2. Tournaments and Semicomplete Digraphs

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The class of tournaments is by far the most well-studied class of digraphs with many deep and important results. Since Moon's pioneering book in 1968 [146], the study of tournaments and their properties has flourished. A search in May 2017 on MathSciNet for 'tournament' and 05C20 gives more than 900 hits. Clearly we can only cover a small fraction of the research on tournaments, but we believe that our coverage will stimulate new research on this beautiful class of digraphs.

Being a super-class of tournaments, the class of semicomplete digraphs inherits many of the properties of tournaments, but there are important differences and we shall try to point out such when relevant. Due to space limitations we will not mention all places where a result for tournaments extends to semicomplete digraphs. Note that the results of Section 2.3 imply that results for k-strong tournaments often imply similar results for $(3k-2)$-strong semicomplete digraphs.

In Section 2.1 we introduce some special tournaments that occur in several proofs and results in the chapter. Section 2.2 gives some basic properties of tournaments and semicomplete digraphs such as the fact that they are always traceable. The short Section 2.3 is about spanning tournaments of high connectivity in highly connected semicomplete digraphs. In Section 2.4 we give two very different proofs for the tournament case of the conjecture of Seymour (and Dean in the case of tournaments) that every oriented graph has a vertex with distance 2 to at least as many vertices as it has out-neighbours. Section 2.5 deals with linkages and disjoint cycles in tournaments and semicomplete digraphs. In Section 2.6 we discuss further topics related to Hamiltonian paths and cycles and give a proof of Redéi's theorem that every tournament has an odd number of Hamiltonian paths. Section 2.7 is devoted to oriented subgraphs in tournaments, in particular to oriented Hamiltonian paths and cycles in tournaments. In Section 2.8 we study vertex-partitions

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of semicomplete digraphs where each part has to have certain properties, e.g. being strongly connected or being acyclic. Section 2.9 deals with results of feedback sets, that is, sets of vertices or arcs whose deletion makes the resulting digraph acyclic. Even for tournaments, finding such a set of minimum cardinality is $\mathcal{NP}$-complete. In Section 2.10 we study the problem of how many arcs one may delete from a k-(arc)-strong tournament without reducing the connectivity of the resulting digraph. The answer is that we may delete surprisingly many. Section 2.11 is also on connectivity, but this time the operation we consider is that of either reversing arcs or of deorienting arcs, that is, adding an arc oppositely oriented to an existing arc. In Section 2.12 we consider arc-disjoint spanning subdigraphs of semicomplete digraphs. This includes the famous Kelly conjecture that the arc set of every regular tournament decomposes into Hamiltonian cycles. Section 2.13 is on minors of semicomplete digraphs. It turns out that for this class of digraphs the notion of a minor, defined as being any digraph that can be obtained by contracting strong subdigraphs, leads to results in the same vein as the graph minor theory of Roberson and Seymour. Finally, in Section 2.14 we briefly survey a few further topics on tournaments.

We will use the shorthand names **n-tournament** and **n-semicomplete** digraph for a tournament, resp. semicomplete digraph on n vertices. Throughout this chapter, except for Section 2.7, paths and cycles are always assumed to be directed.

2.1 Special Tournaments

We first define a number of special tournaments that will be referred to later. Let $n \geq 1$ be an integer. The unique acyclic n-tournament is the **transitive tournament**, denoted TT_n. This has an ordering $(v_1, v_2, \dots, v_n)$ of its vertices so that v_iv_j is an arc whenever $1 \leq i < j \leq n$.

A tournament is **almost transitive** if it is obtained from a transitive tournament with acyclic ordering $(v_1, v_2, \dots, v_n)$ (i.e., $v_i \rightarrow v_j$ for all $1 \leq i < j \leq n$) by reversing the arc v_1v_n.

The **random n-tournament** RT_n is the (random) digraph one obtains from the complete graph K_n by choosing one from each of the two possible orientations of each edge uv of K_n with probability $\frac{1}{2}$ for each of the two possible orientations.

Recall that an n-tournament is **regular** if $n = 2k + 1$ for some $k \geq 1$ and every vertex has in- and out-degree k. Below we describe two important examples of classes of regular tournaments.

