Chapter 9
Evolution Equations for Defects in Finite Elasto-Plasticity

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Abstract The paper deals with continuous models of elasto-plastic materials with microstructural defects such as dislocations and disclinations. The basic assumptions concern the existence of plastic distortion and so-called plastic connection with metric property and the existence of the free energy function. This is dependent on the Cauchy-Green strain tensor, and its gradient with respect to the plastically deformed anholonomic configuration, and on the dislocation and disclination densities. The defect densities are defined in terms of the incompatibility of the plastic distortion and non-integrability of the plastic connection. The evolution of plastic distortion and disclination tensor has been postulated under the appropriate viscoplastic and dissipative type equations, which are compatible with the principle of the free energy imbalance. The associated small distortion model is provided. The present model and the previous ones have been also compared.

9.1 Introduction

The paper deals with defects in crystalline materials, when the differential geometry description is used in order to characterize lattice defects existing at the micro structural level, see Kröner (1990). Here we restrict ourselves to dislocations and disclinations and we make reference to different continuous descriptions which are close to the background of our finite elasto-plastic model.

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9.1.1 Defects in Linear eElasticity

The elastic model for the defects such as dislocations and disclinations have been described by the solutions of the linear theory of elasticity having the displacement fields discontinuous along cut-off surfaces, see de Wit (1973a,b); Teodosiu (1982). The problems formulated by de Wit (1973c) and Kossecka and de Wit (1977) concern the finding of the elastic basic fields (i.e. strain and bent twist tensors) and the stress, when the plastic fields, namely the defect densities, $\boldsymbol{\alpha}$ for dislocations and $\boldsymbol{\omega}$ for disclinations, have been prescribed (without specifying the nature of these defects). The incompatibilities in linear elasticity were reviewed by de Wit (1970, 1981); Kossecka and de Wit (1977), see also Fressengeas et al (2011). Traditionally the dislocations are determined by Burgers vector, $\boldsymbol{b}$, which is equal to the translational displacement, and the disclinations are characterized by Frank vector, $\boldsymbol{\Omega}$, which is equal to the rotational displacement, see de Wit (1973a,c); Kossecka and de Wit (1977). The internal mechanical state of solids with defects leads Kröner (1992), to solve the elastic problems with given incompatibilities.

9.1.2 Defects in Non-Linear Elasticity

Yavari and Goriely (2013) considered the cases of a single wedge disclination and a parallel cylindrically-symmetric distribution of wedge disclination, respectively. They solved the problem of existence of the residual stress: find the stress distribution in an Neo-Hookean material, which is stress free, namely zero traction is applied on the outer radius of the cylinder, when the wedge disclination densities, as those mentioned above, have been given. First Yavari and Goriely (2013) construct a Riemannian material manifold which is metric compatible having zero torsion and non-zero curvature, for the given disclination density identified with the curvature tensor. The manifold corresponds to Volterra's geometrical description of wedge disclination in a cylindrical body: cut, following by removing or inserting material, and weld operations. As an example of Riemann-Cartan geometry Yavari and Goriely (2012) built the material manifold which is dependent on the distribution of dislocations, namely for a screw and a cylindrically-symmetric distribution of parallel screw dislocations, respectively. The material manifold has torsion and vanishing non-metricity, and corresponds to Volterra's description of screw dislocation: cut a half axial plane of the cylinder followed by the displacement with Burgers vector along the symmetry axis, and weld procedure with removing the axis. The authors found the residual stress for a generalized Neo-Hookean elastic body induced by several distributed dislocations, including also the radially-symmetric distributed edge dislocations.

9.1.3 Defects in Nonlocal Elasticity

An improvement has been obtained using the nonlocal elasticity instead of the classical one, namely the stress and strain energy singularities, which are present in classical elasticity, have been eliminated. The nonlocal elasticity is given by an integral type constitutive equations, which are characterized by the so-called nonlocal kernel. The approaches to nonlocal elasticity proposed and discussed in Eringen (2002) replace the integral operators with a special class of kernels by certain differential operators. The integral representation of the stress is given in terms of the Hookean stress. The displacement fields are identical with the classical forms obtained by integrating the stress-strain relations of linear elasticity. Solutions for screw dislocation, edge dislocation and wedge disclination have been described and analyzed within the nonlocal elasticity, with Gaussian kernel, and for a special class of kernels, which are Green functions of the Helmholtz equation in Eringen (2002) and for bi-Helmholtz equation in Lazar et al (2006). Lazar and Maugin (2004a,b), developed first a constitutive framework of gradient micropolar isotropic elasticity, which was connected to the nonlocal micropolar elasticity given by Eringen (2002). Second, Lazar and Maugin examined the mentioned defects in gradient micropolar elasticity. The micropolar distortion and bent twist tensors satisfy the appropriate inhomogeneous Helmholtz equations, with the inhomogeneities identified with classical elastic expressions for the stress and couple stress tensor. The authors did not derive the associate boundary conditions because they considered an infinite extended medium. Only in a small region in the vicinity of $r = 0$, the stress calculated in nonlocal elasticity of Helmholtz or bi-Helmholtz type is different, both of them being zero at $r = 0$. Eringen's results were recovered too.

9.1.4 Elasto-Plastic Models for Defects

Continuum models of these defects involve the couple stresses within the micropolar materials and Cosserat continuum, see Clayton et al (2006); Fressengeas et al (2011). In the models developed by Arsenlis and Parks (1999); Gurtin (2002) the Burgers vector has been defined by the geometrically necessary dislocation (GND) tensor $\boldsymbol{G} = \boldsymbol{F}^p \mathrm{curl} \boldsymbol{F}^p$ in the lattice space. The GND density tensor is decomposed in the appropriate edge and screw dislocations. Clayton et al (2006) introduced the geometrically necessary defect density tensors in the deformed configuration, $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$, accounting for the incompatibilities induced by the torsion and curvature tensor, which are associated with the connection $\boldsymbol{\Gamma}$. The connection coefficients were defined as in Minagawa (1979) in terms of non-Riemannian's type connection (using $(\boldsymbol{F}^{\mathscr{L}})^{-1}$) and a third order tensor field $\boldsymbol{Q}$ with assigned skew-symmetry. We recall the multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}^{\mathscr{L}} \boldsymbol{F}^p$ assumed by Clayton et al (2006), where the lattice part $\boldsymbol{F}^{\mathscr{L}}$ is given by $\boldsymbol{F}^{\mathscr{L}} = \boldsymbol{F}^e \boldsymbol{F}^i$, $\boldsymbol{F}^e$ is the elastic part, $\boldsymbol{F}^i$ is the residual part due to the micro-heterogeneity in the presence in the of lattice defects. The free energy function in the intermediate configuration is dependent on elastic Cauchy-

Green strain tensor (expressed in terms of $\boldsymbol{F}^e$), defect density tensors pulled back to the intermediate configuration by $\boldsymbol{F}^e$, i.e. $\widetilde{\boldsymbol{\alpha}}$ and $\widetilde{\boldsymbol{\theta}}$, the symmetric and positive definite part of $\boldsymbol{F}^i$, and so on.

9.1.5 Aim of this Paper

Fressengeas et al (2011) proposed a *field defect* (dislocation and disclination), restricted to small strains. The non-symmetric Cauchy stress, $\boldsymbol{T}$, and couple- stress tensor, $\boldsymbol{m}$, are described in terms of elastic strain and bent-twist, $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\kappa}^e$, with macro forces, $\boldsymbol{T}$ and $\boldsymbol{m}$, satisfying the balance equation formulated by Fleck et al (1994). The evolution equations for basic plastic fields, $\boldsymbol{\varepsilon}^p$, $\boldsymbol{\kappa}^p$, are dependent on the density of dislocations and disclinations, $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$, and on the macro forces. The dislocation density $\boldsymbol{\theta}$ generates a Frank vector $\boldsymbol{\Omega}$.

In this paper we propose a model for structural defects such as dislocations and disclinations, which can be viewed as an improvement of the models provided by Cleja-Ţigoiu (2014); Cleja-Ţigoiu et al (2016), within the constitutive framework developed by Cleja-Ţigoiu (2007, 2010). The key point is related to the expression of the free energy density, this time also dependent on the gradient of the Cauchy-Green elastic strain with respect to the plastically deformed configuration or the so-called configuration with torsion. The basic assumptions concern the existence of plastic distortion and so-called plastic connection with metric property and the existence of the free energy function. This function is dependent on the Cauchy-Green strain tensor and its gradient with respect to the plastically deformed anholonomic configuration, and on dislocation and disclination densities. The defect densities are defined in terms of the incompatibility of the plastic distortion and non-integrability of the plastic connection, respectively. The free energy imbalance principle is postulated in a similar form with those presented by Cleja-Ţigoiu (2007, 2010), following the ideas given by Gurtin (2002); Gurtin et al (2010). The balance equation for micro forces have been considered in the form provided by Cleja-Ţigoiu (2007, 2017). The constitutive and evolution equations for plastic distortion and disclination tensor are derived to be compatible with the free energy imbalance principle. The evolution of plastic distortion and disclination tensor has been postulated under the appropriate viscoplastic and dissipative type equations. The associated small distortion model is also provided. The proposed model is compared with the previous models discussed in Cleja-Ţigoiu et al (2016); Cleja-Tigoiu and Maugin (2000).

9.1.6 List of Notations

Further the following notations will be used:
$\mathscr{E}$ - the three dimensional Euclidean space, with the vector space of translations $\mathscr{V}$;
Lin - the set of the linear mappings from $\mathscr{V}$ to $\mathscr{V}$, i.e the set of second order tensor,

$Sym_2, Skw_2 \subset Lin$ are the sets of all symmetric and skew-symmetric second order tensors, respectively; $\boldsymbol{u} \cdot \boldsymbol{v}, \boldsymbol{u} \times \boldsymbol{v}, \boldsymbol{u} \otimes \boldsymbol{v}$ denote scalar, cross and tensorial products of vectors; $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}) := (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{z}$ is the mixed product of the vectors from $\mathscr{V}$. $\boldsymbol{a} \otimes \boldsymbol{b}$ and $\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c}$ are defined to be a second order tensor and a third order tensor by $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{u} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{u}), (\boldsymbol{a} \otimes \boldsymbol{b} \otimes \boldsymbol{c})\boldsymbol{u} = (\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \cdot \boldsymbol{u})$, for all vectors $\boldsymbol{u}$.

For $\boldsymbol{A} \in Lin-$ a second order tensor, we introduce:
the notations $\{\boldsymbol{A}\}^S, \{\boldsymbol{A}\}^a$ for the symmetric and skew-symmetric parts of the tensor; definition of the trace: $\mathrm{tr}\boldsymbol{A}((\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{z}) = (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}) + (\boldsymbol{u}, \boldsymbol{A}\boldsymbol{v}, \boldsymbol{z}) + (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}\boldsymbol{z})$. $\boldsymbol{I}$ is the identity tensor in Lin, $\boldsymbol{A}^T$ denotes the transpose of $\boldsymbol{A} \in Lin$, $\partial_{\boldsymbol{A}}\phi(x)$ denotes the partial differential of the function ϕ with respect to the field $\boldsymbol{A}$.

Let $\chi : \mathscr{B} \times \boldsymbol{R} \to \mathscr{V}$ defines the motion of the body $\mathscr{B}$. The deformation gradient and its gradient are expressed in coordinate systems by

$$\boldsymbol{F}(\boldsymbol{X},t) = \nabla\chi(\boldsymbol{X},t) = \frac{\partial x^i}{\partial X^j}\boldsymbol{g}_i \otimes \boldsymbol{G}^j, \quad \nabla\boldsymbol{F}(\boldsymbol{X},t) = \frac{\partial^2 x^i}{\partial X^j \partial X^k}\boldsymbol{g}_i \otimes \boldsymbol{G}^j \otimes \boldsymbol{G}^k. \tag{9.1}$$

Here $\{\boldsymbol{g}_i\}_{i=1,2,3}$ and $\{\boldsymbol{G}_i\}_{i=1,2,3}$ are local bases in the actual and reference configurations, respectively.

