Chapter 32
About Collaborative Work: Exploring the Functional World in a Computer-Enriched Environment

Carmen Sessa

Abstract The purpose of this paper is to address two main concerns in mathematics education. The first is finding ways of bridging the gap between the worldviews of a university research team and secondary school mathematics teachers. The second is meaningful and implementable ways of introducing technological tools in regular classrooms in order to teach and explore functional relationships. Whereas these two issues have been discussed in the literature, this contribution blends these two issues in the context of Argentina while proposing general insights for the mathematics education community at large. This paper outlines and describes the different stages of the formation and functioning of a collaborative team of researchers and teachers and discusses some didactical complexities encountered.

Keywords Collaborative group · Integration of ICT · Design proposal

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32.1 The Journey that Led Me to Work in a Collaborative Group[1]

I have been working in the area of didactics in mathematics for the last 25 years. I originally trained in mathematics, and my shift towards didactics was accompanied by an important change in the conception of teaching that I held at the time. I went from considering the problem of teaching as a question of organizing a discourse (logically organized and attractive for the listener/student) to considering it as a double interaction process: the interaction between the students and a problem or task and the interaction between teachers and their students regarding their productions.[2]

Looking at teaching in this way obliges one to attend to various matters: first, the need for powerful tasks that promote varied and rich productions from the students and, second, a teacher's management of classroom activities that fosters the students' work and, based on their production, is able to build relations between this work and the knowledge wished to be consolidated in the classroom.

The beginning in didactics of mathematics. The first years of research into the didactics of mathematics were the result of teamwork with many colleagues[3] and were centered on understanding how the teaching system works at the beginning of the learning of algebra. This led us to get in touch with teachers and students and to analyze and interpret different programs and curricular documents. This was a very rich period during which the findings of research in this field provided us with conceptual tools for the analysis.

The didactic engineering[4] stage. Not long after starting, we needed to test and study how the situations we created with the objective of provoking specific work from the students would function in a classroom situation. Briefly, the work process was the following: First, we produced a teaching situation or sequence, then we contacted various teachers who were willing to teach it in their classrooms. Next, we undertook to communicate to the teachers the greater and smaller objectives of the overall sequence and of its individual parts. (This instance was usually very delicate. How to convey the subtlest objectives of each part of the sequence, without imposing the teacher as a script to follow? Then there was the fact that the

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[1]These are the teachers who now form the collaborative group in which I work: Marina Andrés, María Brunand, Marité Coronel, Rosa Escayola, Claudia Kerlakian, Sabrina Maffei, Esteban Romañuk, Débora Sanguinetti, Marina Torresi, and Martín Tornay. Gema Fioriti, from the National University of General San Martin, has participated in this group since its very beginnings and in all these years; the interaction with her has enriched my work. I must mention too that most of the ideas on this paper came about in interaction with the members of the research team at the Pedagogical University: Betina Duarte, Enrique di Rico, Mara Cedrón, Valeria Borsani, Juan Pablo Luna, and Rosa Cicala.

[2]These are the central issues of Guy Brousseau's Theory of Situations. See, for example, Brousseau (1997).

[3]Among them, I want to mention Patricia Sadovsky and Mabel Panizza.

[4]See, for example, Artigue (1998).

teacher was time for joint meetings). After this, we went to the classroom to take notes on the set up of the situation, and finally we analyzed the work produced with a small group of "specialists."

This research model proved to be unsatisfactory. We observed it locked inside our academic university environment. Although the work spoke of the classroom and of teaching, it was far away from it. We were not satisfied with the kind of relation we had with the teachers: It was always respectful but distant.

The emergence of a collaborative group: Associate teachers with our research. In 2006, we could form a group made up of secondary school teachers, academics specializing in mathematical didactics, and students in pre-service teacher training, with the objective of thinking together about the teaching of mathematics. The group's work was based on the design and analysis of teaching situations that the group's teachers then implemented in their classrooms. The practical realization of these situations was then once again analyzed by the group. Our work could be considered a kind of collaborative didactics engineering (Sensevy 2011). I have been working in this collaborative group ever since, and it has undergone different periods and continues to undergo changes in its conformation.

32.2 Different Stages in the Consolidation of the Collaborative Group

We can identify four stages in the development of the collaborative group's work.

Stage 1. This stage corresponds to a period in which the usual work environment in the classrooms of Argentina involved pencil and paper and blackboard. During this stage, we developed a teaching proposal centered on quadratic functions. The proposal and its findings were set out in a curricular document: Mathematics, Quadratic Functions, Parabolas, and Second-Degree Equations. Contributions to Teaching. Middle School. the document was completed in 2009 and was published in 2014 (Sessa et al. 2014).