Let $\mathbb{Z}_{2k+1}$ be the set of integers modulo $2k + 1$ and let J be a subset of $\mathbb{Z}_{2k+1} \setminus \{0\}$ such that for every $i \in \mathbb{Z}_{2k+1} \setminus \{0\}$, we have $i \in J$ if and only if $-i \notin J$. Then the **circulant tournament** $CT_{2n+1}(J)$ is the tournament whose vertex set is $\mathbb{Z}_{2k+1}$ and ij is an arc if and only if $j - i \in J$. For some examples of papers on circulant tournaments, see [14, 47, 136, 149].

For each prime power q of the form $q = 4k+3$, the **Paley tournament** $\mathbb{P}_q$ is the q-tournament whose vertices are the elements of the finite field $GF(q)$ with q elements. There is an arc from x to y if and only if $y-x$ is a non-zero square in the field. E.g. when $q = 7$ the vertex set of $\mathbb{P}_7$ is $\{0,1,2,3,4,5,6\}$ and ij is an arc of $\mathbb{P}_7$ if and only if $((j-i) \mod 7) \in \{1,2,4\}$. For examples of papers dealing with Paley tournaments, see e.g. [44, 46, 47, 51].

2.2 Basic Properties of Tournaments and Semicomplete Digraphs

We start with a very simple but important observation which is proved by a simple counting argument.

Proposition 2.2.1 *Every semicomplete digraph on n vertices contains a vertex with out-degree at least $\lfloor \frac{n}{2} \rfloor$ and a vertex with in-degree at least $\lfloor \frac{n}{2} \rfloor$.*

Proof: Let T be a semicomplete digraph on n vertices. We have

$$\sum_{v \in V(T)} d^+(v) = \sum_{v \in V(T)} d^-(v) = |A(T)| \geq \binom{n}{2} = n \cdot \frac{n-1}{2}.$$

Thus there is a vertex with out-degree (resp. in-degree) at least $\lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$. □

Proposition 2.2.2 *Let k be a positive integer. Every semicomplete digraph has at most $2k-1$ vertices of out-degree less than k.*

Proof: Let D be a semicomplete digraph and let X be the set of vertices of out-degree less than k in T. The number of arcs in the subdigraph $D[X]$ is at most $|X|(k-1)$. On the other hand, $D[X]$ has at least $\binom{|X|}{2}$ arcs. Hence,

$$\frac{|X|(|X|-1)}{2} \leq |A(D[X])| \leq |X|(k-1),$$

implying that $|X| \leq 2k-1$.

Using Proposition 2.2.1 we can now give a lower bound on the largest transitive subtournament in any tournament.

Proposition 2.2.3 *Every n-tournament contains a transitive subtournament TT_k with $k \geq \lceil \log n \rceil$.*

Proof: The following algorithm produces such a transitive subtournament: Let $T' := T$ and $R = \emptyset$. While T' has at least one vertex: let v be a vertex of maximum out-degree in T' and let $R := R \cup \{v\}$. By Proposition 2.2.1,

$|N^+_{T'}(v)| \geq \lfloor \frac{n}{2} \rfloor$. Hence, letting $T' := T'[N^+(v)]$, the new T' has size at least $\lfloor \frac{n}{2} \rfloor$. Repeat the step above for T'.

Clearly the set R returned by this algorithm induces a transitive subtournament of T. To see that R has size at least $\lceil \log n \rceil$, consider the integer r satisfying $2^r \leq n < 2^{r+1}$. After step number i in the algorithm above we have $2^{r-i} \leq |V(T')|$, from which it follows that $|R| \geq r + 1 \geq \lceil \log n \rceil$ holds at the end. □

One of the first results on tournaments is the following, due to Rédei. See Section 2.6 for a beautiful generalization of this, also due to Rédei.

Theorem 2.2.4 (Rédei's Theorem [158]**)** *Every tournament contains a Hamiltonian dipath.*

Proof: By induction on the number of vertices. The statement is trivial for the 1-tournament. Let $n \geq 2$, let T be an n-tournament and let v be a vertex of T. By the induction hypothesis, $T\langle N^-(v)\rangle$ and $T\langle N^+(v)\rangle$ have Hamiltonian directed paths P^- and P^+. Thus P^-vP^+ is a Hamiltonian dipath of T[1]. □

Since we can obtain a tournament from a semicomplete digraph by removing an arbitrary arc from each 2-cycle, we obtain that Theorem 2.2.4 also holds for semicomplete digraphs (and this can also be proved directly with the same proof as above).