In what follows the anholonomic basis vectors, in the so-called plastically deformed configuration or the configuration with torsion, generically denoted by $\mathscr{K}$, are related with *the crystal* and defined by $\boldsymbol{e}_j = \boldsymbol{F}^p\boldsymbol{G}_j$. Let $\{\boldsymbol{G}^i\}_{i=1,2,3}$ be the reciprocal basis in the reference configuration. The plastic connection is represented by its coefficients in a component representation given by

$$\overset{(\mathrm{p})}{\boldsymbol{\Gamma}} = \Gamma^{\alpha}_{\beta\,\gamma}\boldsymbol{G}_\alpha \otimes \boldsymbol{G}^\beta \otimes \boldsymbol{G}^\gamma. \tag{9.2}$$

The differential of smooth field $\boldsymbol{A}$, with respect to the anholonomic configuration $\mathscr{K}$, obeys the rule

$$\nabla_{\mathscr{K}}\boldsymbol{A} = (\nabla\boldsymbol{A})(\boldsymbol{F}^p)^{-1}. \tag{9.3}$$

curl of a second order tensor field $\boldsymbol{A}$ is defined by the second order tensor field

$$\begin{aligned} &(curl\boldsymbol{A})(\boldsymbol{u} \times \boldsymbol{v}) := (\nabla\boldsymbol{A}(\boldsymbol{u}))\boldsymbol{v} - (\nabla\boldsymbol{A}(\boldsymbol{v}))\boldsymbol{u} \quad \forall\boldsymbol{u},\boldsymbol{v} \in \mathscr{V}, \\ &(curl\boldsymbol{A})_{pi} = \epsilon_{ijk}\frac{\partial A_{pk}}{\partial x^j} \end{aligned} \tag{9.4}$$

are the components of *curl*$\boldsymbol{A}$ given in a Cartesian basis.

The transpose of the third order tensor tensor field is defined by

$$\mathscr{A}^T(\boldsymbol{u}) := (\mathscr{A}\boldsymbol{u})^T, \quad \forall\boldsymbol{u} \in \mathscr{V}. \tag{9.5}$$

The third order tensors, denoted by $\mathscr{A}$, are linear mapping defined as element of the set $Lin\{\mathscr{V}, Lin\}$, which are represented in a Cartesian basis $\{\boldsymbol{e}_i\}_{i=1,2,3}$ as

$$\mathscr{A} := \mathscr{A}_{ijk}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k. \tag{9.6}$$

The scalar product of second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, and of the third order tensors $\mathscr{A}$ and $\mathscr{B}$, are defined in terms of their Cartesian components by

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{B} &= A_{ij}B_{ij}, \quad \forall\, \boldsymbol{A},\boldsymbol{B}\in Lin, \\ \mathscr{A}\cdot\mathscr{B} &= \mathscr{A}_{ijk}\mathscr{B}_{ijk}, \quad \forall\, \mathscr{A},\mathscr{B}\in Lin\{\mathscr{V},Lin\}. \end{aligned} \tag{9.7}$$

For any $\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2 \in Lin$ we define a third order tensor associated with them, denoted $\boldsymbol{\Lambda}_1\times\boldsymbol{\Lambda}_2$, by

$$((\boldsymbol{\Lambda}_1\times\boldsymbol{\Lambda}_2)\boldsymbol{u})\boldsymbol{v} = (\boldsymbol{\Lambda}_1\boldsymbol{u})\times(\boldsymbol{\Lambda}_2\boldsymbol{v}), \quad \forall \quad \boldsymbol{u},\boldsymbol{v}. \tag{9.8}$$

The notations $Sym\mathscr{A}$ and $Skw\mathscr{A}$ are introduced for all $\mathscr{A}\in Lin\{\mathscr{V},Lin,\}$ by

$$\begin{aligned} &Sym\mathscr{A}\in Lin\{\mathscr{V},Lin\}, \quad Sym(\mathscr{A}) = \mathscr{A}+\mathscr{A}^T, \\ &Skw\mathscr{A}\in Lin\{\mathscr{V},Lin\}, \quad ((Skw\mathscr{A})\boldsymbol{u})\boldsymbol{v} := (\mathscr{A}\boldsymbol{u})\boldsymbol{v}-(\mathscr{A}\boldsymbol{v})\boldsymbol{u} \quad \forall \boldsymbol{u}\in\mathscr{V}. \end{aligned} \tag{9.9}$$

The following identity holds

$$\mathscr{A}\cdot Sym(\boldsymbol{C}\mathscr{B}) = (\boldsymbol{C}(Sym\mathscr{A}))\cdot\mathscr{B}\,\forall\,\boldsymbol{C}\in Sym_2,\ \mathscr{A},\mathscr{B}\in Lin\{\mathscr{V},Lin\}. \tag{9.10}$$

The third order tensor, denoted by $\mathscr{A}[\boldsymbol{F}_1,\boldsymbol{F}_2]$, is associated to the set of tensors $\mathscr{A}\in Lin(\mathscr{V},Lin)$, and $\boldsymbol{F}_1,\boldsymbol{F}_2\in Lin$, and is defined by

$$((\mathscr{A}[\boldsymbol{F}_1,\boldsymbol{F}_2])\boldsymbol{u})\boldsymbol{v} = (\mathscr{A}(\boldsymbol{F}_1\boldsymbol{u}))\boldsymbol{F}_2\boldsymbol{v}, \quad \forall\,\boldsymbol{u},\boldsymbol{v}\in\mathscr{V}. \tag{9.11}$$

or by its Cartesian components $(\mathscr{A}[\boldsymbol{F}_1,\boldsymbol{F}_2])_{ilq} = \mathscr{A}_{ijk}(\boldsymbol{F}_1)_{jl}(\boldsymbol{F}_2)_{kq}$. Two types of second order tensors that can be associated with any pair of third order tensors, $\mathscr{A},\mathscr{B}$, following the rules written below

$$\begin{aligned} (\mathscr{A}\odot\mathscr{B})\cdot\boldsymbol{L} &= \mathscr{A}[\boldsymbol{I},\boldsymbol{L}]\cdot\mathscr{B} = \mathscr{A}_{isk}L_{sn}\mathscr{B}_{ink}, \\ (\mathscr{A}\ {}_r\!\odot\mathscr{B})\cdot\boldsymbol{L} &= \mathscr{A}\cdot(\boldsymbol{L}\mathscr{B}) = \mathscr{A}_{ijk}L_{in}\mathscr{B}_{njk}, \quad \boldsymbol{L}\in Lin. \end{aligned} \tag{9.12}$$

9.2 Elasto-Plastic Materials with Lattice Defects

We recall our basic relationships which characterize the elasto-plastic material from the geometrical point of view. The motion of the body, χ, induces a second order deformation which is defined by $(\boldsymbol{F},\boldsymbol{\Gamma} := (\boldsymbol{F})^{-1}(\nabla\boldsymbol{F}))$, $\boldsymbol{\Gamma}$ is called the motion connection.

The *multiplicative decomposition* of the deformation gradient $\boldsymbol{F}$ into its elastic and plastic components, $\boldsymbol{F}^e$ and $\boldsymbol{F}^p$, called distortions, namely

$$\boldsymbol{F} = \boldsymbol{F}^e\boldsymbol{F}^p \tag{9.13}$$

is considered.

Definition 9.1. The Cauchy-Green elastic strain tensor, $\boldsymbol{C}^e$, and Cauchy-Green plastic strain tensor, $\boldsymbol{C}^p$, are expressed by

$$\begin{aligned}&\boldsymbol{C}^e = (\boldsymbol{F}^e)^T\boldsymbol{F}^e, \quad \Longrightarrow \quad \boldsymbol{C}^e = (\boldsymbol{F}^p)^{-T}\boldsymbol{C}(\boldsymbol{F}^p)^{-1}, \quad \text{where} \quad \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F},\\ &\boldsymbol{C}^p = (\boldsymbol{F}^p)^T\boldsymbol{F}^p.\end{aligned} \tag{9.14}$$

The gradient of $\boldsymbol{C}^e$ with respect to the configuration $\mathscr{K}$ can be expressed by

$$\nabla_{\mathscr{K}}\boldsymbol{C}^e = (\boldsymbol{F}^p)^{-T}\left(\nabla\boldsymbol{C} - Sym\{\boldsymbol{C}\overset{(p)}{\mathscr{A}}\}\right)[(\boldsymbol{F}^p)^{-1},(\boldsymbol{F}^p)^{-1}] \tag{9.15}$$

As a direct consequence of the multiplicative decomposition formula (9.19) the following relationships hold

$$\boldsymbol{L} = \boldsymbol{L}^e + \boldsymbol{F}^e\boldsymbol{L}^p(\boldsymbol{F}^e)^{-1}, \quad \text{where} \quad \boldsymbol{L}^e = \dot{\boldsymbol{F}}^e(\boldsymbol{F}^e)^{-1}, \quad \boldsymbol{L}^p = \dot{\boldsymbol{F}}^p(\boldsymbol{F}^p)^{-1}. \tag{9.16}$$

The plastic rate tensors with respect to the plastically deformed and reference configurations, respectively, and denoted by $\boldsymbol{L}^p$ and $\underline{\boldsymbol{l}}^p$, are related by

$$\boldsymbol{L}^p = \boldsymbol{F}^p\underline{\boldsymbol{l}}^p(\boldsymbol{F}^p)^{-1}, \quad \text{where} \quad \underline{\boldsymbol{l}}^p = (\boldsymbol{F}^p)^{-1}\dot{\boldsymbol{F}}^p. \tag{9.17}$$

The following time derivatives can be computed

$$\begin{aligned}&\frac{d}{dt}(\boldsymbol{C}^p)^{-1} = -\underline{\boldsymbol{l}}^p(\boldsymbol{C}^p)^{-1} - (\boldsymbol{C}^p)^{-1}(\underline{\boldsymbol{l}}^p)^T,\\ &\dot{\boldsymbol{C}} = 2\boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F}, \quad \text{where} \quad \boldsymbol{D} = \{\boldsymbol{L}\}^S,\\ &\nabla\dot{\boldsymbol{C}} = (\dot{\boldsymbol{C}}\boldsymbol{\Gamma})^T + \boldsymbol{F}^T(\nabla_\chi\boldsymbol{D})[\boldsymbol{F},\boldsymbol{F}] + \dot{\boldsymbol{C}}\boldsymbol{\Gamma}, \quad \text{where} \quad \boldsymbol{\Gamma} = \boldsymbol{F}^{-1}\nabla\boldsymbol{F},\\ &\frac{d}{dt}\{(\boldsymbol{C}^p)^{-1}(\boldsymbol{\Lambda}\times\boldsymbol{I})\} = -(\underline{\boldsymbol{l}}^p(\boldsymbol{C}^p)^{-1} + (\boldsymbol{C}^p)^{-1}(\underline{\boldsymbol{l}}^p)^T)(\boldsymbol{\Lambda}\times\boldsymbol{I}) +\\ &\qquad\qquad + (\boldsymbol{C}^p)^{-1}(\dot{\boldsymbol{\Lambda}}\times\boldsymbol{I}).\end{aligned} \tag{9.18}$$

Here $\boldsymbol{\Lambda}$ is a second order tensor.

Proposition 9.1. *Under the hypothesis of the multiplicative decomposition of the deformation gradient* $\boldsymbol{F}$*, postulated in (9.19), we get the composition rule of the motion connection*

$$\begin{aligned}&\boldsymbol{\Gamma} = (\boldsymbol{F}^p)^{-1}\overset{(e)}{\mathscr{A}}_{\mathscr{K}}[\boldsymbol{F}^p,\boldsymbol{F}^p] + \overset{(p)}{\mathscr{A}},\\ &\textit{where} \quad \overset{(e)}{\mathscr{A}}_{\mathscr{K}} := (\boldsymbol{F}^e)^{-1}\nabla_{\mathscr{K}}\boldsymbol{F}^e, \quad \overset{(p)}{\mathscr{A}} := (\boldsymbol{F}^p)^{-1}\nabla\boldsymbol{F}^p.\end{aligned} \tag{9.19}$$

$\overset{(e)}{\mathscr{A}}_{\mathscr{K}}$ and $\overset{(p)}{\mathscr{A}}$ define Bilby's type connections (see Bilby, 1960) with respect to the so-called configuration with torsion, and initial one, respectively.

