8. Quasi-Transitive Digraphs and Their Extensions

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8.1 Introduction

8.1.1 General Overview

Initially, quasi-transitive digraphs were studied by Ghouila-Houri in [39] because of their relation to comparability graphs.[1] Nonetheless, in their seminal paper [17] of 1995, Bang-Jensen and Huang began the study of this family in its own right. Through the last 20 years, quasi-transitive digraphs have gained a place among the most studied and better understood families of digraphs. Probably the main reason is the characterization theorem found in [17], which has led to solutions of many (usually difficult) problems.

Also, this is a family containing two very well known classes of digraphs: tournaments (and semicomplete digraphs) and transitive digraphs. It is well known that some interesting problems are very easy to solve for both families, *e.g.*, determining hamiltonicity. The appeal of quasi-transitive digraphs comes from the fact that a lot of problems are hard enough to be interesting, but it is still possible to find results similar to those of tournaments or transitive digraphs, yet, it is by no means trivial to do it.

Since a fair number of the classical problems for digraphs have already been studied for the family of quasi-transitive digraphs, it was a natural step to introduce a new class of digraphs generalizing it. Bang-Jensen introduced the family of 3-quasi-transitive digraphs in the context of strong arc-locally semicomplete digraphs [6]. Afterwards, in the context of k-kernels of digraphs, Galeana-Sánchez and Hernández-Cruz began in [48] the study of k-quasi-transitive digraphs. It came as a surprise that many nice structural properties of quasi-transitive digraphs have a natural generalization to k-quasi-transitive digraphs. This made it possible to generalize some of the classical results of

[1] He proved that a graph G admits a quasi-transitive orientation if and only if it admits a transitive orientation if and only if it is a comparability graph.

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quasi-transitive digraphs to k-quasi-transitive digraphs, proving the latter to be an interesting family of digraphs.

Despite this fact, k-quasi-transitive digraphs are harder to handle than quasi-transitive digraphs. For larger values of k, their structure becomes increasingly complicated; as a matter of fact, the structure of strong 4-quasi-transitive digraphs is not completely understood. In view of this difficulty, Hernández-Cruz studied the classes of 3- and 4-transitive digraphs, [46, 47] obtaining a complete structural characterization of strong 3-transitive and 4-transitive digraphs. In [61], Wang and Wang proved that 3-quasi-transitive digraphs and 3-transitive digraphs are related in the same way as quasi-transitive and transitive digraphs: the underlying graphs of 3-quasi-transitive digraphs can be oriented as 3-transitive digraphs. This motivated the study of k-transitive digraphs on their own.

Finally, after reaching the most general case of the k-quasi-transitive digraphs and going back through the k-transitive digraphs, very recently the class of transitive digraphs has been considered again in the context of digraph homomorphisms. In [28], Feder, Hell and Hernández-Cruz showed that although many classical problems for digraphs are trivially solved in the class of transitive digraphs, there are many natural problems that are $\mathcal{NP}$-complete when restricted to this family. It is to be expected that both transitive and quasi-transitive digraphs will receive renewed attention in the near future.

As is usual with many mathematical concepts, k-quasi-transitive digraphs are not the only interesting generalization of quasi-transitive digraphs. On one hand we have k-quasi-transitive digraphs, which are obtained by generalizing the definition of a quasi-transitive digraph. As we have already mentioned, no nice structural characterizations of k-quasi-transitive digraphs are known for $k \geq 3$. So, on the other hand, instead of generalizing the definition of quasi-transitive digraphs, we can generalize the structure obtained by the characterization theorem. Following this idea, the notion of totally Φ-decomposable digraphs was first introduced by Bang-Jensen and Gutin in [14], precisely as a tool to study quasi-transitive digraphs. Nonetheless, we can trace the basic idea of this family back to [41], where Gutin used a simpler version of the Φ-decomposable digraphs to find a polynomial algorithm to solve the minimum path factor problem for quasi-transitive digraphs. It has turned out that this family is a common generalization of many interesting classes of digraphs, e.g., quasi-transitive digraphs, round decomposable graphs, directed cographs, etc.

8.1.2 Chapter Overview

In Subsection 8.1.3 some terminology and notation is introduced that will be used throughout the rest of the chapter. In Section 8.2 a brief overview of transitive digraphs is presented, including some open problems on digraph homomorphisms. Section 8.3 is devoted to presenting structural properties of quasi-transitive digraphs and some of their generalizations, including the canonical decomposition in Subsection 8.3.1, some structural properties of strong k-quasi-transitive digraphs in Subsection 8.3.2 and of k-transitive digraphs in Subsection 8.3.3, and recognition theorems of totally Φ-decomposable digraphs for some choices of Φ. Section 8.4 deals with paths and cycles; Subsection 8.4.1 reviews the few known results for hamiltonicity and traceability for k-transitive and k-quasi-transitive digraphs; Hamiltonicity of quasi-transitive and totally Φ-decomposable digraphs is studied in Subsection 8.4.2 and some variants of vertex-cheapest paths and cycles for quasi-transitive digraphs are studied in Subsections 8.4.3, 8.4.4, 8.4.5, and 8.4.6. The linkage problem is covered in Section 8.5; Subsection 8.5.1 is devoted to k-linkages, and Subsection 8.5.2 to weak k-linkages. The topic of Section 8.6 is kings and kernels; k-kings are covered in Subsection 8.6.1 and k-kernels in Subsection 8.6.2. Section 8.7 deals with the Path Partition Conjecture, it has two subsections, Subsection 8.7.1 presents the conjecture and some of its known variants, and Subsection 8.7.2 deals with the known results for them. The last section of the chapter, Section 8.8 covers miscellaneous topics; vertex pancyclicity is covered in Subsection 8.8.1, acyclic spanning subdigraphs in Subsection 8.8.2, orientations of digraphs almost preserving the original diameter in Subsection 8.8.3, sparse subdigraphs with prescribed connectivity in Subsection 8.8.4, and arc-disjoint in-and out-branchings in Subsection 8.8.5.

8.1.3 Terminology and Notation

In this subsection, for the reader's convenience, we will recall some terminology and notation that will be used throughout this chapter. Only general concepts will be introduced here; more specific ones will be recalled whenever needed.

Throughout this chapter, walks, paths and cycles in a digraph are always meant to be directed. Let D be a digraph. An arc uv of D is **symmetric** if vu is also an arc of D, and **asymmetric** otherwise. Notice that a symmetric arc uv together with the arc vu form a 2-cycle of D; both this 2-cycle and the arc uv will sometimes be referred to as a **digon**. When u, v are adjacent vertices of D, we will write $\overline{uv}$.

If X and Y are disjoint subsets of vertices of D, then $X \to Y$ means that X **dominates** Y, that is, every vertex of X dominates every vertex of Y. If additionally there is no arc from Y to X, then we say that X **completely dominates** Y and denote this by $X \mapsto Y$. We shall use the same notation when X and Y are disjoint subdigraphs rather than subsets of vertices.

Let k be an integer, $k \geq 2$. A digraph D is k**-quasi-transitive** if for every pair of vertices u, v of D, the existence of a (u, v)-path of length k in D implies that $\overline{uv}$. A **quasi-transitive digraph** is a 2-quasi-transitive digraph. A digraph D is k**-transitive** if for every pair of vertices u, v of D, the existence of a (u, v)-path of length k in D implies $u \to v$. A **transitive digraph** is a 2-transitive digraph. Recall that if R is a digraph on r vertices $v_1, \ldots, v_r$ and $L_1, \ldots, L_r$ is a collection of distinct (but possibly isomorphic) digraphs, then we denote by $D = R[L_1, \ldots, L_r]$ the digraph with vertex set $V(L_1) \cup V(L_2) \cup \ldots \cup V(L_r)$ and arc set $(\bigcup_{i=1}^{r} A(G_i)) \cup \{g_i g_j : g_i \in V(G_i), g_j \in V(G_j), v_i v_j \in A(D)\}$. If $D = R[L_1, \ldots, L_r]$, then $R, L_1, \ldots, L_r$ are induced subdigraphs of D and we say that D is **decomposable** (into $R, L_1, \ldots, L_r$). Let Φ be a class of digraphs. A digraph D is **Φ-decomposable** if D is a member of Φ or $D = H[S_1, \ldots, S_h]$ for some $H \in \Phi$ with $h = |V(H)| \geq 2$ and some choice of digraphs $S_1, S_2, \ldots, S_h$ (we call this decomposition a **Φ-decomposition**). A digraph D is called **totally Φ-decomposable** if either $D \in \Phi$ or there is a Φ-decomposition $D = H[S_1, \ldots, S_h]$ such that $h \geq 2$, and each S_i is totally Φ-decomposable. In this case, if $D \notin \Phi$, a Φ-decomposition of D, Φ-decompositions $S_i = H_i[S_{i1}, \ldots, S_{ih_i}]$ of all S_i which are not in Φ, Φ-decompositions of those of S_{ij} which are not in Φ, and so on, form a sequence of decompositions which will be called a **total Φ-decomposition** of D. If $D \in \Phi$, we assume that the (unique) total Φ-decomposition of D consists of itself.

If D is a digraph on n vertices, and $S_1, \ldots, S_n$ are digraphs with no arcs, then we say that the composition $H = D[S_1, \ldots, S_n]$ is an **extension of** D, or we say that H is a **D-extension**. When D belongs to some well-known class of digraphs, we will say that H is an **extended** member of the class, e.g., if D is a semicomplete digraph, we will say that H is an extended semicomplete digraph.

A **k-path-q-cycle subdigraph** (**k-path-q-cycle factor**), $\mathcal{F}$, of a digraph D is a (spanning) collection of k paths and q cycles, all disjoint. When $k = 0$, $\mathcal{F}$ is a **q-cycle subdigraph** (and a **q-cycle factor** if it is spanning) and when $q = 0$, $\mathcal{F}$ is a **k-path-subdigraph** (and a **k-path-factor** if it is spanning). A k-path-q-cycle subdigraph in which q may be arbitrary (including zero) is called a **k-path-cycle subdigraph**.

A longest path in a digraph D is called a **detour** of D. The order of a detour of D is called the **detour order** of D and is denoted by $\mathsf{do}(D)$. For a given digraph D, let $\mathsf{do}_k(D)$ denote the maximum number of vertices contained in a k-path subdigraph of D. A k-path subdigraph of D which covers $\mathsf{do}_k(D)$ vertices is called a **maximum** k-path subdigraph of D. Note that $\mathsf{do}_1(D) = \mathsf{do}(D)$.

The **path-covering number** of a digraph D (denoted by $pc(D)$) is the least positive integer k such that D has a k-path factor. The **path-cycle-covering number** of a digraph D (denoted by $pcc(D)$) is the least positive integer k such that D has a k-path-cycle factor. The path-cycle-covering number of a digraph can easily be found in polynomial time using, in particular, algorithms on flows in networks [10, 14, 41]. The path-covering number is hard to calculate: note that $pc(D) = 1$ if and only if D has a Hamiltonian path. Thus, the path-covering number problem generalizes the Hamiltonian path problem.

Given a fixed digraph H, an **H-colouring** of a digraph D is a **homomorphism of D to H**, i.e., a mapping $f : V(D) \to V(H)$ such that $f(u)f(v)$ is an arc of H whenever uv is an arc of D. The H**-colouring problem** asks whether an input digraph D admits an H-colouring. In the **list** H**-colouring problem** the input D comes equipped with lists $L(u) \subseteq V(H), u \in V(D)$, and the homomorphism f must also satisfy $f(u) \in L(u)$ for all vertices u. Finally, the H**-retraction problem** is a special case of the list H-colouring problem, in which each list is either $L(u) = \{u\}$ or $L(u) = V(H)$. Note that the H-colouring problem is a special case of the H-retraction problem, in which each $L(u) = V(H)$. The **dichromatic number** of a digraph D is the least integer $\chi(D)$ such that $V(D)$ admits a partition into $\chi(D)$ acyclic sets. Notice that if every arc of D is symmetric, then the dichromatic number of D coincides with the (usual) chromatic number of the underlying graph of D.

8.2 Transitive Digraphs

A digraph D is defined to be **transitive** if for any three *distinct* vertices u, v, w, the existence of the arcs uv, vw implies the existence of the arc uw. Note that an acyclic digraph is transitive if and only if its arcs define a transitive relation in the usual sense. However, a digraph with a directed cycle is transitive if and only if its reflexive closure (i.e., adding all loops) defines a transitive relation. This peculiarity means that, for instance, when taking a transitive closure of a digraph we omit any loops that would exist in a transitive closure as a binary relation.

Acyclic transitive digraphs have a particularly nice structure, namely, they are exactly those digraphs whose reflexive closure is a reflexive partial order. It is well known that each transitive digraph D is obtained from an acyclic transitive digraph J by **replication**, whereby each $j \in V(J)$ is replaced by $k_j \geq 0$ vertices forming a complete digraph, so that if ij is an arc in J, then all k_i vertices replacing i dominate in D all k_j vertices replacing j. Note that all k_j vertices replacing j have exactly the same in- and out-neighbours in D (except that each of them does not dominate itself). Note that the strong components of a transitive digraph D are complete digraphs.

The observations in the preceding paragraph are often stated in terms of contraction[2] of the strong components of a transitive digraph, in order to obtain an acyclic transitive digraph, rather than using the replication operation to obtain an arbitrary transitive digraph from an acyclic one. Of course, both points of view are equivalent, but usually this observation is stated in the following way.

Proposition 8.2.1 *Let D be a digraph with an acyclic ordering $D_1, \ldots, D_p$ of its strong components. The digraph D is transitive if and only if the following holds:*

[2] Contraction is defined in Section 1.4 for directed multigraphs. We can obtain a digraph instead of a directed multigraph by deleting spare parallel arcs after contraction.

1. *Each digraph D_i, $i \in [p]$ is complete,*
2. *the digraph H obtained from D by contraction of $D_1, \ldots, D_p$ is a transitive oriented graph, and*
3. *$D = H[D_1, \ldots, D_p]$, where $p = |V(H)|$.*

Notice that Proposition 8.2.1 can be restated as saying that every transitive digraph is totally Ψ_0-decomposable, where Ψ_0 is the family of all acyclic digraphs and all the complete digraphs. Obviously, for a digraph D, the digraph H of Proposition 8.2.1 (which is the same as the digraph J in the above construction by replication), is simply the strong component digraph of D. From here, and using the fact that the strong components of a transitive digraph are complete digraphs, one can directly verify that some problems are easy to solve when restricted to transitive digraphs. Recall that the strong component digraph can be constructed in $O(|V| + |A|)$-time. A necessary condition for a digraph D to be Hamiltonian is that D is strong. In the case of transitive digraphs, this condition is also sufficient, since every transitive strong digraph is a complete digraph, and thus Hamiltonian. Hence, hamiltonicity can be verified in linear time for transitive digraphs. Every transitive digraph D has a kernel; to construct one, it suffices to choose one vertex from every terminal component of D. Thus, it can be verified in constant time whether a transitive digraph has a kernel, one can be constructed in linear time, and the exact number of different kernels can be calculated in linear time. An acyclic transitive digraph J clearly has dichromatic number equal to one, and it follows from the description of the structure of an arbitrary transitive digraph given by replication that the dichromatic number of an arbitrary transitive digraph D obtained from an acyclic transitive J by vertex substitutions is equal to the maximum value k_j of the size of any replacing set of vertices. Therefore, the dichromatic number of a transitive digraph equals the size of its largest strong component. Again, the dichromatic number of a transitive digraph can be determined in linear time. We could go on, enumerating problems which are $\mathcal{NP}$-complete in the general digraph case and become polynomial time solvable when restricted to transitive digraphs. Nonetheless, it is more revealing to exhibit a very natural problem that remains $\mathcal{NP}$-complete even when restricted to transitive digraphs.

In [29], it is shown that there are bipartite graphs H such that the H-retraction problem is $\mathcal{NP}$-complete. Hence, the following result of Feder, Hell and Hernández-Cruz shows that there are digraphs D such that the D-homomorphism problem is $\mathcal{NP}$-complete, even when restricted to transitive inputs.

Theorem 8.2.2 ([28]) *If H is a bipartite graph such that the H-retraction problem is $\mathcal{NP}$-complete, then there exists a digraph H' such that the H'-homomorphism problem is $\mathcal{NP}$-complete, even when restricted to transitive digraphs.*

Before proving Theorem 8.2.2, we will describe how the digraph H' can be obtained from a bipartite graph H. Let H be a bipartite graph with its bipartition given by a set of white vertices and a set of black vertices, with at most n black and at most n white vertices. We form the digraph H' as follows (see Figure 8.1). We first orient all edges of H from the white vertices to the black vertices. Let P_i be a directed path with $n+2$ vertices, in which the first, and the $(i+1)$-st, vertex have been duplicated (replicated once). Let R_i also be a directed path with $n+2$ vertices, in which the the last, and the $(i+1)$-st, vertex have been duplicated. We identify the last vertex of each P_i with the i-th white vertex (if any) of H and the first vertex of each R_i with the i-th black vertex (if any) of H. Then H' is obtained from the resulting digraph by taking the transitive closure. It is easy to see that the added paths ensure that the only homomorphism of H' to itself is the identity. Also consider a directed path P with $n+2$ vertices with only the first vertex duplicated, and a directed path R with $n+2$ vertices and only the last vertex duplicated. Note that P admits a homomorphism to each P_i and R admits a homomorphism to each R_i. For future reference, we define the *level* of the j-th vertex of P or P_i to be j, and the *level* of the j-th vertex of R or R_i to be $n+2+j$; in this we assume the duplicated vertices to have the same level. Note that this forces all white vertices to have level $n+2$ and all black vertices to have level $n+3$.

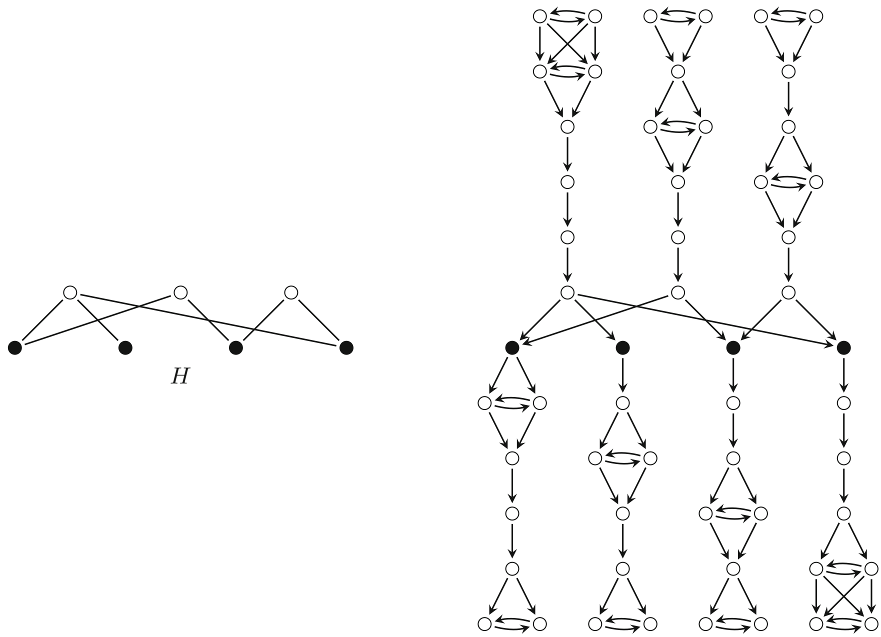

Figure 8.1 The construction of H' from H used for Theorem 8.2.2. The digraph H' is obtained by taking the transitive closure of the digraph on the right.

Proof of Theorem 8.2.2: Suppose G is an instance of the H-retraction problem, i.e., a bipartite graph containing H as a subgraph with lists $\{x\}$ for each (black and white) vertex x of H, and lists $V(H)$ for all other (black and white) vertices of G. We construct an instance G' of the H'-colouring problem by orienting all edges of G from the white vertices to the black vertices, attaching paths P_i and R_j to the vertices of H as in the construction of H', and then (for the vertices not in H) we identify the last vertex of a (separate) copy of P to each white vertex of G not in H, and identify the first vertex of a (separate) copy of R to each black vertex of G not in H, and finally we take the transitive closure. Now it is easy to see that each homomorphism of G' to H' preserves the level of vertices, and that G' admits an H'-colouring if and only if G admits a retraction to H.

Moreover, the above construction of H' ensures that it is itself transitive. Thus we have the following fact.

Corollary 8.2.3 ([28]) *There exists a transitive digraph H' such that the H'-homomorphism problem is $\mathcal{NP}$-complete even when restricted to transitive digraphs.*

In view of Corollary 8.2.3, a natural interesting problem is the following.

Problem 8.2.4 *Characterize the transitive digraphs H such that the H-homomorphism problem restricted to transitive inputs is polynomial time solvable.*

Although Problem 8.2.4 may look innocuous, it may be very hard indeed. Recall that Feder and Vardi proved in [29] that in order to classify all constraint satisfaction problems, it is enough to classify all the digraph homomorphism problems. In [28], Feder, Hell and Hernández-Cruz propose the problem of determining whether for any relational structure H, a (transitive) digraph H' exists such that the constraint satisfaction problem for H is polynomially equivalent to the H'-homomorphism problem for transitive digraphs.

8.3 Structural Properties

As mentioned before, the main appeal of quasi-transitive digraphs comes from the fact that their structure is very well understood. Throughout this section, we will consider structural properties of quasi-transitive, k-transitive and k-quasi-transitive digraphs. Also, some results regarding the recognition of Φ-decomposable digraphs for particular cases of Φ are included. We begin by presenting the classical results due to Bang-Jensen and Huang from [17].

8.3.1 Quasi-Transitive Digraphs

The nice results that have been obtained for quasi-transitive digraphs and all the attention this family and its generalizations have received are principally a consequence of the recursive characterization theorem given by Bang-Jensen and Huang in [17]. The main purpose of this subsection is to reproduce the proof of this theorem, including the lemmas needed, many of which are interesting on their own.