Corollary 2.2.5 *Every semicomplete digraph has a Hamiltonian path.*

There is no analogue to Theorem 2.2.4 for Hamiltonian dicycles since the transitive tournaments are acyclic and in particular have no Hamiltonian dicycle. More generally, no non-strong tournament has a Hamiltonian dicycle because it has a vertex-partition (L, R) such that $L \mapsto R$ (e.g. if we take L to be the vertices of the initial strong component and L to be the remaining vertices). In contrast, all strong tournaments have a Hamiltonian directed cycle as shown by Camion [56].

Theorem 2.2.6 (Camion's Theorem [56]**)** *Every strong tournament has a Hamiltonian dicycle.*

A simple proof of Camion's Theorem, due to Moon [144], actually proves a stronger result.

Theorem 2.2.7 (Moon's Theorem [144]**)** *Every strong tournament is vertex-pancyclic.*

[1] Note that here we allowed one of the two tournaments to be empty, in which case the corresponding path is also empty

Proof: Let x be a vertex in a strong tournament T on $n \geq 3$ vertices. The proof is by induction on k. We first prove that T has a 3-cycle through x. Since T is strong, each of the sets $O = N^+(x)$ and $I = N^-(x)$ are non-empty and the set (O, I) is also non-empty. Let $yz \in (O, I)$. Then $xyzx$ is a 3-cycle through x. Let $C = x_0x_1 \ldots x_t$ be a dicycle in T with $x = x_0 = x_t$ and $t \in \{3, 4, \ldots, n-1\}$. We prove that T has a $(t+1)$-cycle through x.

If there is a vertex $y \in V(T) \setminus V(C)$ which dominates a vertex in C and is dominated by a vertex in C, then it is easy to see that there exists an index i such that $x_i \to y$ and $y \to x_{i+1}$. Therefore, $C[x_0, x_i]yC[x_{i+1}, x_t]$ is a $(t+1)$-cycle through x. Thus, we may assume that every vertex outside of C either dominates every vertex in C or is dominated by every vertex in C. The vertices of $V(T) \setminus V(C)$ that dominate all vertices of $V(C)$ form a set R; the rest of the vertices in $V(T) \setminus V(C)$ form a set S. Since T is strong, both S and R are non-empty and the set (S, R) is non-empty. Hence, taking $sr \in (S, R)$, we see that $x_0srC[x_2, x_t]$ is a $(t+1)$-cycle through $x = x_0$. □

The following is an easy consequence of Theorem 1.7.3.

Proposition 2.2.8 *Every strong semicomplete digraph on $n \geq 3$ vertices contains a strong spanning tournament.*

Together with Moon's theorem, Proposition 2.2.8 implies the following.

Theorem 2.2.9 *Every strong semicomplete digraph is vertex-pancyclic.* □

This easily implies the following.

Corollary 2.2.10 *Every strong semicomplete digraph D on at least four vertices has two distinct vertices v_1, v_2 such that $D - v_i$ is strong for $i \in [2]$.*

This is the best possible as shown by the tournament that one obtains from a transitive tournament TT_k on at $k \geq 3$ vertices by reversing the arcs of the unique Hamiltonian path.

2.2.1 Median Orders, a Powerful Tool

Now we introduce a very useful tool for proving results about tournaments and other classes of digraphs.

A **median order** of a digraph D is a linear order $(v_1, v_2, \ldots, v_n)$ of its vertex set such that $|\{(v_i, v_j) : i < j\}|$ (the number of arcs directed from left to right) is as large as possible. In the case of a tournament, such an order can be viewed as a ranking of the players which minimizes the number of upsets (matches won by the lower-ranked player). As we shall see, median orders of tournaments reveal a number of interesting structural properties.

Let us first note two basic properties of median orders of tournaments whose easy proofs are left to the reader.

Lemma 2.2.11 *Let T be a tournament and $(v_1, v_2, \dots, v_n)$ a median order of T. Then, for any two indices i, j with $1 \le i < j \le n$:*

(M1) The suborder $(v_i, v_{i+1}, \dots, v_j)$ is a median order of the induced subtournament $T\langle\{v_i, v_{i+1}, \dots, v_j\}\rangle$;
(M2) The vertex v_i dominates at least half of the vertices $v_{i+1}, v_{i+2}, \dots, v_j$, and vertex v_j is dominated by at least half of the vertices $v_i, v_{i+1}, \dots, v_{j-1}$.