Based on the definitions for elastic strain $\boldsymbol{C}^e$ and its gradient $\nabla_{\mathscr{K}}\boldsymbol{C}^e$ the following relation holds

$$\nabla_{\mathscr{K}}\boldsymbol{C}^e = (\boldsymbol{C}^e \overset{(e)}{\mathscr{A}}_{\mathscr{K}})^T + \boldsymbol{C}^e(\overset{(e)}{\mathscr{A}}_{\mathscr{K}}), \tag{9.20}$$

see (9.19) together with (9.14).

The formula (9.20) states that the so-called elastic Bilby's connection $\overset{(e)}{\mathscr{A}}_{\mathscr{K}}$ has metric property in $\mathscr{K}$ with respect to the elastic metric tensor $\boldsymbol{C}^e$, using a definition given by Schouten (1954), see also Yavari and Goriely (2012).

9.2.1 Plastic Connection with Metric Property

We accepted the existence of the plastic connection with metric property, see Cleja-Ţigoiu (2010).

Definition 9.2. The connection $\overset{(p)}{\boldsymbol{\Gamma}}$ has metric property if and only if the following relationship holds

$$\nabla\boldsymbol{C}^p = (\boldsymbol{C}^p \overset{(p)}{\boldsymbol{\Gamma}})^T + \boldsymbol{C}^p \overset{(p)}{\boldsymbol{\Gamma}}\,. \tag{9.21}$$

The relationship (9.21) is similarly to (9.20).

Proposition 9.2. *The plastic connection, which is metric compatible with respect to the metric tensor $\boldsymbol{C}^p$, is represented under the form*

$$\overset{(p)}{\boldsymbol{\Gamma}} = \overset{(p)}{\mathscr{A}} + (\boldsymbol{C}^p)^{-1}(\boldsymbol{\Lambda}\times\boldsymbol{I}), \tag{9.22}$$

where the third order tensor $\boldsymbol{\Lambda}\times\boldsymbol{I}$ is generated by the second order (covariant) tensor $\boldsymbol{\Lambda}$, which is called disclination tensor.

Proof. The proof can be found in Cleja-Ţigoiu (2010).

Definition 9.3. The (Cartan) torsion associated with the plastic connection (9.22) is calculated in a given coordinate system by the skew-symmetric part of the connection, as it follows

$$(\boldsymbol{S}^p\boldsymbol{v})\boldsymbol{u} \equiv (\overset{(p)}{\boldsymbol{\Gamma}}\,\boldsymbol{v})\boldsymbol{u} - (\overset{(p)}{\boldsymbol{\Gamma}}\,\boldsymbol{u})\boldsymbol{v} \equiv ((Skw\boldsymbol{\Gamma})\boldsymbol{u})\boldsymbol{v}. \tag{9.23}$$

Let us remark that the Cartan torsion (9.23) can be expressed as a consequence of the formula (9.22) by

$$\boldsymbol{S}^p = Skw\,\overset{(p)}{\mathscr{A}} + Skw((\boldsymbol{C}^p)^{-1}(\overset{(p)}{\boldsymbol{\Lambda}}\times\boldsymbol{I})). \tag{9.24}$$

Proposition 9.3. *The second order torsion tensor $\mathscr{N}^p$, which is associated with Cartan torsion (9.24), is expressed by*

$$\begin{aligned} &\mathscr{N}^p = (\boldsymbol{F}^p)^{-1} curl\boldsymbol{F}^p + (\boldsymbol{C}^p)^{-1}\big((tr\,\boldsymbol{\Lambda})\boldsymbol{I} - (\boldsymbol{\Lambda})^T\big), \\ &where \quad (\boldsymbol{S}^p\boldsymbol{u})\boldsymbol{v} = \mathscr{N}^p(\boldsymbol{u}\times\boldsymbol{v}). \end{aligned} \tag{9.25}$$

Proof. The formula (9.31) follows directely from the mentioned calculus rules:

1. $\forall \mathscr{N} \in Lin(\mathscr{V}, Lin)$ there exists

$$\begin{aligned} &\boldsymbol{\Omega}_{<\mathscr{N}>} \in Skw_2, \quad \text{such that} \quad Skw\mathscr{N} = \boldsymbol{\Omega}_{<\mathscr{N}>}(\boldsymbol{I}\times\boldsymbol{I}), \\ &\text{which means} \quad ((Skw\mathscr{N})\boldsymbol{u})\boldsymbol{v} = \boldsymbol{\Omega}_{<\mathscr{N}>}(\boldsymbol{u}\times\boldsymbol{v}),\ \forall\, \boldsymbol{u},\boldsymbol{v}\in\mathscr{V}. \end{aligned} \tag{9.26}$$

2. The following component representations hold

$$\begin{aligned} &(Skw\mathscr{N})_{ijk} = (\boldsymbol{\Omega}_{<\mathscr{N}>})_{im}\epsilon_{mkj}, \\ &2(\boldsymbol{\Omega}_{<\mathscr{N}>})_{im} = (Skw\mathscr{N})_{ijk}\epsilon_{mkj}. \end{aligned} \tag{9.27}$$

   Here ϵ_{mkj} denotes components of Ricci's permutation tensor, namely $\boldsymbol{\epsilon}$.
3. The following results can be proved

$$\begin{aligned} &Skw \overset{(p)}{\mathscr{A}} = curl\boldsymbol{F}^p(\boldsymbol{I}\times\boldsymbol{I}), \quad \text{that is} \\ &((Skw \overset{(p)}{\mathscr{A}})\boldsymbol{u})\boldsymbol{v} = (curl\boldsymbol{F}^p)(\boldsymbol{u}\times\boldsymbol{v}),\ \forall\, \boldsymbol{u},\boldsymbol{v}\in\mathscr{V}, \\ &Skw(\boldsymbol{\Lambda}\times\boldsymbol{I}) = (tr\boldsymbol{\Lambda}\boldsymbol{I} - \boldsymbol{\Lambda}^T)(\boldsymbol{I}\times\boldsymbol{I}), \quad \text{namely} \\ &(Skw(\boldsymbol{\Lambda}\times\boldsymbol{I})\boldsymbol{u})\boldsymbol{v} = (tr\boldsymbol{\Lambda}\boldsymbol{I} - \boldsymbol{\Lambda}^T)(\boldsymbol{u}\times\boldsymbol{v}). \end{aligned} \tag{9.28}$$

9.2.2 Measure of Defects

The disclination tensor with respect to the configuration with torsion will be denoted by $\widetilde{\boldsymbol{\Lambda}}$, see Cleja-Ţigoiu (2010), and it will be introduced here through

$$\widetilde{\boldsymbol{\Lambda}} = \boldsymbol{F}^p\boldsymbol{\Lambda}(\boldsymbol{F}^p)^{-1}. \tag{9.29}$$

We define *Burgers* and *Frank vectors* in terms of the plastic distortion $\boldsymbol{F}^p$ and disclination tensor $\widetilde{\boldsymbol{\Lambda}}$. Both vectors are associated with a circuit C_0. Let $\mathscr{A}_0$ be a surface with normal $\boldsymbol{N}$, which is surrounded by C_0 in the reference configuration.

Definition 9.4. The Frank vector associated with a circuit C_0 is defined by

$$\boldsymbol{\Omega}_{\mathscr{K}} = \int_{C_{\mathscr{K}}} \widetilde{\boldsymbol{\Lambda}}\, d\boldsymbol{x}_{\mathscr{K}} = \int_{C_0} \widetilde{\boldsymbol{\Lambda}}\boldsymbol{F}^p\, d\boldsymbol{X} = \int_{\mathscr{A}_0} curl(\boldsymbol{F}^p\boldsymbol{\Lambda})\boldsymbol{N} dA. \tag{9.30}$$

Definition 9.5. The *disclination density tensor* with respect to the reference configuration is defined by

$$\boldsymbol{\omega} = curl(\boldsymbol{F}^p \boldsymbol{\Lambda}). \tag{9.31}$$

The Burgers vector is defined in terms of the plastic distortion $\boldsymbol{F}^p$.

Definition 9.6. The Burgers vector associated with the circuit C_0 is defined by

$$\boldsymbol{b}_{\mathscr{K}} = \int_{C_0} \boldsymbol{F}^p \, d\boldsymbol{X} = \int_{\mathscr{A}_0} (curl \boldsymbol{F}^p) \boldsymbol{N} dA. \tag{9.32}$$

The *dislocation density tensor* $\boldsymbol{\alpha}$ is expressed by

$$\boldsymbol{\alpha} := (\boldsymbol{F}^p)^{-1}(curl\, \boldsymbol{F}^p), \quad \text{or} \quad \boldsymbol{\alpha}(\boldsymbol{I} \times \boldsymbol{I}) = Skw(\boldsymbol{S}^p). \tag{9.33}$$

$\boldsymbol{\alpha}$ is a *measure of the incompatibility* of the plastic distortion $\boldsymbol{F}^p$, and its expression is involved in (9.25).

Note 9.1. Starting from the definition of the Cartan torsion $\boldsymbol{S}^p$, via the second order torsion tensor $\mathscr{N}^p$ expressed by (9.25), and using the definitions of the defect densities, (9.31) and (9.33), we can say that $\boldsymbol{S}^p$ is a *measure of the coupling between continuously distributed dislocations and disclinations*.

9.3 Free Energy Imbalance Principle Formulated in $\mathscr{K}$

The local free energy imbalance is formulated with respect to the configuration with torsion $\mathscr{K}$, since the defects are relevant at the level of the lattice microstructure. First we introduce the expression of the free energy density postulated with respect to the configuration with torsion.

9.3.1 Free Energy Function

We assume that the free energy density in $\mathscr{K}$ is dependent on the second order elastic deformation in terms of $(\boldsymbol{C}^e, \nabla_{\mathscr{K}} \boldsymbol{C}^e)$, and it is also influenced by the state of defects, i.e. $\boldsymbol{S}^p_{\mathscr{K}}$, $\boldsymbol{\Lambda}$ and $\nabla_{\mathscr{K}} \boldsymbol{\Lambda}$. The Cartan torsion $\boldsymbol{S}^p$ pushed away to the plastically deformed configuration is related to $\boldsymbol{S}^p_{\mathscr{K}}$, in terms of the plastic distortion as it follows

$$\boldsymbol{S}^p_{\mathscr{K}} = -\boldsymbol{F}^p \boldsymbol{S}^p[(\boldsymbol{F}^p)^{-1}, (\boldsymbol{F}^p)^{-1}]. \tag{9.34}$$

Axiom 9.1 *The free energy density with respect to the plastically deformed configuration, $\mathscr{K}$, is postulated to be a function of the second order elastic deformation, which is also dependent on the defects, and it is given by*

$$\psi = \psi_{\mathscr{K}}(\boldsymbol{C}^e - \boldsymbol{I}, \nabla_{\mathscr{K}}\boldsymbol{C}^e, \boldsymbol{S}^p_{\mathscr{K}}, \widetilde{\boldsymbol{\Lambda}}, \nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}}). \tag{9.35}$$

Now we compare the free energy density function postulated in the paper Cleja-Ţigoiu (2014) and the expression (9.35) considered herein. We recall here the basic relationships between $\boldsymbol{\Gamma}$, $\overset{(p)}{\boldsymbol{\Gamma}}$ and the so-called elastic connection in $\mathscr{K}$, $\overset{(e)}{\boldsymbol{\Gamma}}_{\mathscr{K}}$

$$\begin{aligned} \boldsymbol{\Gamma} \quad &= \overset{(p)}{\boldsymbol{\Gamma}} + (\boldsymbol{F}^p)^{-1}\overset{(e)}{\boldsymbol{\Gamma}}_{\mathscr{K}}[\boldsymbol{F}^p, \boldsymbol{F}^p], \\ \overset{(e)}{\boldsymbol{\Gamma}}_{\mathscr{K}} &= \overset{(e)}{\mathscr{A}}_{\mathscr{K}} - \boldsymbol{\Lambda}_{\mathscr{K}} \times \boldsymbol{I}, \quad \text{where} \quad \boldsymbol{\Lambda}_{\mathscr{K}} = \frac{1}{det\boldsymbol{F}^p}\widetilde{\boldsymbol{\Lambda}}, \end{aligned} \tag{9.36}$$

that can be found in a detailed presentation in Cleja-Ţigoiu (2007) and Cleja-Tigoiu and Maugin (2000).