It was the founding moment for associating teachers with our research. In the development of didactic engineering, the university team provided important experiences and didactic proposals. The necessary symmetry[5] was in the process of building.

Stage 2. Beginning in 2009 and for 2 years, the group worked in one of its most autonomous periods and decided to call itself the "Grupo de los lunes" ("Monday Group") because of the day it used to meet on. At the same time, the nationwide distribution of netbooks in secondary schools begins to take place, with the idea of

[5]We take from Sensevy (2011) the central idea that to install collaboration, it is necessary to build a symmetry between researchers and teachers. This is based on an equalization of legitimacy in relation to the work carried out, rather than in the denial of differences.

providing “one for each student” and with the manifest intention of “closing the technological gap” among different sectors of society. Computers started to reach schools, and the Monday Group wanted to study how to incorporate them into the proposals it was making. Considering this challenge, we elaborated a didactic proposal about an introduction to polynomial functions (Fioriti and Sessa 2015).

With different formations and concerns, the problem we faced was new for all; achieving real progress in building the symmetry required for the group work.

Each person’s contribution was merged in the production and analysis of the elaborated proposal. The dynamism of the GeoGebra program and the possibility of obtaining multiple and linked representations on the screen were the two potentialities of this program that we “learned” during this work. On the other hand, we were able to identify in this stage new concerns, including how to manage the collective spaces of work in the classroom and how the students would keep a record of their work on the computers.

In 2012 the collaborative group found a place within the Universidad Pedagógica, the institution where I have been working for the last 5 years. Two more stages took place in this period.

Stage 3. Once in the institutional framework of the Universidad Pedagógica, the group reconsidered the proposal it had elaborated in Stage 1, which was originally designed to teach the quadratic function in a paper and pencil context, and worked on adapting it so that the computer could be incorporated into the students’ work. The modification of the original proposal was thought of not only in terms of new tasks designed to work with GeoGebra files, but fundamentally in relation to the new aspects that had to be taken into account for the students’ and teachers’ work.

A fundamental concern was to preserve the didactic intentions of the original proposal or to eventually enrich them, but, while teachers felt secure and in control of the proposal that had been elaborated at Stage 1 with paper, pencil, and blackboards, the adaptation that we developed was done in a constant backdrop of uncertainty, a sensation provoked by the process of migrating towards work being done using the computer.

Many questions surfaced at this stage: How will mathematical knowledge be transformed? How can the work and interaction with the software be done independently by the students and be linked and integrated to the moments of collective dialogue and discussion? How can we solve unforeseen situations that will surely arise in the students’ work in their interaction with the software, about which we had no previous repertoire? Some of these questions were anticipated while others arose from the classroom work with this new presence. In the classroom, we could see the need for a teaching action to organize and sustain the students’ work, both individually and collectively. The notion of instrumental orchestration (Trouche 2004a; Drijvers and Trouche 2008) gave us tools to conceptualize this space of teachers’ decisions.

Stage 4. This stage is currently underway. From new additions in Monday’s Group, we decided to think of situations for the introduction to working with functions that involved some aspect of modeling and were directed towards students in the initial years of secondary school. The research of Arcavi and Hadas (2000) and Arcavi (2008) was discussed in the collaborative group and served as inspiration for the proposals we made.

We want to pay special attention to the students' mathematical work; in particular, we will try to identify the existence of knowledge more closely related to technological contexts and more anchored in mathematics. In relation to the teachers' work and considering our earlier stages, we hope to design a possible orchestration that will take into account teachers' room for movement in the management of their classes. In particular, we want to develop didactic techniques that will allow teachers to recover their students' productions made with GeoGebra.

Some remarks about the Monday Group and the incorporation of the GeoGebra

Looking at all the work in retrospect and before commenting on some specific examples, we will add some general questions regarding the Monday Group's work.

Our shared vision of a math class. The members of the Monday Group share some principles regarding mathematical work in the classroom. Our objective is to involve students in the real activity of producing knowledge. To do so it will be necessary to propose challenging problems to the students and generate an environment in the classroom that will encourage them to explore, produce different solutions, and contribute ideas. Attempts, solutions, and ideas are the raw material with which a teacher organizes classroom interaction. The collective space for discussion is appropriate for studying the validity of reasoning processes and procedures, advancing in terms of precision, presenting new problems, speculating, and making conjectures and studying them. In this space, students can get involved in the elaboration of mathematical theory.

Where do these preoccupations lead us when we consider the incorporation of a program like GeoGebra into classroom work?

Changes to take into account. The inclusion of work that uses educational software in the teaching and learning processes establishes the need to take into account changes in relation to the students' mathematical work and the mathematical-didactic work of the teachers.

When referring to the students' activity, changes appear in both the problems and tasks that can be proposed and in the possible techniques that are constituted. Tasks will be created that are unthinkable without the computer.