In particular, each vertex v_i, $1 \le i < n$, dominates its successor v_{i+1}. The sequence $v_1 v_2 \dots v_n$ is thus a Hamiltonian directed path, providing an alternative proof of Rédei's Theorem (Theorem 2.2.4).

2.2.2 Kings

The **second out-neighbourhood** of a vertex v in a digraph D, denoted by $N_D^{++}(v)$ or simply $N^{++}(v)$, is the set of vertices at distance 2 from v. In other words, it is the set of vertices that are dominated by an out-neighbour of v and are not in $v \cup N^+(v)$. The dual notion of **second in-neighbourhood** of a vertex v in a D is defined similarly and is denoted by $N_D^{--}(v)$ or simply $N^{--}(v)$.

A **king** in a tournament T is a vertex v such that $\{v\} \cup N^+(v) \cup N^{++}(v) = V(T)$. Landau [129] proved that every tournament has a king.

Theorem 2.2.12 ([129]) *Every tournament has a king. More precisely, every vertex with maximum out-degree is a king.*

Proof: Let v be a vertex of maximum out-degree in a tournament T. Suppose by way of contradiction that v is not a king. Then there exists a vertex w in T that is dominated by no vertex of $N^+(v) \cup \{v\}$. Hence w dominates $N^+(v) \cup \{v\}$ and $d^+(w) \ge d^+(v) + 1$, a contradiction. □

Havet and Thomassé demonstrated that the existence of a king in a tournament can also be proved using median order.

Lemma 2.2.13 ([109]) *Let T be a tournament. If $(v_1, v_2, \dots, v_n)$ is a median order of T, then v_1 is a king of T.*

Proof: Consider v_i for $2 \le i \le n$. We shall prove that $v_i \in N^+(v_1) \cup N^{++}(v_1)$. Assume that v_i is not in $N^+(v_1)$. Then it dominates v_1. By the property (M2) of Lemma 2.2.11, v_1 dominates at least half of the vertices $\{v_2, \dots, v_i\}$, and so, since v_1 is dominated by v_i, it dominates more than half the vertices of $\{v_2, \dots, v_{i-1}\}$. Similarly, v_i is dominated by more than half the the vertices of $\{v_2, \dots, v_{i-1}\}$. Therefore, there is a vertex in $\{v_2, \dots, v_{i-1}\}$, which dominates v_i and is dominated by v_1. Hence $v_i \in N^{++}(v_1)$. □

Since every tournament admits a median order, Lemma 2.2.13 directly implies Theorem 2.2.12. Moon [145] proved that a tournament has at least three kings, provided that it has no **source** (that is, a vertex with in-degree 0 and thus dominating all other vertices). Observe this condition is necessary: if a tournament contains a source, then this vertex is its unique king.

Corollary 2.2.14 ([145]) *Every tournament T with $\delta^-(T) \geq 1$ has at least three kings.*

Proof:
We give a proof due to Havet and Thomassé [109]. Assume that $\delta^-(T) \geq 1$. Let $(v_1, v_2, \ldots, v_n)$ be a median order of T. By Lemma 2.2.13, vertex v_1 is a king. Let i be the smallest index such that v_i is an in-neighbour of v_1, and let j be the smallest index such that v_j is an in-neighbour of v_i. Those vertices exist since T has no source. We claim that both v_i and v_j are kings of T. First, observe that $1 < j < i$ by (M2). Now, by (M1), $v_i, \ldots, v_n$ is a median order of $T' = T\langle\{v_i, \ldots, v_n\}\rangle$, and so, by Lemma 2.2.13, v_i is a king of T'. Moreover, via v_1, which dominates all vertices in $v_2, \ldots, v_{i-1}$ (by the choice of i), v_i is also a king of $T\langle\{v_1, \ldots, v_i\}\rangle$. Hence v_i is a king of T. Similarly, v_j is a king of $T\langle\{v_j, \ldots, v_n\}\rangle$, and, via v_i, which dominates all vertices in $v_1, \ldots, v_{j-1}$ (by the choice of j), is a king of T. □

The above results have been generalized to arc-coloured tournaments. A **monochromatic king** in an arc-coloured tournament is a vertex v such that for every vertex w, one can find a monochromatic (v, w)-dipath. There are many examples of arc-coloured tournaments with no monochromatic king. Firstly, a tournament with no source and with all its arcs coloured differently has no monochromatic king. Secondly, if there is a partition (V_1, V_2, V_3) of the vertex set of a tournament T such that $V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_1$, then T has no monochromatic king. Shen gave a simple necessary condition for the existence of a monochromatic king in arc-coloured tournaments.