The torsion of the elastic type connection $\overset{(e)}{\boldsymbol{\Gamma}}_{\mathscr{K}}$ is defined by

$$\boldsymbol{S}^e_{\mathscr{K}} = Skw\overset{(e)}{\boldsymbol{\Gamma}}_{\mathscr{K}} \tag{9.37}$$

and the relationship between torsions of the appropriate connections

$$\boldsymbol{S}^e_{\mathscr{K}} = -\boldsymbol{S}^p_{\mathscr{K}} \tag{9.38}$$

can be proved as a direct consequences of the formulae (9.36), (9.37) together with (9.29) and (9.37).

Proposition 9.4. *The constitutive representation for the free energy density as dependent on the second order elastic deformation in terms of* $(\boldsymbol{C}^e, \overset{(e)}{\mathscr{A}}_{\mathscr{K}})$ *and on the defects through* $(\boldsymbol{S}^e_{\mathscr{K}}, \widetilde{\boldsymbol{\Lambda}})$*, namely*

$$\psi = \psi_{\mathscr{K}}(\boldsymbol{C}^e, \overset{(e)}{\mathscr{A}}, \boldsymbol{S}^e_{\mathscr{K}}, \widetilde{\boldsymbol{\Lambda}}), \tag{9.39}$$

which has been postulated in Cleja-Ţigoiu (2014), can be viewed as a function of arguments given by (9.35) if the dependence on $\nabla\boldsymbol{\Lambda}$ *is ignored.*

Proof. In order to justify the statement, first we recall the following theorem referring to the compatible connection and which is written in component representations.

Theorem 9.1. *The plastic Bilby connection allows the following representation*

$$\begin{aligned} \overset{(p)}{\mathscr{A}} \quad &= \boldsymbol{\gamma}^p + \boldsymbol{W}^p, \\ ((\boldsymbol{\gamma}^p\boldsymbol{u})\boldsymbol{v}) \cdot \boldsymbol{z} &= \frac{1}{2}(\boldsymbol{C}^p)^{-1}\big[((\nabla\boldsymbol{C}^p)\boldsymbol{u})\boldsymbol{v} \cdot \boldsymbol{z} + ((\nabla\boldsymbol{C}^p)\boldsymbol{v})\boldsymbol{u} \cdot \boldsymbol{z} - ((\nabla\boldsymbol{C}^p)\boldsymbol{z})\boldsymbol{u} \cdot \boldsymbol{v}\big], \\ (\boldsymbol{W}^p\boldsymbol{u})\boldsymbol{v} \quad &= \frac{1}{2}((\boldsymbol{S}^p)\boldsymbol{u})\boldsymbol{v} - \frac{1}{2}(\boldsymbol{C}^p)^{-1}\big((\boldsymbol{C}^p\boldsymbol{S}^p\boldsymbol{u})^T\boldsymbol{v} + (\boldsymbol{C}^p\boldsymbol{S}^p\boldsymbol{v})^T\boldsymbol{u}\big), \end{aligned} \tag{9.40}$$

defined for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}\in\mathscr{V}$ or in component representation the Levi-Civita (plastic) connection is given by

$$(\boldsymbol{\gamma}^p)^s_{jk}=\frac{1}{2}G^{si}\Big(\frac{\partial G_{ik}}{\partial X^j}+\frac{\partial G_{ij}}{\partial X^k}-\frac{\partial G_{jk}}{\partial X^i}\Big),$$

and the (plastic) contorsion is expressed in terms of the torsion components as

$$(\boldsymbol{W}^p)^i_{jk}=\frac{1}{2}(\boldsymbol{S}^p)^i_{jk}-\frac{1}{2}G^{is}\big(G_{jm}(\boldsymbol{S}^p)^m_{sk}+G_{km}(\boldsymbol{S}^p)^m_{sj}\big).$$

Here the components of the plastic metric tensors $\boldsymbol{C}^p$ and $(\boldsymbol{C}^p)^{-1}$ are denoted by G_{ij} and G^{ij}, respectively. We mention here the symmetry properties of the fields defined above

$$\begin{aligned}&((\boldsymbol{\gamma}^p)\boldsymbol{u})\boldsymbol{v}=(\boldsymbol{\gamma}^p\boldsymbol{v})\boldsymbol{u}\quad\forall\,\boldsymbol{u},\boldsymbol{v}\in\mathscr{V},\\&\boldsymbol{W}^p\boldsymbol{u}\in Skew_2\quad\forall\,\boldsymbol{u}\in\mathscr{V}.\end{aligned}\tag{9.41}$$

The proof can be found, for instance, in Schouten (1954), see also Yavari and Goriely (2012). In addition, we apply the formulae (9.40) to the connection $\overset{(e)}{\mathscr{A}}_{\mathscr{K}}$, with respect to the anholonomic configuration $\mathscr{K}$. This means that gradient ∇ is replaced by $\nabla_{\mathscr{K}}$. Consequently, having in mind the decomposition (9.40) the presence of the fields $(\boldsymbol{C}^e,\nabla_{\mathscr{K}}\boldsymbol{C}^e,\boldsymbol{S}^e_{\mathscr{K}})$ in the formula (9.35) can be justified. Thus the presence of the elastic torsion written with respect to the configuration $\mathscr{K}$ has been replaced by the plastic torsion with respect to the same configuration via the relationship (9.38).

Note 9.2. The free energy density is influenced by the dislocation density $\boldsymbol{\alpha}_{\mathscr{K}}$, which is defined by

$$\begin{aligned}&\boldsymbol{\alpha}_{\mathscr{K}}(\tilde{\boldsymbol{u}}\times\tilde{\boldsymbol{v}})=(Skw(\overset{(e)}{\mathscr{A}}_{\mathscr{K}})\tilde{\boldsymbol{u}})\tilde{\boldsymbol{v}},\quad\text{with the property}\\&\boldsymbol{\alpha}_{\mathscr{K}}(\tilde{\boldsymbol{u}}\times\tilde{\boldsymbol{v}})=-\frac{1}{det\boldsymbol{F}^p}\boldsymbol{F}^p\boldsymbol{\alpha}(\boldsymbol{F}^p)^{-1}(\boldsymbol{u}\times\boldsymbol{v}),\quad(\boldsymbol{F}^p)^{-1}(\tilde{\boldsymbol{u}})=\boldsymbol{u},(\boldsymbol{F}^p)^{-1}(\tilde{\boldsymbol{v}})=\boldsymbol{v},\end{aligned}\tag{9.42}$$

and by the disclination tensor $\widetilde{\boldsymbol{\Lambda}}$, both tensors being defined with respect to the configuration with torsion $\mathscr{K}$.

We mention that the elastic strain field

$$\boldsymbol{C}^e-\boldsymbol{I}=(\boldsymbol{F}^p)^{-T}(\boldsymbol{C}-\boldsymbol{C}^p)(\boldsymbol{F}^p)^{-1},\tag{9.43}$$

its gradient formula written in (9.15) together with (9.34), and (9.29) suggest that the free energy density can be rewritten with respect to the reference configuration. When the fields were pulled back to the reference configuration by $(\boldsymbol{F}^p)^{-1}$, the function ψ can be written under the form

$$\psi=\psi(\boldsymbol{C}-\boldsymbol{C}^p,\nabla\boldsymbol{C}-Sym\{\boldsymbol{C}\overset{(p)}{\mathscr{A}}\},\boldsymbol{S}^p,\boldsymbol{\Lambda},\nabla\boldsymbol{\Lambda}).\tag{9.44}$$

As $\boldsymbol{S}^p$ is considered to be a measure of dislocation-disclination interplay we introduce a more general representation, which contains separately the influence of dislocation and disclination defects, given by

$$\begin{aligned}\psi = {} & \psi^e(\boldsymbol{C}-\boldsymbol{C}^p, \nabla\boldsymbol{C} - Sym\{\boldsymbol{C}\overset{(\mathrm{p})}{\mathscr{A}}\})+ \\ & + \underline{\psi}(Skw(\overset{(\mathrm{p})}{\mathscr{A}}), (\boldsymbol{C}^p)^{-1}Skw(\boldsymbol{\Lambda}\times\boldsymbol{I}), \boldsymbol{\Lambda}, \nabla\boldsymbol{\Lambda}).\end{aligned} \tag{9.45}$$

The time derivative of the free density function (9.45) is computed by

$$\begin{aligned}\dot{\psi} = {} & \partial_{\boldsymbol{C}^e}\psi^e\cdot(\dot{\boldsymbol{C}}-\dot{\boldsymbol{C}}^p)+ \\ & +\partial_{\nabla\boldsymbol{C}^e}\psi^e\cdot[\nabla\dot{\boldsymbol{C}} - Sym\{\boldsymbol{C}\frac{d}{dt}(\overset{(\mathrm{p})}{\mathscr{A}})\} - Sym\{\dot{\boldsymbol{C}}\overset{(\mathrm{p})}{\mathscr{A}}\}] + \dot{\underline{\psi}}.\end{aligned} \tag{9.46}$$

The derivatives of the mentioned fields will be replaced by their appropriate expressions.

9.3.2 *Free Energy Imbalance Principle*

The local free energy imbalance states the internal power expended during the elasto-plastic process is equal or greater than the time derivative of the free energy density.

Axiom 9.2 *The elasto-plastic constitutive description of the material is restricted to satisfy in $\mathscr{K}$ the free energy imbalance principle*

$$(\mathscr{P}_{int})_{\mathscr{K}} - \dot{\psi}_{\mathscr{K}} \geq 0, \tag{9.47}$$

for any virtual (isothermal) processes.

The expression of the internal power is the result of the superposed elastic, plastic and defect effects and will be written here in a slightly modified version of the corresponding expression postulated in Cleja-Ţigoiu (2010).

Axiom 9.3 *The internal power in the configuration with torsion is postulated to be given by the expression*

$$\begin{aligned}(\mathscr{P}_{int})_{\mathscr{K}} = {} & \frac{1}{\rho}(\boldsymbol{T}^s)\cdot\boldsymbol{L}^e + \frac{1}{\tilde{\tilde{\rho}}}\boldsymbol{\Upsilon}^p\cdot\boldsymbol{L}^p + \frac{1}{\tilde{\tilde{\rho}}}\boldsymbol{\mu}^p\cdot\nabla_{\mathscr{K}}\boldsymbol{L}^p+ \\ & +\frac{1}{\tilde{\tilde{\rho}}}\boldsymbol{\mu}_{\mathscr{K}}\cdot\left((\boldsymbol{F}^e)^{-1}(\nabla_{\chi}\boldsymbol{L})[\boldsymbol{F}^e,\boldsymbol{F}^e] - \nabla_{\mathscr{K}}\boldsymbol{L}^p\right)+ \\ & +\frac{1}{\tilde{\tilde{\rho}}}\boldsymbol{\Upsilon}^\lambda\cdot\frac{D}{Dt}\tilde{\boldsymbol{\Lambda}} + \frac{1}{\tilde{\tilde{\rho}}}\boldsymbol{\mu}^\lambda\cdot\nabla_{\mathscr{K}}\frac{D}{Dt}\tilde{\boldsymbol{\Lambda}}.\end{aligned} \tag{9.48}$$