Proposition 8.3.1 ([17]) *Let D be a quasi-transitive digraph. Suppose that $P = x_0x_1 \dots x_n$ is a shortest (x_0, x_n)-path. Then, the subdigraph induced by $V(P)$ is a semicomplete digraph and $x_j \to x_i$ for every $1 \leq i+1 < j \leq k$, unless $n = 3$, in which case the arc between x_0 and x_n may be absent.*

Proof: The cases $k \in \{2, 3, 4, 5\}$ are easily verified. The proof for the case $k \geq 6$ is by induction on k with the case $k = 5$ as the basis. By induction, each of $D[\{x_0, \dots, x_{k-1}\}]$ and $D[\{x_1, \dots, x_k\}]$ is a semicomplete digraph and $x_j \to x_i$ for any $1 < j - i < k - 2$. Hence, x_2 dominates x_0 and x_k dominates x_2, and the minimality of P implies that x_k dominates x_0. □

Corollary 8.3.2 ([17]) *If a quasi-transitive digraph D has an (x, y)-path but x does not dominate y, then either $y \to x$, or there exists vertices $u, v \in V(D) - \{x, y\}$ such that $x \to u \to v \to y$ and $y \to u \to v \to x$.*

Proof: Consider a minimal (x, y)-path and apply Proposition 8.3.1. □

Lemma 8.3.3 ([17]) *Suppose that A and B are distinct strong components of a quasi-transitive digraph D with at least one arc from A to B. Then $A \mapsto B$.*

Proof: Suppose A and B are distinct strong components such that there exists an arc from A to B. Then, for every choice of $x \in A$ and $y \in B$, there exists a path from x to y in D. Since A and B are distinct strong components, none of the alternatives in Corollary 8.3.2 can hold, and hence $x \to y$. □

Proposition 8.3.1 and Lemma 8.3.3 will be generalized in the following sections for k-quasi-transitive digraphs. On the other hand, the following lemma does not have any known generalizations for k-quasi-transitive digraphs when $k \geq 3$.

Lemma 8.3.4 ([17]) *Let D be a strong quasi-transitive digraph on at least two vertices. Then the following holds:*

(a) $\overline{UG(D)}$ *is disconnected;*
(b) If S and S' are two subdigraphs of D such that $\overline{UG(S)}$ and $\overline{UG(S')}$ are distinct connected components of $\overline{UG(D)}$, then either $S \mapsto S'$ or $S' \mapsto S$, or both $S \to S'$ and $S' \to S$, in which case $|V(S)| = |V(S')| = 1$.

Proof: The statement *(b)* can be easily verified from the definition of a quasi-transitive digraph and the fact that S and S' are completely adjacent in D. We prove *(a)* by induction on $|V(D)|$. Statement *(a)* is trivially true when $|V(D)| \in \{2, 3\}$. Assume that it holds when $|V(D)| < n$, where $n > 3$.

Suppose that there is a vertex z such that $D - z$ is not strong. Then, there is an arc from (to) every terminal (initial) strong component of $D - z$ to (from) z. Since D is quasi-transitive, the last fact and Lemma 8.3.3 imply that $X \to Y$ for every initial (terminal) strong component X (Y) of $D - z$. Similar arguments show that each strong component of $D-z$ either dominates some terminal component or is dominated by some initial component of $D-z$ (intermediate strong components satisfy both). These facts imply that z is adjacent to every vertex in $D - z$. Therefore, $\overline{UG(D)}$ contains a component consisting of the vertex z, implying that $\overline{UG(D)}$ is disconnected, and *(a)* follows.

Assume that there is a vertex v such that $D - v$ is strong. Since D is strong, it contains an arc vw from v to $D - v$. By induction, $\overline{UG(D - v)}$ is not connected. Let S and S' be connected components of $\overline{UG(D - v)}$ such that $w \in S$ and $S \to S'$ (here we use *(b)* and the fact that $D - v$ is strong). Then v is completely adjacent to S' in D (as $v \to w$). Hence, $\overline{UG(S')}$ is a connected component of $\overline{UG(D)}$ and the proof is complete. □

In the following subsections we will see that, for some values of k, there are nice characterizations of strong k-transitive and k-quasi-transitive digraphs. Also it is even possible to show that the strong components of, for example, a 3-quasi-transitive digraph, are related in a very special way. Nonetheless, it is difficult to obtain a characterization fully describing the structure of those families, mainly because, for sufficiently small induced subdigraphs, the k-quasi-transitivity becomes irrelevant. The following theorem gives a complete characterization of quasi-transitive digraphs, which makes members of this family easier to deal with. Notice that, since the characterization is recursive, it provides an excellent structure to apply mathematical induction in this class of digraphs.

Theorem 8.3.5 (Bang-Jensen, Huang [17]) *Let D be a digraph which is quasi-transitive.*

- *If D is not strong, then there exists a transitive oriented graph T with vertices $\{u_1, u_2, \ldots, u_t\}$ and strong quasi-transitive digraphs $H_1, H_2, \ldots, H_t$ such that $D = T[H_1, H_2, \ldots, H_t]$, where H_i is substituted for u_i, $i \in \{1, 2, \ldots, t\}$.*
- *If D is strong, then there exists a strong semicomplete digraph S with vertices $\{v_1, v_2, \ldots, v_s\}$ and quasi-transitive digraphs $Q_1, Q_2, \ldots, Q_s$ such that Q_i is either a vertex or is non-strong and $D = S[Q_1, Q_2, \ldots, Q_s]$, where Q_i is subsituted for v_i, $i \in \{1, 2, \ldots, s\}$.*

Proof: Suppose that D is not strong and let $H_1, \ldots, H_t$ be the strong components of D. According to Lemma 8.3.3, if there is an arc between H_i and H_j, then either $H_i \mapsto H_j$ or $H_j \mapsto H_i$. Now, if $H_i \mapsto H_j \mapsto H_k$, then, by quasi-transitivity, $H_i \mapsto H_k$. So, by contracting each H_i to a vertex h_i, we get a transitive oriented graph T with vertices $h_1, \ldots, h_t$. This shows that $D = T[H_1, \ldots, H_t]$.

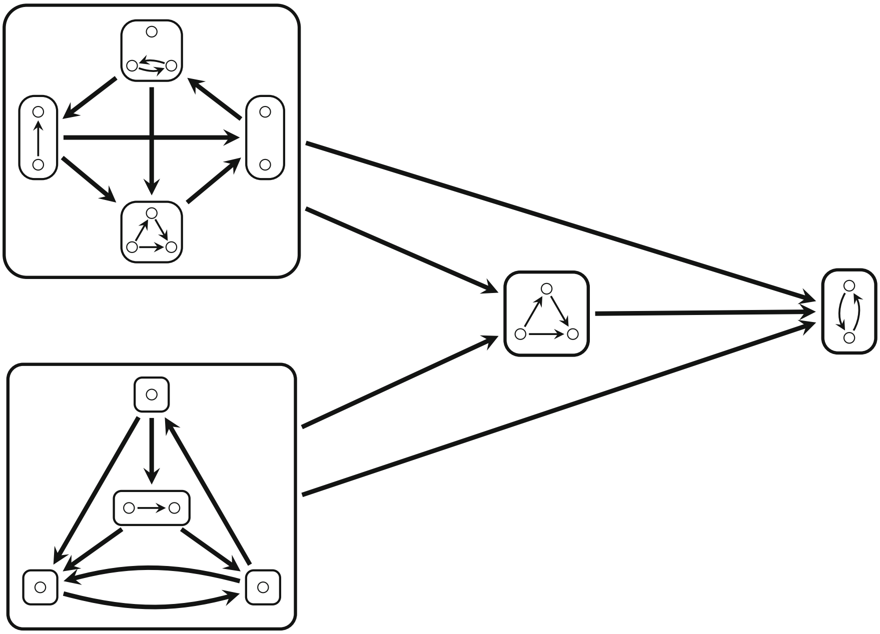

Figure 8.2 The canonical decomposition of a non-strong quasi-transitive digraph. Big arcs between different boxed sets indicate that there is a complete domination in the direction shown.

Suppose that D is strong. Let $Q_1, \ldots, Q_s$ be the subdigraphs of D such that each $\overline{UG(Q_i)}$ is a connected component of $\overline{UG(D)}$. According to Lemma 8.3.4*(a)*, each Q_i is either non-strong or just a single vertex. By Lemma 8.3.4 *(b)*, we obtain a strong semicomplete digraph S if each Q_i is contracted to a vertex. This shows that $D = S[Q_1, \ldots, Q_s]$. □

The decomposition described by Theorem 8.3.5 is called the **canonical decomposition** of the quasi-transitive digraph D. The canonical decomposition of a non-strong quasi-transitive digraph is illustrated in Figure 8.2.

8.3.2 *k*-Quasi-Transitive Digraphs

So far, there are no known characterizations of k-quasi-transitive digraphs for $k \geq 3$. Even if we restrict ourselves to strong digraphs, only strong 3-quasi-transitive digraphs have a simple complete characterization. Despite this fact, there are some structural results valid for any $k \geq 3$ that have been useful to study k-quasi-transitive digraphs.

Despite its simplicity, it could be said that the following result is the cornerstone of the study of k-quasi-transitive digraphs; it was proved by Galeana-Sánchez and Hernández-Cruz in [48]. Notice that it can be regarded as a generalization of Corollary 8.3.2.

Lemma 8.3.6 ([48]) *Let k be an integer with $k \geq 2$. If D is a k-quasi-transitive digraph, and for $u, v \in V(D)$ there is a (u, v)-path in D, then each of the following holds:*

1. *If $d(u, v) = k$, then $d(v, u) = 1$.*
2. *If $d(u, v) = k + 1$, then $d(v, u) \leq k + 1$.*
3. *Assume $d(u, v) = r \geq k + 2$. If k is even or k and r are both odd, then $d(v, u) = 1$; if k is odd and r is even, then $d(v, u) \leq 2$.*

Proof: Let $P = x_0, \ldots, x_r$ be a path of length $r = k + j$, $j \geq 0$. Observe that the k-quasi-transitivity of D and the fact that $d(u, v) = r$ imply that $x_r \to x_j$. This handles *1.* and *2.*

To prove *3.*, we will proceed by induction on j. For $j = 2$, the existence of the k-path $x_r x_2 P x_k x_0$ implies $x_r \to x_0$. For $j = 3$, the existence of the k-path $x_r x_3 P x_{k+1} x_1$ implies $x_r x_1$. Considering the k-path $x_r x_1 P x_k$, we get $x_r \to x_k$. When k is odd, we already have $d(x_r, x_0) \leq 2$. For even k, we will prove by induction on i that $x_r \to x_{k-2i}$ for every $0 \leq i \leq \frac{k}{2}$. We already have $x_r \to x_k$, so suppose that $x_r \to x_{k-2i}$ for some $0 < i < \frac{k}{2}$. Now, the existence of the k-path $x_r x_{k-2i} P x_k x_0 P x_{k-2(i+1)}$ implies $x_r \to x_{k-2(i+1)}$. In particular, $x_r \to x_0$

So, suppose $j > 3$. By the induction hypothesis, if k is even, or both k and r are odd, we obtain $x_r \to x_2$. Hence, $x_r x_2 P x_k x_0$ is a k-path, and thus $x_r \to x_0$. If k is odd and r is even, by the induction hypothesis we have $x_r \to x_1$. So, $x_r x_1 P x_k$ is a k-path, the existence of which implies $x_r \to x_k$. Since we already had $x_k \to x_0$, we conclude $d(x_r, x_0) \leq 2$. □

Proposition 8.3.1 was generalized to k-quasi-transitive digraphs by Wang and Zhang (when k is even) [62] and by Alva-Samos and Hernández-Cruz (when k is odd) [1]. Its proof is long and technical, and thus will be omitted.

Proposition 8.3.7 ([1, 62]) *Let $k \geq 3$ be an integer and let D be a k-quasi-transitive digraph. Suppose that $P = x_0 x_1 \ldots x_r$ is a shortest (x_0, x_r)-path with $r \geq k + 2$ in D.*

- *If k is even, then $D[V(P)]$ is a semicomplete digraph and $x_j \to x_i$ for $1 \leq i + 1 < j \leq r$.*
- *If k is odd, then $D[V(P)]$ is either a semicomplete digraph and $x_j \to x_i$ for $1 \leq i + 1 < j \leq r$, or $D[V(P)]$ is a semicomplete bipartite digraph and $x_j \to x_i$ for $1 \leq i + 1 < j \leq r$ and $i \not\equiv j$ (mod 2).* □

In a quasi-transitive digraph D, Lemma 8.3.3 tells us that for two different strong components A and B, if A reaches B, then $A \mapsto B$. Unfortunately, this is not true for k-quasi-transitive digraphs when $k \geq 3$. Nonetheless, there are some results resembling this behaviour. The following simple (but very useful) result was originally proved by Hernández-Cruz while studying k-transitive digraphs, [47].

Lemma 8.3.8 ([47]) *Let k be an integer, $k \geq 2$, let D be a k-quasi-transitive digraph, and let $C = v_0 v_1 \dots v_{r-1} v_0$ be a directed cycle in D with $r \geq k$. For any $v \in V(D) - V(C)$, if $v \to v_i$ and $(V(C), v) = \varnothing$, then $v \to v_{i+(k-1)}$; if $v_i \to v$ and $(v, V(C)) = \varnothing$, then $v_{i-(k-1)} \to v$, where the subscripts are taken modulo r.*

Proof: It suffices to prove the first statement, the second one is obtained by noting that reversing every arc of a k-quasi-transitive digraphs yields a k-quasi-transitive digraph.

The path $v v_i C v_{i+(k-1)}$ has length exactly k, and thus, $\overline{v v_{i+(k-1)}}$. But $(V(C), v) = \varnothing$, hence $v \to v_{i+(k-1)}$. □

Our previous lemma is complemented by the following result due to Wang and Zhang. Although both results have very simple proofs, they have some very nice consequences on the structure of k-quasi-transitive digraphs.

Lemma 8.3.9 ([62]) *Let k be an integer with $k \geq 2$, and let D be a strong k-quasi-transitive digraph. Suppose that $C = v_0 v_1 \dots v_{r-1} v_0$ is a cycle of length r, with $r \geq k$, in D. Then, for any $v \in V(D) - V(C)$, v and C are adjacent.*

Proof: Since D is strong, v must reach C and vice versa. Let P be a shortest path from v to C, and assume without loss of generality that the endpoint of P is v_0. If the length of P is s, and $s \leq k$, then $v P v_0 C v_{k-s}$ is a k-path, which implies $\overline{v v_{k-s}}$. If $k < s$, then by Lemma 8.3.6, v_0 reaches v at distance at most two. If $v_0 \to v$, then we are done. Otherwise, there is a vertex u in D such that $v_0 \to u \to v$. Either $u \in V(C)$, and the desired result is obtained, or $v_{r-(k-2)} C v_0 u v$ is a k-path in D, implying $v_{r-(k-2)} \to v$. □

As an example of how the previous two lemmas can be used to obtain nice structural results for k-quasi-transitive digraph, we present the following proposition, which is their immediate consequence.

Proposition 8.3.10 ([62]) *Let k be an integer with $k \geq 2$, let D be a strong k-quasi-transitive digraph, and let $C = v_0 v_1 \dots v_{r-1} v_0$ be a cycle of length r with $r \geq k$ in D. Suppose that r and $k-1$ are coprime. For any $v \in V(D) - V(C)$, if $(V(C), v) = \varnothing$, then $v \mapsto V(C)$; if $(v, V(C)) = \varnothing$, then $V(C) \mapsto v$.*

We finish our discussion of general k-quasi-transitive digraphs with some results that give us a lot of information on the structure of k-quasi-transitive digraphs with diameter at least $k+2$. Unfortunately, the proofs of these results are long and technical and thus will be omitted.

Lemma 8.3.11 ([62]) *Let k be an even integer with $k \geq 4$, and let D be a strong k-quasi-transitive digraph. Suppose that $P = v_0 v_1 \dots v_{k+2}$ is a shortest (v_0, v_{k+2})-path in D. For any $v \in V(D) - V(P)$, if $(v, V(P)) \neq \varnothing$ and $(V(P), v) \neq \varnothing$, then either v is adjacent to every vertex of $V(P)$, or*

$\{v_{k+2}, v_{k+1}, v_k, v_{k-1}\} \mapsto v \mapsto \{v_0, v_1, v_2, v_3\}$. *In particular, if* $k = 4$, *then* v *is adjacent to every vertex of* $V(P)$. □

Theorem 8.3.12 ([62]) *Let* k *be an even integer with* $k \geq 4$, *and let* D *be a strong* k*-quasi-transitive digraph. Suppose that* $P = v_0 \ldots v_{k+2}$ *is a shortest* (v_0, v_{k+2})*-path. Then, the subdigraph induced by* $V(D) - V(P)$ *is a semicomplete digraph.* □

Notice that, in particular, it follows from Proposition 8.3.7, Lemma 8.3.11, and Theorem 8.3.12, that a 4-quasi-transitive digraph of diameter at least 6 is a semicomplete digraph. As a more general case, the previous results can be condensed in the following theorem.

Theorem 8.3.13 *Let* k *be an even integer with* $k \geq 4$, *and let* D *be a strong* k*-quasi-transitive digraph. Then,* $V(D)$ *admits a partition* (V_1, V_2) *such that* V_i *induces a semicomplete digraph for* $i \in \{1, 2\}$, *and* $D[V_1]$ *is Hamiltonian.*

When k is odd, Alva-Samos and Hernández-Cruz [1], through a similar development of technical lemmas, obtained the following analogue of Theorem 8.3.13.

Theorem 8.3.14 *Let* k *be an odd integer,* $k \geq 3$, *and let* D *be a strong* k*-quasi-transitive digraph. Then,* $V(D)$ *admits a partition* (V_1, V_2) *such that:*

- *If* D *is bipartite, then* $D[V_i]$ *is a semicomplete bipartite digraph for* $i \in \{1, 2\}$;
- *Else,* $D[V_i]$ *is a semicomplete digraph,* $i \in \{1, 2\}$.

In either case, $D[V_1]$ *is Hamiltonian.*

In particular, it is also noted in [1] that a strong 5-quasi-transitive digraph of diameter at least 7 is either a semicomplete bipartite digraph or a semicomplete digraph.

To finish our discussion of the structure of k-quasi-transitive digraphs, we present the well understood structure of 3-quasi-transitive digraphs. Although a complete characterization telling us the exact structure of 3-quasi-transitive digraphs does not exist, a lot of information can be put together from the existing characterization of strong 3-quasi-transitive digraphs from [34], and the way the strong components relate to each other described in [64].

Let F_i be the graph on $i + 3$ vertices, consisting of a directed 3-cycle $xyzx$, together with i vertices, $v_1, \ldots, v_i$, such that yv_jz is a directed path for each $1 \leq j \leq i$, see Figure 8.3. Define the family $\mathcal{F}$ as $\mathcal{F} = \{F_i : i \geq 1\}$. Due to space constraints, we will not give the proof of the following theorem, originally proved by Galeana-Sánchez, Goldfeder, and Urrutia.

Theorem 8.3.15 ([34]) *Let* D *be a strong* 3*-quasi-transitive digraph. Then* D *is one of the following.*

1. *A semicomplete digraph.*
2. *A semicomplete bipartite digraph.*
3. *An element of the family $\mathcal{F}$ described above.* □

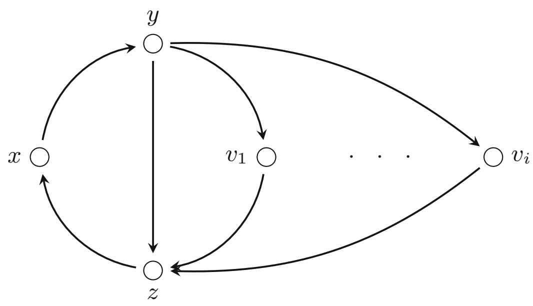

Figure 8.3 The digraph F_i of the family $\mathcal{F}$.

Theorem 8.3.15 can be complemented with the following result, due to Wang and Wang, found in [64].

Lemma 8.3.16 ([64]) *Let D_1 and D_2 be two distinct non-trivial strong components of a 3-quasi-transitive digraph, with at least one arc from D_1 to D_2. Then, either $D_1 \mapsto D_2$, or $D_1 \cup D_2$ is a semicomplete bipartite digraph.* □

It is not hard to see that, given two strong components D_1 and D_2 of a 3-quasi-transitive digraph D such that D_1 reaches D_2, there is an arc from D_1 to D_2 unless D_1 reaches D_2 in distance exactly 2, and both D_1 and D_2 consist of a single vertex. Thus, Lemma 8.3.16 becomes very useful when dealing with non-strong 3-quasi-transitive digraphs.

8.3.3 *k*-Transitive Digraphs

It is clear from the definition of both k-transitive and k-quasi-transitive digraphs that members of these classes having a small order do not really have any organized structure. Nonetheless, as the order increases, a nice structure emerges. As we have seen in the previous subsection, for k-quasi-transitive digraphs, the existence of two vertices at distance $k+2$ is sufficient for the rest of the digraph to organize as almost a semicomplete digraph (when k is even). In this section we will see that the tipping point for a k-transitive digraph D seems to be the existence of a "long enough" cycle; this will be sufficient for the digraph to be a complete digraph, or an extended cycle.

For $k = 3$, this point is easily reached, and thus, the structure of 3-transitive digraphs is easy to describe. But even for $k = 4$, it becomes hard to obtain a complete description of all 4-transitive digraphs; a classification of 4-transitive strong digraphs is given in this case. We begin with a couple of results that show the importance of cycles in k-transitive digraphs.

Results on 3- and 4-transitive digraphs are due to Hernández-Cruz. In this subsection we will present a new, shorter proof of Theorem 8.3.19.

Observe that the proof of Lemma 8.3.8 also yields the following result.