Theorem 2.2.15 ([169]) *If we colour the arcs of a tournament T in such a way that no subtournament of order* 3 *gets three different colours on its arcs, then there exists a monochromatic king.*

Proof: The proof is by induction on the number of vertices. Remove a vertex x_1 from T. By the induction hypothesis, one can find a monochromatic king x_2 in $T - x_1$. If $x_2 \rightarrow x_1$, then x_2 is a monochromatic king in T. Therefore, we may assume $x_1 \rightarrow x_2$. Repeating the process for x_2, and so on, either we find a monochromatic king in T, or we find a directed cycle $C = x_k \ldots x_\ell x_k$ such that x_i is a monochromatic king in $T - x_{i-1}$ (with $x_{k-1} = x_\ell$). If C does not span T, then by the induction hypothesis, there is a monochromatic king in $T\langle C \rangle$, say x_i. Thus there is a monochromatic (x_i, x_{i-1})-dipath in $T\langle C \rangle$. Because, x_i is a monochromatic king in $T - x_{i-1}$, it follows that x_i is also a monochromatic king in T. Henceforth, we assume that $C = x_1 \ldots x_n x_1$

is Hamiltonian in T. If the arcs of C are monochromatic, the conclusion holds, so there is one particular x_i such that $x_{i-1}x_i$ and x_ix_{i+1} have different colours, say c_1 and c_2. By the induction hypothesis, there is a monochromatic dipath P from x_{i+1} to x_{i-1}. If P is coloured by c_1 or c_2, then either x_{i+1} or x_i respectively is a monochromatic king in T. Henceforth, we may assume that P is coloured by c_3. Set $P = y_1 \dots y_q$ with $y_1 = x_{i+1}$ and $y_q = x_{i-1}$. Let j be the smallest index such that the arc a_j between x_i and y_j is not coloured c_2. Such a j exists because y_qx_i is coloured c_1. Since $T\langle\{y_{j-1}, y_j, x_i\}\rangle$ does not have three different colours on its arcs, necessarily a_j is coloured c_3. If $a_j = y_jx_i$ (resp. $a_j = x_iy_j$), then there is a c_3-monochromatic (x_{i+1}, x_i)-dipath (resp. (x_i, x_{i-1})-dipath) and x_{i+1} (resp. x_i) is a monochromatic king in T. □

In Shen's paper the following question was asked: is it true that no matter how we colour the arcs of a tournament, there is either a trichromatic 3-cycle or a monochromatic king. This was disproved by Galeana-Sánchez and Rojas-Monroy in [93].

2.2.3 Scores and Landau's Theorem

Let T be a tournament. Its **score sequence** is the sequence of the out-degrees of its vertices in non-decreasing order. Hence, if $V(T) = \{v_1, v_2, \dots, v_n\}$ with $d^+(v_1) \le d^+(v_2) \le \dots \le d^+(v_n)$, then the score sequence of T is $(d^+(v_1), d^+(v_2), \dots, d^+(v_n))$.

Consider a score sequence $\mathbf{s}$ of some n-tournament T. Any k vertices of T induce a subtournament S and, hence, the sum of the scores in T of these k vertices must be at least the sum of their scores in S, which is just the total number of arcs in S, that is, $\binom{k}{2}$. Hence $\sum_{i\in I} s_i \ge \binom{|I|}{2}$ for all $I \subseteq \{1, 2, \dots, n\}$, with equality for $I = \{1, 2, \dots, n\}$. In particular, $\sum_{i=1}^{k} s_i \ge \binom{k}{2}$, for all $1 \le k \le n$ with equality for $k = n$. Landau proved that this obvious necessary condition is actually also sufficient.