Note 9.3. The gradient of the plastic rate $\boldsymbol{L}^p = \dot{\boldsymbol{F}}^p(\boldsymbol{F}^p)^{-1}$ with respect to the plastically deformed configuration is related to the time derivative of $\overset{(\mathrm{p})}{\mathscr{A}} = (\boldsymbol{F}^p)^{-1}\nabla\boldsymbol{F}^p$, and moreover $\nabla \underline{\boldsymbol{l}}^p$ is involved in the expression of the previous one, as can be seen from the following formulae:

$$\frac{d}{dt}\big(\overset{(\mathrm{p})}{\mathscr{A}}\big) = (\boldsymbol{F}^p)^{-1}\big(\nabla_{\mathscr{K}}\boldsymbol{L}^p\big)[\boldsymbol{F}^p,\boldsymbol{F}^p] = \nabla\underline{\boldsymbol{l}}^p - \underline{\boldsymbol{l}}^p\,\overset{(\mathrm{p})}{\mathscr{A}} + \overset{(\mathrm{p})}{\mathscr{A}}\,[\boldsymbol{I},\underline{\boldsymbol{l}}^p]. \tag{9.49}$$

We represent now the stresses and stress momenta as associated measures with respect to the reference configuration by pulled back procedure, see Cleja-Ţigoiu et al (2016). For instance the Mandel stress tensors, associated with plastic and disclination behaviour, are introduced with respect to the reference configuration by

$$\frac{1}{\rho_0}\boldsymbol{\Sigma}_0^p = \frac{1}{\widetilde{\widetilde{\rho}}}(\boldsymbol{F}^p)^T\boldsymbol{\Upsilon}^p(\boldsymbol{F}^p)^{-T}, \quad \frac{1}{\rho_0}\boldsymbol{\Sigma}_0^\lambda = \frac{1}{\widetilde{\widetilde{\rho}}}(\boldsymbol{F}^p)^T\boldsymbol{\Upsilon}^\lambda(\boldsymbol{F}^p)^{-T}, \tag{9.50}$$

while the appropriate micro stress momenta with respect to the reference configuration are given by

$$\begin{aligned}
\frac{1}{\rho_0}\boldsymbol{\mu}_0 &= (\boldsymbol{F}^p)^T\frac{1}{\widetilde{\widetilde{\rho}}}\boldsymbol{\mu}_{\mathscr{K}}[(\boldsymbol{F}^p)^{-T},(\boldsymbol{F}^p)^{-T}],\\
\frac{1}{\rho_0}\boldsymbol{\mu}_0^p &= (\boldsymbol{F}^p)^T\frac{1}{\widetilde{\widetilde{\rho}}}\boldsymbol{\mu}^p[(\boldsymbol{F}^p)^{-T},(\boldsymbol{F}^p)^{-T}],\\
\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda &= (\boldsymbol{F}^p)^T\frac{1}{\widetilde{\widetilde{\rho}}}\boldsymbol{\mu}^\lambda[(\boldsymbol{F}^p)^{-T},(\boldsymbol{F}^p)^{-T}].
\end{aligned} \tag{9.51}$$

Proposition 9.5. *The internal power postulated by (9.48) is reformulated in terms of the stresses and stress momenta associated with the reference configuration, (9.50) and (9.51), under the form*

$$\begin{aligned}
&\frac{1}{\rho}(\boldsymbol{T}^s)\cdot(\boldsymbol{L}-\boldsymbol{F}\underline{\boldsymbol{l}}^p\boldsymbol{F}^{-1}) - 2\partial_{\boldsymbol{C}^e}\psi^e\cdot\boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F} + \frac{1}{\rho_0}\boldsymbol{\Sigma}_0^p\cdot\underline{\boldsymbol{l}}^p + \frac{1}{\rho_0}\boldsymbol{\mu}_0^p\cdot\frac{d}{dt}(\overset{(\mathrm{p})}{\mathscr{A}})+\\
&+\frac{1}{\widetilde{\rho}_0}\boldsymbol{\mu}_0\cdot\big(\big(\boldsymbol{F}^{-1}(\nabla_\chi\boldsymbol{L})[\boldsymbol{F},\boldsymbol{F}] - \frac{d}{dt}(\overset{(\mathrm{p})}{\mathscr{A}})\big) + \partial_{\boldsymbol{C}^e}\psi^e\cdot[\boldsymbol{C}^p\underline{\boldsymbol{l}}^p + (\underline{\boldsymbol{l}}^p)^T\boldsymbol{C}^p] -\\
&-\partial_{\nabla\boldsymbol{C}^e}\psi^e\cdot\{Sym(\dot{\boldsymbol{C}}\boldsymbol{\Gamma}) + \boldsymbol{F}^T(\nabla_\chi\boldsymbol{D})[\boldsymbol{F},\boldsymbol{F}]\}+\\
&+\partial_{\nabla\boldsymbol{C}^e}\psi^e\cdot[Sym\{\boldsymbol{C}\frac{d}{dt}(\overset{(\mathrm{p})}{\mathscr{A}})\} + Sym\{\dot{\boldsymbol{C}}\,\overset{(\mathrm{p})}{\mathscr{A}}\}]+\\
&+\frac{1}{\rho_0}\boldsymbol{\Sigma}_0^\lambda\cdot\dot{\boldsymbol{\Lambda}} + \frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda\cdot\{\overset{(\mathrm{p})}{\mathscr{A}}\,[\boldsymbol{I},\dot{\boldsymbol{\Lambda}}] + \nabla\dot{\boldsymbol{\Lambda}} - \dot{\boldsymbol{\Lambda}}\,\overset{(\mathrm{p})}{\mathscr{A}}\} - \underline{\dot{\psi}} \geq 0.
\end{aligned} \tag{9.52}$$

In order to derive the consequences that follow from the dissipation inequality (9.52) we introduce certain identities involving the operators defined by (9.12).

The following identities written for any $\mathscr{A}$ and $\mathscr{B} \in Lin(\mathscr{V},Lin)$ are direct consequences of the given definitions

$$
\begin{aligned}
&(\mathscr{A} \odot \mathscr{B})^T = \mathscr{B} \odot \mathscr{A}, \quad (\mathscr{A}\ {}_r\!\odot \mathscr{B})^T = \mathscr{B}\ {}_r\!\odot \mathscr{A},\\
&\{Sym(\mathscr{A} \odot \mathscr{B})\}^s = \frac{1}{2}(\mathscr{A} \odot \mathscr{B} + \mathscr{A}^T \odot \mathscr{B} + \mathscr{B} \odot \mathscr{A} + \mathscr{B} \odot \mathscr{A}^T),\\
&\{Sym(\mathscr{A}\ {}_r\!\odot \mathscr{B})\}^s = \frac{1}{2}(\mathscr{A}\ {}_r\!\odot \mathscr{B} + \mathscr{A}^T\ {}_r\!\odot \mathscr{B} + \mathscr{B}\ {}_r\!\odot \mathscr{A} + \mathscr{B}\ {}_r\!\odot \mathscr{A}).
\end{aligned}
\tag{9.53}
$$

$$
\begin{aligned}
&(Skw\mathscr{A}) \odot (Skw\mathscr{B}) = (\boldsymbol{\Omega}_{<\mathscr{A}>} \cdot \boldsymbol{\Omega}_{<\mathscr{B}>})\boldsymbol{I} + (\boldsymbol{\Omega}_{<\mathscr{B}>})^T \boldsymbol{\Omega}_{<\mathscr{A}>},\\
&(Skw\mathscr{A})\ {}_r\!\odot (Skw\mathscr{B}) = 2\boldsymbol{\Omega}_{<\mathscr{A}>}(\boldsymbol{\Omega}_{<\mathscr{B}>})^T,
\end{aligned}
\tag{9.54}
$$

Moreover

$$
\mathscr{A} \cdot \mathscr{B} = 2\boldsymbol{\Omega}_{<\mathscr{A}>} \cdot \boldsymbol{\Omega}_{<\mathscr{B}>}. \tag{9.55}
$$

9.4 Constitutive Restrictions Imposed by the Imbalance Free Energy Principle

First we derive the elastic type constitutive equations, starting from the supposition that no variation of the irreversible behaviour can occur.

9.4.1 Elastic Type Constitutive Equations

Proposition 9.6. *We suppose that* $\boldsymbol{L}^p = 0$ *or* $\underline{\boldsymbol{l}}^p = 0$ *(then* $\boldsymbol{L}^e = \boldsymbol{L}$*) and* $\dot{\boldsymbol{\Lambda}} = 0$*. Thus the imbalance free energy relation is reduced to the following inequality*

$$
\begin{aligned}
&(\frac{1}{\rho}(\boldsymbol{T}^s) - 2\boldsymbol{F}\partial_{\boldsymbol{C}^e}\psi^e\boldsymbol{F}^T)\cdot\boldsymbol{D} + \frac{1}{\rho_0}\boldsymbol{\mu}_0\cdot\boldsymbol{F}^{-1}(\nabla_\chi\boldsymbol{L})[\boldsymbol{F},\boldsymbol{F}]) -\\
&-\partial_{\nabla\boldsymbol{C}^e}\psi^e\cdot\{Sym(\dot{\boldsymbol{C}}\boldsymbol{\Gamma}) + \boldsymbol{F}^T\nabla_\chi\boldsymbol{D}[\boldsymbol{F},\boldsymbol{F}]\} + \partial_{\nabla\boldsymbol{C}^e}\psi^e\cdot Sym\{\dot{\boldsymbol{C}}\overset{(p)}{\mathscr{A}}\} \geq 0,
\end{aligned}
\tag{9.56}
$$

which holds for any $\boldsymbol{L}$ *and* $\nabla_\chi\boldsymbol{L}$*.*

Theorem 9.2. *The elastic free energy, denoted by* ψ^e *is potential for the macro stress and macro stress momentum, respectively, related to the reference configuration, namely*

$$
\begin{aligned}
&\frac{1}{2\rho}\boldsymbol{\pi}_0 = \partial_{\boldsymbol{C}^e}\psi^e + \{Sym(\partial_{\nabla\boldsymbol{C}^e}\psi^e)\ {}_r\!\odot(\boldsymbol{\Gamma} - \overset{(p)}{\mathscr{A}})\}^S,\\
&\frac{1}{\rho_0}\boldsymbol{\mu}_0 = Sym(\boldsymbol{C}\partial_{\nabla\boldsymbol{C}^e}\psi^e),
\end{aligned}
\tag{9.57}
$$

were $\boldsymbol{\Gamma} - \overset{(p)}{\mathscr{A}} \equiv (\boldsymbol{F}^p)^{-1}\overset{(e)}{\mathscr{A}}[\boldsymbol{F}^p,\boldsymbol{F}^p]$*.*

Proof. In order to compare the terms written in (9.56) we use the rule (9.12)

$$\partial_{\nabla \boldsymbol{C}^e}\psi^e \cdot Sym\big(\dot{\boldsymbol{C}}(\boldsymbol{\Gamma} - \overset{(p)}{\mathscr{A}})\big) = \big((Sym\partial_{\nabla \boldsymbol{C}^e}\psi^e)\,_r \odot (\boldsymbol{\Gamma} - \overset{(p)}{\mathscr{A}})\big)\cdot \dot{\boldsymbol{C}}, \tag{9.58}$$

and we pass from $\boldsymbol{D}$ to $\dot{\boldsymbol{C}}$ via the relation $(9.18)_2$. The inequality (9.56) is written finally under the form

$$\begin{aligned} &\{\frac{1}{2\rho}\boldsymbol{\pi}_0 - \partial_{\boldsymbol{C}^e}\psi^e)\cdot\dot{\boldsymbol{C}} - \big((Sym\partial_{\nabla \boldsymbol{C}^e}\psi^e)_r \odot (\boldsymbol{\Gamma} - \overset{(p)}{\mathscr{A}})\big)\}\cdot\dot{\boldsymbol{C}} + \\ &+\frac{1}{\rho_0}\boldsymbol{F}^{-T}\big(\boldsymbol{\mu}_0 - Sym(\boldsymbol{C}\partial_{\nabla \boldsymbol{C}^e}\psi^e)\big)[\boldsymbol{F}^T,\boldsymbol{F}^T]\cdot\nabla_\chi \boldsymbol{L} \geq 0, \end{aligned} \tag{9.59}$$

which holds for any $\boldsymbol{L}$ and $\nabla_\chi \boldsymbol{L}$. In (9.59) the expression of the Piola-Kirchhoff stress tensor with respect to the reference configuration, $\boldsymbol{\pi}_0$, has been introduced

$$\frac{1}{\rho_0}\boldsymbol{\pi}_0 = \frac{1}{\rho}\boldsymbol{F}^{-1}\boldsymbol{T}\boldsymbol{F}^{-T}, \tag{9.60}$$

Note 9.4. The non-symmetric Cauchy stress, $\boldsymbol{T}$, and couple-stress tensor, $\boldsymbol{m}$, satisfy the balance equations formulated by Fleck et al (1994), see also Cleja-Ţigoiu and Ţigoiu (2011). In this model $\boldsymbol{\mu}_0 \boldsymbol{z} \in Sym_2$ and consequently $\boldsymbol{T}^a$ is vanishing. The (equilibrium) balance equations for macro forces, say $(\boldsymbol{T},\boldsymbol{\mu})$ in the actual configuration is reduced to the classical one, $div\boldsymbol{T} = 0$, if the mass density of the body and couple forces are neglected.