Proposition 8.3.17 ([47]) *Let $k \geq 2$ be an integer, D a k-transitive digraph and $C = v_0v_1 \ldots v_{r-1}v_0$ a directed cycle in D with $r \geq k$. If $v \in V(D) - V(C)$ is such that $v \to v_0$, then $v \to S = \{v_i \big| i \in (k-1)\mathbb{Z}_r\}$.*

Observe that under the same assumptions as in Proposition 8.3.17, if $v_0 \to v$, we can conclude that $S \to v$. This follows from the fact that reversing all the arcs of a k-transitive digraph yields a k-transitive digraph, and applying Proposition 8.3.17. So, in this subsection we will refer to either result as Proposition 8.3.17.

Lemma 8.3.18 *Let D be a strong digraph. If the circumference of D is 2, then the underlying graph of D is a tree.*

Proof: Assuming that the circumference of D is 2, it is easy to verify that every arc of D is a digon. Thus, between any pair of vertices there is exactly one path, and hence, the underlying graph of D is a tree. □

Recall that $\vec{C}_3$ is the directed cycle on three vertices, and let $C_3^\star$ and $C_3^{\star\star}$ be the directed 3-cycle with exactly one symmetric arc and the directed 3-cycle with exactly two symmetric arcs, respectively. Now we give the characterization of strong 3-transitive digraphs due to Hernández-Cruz, although with a new, simpler proof.

Theorem 8.3.19 ([46]) *If D is a 3-transitive strong digraph, then D is one of the following:*

1. *A complete biorientation of a complete graph;*
2. *A complete biorientation of a complete bipartite graph; or*
3. *$\vec{C}_3$, $C_3^\star$ or $C_3^{\star\star}$.*

Proof: We begin by observing that every strong digraph with fewer than four vertices is either complete, complete bipartite or one of $\vec{C}_3$, $C_3^\star$ or $C_3^{\star\star}$. Thus, we can assume that D has at least four vertices.

Claim 1. If the circumference of D is 2, then D is a complete biorientation of a star, and hence, a complete biorientation of a complete bipartite graph.
Proof of Claim 1. It follows from Lemma 8.3.18 that D is a complete biorientation of a tree. Since D is 3-transitive, the diameter of D should be strictly less than 3. Hence, the underlying graph of D is a tree of diameter 2, i.e., a star. □

Claim 2. If D contains a directed odd cycle, then D is a complete biorientation of a complete graph.

Proof of Claim 2. It can be proved inductively that if D contains an odd cycle, then it contains a directed 3-cycle, C. Since D has at least four vertices, there exists a vertex $v \in V(D) \setminus V(C)$. Since D is 3-transitive and strong, there must be an arc from v to C and one arc from C to v. It follows from Proposition 8.3.17 that $v \to C$ and $C \to v$. But now, any two vertices of C together with v induce a 3-cycle (with some symmetric arcs), and the same argument can be used to prove that v is adjacent to any vertex in D through a digon. Since v was chosen arbitrarily outside a 3-cycle, D is a complete biorientation of a complete digraph. □

Claim 3. If every directed cycle of D is even, then D is a complete biorientation of a complete bipartite graph.

Proof of Claim 3. First, notice that under these assumptions, D is bipartite.

By Claim 1., we may assume that D contains a cycle of length at least 4. Again, it can be proved inductively that D contains a 4-cycle, C. One can directly verify that every arc in a 4-cycle of a 3-transitive digraph is a digon. Consider a 2-colouring of C with colours black and white. If there are no more vertices in D, then we are done. Otherwise, let v be a vertex of D not in C. Since D is 3-transitive and strong, then there is at least one arc from v to C and vice versa. Observe that both arcs join v to only black or only white vertices, otherwise D would not be bipartite. Suppose without loss of generality that v is adjacent to a black vertex in C. We will recursively colour all the vertices of D to obtain a bipartition such that every white vertex is adjacent through digons to every black vertex. Proposition 8.3.17 implies that there are digons between v and every black vertex in C, so, colour v white. Now, any four vertices of D already coloured, two black and two white, induce a symmetric 4-cycle in D. Repeating the argument, it can be shown that every vertex of D not already coloured is either adjacent through digons to every black vertex, and we colour it white, or to every white vertex, and we colour it black. □

Since the cases are exhaustive, the result now follows from Claims 1–3. □

Although more complicated than classifying strong transitive digraphs, strong 3-transitive digraphs are still easy to classify. Nonetheless, as the value of k grows, this task becomes increasingly difficult. In fact, 4 is the largest value of k such that strong k-transitive digraphs are characterized. Next, we reproduce the characterization theorem due to Hernández-Cruz found in [47]. The proof, although not very difficult, is lengthy and technical, so we omit it.

Theorem 8.3.20 ([47]) *Let D be a strong 4-transitive digraph. Then exactly one of the following possibilities holds.*

1. *D is a complete digraph.*
2. *D is a 3-cycle extension.*
3. *D has circumference 3, a 3-cycle extension as a spanning subdigraph with cyclical partition $\{V_0, V_1, V_2\}$, at least one symmetrical arc exists in D and for every symmetrical arc $v_i v_{i+1} \in A(D)$, with $v_j \in V_j$ for $j \in \{i, i+1\}$ (mod 3), $|V_i| = 1$ or $|V_{i+1}| = 1$.*
4. *D has circumference 3, $UG(D)$ is not 2-edge-connected and $\{S_1, S_2, \ldots, S_r\}$ are the vertex sets of the maximal 2-edge connected subgraphs of $UG(D)$, with $S_i = \{u_i\}$ for every $2 \leq i \leq r$ and such that $D[S_1]$ has a 3-cycle extension with cyclical partition $\{V_0, V_1, V_2\}$ as a spanning subdigraph. A vertex $v_0 \in V_0$ (without loss of generality) exists such that $v_0 u_j, u_j v_0 \in A(D)$ for every $2 \leq j \leq n$. Also $|V_0| = 1$ and $D[S_1]$ has the structure described in 1. or 2., depending on the existence of symmetrical arcs.*
5. *A complete biorientation of a 5-cycle.*
6. *D is a complete biorientation of the star $K_{1,r}$, $r \geq 3$.*
7. *D is a complete biorientation of a tree with diameter 3.*
8. *D is a strong digraph of order less than or equal to 4 not included in the previous families.*

For values of k greater than 4, there are no known structural characterizations for strong k-transitive digraphs. As we have already mentioned above, this situation may be a consequence of the fact that every digraph on less than $k+1$ vertices, and every digraph without paths of length k, are k-transitive digraphs, so small k-transitive digraphs are difficult to characterize. In spite of this fact, it has been observed that the existence of some structures in a strong k-transitive digraph is enough to guarantee that the whole digraph will have a nice structure. Hernández-Cruz and Montellano-Ballesteros proved that k-transitive digraphs with cycles of length at least k have a very nice structure. The proofs of the following theorems are several pages long, so they will be omitted; it would be a nice problem to find short proofs for both of them.

Theorem 8.3.21 ([49]) *Let $k \geq 2$ be an integer, and let D be a strong k-transitive digraph. Suppose that D contains a cycle of length r such that the g.c.d. of r and $k-1$ is d, and $r \geq k+1$. Then the following hold.*

1. *If $d = 1$, then D is a complete digraph.*
2. *If $d \geq 2$, then D is either a complete digraph, a complete bipartite digraph, or a d-cycle extension.* □

Theorem 8.3.22 ([49]) *Let $k \geq 2$ be an integer, and let D be a strong k-transitive digraph of order at least $k+1$. If D contains a cycle of length k, then D is a complete digraph.* □

It follows from Theorems 8.3.21 and 8.3.22 that a strong k-transitive digraph is not likely to grow disorganizedly. On one hand, we have that every "sufficiently small" digraph is k-transitive. On the other hand, if a strong k-transitive digraph has a large enough circumference, its structure becomes very well determined. So a natural question arises: what happens if a strong k-transitive digraph has circumference less than k but at least $k+1$ vertices? Is there a proportion between order and circumference which allows us to say something about the structure of a strong k-transitive digraph? Theorem 8.3.22 seems to be the most simple case of such a result. Following this idea, there is a partial result due to Wang.

Theorem 8.3.23 ([59]) *Let D be a strong k-quasi-transitive digraph with $k \geq 4$, and let C be a cycle of length $k-1$. Then, for every $v \in V(D) \setminus V(C)$, the sets $(v, V(C))$ and $(V(C), v)$ are non-empty.* □

Proof: Since reversing every arc of a k-transitive digraph yields a k-transitive digraph, we only need to show $(v, V(C)) \neq \varnothing$. Let $C = v_0 \dots v_{k-2} v_0$ be a $(k-1)$-cycle. Since D is strong, there exists a path from v to C. Let $P = u_0 \dots u_s$ be a shortest path from v to C, where $s \geq 1$, $u_0 = v$ and $u_s \in V(C)$. Without loss of generality, assume that $u_s = v_0$. We prove that u_0 dominates some vertex of $V(C)$ by induction on the length s of P. It clearly holds for $s = 1$. Thus, we assume that $s \geq 2$. Note that $u_1 \dots u_s$ is a path of length $s-1$. By the induction hypothesis, there is a vertex $v_i \in V(C)$ such that $u_1 \to v_i$. Then $u_0 u_1 v_i C v_{i-1}$ is a path of length k in D, which implies $u_0 \to v_{i-1}$. □

8.3.4 Totally Φ-Decomposable Digraphs

The structure of totally Φ-decomposable digraphs is already determined from its definition and the choice of Φ. Thus, instead of studying their structure, we will show that for some choices of Φ, totally Φ-decomposable digraphs can be recognized in polynomial time.

As we will have already mentioned, Theorem 8.3.5 is the turning point on the study of quasi-transitive digraphs; it will let us construct polynomial algorithms for Hamiltonian paths and cycles in quasi-transitive digraphs, and also solve more general problems in this class of digraphs. This theorem shows that quasi-transitive digraphs are totally Φ-decomposable, where Φ is the union of extended semicomplete and transitive digraphs. Since both extended semicomplete digraphs and transitive digraphs are special subclasses of much wider classes of digraphs, it is natural to study totally Φ-decomposable digraphs, where Φ is a much more general class of digraphs than the union of extended semicomplete and transitive digraphs. However, our choice of candidates for the class Φ should be restricted in such a way that we can still construct polynomial algorithms for some important problems such as the Hamiltonian cycle problem, using properties of digraphs in Φ.

This idea was first used by Bang-Jensen and Gutin [13] to introduce the following classes of digraphs:

Definition 8.3.24

- Φ_0 *is the union of all semicomplete multipartite digraphs, all connected extended locally semicomplete digraphs and all acyclic digraphs,*
- Φ_1 *is the union of all semicomplete bipartite digraphs, all connected extended locally semicomplete digraphs and all acyclic digraphs,*
- Φ_2 *is the union of all connected extended locally semicomplete digraphs and all acyclic digraphs, and*
- Φ_3 *is the union of all semicomplete digraphs and all acyclic digraphs.*

Note that we have $\Phi_3 \subset \Phi_2 \subset \Phi_1 \subset \Phi_0$ and that all four classes are closed under taking extensions.

A class Φ of digraphs is **hereditary** if $D \in \Phi$ implies that every induced subdigraph of D is in Φ. Observe that every Φ_i, $0 \leq i \leq 3$, is a hereditary class. The following results are due to Bang-Jensen and Gutin.

Lemma 8.3.25 ([13]) *Let Φ be a hereditary class of digraphs. If a given digraph D is totally Φ-decomposable, then every induced subdigraph D' of D is totally Φ-decomposable. In other words, total Φ-decomposability is a hereditary property.*

Proof: By induction on the number of vertices of D. The claim is obviously true if D has fewer than 3 vertices.

If $D \in \Phi$, then our claim follows from the fact that Φ is hereditary. So, we may assume that $D = R[H_1, \ldots, H_r]$, $r \geq 2$, where $R \in \Phi$ and each of $H_1, \ldots, H_r$ is totally Φ-decomposable.

Let D' be an induced subdigraph of D. If there is an index i such that $V(D') \subseteq V(H_i)$, then D' is totally Φ-decomposable by induction. Otherwise, $D' = R'[T_1, \ldots, T_{r'}]$, where $r \geq 2$ and $R' \in \Phi$, is the subdigraph of R induced by those vertices i of R, whose H_i has a non-empty intersection with $V(D')$ and the T_j's are the corresponding H_i's restricted to the vertices of D'. Observe that $R' \in \Phi$, since Φ is hereditary. Moreover, by induction, each T_j is totally Φ-decomposable, hence so is D'. □

The following result gives a polynomial time algorithm for verifying Φ_i-decomposability, $i \in \{0, 1, 2, 3\}$. Its proof can be found in [9].

Lemma 8.3.26 ([13]) *There exists an $O(mn + n^2)$-algorithm for checking if a digraph D with n vertices and m arcs has a decomposition $D = R[H_1, \ldots, H_r]$, $r \geq 2$, where H_i is an arbitrary digraph and the digraph R_i is either acyclic or semicomplete multipartite or semicomplete bipartite or connected extended locally semicomplete.* □

The previous lemma can now be used to obtain the main result of this section. Again, its proof can be found on [9].

Theorem 8.3.27 ([13]) *There exists an $O(n^2 m + n^3)$-algorithm for checking if a digraph with n vertices and m arcs is totally Φ_i-decomposable, for $i \in \{0, 1, 2, 3\}$.*

8.4 Hamiltonian, Longest and Vertex-Cheapest Paths and Cycles

In this section we will study the Hamiltonian path and cycle problems, as well as some problems in weighted digraphs generalizing them. The subsections on quasi-transitive digraphs and totally Φ-decomposable digraphs in this section are strongly based on Sections 6.7 and 6.8 of [9], where this subject has received a full treatment. We begin by considering the few existing results for k-transitive and k-quasi-transitive digraphs.

8.4.1 *k*-Transitive and *k*-Quasi-Transitive Digraphs

Since strong 3-transitive and 4-transitive digraphs are completely characterized, it suffices to make a case by case analysis for these families of digraphs (using Theorems 8.3.19 and 8.3.20) to completely characterize Hamiltonian 3- and 4-transitive digraphs. This analysis can be summarized in the following result. We say that a k-cycle extension $D = C_k[S_1, \dots, S_k]$ is **balanced** if $|S_i| = |S_j|$ for every $i \neq j$, and non-balanced, otherwise.

Theorem 8.4.1 *If D is a strong 3-transitive digraph, then D is Hamiltonian if and only if it is not a complete bipartite digraph $D = (X, Y)$ with $|X| \neq |Y|$.*

If D is a strong 4-transitive digraph, then D is Hamiltonian if and only if it is a complete digraph, a balanced 3-cycle extension, a symmetrical 5-cycle, or a semicomplete digraph on at most 4 vertices. □

It follows from Theorem 8.4.1 that hamiltonicity for 3-transitive and 4-transitive digraphs can be determined in linear time: Hamiltonian members of these families can be easily recognized through their in-degree and out-degree sequences. In view of this fact, the following problem is proposed.

Problem 8.4.2 *For all values of $k \geq 5$, determine the complexity of the Hamiltonian cycle problem for the class of k-transitive digraphs.*

Considering the results for $k \in \{2, 3, 4\}$, it does not seem too adventurous to conjecture that hamiltonicity of a k-transitive digraph could be determined in linear time for every integer $k \geq 2$. From Theorems 8.3.21 and 8.3.22, easy to verify sufficient conditions for the existence of a Hamiltonian cycle in a strong k-transitive digraph can be derived: A k-transitive digraph containing a cycle of length at least k is Hamiltonian unless it is a non-balanced extended cycle.

For 3-quasi-transitive digraphs, Theorem 8.3.15 also provides enough information to completely characterize Hamiltonian members of this family.

Theorem 8.4.3 *If D is a strong 3-quasi-transitive digraph, then D is Hamiltonian if and only if one of the following hold:*

- *D is semicomplete,*
- *D is semicomplete bipartite with a cycle factor, or*
- *D is the member of the family $\mathcal{F}$ of order 4 (see Figure 8.3).*

Proof: Clearly, all the digraphs mentioned in the statement of the theorem are Hamiltonian. Using Theorem 8.3.15 we can rule out the remaining cases for a strong 3-quasi-transitive digraph.

We know that a strong semicomplete bipartite digraph is Hamiltonian if and only if it has a cycle factor (see Theorem 7.4.1), and clearly, every digraph in $\mathcal{F}$ of order greater than 4 is not Hamiltonian. □

It follows from Theorems 8.4.3 and 7.4.1 that hamiltonicity can be verified for 3-quasi-transitive digraphs in time $O(n^{2.5})$. So, the following question comes to mind.

Problem 8.4.4 *Let k be an integer, $k \geq 4$. Is it true that hamiltonicity can be determined for the class of k-quasi-transitive digraphs in polynomial time?*

Regarding Hamiltonian paths, Wang and Zhang gave a sufficient condition for traceability when k is even, [62].

Theorem 8.4.5 ([62]) *Let k be an even integer with $k \geq 4$ and D be a strong k-quasi-transitive digraph. If $\mathrm{diam}(D) \geq k+2$, then D has a Hamiltonian path.*

Proof: Since $\mathrm{diam}(D) \geq k+2$, there exist $u, v \in V(D)$ such that $d(u,v) = k+2$. Let $P = x_0 \dots x_{k+2}$ be a shortest (u,v)-path where $u = x_0$ and $v = x_{k+2}$. By Lemma 8.3.6, $x_{k+2} \rightarrow x_0$. Let C be the cycle $C = x_0 \dots x_{k+2} x_0$ and $H = D[V(D) - V(C)]$. By Proposition 8.3.7 and Theorem 8.3.12, $D[V(C)]$ and H are both semicomplete digraphs. It is well known that there is a Hamiltonian path in every semicomplete digraph. Let $Q = y_0 \dots y_p$ be a Hamiltonian path in H. By Lemma 8.3.9, for any $y_i \in V(Q)$, y_i is adjacent to C. If there exists an $x_j \in V(C)$ such that $x_j \rightarrow y_0$, then $x_{j+1} C x_j y_0 Q$ is a Hamiltonian path in D. Now assume $(V(C), y_0) = \varnothing$. Note that $k-1$ and $k+3$ are coprime.[3] According to Proposition 8.3.10, $y_0 \mapsto V(C)$, and thus, either there is an $x_j \in V(C)$ such that $x_j \rightarrow y_1$ and therefore $y_0 x_{j+1} C x_j y_1 Q$ is a Hamiltonian path in D, or $y_1 \mapsto V(C)$. Continuing in this way, we can conclude that either D has a Hamiltonian path, or $V(H) \mapsto V(C)$. But since D is strong, $(V(C), V(H)) \neq \varnothing$. So D has a Hamiltonian path. □

Notice that, since every complete bipartite digraph is k-quasi-transitive for any odd integer $k \geq 3$, it is not possible to obtain a result similar to Theorem 8.4.5 for odd values of k.

[3] Recall that the g.c.d. of two integers is their least positive linear combination. Clearly, 4 is a linear combination of $k-1$ and $k+3$, but since k is even, and $k-1 \not\equiv k+3 \pmod 3$, the least positive linear combination of $k-1$ and $k+3$ is 1.

The following question was proposed by the same authors.

Problem 8.4.6 ([62]) *Let k be an even integer, $k \geq 4$, and suppose that $diam(D) \geq k+2$. Is there a Hamiltonian cycle in D?*

It follows from the remark after Theorem 8.3.12 that Problem 8.4.6 has a positive answer for $k = 4$.

8.4.2 Hamiltonian Cycles in quasi-transitive digraphs and Totally Φ-Decomposable Digraphs

Hamiltonicity is one of the most studied topics in both graphs and digraphs. Having a family as nice as quasi-transitive digraphs, it is natural to have a lot of results for this class regarding both Hamiltonian paths and cycles, many of which come from the study of semicomplete digraphs and its corresponding hamiltonicity results. Since Chapter 2 is devoted to tournaments and semicomplete digraphs, we will not elaborate on the results regarding these digraph classes, but we will restate some of them.

As mentioned in the introduction to this chapter, totally Φ-decomposable digraphs generalize the structure of quasi-transitive digraphs. Thus, it is common to find that the techniques used to prove certain results for quasi-transitive digraphs can be adapted to study this more general family of digraphs. In particular, the methods developed in [17] by Bang-Jensen and Huang, and in [41] by Gutin, to characterize Hamiltonian and traceable quasi-transitive digraphs as well as to construct polynomial algorithms for verifying the existence of Hamilton paths and cycles in quasi-transitive digraphs, can be easily generalized to much wider classes of digraphs [11]. Thus, in this subsection, along with quasi-transitive digraphs, we consider totally Φ-decomposable digraphs for various families Φ of digraphs.

Recall that a digraph D is an **extended semicomplete digraph** if it can be obtained from some semicomplete digraph S by substituting independent sets for the vertices of S.

Recall that the decompositions given by Theorem 8.3.5 are called canonical decompositions. The following characterization of Hamiltonian quasi-transitive digraphs is due to Bang-Jensen and Huang [17].

Theorem 8.4.7 ([17]) *A strong quasi-transitive digraph D with canonical decomposition $D = S[Q_1, Q_2, \ldots, Q_s]$ is Hamiltonian if and only if it has a cycle factor $\mathcal{F}$ such that no cycle of $\mathcal{F}$ is a cycle of some Q_i.*

Proof: Clearly, a Hamilton cycle in D crosses every Q_i. Thus, it suffices to show that if D has a cycle factor $\mathcal{F}$ such that no cycle of $\mathcal{F}$ is a cycle of some Q_i, then D is Hamiltonian. Observe that $V(Q_i) \cap \mathcal{F}$ is a path factor $\mathcal{F}_i$ of Q_i for every $i \in [s]$. For every $i \in [s]$, delete the arcs between end-vertices of all paths in $\mathcal{F}_i$ except for the paths themselves, and then perform the operation of path-contraction for all paths in $\mathcal{F}_i$. As a result, one obtains an extended semicomplete digraph S' (since S is semicomplete). Clearly, S' is strong and has a cycle factor. Hence, by Theorem 7.10.1, S' has a Hamilton cycle C. After replacing every vertex of S' with the corresponding path from $\mathcal{F}$, we obtain a Hamilton cycle in D. □

Similarly to Theorem 8.4.7, one can prove the following characterization of traceable quasi-transitive digraphs. This result is also due to Bang-Jensen and Huang.