Theorem 2.2.16 (Landau [129]) *The sequence $\mathbf{s} = (s_1 \le s_2 \le \dots \le s_n)$ of integers is the score sequence of an n-tournament if and only if*

$$\sum_{i=1}^{k} s_i \ge \binom{k}{2}, \text{ for all } 1 \le k \le n, \quad \text{with equality for } k = n. \tag{2.1}$$

There are many known proofs of Landau's theorem (see [52, 97, 140, 160, 185]). Many of these proofs are discussed in the survey [160] by Reid. The proof we present here is due to Griggs and Reid [97].

Proof: The specific sequence $\mathbf{t} = (0, 1, 2, \dots, n-1)$ satisfies conditions (2.1) as it is the score sequence of the transitive n-tournament. If a sequence $\mathbf{s} \ne \mathbf{t}$ satisfies (2.1), then since $s_1 \ge 0$ and $s_n \le n-1$, s contains a repeated term.

The object of this proof is to produce a new sequence $\mathbf{s}'$ from $\mathbf{s}$ which also satisfies (2.1), is 'closer' to $\mathbf{t}$ than is $\mathbf{s}$, and is a score sequence if and only if $\mathbf{s}$ is a score sequence. Toward this end, define j to be the smallest index for which $s_j = s_{j+1}$, and define m to be the number of occurrences of the term s_j in $\mathbf{s}$. Note that $j \geq 1$ and $m \geq 2$, and that either $j + m - 1 = n$ or $s_j = s_{j+1} = \cdots = s_{j+m-1} < s_{j+m}$. Define $\mathbf{s}'$ as follows:

$$\text{for } 1 \leq i \leq n, \quad s'_i = \begin{cases} s_i - 1, \text{ if } i = j, \\ s_i + 1, \text{ if } i = j + m - 1, \\ s_i, \qquad \text{otherwise.} \end{cases}$$

Clearly, $s'_1 \leq s'_2 \leq \cdots \leq s'_n$.

Let us show that $\mathbf{s}'$ a score sequence if and only if $\mathbf{s}$ is a score sequence. If $\mathbf{s}'$ is the score sequence of some n-tournament T' in which vertex v_i has out-degree s'_i, $1 \leq i \leq n$, then, since $s'_{j+m-1} > s'_j$, there is a vertex in T', say v_p, for which $v_{j+m-1} \to v_p$ and $v_p \to v_j$. The reversal of those two arcs in T' yields an n-tournament with score sequence $\mathbf{s}$. Conversely, if $\mathbf{s}$ is the score sequence of some n-tournament T in which vertex v_i has score s_i, $1 \leq i \leq n$, then we may suppose that v_j dominates v_{j+m-1} in T, for otherwise, interchanging the labels on these two vertices does not change $\mathbf{s}$. The reversal of the arc $v_j v_{j+m-1}$ in T yields an n-tournament with score sequence $\mathbf{s}'$.

To conclude the inductive proof, since $\mathbf{s}'$ is closer to $\mathbf{t}$ than $\mathbf{s}$, it remains to show that $\mathbf{s}'$ satisfies (2.1). By definition of $\mathbf{s}'$, one needs to show that $\sum_{i=1}^{k} s_i \geq \binom{k}{2} + 1$ for all $j \leq k \leq j + m - 2$. The proof is by induction on $k \geq j$. The case $k = j$ is very similar to the induction step and is omitted. Suppose that for some k, $j \leq k < j + m - 2$, $\sum_{i=1}^{k} s_i \geq \binom{k}{2} + 1$. We shall prove that $\sum_{i=1}^{k+1} s_i \geq \binom{k+1}{2} + 1$. Suppose by way of contradiction that this is not the case. Then by (2.1),

$$\sum_{i=1}^{k+1} s_i = \binom{k+1}{2}. \tag{2.2}$$

Now since $j < k + 2 \leq j + m - 1$, by definition of j and m and the above equation, we have

$$s_{k+1} = s_{k+2} = \sum_{i=1}^{k+2} s_i - \sum_{i=1}^{k+1} s_i \geq \binom{k+2}{2} - \binom{k+1}{2} = k + 1.$$

Consequently, by the induction hypothesis,

$$\sum_{i=1}^{k+1} s_i = s_{k+1} + \sum_{i=1}^{k} s_i \geq s_{k+1} + \binom{k}{2} + 1 \geq k + 1 + \binom{k}{2} + 1 \geq \binom{k+1}{2} + 1.$$

This contradicts (2.2). □

2.3 Spanning k-Strong Subtournaments of Semicomplete Digraphs

Theorem 1.7.3 asserts that every strong digraph D without a bridge contains a spanning strong oriented graph (obtained by deleting one arc from every 2-cycle in D). It is then natural to ask whether there exists, for each non-negative integer k, a minimum integer $f(k)$ such that every $f(k)$-strong digraph contains a spanning k-strong oriented graph. Because every k-strong oriented graph has at least $2k+1$ vertices and the complete digraph on $r+1$ vertices is r-strong, we have $f(k) \geq 2k$ for all $k \geq 2$. Jackson and Thomassen (see [178]) conjectured that this lower bound is indeed tight.