The instrumental approach, which recognizes the complexity of the teaching of mathematics mediated by technology, gives us theoretical elements with which to think about our work. In this approach, the use of a technological tool implies a process of instrumental genesis in which the object or artifact becomes an instrument. This instrument is a psychological construct, combining the artifact and schemas (in the sense of Vergnaud 1990) that the user develops to use for specific types of tasks (Drijver et al. 2010).

The construction of the instrument must be understood in a double movement: a movement directed towards the artifact, where users take the artifact in their hands and adapt it to their work habits (*instrumentalization*), and a user-oriented movement in which both the limitations imposed by the device and the possibilities offered by it contribute to structuring user activity (*instrumentation*; Trouche 2004b).

In terms of the teachers' work, new spaces requiring decision taking have appeared in collective planning and other more personal spaces have come into play in the management of each teacher's classroom. We found the idea of *instrumental orchestration* (Drijvers and Trouche 2008) relevant in order to pay attention to these teacher decision-taking spaces when working with the inclusion of computers. This notion includes both those spaces related to the tasks and the ways of solving them (which includes the methods and techniques that the students are expected to develop) as well as those related to the instruments and their organization for individual and group work.

In terms of the way the group works, I would like to highlight the fact that the production of classroom activities is developed in interaction with the teachers' work. The presence of teachers-in-activity as part of the research group makes it possible to constantly question the feasibility of what is being proposed.

I have borrowed words and concepts from Fernanda Delprato, a young Argentine woman researcher at the University of Cordoba, when we talk about the search for the (re)signification and reciprocity of different knowledges and meanings that are made possible by the mutual recognition of different visions arising from the spaces occupied by each member of the group (be they teachers or researchers (Delprato 2013).

These are ideas that we find once again in authors such as Gerard Sensevy, in discussions about cooperative engineering. Sensevy criticizes a position of a certain duality that exists in the world of education regarding how teachers and researchers are considered. "According to this duality, teachers are seen as 'practical agents' trapped in a practical relation with their work, while investigators uphold a theoretical position. In this division of work, educational research must be an applied research (in which the practical agents must apply the 'scientific results' to their practice)" (Sensevy et al. 2013, p. 1032).

The paradigm of design-based research—which the author places in cooperative engineering—positions itself in contrast to this duality, proposing a different form of relationship between teachers and researchers.

Using the words of these authors, I would say that, for the Monday Group, the idea of the cooperative elaboration of a proposal and the posterior analysis of how it was developed in certain classrooms supposes eliminating the classic duality about persons who "think" and persons who "act," because all the participants get involved in the conceptual work and, at the same time, think about its concrete realization.

Looking in perspective, we can identify three dimensions in the production of the collaborative work:

- The collaborative production of a teaching sequence for a particular curriculum subject, with the incorporation of the computer into the mathematical work of students in the classroom.
- The study of didactical phenomena associated to computer mathematical work in the secondary school classroom.
- The reflection on the collaborative working device itself, a device that is modified in the search for the genuine conformation of a collaborative group in which we are included.

Based on the last two dimensions, necessarily imbricated, I have tried so far to point out different questions and challenges that the group has to face in the different stages and the ways they faced them. Although both themes have been studied in the literature, the expected contribution of this paper is to reflect on the convergence of both and the synergy and problems that occur when the two perspectives are merged for a common production.

In the next three examples, I will try to show some didactical complexities encountered in the work of the Monday Group.

32.3 Three Examples Which Illustrate Products and Processes in the Working Group

In the rest of this paper, we will present three examples, each one chosen from each one of the three stages during which the Monday Group has worked with computers. We will illustrate some areas of both products and processes through the work of the Monday group.

First example: New tasks for students involving new didactic questions that teaching must consider

This example corresponds to our first encounter with computers being used in the mathematical work of the students. The teaching proposal we developed to introduce students to working with polynomials and polynomial functions focused on:

- The production of higher degree functions as a result of the product of two lower degree functions. Basically, we created higher degree functions as a result of linear and quadratic functions.
- The strong presence of graphs, to the point that both the students and the teachers/researchers talked of “multiplying” straight lines and parabolas.
- The possibility of working with high degree polynomials with roots of multiplicity 2, 3, or more, their formulas and their graphs.

Within this context and after many days of work, the students were expected to come up with the formula and the graph of a sextic function without zeroes or be able to justify that the required function did not exist.[6] Although the students had no problem coming up with functions like this by multiplying three parabolas without zeroes, they were not always able to produce a screen in which the factor parabolas and the resulting sixth-degree function could be seen simultaneously. Let's see three different students' answers to this.

– The first two students explained:

A sixth-degree function can, in fact, not cross the *x* axis because it contains three parabolas that can never have a zero. An example is shown in Fig. 32.1.