Theorem 8.4.8 ([17]) *A quasi-transitive digraph D with at least two vertices and with canonical decomposition $D = R[G_1, G_2, \ldots, G_r]$ is traceable if and only if it has a 1-path-cycle factor $\mathcal{F}$ such that no cycle or path of $\mathcal{F}$ is completely in some $D[V(G_i)]$.* □

Theorems 8.4.7 and 8.4.8 do not imply polynomial algorithms to verify hamiltonicity and traceability, respectively. The following characterization of Hamiltonian quasi-transitive digraphs is given implicitly in the paper [41] by Gutin:

Theorem 8.4.9 (Gutin [41]) *Let D be a strong quasi-transitive digraph with canonical decomposition $D = S[Q_1, Q_2, \ldots, Q_s]$. Let $n_1, \ldots, n_s$ be the orders of the digraphs $Q_1, Q_2, \ldots, Q_s$, respectively. Then D is Hamiltonian if and only if the extended semicomplete digraph $S' = S[\overline{K}_{n_1}, \overline{K}_{n_2}, \ldots, \overline{K}_{n_s}]$ has a cycle subdigraph which covers at least $\mathrm{pc}(Q_j)$ vertices of $\overline{K}_{n_j}$ for every $j \in [s]$.*

Proof: Suppose that D has a Hamilton cycle H. For every $j \in [s]$, $V(Q_j) \cap H$ is a k_j-path factor $\mathcal{F}_j$ of Q_j. By the definition of the path covering number, we have $k_j \geq \mathrm{pc}(Q_j)$. For every $j \in [s]$, the deletion of the arcs between end-vertices of all paths in $\mathcal{F}_j$ except for the paths themselves, and then path-contraction of all paths in $\mathcal{F}_j$, transforms H into a cycle of S' having at least $\mathrm{pc}(Q_j)$ vertices of $\overline{K}_{n_j}$ for every $j \in [s]$.

Suppose now that S' has a cycle subdigraph $\mathcal{L}$ containing $p_j \geq \mathrm{pc}(Q_j)$ vertices of $\overline{K}_{n_j}$ for every $j \in [s]$. Since S' is a strong extended semicomplete digraph, by Theorem 7.10.2, S' has a cycle C such that $V(C) = V(\mathcal{L})$. Clearly, every Q_j has a p_j-path factor $\mathcal{F}_j$. Replacing, for every $j \in [s]$, the p_j vertices of $\overline{K}_{n_j}$ in C with the paths of $\mathcal{F}_j$, we obtain a Hamiltonian cycle in D. □

Theorem 8.4.9 can be used to show that the Hamilton cycle problem for quasi-transitive digraphs is polynomial time solvable.

Theorem 8.4.10 (Gutin [41]) *There is an $O(n^4)$ algorithm which, given a quasi-transitive digraph D, either returns a Hamiltonian cycle in D or verifies that no such cycle exists.* ◇

The approach used in the proofs of Theorems 8.4.9 and 8.4.10 in [41] can be generalized to a much wider class of digraphs, as was observed by Bang-Jensen and Gutin [11]. We follow the main ideas of [11].

Recall the definition of $\Phi_0, \Phi_1, \Phi_2, \Phi_3$ in Definition 8.3.24 and the fact that, for each of these classes, in time $O(n^4)$, one can check if a given digraph D is totally Φ_i-decomposable ($i \in \{0,1,2,3\}$) and (in case it is so) construct a total decomposition of D. Moreover, Theorem 8.3.5 implies that every quasi-transitive digraph is totally Φ_3-decomposable.

Theorem 8.4.11 *Let Φ be an extension-closed class of digraphs, i.e., $\Phi^{ext} = \Phi$, including the trivial digraph $\overline{K}_1$ on one vertex. Suppose that for every digraph $H \in \Phi$ we have $\mathrm{pcc}(H) = \mathrm{pc}(H)$. Let D be a totally Φ-decomposable digraph. Then, given a total Φ-decomposition of D, the path covering number of D can be calculated and a minimum path factor found in time $O(n^4)$.*

Proof: We prove this theorem by induction on n. For $n = 1$ the claim is trivial.

Let D be a totally Φ-decomposable digraph and let $D = R[H_1, \dots, H_r]$ be a Φ-decomposition of D such that $R \in \Phi$, $r = |V(R)|$ and every H_i (of order n_i) is totally Φ-decomposable. A $\mathrm{pc}(D)$-path factor of D restricted to every H_i corresponds to a disjoint collection of some p_i paths covering $V(H_i)$. Hence, we have $\mathrm{pc}(H_i) \le p_i \le n_i$. Therefore, arguing similarly to the proof of Theorem 8.4.9, we obtain

$$\mathrm{pc}(D) = \min\{\mathrm{pc}(R[\overline{K}_{p_1}, \dots, \overline{K}_{p_r}]) : \ \mathrm{pc}(H_i) \le p_i \le n_i, \ i \in [r]\}.$$

Since Φ is extension-closed, and since, for every digraph $Q \in \Phi$, $\mathrm{pc}(Q) = \mathrm{pcc}(Q)$, we obtain

$$\mathrm{pc}(D) = \min\{\mathrm{pcc}(R[\overline{K}_{p_1}, \dots, \overline{K}_{p_r}]) : \ \mathrm{pc}(H_i) \le p_i \le n_i, \ i \in [r]\}. \quad (8.1)$$

Given the lower and upper bounds $\mathrm{pc}(H_i)$ and n_i ($i \in [r]$), the recursive formula (8.1) allows us to find $\mathrm{pc}(D)$ in time $O(n^3)$. To show this, it suffices to demonstrate how to find, in time $O(n^3)$, the minimum in formula 8.1 given all the values of $pc(H_i)$ (and n_i). Construct a network N_R containing the digraph R and two additional vertices (source and sink) s and t such that s and t are adjacent to every vertex of $V(R)$ and the arcs between s (t, respectively) and R are oriented from s to R (from R to t, respectively). Associate with each vertex v_i of R (corresponding to H_i in D) the lower and upper bounds $pc(H_i)$ and n_i ($1 \le i \le r$) on the amount of flow that can pass through v_i. It is not difficult to see that the minimum value, m, of a feasible flow from s to t in N_R, is related to the minimum in 8.1, i.e. $pc(D)$, as follows: $pc(D) = \max\{1, m\}$ (for further details, see [41]).

Let $T(n)$ be the time needed to find the path covering number of a totally Φ-decomposable digraph of order n. Then, by (8.1),

$$T(n) = O(n^3) + \sum_{i=1}^{r} T(n_i).$$

Furthermore, $T(1) = O(1)$. Hence $T(n) = O(n^4)$. □

As we know, $\text{pc}(D) = \text{pcc}(D)$ for every semicomplete multipartite digraph D (see Theorem 7.5.2), for every extended locally semicomplete digraph D (by Theorem 5.8.1 in [8]) and every acyclic digraph D (which is trivial). Therefore, Theorems 8.4.11 and 8.3.27 imply the following theorem of Bang-Jensen and Gutin:

Theorem 8.4.12 ([12]) *The path covering number can be calculated in time $O(n^4)$ for digraphs that are totally Φ_0-decomposable.* □

Corollary 8.4.13 ([12]) *One can verify whether a totally Φ_1-decomposable digraph is Hamiltonian in time $O(n^4)$.*

Proof: Let $D = R[H_1, \dots, H_r]$, $r = |R|$, be a decomposition of a strong digraph D ($r \geq 2$). Then, D is Hamiltonian if and only if the following family $\mathcal{S}$ of digraphs contains a Hamiltonian digraph:

$$\mathcal{S} = \{R[\overline{K}_{p_1}, \dots, \overline{K}_{p_r}] : \ \text{pc}(H_i) \leq p_i \leq |V(H_i)|, \ i \in [r]\}.$$

Now suppose that D is a totally Φ_1-decomposable digraph. Then, every digraph of the form $R[\overline{K}_{p_1}, \dots, \overline{K}_{p_r}]$ is in Φ_1. We know (see Theorem 7.4.1 and Theorem 5.8.1 in [8]) that every digraph in Φ_1 is Hamiltonian if and only if it is strong and contains a cycle factor. Thus, all we need is to verify whether there is a digraph in $\mathcal{S}$ containing a cycle factor. It is easily seen that there is a digraph in $\mathcal{S}$ containing a cycle factor if and only if there is a circulation in the network formed from R by adding lower bounds $\text{pc}(H_i)$ and upper bounds $|V(H_i)|$ to the vertex v_i of R for every $i \in [r]$. Since the lower bounds can be found in time $O(n^4)$ (see Theorem 8.4.11) and the existence of a circulation checked in time $O(n^3)$ (a feasible circulation, if one exists, can be found by just one max flow calculation in an (s,t)-flow network obtained from our network, see [9, Exercise 4.31]), we obtain the required complexity $O(n^4)$. □

Since every quasi-transitive digraph is totally Φ_1-decomposable this theorem immediately implies Theorem 8.4.10. Note that the minimum path factors in Theorem 8.4.11 can be found in time $O(n^4)$. Also, a Hamiltonian cycle in a Hamiltonian totally Φ_1-decomposable digraph can be constructed in time $O(n^4)$.

8.4.3 Vertex-Cheapest Paths and Cycles

For the remainder of this section, we consider problems that generalize the Hamilton path and cycle problems in a significant way. We prove that the problems of finding vertex-cheapest paths and cycles in vertex-weighted quasi-transitive digraphs are polynomial time solvable. The values of the weights can be any reals, positive or negative. Thus, we can conclude that the longest and shortest path and cycle problems for quasi-transitive digraphs are polynomial time solvable. The same result holds for acyclic digraphs as the only non-trivial problem from the above four is the longest path problem and it is well-known that it can be solved in polynomial time (see e.g. [9, Theorem 3.3.5]). Notice that for the quasi-transitive digraphs three of the above four problems are non-trivial (the shortest and longest cycles and longest path) and, in fact, much more difficult than the longest path problem for acyclic digraphs as the reader can see in the rest of this subsection. It appears that the problems are non-trivial even for semicomplete digraphs. Theorems 7.10.4 and 6.17.16 were proved by Bang-Jensen, Gutin and Yeo for extended semicomplete and locally semicomplete digraphs.

The approach described in the previous subsection seems too weak to allow us to construct polynomial time algorithms for vertex-cheapest paths and cycles in quasi-transitive digraphs. A more powerful method that leads to such algorithms was first suggested by Bang-Jensen, Gutin and Yeo [15] and, in the rest of this section, we describe this method.

Recall that the cost of a subset of vertices is the sum of the costs of its vertices and the cost of a subdigraph is the sum of the costs of its vertices. For a digraph D of order n and $i \in [n]$ we define $mp_i(D)$ ($mpc_i(D)$) to be the minimum cost of an i-path (i-path-cycle) subdigraph of D. We set $mp_0(D) = 0$ and $mpc_0(D)$ is zero if D has no negative cycle and otherwise it is the minimum cost of a cycle subdigraph in D which can be found using minimum cost flows. Note that $mp_0(D)$ and $mpc_0(D)$ always exist as we may take single vertices as paths and we always have $mpc_i(D) \leq mp_i(D)$. For any digraph D with at least one cycle we denote by $mc(D)$ the minimum cost of a cycle in D.

Let $D = (V, A)$ be a digraph and let X be a non-empty subset of V. We say that a cycle C in D is an X**-cycle** if C contains all vertices of X. In the remaining subsections, we consider the following problems for a digraph $D = (V, A)$ with n vertices and real-valued costs on the vertices:

(P1) Determine $mp_i(D)$ for all $i \in [n]$.
(P2) Find a cheapest cycle in D or determine that D has no cycle.

Clearly, problems (P1) and (P2) are $\mathcal{NP}$-hard as determining the numbers $mp_1(D)$ and $mc(D)$ generalize the Hamiltonian path and cycle problems (assign cost -1 to each vertex of D). The problem (P2) can be solved in time $O(n^3)$ when all costs are non-negative using an all pairs shortest path calculation. The problems (P1) and (P2) were solved in [14] for the special case when all costs are non-negative. However, the approach of [14] cannot be used or modified to work with negative costs. Bang-Jensen, Gutin and Yeo [15] managed to obtain an approach suitable for arbitrary real costs.

8.4.4 Minimum Cost k-Path-Cycle Subdigraphs

Although this chapter is intended to be almost self-contained, in order to present the main results of this subsection, we need certain notions and results on network flows. We refer the reader to Section 1.9 of this book for basic terminology, and to chapter 4 of [9] for the proofs of the results we will state. As in the aforementioned chapter of [9], we will allow capacities and costs on the vertices in our networks. This makes it easier to model certain problems for digraphs and it is easy to transform such a network into one where all capacities and costs are on the arcs (see Subsection 4.2.4 of [9] for details). With these remarks in mind, the following lemma of Bang-Jensen, Gutin and Yeo follows directly from Lemma 4.2.4 and Proposition 4.10.7 in [9].

Lemma 8.4.14 ([15]) *Let $N = (V, A)$ be a network with source s and sink t, capacities on arcs and vertices and a real-valued cost $c(v)$ for each vertex $v \in V$. For all integers i such that there exists a feasible (s,t)-flow of value i in N, let f_i be a minimum cost (s,t)-flow in N of value i and let $c(f_i)$ be the cost of f_i. Then, for all i where all of f_{i-1}, f_i, f_{i+1} exist, we have*

$$c(f_{i+1}) - c(f_i) \geq c(f_i) - c(f_{i-1}). \tag{8.2}$$

◇

Recall that a cycle subdigraph of a digraph D is a collection of vertex-disjoint cycles of D. The following two results are also due to Bang-Jensen, Gutin and Yeo.

Lemma 8.4.15 ([15]) *Let $D = (V, A)$ be a digraph with real-valued cost function c on the vertices. In time $O(n(m+n\log n))$ we can determine the number $mpc_0(D)$ and find a cycle subdigraph of cost $mpc_0(D)$ if $mpc_0(D) < 0$.*

Proof: Let $H(w)$ be the digraph on 4 vertices w_1, w_2, w_3, w_4 and the following arcs $w_1w_2, w_2w_1, w_2w_3, w_3w_4, w_4w_3$. Let $D^* = (V^*, A^*)$ be obtained from D as follows: replace every vertex v by the digraph $H(v)$. Furthermore, for every original arc $uv \in A$, D^* contains the arc u_4v_1. There are no costs on the vertices and all arcs have cost 0 except the arcs of the form v_2v_3 which have cost $c(v)$. Observe that $mpc_0(D)$ is precisely the minimum cost of a spanning cycle subdigraph in D^*. Let $V^* = \{x_1, x_2, \ldots, x_{4n}\}$. Construct a bipartite graph B with partite sets $L = \{\ell_1, \ldots, \ell_{4n}\}$ and $R = \{r_1, \ldots, r_{4n}\}$, in which $\ell_i r_j$ is an edge if and only if $x_ix_j \in A^*$. Moreover, the cost of $\ell_i r_j$ is equal to the cost of x_ix_j. Observe that a minimum cost perfect matching in B corresponds to a minimum cost cycle subdigraph in D^*. We can find a minimum cost perfect matching in B in time $O(n(m + n\log n))$, see the remark after the proof of Theorem 11.1 in [51]. Using the transformation from B to D^*, we can compute the minimum cost of a spanning cycle subdigraph F in D^* in time $O(n(m + n\log n))$. If this cost is negative, we can find a minimum cost cycle subdigraph of D within the same time. □

Lemma 8.4.16 ([15]) *Let $D = (V, A)$ be a vertex-weighted digraph.*

(a) In total time $O(n^2m + n^3)$ we can determine the numbers $\{mpc_1(D), mpc_2(D), \ldots, mpc_n(D)\}$ and find j-path-cycle subdigraphs F_j, $j \in \{1, 2, \ldots, n\}$, where F_j has cost $mpc_j(D)$.
(b) The costs $mpc_i(D)$ satisfy the following inequality for every $i \in [n-1]$:

$$mpc_{i+1}(D) - mpc_i(D) \geq mpc_i(D) - mpc_{i-1}(D). \tag{8.3}$$

Proof: Form a network $N(D)$ from D by adding a pair s, t of new vertices along with arcs $\{(s, v), (v, t) : v \in V\}$. Let all vertices and all arcs of D have lower bound 0 and capacity 1. Let $c(s) = c(t) = 0$, let each other vertex of $N(D)$ inherit its cost from D and let all arcs have cost 0.

Suppose F_j is a j-path-cycle subdigraph of D. Using F_j we can obtain a feasible flow f_j of value j in $N(D)$ if we assign $f_j(a) = 1$ to all arcs a in F_j and those arcs a of $N(D)$ that start (terminate) at s (t) and terminate (start) at the initial (terminal) vertex of a path in F_j, and $f_j(a) = 0$ for all other arcs of $N(D)$. Similarly, we can transform a feasible integer-valued (s, t)-flow of value j in $N(D)$ into a j-path-cycle subdigraph of D (see Theorem 4.3.1 in [9]).

Notice that $N(D)$ has a feasible integer-valued (s, t)-flow of value k for any integer $k \in \{0, 1, \ldots, n\}$. Thus it follows from the observations above that for every $j \in \{0, 1, \ldots, n\}$ the value $mpc_j(D)$ is exactly the minimum cost of a flow of value j in $N(D)$. Now (8.2) implies that the inequality (8.3) is valid.

It remains to prove (a). It follows from Lemma 8.4.15 that we can find a minimum cost flow f of value 0 in time $O(n^3)$. Now we can use the Buildup algorithm from Subsection 4.10.2 in [9] starting from f. Using the Buildup algorithm we can find feasible integer-valued flows f_j for all $j \in [n]$, such that f_j is a minimum cost feasible (s, t)-flow of value j in $N(D)$, in total time $O(n^2m)$ (the complexity of obtaining f_{j+1} starting from f_j is $O(nm)$). This proves (a). □

8.4.5 Cheapest i-Path Subdigraphs in Quasi-Transitive Digraphs

Theorem 7.5.4, regarding semicomplete multipartite digraphs, will play an important role in our algorithms. The next theorem due to Bang-Jensen, Gutin and Yeo shows that (P1) is polynomially solvable for quasi-transitive digraphs.

Theorem 8.4.17 ([15]) *Let $D = (V, A)$ be a vertex-weighted quasi-transitive digraph. Then the following holds:*

(a) In total time $O(n^2m + n^3)$ we can find for every $i \in [n]$, the value of $mp_i(D)$ and an i-path subdigraph F_i of cost $mp_i(D)$.

(b) For all $i \in [n-1]$ we have

$$mp_{i+1}(D) - mp_i(D) \geq mp_i(D) - mp_{i-1}(D). \tag{8.4}$$

Proof: We prove (b) by induction on n. The statement vacuously holds for $n = 1$ and is easy to verify for $n = 2$ (recall that, by definition, $mp_0(D) = 0$). This proves the basis of induction and we now assume that $n \geq 3$.

By Theorem 8.3.5, D has a decomposition $D = T[Q_1, \ldots, Q_t]$, $t = |T| \geq 2$, where T is an acyclic digraph or a semicomplete digraph. Let $D' = T[\overline{K}_{n_1}, \ldots, \overline{K}_{n_t}]$ be obtained from D by deleting all arcs inside each Q_i, $i \in [t]$. Assign costs to the vertices $v_1^k, \ldots, v_{n_k}^k$ of $\overline{K}_{n_k}$, as follows:

$$c'(v_j^k) = mp_j(Q_k) - mp_{j-1}(Q_k).$$

By the induction hypothesis (b) holds for Q_k implying that we have

$$c'(v_j^k) \leq c'(v_{j+1}^k) \text{ for every } j \geq 1. \tag{8.5}$$

Let F_i' be an i-path-cycle subdigraph of D'. If T is acyclic, then D' is acyclic and, thus, F_i' is an i-path subdigraph of D'. If T is semicomplete, then D' is extended semicomplete and, thus, by Theorem 7.5.1 and Theorem 7.5.4, we may assume that F_i' is an i-path subdigraph of D'. Hence, $mp_i(D') = mpc_i(D')$ and it follows from Lemma 8.4.16(b) that (8.4) holds for D'. Thus it suffices to prove that $mp_i(D) = mp_i(D')$.

Let F_i' be an i-path subdigraph of D' and let p_k denote the number of vertices from $\overline{K}_{n_k}$ which are covered by F_i'. Since all vertices of $\overline{K}_{n_k}$ are similar it follows from (8.5) that we may assume (by making the proper replacements if necessary) that F_i' includes $v_1^k, \ldots, v_{p_k}^k$. For each k, replace the vertices $v_1^k, \ldots, v_{p_k}^k$ in F_i' by a p_k-path subdigraph of Q_k with cost $mp_{p_k}(Q_k) = \sum_{i=1}^{p_k} c'(v_i^k)$. As a result, we obtain, from F_i', an i-path subdigraph F_i of D for which we have $c'(F_i') = \sum_{k=1}^{t} mp_{p_k}(Q_k) = c(F_i)$ and, thus, $c(F_i) = c'(F_i')$. Reversing the process above it is easy to get, from an i-path subdigraph of D, an i-path subdigraph F_i' of D' such that $c(F_i) = c'(F_i')$. This shows that $mp_i(D) = mp_i(D')$ and hence (8.4) holds for D by the remark above.

