Chapter 25
A Consistent Dynamic Finite-Strain Plate Theory for Incompressible Hyperelastic Materials

Yuanyou Li and Hui-Hui Dai

Abstract In this chapter, a dynamic finite-strain plate theory for incompressible hyperelastic materials is deduced. Starting from nonlinear elasticity, we present the three-dimensional (3D) governing system through a variational approach. By series expansion of the independent variables about the bottom surface, we deduce a 2D vector dynamic plate system, which preserves the local momentum-balance structure. Then we propose appropriate position and traction boundary conditions. The 2D plate equation guarantees that each term in the variation of the generalized potential energy functional attains the required asymptotic order. We also consider the associated weak formulations of the plate model, which can be applied to different types of practical edge conditions.

25.1 Introduction

Plate structures are defined as plane elements with one small thickness dimension compared with the other two planar dimensions. The theory of plates has been widely studied by scientists in both mathematical and engineering communities since the nineteenth century. The literature in this field is extremely plentiful, including theories based on engineering intuitions and assumptions, derived theories from three-dimensional elasticity, as well as direct theories (Timoshenko and Woinowsky-Krieger, 1959; Naghdi, 1972; Reddy, 2007; Altenbach et al, 2010). We also refer the readers to Dai and Song (2014) for a review of some selected works. For derived plate theories, we usually focus on how to reduce the original 3D elasticity theory to a two-dimensional (2D) approximate model while the fundamental mechanical properties of plate structures can be appropriately captured.

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Early attempts on plate theories relied on a priori hypotheses, either of geometrical and mechanical nature or on the specific form of the solution (displacements and stresses). Classical plate theories derived in this way include the Kirchhoff-Love theory (Kirchhoff, 1850; Love, 1888), the von Kármán theory (von Kármán, 1910), the Mindlin-Reissner theory (Mindlin, 1951). Despite the success of these theories to various specific situations, their applicability for relatively thick plates and general loadings (say, under shear tractions) may be limited.

One approach involves no explicit kinematic assumptions on displacements (or deformed positions) except general power (or other function) series expansions

$$\boldsymbol{x}(\boldsymbol{r},Z) = \sum_{j=0}^{N} Z^j \boldsymbol{x}^j(\boldsymbol{r}). \tag{25.1}$$

Normally, all the coefficients $\boldsymbol{x}^j$ ($j \geqslant 1$) are treated as independent unknowns. By first integrating out the Z variable and conducting a truncation, the 2D potential energy of the plate is formulated. Then the governing equations are derived from the two-dimensional variational or virtual work principle. Such an approach has been adopted by Kienzler (2002) based on linear elasticity. Based on nonlinear elasticity, Meroueh (1986); Steigmann (2007) adopted Legendre polynomials of Z in (25.1) and formulated a system in terms of generalized (high order) stress resultants for finite-strain problems. It is worth mentioning that by imposing restrictions on the high-order coefficients in (25.1) instead of treating them as independent unknowns, Steigmann (2013) derived more proper plate and shell models which incorporate both stretching and bending.

Based on a priori scalings between the plate thickness and the deformations (or applied loads), some consistent mathematical approaches are utilized for deriving asymptotically correct plate theories. The method of Gamma convergence (Friesecke et al, 2002) is concerned with the two-dimensional variational problem in the limit of small thickness, but it cannot be used to study dynamic problems and derive plate theories incorporating both bending and stretching. The method of asymptotic analysis, which aims at developing the leading-order weak formulation by formal expansions with the thickness as the small parameter, was used to derive the von Kármán plate equations from the 3D weak formulation in Ciarlet (1980). In Millet et al (2001), based on the 3D differential formulation, a hierarchy of leading-order plate equations were derived.

Most of the works in the literature consider compressible materials, the existing plate theories for incompressible materials are much fewer. With the Gamma convergence method, Trabelsi (2005) formulated a nonlinear elastic thin membrane model for incompressible materials, while Conti and Dolzmann (2008) extended the plate theory derived in Friesecke et al (2002) to the case of incompressible materials. By using the principle of virtual work, Batra (2007) proposed a compatible shear and normal deformable theory for a plate made of an incompressible linear elastic material, in which the orthonormal Legendre polynomials were adopted to derive the high-order plate theory.

Nowadays, soft materials and biological materials have attracted attentions of researchers of different fields. It happens that most of soft materials are incompressible. In this chapter, we intend to provide a dynamic plate theory for incompressible materials. Taking the incompressibility constraint into account, we extend the consistent plate theory proposed in Song and Dai (2016) to the case of incompressible hyperelastic materials.

We organize this chapter as follows. In Sect. 25.2, the 3D governing system of incompressible materials is derived through conventional variational approach. In Sect. 25.3, according to the criterion of consistency we derive the 2D vector plate equation and propose some proper edge boundary conditions as well. In Sect. 25.4, we consider the associated weak formulations of the 2D plate equation and adopt them to distinct types of boundary conditions. Finally, we make some concluding remarks.

25.2 The 3D Governing Equations

In this section, we consider a homogeneous thin plate of constant thickness, which is composed of an incompressible hyperelastic material. A material point of the plate in the reference configuration $\kappa = \Omega \times [0, 2h]$ is denoted by $\boldsymbol{X} = (\boldsymbol{r}, Z)$, where the thickness $2h$ of the plate is small compared with the planar dimensions of the top (or bottom) surface Ω. The coordinates of a material point in the current configuration κ_t is denoted as $\boldsymbol{x}$. Throughout the paper, symbols with typefaces a, $\boldsymbol{a}$, $\mathbb{A}$, $\mathscr{A}$ represent scalar, vector, second-order tensor (matrix) and higher-order tensor, respectively. In component forms, we adopt the convention that Latin indices run from 1 to 3 whereas Greek indices run from 1 to 2, repeated summation convention is used and the index after the comma indicates differentiation.

The deformation gradient tensor of a material point in the plate can then be represented by

$$\mathbb{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}} + \frac{\partial \boldsymbol{x}}{\partial Z} \otimes \boldsymbol{k} = \nabla \boldsymbol{x} + \frac{\partial \boldsymbol{x}}{\partial Z} \otimes \boldsymbol{k}, \tag{25.2}$$

where ∇ is the in-plane two-dimensional gradient and $\boldsymbol{k}$ is the unit outward normal vector of the reference top surface Ω. More precisely, with rectangular Cartesian coordinates $\boldsymbol{r} = X_1 \boldsymbol{E}_1 + X_2 \boldsymbol{E}_2$, we have

$$\nabla \boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial X_1} \otimes \boldsymbol{E}_1 + \frac{\partial \boldsymbol{x}}{\partial X_2} \otimes \boldsymbol{E}_2.$$

Besides, we consider the following incompressibility constraint equation

$$R(\mathbb{F}) = \text{Det}(\mathbb{F}) - 1 = 0, \quad \text{in } \Omega \times [0, 2h]. \tag{25.3}$$

Suppose the material has the strain-energy density function $\Phi(\mathbb{F})$, the associated first and second order elastic moduli are defined by

$$\mathscr{A}^1(\mathbb{F}) = \frac{\partial^2 \Phi}{\partial \mathbb{F} \partial \mathbb{F}} \left(\mathscr{A}^1_{ijkl} = \frac{\partial^2 \Phi}{\partial F_{ji} \partial F_{lk}} \right), \quad \mathscr{A}^2(\mathbb{F}) = \frac{\partial^3 \Phi}{\partial \mathbb{F} \partial \mathbb{F} \partial \mathbb{F}}. \tag{25.4}$$