9.4.2 Dissipation Inequality

In order to derive the restrictions imposed by the free energy imbalance related to the plastic behaviour, we return to the inequality (9.52).

Theorem 9.3. *The reduced dissipation inequality is derived under the form*

$$\begin{aligned} &\{\frac{1}{\rho_0}(\boldsymbol{\mu}_0^p - \boldsymbol{\mu}_0) + \boldsymbol{C}Sym(\partial_{\nabla \boldsymbol{C}^e}\psi^e) - Skw(\partial_{\mathscr{Z}_1}\underline{\psi})\}\cdot\frac{d}{dt}\overset{(p)}{\mathscr{A}} + \\ &+\{\frac{1}{\rho_0}\boldsymbol{\Sigma}_0^p + 2\boldsymbol{C}^p\partial_{\boldsymbol{C}^e}\psi^e + \partial_{\mathscr{Z}_2}\underline{\psi}\,_r\odot(\boldsymbol{C}^p)^{-1}Skw(\boldsymbol{\Lambda}\times\boldsymbol{I}) + \\ &+Skw(\boldsymbol{\Lambda}\times\boldsymbol{I})\,_r\odot(\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi}\}\cdot\underline{\boldsymbol{l}}^p + (\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda - \partial_{\nabla\boldsymbol{\Lambda}}\underline{\psi})\cdot\nabla\dot{\boldsymbol{\Lambda}} + \\ &+(\frac{1}{\rho_0}\boldsymbol{\Sigma}_0^\lambda - \partial_{\boldsymbol{\Lambda}}\underline{\psi} + \overset{(p)}{\mathscr{A}}\odot\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda - \frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda{}_r\odot\overset{(p)}{\mathscr{A}})\cdot\dot{\boldsymbol{\Lambda}} + \\ &+\{\boldsymbol{\epsilon}\cdot\big((\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi}\big)\boldsymbol{I} - \big(\boldsymbol{\epsilon}_r\odot((\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi})\big)\}\cdot\dot{\boldsymbol{\Lambda}} \geq 0, \end{aligned} \tag{9.61}$$

when we put into evidence the terms which contain the rates of the appropriate fields and their gradients, namely $\underline{\boldsymbol{l}}^p$, $\frac{d}{dt}(\overset{(p)}{\mathscr{A}})$, $\dot{\boldsymbol{\Lambda}}$ *and* $\nabla\dot{\boldsymbol{\Lambda}}$. *Here* $\boldsymbol{\epsilon}$ *denotes Ricci's permutation tensor.*

Proof. First we introduce the expression for the time derivative of non-elastic free energy function $\underline{\psi}$, in which we use the notations mentioned below

$$\underline{\psi} = \underline{\psi}(Skw(\overset{(p)}{\mathscr{A}}), (\boldsymbol{C}^p)^{-1}Skw(\boldsymbol{\Lambda}\times\boldsymbol{I}), \boldsymbol{\Lambda}, \nabla\boldsymbol{\Lambda}) \equiv \underline{\psi}(Skw(\mathscr{Z}_1, \mathscr{Z}_2, \boldsymbol{\Lambda}, \nabla\boldsymbol{\Lambda}), \quad \text{where} \quad \mathscr{Z}_1 = Skw(\overset{(p)}{\mathscr{A}}), \quad \mathscr{Z}_2 = (\boldsymbol{C}^p)^{-1}Skw(\boldsymbol{\Lambda}\times\boldsymbol{I}). \tag{9.62}$$

The time derivative of the non elastic free energy function (9.62) is expressed as

$$\dot{\underline{\psi}} = \partial_{\mathscr{Z}_1}\underline{\psi}\cdot Skw(\frac{d}{dt}(\overset{(p)}{\mathscr{A}}) + \partial_{\mathscr{Z}_2}\underline{\psi}\cdot Skw\big(\frac{d}{dt}\big((\boldsymbol{C}^p)^{-1}(\boldsymbol{\Lambda}\times\boldsymbol{I})\big) + +\partial_{\boldsymbol{\Lambda}}\underline{\psi}\cdot\dot{\boldsymbol{\Lambda}} + \partial_{\nabla\boldsymbol{\Lambda}}\underline{\psi}\cdot\nabla\dot{\boldsymbol{\Lambda}}. \tag{9.63}$$

Using the time derivative formula written in $(9.18)_4$, $(9.7)_4$ and the rules (9.12) we obtain

$$\partial_{\mathscr{Z}_2}\underline{\psi}\cdot Skw\frac{d}{dt}\big((\boldsymbol{C}^p)^{-1}(\boldsymbol{\Lambda}\times\boldsymbol{I})\big) = -\big(\partial_{\mathscr{Z}_2}\underline{\psi}_r \odot Skw(\boldsymbol{C}^p)^{-1}(\boldsymbol{\Lambda}\times\boldsymbol{I})\big)\cdot\underline{\boldsymbol{l}}^p - -\big((\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi}_r \odot Skw(\boldsymbol{\Lambda}\times\boldsymbol{I})\big)\cdot(\underline{\boldsymbol{l}}^p)^T - -\boldsymbol{\epsilon}\cdot\big((\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi})\boldsymbol{I}\cdot\dot{\boldsymbol{\Lambda}} + \big(\boldsymbol{\epsilon}_r\odot((\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi})\big)\cdot\dot{\boldsymbol{\Lambda}}. \tag{9.64}$$

The reduced dissipation inequality is derived from (9.52) together with the elastic type constitutive relation $(9.57)_1$, were $\dot{\underline{\psi}}$ is given by the formulae (9.63) together with (9.64).

9.5 Viscoplastic Type Evolution Equations for Plastic Distortion and Disclination Tensor

Hypotheses. The energetic type constitutive equations will be defined for micro momenta related to the plastic and disclination mechanism, namely

$$\frac{1}{\rho_0}\boldsymbol{\mu}_0^p = \frac{1}{\rho_0}\boldsymbol{\mu}_0 - \boldsymbol{C}Sym(\partial_{\nabla\boldsymbol{C}^e}\psi^e) + Skw(\partial_{\mathscr{Z}_1}\underline{\psi}), \quad \frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda = \partial_{\nabla\boldsymbol{\Lambda}}\underline{\psi}. \tag{9.65}$$

Theorem 9.4. *Under the hypotheses formulated by (9.65) the reduced dissipation inequality can be written under the form*

$$
\begin{aligned}
&\{\frac{1}{\rho_0}\boldsymbol{\Sigma}_0^p + 2Skw\partial_{\mathscr{Z}_2}\underline{\psi}\ {}_r\odot (\boldsymbol{C}^p)^{-1}(\boldsymbol{\Lambda}\times\boldsymbol{I})+\\
&+(\boldsymbol{\Lambda}\times\boldsymbol{I})\ {}_r\odot\big((\boldsymbol{C}^p)^{-1}Skw(\partial_{\mathscr{Z}_2}\underline{\psi})\big)+2\boldsymbol{C}^p\partial_{\boldsymbol{C}^e}\psi^e\}\cdot\underline{\boldsymbol{l}}^p+\\
&+\big(\frac{1}{\rho_0}\boldsymbol{\Sigma}_0^\lambda-\partial_{\boldsymbol{\Lambda}}\underline{\psi}+\overset{(\mathrm{p})}{\mathscr{A}}\odot\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda-\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda{}_r\odot\overset{(\mathrm{p})}{\mathscr{A}}\big)\cdot\dot{\boldsymbol{\Lambda}}+\\
&+\{\boldsymbol{\epsilon}\cdot\big((\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi}\big)\boldsymbol{I}-\big(\boldsymbol{\epsilon}_r\odot((\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi})\big)\}\cdot\dot{\boldsymbol{\Lambda}}\geq 0
\end{aligned}
\tag{9.66}
$$

Proof. As a direct consequence of (9.61) together with (9.65) the inequality (9.66) follows at once. The variables $\mathscr{Z}_1$ and $\mathscr{Z}_2$ have been defined in (9.62). Here Mandel's type stress tensor, $\frac{1}{\rho_0}\boldsymbol{\Sigma}_0^p$, appears to be power conjugate to the rate of plastic distortion $\underline{\boldsymbol{l}}^p$. For physical meaning and properties of Mandel and Eshelby stress tensors see for instance Maugin (1994); Cleja-Tigoiu and Maugin (2000).

Axiom 9.4 *The evolution equations for plastic distortion and disclination tensor are supposed to be given by*

$$
\begin{aligned}
\xi_1\,\underline{\boldsymbol{l}}^p &= \frac{1}{\rho_0}\boldsymbol{\Sigma}_0^p + Skw(\partial_{\mathscr{Z}_2}\underline{\psi})\ {}_r\odot\big((\boldsymbol{C}^p)^{-1}Skw(\boldsymbol{\Lambda}\times\boldsymbol{I})\big)+\\
&\quad+Skw(\boldsymbol{\Lambda}\times\boldsymbol{I})\ {}_r\odot\big((\boldsymbol{C}^p)^{-1}Skw(\partial_{\mathscr{Z}_2}\underline{\psi})\big)+2\boldsymbol{C}^p\partial_{\boldsymbol{C}^e}\psi^e,\\
\xi_2\,\dot{\boldsymbol{\Lambda}} &= \frac{1}{\rho_0}\boldsymbol{\Sigma}_0^\lambda-\partial_{\boldsymbol{\Lambda}}\underline{\psi}+\overset{(\mathrm{p})}{\mathscr{A}}\odot\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda-\frac{1}{\rho_0}\boldsymbol{\mu}_0^\lambda\ {}_r\odot\overset{(\mathrm{p})}{\mathscr{A}}+\\
&\quad+\boldsymbol{\epsilon}\cdot\big((\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi}\big)\boldsymbol{I}-\boldsymbol{\epsilon}\odot((\boldsymbol{C}^p)^{-1}\partial_{\mathscr{Z}_2}\underline{\psi}).
\end{aligned}
\tag{9.67}
$$

As a direct consequence of (9.67) the dissipation inequality (9.66) becomes

$$
\xi_1\,\underline{\boldsymbol{l}}^p\cdot\underline{\boldsymbol{l}}^p+\xi_2\,\dot{\boldsymbol{\Lambda}}\cdot\dot{\boldsymbol{\Lambda}}\geq 0. \tag{9.68}
$$

The last inequality holds for any non-negative constitutive functions ξ_1,ξ_2.

Concerning the expression of the Mandel type stress tensors, $\frac{1}{\rho_0}\boldsymbol{\Sigma}^p$ and $\frac{1}{\rho_0}\boldsymbol{\Sigma}^\lambda$ defined by (9.50), we shall use the balance equations for micro forces provided in the paper by Cleja-Ţigoiu (2017). We recall the micro balance equation for the plastic mechanism

$$
\frac{1}{\tilde{\tilde{\rho}}}\big(\boldsymbol{\Upsilon}^p-\boldsymbol{\Sigma}_{\mathscr{K}}\big)=\mathrm{div}\,\big(\frac{1}{\tilde{\tilde{\rho}}}(\boldsymbol{\mu}^p-\boldsymbol{\mu}_{\mathscr{K}})(\boldsymbol{F}^p)^{-T}\big)+\boldsymbol{B}^p, \tag{9.69}
$$

and the appropriate micro balance equation related to the disclination mechanism

$$
\frac{1}{\tilde{\tilde{\rho}}}\boldsymbol{\Upsilon}^\lambda=\mathrm{div}\big(\frac{1}{\tilde{\tilde{\rho}}}\boldsymbol{\mu}^\lambda\ (\boldsymbol{F}^p)^{-T}\big)+\boldsymbol{B}^\lambda. \tag{9.70}
$$

Here $\tilde{\rho}\boldsymbol{B}^p$ and $\tilde{\rho}\boldsymbol{B}^\lambda$ are mass densities of couple body forces.