We prove the complexity by induction on n. Let m' be the number of arcs in D' and recall that all these arcs are also in D. Clearly when a digraph H has $|V(H)| \leq 2$ we can choose a constant c_1 so that we can determine the numbers $mp_i(H)$, $i = 1, 2, \ldots, |V(H)|$, in time at most $c_1|V(H)|^2(|A(H)| + |V(H)|)$. Now assume by induction that for each Q_i we can determine the desired numbers inside Q_i in time at most $c_1 n_i^2(m_i + n_i)$. This means that we can find the numbers $mp_i(Q_j)$ for all $j \in [t]$ and $i \in [n_j]$ in total time

$$\sum_{j=1}^{t} c_1 n_j^2(m_j + n_j) \leq c_1 n^2 \sum_{j=1}^{t}(m_j + n_j) = c_1 n^2(m - m' + n).$$

By Lemma 8.4.16(a), Theorems 7.5.1 and 7.5.4, there is a constant c_2 such that in total time at most $c_2n^2(m'+n)$ we can find, for every $j \in [n]$, a j-path-cycle subdigraph of cost $mp_j(D')$ in D'. It follows from the way we construct F_i above from F'_i that if we are given for each $k \in [t]$ and each $1 \leq j \leq n_k$ a j-path subdigraph in Q_k of cost $mp_j(Q_k)$, then we can construct all the path subdigraphs F_r, $1 \leq r \leq n$, in time at most c_3n^3 for some constant c_3. Hence the total time needed by the algorithm is at most

$$c_1n^2(m - m' + n) + c_2n^2(m' + n) + c_3n^3 =$$
$$c_1n^2(m + n) + (c_2 - c_1)n^2m' + (c_2 + c_3)n^3,$$

which is at most $c_1n^2(m + n)$ for c_1 sufficiently large. □

The next theorem, also due to Bang-Jensen, Gutin and Yeo, is an easy consequence of Theorem 8.4.17 (assign all vertices cost -1).

Theorem 8.4.18 ([15]) *One can find a longest path in any quasi-transitive digraph in time $O(n^2m + n^3)$.* □

Sometimes, one is interested in finding path subdigraphs that include a maximum number of vertices from a given set X or avoid as many vertices of X as possible. We consider a minimum cost extension of this problem in the next result.

Theorem 8.4.19 ([15]) *Let $D = (V, A)$ be a vertex-weighted quasi-transitive digraph and let $X \subseteq V$ be non-empty. Let p_j be the maximum possible number of vertices from X in a j-path subdigraph and let q_j be the maximum possible number of vertices from X not in a j-path subdigraph. In total time $O(n^2m+n^3)$ we can find, for all $j \in [n]$, a cheapest j-path subdigraph which includes p_j (avoids q_j, respectively) vertices of X.*

Proof: Let $C = \sum_{v \in V} |c(v)|$ and subtract $C+1$ from the cost of every vertex in X. Now, for each $j \in [n]$, every cheapest j-path subdigraph F_j must cover as many vertices from X as possible, i.e., p_j vertices. Furthermore, since the new cost of F_j is exactly the original one minus $p_j(C + 1)$, cheapest j-path subdigraphs covering p_j vertices from X are preserved under this transformation. Now the 'including' part of the claim follows from Theorem 8.4.17(a). The 'avoiding' part can be proved similarly, by adding $C + 1$ to every vertex of X. □

8.4.6 Finding a Cheapest Cycle in a Quasi-Transitive Digraph

Bang-Jensen, Gutin and Yeo obtained the following:

Theorem 8.4.20 ([15]) *For quasi-transitive digraphs with vertex-weights the minimum cost cycle problem can be solved in time $O(n^5 \log n)$.*

Proof: Let D be a quasi-transitive digraph. If D is not strong, then we simply look at the strong components, so assume that D is strong. By Theorem 8.3.5, $D = T[Q_1, \ldots, Q_t]$, where T is a strong semicomplete digraph, and each Q_i is either a single vertex or a non-strong quasi-transitive digraph.

Suppose we have found a minimum cost cycle C_i in each Q_i which contains a cycle. Then clearly the minimum cost of a cycle in D is given by $\min(\min_i(c(C_i)), c(C))$, where C is a minimum cost cycle among those intersecting at least two Q_i's. Hence it follows that applying this approach recursively we can find the minimum cost cycle in D. Now we show how to compute a minimum cost cycle C as above.

Let D' be defined as in the proof of Theorem 8.4.17 including the vertex-costs. It is easy to show using the same approach as when we converted between i-path subdigraphs of D' and D in the proof of Theorem 8.4.17, that the cost of C is precisely $mc(D')$. Now it follows from Theorem 7.10.4 that we can find the cycle C in time $O(n^3m + n^4 \log n)$.

Since we can construct D', including finding the costs for all the vertices in time $O(n^2m + n^3)$ by Theorem 8.4.17, and there are at most $O(n)$ recursive calls, the approach above will lead to a minimum cost cycle of D in time $O(n^4m+n^5 \log n)$. In fact, we can bound the first term as we did in the proof of Theorem 8.4.17 and obtain $O(n^3m + n^5 \log n) = O(n^5 \log n)$ rather than $O(n^4m + n^5 \log n)$. This completes the proof. □

8.5 Linkages

It is a well-known fact that it is easy to check (e.g., using flows) whether a directed multigraph $D = (V, A)$ has k (arc)-disjoint paths $P_1, \ldots, P_k$ from a subset $X \subseteq V$ to another subset $Y \subseteq V$, and we can also find such paths efficiently. On many occasions (e.g., in practical applications) we need to be able to specify the initial and terminal vertices of each P_i, $1 \leq i \leq k$, that is, we wish to find a so-called **linkage** from $X = \{x_1, \ldots, x_k\}$ to $Y = \{y_1, \ldots, y_k\}$ such that P_i is an (x_i, y_i)-path for every $1 \leq i \leq k$. This problem is considerably more difficult and is in fact $\mathcal{NP}$-complete already when $k = 2$.

Recall that, for a digraph $D = (V, A)$ with distinct vertices x, y we denote by $\kappa_D(x, y)$ the largest integer k such that D contains k internally disjoint (x, y)-paths. When discussing intersections between paths P, Q we will often use the phrase 'let u be the first (last) vertex on P which is on Q'. By this we mean that if, say, P is an (x, y)-path, then u is the only vertex of $P[x, u]$ ($P[u, y]$) which is also on Q.

Let $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ be distinct vertices of a digraph D. A ***k*-linkage** from $(x_1, x_2, \ldots, x_k)$ to $(y_1, y_2, \ldots, y_k)$ in D is a system of vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is an (x_i, y_i)-path in D.[4] A digraph

[4] Sometimes we allow that the paths may share one or both of their end-vertices, i.e., $V(P_i) \cap V(P_j) \subseteq \{x_i, y_i, x_j, y_j\}$ whenever $i \neq j$, where $x_i = y_j$ or $x_i = x_j$ is possible.

$D = (V, A)$ is ***k*-linked** if it contains a k-linkage from $(x_1, x_2, \ldots, x_k)$ to $(y_1, y_2, \ldots, y_k)$ for every choice of distinct vertices $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$. The k-Disjoint paths problem is defined as follows.

k-Disjoint paths **Input:** A digraph $D = (V, A)$ and distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$. **Question:** Does D contain vertex disjoint paths $P_1, \ldots, P_k$ such that P_i is an (s_i, t_i)-path for $i \in [k]$?

Fortune, Hopcroft and Wyllie [30] showed that if we impose no restriction on the input, then the k-Disjoint paths problem is $\mathcal{NP}$-complete already for $k = 2$. This motivates the study of subclasses of digraphs for which the problem is polynomial-time solvable.

From the algorithmic point of view, the 2-Disjoint paths problem for semicomplete digraphs has already been solved by Bang-Jensen and Thomassen in Theorem 2.5.6. The proof of this result in [21] is highly non-trivial. The basic approach is divide and conquer and several non-trivial results and steps are needed to make the algorithm work. Now we show that the 2-Disjoint paths problem can be solved in polynomial time for quite large classes of digraphs which can be obtained by starting from semicomplete digraphs and then performing certain substitutions. The algorithm we describe uses the polynomial algorithm from Theorem 2.5.6 for the case of semicomplete digraphs as a subroutine. The results in this section are due to Bang-Jensen [5].

Theorem 8.5.1 ([5]) *Let $D = F[S_1, S_2, \ldots, S_f]$ where F is a strong digraph on $f \geq 2$ vertices and each S_i is a digraph with n_i vertices and let x_1, x_2, y_1, y_2 be distinct vertices of D. There exist semicomplete digraphs $T_1, \ldots, T_f$ such that $V(T_i) = V(S_i)$ for all $i \in [f]$, and the digraph $D' = F[T_1, T_2, \ldots, T_f]$ has vertex-disjoint (x_1, y_1)-, (x_2, y_2)-paths if and only if D has such paths. Furthermore, given D and x_1, x_2, y_1, y_2, D' can be constructed in time $O(n^2)$, where n is the number of vertices of D.*

Proof: If D has the desired paths, then so does any digraph obtained from D by adding arcs. Hence if D has the desired paths, then trivially D' exists and can be constructed in time $O(n^2)$ once we know a pair of disjoint (x_1, y_1)-, (x_2, y_2)-paths.

If no S_i contains both of x_1, y_1 or both of x_2, y_2, then it is easy to see that D has the desired paths if and only if it has such paths which do not use an arc inside any S_j. Thus in this case we can add arcs arbitrarily inside each S_i to obtain a D' which satisfies the requirement.

Suppose next that some S_i contains all of the vertices x_1, x_2, y_1, y_2. If there is an (x_j, y_j)-path P in $S_i - \{x_{3-j}, y_{3-j}\}$, $j \in \{1, 2\}$, then it follows from that fact that F is strong that D has the desired paths and we can find such a pair in time $O(n^2)$. Thus, by our initial remark, we may assume that there is no (x_j, y_j)-path P in $S_i - \{x_{3-j}, y_{3-j}\}$ for $j \in \{1, 2\}$. Now it is easy to see that D has the desired paths if and only if it has such paths which do not use an arc inside any S_j. Thus we can replace S_i by a tournament in which x_1 and x_2 both have no out-neighbours in $S_i - \{x_1, x_2\}$ and every other S_k by an arbitrary tournament on the same vertex set. Clearly the digraph D' obtained in this way satisfies the requirement.

Suppose now without loss of generality that $x_1, y_1 \in V(S_j)$ for some j but $x_2 \notin V(S_j)$. Suppose first that $y_2 \in V(S_j)$. If there is no (x_1, y_1)-path in $S_j - y_2$, then D has the desired paths if and only if it has such paths which do not use an arc inside any S_i and we can construct D' by adding arcs in S_j in such a way that no (x_1, y_1)-path avoiding y_2 is created (that is, y_2 will still separate x_1 from y_1 in $D'[V(S_j)]$) and arbitrary arcs in every other S_i. On the other hand, if $S_j - y_2$ contains an (x_1, y_1)-path avoiding y_2, then it follows from the fact that F is strong that D has the desired paths and hence D' exists, as remarked above. Hence we may assume that $y_2 \notin V(S_j)$.

If S_j contains an (x_1, y_1)-path which does not cover all the vertices of S_j, then it follows from the fact that F is strong that D has the desired paths. Thus we may assume that either S_j has no (x_1, y_1)-path, or every (x_1, y_1)-path in S_j contains all the vertices of S_j. In the last case we may assume that $V(S_j)$ separates x_2 from y_2. Now D has the desired paths if and only if it has such a pair which does not use any arcs from S_j. Thus in both cases we can construct D' by replacing S_j by a tournament with no (x_1, y_1)-path and every other S_i by an arbitrary tournament on the same vertex set, except in the case when x_2 and y_2 belong to some S_i, $i \neq j$. In this case we replace that S_i by a tournament with no (x_2, y_2)-path (by the remark above we may assume that S_i has no (x_2, y_2)-path).

It follows from the considerations above that D' can be constructed in time $O(n^2)$. □

Recall that Theorem 8.3.5 gives the canonical decomposition for quasi-transitive digraphs. Hence we can apply Theorem 8.5.1 to these digraphs.

Theorem 8.5.2 ([5]) *There exists a polynomial-time algorithm for the* 2-Disjoint paths *problem for quasi-transitive digraphs.*

Proof: Let D be a quasi-transitive digraph and x_1, x_2, y_1, y_2 specified distinct vertices for which we want to determine the existence of vertex-disjoint (x_1, y_1)-,(x_2, y_2)-paths. First check that $D - \{x_i, y_i\}$ contains an (x_{3-i}, y_{3-i})-path for $i \in \{1, 2\}$. If not, then we stop. Now it follows from Theorem 8.3.5 that either x_1, x_2, y_1, y_2 are all in the same strong component of D, or the paths exist. For example, if D is not strong and y_1, say, is not in the same strong component as x_1 then, by Theorem 8.3.5, x_1 and y_1 belong to different sets W_i, W_j in the canonical decomposition $D = Q[W_1, \dots, W_{|Q|}]$, where Q is a transitive digraph. Hence $x_1 \rightarrow y_1$ and the desired paths clearly exist.

Thus we may assume that D is strong. Let $D = S[W_1, W_2, \dots, W_{|S|}]$ be the canonical decomposition of D. Now apply Theorem 8.5.1 and construct the digraph D' which has the desired paths if and only if D does. As remarked in Theorem 8.5.1, D' can be constructed in polynomial time. By the construction of D' (replacing each W_i by a semicomplete digraph) it follows that D' is a semicomplete digraph and hence we can apply the polynomial algorithm of Theorem 2.5.6 to D' in order to decide the existence of the desired paths in D. The algorithm of Theorem 2.5.6 can be used to find vertex-disjoint (x_1, y_1)-, (x_2, y_2)-paths in D' if they exist and given these paths it is easy to construct the corresponding paths in D (it suffices to take minimal paths). □

By inspecting the proof of Theorem 8.5.1 it is not difficult to see that the following much more general result is true. The main point is that in the proof of Theorem 8.5.1 we either find the desired paths or decide that they exist if and only if there are such paths that use no arcs inside any S_i. Hence instead of making each T_i semicomplete, we may just as well make it an independent set, by deleting all arcs inside S_i.

Theorem 8.5.3 ([5]) *Let Φ be a class of strongly connected digraphs, let Φ^{ext} denote the class of all extensions of graphs in Φ and let*

$$\Phi^* = \{F[D_1, \ldots, D_{|F|}] : F \in \Phi,\ \textit{each } D_i \textit{ is an arbitrary digraph}\}.$$

There is a polynomial algorithm for the 2-Disjoint paths *problem in Φ^* if and only if there is a polynomial algorithm for the* 2-Disjoint paths *problem for all digraphs in Φ^{ext}.* □

This result shows that studying extensions of digraphs can be quite useful.

One example of such a class Φ, for which Theorem 8.5.3 applies, is the class of strong semicomplete digraphs. This follows from the fact that we can reduce the 2-Disjoint paths problem for extended semicomplete digraphs to the case of semicomplete digraphs in the same way as we did for quasi-transitive digraphs in the proof of Theorem 8.5.2. Hence the 2-Disjoint paths problem is polynomially solvable for all digraphs that can be obtained from strong semicomplete digraphs by substituting arbitrary digraphs for vertices. It is important to note here that Φ must consist only of strong digraphs, since it is not difficult to reduce the 2-Disjoint paths problem for arbitrary digraphs (which is $\mathcal{NP}$-complete) to the 2-Disjoint paths problem for those digraphs that can be obtained from the digraph H consisting of just an arc uv by substituting arbitrary digraphs for the vertex v.

The proof of the following easy lemma is left to the reader. Note that four is the best possible, as can be seen from the complete biorientation of the undirected graph consisting of a 4-cycle $x_1x_2y_1y_2x_1$ and a vertex z joined to each of the four other vertices.

Lemma 8.5.4 *Let D be a digraph of the form $D = \vec{C}_2[S_1, S_2]$, where S_i is an arbitrary digraph on n_i vertices, $i = 1, 2$. If D is 4-strong, then D is 2-linked.* ◇

Theorem 2.5.1 gives a sufficient condition for a semicomplete digraph to be 2-linked in terms of its strong connectivity. The same condition turns out to be sufficient for quasi-transitive digraphs.

Before proving our final results of this subsection, we will be needing a structural theorem regarding k-strong digraphs due to Bang-Jensen.

Lemma 8.5.5 ([5]) *Let $D = F[S_1, \ldots, S_f]$ where F is a strong digraph on $f \geq 2$ vertices, each S_i is a digraph with n_i vertices, and F has as few vertices as possible among all non-trivial decompositions of D of this kind. Let $D_0 = F[\overline{K}_{n_1}, \ldots, \overline{K}_{n_f}]$ be the digraph obtained from D by deleting every arc which lies inside some S_i, and let S be a minimal (with respect to inclusion) separating set of D_0. Then S is also a separating set of D, unless each of the following holds:*

(a) $S = \bigcup_{j \neq i} V(S_j)$ *for some* $1 \leq i \leq f$,
(b) $D[S_i]$ *is a strong digraph, and*
(c) $D = \vec{C}_2[S, S_i]$.

In particular, if F has at least three vertices, then D is k-strong if and only if D_0 is k-strong.

Theorem 8.5.6 ([5]) *Let $k \geq 4$ be a natural number and let F be a digraph on $f \geq 2$ vertices with the property that every k-strongly connected digraph of the form $F[T_1, T_2, \ldots, T_f]$, where each T_i, $i \in [f]$, is a semicomplete digraph, is 2-linked. Let $D = F[S_1, S_2, \ldots, S_f]$, where S_i is an arbitrary digraph on n_i vertices for all $i \in [f]$. If D is k-strongly connected, then D is 2-linked.*

Proof: Let $D = F[S_1, S_2, \ldots, S_f]$, where S_i is an arbitrary digraph on n_i vertices for each $i \in [f]$, be given. By Lemma 8.5.4 we may assume that D cannot be decomposed as $D = \vec{C}_2[R_1, R_2]$, where R_1 and R_2 are arbitrary digraphs. Construct D' as described in Theorem 8.5.1. Note that by Lemma 8.5.5, $\kappa(D') = \kappa(D)$. Thus D' is k-strong and using Theorem 8.5.1 and the assumption of the theorem we conclude that D is 2-linked. □

Corollary 8.5.7 ([5]) *Every 5-strong quasi-transitive digraph is 2-linked.*

Proof: By Theorem 8.3.5, every strong quasi-transitive digraph is of the form $D = F[S_1, S_2, \ldots, S_f]$, $f = |F|$, where F is a strong semicomplete digraph and each S_i is a non-strong quasi-transitive digraph on n_i vertices. By Lemma 8.3.4 and the connectivity assumption, $|F| \geq 3$. Note that for any choice of semicomplete digraphs $T_1, \ldots, T_f$ the digraph $D' = F[T_1, T_2, \ldots, T_f]$ is semicomplete. Hence the claim follows from Theorem 8.5.6 and the fact that, by Theorem 2.5.12, every 5-strong semicomplete digraph is 2-linked. (Since F has at least three vertices, it follows from Lemma 8.5.5 that $\kappa(D') = \kappa(D)$.) □

8.5.1 *k*-Linkages

As mentioned at the beginning of this section, since the k-Disjoint paths problem is already $\mathcal{NP}$-complete for $k = 2$, the restriction of this problem to particular classes of digraphs has been studied by many authors. It turns out that, for some families, the problem can be solved in polynomial time when k is fixed. For example, consider Theorems 3.4.1, 2.5.7, and 2.5.11.

Recall that a digraph D is decomposable if there exist a digraph R on r vertices, and distinct (but possibly isomorphic) digraphs $L_1, \ldots, L_r$, such that $D = R[L_1, \ldots, L_r]$. In this section we will study the k-Disjoint paths problem in decomposable digraphs. As a consequence, we will obtain polynomial algorithms to solve the k-Disjoint paths problem in quasi-transitive digraphs and extended semicomplete digraphs. The results of this section are due to Bang-Jensen, Christiansen, and Maddaloni [7].

Let $D = S[M_1, ..., M_s]$ be a decomposable digraph and let P be a path in D. We say that P is ***D*-internal** if $P \subseteq M_i$ for some i, and we say that P is ***D*-external** otherwise. When D is clear from the context we just call the path **internal** or **external**. Similarly we say that a pair $(s, t) \in V(D) \times V(D)$ is internal if $s, t \in V(M_i)$ for some i, and is external otherwise.

Let $\Pi = \{(s_1, t_1), ..., (s_k, t_k)\}$ be a set of k pairs of distinct terminals. A **Π-linkage** is a collection L of k disjoint paths P_i, $i \in [k]$, such that P_i is an (s_i, t_i)-path. If a Π-linkage L exists in the digraph D we say that L is a linkage for (D, Π)

Lemma 8.5.8 ([7]) *Let $D = S[M_1, ..., M_s]$ be a decomposable digraph and Π a set of pairs of terminals. Then (D, Π) has a linkage if and only if it has a linkage whose external paths do not use any arc of $D[M_i]$ for $i \in [s]$.* □

Let D be a digraph with vertex set $v_1, v_2, \ldots, v_n$ and let K be another digraph. By **blowing up $\mathbf{v_i}$ into K in D** we mean the operation that substitutes the digraph K for the vertex v_i in D, that is, creates the digraph $D' = D[\{v_1\}, \ldots, \{v_{i-1}\}, K, \{v_{i+1}\}, \ldots, \{v_n\}]$. We say that a class of digraphs Φ is **closed with respect to blow-up** if for any $D \in \Phi$, for every integer m and for every $v \in V(D)$, there exists a digraph K on m vertices such that the blowing up of v into K results in a digraph which is still in Φ.

Lemma 8.5.9 ([7]) *If the class Φ is closed with respect to the blowing-up operation, $S \in \Phi$ and $D = S[M_1, ..., M_s]$, then it is possible to replace the arcs inside each M_i, $i \in [s]$, with other arcs, so that the resulting digraph is in Φ.* □

We say that a class of digraphs Φ is **a linkage ejector** if

1. There exists a polynomial algorithm $\mathcal{A}_\Phi$ to find a total Φ-decomposition of every totally Φ-decomposable digraph.
2. There exists a polynomial algorithm $\mathcal{B}_\Phi$ for solving the k-Disjoint paths problem on Φ. The running time depends (possibly exponentially) on k but the algorithm is polynomial when k is fixed.

3. The class Φ is closed with respect to blow-up and there exists a polynomial algorithm $\mathcal{C}_\Phi$ wich given a totally Φ-decomposable digraph $D = S[M_1, ..., M_s]$, constructs a digraph of Φ by replacing the arcs inside each of the M_i's, as in Lemma 8.5.9.