It is assumed that the strain energy function for the deformations concerned satisfies the strong-ellipticity condition

$$\boldsymbol{a} \otimes \boldsymbol{b} : \mathscr{A}^1(\mathbb{F})[\boldsymbol{a} \otimes \boldsymbol{b}] > 0, \quad \text{for all } \boldsymbol{a} \otimes \boldsymbol{b} \neq 0, \tag{25.5}$$

where the colon between second-order tensors means a scalar tensor product defined by $\mathbb{A} : \mathbb{B} = A_{kl} B_{lk}$ and the square bracket after a higher-order modulus tensor represents the operations

$$\left\{ \mathscr{A}^1[\mathbb{A}] \right\}_{ij} = \mathscr{A}^1_{ijkl} A_{lk}, \quad \left\{ \mathscr{A}^2[\mathbb{A}, \mathbb{B}] \right\}_{ij} = \mathscr{A}^2_{ijklmn} A_{lk} B_{nm}. \tag{25.6}$$

For the dynamic case with dead-loading on the boundary, suppose $\boldsymbol{q}_b$ is the body force, $\boldsymbol{q}^{\pm}$ are the applied tractions on the top and bottom surfaces, and $\boldsymbol{q}$ is the applied traction on $\partial \Omega_q$. The kinetic energy, the strain energy and the load potential are given by

$$\begin{aligned}
\underline{K} &= \int_{\kappa} \frac{1}{2} \rho \dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} d\boldsymbol{X}, \\
\underline{\Phi} &= \int_{\kappa} \Phi(\mathbb{F}) d\boldsymbol{X}, \\
\underline{V} &= -\int_{\kappa} \boldsymbol{q}_b(\boldsymbol{X}) \cdot \boldsymbol{x}(\boldsymbol{X}) d\boldsymbol{X} - \int_{\Omega} \boldsymbol{q}^-(\boldsymbol{r}) \cdot \boldsymbol{x}(\boldsymbol{r}, 0) + \boldsymbol{q}^+(\boldsymbol{r}) \cdot \boldsymbol{x}(\boldsymbol{r}, 2h) d\boldsymbol{r} \\
&\quad - \int_{\partial \Omega_q} \int_0^{2h} \boldsymbol{q}(s, Z) \cdot \boldsymbol{x}(s, Z) dZ ds,
\end{aligned}$$

where ρ is the mass density of the plate material and the overhead dot means time derivative. The lateral surface of the plate, which is denoted as $\partial \Omega$, is composed of the position boundary $\partial \Omega_0$ and the traction boundary $\partial \Omega_q$. All the quantities are defined in the reference configuration.

In order to calculate the minimum of the potential energy functional under the constraint condition (25.3), we consider the following generalized potential energy functional

$$\Psi(\boldsymbol{x}, p; \boldsymbol{X}) = \int_{t_1}^{t_2} \left\{ \underline{\Phi} + \underline{V} - \underline{K} - \int_{\kappa} p(\boldsymbol{X}) R(\mathbb{F}) d\boldsymbol{X} \right\} dt,$$

where $p(\boldsymbol{X})$ plays the role of the Lagrangian multiplier. Next, the governing system of the current plate model will be derived by calculating the variations of Ψ with respect to the independent variables $\boldsymbol{x}$ and p.

First, from the Hamilton's principle and upon using the divergence theorem, we obtain the variation of Ψ with respect to $\boldsymbol{x}$

$$\frac{\delta\Psi}{\delta \boldsymbol{x}} = \int_{t_1}^{t_2} \Big\{ \int_{\kappa} (-\mathrm{Div}\mathbb{S} - \boldsymbol{q}_b + \rho\ddot{\boldsymbol{x}}) \cdot \delta\boldsymbol{x} d\boldsymbol{X} - \int_{\Omega} (\mathbb{S}^T \boldsymbol{k}|_{Z=0} + \boldsymbol{q}^-) \cdot \delta\boldsymbol{x}(\boldsymbol{r}, 0) d\boldsymbol{r}$$
$$+ \int_{\Omega} (\mathbb{S}^T \boldsymbol{k}|_{Z=2h} - \boldsymbol{q}^+) \cdot \delta\boldsymbol{x}(\boldsymbol{r}, 2h) d\boldsymbol{r} + \int_{\partial\Omega_0} \int_0^{2h} \mathbb{S}^T \boldsymbol{N} \cdot \delta\boldsymbol{x}(s, Z) dZ ds$$
$$+ \int_{\partial\Omega_q} \int_0^{2h} (\mathbb{S}^T \boldsymbol{N} - \boldsymbol{q}) \cdot \delta\boldsymbol{x}(s, Z) dZ ds \Big\} dt, \tag{25.7}$$

where

$$\mathbb{S} = \frac{\partial \Phi}{\partial \mathbb{F}} - p \frac{\partial R}{\partial \mathbb{F}}, \tag{25.8}$$

is the nominal stress tensor of the incompressible material (Ogden, 1984)), $\boldsymbol{N}$ is the unit outward normal to the lateral surface, and $\dot{\boldsymbol{x}}\delta\boldsymbol{x}$ is assumed to vanish at both t_1 and t_2. Due to the arbitrariness of $\delta\boldsymbol{x}$ in (25.7), the equations of motion for any $t \in (t_1, t_2)$ together with boundary conditions are

$$\begin{array}{ll} \mathrm{Div}\mathbb{S} + \boldsymbol{q}_b = \rho\ddot{\boldsymbol{x}}, & \text{in } \Omega \times [0, 2h], \\ \mathbb{S}^T \boldsymbol{k}|_{Z=0} = -\boldsymbol{q}^-, & \text{in } \Omega, \\ \mathbb{S}^T \boldsymbol{k}|_{Z=2h} = \boldsymbol{q}^+, & \text{in } \Omega, \\ \boldsymbol{x} = \boldsymbol{b}(s, Z), & \text{on } \partial\Omega_0 \times [0, 2h], \\ \mathbb{S}^T \boldsymbol{N} = \boldsymbol{q}(s, Z), & \text{on } \partial\Omega_q \times [0, 2h], \end{array} \tag{25.9}$$

where $\boldsymbol{b}$ is the prescribed position on the boundary $\partial\Omega_0$, and we omit the argument t in all the above quantities. Next, we obtain the variation of Ψ with respect to p

$$\frac{\delta\Psi}{\delta p} = -\int_{t_1}^{t_2} \int_{\kappa} R(\mathbb{F})\, \delta p\, d\boldsymbol{X} dt. \tag{25.10}$$

We can obtain the constraint Eq.(25.3) from the above equation. Now we have formulated the 3D governing system Eq.(25.3) and Eq.(25.9), which contains two independent variables $\boldsymbol{x}$ and p.

25.3 The 2D Dynamic Plate Theory

In this section, we derive the 2D plate theory for incompressible materials from the previous 3D governing partial differential equations system, including the consistent dynamic plate equations, the boundary conditions as well as the associated weak formulation. It is a general approach to make approximations to eliminate the Z variable. Here we use the same consistency criterion proposed in Dai and Song (2014):

For all loadings that satisfy some smooth requirements, each term in the first variation of the energy functional should be of a required asymptotic order (say, $O(h^4)$) separately for the plate approximation.

Without loss of generality, it is assumed that all the spatial variables are scaled by the typical dimension of the in-plane surface, then $2h$ in fact represents the thickness ratio of the plate. The derivation follows the similar lines proposed in Song and Dai (2016); Wang et al (2016), but here we take into account the incompressibility constraint. We start from the series expansion of the independent variables with respect to Z.