The sextic function has values for "y" that are so negative that they can't be seen by the naked eye.

– A second pair of students was able to zoom in sufficiently so as to be able to see the sextic equation on the screen (Fig. 32.2).
– A third pair of students explained that they modified the coefficients of the parabolas so that they became a little flatter, thus making it possible to see the sextic equation on the screen (also by zooming in a little; Fig. 32.3).

I bring this first example to identify a new task to be solved in the classroom, inherent in the work carried out by the computer: to make something look good on the screen. The previous examples illustrate a variety of positions that can be taken when faced with this task:

– Not dealing with it.
– Solve it using the program's tools.
– Solve it using mathematical knowledge.

Although in this case the work was done in written form and included individual feedback, the example shows the new questions that teachers will have to deal with in the classroom: handling the interaction among student's solutions when they were elaborated from such different positions.

A more general reflection as a result of this specific example. Because of our work throughout the years and on different teaching proposals, we have run into very different and often unexpected answers and ways of solving problems proposed by students with computers. They range, as in this example, from answers based on computer skills and program-provided tools to answers based on mathematical knowledge specified by the students. And in between there are a wide range of gray areas and variations. It is then up to the teacher to find ways of working in the classroom to relate these varied solutions. Regarding this, we find ourselves confronted with the following questions: How can we make a mathematical question about a student's answer that focuses mainly on the program's

[6]This represents a new task in the high school classroom. Teachers stated they had never worked on high degree functions, even less so if they were factorized and were presented with their graphs.

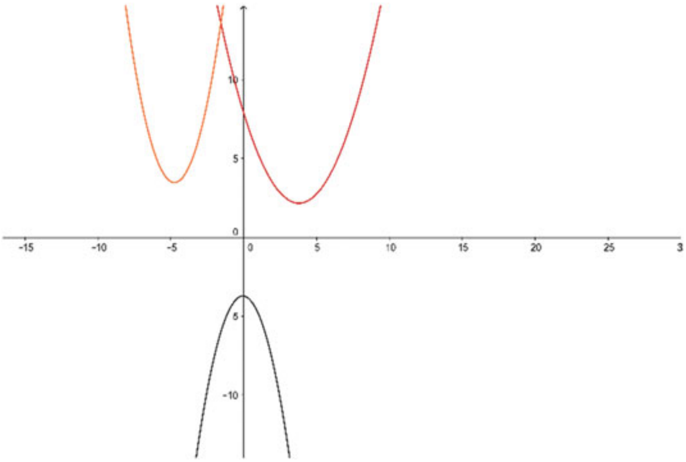

Fig. 32.1 The image presented by the first pair of students

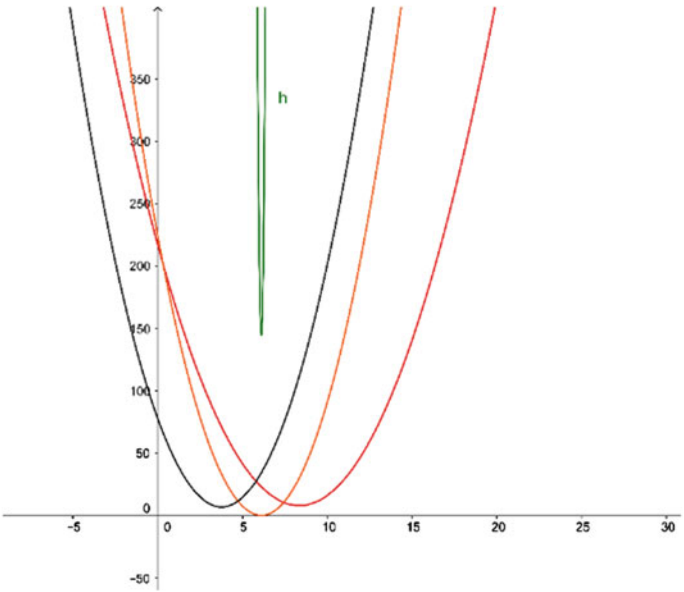

Fig. 32.2 Screen the second pair of students got

tools and is described in terms of actions on the computer? How can we move the most non-mathematical actions and discourses towards others that incorporate mathematical relations?

We will return to this question in our third example.

