7 Analytical Value-at-Risk

A second approach, already mentioned at the beginning, to calculate the VaR is an analytical one. It is only approximate, as its assumptions don't always hold in practice, but it involves fewer computational steps because it relies on sensitivities and avoids the 1000×10^6 full position pricings. Often it is very close to the VaR obtained in the historical simulation, which makes it a useful sanity-check. It also clearly exposes the relation between the VaR and the sensitivities, volatilities, and correlations. Even more importantly, it provides some helpful analysis tools in dealing with the questions we're most interested in: How does the VaR react if we change our positions? What risk factors contribute most to the VaR? What is the reason for a particular VaR change?

This small chapter is mathematically the most involved. At first reading, you could also just skip it to avoid getting bogged down. Maybe return to it later if you find, e.g., the VaR-contribution helpful in dealing with the model's daily operation, as described in Chap. 17 entitled "Nine to Five." (If you tackle it, you can find some background on multiple randoms, the covariance matrix, and normal quantiles in Appendix A.)

7.1 Approximate VaR

We start off with our familiar 2200×1000 matrix of returns [**R**] (recall it from Fig. 4.3). It represents, after some rescaling and mirroring, our view of the history of asset or risk factor returns. Each row is associated to a risk factor and represents a historical sample of its returns.

M. Auer, *Hands-On Value-at-Risk and Expected Shortfall*, Management for Professionals, https://doi.org/10.1007/978-3-319-72320-4_7

We can also view each risk factor as a normal random variable x_i instead and denote all risk factors as

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2200} \end{pmatrix}.$$

These random variables have certain properties that can be estimated from their corresponding historical samples. Each x_i has a variance $\mathbb{V}\mathrm{ar}[x_i]$, and each pair of x_i, x_j has a covariance $\mathbb{C}\mathrm{ov}[x_i, x_j]$. Their estimates form the symmetric covariance matrix $[\mathbf{C}]$.[1]

Now assume that the current price of a position or portfolio is p_0 and that its price under various returns is a function of $\mathbf{X}$, denoted as $p(\mathbf{X})$. We are, as ever, interested in price *changes* or PnLs, which we can express as

$$\Delta(\mathbf{X}) = p(\mathbf{X}) - p_0.$$

This is a function of the random variables $\mathbf{X}$, and we'd like to get a handle on its volatility, i.e., standard deviation. For this, we first simplify Δ in a first-order Taylor expansion (we'll soon simply drop the anxious "$\approx$" and let the linearity assumption take over):

$$\Delta(\mathbf{X}) \approx p_0 + x_1 \frac{\partial p}{\partial x_1}(0) + x_2 \frac{\partial p}{\partial x_2}(0) + \cdots - p_0.$$

If we denote the partial derivatives here with d_i, we may write:

$$\Delta(\mathbf{X}) = d_1 x_1 + d_2 x_2 + \ldots$$

This Δ is a weighted sum of random variables. With $\mathbf{d} = \big(d_1, d_2, \ldots\big)$, its variance and standard deviation become

$$\begin{aligned} \mathbb{V}\mathrm{ar}[\Delta] &= \mathbf{d}[\mathbf{C}]\mathbf{d}^\top, \\ \mathrm{std}[\Delta] &= \sqrt{\mathbb{V}\mathrm{ar}[\Delta]} = \sqrt{\mathbf{d}[\mathbf{C}]\mathbf{d}^\top}. \end{aligned}$$

[1] Hence this method's alternative name of *variance-covariance* approach.

Because Δ, as sum of normals, is normal itself, the 1%-quantile or VaR[2] can now be directly derived from its standard deviation:

$$\text{VaR}(\mathbf{d}) = \sqrt{\mathbf{d}[\mathbf{C}]\mathbf{d}^\top} \times (-2.33..).$$

We don't want to compute the partial derivatives. Luckily, for any position or portfolio we usually have ready access to a close relative of their derivatives d_i—their sensitivities s_i, collected in the vector $\mathbf{s}$. We have already seen that $\mathbf{d} = 10^4\mathbf{s}$, which gets us the following useful expression for the VaR as a function of our sensitivities:

$$\begin{aligned}\text{VaR}(\mathbf{s}) &= \sqrt{(10^4\mathbf{s})[\mathbf{C}](10^4\mathbf{s}^\top)} \times (-2.33..)\\ &= -2.33.. \times 10^4 \times \sqrt{\mathbf{s}[\mathbf{C}]\mathbf{s}^\top}\\ &= c\sqrt{\mathbf{s}[\mathbf{C}]\mathbf{s}^\top}.\end{aligned}$$

We hereby have a fast way to estimate the VaR for each position or portfolio whose sensitivity vector we know.