25.3.1 Dynamic 2D Vector Plate Equation

Suppose that both $\boldsymbol{x}(\boldsymbol{X})$ and $p(\boldsymbol{X})$ are C^5 functions in their arguments, then we obtain the following series expansions:

$$\begin{aligned}\boldsymbol{x}(\boldsymbol{X}) &= \boldsymbol{x}^{(0)}(\boldsymbol{r}) + Z\boldsymbol{x}^{(1)}(\boldsymbol{r}) + \frac{1}{2}Z^2\boldsymbol{x}^{(2)}(\boldsymbol{r}) \\ &\quad + \frac{1}{6}Z^3\boldsymbol{x}^{(3)}(\boldsymbol{r}) + \frac{1}{24}Z^4\boldsymbol{x}^{(4)}(\boldsymbol{r}) + O(Z^5),\end{aligned} \tag{25.11}$$

$$\begin{aligned}p(\boldsymbol{X}) &= p^{(0)}(\boldsymbol{r}) + Zp^{(1)}(\boldsymbol{r}) + \frac{1}{2}Z^2p^{(2)}(\boldsymbol{r}) \\ &\quad + \frac{1}{6}Z^3p^{(3)}(\boldsymbol{r}) + \frac{1}{24}Z^4p^{(4)}(\boldsymbol{r}) + O(Z^5),\end{aligned} \tag{25.12}$$

where $(\cdot)^{(n)} = \partial^n(\cdot)/\partial Z^n|_{Z=0}$ $(n = 1,\ldots,4)$. According to the expansion of $\boldsymbol{x}$, the deformation gradient tensor can also be expanded as

$$\mathbb{F}(\boldsymbol{X}) = \mathbb{F}^{(0)}(\boldsymbol{r}) + Z\mathbb{F}^{(1)}(\boldsymbol{r}) + \frac{1}{2}Z^2\mathbb{F}^{(2)}(\boldsymbol{r}) + \frac{1}{6}Z^3\mathbb{F}^{(3)}(\boldsymbol{r}) + O(Z^4). \tag{25.13}$$

By substituting (25.11) into (25.2) and comparing with (25.13), we obtain the following relations

$$\mathbb{F}^{(n)} = \nabla\boldsymbol{x}^{(n)} + \boldsymbol{x}^{(n+1)} \otimes \boldsymbol{k}, \quad n = 0,1,2,3. \tag{25.14}$$

An observation from (25.14) is that the dependence of $\mathbb{F}^{(n)}$ on $\boldsymbol{x}^{(n+1)}$ is linearly algebraic. We also suppose the strain energy $\Phi(\mathbb{F})$ is C^5 functions in their arguments, so the nominal stress tensor can be expanded as

$$\mathbb{S}(\boldsymbol{X}) = \mathbb{S}^{(0)}(\boldsymbol{r}) + Z\mathbb{S}^{(1)}(\boldsymbol{r}) + \frac{1}{2}Z^2\mathbb{S}^{(2)}(\boldsymbol{r}) + \frac{1}{6}Z^3\mathbb{S}^{(3)}(\boldsymbol{r}) + O(Z^4). \tag{25.15}$$

Besides, from (25.8) and by using the chain rule, we obtain

$$\begin{aligned}\mathbb{S} = {} & \mathbb{A}^{(0)}(\mathbb{F}^{(0)}) + \mathscr{A}^{(1)}(\mathbb{F}^{(0)})[\mathbb{F}-\mathbb{F}^{(0)}] + \frac{1}{2}\mathscr{A}^{(2)}(\mathbb{F}^{(0)})[\mathbb{F}-\mathbb{F}^{(0)},\mathbb{F}-\mathbb{F}^{(0)}] \\ & + \frac{1}{6}\mathscr{A}^{(3)}(\mathbb{F}^{(0)})[\mathbb{F}-\mathbb{F}^{(0)},\mathbb{F}-\mathbb{F}^{(0)},\mathbb{F}-\mathbb{F}^{(0)}] + \cdots \\ & - \left\{ p^{(0)}(\boldsymbol{r}) + Zp^{(1)}(\boldsymbol{r}) + \frac{1}{2}Z^2 p^{(2)}(\boldsymbol{r}) + \frac{1}{6}Z^3 p^{(3)}(\boldsymbol{r}) + \cdots \right\} \\ & \times \left\{ \mathbb{R}^{(0)}(\mathbb{F}^{(0)}) + \mathscr{R}^{(1)}(\mathbb{F}^{(0)})[\mathbb{F}-\mathbb{F}^{(0)}] + \frac{1}{2}\mathscr{R}^{(2)}(\mathbb{F}^{(0)})[\mathbb{F}-\mathbb{F}^{(0)},\mathbb{F}-\mathbb{F}^{(0)}] \right. \\ & \left. + \frac{1}{6}\mathscr{R}^{(3)}(\mathbb{F}^{(0)})[\mathbb{F}-\mathbb{F}^{(0)},\mathbb{F}-\mathbb{F}^{(0)},\mathbb{F}-\mathbb{F}^{(0)}] + \cdots \right\},\end{aligned} \tag{25.16}$$

where $\mathscr{A}^{(i)}$ $(i = 1,2,3)$ are elastic moduli associated with $\boldsymbol{\Phi}$, the similar moduli $\mathscr{R}^{(i)}$ $(i = 1,2,3)$ are defined by replacing $\boldsymbol{\Phi}$ with the constraint R, and

$$\mathbb{A}^{(0)}(\mathbb{F}^{(0)}) = \left.\frac{\partial \boldsymbol{\Phi}}{\partial \mathbb{F}}\right|_{\mathbb{F}=\mathbb{F}^{(0)}},$$
$$\mathbb{R}^{(0)}(\mathbb{F}^{(0)}) = \left.\frac{\partial R}{\partial \mathbb{F}}\right|_{\mathbb{F}=\mathbb{F}^{(0)}} = \mathrm{Det}(\mathbb{F}^{(0)})(\mathbb{F}^{(0)})^{-1}.$$

By substituting (25.13) into (25.16) and comparing with (25.15), we obtain the following expressions for $\mathbb{S}^{(n)}$

$$\begin{aligned}\mathbb{S}^{(0)}(\boldsymbol{r}) = {} & \mathbb{A}^{(0)} - p^{(0)}\mathbb{R}^{(0)}, \\ \mathbb{S}^{(1)}(\boldsymbol{r}) = {} & \underline{\mathscr{A}}[\mathbb{F}^{(1)}] - p^{(1)}\mathbb{R}^{(0)}, \\ \mathbb{S}^{(2)}(\boldsymbol{r}) = {} & \underline{\mathscr{A}}[\mathbb{F}^{(2)}] - p^{(2)}\mathbb{R}^{(0)} + \mathscr{A}^{(2)}[\mathbb{F}^{(1)},\mathbb{F}^{(1)}] - p^{(0)}\mathscr{R}^{(2)}[\mathbb{F}^{(1)},\mathbb{F}^{(1)}] - 2p^{(1)}\mathscr{R}^{(1)}[\mathbb{F}^{(1)}], \\ \mathbb{S}^{(3)}(\boldsymbol{r}) = {} & \underline{\mathscr{A}}[\mathbb{F}^{(3)}] - p^{(3)}\mathbb{R}^{(0)} + 3\mathscr{A}^{(2)}[\mathbb{F}^{(1)},\mathbb{F}^{(2)}] + \mathscr{A}^{(3)}[\mathbb{F}^{(1)},\mathbb{F}^{(1)},\mathbb{F}^{(1)}] \\ & - 3p^{(0)}\mathscr{R}^{(2)}[\mathbb{F}^{(1)},\mathbb{F}^{(2)}] - p^{(0)}\mathscr{R}^{(3)}[\mathbb{F}^{(1)},\mathbb{F}^{(1)},\mathbb{F}^{(1)}] - 3p^{(1)}\mathscr{R}^{(1)}[\mathbb{F}^{(2)}] \\ & - 3p^{(1)}\mathscr{R}^{(2)}[\mathbb{F}^{(1)},\mathbb{F}^{(1)}] - 3p^{(2)}\mathscr{R}^{(1)}[\mathbb{F}^{(1)}],\end{aligned} \tag{25.17}$$

where the function of the new combined modulus

$$\underline{\mathscr{A}} = \mathscr{A}^{(1)} - p^{(0)}\mathscr{R}^{(1)}$$

in this incompressible case resembles $\mathscr{A}^{(1)}$ in the compressible case, and the argument $\mathbb{F}^{(0)}$ in $\mathbb{A}^{(0)}$, $\mathbb{R}^{(0)}$, $\mathscr{A}^{(i)}$ and $\mathscr{R}^{(i)}$ $(i = 1,\ldots,3)$ is omitted for brevity. Due to the series expansions (25.11) and (25.12), we obtain totally 19 unknowns in the governing system (including five vectors $\boldsymbol{x}^{(n)}$ $(n = 0,\ldots,4)$ and four scalars $p^{(n)}$ $(n = 0,\ldots,3)$), which are necessary in formulating a closed system by some consistent truncations of the 3D system. In addition, some equations in (25.9) serves to eliminate most of the unknowns, leading to a single vector plate equation. From

(25.14) and (25.17), we can observe that $\mathbb{S}^{(i)}$ depend linearly on $\boldsymbol{x}^{(i+1)}$ $(i = 1,2,3)$ as well, which plays a fundamental role in deriving the following recursion relations.