Conjecture 2.3.1 (Jackson and Thomassen [178]) *Every $2k$-strong digraph contains a spanning k-strong oriented graph.*

This conjecture is still widely open for general digraphs, even in the case when $k = 2$. It was verified by Thomassen [186] for the special case when $k = 2$ and D is a symmetric digraph (all arcs are in 2-cycles), thus improving on a result of Jordán [115] establishing the existence of a spanning 2-strong oriented graph in every 18-strong symmetric digraph. For all $k \geq 3$ it is still open whether there is a function $g(k)$ such that every $g(k)$-strong symmetric digraph has a spanning k-strong oriented subdigraph.

Even for the class of semicomplete digraphs the conjecture is open when $k \geq 3$. The case $k = 2$ and D semicomplete follows from the next result.

Improving an earlier bound of $5k$, due to Bang-Jensen and Thomassen, Guo proved the following, which implies that the case $k = 2$ of Conjecture 2.3.1 holds for semicomplete digraphs.

Theorem 2.3.2 ([99]) *Let k be a positive integer. Every $(3k-2)$-strong tournament contains a spanning k-strong tournament.*

Bang-Jensen and Jordán proved that the function $3k - 2$ is not the best possible when $k = 2$.

Theorem 2.3.3 ([30]) *Every 3-strong semicomplete digraph on at least 5 vertices contains a spanning 2-strong tournament. There is a polynomial algorithm for constructing a spanning 2-strong tournament of a given 3-strong semicomplete digraph.*

Bang-Jensen and Jordán conjectured that the bound $(3k - 2)$ can be improved as follows.

Conjecture 2.3.4 ([30]) *For each $k \geq 1$, every $(2k-1)$-strong semicomplete digraph on at least $2k+1$ vertices contains a spanning k-strong tournament.*

The number $(2k-1)$ would be the best possible as seen from the following construction from [30]: Let $k \geq 2$ be an integer, let U and W be disjoint copies of the complete digraph $\overleftrightarrow{K}_{2k-2}$ with vertex sets $\{u_1, \ldots, u_{2k-2}\}$ and $\{w_1, \ldots, w_{2k-2}\}$, respectively, and let H' be the semicomplete digraph obtained from these by adding the arcs of a matching $\{u_i w_i | i \in [2k-2]\}$ oriented from U to W and the arcs $\{w_i u_j | i, j \in [n] \text{ and } i \neq j\}$ from W to U. It is easy to check that H' is $(2k-2)$-strong and that H' cannot contain a spanning k-strong tournament, because when we delete one arc from every 2-cycle there is some vertex of U which will have out-degree at most $k-1$. By taking an arbitrary tournament C and adding all arcs from W to C and from C to U, we obtain an infinite family of $(2k-2)$-strong semicomplete digraphs containing no spanning k-strong tournament.

2.4 The Second Neighbourhood Conjecture

One of the (apparently) simplest open questions concerning digraphs is Seymour's Second Neighbourhood Conjecture, asserting that one can always find, in an oriented graph D, a vertex whose second out-neighbourhood is at least as large as its out-neighbourhood (see [69]).

Conjecture 2.4.1 (Seymour's Second Neighbourhood Conjecture) *In every oriented graph D, there exists a vertex x such that $|N_D^+(x)| \leq |N_D^{++}(x)|$.*

Observe that this conjecture is false for digraphs in general. Consider for example $\overleftrightarrow{K}_n$, the complete digraph on n vertices: for every vertex v, $N^+(v) = V(\overleftrightarrow{K}_n) \setminus \{v\}$ while $N^{++}(v) = \emptyset$.