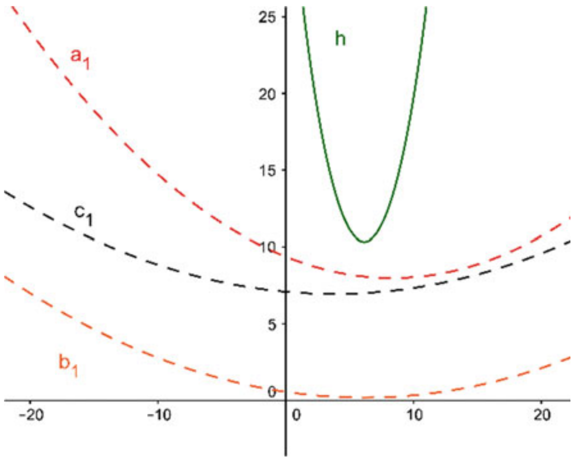

Fig. 32.3 Screen the third pair of students got

Second example: The rounding and the autonomy of the students

In the next stage, while working on the transformation of the proposal we had already thought up for quadratic functions, we ran into problems due to the "rounding off" that the program does. We show two episodes where the group had to make decisions about it. They were different, not only because of the mathematical activity required in each case, but also because of our growing confidence in mathematical work with the program.

Second example, first episode. The unexpected appearance of the "rounding off" took place in a classroom while students were working with the first problem of the proposal. That obliged us to take some "controlling measures."

Students were studying how the area of rectangles inscribed in a right-angle isosceles triangle with sides measuring 11 varied (Fig. 32.4). They had a dynamic model of this on a GeoGebra screen to explore.

At one point, they must calculate the area of a rectangle having "base 2." Using GeoGebra, some of them came up with the following screens (Figs. 32.5 and 32.6):

In both cases, they saw a rectangle with base 2 and therefore with a height of 9 and, nevertheless, the program gives an area different from 18. Since the calculation was easy to do by hand, it was very clear for the students that the program had made a mistake. This produced a chaos in the classroom! Some students even went so far as to shut their computers off.

In many cases, the area values that the program displayed when the different rectangles were dragged in did not match with the base values that it showed.

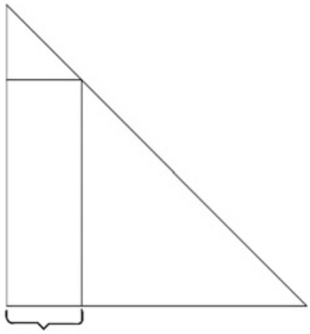

Fig. 32.4 Drawing of one of the rectangles of the family

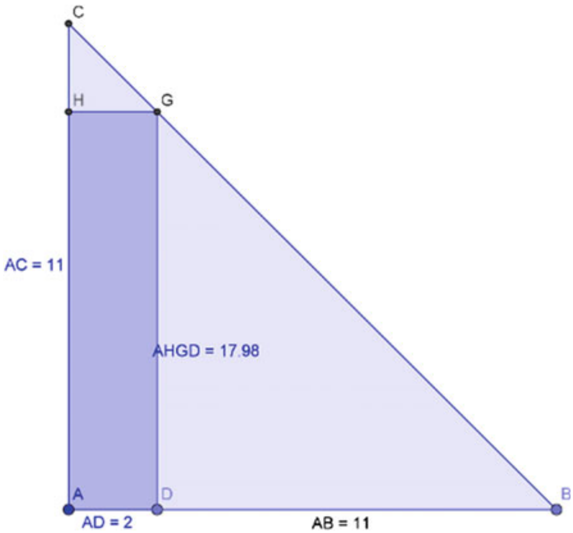

Fig. 32.5 Screen with a rectangle of base 2 and area different from 18

The machine used many more decimal digits to calculate than it showed on the screen and, in fact, the students did not reach the value of 2 but rather some value sufficiently close to 2.[7]

As the whole purpose of the activity was to study the variation of the rectangle's area as a function of the length of one of its sides, measurements played an important role. When thinking about students being introduced to the study of the variation of sizes via dynamic models, these difficulties can contribute to the fact that they will not use the model to answer the questions they are asked, as occurred in the classes we just mentioned.

[7]The issue was that the problem could not be fixed by asking the program to display more rounding off decimal digits. Even if we allow more decimal digits for the base values, the program would calculate the area using more digits than that anyway. So it would once again display a result for the area that would not be the right one.