First, by substituting $(25.17)_1$ into the bottom traction condition $(25.9)_2$, we obtain

$$\begin{aligned}
\{\mathbb{S}^{(0)}\}^T\boldsymbol{k} &= \left\{\mathbb{A}^{(0)}(\mathbb{F}^{(0)}) - p^{(0)}\mathbb{R}^{(0)}(\mathbb{F}^{(0)})\right\}^T\boldsymbol{k} \\
&= \left\{\mathbb{A}^{(0)}(\nabla\boldsymbol{x}^{(0)} + \boldsymbol{x}^{(1)}\otimes\boldsymbol{k}) - p^{(0)}\mathbb{R}^{(0)}(\nabla\boldsymbol{x}^{(0)} + \boldsymbol{x}^{(1)}\otimes\boldsymbol{k})\right\}^T\boldsymbol{k} \\
&= -\boldsymbol{q}^-.
\end{aligned} \tag{25.18}$$

Equation (25.18) provides three algebraic equations for the unknowns $\boldsymbol{x}^{(1)}$ and $p^{(0)}$. In order to ease the sequel derivations, we define

$$\begin{aligned}
\boldsymbol{g}(\boldsymbol{x}^{(0)}) &\triangleq \mathbb{R}^{(0)T}\boldsymbol{k} = \mathrm{Det}(\mathbb{F}^{(0)})(\mathbb{F}^{(0)})^{-T}\boldsymbol{k} = \mathbb{F}^{(0)^*}\boldsymbol{k} = \mathbb{F}^{(0)^*}(\boldsymbol{E}_1\wedge\boldsymbol{E}_2) \\
&= (\mathbb{F}^{(0)}\boldsymbol{E}_1)\wedge(\mathbb{F}^{(0)}\boldsymbol{E}_2) = \boldsymbol{x}^{(0)}_{,1}\wedge\boldsymbol{x}^{(0)}_{,2},
\end{aligned}$$

where '$*$' represents the adjugate and '$\wedge$' means the cross product (Chadwick, 1999). So (25.18) reduces to

$$\mathbb{A}^{(0)T}(\mathbb{F}^{(0)})\boldsymbol{k} - p^{(0)}\boldsymbol{g}(\boldsymbol{x}^{(0)}) = -\boldsymbol{q}^-. \tag{25.19}$$

Next, vanishing of the coefficients of Z^n from $(25.9)_1$ yields that

$$\nabla\cdot\mathbb{S}^{(n)} + \mathbb{S}^{(n+1)T}\boldsymbol{k} + \boldsymbol{q}_b^{(n)} = \rho\ddot{\boldsymbol{x}}^{(n)}, \quad n = 0,1,2. \tag{25.20}$$

Equation (25.20) provides three linear algebraic equations for the unknowns $\boldsymbol{x}^{(n+1)}$ and $p^{(n)}$ $(n = 1,2,3)$.

Furthermore, by substituting the series expansion (25.13) into the constraint Eq. (25.3), we obtain

$$\begin{aligned}
R(\mathbb{F}^{(0)}) &+ \mathbb{R}^{(0)} : \left\{Z\mathbb{F}^{(1)} + \frac{1}{2}Z^2\mathbb{F}^{(2)} + \frac{1}{6}Z^3\mathbb{F}^{(3)}\right\} \\
&+ \frac{1}{2}\left\{Z\mathbb{F}^{(1)} + \frac{1}{2}Z^2\mathbb{F}^{(2)}\right\} : \mathscr{R}^{(1)}\left[Z\mathbb{F}^{(1)} + \frac{1}{2}Z^2\mathbb{F}^{(2)}\right] \\
&+ \frac{1}{6}Z\mathbb{F}^{(1)} : \mathscr{R}^{(2)}[Z\mathbb{F}^{(1)}, Z\mathbb{F}^{(1)}] + O(Z^4) = 0.
\end{aligned}$$

The vanishing of the coefficients of Z^n $(n = 0,1,2,3)$ in the above equation leads to

$$
\begin{aligned}
&R(\mathbb{F}^{(0)}) = (\boldsymbol{x}^{(0)}_{,1} \wedge \boldsymbol{x}^{(0)}_{,2}) \cdot \boldsymbol{x}^{(1)} - 1 = \boldsymbol{g} \cdot \boldsymbol{x}^{(1)} - 1 = 0,\\
&\mathbb{R}^{(0)} : \mathbb{F}^{(1)} = \boldsymbol{g} \cdot \boldsymbol{x}^{(2)} + \mathbb{R}^{(0)} : \nabla \boldsymbol{x}^{(1)} = 0,\\
&\mathbb{R}^{(0)} : \mathbb{F}^{(2)} + \mathbb{F}^{(1)} : \mathscr{R}^{(1)}[\mathbb{F}^{(1)}] = 0,\\
&\mathbb{R}^{(0)} : \mathbb{F}^{(3)} + 3\mathbb{F}^{(2)} : \mathscr{R}^{(1)}[\mathbb{F}^{(1)}] + \mathbb{F}^{(1)} : \mathscr{R}^{(2)}[\mathbb{F}^{(1)}, \mathbb{F}^{(1)}] = 0,
\end{aligned}
\tag{25.21}
$$

which provide the additional (linear) equations for the unknowns $\boldsymbol{x}^{(n)}$ $(n = 1,2,3,4)$ and $p^{(n)}$ $(n = 1,2,3)$.

These equations can be used to derive the recursion relations for $\boldsymbol{x}^{(n+1)}$ and $p^{(n)}$ $(n = 1,2,3)$. For instance, substituting $(25.17)_2$ into (25.20) $(n = 0)$ furnishes

$$
\mathbb{B}\boldsymbol{x}^{(2)} + \boldsymbol{f}^{(2)} - p^{(1)}\boldsymbol{g} = \rho \ddot{\boldsymbol{x}}^{(0)},
$$

where the second-order (acoustic) tensor $\mathbb{B}$ and the vector $\boldsymbol{f}^{(2)}$ are defined as

$$
\begin{aligned}
&\mathbb{B}\boldsymbol{x} = \{\underline{\mathscr{A}}[\boldsymbol{x} \otimes \boldsymbol{k}]\}^T \boldsymbol{k}, \quad \Rightarrow (\mathbb{B})|_{ij} = \underline{\mathscr{A}}_{3i3j},\\
&\boldsymbol{f}^{(2)} = \left\{\underline{\mathscr{A}}[\nabla \boldsymbol{x}^{(1)}]\right\}^T \boldsymbol{k} + \nabla \cdot \mathbb{S}^{(0)} + \boldsymbol{q}_b^{(0)}.
\end{aligned}
$$

By the strong-ellipticity condition in (25.5), $\mathbb{B}$ is invertible and positive-definite and we obtain

$$
\boldsymbol{x}^{(2)} = -\mathbb{B}^{-1}\boldsymbol{f}^{(2)} + p^{(1)}\mathbb{B}^{-1}\boldsymbol{g} + \mathbb{B}^{-1}\rho\ddot{\boldsymbol{x}}^{(0)}. \tag{25.22}
$$