Theorem 8.5.10 *Let Φ be a linkage ejector. For every fixed k, there exists a polynomial algorithm to solve the k-*DISJOINT PATHS *problem on totally Φ-decomposable digraphs.* □

Recall that, by Theorem 8.3.5, quasi-transitive digraphs are totally Φ_3-decomposable. The following result of Bang-Jensen, Christiansen, and Maddaloni deals with this class of digraphs, which also includes, for example, extended semicomplete digraphs.

Lemma 8.5.11 ([7]) *The class Φ_3 is a linkage ejector.* □

We thus obtain the following corollary of Theorem 8.5.10

Theorem 8.5.12 *For every fixed k, there exists a polynomial algorithm to solve the k-*DISJOINT PATHS *problem on quasi-transitive digraphs and extended semicomplete digraphs.*

8.5.2 Weak k-Linkages

Note that for this subsection we allow both parallel arcs and loops and (for simplicity) we still use the name digraph rather than directed pseudograph.

Let $D = (V, A)$ be a digraph and let $s_1, \ldots, s_k, t_1, \ldots, t_k$ be a collection of (not necessarily distinct) vertices of D. A **weak k-linkage** from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$ is a collection of k arc-disjoint paths $P_1, \ldots, P_k$ such that, for each $i \in [k]$, P_i is an (s_i, t_i)-path if $s_i \neq t_i$ and a proper cycle containing s_i if $s_i = t_i$.

WEAK k-LINKAGE
Input: A digraph $D = (V, A)$ and not necessarily distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$.
Question: Does D contain a weak k-linkage from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$?

It is well-known that the WEAK k-LINKAGE problem is $\mathcal{NP}$-complete already when $k = 2$ [30].

Until recently, results regarding the WEAK k-LINKAGE problem were limited, both in number and depth. In Section 3.4, the case of acyclic digraphs is discussed, and Section 2.5 presents a brief evolution of this problem, with an obvious emphasis on the class of semicomplete digraphs. In particular, the results of Fradkin and Seymour found in [31] (and included in Section 2.5) mark a turning point in the scope of families for which nice results can be obtained. In this section we present the results obtained by Bang-Jensen and Maddaloni in [19] regarding the WEAK k-LINKAGE problem on decomposable digraphs. We begin with a result which is implicitly stated in [31]. See Section 2.5 for the definition and a brief discussion of the concept of cutwidth.

Theorem 8.5.13 (Fradkin–Seymour [31]) *For every natural number θ the* WEAK k-LINKAGE *problem is polynomial for every fixed k, when we consider digraphs with cutwidth at most θ.*

The following easy consequence will be used in our algorithms.

Theorem 8.5.14 *For every natural number p the* WEAK k-LINKAGE *problem is polynomial, for every fixed k, when we consider digraphs with at most p directed cycles.*

Proof: Let D be a digraph with at most p directed cycles. Then the cutwidth of D is at most p: we may delete an arbitrary arc from each of the at most p cycles to get a digraph with cutwidth 0, so D has cutwidth at most p. Now the claim follows from Theorem 8.5.13. □

Assume we want to decide the existence of a weak k-linkage from the vertices $(s_1, ..., s_k)$ to the vertices $(t_1, ..., t_k)$. We will denote by Π the list of pairs[5] $(s_1, t_1), \ldots, (s_k, t_k)$. In the rest of this subsection we will think of Π both as a list of k pairs and as a collection of all the terminals $s_1, \ldots, s_k, t_1, \ldots, t_k$.

We say that D has a weak Π-linkage if it contains a weak k-linkage from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$. We sometimes also say that (D, Π) has a weak linkage.

As in the previous subsection, we will use the term **blow up** of v_i into a digraph K (in D) meaning the composition $D[v_1, ..., v_{i-1}, K, v_{i+1}, ..., v_n]$.

Recall that we allow multiple arcs (and loops) in our digraphs, and also that $\mu_D(u, v)$ denotes the number of arcs from a vertex u to a vertex v. We will assume throughout the rest of this subsection, unless otherwise stated, that k denotes the number of pairs to be linked. An instance of the problem (D, Π) is equivalent to (D', Π) where $V(D') = V(D)$ and for every $u, v \in V(D')$ one has $\mu_{D'}(u, v) = \min(\mu_D(u, v), k)$. Therefore from now on **we will only consider digraphs D with**

$$\mu_D(u, v) \leq k \quad \forall u, v \in V(D)$$

while studying the WEAK k-LINKAGE problem.

Let $D = (V, A)$ be a digraph and H an induced subdigraph of D. We say that H is a **module** if for every $a, b \in V(H)$, $v \in V(D \setminus H)$ we have that $\mu_D(v, a) = \mu_D(v, b)$ and $\mu_D(a, v) = \mu_D(b, v)$. We say that H is a **clean**

[5] Note that the same pair (or the same vertex) may appear more than once in the list and we may have $s_i = t_i$.

module with respect to Π if it is a module containing no terminals of Π. The concept of module yields an alternative definition of a decomposable graph. A digraph D is **decomposable** if $D = S[H_1, ..., H_s]$, for some digraph S, with $s = |V(S)| \geq 2$ and some choice of disjoint modules $H_1, ..., H_s$. In this case S is called the **quotient digraph** (of D) induced by $H_1, ..., H_s$.

The algorithms developed in this subsection rely on the following fundamental fact, the proof of which we will omit: a weak linkage need not use any arc inside clean modules. As mentioned earlier, the results of this section are due to Bang-Jensen and Maddaloni.

Lemma 8.5.15 ([19]) *Let D be a digraph, Π a list of k terminal pairs and $H \subset D$ a clean module with respect to Π. Let D' be the contraction of H into a single vertex h. Then D has a weak Π-linkage if and only if D' has a weak Π-linkage.* □

The following result is an immediate consequence of the proof of Lemma 8.5.15 (see [19]).

Lemma 8.5.16 ([19]) *Let Π be a list of terminal pairs and $H \subset D$ be a clean module with respect to Π. For every weak linkage $P'_1, ..., P'_k$ of (D, Π), there exists another weak linkage $P_1, ..., P_k$ such that $P'_i = P_i$ on $D \setminus H$, and for $i = 1, ..., k$, $A(P_i \cap H) = \varnothing$.*

As in the previous subsection, given a decomposable digraph $D = S[H_1, ..., H_s]$ and a path P we say that P is **internal** if $P \subseteq H_j$ for some H_j, and we say that P is **external** otherwise.[6]

Similarly, we say that a pair (s, t) is an **internal pair** if $s, t \in H_j$ for some j, and we say that (s, t) is an **external pair** otherwise.

If a module H is not clean, i.e. it contains terminals, then some of the arcs in $A(H)$ may be necessary to guarantee a weak linkage. See Figure 8.4. The following lemma shows that, in a precise sense, a weak linkage need not use too many arcs inside a given module. Together with Lemma 8.5.16, this will allow a polynomial brute-force algorithm (Theorem 8.5.19).
For technical reasons that will become clear later, we consider the more general case where a set of arcs F has been deleted from D.

Lemma 8.5.17 ([19]) *Let $D = S[H_1, ..., H_s]$ be a decomposable digraph, let Π' be a list of h terminal pairs and let F be a set of arcs in D satisfying $d^-_F(v), d^+_F(v) \leq r$ for all $v \in V(D)$. If $(D \setminus F, \Pi')$ has a weak linkage, then it has a weak linkage $P_1, ..., P_h$ such that we have $|V(\bigcup_{i\in\mathcal{E}} P_i \cap H_j)| \leq 2h(h+r)$, for every $j \in \{1, ..., s\}$, where $\mathcal{E}$ denotes the set of indices i for which P_i is external.*

Note that from the previous proof we have that for every $j \in \{1, \ldots, s\}$ and every $i \in \mathcal{E}$, $|A(P_i \cap H_j)| < 2(h + r)$.

---

[6] Note that an external path may still start and end in the same module H_j.

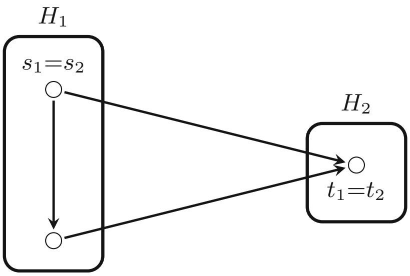

Figure 8.4 An example with $|\Pi| = 2$, the only weak Π-linkage uses the arc inside H_1.

Lemma 8.5.18 ([19]) *Let $\mathcal{C}$ be a class of digraphs for which there exists an algorithm $\mathcal{A}$ to decide the* WEAK k-LINKAGE *problem, whose running time is bounded by $f(n,k)$. Let $D = (V,A)$ be a digraph, Π a list of k pairs of terminals and $F \subseteq V \times V$ such that $D' := (V, A \cup F)$ is a member of $\mathcal{C}$. There exists an algorithm $\mathcal{A}^-$, whose running time is bounded by $f(n, k + |F|)$, to decide whether D has a weak Π-linkage.*

Proof: Suppose $F = \{s'_1 t'_1, ..., s'_{k'} t'_{k'}\}$, where $k' = |F|$, and let $\Pi' = (s'_1, t'_1), ..., (s'_{k'}, t'_{k'})$. D has a weak Π-linkage if and only if D' has a weak $(\Pi \cup \Pi')$-linkage, from which the claim follows. Indeed, if D has a weak Π linkage, then this extends to a weak $(\Pi \cup \Pi')$-linkage of D' by simply taking the arcs $s'_i t'_i$ as (s'_i, t'_i)-paths. If D' has a weak $\Pi \cup \Pi'$-linkage L, it is easy to see that $A(L) \setminus F$ contains a weak Π-linkage of D. □

Given a digraph D and a non-negative integer c, let $D(c)$ denote the set of digraphs that can be obtained from D by first adding any number of arcs parallel to the already existing ones and then blowing up b vertices, with $0 \le b \le c$, to digraphs of size less than or equal to c each. We say that a class of digraphs Φ is **bombproof** if there exists a polynomial algorithm $\mathcal{A}_\Phi$ to find a total Φ-decomposition of every totally Φ-decomposable digraph and, for every integer c, there exists a polynomial algorithm[7] $\mathcal{B}_\Phi$ to decide the WEAK k-LINKAGE problem for the class

$$\Phi(c) := \bigcup_{D \in \Phi} D(c).$$

The following theorem of Bang-Jensen and Maddaloni is the main result in [19].

Theorem 8.5.19 ([19]) *Let Φ be a bombproof class of digraphs. There is a polynomial algorithm $\mathcal{M}$ which takes as input a 5 tuple $[D, k, k', \Pi, F]$, where D is a totally Φ-decomposable digraph, k, k' are natural numbers with $k' \le k$, Π is a list of k' terminal pairs and $F \subseteq A(D)$ is a set of arcs satisfying*

[7] Note that the running time of $\mathcal{B}_\Phi$ may depend heavily on c.

$$d_F^-(v), d_F^+(v) \leq k - k' \text{for all } v \in V(D).$$
$$|F| \leq (k - k')2k$$

and decides whether $D \setminus F$ contains a weak Π-linkage. □

For the sake of brevity we will omit the proof of Theorem 8.5.19. Nonetheless, we present a description for the proposed polynomial algorithm $\mathcal{M}$.

1. If $\Pi = \varnothing$ output that a solution exists and return.
2. Run $\mathcal{A}_\Phi$ to find a total Φ-decomposition of $D = S[H_1, ..., H_s]$.
3. If this decomposition is trivial, that is $D = S$, then $D \in \Phi \subseteq \Phi(1)$, so run $\mathcal{B}_\Phi^-$ on $(D \setminus F, \Pi)$ to decide the problem and return.
4. Find among $H_1, ..., H_s$ those modules $K_1, ..., K_l$ that contain at least one terminal. Let D' be obtained by contracting all the modules distinct from $K_1, ..., K_l$. Let F' be the set of arcs obtained from F after the contraction.
5. Let $\Pi^e \subseteq \Pi$ ($\Pi^i \subseteq \Pi$) be the list of external (internal) pairs (s_q, t_q) in Π.
6. For every partition of $\Pi^i = \Pi_1 \cup \Pi_2$ look for external paths linking the pairs in $\Pi^e \cup \Pi_1$ and internal paths linking the pairs in Π_2. This is done in the following way:

   a) If $\Pi^e \cup \Pi_1 = \varnothing$, then for $i = 1, ..., l$: run $\mathcal{M}$ recursively on input $[K_i, k, k_i', \Pi \cap K_i, F \cap A(K_i)]$, where $\Pi \cap K_i$ denotes the list of terminal pairs that lie inside K_i and k_i' is the number of those pairs.
   b) If $\Pi^e \cup \Pi_1 \neq \varnothing$, let k_i' be the number of pairs in $\Pi_2 \cap K_i$. We do the following for each possible choice of l vertex sets $W_i \subseteq V(K_i)$, $i = 1, ..., l$, of size $\min\{|V(K_i)|, 2(k' - k_i')(k - k')\}$ and arc sets[8] $F_i \subseteq A(K_i[W_i]) \setminus F$, $i = 1, ..., l$, with F_i satisfying

   $$d^-_{(F \cap A(K_i)) \cup F_i}(v), d^+_{(F \cap A(K_i)) \cup F_i}(v) \leq k' - k_i'.$$
   $$|F_i| \leq 2(k' - k_i')(k - k').$$

   - For every module K_i remove all the vertices of $V(K_i) \setminus W_i$ and then all remaining arcs except those in F_i.
   - Define D'' to be the digraph obtained from D' with this procedure.
   - Run $\mathcal{B}_\Phi^-$ on $(D'' \setminus F', \Pi^e \cup \Pi_1)$.
   - For $i = 1, ..., l$, run $\mathcal{M}$ recursively on input $[K_i, k, k_i', \Pi_2 \cap K_i, (F \cap A(K_i)) \cup F_i]$.

   If at step 6(a) all the instances examined are linked or at step 6(b), there is a choice of $W_i, F_i,\ i = 1, ..., l$, such that all instances examined are linked, then output that a weak linkage exists and return.
7. If all choices of Π_1, Π_2 have been considered, without verifying the existence of any weak linkage, then output that no weak linkage exists.

[8] $K_i[W_i]$ is the subdigraph of K_i induced by W_i.

Taking $k' = k$ and running the previous algorithm on input $[D, k, k, \Pi, \varnothing]$ where D is any totally Φ-decomposable digraph and Π is a list of k terminal pairs from $V(D)$, we obtain the main result of this subsection.

Theorem 8.5.20 *Let Φ be a bombproof class of digraphs. For every fixed k there exists a polynomial algorithm for the* WEAK k-LINKAGE *problem for the totally Φ-decomposable digraphs.*

Based on the recursive structure given by the canonical decomposition for quasi-transitive digraphs (Theorem 8.3.5), Bang-Jensen and Maddaloni proved that there is a polynomial algorithm for the WEAK k-LINKAGE problem on quasi-transitive digraphs [19]. Recall that Theorem 8.3.5 can be restated to say that quasi-transitive digraphs are totally Φ_3-decomposable.

Lemma 8.5.21 *The class Φ_3 is bombproof.*

Proof: We can get a polynomial algorithm for the total Φ_3-decomposition from Theorem 8.3.27. Given a positive integer c and a digraph $D \in \Phi_3$, consider a digraph in $D' \in D(c)$: if D is semicomplete, then D' misses no more than c^3 arcs to be semicomplete. If D is acyclic, then D' has at most $O(c^{c+1})$ cycles or $O(c \cdot (ck)^c)$ if there are (at most k) parallel arcs, because all the cycles must lie in one of the blown up subdigraphs. By Theorem 2.5.5 and Lemma 8.5.18 in the first case and Theorem 8.5.14 in the second case, there is a polynomial algorithm to decide the WEAK k-LINKAGE problem in $D(c)$ and hence in $\Phi_3(c)$. Thus we can conclude that Φ_3 is bombproof. □

Theorem 8.5.22 *For every fixed k there exists a polynomial algorithm for the* WEAK k-LINKAGE *problem for quasi-transitive digraphs.*

Proof: It follows from Theorem 8.3.5 that quasi-transitive digraphs are totally Φ_3-decomposable. By Lemma 8.5.21 Φ_3 is bombproof, hence we can apply Theorem 8.5.20. □

We can apply Theorem 8.5.20 to another class of digraphs; extended semicomplete digraphs are clearly totally Φ_3-decomposable. Hence, from Theorem 8.5.20, we have the following

Theorem 8.5.23 *For every fixed k there exists a polynomial algorithm for the* WEAK k-LINKAGE *problem for extended semicomplete digraphs.* □

8.6 Kings and Kernels

The existence of k-kings was one the first problems to be explored for quasi-transitive digraphs. As a matter of fact, the concept of k-king was first introduced in [16] for the purpose of studying quasi-transitive digraphs. In families of digraphs closed under the reversal of every arc, like quasi-transitive digraphs, the study of k-kings is closely related to the study of $(k+1)$-kernels: a k-king in the reversal of D is a $(k+1)$-kernel of D.

After spending some years dormant, this subject has received a lot of attention lately. Surprisingly, many of the nice existing results for kings in quasi-transitive digraphs admit natural generalizations to k-quasi-transitive digraphs.

8.6.1 Kings

A k-**king** in a digraph D is a vertex u such that $d(u,v) \leq k$ for every $v \in V(D) - u$ (it is a k-**dominating vertex**). A **king** is a 2-king. The study of kings in digraphs began with the mathematical sociologist Landau, who proved that every vertex of maximum out-degree in a tournament is a king, [53] (see Theorem 2.2.12). Nonetheless, the term king was introduced by Maurer in [54], where he used tournaments to model dominance in flocks of chickens. Some of the classical results on k-kings in digraphs can be consulted in [9], and Section 2.2 includes the most relevant results for tournaments.

Most of the main results in this section rely on several technical lemmas, so we prefer to omit them for the sake of presentation.

In [16], Theorem 8.3.5 is used extensively by Bang-Jensen and Huang to prove the first results on the existence and number of 3-kings in quasi-transitive digraphs. The main results can be condensed in the following theorem.

Theorem 8.6.1 ([16]) *Let D be a quasi-transitive digraph. Then we have*

1. *D has a 3-king if and only if it has an out-branching.*
2. *If D has a 3-king, then the following holds:*
   a) *Every vertex in D of maximum out-degree is a 3-king.*
   b) *If D has no vertex of in-degree zero, then D has at least two 3-kings.*
   c) *If the unique initial strong component of D contains at least three vertices, then D has at least three 3-kings.*

Sketch of Proof. Clearly, the existence of an out-branching is necessary. To prove the converse, assume that D has an out-branching. This implies that D has a unique initial strong component. Since the strong components digraph of a quasi-transitive digraph is transitive, a vertex is a 3-king of D if and only if it is a 3-king of the unique initial component of D. So, we may assume that D is strong.

Let $D = S[Q_1, \ldots, Q_s]$ be the decomposition of D given by Theorem 8.3.5. Since S is semicomplete, every vertex of S belongs to a 3-cycle of S. Thus, for every $1 \leq i \leq s$, each vertex in every Q_i has distance at most 3 to every other vertex in Q_i. Assume without loss of generality that Q_1 corresponds to a vertex s_1 of maximum out-degree in S. Then s_1 is a 2-king in S, and hence every vertex in Q_1 is a 3-king of D.

Observe that the vertices of maximum out-degree in D must belong to the Q_i's corresponding to the vertices of maximum out-degree in D. If there are no vertices of in-degree zero in D, there are at least two vertices of maximum out-degree in S. □

From the previous argument it can also be observed that D has a 2-king if and only if $|V(Q_i)| = 1$ for some Q_i corresponding to a 2-king of S. Also, similar argumentation leads to other, more specific, results regarding the distribution of 3-kings in a quasi-transitive digraphs. As an example consider the following proposition from [16]; a **non-king** is a vertex which is not a 3-king.

Proposition 8.6.2 *Let D be a quasi-transitive digraph which contains a 3-king but no vertex of in-degree zero. Every non-king is dominated by at least three 3-kings, unless the initial component of D is a 2-cycle, in which case every non-king is dominated by exactly two 3-kings.* □

After this first wave of results, most of the study of k-kings was restricted to multipartite tournaments for several years. It was not until 2012 that k-quasi-transitive digraphs were introduced in [48], and the following generalization of the first item of Theorem 8.6.1 was proved between [48] and [37] by Galeana-Sánchez, Hernández-Cruz and Juárez-Camacho.

Proposition 8.6.3 ([37, 48]) *Let $k \geq 2$ be an integer. If D is a k-quasi-transitive digraph, then D has a $(k+1)$-king if and only if it has a unique initial strong component.* □

Proposition 8.6.3 was then the starting point for studying kings in k-quasi-transitive digraphs. Further generalizations to Theorem 8.6.1 were obtained, but also some strengthenings. Recall that we know exactly when a quasi-transitive digraph has a 2-king; a similar situation was described by Wang and Meng for k-quasi-transitive digraphs.

Theorem 8.6.4 ([60]) *Let $k \geq 4$ be an integer. If D is a k-quasi-transitive digraph, then D has a k-king if and only if it has a unique strong component which is not isomorphic to an extended $(k+1)$-cycle $\vec{C}[E_0, \ldots, E_k]$, where each E_i is an independent set on at least two vertices.* □

Now that we know exactly when a k-king exists, it is natural to ask for the minimum number of k-kings in a k-quasi-transitive digraph. The following theorem of Wang and Zhang deals with this question.

Theorem 8.6.5 ([63]) *Let $k \geq 5$ be an integer, and let D be a strong k-quasi-transitive digraph with at least two vertices. If D is not isomorphic to an extended $(k+1)$-cycle, then D has at least two k-kings.* □

It should be noted that this is the best possible result in terms of the number of k-kings in a k-quasi-transitive digraph. Consider the digraph

$H = \vec{C}_{k+1}[\{x_0\}, \{x_1\}, E_2, E_3, \dots, E_k]$, where E_i is an independent set with at least two vertices. Let D be obtained from H by adding the arc x_1x_0. Clearly D is a k-quasi-transitive digraph and it is not isomorphic to an extended $(k+1)$-cycle. It is not difficult to check that there are exactly two k-kings in D, namely, x_0 and x_1.