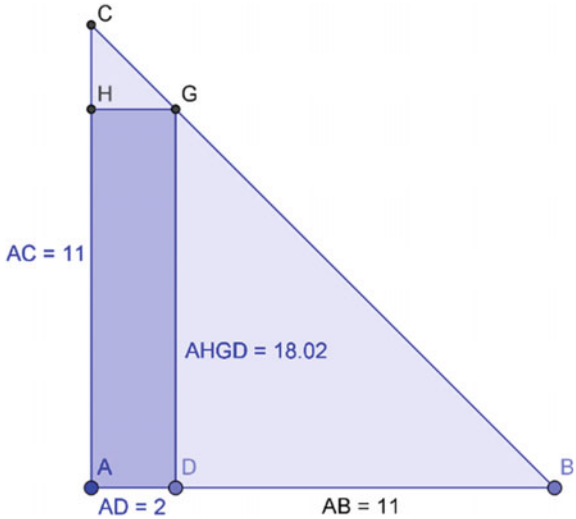

Fig. 32.6 Other screen with a rectangle of base 2 and area different from 18

Faced with this situation, for the following presentation of this problem in another classroom, we decided to alter the file so that everything would turn out happily exact. We established the working area: We regulated the mouse's movement by setting up a sufficiently small grid and we included attractors in it and then we fixed the number of decimal places it would display. Additionally, we offered the students a file with the pre-constructed triangle to insure a certain position on the screen. In this way, the rectangles—that they built to begin the work—would necessarily have sides on the grid. In this way, we were able to control the values the students passed over when moving the mouse and the values the program would use to do the calculations for the values displayed on the screen. Finally, it displayed the complete result without any rounding off. We decided to share with the students some superficial information about this file.[8]

Second example, second episode. One of the problems we designed to work with GeoGebra was assigned at an exploratory place, but, as a way of making some of the mathematical relations in play more explicit, we introduced some numeric values into the problem so that the final answer could not be completed by the program and would require some paper and pencil work. Below is the specific problem from the proposal.

[8]For more detail see Sessa et al. (2015).

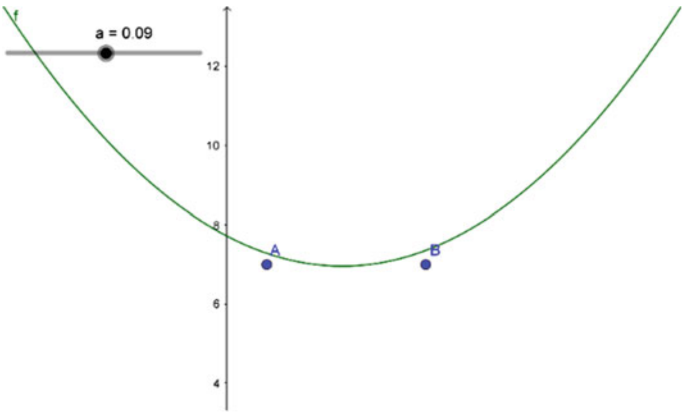

Fig. 32.7 An approximation to the requested parabola

Problem 5 (to be done with GeoGebra)

- Enter the points (1; 7) and (5; 7).
- Enter the parameters a and c and the function $f(x) = a(x - 2.9)^2 + c$
- Modify the values for a and c so that, if possible, the graph of $f(x)$ passes through the given points.

We wanted the students to explore this on a GeoGebra screen by moving parameters. In Fig. 32.7, you can see an attempt that gives an approximate answer.

They might be able to "visualize" a solution in the graphic view (Fig. 32.8).

They would then go to the algebraic view to verify whether the function really had a value of 7 at 1 and at 5. We expected that the algebraic view would show that

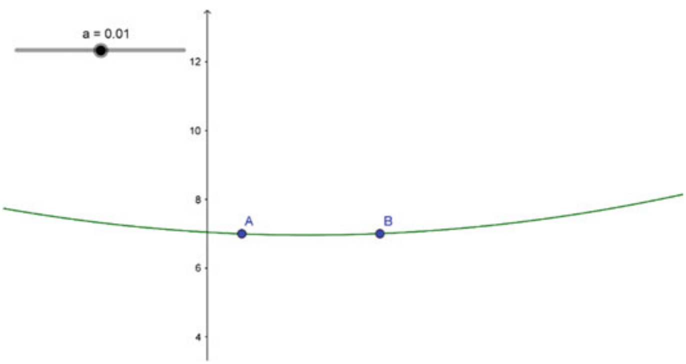

Fig. 32.8 Screen with an apparent solution

it did not happen. So the idea was that they should leave the program and find the reasons why it could not be done. However, when we asked GeoGebra for the evaluation of the function obtained from $f(x) = 0.01\ (x - 2.9)^2 + 6.96$ at 1 and 5, the program answered 7 for both values. We were now once again facing the problem of incompatibility of information between the graphic view and the algebraic one, and even within the algebraic view. The function, of course, is not a solution, and if we zoomed in more, we would that in fact its graph does not pass through the points in question.

This time, we thought we could control this problem if we could anticipate the number of digits the calculation would have. To do this, we decided to set the file with a parameter of a 2-decimal digit and "rounding off" using 4 decimal places. In this way, we would be able to make it display the complete result. Doing that in the previous function, we get $f(1) = 6.991$ and $f(5) = 7.0041$.

This time, unlike the episode above, we thought that we could share these decisions with the students. We feel that the fact that they can estimate the number of figures in the result has a formative value both in mathematics as well as in the mathematical work done with the program. This would allow the students to understand more about how the machine works in order to prepare it to respond to the working necessities in each future problem. The decision to share this with the students was also the result of the more solid position that the group had in referring to the mathematical work with the program.