By substituting (25.22) into $(25.21)_2$, we easily derive the expression of $p^{(1)}$:

$$
p^{(1)} = \frac{1}{g}\left(\boldsymbol{g} \cdot \mathbb{B}^{-1}\boldsymbol{f}^{(2)} - \boldsymbol{g} \cdot \mathbb{B}^{-1}\rho\ddot{\boldsymbol{x}}^{(0)} - \mathbb{R}^{(0)} : \nabla \boldsymbol{x}^{(1)}\right), \text{ with } g = \boldsymbol{g} \cdot \mathbb{B}^{-1}\boldsymbol{g}, \tag{25.23}
$$

where $g > 0$ due to the positive-definiteness of $\mathbb{B}$. Similarly, we obtain the following expressions of $\boldsymbol{x}^{(3)}$ and $p^{(2)}$

$$
\boldsymbol{x}^{(3)} = -\mathbb{B}^{-1}\boldsymbol{f}^{(3)} + p^{(2)}\mathbb{B}^{-1}\boldsymbol{g} + \mathbb{B}^{-1}\rho\ddot{\boldsymbol{x}}^{(1)}, \tag{25.24}
$$

$$
p^{(2)} = \frac{1}{g}\left(\boldsymbol{g} \cdot \mathbb{B}^{-1}\boldsymbol{f}^{(3)} - \boldsymbol{g} \cdot \mathbb{B}^{-1}\rho\ddot{\boldsymbol{x}}^{(1)} - \mathbb{R}^{(0)} : \nabla \boldsymbol{x}^{(2)} - \mathbb{F}^{(1)} : \mathscr{R}^{(1)}[\mathbb{F}^{(1)}]\right), \tag{25.25}
$$

where

$$
\begin{aligned}
\boldsymbol{f}^{(3)} = &\left\{\underline{\mathscr{A}}[\nabla \boldsymbol{x}^{(2)}]\right\}^T \boldsymbol{k} + \nabla \cdot \mathbb{S}^{(1)} + \boldsymbol{q}_b^{(1)}\\
&+ \left\{(\mathscr{A}^{(2)} - p^{(0)}\mathscr{R}^{(2)})[\mathbb{F}^{(1)}, \mathbb{F}^{(1)}] - 2p^{(1)}\mathscr{R}^{(1)}[\mathbb{F}^{(1)}]\right\}^T \boldsymbol{k}.
\end{aligned}
$$

The recursion relations for $p^{(3)}$ and $\boldsymbol{x}^{(4)}$ are not needed in the following derivations, however the relation (25.20) with $n = 2$ as a whole will be used to eliminate them.

Finally, the top traction condition in $(25.9)_3$ states

$$
\mathbb{S}^{(0)T}\boldsymbol{k} + 2h\mathbb{S}^{(1)T}\boldsymbol{k} + 2h^2\mathbb{S}^{(2)T}\boldsymbol{k} + \frac{4}{3}h^3\mathbb{S}^{(3)T}\boldsymbol{k} + O(h^4) = \boldsymbol{q}^+, \tag{25.26}
$$

which contains all the unknowns $\boldsymbol{x}^{(n)}$ $(n=0,\ldots,4)$ and $p^{(n)}$ $(n=0,\ldots,3)$.

As for $\boldsymbol{x}^{(1)}$ and $p^{(0)}$, it can be seen from (25.19) and $(25.21)_1$ that they only depend on $\nabla\boldsymbol{x}^{(0)}$. However, (25.19) and $(25.21)_1$ are nonlinear algebraic equations of $\boldsymbol{x}^{(1)}$ and $p^{(0)}$, which can only be solved when the concrete form of the strain-energy density $\Phi(\mathbb{F})$ is given. The strong-ellipticity condition and the implicit function theorem imply that they can uniquely determined in terms of $\nabla\boldsymbol{x}^{(0)}$, as shown in Wang et al (2016).

Finally, by subtracting (25.18) from (25.26) and further using (25.20), we can obtain the dynamic 2D vector plate equation

$$\nabla\cdot\underline{\mathbb{S}}+\underline{\boldsymbol{q}}=\rho\ddot{\underline{\boldsymbol{x}}}, \tag{25.27}$$

where

$$\underline{\boldsymbol{q}}=\frac{\boldsymbol{q}^{+}+\boldsymbol{q}^{-}}{2h}+\underline{\boldsymbol{q}}_b,$$
$$\underline{\boldsymbol{x}}=\frac{1}{2h}\int_0^{2h}\boldsymbol{x}dZ=\boldsymbol{x}^{(0)}+h\boldsymbol{x}^{(1)}+\frac{2}{3}h^2\boldsymbol{x}^{(2)}+O(h^3),$$

and $\underline{\mathbb{S}}$ and $\underline{\boldsymbol{q}}_b$ are defined in the same way as $\underline{\boldsymbol{x}}$. The dynamic plate equation, after substituting the recursion relations, becomes a fourth-order differential equation for $\boldsymbol{x}^{(0)}$ with an error of $O(h^3)$.

25.3.2 Edge Boundary Conditions

Besides the vector plate equation, we shall also reduce the original 3D lateral surface conditions to appropriate boundary conditions for 2D equation. Since the plate equation is of fourth order in spatial derivatives, on either the position boundary $\partial\Omega_0$ or the traction boundary $\partial\Omega_q$ two conditions regarding $\boldsymbol{x}^{(0)}$ or its derivatives are required. Some conditions might involve time-derivative of $\boldsymbol{x}^{(0)}$, which is different from the boundary conditions proposed in Dai and Song (2014).

25.3.2.1 Case 1. Prescribed Position in the 3D Formulation

Suppose that on $\partial\Omega_0\times[0,2h]$ the position $\boldsymbol{b}$ is prescribed, then we adopt the following two boundary conditions

$$\begin{aligned}&\boldsymbol{x}^{(0)}=\boldsymbol{b}^{(0)}(s),\\&\underline{\boldsymbol{x}}=\underline{\boldsymbol{b}}\quad\text{on }\partial\Omega_0\\\Leftrightarrow\;&\boldsymbol{x}^{(1)}+\frac{2}{3}h\boldsymbol{x}^{(2)}+\frac{1}{3}h^2\boldsymbol{x}^{(3)}+O(h^3)=\frac{1}{h}(\underline{\boldsymbol{b}}-\boldsymbol{b}^{(0)}),\end{aligned} \tag{25.28}$$

where

$$\boldsymbol{b}^{(0)} = \boldsymbol{b}|_{Z=0}$$

and $\underline{\boldsymbol{b}}$ represents the prescribed averaged position. The second condition contains up to the third-order spatial-derivatives and third-order time-derivatives of $\boldsymbol{x}^{(0)}$ upon using the recursion relations.

25.3.2.2 Case 2. Prescribed traction in the 3D formulation

Suppose that on $\partial\Omega_q \times [0, 2h]$ the traction $\boldsymbol{q}$ is specified and is C^4 in Z, then we adopt the following two boundary conditions

$$\begin{aligned}
&\mathbb{S}^{(0)T}\boldsymbol{N} = \boldsymbol{q}^{(0)},\\
&\underline{\mathbb{S}}^{T}\boldsymbol{N} = \frac{1}{2h}\int_0^{2h} \mathbb{S}^{T}\boldsymbol{N}dZ = \frac{1}{2h}\int_0^{2h} \boldsymbol{q}dZ = \boldsymbol{q}_0 \quad \text{on } \partial\Omega_q\\
\Leftrightarrow &\left(\mathbb{S}^{(0)} + h\mathbb{S}^{(1)} + \frac{2}{3}h^2\mathbb{S}^{(2)} + \frac{1}{3}h^3\mathbb{S}^{(3)} + O(h^4)\right)^T \boldsymbol{N} = \boldsymbol{q}_0,
\end{aligned} \tag{25.29}$$

where

$$\boldsymbol{q}^{(0)} = \boldsymbol{q}|_{Z=0}$$

and $\boldsymbol{q}_0$ is the averaged traction along the thickness of the plate. As for the second condition, we may utilize the traction condition at an arbitrary Z, or alternatively, use the specified moment about the middle line

$$\begin{aligned}
&\frac{1}{2h}\int_0^{2h} (Z-h)\mathbb{S}^{T}\boldsymbol{N}dZ = \frac{1}{2h}\int_0^{2h} (Z-h)\boldsymbol{q}dZ = \boldsymbol{m}(s)\\
\Leftrightarrow &\frac{1}{3}\mathbb{S}^{(1)T}\boldsymbol{N} + \frac{1}{3}h\mathbb{S}^{(2)T}\boldsymbol{N} + \frac{1}{5}h^2\mathbb{S}^{(3)T}\boldsymbol{N} + O(h^3) = \frac{\boldsymbol{m}(s)}{h^2},
\end{aligned} \tag{25.30}$$

where $\boldsymbol{m}(s)$ can be expressed in terms of $\boldsymbol{q}^{(i)}$ in the same way as the left-hand side.