(Note: the behavior of this VaR estimate depends on the covariance matrix. If we use rescaled returns to estimate it, this analytical VaR will also be aligned to the most recently observed market volatilities.)

7.2 VaR-Sensitivity

Our VaR function here is expressed as a single-valued function of all the s_i. A natural question to ask is then: how does this VaR change if the underlying sensitivities change (i.e., if our exposures or, effectively, our positions change)? To answer this, we need to look at the partial derivatives of the VaR function itself, called *VaR-sensitivities*. There is one for each risk factor.

To compute these VaR-sensitivities, let us first denote the vector of partial derivatives of any single-valued vector function $f(\mathbf{x})$ as follows:

$$\frac{\partial f}{\partial \mathbf{x}} = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots\right).$$

We recall that for any quadratic form $g(\mathbf{x}) = \mathbf{x}[\mathbf{A}]\mathbf{x}^\top$ we have

$$\frac{\partial g}{\partial \mathbf{x}} = \mathbf{x}([\mathbf{A}] + [\mathbf{A}]^\top).$$

[2]The variance "$\mathbb{V}$ar" is not the value-at-risk "VaR."

Mindful that $[\mathbf{C}] = [\mathbf{C}]^\top$, we proceed to partially differentiate our VaR function by its sensitivities s_i:

$$\begin{aligned}\mathbf{v}(\mathbf{s}) := \frac{\partial\,\mathrm{VaR}}{\partial \mathbf{s}} = \frac{\partial c\sqrt{\mathbf{s}[\mathbf{C}]\mathbf{s}^\top}}{\partial \mathbf{s}} \\ &= \frac{1}{2}\,c\,\frac{1}{\sqrt{\mathbf{s}[\mathbf{C}]\mathbf{s}^\top}}\,\frac{\partial \mathbf{s}[\mathbf{C}]\mathbf{s}^\top}{\partial \mathbf{s}} \\ &= \frac{1}{2}\,c\,\frac{1}{\sqrt{\mathbf{s}[\mathbf{C}]\mathbf{s}^\top}}\,\mathbf{s}([\mathbf{C}] + [\mathbf{C}]^\top) \\ &= c\frac{\mathbf{s}[\mathbf{C}]}{\sqrt{\mathbf{s}[\mathbf{C}]\mathbf{s}^\top}}.\end{aligned}$$

Each entry v_i in the vector of VaR-sensitivities $\mathbf{v}$ describes by how much the VaR changes if the corresponding sensitivity increases by 1$.

Example We would like to increase our exposure to asset i. Its sensitivity $s_i = -12{,}000$. The corresponding VaR-sensitivity $v_i = -8$. Changing the sensitivity by, say, $-1000$$ should then approximately affect the VaR by $-1000 \times -8 = +8000$, so we'd expect the (negative) VaR to change by $+8000$, decreasing in magnitude.

VaR-sensitivities are handy for quickly assessing the sign and magnitude of the VaR impact due to a prospective change in sensitivities, i.e., in positions. For relatively small sensitivity changes, it is also fairly accurate, but we can just as easily employ an exact, full repricing. The most useful application of VaR-sensitivities, however, is described next.

7.3 VaR-Contribution

Our expression for the VaR turns out to be a so-called homogeneous function of order 1, since for any scaling factor a we have

$$\mathrm{VaR}(a\mathbf{s}) = a\,\mathrm{VaR}(\mathbf{s}).$$

This allows us to write, per Euler's homogeneous function theorem,

$$\mathrm{VaR}(\mathbf{s}) = s_1 v_1 + s_2 v_2 + \dots$$

The terms $s_i v_i$ sum up to the VaR-value; we call such a term the *VaR-contribution* $c_i = s_i v_i$ of its risk factor. These VaR-contributions can be positive or negative. Large negative VaR-contributions indicate the risk factors that dominate or drive the VaR's magnitude. An example for a portfolio whose VaR is driven by the euro/dollar exchange rate is given in Table 17.1.