We as teachers also go through a process of *instrumental genesis* (Trouche 2004b). We believe we have advanced a little, but there is much left to do:

– Creating situations in which students can work in spite of coming up with "rounded off" results.
– Working in classrooms in which each student will have a different configuration of their working area.

All of this will occur if we maintain the objective of building greater student autonomy in working with GeoGebra.

Third example. From the actions on the computer to mathematical question

This is an example of our current work. On one hand, there was a change in the subject: We went back into the curriculum. On the other hand, we have incorporated new teachers while others had to discontinue their participation in the group. As the group settled again, some discussions reappeared. But it was never the same discussion because the people were not the same. Some of us already had history in the Monday Group and our ideas were different than they had been in the beginning. Among the new members, there were some teachers who were already using computers, so they brought new experiences to the discussion. Also, there were new members who had not used the computer in their classes yet and came to the Monday Group to see what it would be like. I will refer briefly to a question that arose in the shared planning stage, which is what we are going through at the moment. We are designing a small teaching sequence around a problem that we

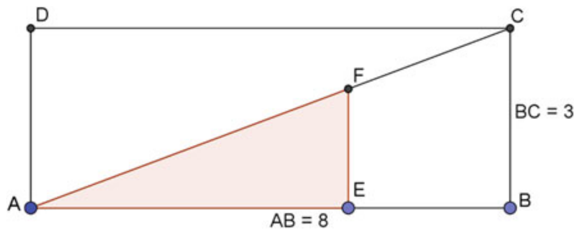

Fig. 32.9 One of the triangles of the dynamic model

read in an article by Arcavi (2008). In a GeoGebra file, a dynamic construct is presented (Fig. 32.9).

Students can move point *E* to obtain other triangles. They must study the function that relates the base of the triangle with its area. They are 13-year-old eighth graders who do not know how to calculate the area of one of these triangles knowing only the base. But they can calculate the area when the base is 4 or 2. We ask them to calculate those values and then invite them to draw those points in the second graphical view by entering the ordered pair in the entry bar. To continue the graph of the function, we decided to present the "dynamic point" tool. This tool allows us to get a representation coordinated with the dynamic figure of the triangle.

The decision to introduce the "dynamic point" tool makes us take into account the complexity of this kind of representation of a function:

- It is linked to a dynamic situation in which two variables have been chosen. Each state of this situation, i.e., each specific triangle, determines both the first and the second coordinate of point *P*.
- As a representation of the function, it always requires some time to become a representation. It is not a representation at any given moment. Moreover, the representation requires our movement. Therefore, it is a representation in action.

In our sequence, we invite students to move *E* in Graphical View 1 and observe the path of the point *P* in Graphical View 2, especially noticing that *P* passes over the points already marked (Figs. 32.10 and 32.11).

At first, we decided not to activate the trace of *P*. We propose the following task:

(5) Explore the path of point *P* in the second graphical view and answer the questions.
(a) What is the area of the triangle when the base $AE = 3$?
(b) What will be the base of a triangle that has an area equal to 6?

In each case, explain how you reached those answers.