25.3.3 Examination of the Consistency

According to the criterion introduced in Sect. 25.3, in order to examine the consistency of the derived 2D dynamic vector plate equation system, we analyze the asymptotic orders of the terms in the variations (25.7) and (25.10).

For the first term on the right hand side (r.h.s.) of (25.7), we consider the series expansions of Div$\mathbb{S}$ in terms of Z. As the first three terms in the series expansion (25.20) have been used together with $(25.21)_{(2-4)}$ to obtain the recursion relations of $\boldsymbol{x}^{(i)}$ $(i=2,3,4)$ and $p^{(i)}$ $(i=1,2,3)$, we have $\mathrm{Div}\mathbb{S}=O(Z^3)$. Thus the first term in (25.7) is of $O(h^4)$. The second term on the r.h.s of (25.7) is exactly equal to zero because Eq. (25.19) together with $(25.21)_1$ have been used to derive the expressions of $\boldsymbol{x}^{(1)}$ and $p^{(0)}$. In Eq. (25.26), $\mathbb{S}^T\boldsymbol{k}$ on the top surface has been expanded to $O(h^4)$, which implies that the third term in (25.7) is $O(h^4)$.

Next, we examine the asymptotic order of the fourth term in (25.7) which involves the prescribed position boundary condition on $\partial\Omega_0\times[0,2h]$. We rewrite the integrand in the following form

$$\begin{aligned}\int_0^{2h}\mathbb{S}^T\boldsymbol{N}\cdot\delta\boldsymbol{x}dZ=&\int_0^{2h}\mathbb{S}^{(0)T}\boldsymbol{N}\cdot\left(\delta\boldsymbol{x}^{(0)}+Z\delta\boldsymbol{x}^{(1)}+\frac{1}{2}Z^2\delta\boldsymbol{x}^{(2)}\right)dZ\\&+\int_0^{2h}Z\mathbb{S}^{(1)T}\boldsymbol{N}\cdot\left(\delta\boldsymbol{x}^{(0)}+Z\delta\boldsymbol{x}^{(1)}\right)dZ\\&+\int_0^{2h}\frac{1}{2}Z^2\mathbb{S}^{(2)T}\boldsymbol{N}\cdot\delta\boldsymbol{x}^{(0)}dZ+O(h^4).\end{aligned}\tag{25.31}$$

From the position boundary conditions (25.28), it can be obtained that $\delta\boldsymbol{x}^{(0)}=0$, $\delta\boldsymbol{x}^{(1)}=O(h)$ and $\delta\boldsymbol{x}^{(1)}+2/3h\delta\boldsymbol{x}^{(2)}=O(h^2)$. By substituting these results into (25.31), it is easy to check that all the three terms on the r.h.s. of (25.31) are of $O(h^4)$. Thus, the fourth term in (25.7) also satisfies the consistency condition.

To examine the asymptotic order of the fifth term in (25.7), we denote

$$\tilde{\boldsymbol{q}}=\mathbb{S}^T\boldsymbol{N}-\boldsymbol{q}.$$

The coefficients of the series expansion of $\tilde{\boldsymbol{q}}$ are represented as $\tilde{\boldsymbol{q}}^{(i)}$ $(i=0,\dots,3)$. Then the integration of the fifth term in (25.7) can be rewritten as

$$\begin{aligned}\int_0^{2h}\tilde{\boldsymbol{q}}\cdot\delta\boldsymbol{x}dZ=&\int_0^{2h}\left(\tilde{\boldsymbol{q}}^{(0)}+Z\tilde{\boldsymbol{q}}^{(1)}+\frac{1}{2}Z^2\tilde{\boldsymbol{q}}^{(2)}\right)\cdot\delta\boldsymbol{x}^{(0)}dZ\\&+\int_0^{2h}Z\left(\tilde{\boldsymbol{q}}^{(0)}+Z\tilde{\boldsymbol{q}}^{(1)}\right)\cdot\delta\boldsymbol{x}^{(1)}dZ\\&+\int_0^{2h}\frac{1}{2}Z^2\tilde{\boldsymbol{q}}^{(0)}\cdot\delta\boldsymbol{x}^{(2)}dZ+O(h^4).\end{aligned}\tag{25.32}$$

From the traction boundary conditions (25.29), it can be obtained that $\tilde{\boldsymbol{q}}^{(0)}=0$, $\tilde{\boldsymbol{q}}^{(1)}=O(h)$ and $\tilde{\boldsymbol{q}}^{(1)}+2/3h\tilde{\boldsymbol{q}}^{(2)}=O(h^2)$. By substituting these results into (25.32), it is easy to check that all the three terms on the r.h.s. of (25.32) are of $O(h^4)$. Thus, the fifth term in (25.7) also satisfies the consistency condition.

For the variation (25.10), we have considered the series expansion of $R(\mathbb{F})$ in terms of Z, where the coefficients of Z^i $(i=0,1,2,3)$ are set to be zero to derive the recursion relations of $\boldsymbol{x}^{(i+1)}$ $(i=0,1,2,3)$. Thus the variation (25.10) also attains $O(h^4)$, which satisfies the consistency condition. To sum up, the 2D vector plate equation (25.27) and the edge boundary conditions (25.28) and (25.29) ensure each term in the variations to be of an asymptotic order of $O(h^4)$, which satisfies the consistency criterion.

25.4 The Associated Weak Formulations

In this section, we shall derive the associated weak formulations of the previous 2D vector plate system to be prepared for future numerical calculations. Furthermore, when the 3D edge conditions are unknown, suitable boundary conditions can be proposed for practical loading cases according to the weak form.

First, by multiplying both sides of the plate equation (25.27) with $\boldsymbol{\xi} = \delta\boldsymbol{x}^{(0)}$ and calculating the integrations over the region Ω, we obtain

$$\begin{aligned}&\int_{\Omega} (\nabla\cdot\underline{\mathbb{S}})\cdot\boldsymbol{\xi}\,d\boldsymbol{r} = -\int_{\Omega}\underline{\boldsymbol{q}}\cdot\boldsymbol{\xi}\,d\boldsymbol{r} + \int_{\Omega}\rho\underline{\ddot{\boldsymbol{x}}}\cdot\boldsymbol{\xi}\,d\boldsymbol{r}\\ \Rightarrow &\int_{\partial\Omega}\left(\underline{\mathbb{S}}^T\boldsymbol{N}\right)\cdot\boldsymbol{\xi}\,ds - \int_{\Omega}\underline{\mathbb{S}}:\nabla\boldsymbol{\xi}\,d\boldsymbol{r} = -\int_{\Omega}\underline{\boldsymbol{q}}\cdot\boldsymbol{\xi}\,d\boldsymbol{r} + \int_{\Omega}\rho\underline{\ddot{\boldsymbol{x}}}\cdot\boldsymbol{\xi}\,d\boldsymbol{r}.\end{aligned} \tag{25.33}$$

Generally, the weak formulation associated with the fourth-order plate equation (25.27) should only contain up to the second-order derivative of $\boldsymbol{x}^{(0)}$. However, the weak formulation (25.33) involves the third-order derivatives, which originates from the terms $\mathbb{F}^{(2)}$ and $p^{(2)}$ in $\mathbb{S}^{(2)}$ and should be eliminated.