Definition 9.7. The Piola-Kirchhoff macroscopic stress tensor, $\boldsymbol{\pi}_0$, the macro stress tensor, $\boldsymbol{\Sigma}_{\mathcal{K}}$, and the Cauchy stress $\boldsymbol{T}$ are related by the following relationships

$$\frac{1}{\tilde{\rho}}\boldsymbol{\Sigma}_{\mathcal{K}} = \frac{1}{\rho_0}(\boldsymbol{F}^p)^{-T}\boldsymbol{C}\boldsymbol{\pi}_0(\boldsymbol{F}^p)^T = \frac{1}{\rho}(\boldsymbol{F}^e)^{-T}\boldsymbol{T}(\boldsymbol{F}^e)^T. \tag{9.71}$$

Note 9.5. Clayton et al (2006) assumed that the geometrically necessary density tensors $\tilde{\boldsymbol{\alpha}}$ and $\tilde{\boldsymbol{\theta}}$ do not contribute to the free energy dissipation, namely

$$\tilde{\sigma} = \frac{\partial\tilde{\psi}}{\partial\tilde{\boldsymbol{\alpha}}}, \quad \tilde{\boldsymbol{\mu}} = \frac{\partial\tilde{\psi}}{\partial\tilde{\boldsymbol{\theta}}},$$

and the microforces with respect to the intermediate configuration satisfy the Fleck et al (1994) type balance equations.

9.5.1 Quadratic Free Energy

We restrict ourself to the case of the free energy function which is quadratic with respect to above mentioned variables, given by

$$\begin{aligned}
\psi &= \psi^e + \underline{\psi}, \\
\psi^e &= \frac{1}{8}\mathscr{E}(\boldsymbol{C}-\boldsymbol{C}^p)\cdot(\boldsymbol{C}-\boldsymbol{C}^p) + \frac{1}{4}\beta_1(\nabla\boldsymbol{C}- \\
&\quad -Sym\{\boldsymbol{C}\overset{(\mathrm{p})}{\mathscr{A}}\})\cdot(\nabla\boldsymbol{C}-Sym\{\boldsymbol{C}\overset{(\mathrm{p})}{\mathscr{A}}\}), \\
\underline{\psi} &= \frac{1}{4}\beta_2\{Skw\overset{(\mathrm{p})}{\mathscr{A}} + \tilde{\beta}(\boldsymbol{C}^p)^{-1}Skw(\boldsymbol{\Lambda}\times\boldsymbol{I})\}\cdot\{Skw\overset{(\mathrm{p})}{\mathscr{A}} + \\
&\quad +\tilde{\beta}(\boldsymbol{C}^p)^{-1}Skw(\boldsymbol{\Lambda}\times\boldsymbol{I})\} + \frac{1}{2}\beta_3\boldsymbol{\Lambda}\cdot\boldsymbol{\Lambda} + \frac{1}{2}\beta_4\nabla\boldsymbol{\Lambda}\cdot\nabla\boldsymbol{\Lambda}.
\end{aligned} \tag{9.72}$$

Note 9.6. If $\tilde{\beta} = 1$ then the non-elastic part of the free energy postulated here coincides with those introduced by Cleja-Ţigoiu et al (2016).

The appropriate partial derivatives of the free energy function (9.72) determine the elastic type constitutive equations (9.57), the energetic representation for the micro stress momenta (9.65) as well as the evolution equations for plastic distortion and disclination tensor (9.67).

We do not provide here the particular constitutive model associated with the free energy function (9.72) for the conciseness of the exposure. We pass directly to the case of small distortions, that follows directly from the finite deformation model.

9.5.2 Elasto-Plastic Model for Dislocations and Disclinations in the Case of Small Distortions

The case of small elastic and plastic distortions is defined by the following conditions

$$\begin{aligned} &\boldsymbol{F}^e = \boldsymbol{I} + \boldsymbol{H}^e, \quad \boldsymbol{F}^p = \boldsymbol{I} + \boldsymbol{H}^p, \quad \boldsymbol{F} = \boldsymbol{I} + \boldsymbol{H}, \\ &\boldsymbol{H} = \nabla \boldsymbol{u}, \quad \boldsymbol{H} = \boldsymbol{H}^e + \boldsymbol{H}^e, \quad \text{for} \quad \| \boldsymbol{H}^e \| \ll 1, \| \boldsymbol{H}^p \| \ll 1, \end{aligned} \tag{9.73}$$

The following approximated formulae can be put into evidence

$$\boldsymbol{C} = \boldsymbol{I} + 2\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T), \quad \boldsymbol{C}^p = \boldsymbol{I} + \boldsymbol{\varepsilon}^p, \tag{9.74}$$

where

$$\begin{aligned} &\boldsymbol{\varepsilon}^p = \frac{1}{2}(\boldsymbol{H}^p + (\boldsymbol{H}^p)^T) \\ &\frac{1}{2}\beta_1 \big(\nabla \boldsymbol{C} - Sym(\{\boldsymbol{C} \overset{(p)}{\mathscr{A}}\})\big) = \nabla \boldsymbol{\varepsilon} - \nabla \boldsymbol{\varepsilon}^p, \\ &\overset{(p)}{\mathscr{A}} = \nabla \boldsymbol{H}^p, \quad \boldsymbol{\Gamma} = \nabla \boldsymbol{H}, \quad \mathscr{Z}_1 = Skw \nabla \boldsymbol{H}^p, \quad \mathscr{Z}_2 = Skw(\boldsymbol{\Lambda} \times \boldsymbol{I}). \end{aligned}$$

The elastic type constitutive equations, namely the formulae (9.57), can be represented under the form

$$\begin{aligned} \frac{1}{\rho_0} \boldsymbol{\pi}_0 &= \mathscr{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \frac{1}{2}\beta_1 \{(\nabla \boldsymbol{\varepsilon} - \nabla \boldsymbol{\varepsilon}^p)\ {}_r\!\odot (\nabla \boldsymbol{H} - \nabla \boldsymbol{H}^p) \\ &\quad + (\nabla \boldsymbol{H} - \nabla \boldsymbol{H}^p)\ {}_r\!\odot (\nabla \boldsymbol{\varepsilon} - \nabla \boldsymbol{\varepsilon}^p)\}, \\ \frac{1}{\rho_0} \boldsymbol{\mu}_0 &= \beta_1 (\nabla \boldsymbol{\varepsilon} - \nabla \boldsymbol{\varepsilon}^p). \end{aligned} \tag{9.75}$$

The energetic expressions for the plastic and disclination momenta are given by

$$\begin{aligned} \frac{1}{\rho_0} \boldsymbol{\mu}^p &= \beta_2 \big(Skw \nabla \boldsymbol{H}^p + \tilde{\beta} Skw(\boldsymbol{\Lambda} \times \boldsymbol{I})\big), \\ \frac{1}{\rho_0} \boldsymbol{\mu}^\lambda &= \beta_4 \nabla \boldsymbol{\Lambda}. \end{aligned} \tag{9.76}$$

The evolution equations for plastic distortion and disclination tensor become

$$\begin{aligned} \xi_1\, \underline{\boldsymbol{l}}^p &= \frac{1}{\rho_0} \boldsymbol{\Sigma}_0^p + \mathscr{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \\ &\quad + \beta_2 Skw(\boldsymbol{\Lambda} \times \boldsymbol{I})\ {}_r\!\odot \big(Skw \nabla \boldsymbol{H}^p + \tilde{\beta} Skw(\boldsymbol{\Lambda} \times \boldsymbol{I})\big) + \\ &\quad + \beta_2 \big(Skw \nabla \boldsymbol{H}^p + \tilde{\beta} Skw(\boldsymbol{\Lambda} \times \boldsymbol{I})\big)\ {}_r\!\odot Skw(\boldsymbol{\Lambda} \times \boldsymbol{I}), \end{aligned} \tag{9.77}$$

$$\begin{aligned}\xi_2\,\dot{\boldsymbol{\Lambda}} =& \frac{1}{\rho_0}\boldsymbol{\Sigma}_0^{\lambda} + \beta_4\,\nabla\boldsymbol{H}^p \odot \nabla\boldsymbol{\Lambda} - \beta_4\nabla\boldsymbol{\Lambda}\,{}_r\odot\nabla\boldsymbol{H}^p - \beta_3\boldsymbol{\Lambda} + \\ &+\frac{1}{2}\beta_2\,\in\cdot\big(Skw\nabla\boldsymbol{H}^p + \tilde{\beta}Skw(\boldsymbol{\Lambda}\times\boldsymbol{I})\boldsymbol{I}- \\ &-\frac{1}{2}\beta_2\in{}_r\odot\big(Skw\nabla\boldsymbol{H}^p + \tilde{\beta}Skw(\boldsymbol{\Lambda}\times\boldsymbol{I})\big).\end{aligned} \tag{9.78}$$

Using the identities (9.54) and (9.55), together with the property $\Omega_{<\in>} = -\boldsymbol{I}$ the evolution equation for the disclination tensor $\boldsymbol{\Lambda}$ will be given by

$$\begin{aligned}\xi_2\,\dot{\boldsymbol{\Lambda}} =& \frac{1}{\rho_0}\boldsymbol{\Sigma}_0^{\lambda} + \beta_4\,\nabla\boldsymbol{H}^p \odot \nabla\boldsymbol{\Lambda} - \beta_4\nabla\boldsymbol{\Lambda}\,{}_r\odot\nabla\boldsymbol{H}^p - \beta_3\boldsymbol{\Lambda} - \\ &-2\beta_2\,tr\,(curl\boldsymbol{H}^p)\boldsymbol{I} - 4\beta_2\,\tilde{\beta}tr(\boldsymbol{\Lambda})\boldsymbol{I} + \beta_2\,(curl\boldsymbol{H}^p)^T + \beta_2\tilde{\beta}\big(tr(\boldsymbol{\Lambda})\boldsymbol{I}-\boldsymbol{\Lambda})\big)\end{aligned} \tag{9.79}$$

Proposition 9.7. *In the case of small distortions the evolution equations for the plastic distortion* $\boldsymbol{H}^p$ *and* $\boldsymbol{\Lambda}$ *are given by*

$$\begin{aligned}\xi_1\,\dot{\boldsymbol{H}}^p =& \frac{1}{\rho_0}\boldsymbol{\Sigma}_0^{p} + \mathscr{E}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p) + \beta_2\,\big((tr\boldsymbol{\Lambda})\boldsymbol{I}-\boldsymbol{\Lambda}^T\big)(curl\boldsymbol{H}^p)^T + \\ &+\beta_2\,(curl\boldsymbol{H}^p)\big((tr\boldsymbol{\Lambda})\boldsymbol{I}-\boldsymbol{\Lambda}\big) + 2\beta_2\tilde{\beta}\,\big((tr\boldsymbol{\Lambda})\boldsymbol{I}-\boldsymbol{\Lambda}^T\big)\big((tr\boldsymbol{\Lambda})\boldsymbol{I}-\boldsymbol{\Lambda}\big), \\ \xi_2\,\dot{\boldsymbol{\Lambda}} =& \frac{1}{\rho_0}\boldsymbol{\Sigma}_0^{\lambda} + \beta_4\,\nabla\boldsymbol{H}^p \odot \nabla\boldsymbol{\Lambda} - \beta_4\nabla\boldsymbol{\Lambda}\,{}_r\odot\nabla\boldsymbol{H}^p - \beta_3\boldsymbol{\Lambda} - \\ &-2\beta_2\,tr\,(curl\boldsymbol{H}^p)\boldsymbol{I} - 4\beta_2\,\tilde{\beta}tr(\boldsymbol{\Lambda})\boldsymbol{I} + \\ &+\beta_2\,(curl\boldsymbol{H}^p)^T + \beta_2\tilde{\beta}\big(tr(\boldsymbol{\Lambda})\boldsymbol{I}-\boldsymbol{\Lambda}\big).\end{aligned} \tag{9.80}$$

