental activity, and proposes that whether or not a proof provides causality, depends on the individual learner's preexisting understanding and mental schemes.

Again, we go back to the series of sum of the natural numbers to illustrate Harel's point. For the first n numbers, we have that $0+1+2+3\ldots+n=\frac{n(n+1)}{2}$. Proof by induction would first start by showing that the statement holds for $n=1$, which is obviously true, as the two sides of the equation would be equal. The inductive step shows that if the statement is true for $n=k$, then it would also be true for $n=k+1$. We assume that the statement is true for some value of k and we must now demonstrate that the statement is true for $k+1$:

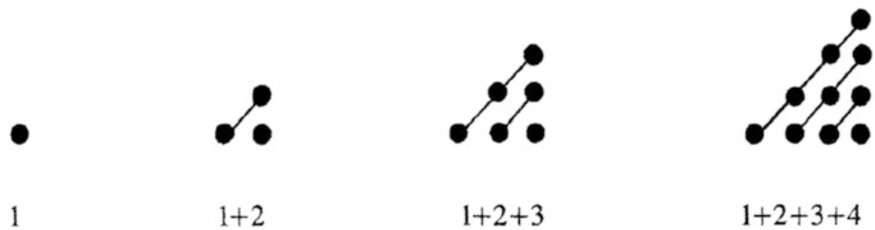
$$(0+1+2+3\ldots+k)+(k+1)=\frac{(k+1)((k+1)+1)}{2}$$

Using the induction hypothesis that the statement holds for $n=k$, the left hand side can be rewritten to:

$$\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)((k+1)+1)}{2}$$

Thereby showing that indeed $n=k+1$ holds.

Now, Hanna (1990) claims that although this proof demonstrates that a certain mathematical statement is true, it does not show why the sum of the first n natural numbers is $\frac{n(n+1)}{2}$. However, if we look at proof by induction as a recursive process, we can illustrate this sequence in the following way:



Here we see that the dots form isosceles right triangles, and if we double them, we get rectangles with $n(n+1)$ dots. The rectangles are exactly twice the size of the corresponding sum, so the sum of the first n numbers is $\frac{n(n+1)}{2}$, and we can do this for $n = 1$, $n = 2$, $n = 3$, and so on. So, as Harel (2013) says, a proof by induction can very much be a proof that also explains—it depends on the individual's preexisting knowledge and how the individual perceives the proof. We now see how a constructionist proof, that is based on human mental activity and human intuition, is in many ways analogous to mathematics educators' call for proofs that explain—both begin with the human mind, and not the laws of logic, as a starting point!

Concluding Points

Mathematics is one single thing. The Platonist, formalist and constructionist views of it are believed because each corresponds to a certain view of it, a view from a certain angle, or an examination with a particular instrument of observation. This view is corroborated by Friend in her thesis on pluralistic views of mathematics being compatible with model building (Friend 2017). Grosholz (2016) gives other examples of this working philosophy through models (examples from celestial mechanics) which are developed simultaneously by different people using completely different methods from analysis that reflect different, even apposite views of the philosophy of mathematics. There are plenty of other examples that can be used to make the case that the realist/anti-realist dichotomy is false. One such classical result is: Gauss' result about the constructability of regular polygons and its relationship to Fermat primes. Most modern books use a realist approach using heavy tools from abstract algebra, whereas Gauss invented those tools very informally as he was tackling the problem from a number theoretic viewpoint. His approach is very anti-realist. More modestly put, the realist/anti-realist dichotomy is reconcilable.

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