By substituting (25.23) and (25.25) into (25.14) and through some manipulations, we decompose $p^{(2)}$ and $\mathbb{F}^{(2)}$ into two parts

$$p^{(2)} = p_1^{(2)} + p_2^{(2)}, \quad \mathbb{F}^{(2)} = \mathbb{F}_1^{(2)} + \mathbb{F}_2^{(2)},$$

where

$$\begin{aligned}p_1^{(2)} =& \frac{1}{g}\Big\{\boldsymbol{g}\cdot\mathbb{B}^{-1}\left\{(\mathscr{A}^{(2)} - p^{(0)}\mathscr{R}^{(2)})[\mathbb{F}^{(1)},\mathbb{F}^{(1)}]\right\}^T\boldsymbol{k} - 2p^{(1)}\boldsymbol{g}\cdot\mathbb{B}^{-1}\left(\mathscr{R}^{(1)}[\mathbb{F}^{(1)}]\right)^T\boldsymbol{k}\\ &+\boldsymbol{g}\cdot\mathbb{B}^{-1}\boldsymbol{q}_b^{(1)} - \boldsymbol{g}\cdot\mathbb{B}^{-1}\rho\ddot{\boldsymbol{x}}^{(1)} - \mathbb{F}^{(1)}:\mathscr{R}^{(1)}[\mathbb{F}^{(1)}]\Big\},\\ p_2^{(2)} =& \frac{1}{g}\Big\{\boldsymbol{g}\cdot\mathbb{B}^{-1}\left(\nabla\cdot\mathbb{S}^{(1)}\right) + \boldsymbol{g}\cdot\mathbb{B}^{-1}\left\{\underline{\mathscr{A}}[\nabla\boldsymbol{x}^{(2)}]\right\}^T\boldsymbol{k} - \mathbb{R}^{(0)}:\nabla\boldsymbol{x}^{(2)}\Big\},\end{aligned}$$

$$
\begin{aligned}
\mathbb{F}_1^{(2)} =& p_1^{(2)}\mathbb{B}^{-1}\boldsymbol{g}\otimes\boldsymbol{k} - \mathbb{B}^{-1}\left\{(\mathscr{A}^{(2)} - p^{(0)}\mathscr{R}^{(2)})[\mathbb{F}^{(1)},\mathbb{F}^{(1)}]\right\}^T\boldsymbol{k}\otimes\boldsymbol{k} \\
&+ 2p^{(1)}\mathbb{B}^{-1}\left(\mathscr{R}^{(1)}[\mathbb{F}^{(1)}]\right)^T\boldsymbol{k}\otimes\boldsymbol{k} - \mathbb{B}^{-1}\boldsymbol{q}_b^{(1)}\otimes\boldsymbol{k} + \mathbb{B}^{-1}\rho\ddot{\boldsymbol{x}}^{(1)}\otimes\boldsymbol{k}, \\
\mathbb{F}_2^{(2)} =& \nabla\boldsymbol{x}^{(2)} - \mathbb{B}^{-1}\left\{\underline{\mathscr{A}}[\nabla\boldsymbol{x}^{(2)}]\right\}^T\boldsymbol{k}\otimes\boldsymbol{k} - \mathbb{B}^{-1}(\nabla\cdot\mathbb{S}^{(1)})\otimes\boldsymbol{k} + p_2^{(2)}\mathbb{B}^{-1}\boldsymbol{g}\otimes\boldsymbol{k}.
\end{aligned}
$$

It can be found that only $p_2^{(2)}$ and $\mathbb{F}_2^{(2)}$ involve the third-order derivative of $\boldsymbol{x}^{(0)}$. Correspondingly, we consider the following decomposition

$$
\underline{\mathbb{S}} : \nabla\boldsymbol{\xi} = \mathscr{W}_1(\nabla\nabla\boldsymbol{x}^{(0)}, \nabla\boldsymbol{\xi}) + \mathscr{W}_2(\nabla\nabla\nabla\boldsymbol{x}^{(0)}, \nabla\boldsymbol{\xi}), \tag{25.34}
$$

where

$$
\begin{aligned}
\mathscr{W}_1 =& (\mathbb{S}^{(0)} + h\mathbb{S}^{(1)}) : \nabla\boldsymbol{\xi} + \frac{2}{3}h^2\left\{\underline{\mathscr{A}}[\mathbb{F}_1^{(2)}] - p_1^{(2)}\mathbb{R}^{(0)}\right\} : \nabla\boldsymbol{\xi} \\
&+ \frac{2}{3}h^2\left\{(\mathscr{A}^{(2)} - p^{(0)}\mathscr{R}^{(2)})[\mathbb{F}^{(1)},\mathbb{F}^{(1)}] - 2p^{(1)}\mathscr{R}^{(1)}[\mathbb{F}^{(1)}]\right\} : \nabla\boldsymbol{\xi}, \\
\mathscr{W}_2 =& \frac{2}{3}h^2\left\{\underline{\mathscr{A}}[\mathbb{F}_2^{(2)}] - p_2^{(2)}\mathbb{R}^{(0)}\right\} : \nabla\boldsymbol{\xi}.
\end{aligned} \tag{25.35}
$$

In order to eliminate the third-order derivative terms of $\boldsymbol{x}^{(0)}$, we then substitute the expressions of $\mathbb{F}_2^{(2)}$ and $p_2^{(2)}$ into (25.35). Further manipulations yield the following result

$$
\mathscr{W}_2 = \frac{2}{3}h^2\left\{\mathscr{S}_0 : \nabla\boldsymbol{x}^{(2)} + \boldsymbol{\eta}\cdot(\nabla\cdot\mathbb{S}^{(1)})\right\},
$$

and

$$
\begin{aligned}
\mathscr{S}_0 &= \underline{\mathscr{A}}[\nabla\boldsymbol{\xi} + \boldsymbol{\eta}\otimes\boldsymbol{k}] - \zeta\mathbb{R}^{(0)}, \\
\boldsymbol{\eta} &= -\mathbb{B}^{-1}\{\underline{\mathscr{A}}[\nabla\boldsymbol{\xi}]\}^T\boldsymbol{k} + \zeta\mathbb{B}^{-1}\boldsymbol{g}, \\
\zeta &= \frac{1}{g}\left(\underline{\mathscr{A}}[\mathbb{B}^{-1}\boldsymbol{g}\otimes\boldsymbol{k}] - \mathbb{R}^{(0)}\right) : \nabla\boldsymbol{\xi} \triangleq \mathbb{P} : \nabla\boldsymbol{\xi}.
\end{aligned} \tag{25.36}
$$

In fact, it can be proved that

$$
\delta p^{(0)} = \zeta, \quad \delta\boldsymbol{x}^{(1)} = \boldsymbol{\eta}, \quad \delta\mathbb{S}^{(0)} = \mathscr{S}_0.
$$

Then, integration by parts leads to

$$
\begin{aligned}
\int_\Omega \mathscr{W}_2 d\boldsymbol{r} &= \frac{2}{3}h^2\int_{\partial\Omega}(\mathscr{S}_0^T\boldsymbol{N})\cdot\boldsymbol{x}^{(2)} + (\mathbb{S}^{(1)T}\boldsymbol{N})\cdot\boldsymbol{\eta}\,ds + \int_\Omega \mathscr{W}_3 d\boldsymbol{r}, \\
\mathscr{W}_3 &= -\frac{2}{3}h^2\left\{(\nabla\cdot\mathscr{S}_0)\cdot\boldsymbol{x}^{(2)} + \mathbb{S}^{(1)} : \nabla\boldsymbol{\eta}\right\}.
\end{aligned} \tag{25.37}
$$

It can be seen from (25.37) that the third-order derivatives of $\boldsymbol{x}^{(0)}$ have been eliminated. Combining the results (25.33), (25.34) and (25.37), the following 2D weak formulation can be derived

$$\int_{\Omega} \left(\mathscr{W}_1 + \mathscr{W}_3 + (\rho \underline{\ddot{\boldsymbol{x}}} - \underline{\boldsymbol{q}}) \cdot \boldsymbol{\xi}\right) d\boldsymbol{r} = \int_{\partial\Omega} \left(\underline{\mathbb{S}}^T \boldsymbol{N} \cdot \boldsymbol{\xi} - \frac{2}{3} h^2 \mathbb{S}^{(1)T} \boldsymbol{N} \cdot \boldsymbol{\eta} - \frac{2}{3} h^2 \mathscr{S}_0^T \boldsymbol{N} \cdot \boldsymbol{x}^{(2)} \right) ds. \tag{25.38}$$

In the following, by considering the boundary conditions, the 2D weak formulation (25.38) can be further simplified. We adapt it to two distinct types of boundary conditions, i.e., the previous cases in Sect. 25.3.2 and other practical cases when 3D edge conditions are unclear.