Given the previous discussion, it is natural to give further consideration to $(k+1)$-kings in k-quasi-transitive digraphs. Unfortunately, unlike the case of quasi-transitive digraphs, it is not true that every vertex of maximum out-degree in a k-quasi-transitive digraph is a $(k+1)$-king. As noted in [37], the only vertex of maximum out-degree in the 4-transitive digraph with vertex set $\{v_1, v_2, v_3, v_4\}$, and arc set $\{v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_4v_3\}$, is not a 5-king. Nonetheless, there are some simple conditions that will ensure that every vertex of maximum out-degree in a k-quasi-transitive digraphs is a $(k+1)$-king. The following condition was given by Wang and Zhang in [63]; recall that a k-king u in a digraph D is *strict* if there exists a vertex v such that $d(u, v) = k$.

Theorem 8.6.6 ([63]) *Let $k \geq 2$ be an integer and let D be a k-quasi-transitive digraph.*

1. *If D is strong, then every vertex of maximum out-degree in D is a $(k+1)$-king.*
2. *If D has a strict $(k+1)$-king, then every vertex of maximum out-degree in D is a $(k+1)$-king.*

□

As can be observed in Theorem 8.6.8, the number of $(k+1)$-kings in k-quasi-transitive digraphs can be very large, compared to the number of k-kings. The proof uses the following theorem regarding 4-kings in semicomplete bipartite digraphs, which can be obtained from the analogous result due to Koh and Tan, on bipartite tournaments (Theorem 7.12.2), [50].

Theorem 8.6.7 *Let D be a semicomplete bipartite digraph with a unique initial strong component. If there is no 3-king in D, then there are at least eight 4-kings in D.*

Let us point out that Theorem 8.6.7 was first stated by Wang in [63], and she cited [50] by Koh and Tan as the source of this result. Nonetheless, as mentioned above, Koh and Tan only proved this result for bipartite tournaments (Theorem 7.12.2). Wang does not give any argument in [63] to extend this result to semicomplete bipartite digraphs, so we will give a short one here.

Let D be a semicomplete bipartite digraph with only one initial strong component S, and without 3-kings. By Theorem 7.1.1, every strong component C of D contains a strong spanning subgraph C', which is a bipartite tournament. Replacing every strong component C by C' in D results in a bipartite tournament with a unique strong component and without a 3-king. We can now apply Theorem 7.12.2 to this bipartite tournament to obtain the desired eight 4-kings. When we put back the deleted arcs, we will still have eight 4-kings in D, and, since they are in an initial strong component, they are 4-kings of all of D.

The following result was proved by Galeana-Sánchez, Hernández-Cruz and Juárez-Camacho, for $k = 2$, and by Wang and Zhang for $k \geq 3$.

Theorem 8.6.8 ([37, 63]) *Let $k \geq 2$ be an integer, and let D be a k-quasi-transitive digraph. If there is no k-king in D, then the number of $(k+1)$-kings in D is at least $2k + 2$.*

Proof: It suffices to prove the result for strong digraphs. For $k = 2$, consider the decomposition $D = S[Q_1, \ldots, Q_s]$ given by Theorem 8.3.5. Since D is strong, the semicomplete digraph S is also strong, and thus, by Corollary 2.2.14, it has at least three 2-kings. Since D does not have 2-kings, it follows from the observation made after Theorem 8.6.1 that every Q_i corresponding to a 2-king of S has at least two vertices. Each of these vertices is a 3-king, and thus, D has at least six 3-kings.

The case $k = 3$ can be directly verified using Theorems 8.3.15 and 8.6.7. Finally, for $k \geq 4$, as D has no k-king, it must be isomorphic to an extended $(k+1)$-cycle, by Theorem 8.6.4. Every partite set in this cycle extension must have at least two vertices, otherwise there would be a k-king in D. Since every vertex of D is a $(k + 1)$-king, the number of $(k + 1)$-kings is at least $2k + 2$.
□

8.6.2 (k, ℓ)-Kernels

A **kernel** in a digraph D is an independent set K such that every vertex not in K dominates some vertex in K. Kernels in digraphs were introduced by von Neumann and Morgenstern while studying cooperative games [57]. Since then, digraph kernels have been studied in many contexts, including list colouring, game theory and graph perfectness [25], mathematical logic [22], and complexity theory [58].

There are many generalizations of this concept, one that has been widely studied and which relates to the kings from the previous subsection is the following. A subset K of $V(D)$ is **k-independent** if the distance between every pair of vertices of K is at least k, and it is **ℓ-absorbing** if for every vertex not in K, it reaches a vertex in K at distance at most ℓ; if $\ell = 1$, we simply say that K is **absorbing**. A **(k, ℓ)-kernel** in the digraph D is a k-independent and ℓ-absorbing subset of $V(D)$. A **k-kernel** is a $(k, k - 1)$-kernel, and thus, a 2-kernel is a kernel. The decision problem k-Kernel has an arbitrary digraph D as an input, and asks whether D has a k-kernel. When $k = 2$, the corresponding problem will be referred to only as Kernel.

Chvátal proved that KERNEL is $\mathcal{NP}$-complete [26]. Later Fraenkel proved that this problem remains $\mathcal{NP}$-complete even when restricted to planar digraphs with $\Delta \leq 3$, $\Delta^+, \Delta^- \leq 2$, [32]. Recently, Hell and Hernández-Cruz proved that it is also $\mathcal{NP}$-complete when restricted to digraphs with 3-colourable underlying graph (as opposed to the fact that every bipartite digraph has a kernel) [45]. Given the nice structure of quasi-transitive digraphs, it is not a surprise that members of this family having a kernel admit a simple characterization. One such characterization was given by Hell and Hernández-Cruz in [45].

Theorem 8.6.9 ([45]) *Let D be a strong quasi-transitive digraph. Then D has a kernel if and only if there is an absorbing vertex in D.*

Proof: We only prove the non-trivial implication. Let K be a kernel of D. Since K is independent, it follows from Lemma 8.3.4 that it must be contained in $V(S)$ for some connected component of $\overline{UG(D)}$. Recalling that D is strongly connected, there must be at least one connected component $S' \neq S$ of $\overline{UG(D)}$ such that $V(S) \to V(S')$. Since $K \subseteq S$, it must be the case that $V(S') \to V(S)$. Hence, Lemma 8.3.4 implies that $|V(S)| = 1$, and thus $|K| = 1$. If $K = \{v\}$, then v is an absorbing vertex of D. □

In [48] Galeana-Sánchez and Hernández Cruz observe that, in order for a k-quasi-transitive digraph D to have a k-kernel, it suffices to construct a k-kernel for every terminal strong component of D. In particular, this applies to kernels and quasi-transitive digraphs, and it allows us to conclude the following observation, which appears implicitly in [45].

Corollary 8.6.10 *Let D be a quasi-transitive digraph. Then D has a kernel if and only if every terminal strong component contains an absorbing vertex.*

Hence, we obtain a polynomial time algorithm for the problem KERNEL restricted to the class of quasi-transitive digraphs.

Corollary 8.6.11 *The problem* KERNEL *restricted to the class of quasi-transitive digraphs can be solved in polynomial time. Also, if a kernel exists, it can be constructed in polynomial time.*

Proof: Let $D = (V, A)$ be a digraph such that $|V| = n$ and $|A| = m$. The strong components digraph of D can be obtained in time $O(n + m)$ and it can have at most $O(n)$ terminal strong components. For every terminal component C, it can be verified in time $O(n + m)$ if an absorbing vertex exists: it suffices to construct the out-degree sequence of C. Hence, the kernel problem can be decided in time $O(n^2 + nm)$. If D has a kernel, it can be found in the same time. □

In order to obtain an analogous result for 3-kernels in 3-quasi-transitive digraphs, we need the following result of Hell and Hernández-Cruz.

Proposition 8.6.12 ([45]) *It can be determined in linear time whether a semicomplete bipartite digraph has a* 3*-kernel. Also, if a* 3*-kernel exists, it can be found in linear time.*

This suffices to obtain the desired result. The proof of the following theorem implicitly uses the structure given in Theorem 8.3.15. Recall that the digraph F_n has vertex set $\{x, y, z, v_1 \dots, v_n\}$, and its arcs are such that $xyzx$ is a directed cycle, and yv_iz is a directed path for every $1 \leq i \leq n$ (see Figure 8.3).

Theorem 8.6.13 ([45]) *The problem* 3-KERNEL *restricted to the class of* 3*-quasi-transitive digraphs can be decided in polynomial time. Also, if a* 3*-kernel exists, it can be constructed in polynomial time.*

Proof: Let $D = (V, A)$ be a digraph such that $|V| = n$ and $|A| = m$. The strong components digraph of D can be constructed in time $O(n + m)$ and it can have at most $O(n)$ terminal strong components. For every semicomplete bipartite terminal component, according to Proposition 8.6.12, it can be verified if it has a 3-kernel and, if so, a 3-kernel can be found, both in time $O(n + m)$. For each semicomplete terminal component, a 3-kernel (a 2-king) can be found in time $O(n + m)$. For every terminal component isomorphic to F_n, a 3-kernel can be constructed in constant time. Hence, the 3-kernel problem can be decided in time $O(n^2 + nm)$. If D has a 3-kernel, it can be found in the same time. □

In view of Corollary 8.6.11 and Theorem 8.6.13, the following natural question was stated by Hell and Hernández-Cruz in [45].

Problem 8.6.14 ([45]) *Is k-*KERNEL *polynomial time solvable for k-quasi-transitive digraphs?*

In [48] and [37], the existence of r-kernels for $r \geq k+2$ was proved for every k-quasi-transitive digraph by Galeana-Sánchez, Hernández-Cruz and Juárez-Camacho. It was also proved in [36] that every quasi-transitive digraph has an r-kernel for $r \geq 3$, and in [48] it was proved that every 3-quasi-transitive digraph contains a 4-kernel, so it was natural to conjecture the existence of a $(k + 1)$-kernel for every k-quasi-transitive digraph. This conjecture was later proved by Wang and Zhang [63].

Theorem 8.6.15 ([63]) *Let D be a k-quasi-transitive digraph with $k \geq 2$. Then D has a $(k + 1)$-kernel.*

Proof: We will only prove the case $k \geq 4$. For the cases $k \in \{2, 3\}$ we refer the reader to [36, 48].

Note that it suffices to choose $(k+1)$-kernels for every terminal component of D. To achieve this, consider the digraph $\overleftarrow{D}$ (called the **converse** of D) which is obtained from D by reversing every arc. Theorem 8.6.4 guarantees that every initial component of $\overleftarrow{D}$ either contains a k-king, or is isomorphic to an extended $(k+1)$-cycle. The k-kings in the initial components of $\overleftarrow{D}$ become $(k+1)$-kernels in the terminal components of D. Since the reversal of an extended $(k+1)$-cycle is again an extended $(k+1)$-cycle, and it is clear that any partite set of an extended $(k+1)$-cycle is a $(k+1)$-kernel for it, we can choose a $(k+1)$-kernel for every terminal component of D. □

A $(k+1)$-cycle is a k-quasi-transitive digraph without a k-kernel. Thus, Problem 8.6.14 asks whether the first integer r such that the r-Kernel problem is not trivial when restricted to k-quasi-transitive digraphs yields a polynomial time solvable r-Kernel problem. Notice that k-transitive digraphs always have a k-kernel, so, the first interesting kernel problem for k-transitive digraphs is $(k-1)$-Kernel. Regarding this problem, Hernández-Cruz characterized 3-transitive digraphs with a kernel [46].

Theorem 8.6.16 ([46]) *A 3-transitive digraph has a kernel if and only if none of its terminal components is isomorphic to a 3-cycle.*

Proof: Necessity is trivial to verify, we will only prove sufficiency. We will proceed by induction on the number of strong components of D. If D is strong, the result can be verified directly by exploring the possibilities in Theorem 8.3.19. Let D be a non-strong 3-transitive digraph, and let S be an initial component of D. By induction hypothesis, $D-S$ has a kernel N. If S is not a complete bipartite digraph, then either S consists of a single vertex, or it contains a subdigraph isomorphic to $\vec{C}_3$. In the former case, either the only vertex in S is absorbed by N, and we are done, or it is not, and we can add it to N to obtain a kernel for D. If S contains an isomorphic copy of $\vec{C}_3$, and since at least one vertex from S reaches at least one vertex from some initial component of D, say S', then Proposition 8.3.17 implies that $S \mapsto S'$. But $S' \cap N \neq \varnothing$, thus, every vertex of S is absorbed by N.

If $S=(X,Y)$ is a complete bipartite digraph, we will consider three cases. If neither X nor Y is absorbed by N, then $N \cup X$ is a kernel of D. If some vertex of X is absorbed by N, it follows from Proposition 8.3.17 that every vertex of X is absorbed by N. If Y is also absorbed by N, then N is a kernel of D. Else, none of the vertices of Y is absorbed by D, and thus, $N \cup Y$ is a kernel of D. □

Inspired by Theorem 8.6.16, Wang proved the following general result for strong k-transitive digraphs.

Theorem 8.6.17 ([59]) *Let D be a strong k-transitive digraph with $k \geq 4$. Then D has a $(k-1)$ kernel if and only if it is not isomorphic to a k-cycle.*

This, again observing Theorem 8.6.16, led to the following conjecture.

Conjecture 8.6.18 ([59]) *Let $k \geq 3$ be an integer. If D is a k-transitive digraph, then D has a $(k-1)$-kernel if and only if has no terminal strong component isomorphic to a k-cycle.*

In [38], García-Vázquez and Hernández-Cruz provided support to Conjecture 8.6.18 by proving it true for $k = 4$. Additionally, in the same paper the authors characterized 4-transitive digraphs having a kernel. The characterization relies heavily on a characterization of strong 4-transitive digraphs having a kernel, found in the same paper (see Subsection 8.3.3). It is trivial to observe that a k-kernel for a k-transitive digraph consists of a disjoint union of k-kernels for each of its terminal components. Conjecture 8.6.18 can be reformulated as follows: A k-transitive digraph D has a $(k-1)$-kernel if and only if each of its terminal components has a $(k-1)$-kernel. To prove the aforementioned characterization of 4-transitive digraphs with a kernel, the authors actually prove that a 4-transitive digraph has a kernel if and only if every terminal component has a kernel. So, the following questions come to mind.

Problem 8.6.19 *Let $k \geq 4$ be an integer and let D be a k-transitive digraph. Is it true that D has a $(k-2)$-kernel if and only if every terminal component of D has a $(k-2)$-kernel?*

If so, which is the least value of r for $2 \leq r \leq k-3$ such that D has an r-kernel if and only if every terminal component of D has an r-kernel?

An affirmative answer to the first question in Problem 8.6.19 would imply that it suffices to solve the $(k-2)$-kernel problem for strong k-transitive digraphs to obtain a solution for all k-transitive digraphs.

Finally, another particular case of (k, ℓ)-kernels is that of quasi-kernels. A **quasi-kernel** is simply a $(2, 2)$-kernel. Chvátal and Lovász proved that every digraph has a quasi-kernel. So, a question that has been raised by Gutin, Koh, Tay and Yeo [42] is the following: Which digraphs contain (at least) a pair of disjoint quasi-kernels? Clearly, a digraph which has a pair of disjoint quasi-kernels cannot contain vertices of out-degree zero, since every such vertex is included in every quasi-kernel. Unfortunately, this obvious necessary condition is not sufficient in general for a digraph to have a pair of disjoint quasi-kernels. Examples of digraphs which have neither vertices of out-degree zero nor a pair of disjoint quasi-kernels are given in [42]. Nonetheless, Heard and Huang proved that this condition is indeed sufficient in the class of quasi-transitive digraphs [44]. We need the following result, which is found in [44].

Proposition 8.6.20 ([44]) *Every semicomplete digraph D with no vertices of out-degree zero contains two vertices x, y such that $\{x\}$ and $\{y\}$ are both quasi-kernels of D.*

We begin with strong quasi-transitive digraphs.

Proposition 8.6.21 ([44]) *Every strong quasi-transitive digraph without vertices of out-degree zero contains a pair of disjoint quasi-kernels.*

Proof: Let D be a strong quasi-transitive digraph without vertices of out-degree zero. Let $D = S[H_1, \ldots, H_s]$ be the canonical decomposition of D (Theorem 8.3.5). Since D has no vertices of out-degree zero, S must contain at least two vertices and hence contain no vertices of out-degree zero. By Proposition 8.6.20, there are two vertices x, y such that $\{x\}$ and $\{y\}$ are both quasi-kernels of S. Suppose that H_i and H_j are the two digraphs which substitute x and y, respectively, in the composition. Let Q and Q' be quasi-kernels of H_i and H_j, respectively. Then, it is easy to see that Q and Q' are disjoint quasi-kernels of D. □

We now turn to the non-strong case.

Proposition 8.6.22 ([44]) *Every non-strong quasi-transitive digraph without vertices of out-degree zero contains a pair of disjoint quasi-kernels.*

Proof: Let D be a non-strong quasi-transitive digraph without vertices of out-degree zero. Let $D = T[H_1, \ldots, H_t]$ be the canonical decomposition of D (Theorem 8.3.5). Let $\{u_1, \ldots, u_t\}$ be the vertex set of T, and, without loss of generality, suppose that $u_1, \ldots, u_r$ are the sinks of T. Note that $\{u_1, \ldots, u_r\}$ is a kernel of T. Since D does not contain vertices of out-degree zero, neither do any H_i, $1 \leq i \leq r$. By Proposition 8.6.21, each H_i contains two disjoint quasi-kernels, say $Q_{i,1}$ and $Q_{i,2}$, $1 \leq i \leq r$. It is not hard to verify that $Q_1 = \bigcup_{i=1}^{r} Q_{i,1}$ and $Q_2 = \bigcup_{i=1}^{r} Q_{i,2}$ are disjoint quasi-kernels of D. □

Combining Propositions 8.6.21 and 8.6.22, we have the following:

Theorem 8.6.23 *Every quasi-transitive digraph without vertices of out-degree zero contains a pair of disjoint quasi-kernels.*

8.7 The Path Partition Conjecture

8.7.1 The Conjecture

Recall that a longest path in a digraph D is called a **detour** of D. The order of a detour of D is called the **detour order** of D and is denoted by $\mathsf{do}(D)$. The Gallai–Roy–Vitaver Theorem states that the chromatic number of the underlying graph of a digraph D is at most $\mathsf{do}(D)$. In 1982 Laborde, Payan and Xuong posed the following conjecture, which extends this theorem in a natural way.

Conjecture 8.7.1 ([52]) *Every digraph D contains an independent set X such that* $\mathsf{do}(D - X) < \mathsf{do}(D)$.

Conjecture 8.7.1 has proved to be a very difficult problem, and only a handful of partial results have been obtained. Nonetheless, it has not received as much attention as one of its generalizations. The following conjecture is probably the best known among the related path partition problems, it is known as the **Path Partition Conjecture**.

Conjecture 8.7.2 (Path Partition Conjecture) *[52] For every digraph D and every choice of positive integers ℓ_1, ℓ_2 such that $\mathsf{do}(D) = \ell_1 + \ell_2$, there exists a partition of D into two digraphs, D_1 and D_2, such that $\mathsf{do}(D_i) \leq \ell_i$ for $i \in \{1, 2\}$.*

Clearly, Conjecture 8.7.1 is the particular case of Conjecture 8.7.2 when $\ell_1 = 1$ and $\ell_2 = \mathsf{do}(D) - 1$.

A seemingly stronger version of the conjecture is stated in [24]. Bondy attributes it to Laborde *et al.* [52] although only the undirected version of Conjecture 8.7.2 is explicitly mentioned there.

Conjecture 8.7.3 ([24]) *For every digraph D and every choice of positive integers ℓ_1, ℓ_2 such that $\mathsf{do}(D) = \ell_1 + \ell_2$, there exists a partition of D into two digraphs, D_1 and D_2, such that $\mathsf{do}(D_i) = \ell_i$ for $i \in \{1, 2\}$.*

There is another problem also found in [52] which is a stronger version of Conjecture 8.7.1, but somehow this conjecture, sometimes referred to as the **Strong Laborde–Payan–Xuong Conjecture**, has received even less attention than Conjecture 8.7.1.

Conjecture 8.7.4 ([52]) *Every digraph D contains an independent set X such that that $\mathsf{do}(D - X) < \mathsf{do}(D)$, and has the additional property that every vertex in X is the beginning of some detour of D.*

One further extension of Conjecture 8.7.1 has been considered by Galeana-Sánchez and Gómez in [35]. A path $P = x_0x_1 \ldots x_n$ is non-augmentable if for every $v \in V(D) - V(P)$, and for every $0 \leq i \leq n-1$, $vx_0 \ldots x_n$, $x_0 \ldots x_n v$ and $x_0 \ldots x_i v x_{i+1} \ldots x_n$ are not paths. Clearly, every detour is non-augmentable, so, if true, Conjecture 8.7.1 would be an immediate consequence of the following conjecture, which appears implicitly in the paper of Galeana-Sánchez and Gómez [35] but has never been explicitly stated.

Conjecture 8.7.5 ([35]) *Every digraph D contains an independent set which intersects every non-augmentable path of D.*

8.7.2 Known Results

There are some partial results supporting each of the aforementioned conjectures, principally, Conjectures 8.7.1 and 8.7.2; we refer the reader to [2, 20, 33, 35, 38, 64]. Most of the existing results prove some of these conjectures for restricted families of digraphs; in most cases, generalizations of tournaments.

In [20], Conjecture 8.7.2 is considered for the family of quasi-transitive digraphs. There, Bang-Jensen, Nielsen and Yeo prove the following theorem. Recall that $\mathsf{do}_k(D)$ is the maximum number of vertices contained in a k-path subdigraph of a digraph D.

Theorem 8.7.6 ([20]) *Let D be a quasi-transitive digraph or a strong extended semicomplete digraph, and let q be any positive integer. Then there exists a partition, (A, B), of $V(D)$ such that the following holds:*

1. $\mathsf{do}(D[A]) \leq q$;
2. $\mathsf{do}_k(D[B]) \leq \mathsf{do}_k(D) - q$ *for all* $k \in \{1, \ldots, |V(B)|\}$, *provided* $\mathsf{do}_k - q \geq 0$.