The appropriate Mandel stress tensors are approximated by

$$\boldsymbol{\Sigma}_0^p = \boldsymbol{\Upsilon}^p, \quad \boldsymbol{\Sigma}_0^{\lambda} = \boldsymbol{\Upsilon}^{\lambda}, \quad \boldsymbol{\Sigma}_{\mathscr{K}} = \boldsymbol{\pi}_0 = \boldsymbol{T}, \quad \widetilde{\rho} = \rho_0, \tag{9.81}$$

and are characterized by the micro balance equations (9.69) and (9.70) reduced to the following ones

$$\frac{1}{\rho_0}\big(\boldsymbol{\Upsilon}^p - \boldsymbol{\pi}_0\big) = div\,\frac{1}{\rho_0}(\boldsymbol{\mu}_0^p - \mu), \tag{9.82}$$

where

$$\begin{aligned}&div\,(\frac{1}{\rho_0}\boldsymbol{\mu}_0^p) = -\beta_2 curl\big(curl\boldsymbol{H}^p\big) - \beta_2\tilde{\beta}\,\in\,(\nabla tr\boldsymbol{\Lambda}) + \beta_2\tilde{\beta}\,curl(\boldsymbol{\Lambda}^T), \\ &div\,(\frac{1}{\rho_0}\boldsymbol{\mu}_0) = \beta_1(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^p).\end{aligned}$$

and

$$\frac{1}{\rho_0}\boldsymbol{\Upsilon}^{\lambda} = \beta_4\,\Delta\boldsymbol{\Lambda}, \tag{9.83}$$

when the mass density of couple body forces were neglected.

Theorem 9.5. *Finally, the evolution equations for the unknowns $\boldsymbol{H}^p$ and $\boldsymbol{\Lambda}$ are given by*

$$\begin{aligned}
\xi_1 \dot{\boldsymbol{H}}^p &= \frac{1}{\rho_0}\boldsymbol{T} + \mathscr{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \beta_1(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^p) + \\
&-\beta_2 curl\big(curl\boldsymbol{H}^p\big) - \beta_2\tilde{\beta} \in (\nabla tr\boldsymbol{\Lambda}) + \beta_2\tilde{\beta}\, curl(\boldsymbol{\Lambda}^T) + \\
&+\beta_2 \left((tr\boldsymbol{\Lambda})\boldsymbol{I} - \boldsymbol{\Lambda}^T\right)(curl\boldsymbol{H}^p)^T + \beta_2 (curl\boldsymbol{H}^p)\big((tr\boldsymbol{\Lambda})\boldsymbol{I} - \boldsymbol{\Lambda}\big) + \\
&+2\beta_2\tilde{\beta} \left((tr\boldsymbol{\Lambda})\boldsymbol{I} - \boldsymbol{\Lambda}^T\right)\big((tr\boldsymbol{\Lambda})\boldsymbol{I} - \boldsymbol{\Lambda}\big), \\
\xi_2 \dot{\boldsymbol{\Lambda}} &= \beta_4\, \Delta\boldsymbol{\Lambda} + \beta_4\, \nabla\boldsymbol{H}^p \odot \nabla\boldsymbol{\Lambda} - \beta_4 \nabla\boldsymbol{\Lambda}\,_r \odot \nabla\boldsymbol{H}^p - \beta_3\boldsymbol{\Lambda} - \\
&-2\beta_2\, tr\,(curl\boldsymbol{H}^p)\boldsymbol{I} - 4\beta_2\, \tilde{\beta} tr(\boldsymbol{\Lambda})\boldsymbol{I} + \\
&+\beta_2\, (curl\boldsymbol{H}^p)^T + \beta_2\tilde{\beta}\big(tr(\boldsymbol{\Lambda})\boldsymbol{I} - \boldsymbol{\Lambda}\big).
\end{aligned} \tag{9.84}$$

Here $\boldsymbol{T}$ is given by the relationship (9.75) when the hypothesis of small distortions is accepted.

9.6 Conclusions

The proposed model of structural defects such as dislocations and disclinations appears to be a continuation of the previous ones proposed by Cleja-Ţigoiu (2014); Cleja-Ţigoiu et al (2016).

- The postulated free energy functions contain somehow the same variables describing the defects, excepting the gradient of the disclination tensor which is not involved in Cleja-Ţigoiu (2014).

The elastic constitutive functions have been essentially changed, as follows

- The elastic response is characterized here by the formulae (9.57). The Piola-Kirchhoff stress tensor is expressed in terms of the partial derivatives $\partial_{\boldsymbol{C}^e}\psi, \partial_{\nabla\boldsymbol{C}^e}\psi$, as well as Bilby's elastic connection, $\overset{(e)}{\mathscr{A}}$, while

$$\frac{1}{\rho_0}\boldsymbol{\pi} = \partial_{\boldsymbol{C}^e}\psi$$

  in Cleja-Ţigoiu (2014).
- The macro stress momentum with respect to the reference configuration is given in terms of $\partial_{\nabla\boldsymbol{C}^e}\psi$ via the relation $(9.57)_2$, while in Cleja-Ţigoiu (2014) the macro stress momentum is not a third order symmetric tensor and it depends on $\partial_{\mathscr{A}^e}\psi$, and $\partial_{\boldsymbol{S}^e}\psi$. We used the notations

$$\mathscr{A}^e \equiv \overset{(e)}{\mathscr{A}}_{\mathscr{K}}$$

  and

$$\boldsymbol{S}^e \equiv \boldsymbol{S}^e_{\mathscr{K}}.$$

The evolution equations for plastic distortion, $\boldsymbol{F}^p$, and disclination tensor, $\widetilde{\boldsymbol{\Lambda}}$, were provided to be compatible with the reduced dissipation inequality. We derived also peculiar evolution equations for $\boldsymbol{H}^p$ and $\widetilde{\boldsymbol{\Lambda}}$ within the small distortions framework.

The evolution equations provided here for $\underline{\beta} = 1$ are similar with those derived in Cleja-Ţigoiu et al (2016) for the small strains, apart from the terms induced by the elastic effect, namely the first three terms involved in right-hand side of the evolution equation $(9.84)_1$. In Cleja-Ţigoiu (2014) the micro stress associated with the disclination mechanism remained undefined, and the disclination tensor, $\widetilde{\boldsymbol{\Lambda}}$, was viewed as internal variable, see Maugin (2006). The presence of the gradient $\nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}}$ in the free energy function allowed us to define the micro stress $\boldsymbol{\Upsilon}^{\lambda}$, introduced in (9.48) via (9.70). Thus both evolution equations are viscoplastic and diffusion type.

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References

Arsenlis A, Parks DM (1999) Crystallographic aspects of geometrically-necessary and statistically-stored dislocation density. Acta Materialia 47(5):1597 – 1611

Bilby BA (1960) Continuous distribution of dislocations. In: Sneddon IN, Hill R (eds) Progress in Solid Mechanics, North-Holland, Amsterdam, pp 329–398

Clayton JD, McDowell DL, Bammann DJ (2006) Modeling dislocations and disclinations with finite micropolar elastoplasticity. International Journal of Plasticity 22(2):210 – 256

Cleja-Ţigoiu S, Ţigoiu V (2011) Strain gradient effect in finite elasto-plastic damaged materials. International Journal of Damage Mechanics 20(4):484–514

Cleja-Ţigoiu S, Paşcan R, Ţigoiu V (2016) Interplay between continuous dislocations and disclinations in elasto-plasticity. International Journal of Plasticity 79:68 – 110

Cleja-Ţigoiu S (2007) Material forces in finite elasto-plasticity with continuously distributed dislocations. International Journal of Fracture 147(1):67–81

Cleja-Ţigoiu S (2010) Elasto-plastic materials with lattice defects modeled by second order deformations with non-zero curvature. International Journal of Fracture 166(1):61–75

Cleja-Ţigoiu S (2014) Dislocations and disclinations: continuously distributed defects in elasto-plastic crystalline materials. Archive of Applied Mechanics 84(9):1293–1306

Cleja-Ţigoiu S (2017) Finite elasto-plastic models for lattice defects in crystalline materials. In: dell'Isola F, Sofonea M, Steigmann D (eds) Mathematical Modelling in Solid Mechanics, Springer, Singapore, Advanced Structured Materials, vol 69, pp 43–57

Cleja-Tigoiu S, Maugin GA (2000) Eshelby's stress tensors in finite elastoplasticity. Acta Mechanica 139(1):231–249

de Wit R (1970) Linear theory of static disclinations. In: Simmons JA, de Wit R, Bullough R (eds) Fundamental Aspects of Dislocation Theory, Nat. Bur. Stand. (U.S.) Spec. Publ., vol 317, I, pp 651–673

de Wit R (1973a) Theory of disclinations: II. Continuous and discrete disclinations in anisotropic elasticity. J Res Nat Bur Stand - A Phys Chem 77A(1):49 – 100

de Wit R (1973b) Theory of disclinations: III. Continuous and discrete disclinations in isotropic elasticity. J Res Nat Bur Stand - A Phys Chem 77A(3):359 – 368

de Wit R (1973c) Theory of disclinations: IV. Straight disclinations. J Res Nat Bur Stand - A Phys Chem 77A(5):607 – 658 de Wit R (1981) A view of the relation between the continuum theory of lattice defects and non-euclidean geometry in the linear approximation. International Journal of Engineering Science 19(12):1475 – 1506

Eringen AC (2002) Nonlocal Continuum Field Theories. Springer, New York

Fleck NA, Muller GM, Ashby MF, Hutchinson JW (1994) Strain gradient plasticity: Theory and experiment. Acta Metallurgica et Materialia 42(2):475 – 487

Fressengeas C, Taupin V, Capolungo L (2011) An elasto-plastic theory of dislocation and disclination fields. International Journal of Solids and Structures 48(25):3499 – 3509

Gurtin ME (2002) A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations. Journal of the Mechanics and Physics of Solids 50(1):5 – 32

Gurtin ME, Fried E, Anand L (2010) The Mechanics and Thermodynamics of Continua. University Press, Cambridge

Kossecka E, de Wit R (1977) Disclination kinematics. Archives of Mechanics 29:633 – 650

Kröner E (1990) The differential geometry of elementary point and line defects in Bravais crystals. International Journal of Theoretical Physics 29(11):1219–1237

Kröner E (1992) The internal mechanical state of solids with defects. International Journal of Solids and Structures 29(14):1849–1857

Lazar M, Maugin GA (2004a) Defects in gradient micropolar elasticity. I: screw dislocation. Journal of the Mechanics and Physics of Solids 52(10):2263 – 2284

Lazar M, Maugin GA (2004b) Defects in gradient micropolar elasticity. II: edge dislocation and wedge disclination. Journal of the Mechanics and Physics of Solids 52(10):2285 – 2307

Lazar M, Maugin GA, Aifantis EC (2006) On a theory of nonlocal elasticity of bi-Helmholtz type and some applications. International Journal of Solids and Structures 43(6):1404 – 1421

Maugin GA (1994) Eshelby stress in elastoplasticity and ductile fracture. International Journal of Plasticity 10(4):393 – 408

Maugin GA (2006) Internal variables and dissipative structures. J Non-Equilib Thermodyn 15:173 – 192

Minagawa S (1979) A non-Riemannian geometrica theory of imperfections in a Cosserat continuum. Arch Mech 31(6):783 – 792

Schouten J (1954) Ricci Calculus. Springer, Berlink

Teodosiu C (1982) Elastic Models of Crystal Defects. Springer, Berlin, Heidelberg, New York

Yavari A, Goriely A (2012) Riemann–Cartan geometry of nonlinear dislocation mechanics. Archive for Rational Mechanics and Analysis 205(1):59–118

Yavari A, Goriely A (2013) Riemann-Cartan geometry of nonlinear disclination mechanics. Mathematics and Mechanics of Solids 18(1):91–102