25.4.0.1 Case 1. Edge position and traction in the 3D formulation are known

From (25.28), it is easy to deduce that $\boldsymbol{\xi} = \delta \boldsymbol{x}^{(0)} = \boldsymbol{0}$ and $\boldsymbol{\eta} = \delta \boldsymbol{x}^{(1)} = O(h)$ on $\partial\Omega_0$, which together with (25.36) further implies that $\nabla \boldsymbol{\xi} = O(h)$ and $\mathscr{S}_0 = O(h)$. Consequently, the boundary integral on $\partial\Omega_0$ in (25.38) is of $O(h^3)$ and can be neglected.

While on $\partial\Omega_q$, it follows from conditions (25.29), (25.30) that

$$\mathscr{S}_0^T \boldsymbol{N} = \delta[\mathbb{S}^{(0)T} \boldsymbol{N}] = O(h^2).$$

Thus, the third term in the boundary integral can be neglected. Besides, replacing $\mathbb{S}^{(1)T}\boldsymbol{N}$ by the condition in (25.30) only causes a higher-order correction. Then, the 2D weak formulation (25.38) reduces to

$$\begin{aligned} \int_{\Omega} \left(\mathscr{W}_1 + \mathscr{W}_3 + (\rho \underline{\ddot{\boldsymbol{x}}} - \underline{\boldsymbol{q}}) \cdot \boldsymbol{\xi}\right) d\boldsymbol{r} &= \int_{\partial\Omega_q} \left(\underline{\mathbb{S}}^T \boldsymbol{N} \cdot \boldsymbol{\xi} - \frac{2}{3} h^2 \mathbb{S}^{(1)T} \boldsymbol{N} \cdot \boldsymbol{\eta} \right) ds \\ &= \int_{\partial\Omega_q} \boldsymbol{q}_0 \cdot \boldsymbol{\xi} - 2\boldsymbol{m} \cdot \boldsymbol{\eta}\, ds. \end{aligned}$$

25.4.0.2 Case 2. Edge position and traction in the 3D formulation are unknown

In many practical situations, where the edge traction distribution (e.g. a pinned edge) or displacement distribution (e.g. a clamped edge) is unknown, we should propose the so-called natural boundary conditions according to the weak formulation. For this purpose, we shall recast the boundary integral in (25.38) in terms of $\boldsymbol{\xi}$ and its normal derivative $\boldsymbol{\xi}_{,N}$.

For convenience, considering the last two terms in the boundary integral of (25.38), we introduce a third-order tensor $\mathscr{M}$ through

$$-\frac{2}{3}h^2\left(\mathbb{S}^{(1)T}\boldsymbol{N}\cdot\boldsymbol{\eta}+\mathscr{S}_0^T\boldsymbol{N}\cdot\boldsymbol{x}^{(2)}\right)$$
$$=\frac{2}{3}h^2\left\{\underline{\mathscr{A}}\left[\boldsymbol{t}\otimes\boldsymbol{k}-\boldsymbol{x}^{(2)}\otimes\boldsymbol{N}\right]+\left(\boldsymbol{x}^{(2)}\cdot\mathbb{R}^{(0)T}\boldsymbol{N}-\boldsymbol{t}\cdot\boldsymbol{g}\right)\mathbb{P}\right\}:\nabla\boldsymbol{\xi}\triangleq(\mathscr{M}[\boldsymbol{N}])^T:\nabla\boldsymbol{\xi}, \tag{25.39}$$

where

$$\boldsymbol{t}=\mathbb{B}^{-1}\left(\mathbb{S}^{(1)T}\boldsymbol{N}+\mathbb{B}_1\boldsymbol{x}^{(2)}\right),\quad(\mathbb{B}_1)_{ij}=\underline{\mathscr{A}}_{3i\alpha j}N_\alpha.$$

Furthermore, we introduce the decomposition

$$\nabla\boldsymbol{\xi}=\boldsymbol{\xi}_{,s}\otimes\boldsymbol{T}+\boldsymbol{\xi}_{,N}\otimes\boldsymbol{N}, \tag{25.40}$$

where $\boldsymbol{T}$ is the unit tangential vector, and $\boldsymbol{\xi}_{,s}$ is tangential derivative on $\partial\Omega$. Substituting (25.39) and (25.40) into the boundary integral in (25.38) and a simple integration by parts leads to

$$\int_\Omega\left(\mathscr{W}_1+\mathscr{W}_3+(\rho\underline{\ddot{\boldsymbol{x}}}-\underline{\boldsymbol{q}})\cdot\boldsymbol{\xi}\right)d\boldsymbol{r}=\int_{\partial\Omega}\left\{\underline{\mathbb{S}}^T\boldsymbol{N}-(\mathscr{M}[\boldsymbol{N}]\boldsymbol{T})_{,s}\right\}\cdot\boldsymbol{\xi}+\{\mathscr{M}[\boldsymbol{N}]\boldsymbol{N}\}\cdot\boldsymbol{\xi}_{,N}ds. \tag{25.41}$$

If we regard

$$\mathscr{W}:=\mathscr{W}_1+\mathscr{W}_3$$

as the variation of the plate stress work due to the virtual displacement $\boldsymbol{\xi}$, the weak formulation (25.41) can be rewritten as

$$\int_\Omega\left(\mathscr{W}+(\rho\underline{\ddot{\boldsymbol{x}}}-\underline{\boldsymbol{q}})\cdot\boldsymbol{\xi}\right)d\boldsymbol{r}=\int_{\partial\Omega}\hat{\boldsymbol{q}}(s)\cdot\boldsymbol{\xi}+\hat{\boldsymbol{m}}(s)\cdot\boldsymbol{\xi}_{,N}ds, \tag{25.42}$$

where $\hat{\boldsymbol{q}}$ and $\hat{\boldsymbol{m}}$ are respectively the applied generalized traction and bending moment at the edge. Based on (25.41) or (25.42), boundary conditions can be suitably proposed for various practical cases (e.g., clamped, pinned, simply-supported).

25.5 Conclusions

In this chapter, we propose a consistent dynamic finite-strain plate theory for incompressible hyperelastic materials with no special restrictions on loadings or the order of deformations. The developed plate theory follows similar lines as the previous work, except that the dynamic terms and an additional constraint equation are involved into the recursion relations and the final dynamic system. It is consistent with the 3D weak formulation since each term in the variations of the generalized potential energy functional attains the asymptotic order of $O(h^4)$. The current plate theory can recover the 3D displacement and stress fields. For the convenience of numerical calculations, we also derive the weak formulation of the 2D vector plate equation together with the position or traction edge boundary conditions. In comparison with other plate theories, the present one takes into account dynamic, finite-strain, bending and stretching effects together with incompressibility constraint, so it may provide a general framework for studying mechanical behaviour of soft-material plates under various loading conditions.

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