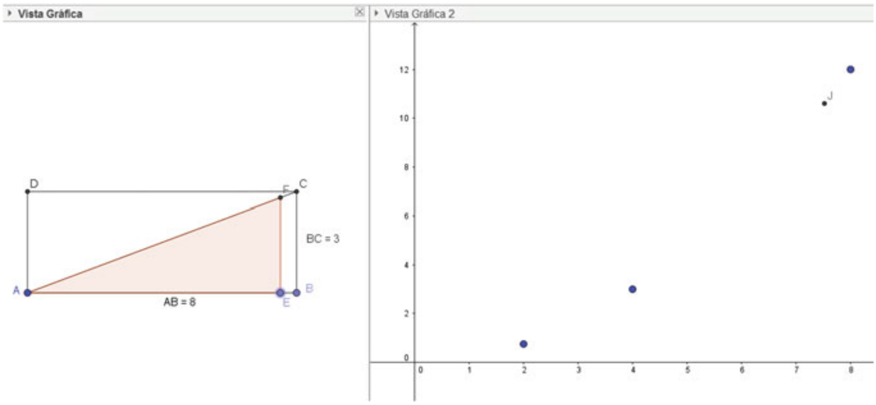

Fig. 32.10 Screen with a triangle and the associated point P

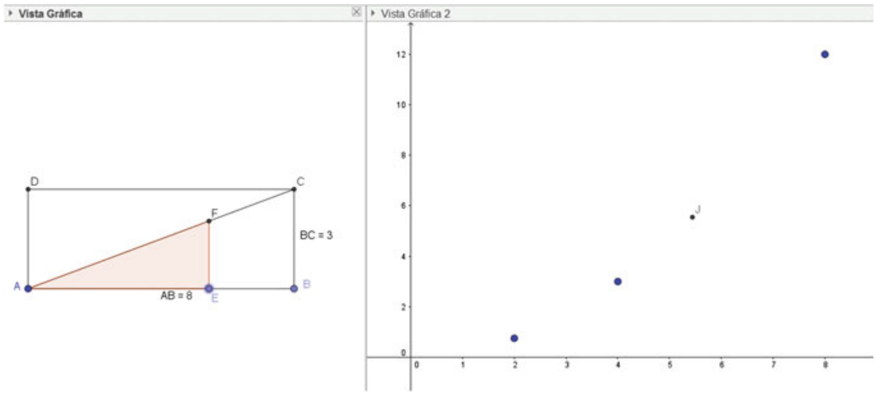

Fig. 32.11 Another triangle and its corresponding point P

The lack of computer presence in middle and high school math classes and our group's work path led to a shortage in our repertoire of student responses to working with the computer. Moreover, in the face of certain tasks we do not have experience that allows us to anticipate how they could be carried out. These two issues reconfigure the planning task, forcing us to pass ourselves through the experience of producing answers by working with the computer. In tune with this, at a meeting of the Monday Group we all set out to explore the task, working in pairs.

The work focused on appealing to tools of the program that allowed visualization and greater precision (zoom, changes in the configuration, "stretching" the axes, putting on a grid, etc.). It was work that took a long time and in which we did not put into play many mathematical relationships, and it left the pairs with a an unpleasant feeling about the task. In the final discussion of Monday Group it seemed that the balance was inclined to the side of giving little epistemic value to both the anticipated gestures of the students and the possibility of discussing them in the classroom.

The university team resumed this discussion at their weekly meeting, trying to identify what could be discussed in the classroom as a result of Problem 5 and the more accurate search work. Along these lines we had separate two issues:

1. On the one hand, in the search for precision, you first have to move the point *E* in Graphical View 1 to achieve a triangle of the family that fulfills the requested conditions: in Part a, the measurement of the base, in Part b, the measurement of the area. These measurements are read in the Cartesian plane that appears in Graphical View 2 from the location of *P* that is determined. Once point *P* is located, the problem involves obtaining information about the other magnitude as accurately as possible. This necessary coordination between a triangle and a point on the Cartesian plane is an opportunity to speak again of the link between the Cartesian graph of the relation and the geometric situation that allows us to define the relation.
2. On the other hand, once we locate point *P*, because we, perhaps visually, have ensured the value of one of the coordinates and we want to know the value of the other, tools such as scaling, the use of zoom, the "stretching" of the axes, and the addition of a grid all bring the appearance of new values on the screen for the coordinates of point *P*. These actions do not modify point *P* (because point *E* was not moved in Graphical View 1); however, some students may consider that they have obtained, for example, different area values for base 3. This last fact would be an opportunity to talk about the necessary uniqueness of the area value for a single triangle (which shows in Graphical View 1). Making explicit these questions in the classroom can contribute to the understanding that the base-area relationship is a function and to the understanding of the concept of a function itself.

These two questions refer to Lagrange (2000), who states, about the actions of a user and the responses produced by software, that it is necessary to distinguish which actions produce changes in the mathematical object and which produce changes in what can be seen from the representation that the program makes of that mathematical object.

We take from semiotic mediation theory (Mariotti 2009) the idea of pivot sign and extend it to the idea of *question pivot*. We think that a teaching intervention promoting a reflection on what is realized by and about the response given by the software allows construction of mathematical meanings in relation to the signs of the artifact. A question from the teacher such as "When we zoom, will point *P* change?" would allow reflection on the most artefactual actions in terms of mathematical objects.

This event refers us to what was formulated in first example:

– How can we make a mathematical question about a student's answer that focuses mainly on the program's tools and that is described in terms of actions on the computer?

– How can we move the most non-mathematical actions and discourses towards others that incorporate mathematical relations?

32.4 Coda

In this paper, we wanted to show some aspects of the didactic complexity that involve the incorporation of the computer in the mathematical work of the students. We did this by showing the close work of a collaborative group of school teachers and university team who faced this problem. We think that the convergence of views and approaches and the diversity of experience that characterize our collaborative group create good conditions for thinking about teaching and learning in key transformations for real and proper integration of TIC.

As part of a process, we raise some issues that also mark a way forward in our study:

– Gaining greater confidence in our work as teachers and greater autonomy in the work of students.
– Thinking about gestures and teacher discourse that allow us to weave bridges in the collective space of the classroom between work that is more focused on the tools of the program and the mathematical knowledge to which it is pointed.
– Advancing in the co-construction of working devices in the collaborative group.[9]

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