Although Theorem 8.7.6 implies that Conjecture 8.7.1 is also true for quasi-transitive digraphs, it does not give us information on any of the other conjectures mentioned in the previous subsection. In [35], Galeana-Sánchez and Gómez proved Conjecture 8.7.5 to be true for quasi-transitive digraphs. Again, the proof of this result relies heavily on Theorem 8.3.5, which is also used by the following necessary lemma.

Lemma 8.7.7 ([35]) *Let H be a digraph such that $H = D[H_1, \ldots, H_n]$, where D is a transitive acyclic digraph with vertex set $\{v_1, \ldots, v_n\}$, and H_i are arbitrary digraphs for $1 \leq i \leq n$. If $\mathcal{I}_i$ is a maximal independent set intersecting every non-augmentable path of H_i, $1 \leq i \leq n$, then $\mathcal{I} = \bigcup_{i=1}^{n} \mathcal{I}_i$ is a maximal independent set that intersects every non-augmentable path in H.*

Proof: Since D is acyclic and transitive, its set of vertices of in-degree zero, S, is a maximal independent set intersecting every non-augmentable path of D.

Let P be a non-augmentable path in H. It is not hard to verify that the contraction[9] $P' = P/\{H_1, \ldots, H_n\}$ is a non-augmentable path of D, hence, S intersects P'. Also, if P uses at least one vertex from H_i, then it should be the case that $P \cap H_i$ is a non-augmentable path of H_i; otherwise, P could be augmented.

Thus, if we let $\mathcal{I}$ be the union of the $\mathcal{I}_j$'s corresponding to the vertices in S, then $\mathcal{I}$ is a maximal independent set intersecting every non-augmentable path of H. □

We will only present the general idea of the proof of the following theorem, due to its length and technical arguments.

Theorem 8.7.8 ([35]) *Let D be a quasi-transitive digraph. There exists a maximal independent set $\mathcal{I}$ of D that intersects every non-augmentable path in D. Moreover, if D is strong with decomposition $D = S[Q_1, \ldots, Q_s]$, and $\mathcal{I}_i \subseteq V(Q_i)$ is a maximal independent set intersecting every non-augmentable path in Q_i, for $1 \leq i \leq s$, then each $\mathcal{I}_i$ is also a maximal independent set intersecting every non-augmentable path in D.*

Idea of Proof. The proof is by induction on $|V(D)|$. If $|V(D)| = 1$, the result is clearly true.

[9] See Section 1.4.

If D is not strong, the result follows from Lemma 8.7.7 and the induction hypothesis.

If D is strong, then, by Theorem 8.3.5, $D = S[Q_1, \dots, Q_s]$, with S strong semicomplete and each Q_i a non-strong quasi-transitive digraph or a single vertex. Let P be a non-augmentable path of D. Recall that a path in a semicomplete digraph is non-augmentable if and only if it is Hamiltonian. Thus, P must intersect every Q_i. If Q_i is a single vertex, then it is a maximal independent set intersecting every non-augmentable path of D. Else, by induction hypothesis there is a maximal independent set S_i intersecting every non-augmentable path in Q_i. The proof finishes with an analysis of cases to show that S_i intersects every non-augmentable path of D. □

Wang and Wang attacked Conjecture 8.7.5 in [64]. Their main result relevant to our interests in this chapter is the following.

Theorem 8.7.9 ([64]) *If D is a 3-quasi-transitive digraph, then there exists an independent set intersecting every non-augmentable path in D.*

Proof: If D is strong, then, using the characterization given in Theorem 8.3.15, it is easy to verify that every maximal independent set intersects every non-augmentable path in D. Therefore, assume that D is not strong and let $D_0, \dots, D_k$ be its strong components. Let $D_0, \dots, D_s$ be the initial strong components and let F_i be a maximal independent set of D_i, for $1 \le i \le s$. Let $Z = V(D) - \bigcup_{i=0}^{s} V(D_s)$ and define W as

$$W = \{x \in Z\colon \text{ there exists a non-augmentable path in } D \text{ starting at } x\}.$$

Observe that W is either independent or empty. If $|W| \le 1$, there is nothing to prove. Assume $|W| \ge 2$, and suppose for a contradiction that there is a pair x, y of adjacent vertices in W. By the definition of W, x and y must belong to the same strong component, say D_j. Since $N^-(V(D_j))$ is non-empty, we may choose a vertex $u \in N^-(V(D_j))$. If D_j is non-bipartite, then, by Lemma 8.3.16, $u \mapsto D_j$, and hence $u \mapsto x$, a contradiction. If D_j is bipartite, then x and y must belong to different parts. Hence, by Lemma 8.3.16, u and one of x and y are adjacent, a contradiction.

Let $F = F_0 \cup \dots \cup F_s \cup W$. It is not difficult to deduce that F is an independent set in D. Let P be a non-augmentable path of D with initial vertex x_0. If x_0 does not belong to any initial component, then $x \in W$. Else, x_0 belongs to an initial component D_0 of D. If D_0 is semicomplete, then it is not hard to observe that $P \cap D_0$ is a Hamiltonian path of D_0, and thus, P must intersect F_0. If D_0 is complete bipartite, then F_0 is some part of D_0, so, F_0 intersects $P \cap D_0$. If D_0 is an element of the family $\mathcal{F}$ (see Theorem 8.3.15), then it is easy to verify that F_0 intersects $P \cap D_0$. □

Theorem 8.7.9 settles Conjecture 8.7.5 (which implies Conjecture 8.7.1) for 3-quasi-transitive digraphs. Arroyo and Galeana-Sánchez proved Conjecture 8.7.2 for strong 3-quasi-transitive digraphs in [2].

Theorem 8.7.10 ([2]) *Let D be a strong 3-quasi-transitive digraph. Consider two positive integers $\ell_1 \geq \ell_2$ such that $\ell_1 + \ell_2 = \mathsf{do}(D)$. Then there exists a partition (A, B) of $V(D)$ such that $\mathsf{do}(D[A]) \leq \ell_1$ and $\mathsf{do}(D[B]) \leq \ell_2$.*

Proof: Since the conjecture is easy to verify for semicomplete and bipartite digraphs, it follows from Theorem 8.3.15 that it only remains to show its validity in the digraphs of the family $\mathcal{F}$.

Let D be a digraph in the family $\mathcal{F}$. Notice that $4 \leq \mathsf{do}(D) \leq 5$, hence, it is easy to verify that, for every choice of ℓ_1, ℓ_2 such that $\ell_1 + \ell_2 = \mathsf{do}(D)$, the partition $(\{y, z\}, V(D) - \{y, z\})$ (see Figure 8.3) has the required property. □

Since every 3-transitive digraph is also 3-quasi-transitive, Theorems 8.7.9 and 8.7.10 also cover the 3-transitive case. Thus, the first interesting case for k-transitive digraphs is $k = 4$. For 4-transitive digraphs only Conjecture 8.7.1 has been explored; García-Vázquez and Hernández-Cruz proved it true for 4-transitive digraphs [38]. Again, the proof of the following theorem involves a technical analysis of various cases, and thus, only an idea of the proof method will be given.

Theorem 8.7.11 ([38]) *For every 4-transitive digraph D there exists an independent set intersecting every longest path of D.*

Idea of Proof. It is possible to prove that a 4-transitive digraph has a kernel if and only if every terminal strong component has a kernel. Also, using Theorem 8.3.20, it is not hard to characterize the strong 4-transitive digraphs having a kernel.

Let D be a 4-transitive digraph. Using the aforementioned characterization of the strong 4-transitive digraphs having a kernel, it is possible to find a minimal subset S of $V(D)$ such that $D - S$ has a kernel K. The set K is precisely the stable set we are looking for. □

To finish this section, we present a table with the values of k for which each of the discussed conjectures is known to be valid in k-transitive and k-quasi-transitive digraphs, and their corresponding strongly connected versions. In the columns of the table, LPX stands for Laborde–Payan–Xuong (Conjecture 8.7.1), SLPX for Strong LPX (Conjecture 8.7.4), NALPX for Non-Augmentable LPX (Conjecture 8.7.5), PPC for the Path Partition Conjecture (Conjecture 8.7.2), and SPPC for Strong PPC (Conjecture 8.7.3).

	LPX	SLPX	NALPX	PPC	SPPC
Strong k-transitive	$k \leq 4$	$k \leq 4$	$k \leq 3$	$k \leq 3$	$k = 2$
k-transitive	$k \leq 4$	$k \leq 3$	$k \leq 3$	$k = 2$	$k = 2$
Strong k-quasi-transitive	$k \leq 3$		$k \leq 3$	$k \leq 3$	
k-quasi-transitive	$k \leq 3$		$k \leq 3$	$k = 2$	

8.8 Miscellaneous

8.8.1 Vertex Pancyclicity

Pancyclicity is one of the properties that first comes to mind when thinking of tournaments.

Recall from Theorem 2.2.9 that every strong semicomplete digraph is vertex-pancyclic. As a generalization of tournaments, and semicomplete digraphs, it is natural to ask whether a Hamiltonian quasi-transitive digraph is vertex-pancyclic. In [17], Bang-Jensen and Huang use the similarities between extended semicomplete digraphs and quasi-transitive digraphs to derive results on pancyclic and vertex-pancyclic quasi-transitive digraphs. In this section we present a brief summary of these results.

A digraph D is **triangular with partition** V_0, V_1, V_2 if the vertex set of D can be partitioned into three disjoint sets V_0, V_1, V_2 with $V_0 \mapsto V_1 \mapsto V_2 \mapsto V_0$. Note that this is equivalent to saying that $D = \vec{C}_3[D[V_0], D[V_1], D[V_2]]$.

Gutin [40] characterized pancyclic and vertex-pancyclic extended semicomplete digraphs. Clearly no extended semicomplete digraph of the form $D = \vec{C}_2[\overline{K}_{n_1}, \overline{K}_{n_2}]$ with at least 3 vertices is pancyclic since all cycles are of even length. Hence we must assume that there are at least 3 parts in order to get a pancyclic extended semicomplete digraph. It is also easy to see that the (unique) strong 3-partite extended semicomplete digraph on 4 vertices is not pancyclic (since it has no 4-cycle). These observations together with Theorem 7.10.8 completely characterize pancyclic and vertex-pancyclic extended semicomplete digraphs. It is not difficult to see that Theorem 7.10.8 extends Theorem 1.5.1, since no semicomplete digraph on $n \geq 5$ vertices satisfies any of the exceptions from (a) and (b).

The next two lemmas of Bang-Jensen and Huang [17] concern cycles in triangular digraphs. They are used in the proof of Theorem 8.8.3, which characterizes pancyclic and vertex-pancyclic quasi-transitive digraphs.

Lemma 8.8.1 ([17]) *Suppose that D is a triangular digraph with a partition V_0, V_1, V_2 and suppose that D is Hamiltonian. If $D[V_1]$ contains an arc xy and $D[V_2]$ contains an arc uv, then every vertex of $V_0 \cup \{x, y, u, v\}$ is on cycles of lengths $3, 4, \dots, n$.* □

Lemma 8.8.2 ([17]) *Suppose that D is a triangular digraph with a partition V_0, V_1, V_2 and D has a Hamiltonian cycle C. If $D[V_0]$ contains an arc of C and a path P of length 2, then every vertex of $V_1 \cup V_2 \cup V(P)$ is on cycles of lengths $3, 4, \dots, n$.* □

It is easy to check that a strong quasi-transitive digraph on 4 vertices is pancyclic if and only if it is a semicomplete digraph. For $n \geq 5$ we have the following characterization due to Bang-Jensen and Huang:

Theorem 8.8.3 ([17]) *Let $D = (V, A)$ be a Hamiltonian quasi-transitive digraph on $n \geq 5$ vertices.*

(a) D is pancyclic if and only if it is not triangular with a partition V_0, V_1, V_2, two of which induce digraphs with no arcs, such that either $|V_0| = |V_1| = |V_2|$, or no $D[V_i]$ $(i = 0, 1, 2)$ contains a path of length 2.

(b) D is not vertex-pancyclic if and only if D is not pancyclic or D is triangular with a partition V_0, V_1, V_2 such that one of the following occurs:

(b1) $|V_1| = |V_2|$, both $D[V_1]$ and $D[V_2]$ have no arcs, and there exists a vertex $x \in V_0$ such that x is not contained in any path of length 2 in $D[V_0]$ (in which case x is not contained in a cycle of length 5).

(b2) one of $D[V_1]$ and $D[V_2]$ has no arcs and the other contains no path of length 2, and there exists a vertex $x \in V_0$ such that x is not contained in any path of length 1 in $D[V_0]$ (in which case x is not contained in a cycle of length 5);

□

8.8.2 Acyclic Spanning Subgraphs

It is well known that a semicomplete digraph T contains an (x, y)-Hamiltonian path if and only if there is a spanning acyclic subgraph S (not necessarily induced) such that S contains an (x, z)-path and a (z, y)-path for each vertex z of T, cf. [56]. This also follows from the fact that semicomplete digraphs are path-mergeable, see [3] and Section 6.2.

It follows from the characterization in Theorem 8.4.7 that a quasi-transitive digraph D may not have a Hamiltonian path even if it is highly connected and has a path P such that $D - P$ has a cycle factor (see [17] for such an example). On the other hand, Bang-Jensen and Huang proved in [17] that if a quasi-transitive digraph has a unique initial and a unique terminal strong component then we can always guarantee the existence of such an acyclic spanning subgraph.

Theorem 8.8.4 ([17]) *Suppose that D is a quasi-transitive digraph having both in- and out-branchings. Then D has a spanning acyclic subgraph S with a source x and a sink y such that for each vertex z of D, D contains an (x, z)-path and a (z, y)-path.* □

Corollary 8.8.5 *Every strong quasi-transitive digraph has a spanning acyclic subdigraph S with a source x and a sink y such that, for each vertex z of D, S contains an (x, z)-path and a (z, y)-path.* □

8.8.3 Orientations of Digraphs Almost Preserving Diameter

Recall that an **orientation** of a digraph D is a spanning subdigraph of D obtained from D by deleting exactly one arc from every 2-cycle. Chvátal and Thomassen [27] proved that the problem of checking whether a given undirected graph has an orientation of diameter 2 is $\mathcal{NP}$-complete, and the upper bound on the diameter of an orientation of an undirected graph obtained in [27] is far from the best possible for many classes of undirected graphs (recall that undirected graphs may be regarded as digraphs where every arc is symmetric).

We have already seen many problems which have very nice solutions for the class of quasi-transitive digraphs, e.g., hamiltonicity, existence of kernels, k-linkages and weak k-linkages, which are $\mathcal{NP}$-complete in the general case, are polynomial time solvable for quasi-transitive digraphs. The study of minimum diameter orientations of quasi-transitive digraphs is not an exception; a surprisingly good bound on the minimum diameter of an orientation of a quasi-transitive digraph holds. Before stating the main results of this section, we will recall a result due to Boesch and Tindell which extends Robbins' Theorem.

Theorem 8.8.6 ([23]) *A strong digraph D has no strong orientation if and only if there is a pair x, y of vertices in D such that the deletion of the arcs xy, yx leaves D disconnected.*

Applying Theorem 8.8.6 it is easy to see that every strong quasi-transitive digraph of order $n \geq 3$ has a strong orientation. For a digraph D, let $\mathrm{diam}_{\min}(D)$ denote the minimum diameter of an orientation of D. The following result is due to Gutin and Yeo [43].

Theorem 8.8.7 ([43]) *If D is a strong quasi-transitive digraph, then*

$$\mathrm{diam}_{\min}(D) \leq \max\{3, \mathrm{diam}(D)\}.$$

The upper bound of this theorem is sharp as one can see from the following example. Let T_k, $k \geq 3$, be a (transitive) tournament with vertices $x_1, x_2, ..., x_k$ and arcs $x_i x_j$ for every $1 \leq i < j \leq k$. Let y be a vertex not in T_k, which dominates all vertices of T_k but x_k and is dominated by all vertices of T_k but x_1. The resulting semicomplete digraph D_{k+1} has diameter 2. However, the deletion of any arc of D_{k+1} between y and the set $\{x_2, x_3, ..., x_{k-1}\}$ leaves a digraph with diameter 3. Indeed, if we delete yx_i, $2 \leq 2 \leq k-1$, then a shortest (x_k, x_i)-path becomes of length 3.

8.8.4 Sparse Subdigraphs with Prescribed Connectivity

A spanning k-(arc)-strong subdigraph D' of a directed multigraph D is called a **certificate** for the k-(arc)-strong connectivity of D. A problem of practical interest is the following. Let $D = (V, A)$ be a k-(arc)-strong directed multigraph and let c be a cost function on A (possibly $c(a) = 1$ for all $a \in A$). What is the minimum cost of a k-(arc)-strong spanning subdigraph D' of D? An **optimal certificate** for k-(arc)-strong connectivity in D is a spanning k-(arc)-strong subdigraph D' of minimum cost. Finding such an optimal certificate is a hard problem already when $k = 1$ and $c \equiv 1$. This follows from the fact that the optimal certificate for the strong connectivity of D has $|V|$ arcs if and only if D has a Hamilton cycle.

When $c \equiv 1$, we have the problem of finding an optimal certificate for strong connectivity. We call this the MINIMUM SPANNING STRONG SUBDIGRAPH problem (MSSS, see [18]).

For the case of quasi-transitive digraphs, we begin with a lower bound. Recall that the path-covering number of a digraph D, $pc(D)$, is the least positive integer k such that D has a k-path factor. For a strong quasi-transitive digraph D we define $\text{pc}^*(D)$ to be equal to 0 if D is Hamiltonian, and $\text{pc}^*(D) = \text{pc}(D)$ otherwise. The optimal solution to the MSSS problem for quasi-transitive digraphs was given by Bang-Jensen, Huang, and Yeo. The proof can be found in [9].

Theorem 8.8.8 ([18]) *Every minimum spanning strong subdigraph of a quasi-transitive digraph has precisely $n + \text{pc}^*(D)$ arcs. Furthermore, we can find a minimum spanning strong subdigraph in time $O(|V|^4)$.*

A **directed cactus** is a strongly connected digraph in which each arc is contained in exactly one cycle.

Palbom [55] studied the complexity of various problems related to spanning directed cactii in digraphs. It is not difficult to check whether a given digraph is a cactus, but Palbom proved that deciding whether a digraph contains a spanning cactus is an $\mathcal{NP}$-complete problem [55].

Since the directed spanning cactus problem (the problem of determining whether a digraph contains a spanning cactus) is trivial for locally in-semicomplete digraphs, and easy for path-mergeable digraphs, but already non-trivial for extended semicomplete digraphs (see, Exercises 12.17 and 12.20 in [9]), the following problem comes as a natural next step in this subject.

Problem 8.8.9 ([9]) *Determine the complexity of the spanning directed cactus problem for quasi-transitive digraphs.*

8.8.5 Arc-Disjoint In- and Out-Branchings

We now consider the problem ARC-DISJOINT IN- AND OUT-BRANCHINGS: Given a digraph D and vertices u, v (not necessarily distinct), decide whether D has a pair of arc-disjoint branchings B_u^+, B_v^- such that B_u^+ is an out-branching rooted at u and B_v^- is an in-branching rooted at v. Recall from Theorem 2.12.19 that Thomassen proved that ARC-DISJOINT IN- AND OUT-BRANCHINGS is $\mathcal{NP}$-complete for general digraphs.

In [4], Bang-Jensen proved that a tournament T has arc-disjoint in- and out-branchings rooted at some vertex v if and only if there is no arc that must be on all out-branchings from v and all in-branchings to v, see Corollary 2.12.21. In [17], Bang-Jensen and Huang considered digraphs having a vertex v which is adjacent to every other vertex; they obtained a characterization of digraphs having arc-disjoint in- and out-branchings rooted at v. As a consequence, they obtained the following result.

Theorem 8.8.10 ([17]) *Let D be a strong digraph and v a vertex of D such that $V(D) = \{v\} \cup N^+(v) \cup N^-(v)$. There is a polynomial algorithm to decide if D has arc-disjoint in- and out-branchings F_v^-, F_v^+ rooted at v.*

The previous result can be combined with the following lemma to obtain a polynomial algorithm to decide if a quasi-transitive digraph D has arc-disjoint in- and out-branchings rooted at a given vertex v.

Lemma 8.8.11 ([17]) *Let D be a quasi-transitive digraph and $v \in V(D)$ a vertex of D. Then D contains arc-disjoint branchings F_v^+, F_v^- rooted at v if and only if $D' = D[\{v\} \cup N^-(v) \cup N^+(v)]$ has arc-disjoint branchings $F_v'^+, F_v'^-$ rooted at v.* □

Theorem 8.8.12 ([17]) *Let D be a strong quasi-transitive digraph, and v a vertex of D. If $B = \{B_1 \dots, B_k\}$ ($C = \{C_1, \dots, C_r\}$) denote the set of terminal (initial) components in $D[N^+(v)]$ ($D[N^-(v)]$), then D contains a pair of arc-disjoint branchings F_v^+, F_v^- such that F_v^+ is an out-branching rooted at v and F_v^- is an in-branching rooted at v if and only if there exist two disjoint arc sets $A_B, A_C \subset A(D)$ such that all arcs in $A_B \cup A_C$ go from $N^+(v)$ to $N^-(v)$ and every component in $B_i \in B$ ($C_j \in C$) is incident with an arc from A_B (A_C).* □

From here, the following result settling the problem for quasi-transitive digraphs is obtained.

Corollary 8.8.13 ([17]) *There is a polynomial algorithm to decide if a quasi-transitive digraph D has arc-disjoint in- and out-branchings rooted at a given vertex v.* □

As noted in Section 2.12, already for semicomplete digraphs, the problem of finding arc-disjoint in- and out-branchings becomes much harder when $u \neq v$. Even the class of semicomplete digraphs is still lacking a polynomial time algorithm to decide this problem when $u \neq v$.

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