Kaneko and Locke [116] proved Conjecture 2.4.1 for oriented graphs with minimum out-degree at most 6. Fidler and Yuster [79] proved that it holds for oriented graphs D with minimum degree $|V(D)|-2$, tournaments minus a star, and tournaments minus the arc set of a subtournament. Cohn, Godbole, Wright Harkness, and Zhang [66] proved that the conjecture holds for random oriented graphs. Gutin and Li proved Conjecture 2.4.1 for quasi-transitive oriented graphs [102].

One approach to Conjecture 2.4.1 is to determine the maximum value λ such that in every oriented graph D, there exists a vertex x such that $|N_D^+(x)| \leq \lambda |N_D^{++}(x)|$. The conjecture is that $\lambda = 1$. Chen, Shen, and Yuster [60] proved that $\lambda \geq \gamma$ where $\gamma = 0.657298...$ is the unique real root of $2x^3 + x^2 - 1 = 0$. They also claim a slight improvement to $\lambda \geq 0.67815...$

For tournaments, Seymour's Second Neighbourhood Conjecture was also known as Dean's conjecture [69] and was first solved by Fisher [80].

Theorem 2.4.2 ([80]) *In any tournament, there is a vertex v such that $|N^+(v)| \leq |N^{++}(v)|$.*

The original proof of Fisher used a sort of weighted version of the problem via probability distributions. It is presented in the next subsection. A more elementary proof using median orders was then given by Havet and Thomassé [109]. Their proof also yields the existence of two vertices v such that $|N^+(v)| \leq |N^{++}(v)|$ under the condition that no vertex is a **sink** (that is, a vertex of out-degree 0). This is detailed in Subsection 2.4.2.

2.4.1 Fisher's Original Proof

A **(probability) distribution** on a digraph D is a function p that assigns to each vertex a non-negative real number such that $p(V(D)) = \sum_{v \in V(D)} p(v) = 1$. For every subset S of $V(D)$, we set $p(S) = \sum_{v \in S)} p(v)$. A distribution is **losing** if $p(N^-(v)) \leq p(N^+(v))$ for all $v \in V(D)$.

Let D be an oriented graph with n vertices $v_1, \dots, v_n$. The **adjacency matrix** of D, denoted by $\mathbf{A}_D$, is the $n \times n$ matrix defined by $(\mathbf{A}_D)_{i,j} = 1$ if $v_i \to v_j$, $(\mathbf{A}_D)_{i,j} = -1$ if $v_j \to v_i$ and $(\mathbf{A}_D)_{i,j} = 0$ otherwise. Observe that $\mathbf{A}_D^T = -A_D$.

We shall use the following well-known lemma, due to Farkas, see e.g. [92, Lemma 1].

Lemma 2.4.3 (Farkas's Lemma) *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix and* $\mathbf{b}$ *an* m*-dimensional real vector. Then exactly one of the following two statements is true:*

1. *There exists a* $\mathbf{x} \in \mathbb{N}^n$ *such that* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *and* $\mathbf{x} \geq \mathbf{0}$*;*
2. *There exists a* $\mathbf{y} \in \mathbb{N}^m$ *such that* $\mathbf{A}^T\mathbf{y} \geq \mathbf{0}$ *and* $\mathbf{b}^T\mathbf{y} < \mathbf{0}$*.*

Theorem 2.4.4 ([80]) *Every digraph has a losing distribution.*

Proof: Let D be a digraph with n vertices $v_1, \dots, v_n$. To each distribution p of D, we can associate the vector $\mathbf{w}_p = (p(v_1), \dots, p(v_n))^T$. Observe that $\mathbf{w}_p \geq \mathbf{0}$ and $\mathbf{1}^T\mathbf{w}_p = p(V(D)) = 1$. Furthermore, $(\mathbf{A}_D\mathbf{w}_p)_{i,} = p(N_D^+(v_i)) - p(N_D^-(v_i))$. Hence p is a losing distribution if $\mathbf{A}_D\mathbf{w}_p \geq \mathbf{0}$.

Suppose D has no losing distribution. Since $\mathbf{A}_D^T = -\mathbf{A}_D$, the following system has no solutions. ($\mathbf{I}$ denotes the identity $n \times n$ matrix.)

$$\begin{bmatrix} \mathbf{A}_D^T & \mathbf{I} \\ \mathbf{1}^T & \mathbf{0}^T \end{bmatrix} \begin{pmatrix} \mathbf{w} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \text{ with } \begin{pmatrix} \mathbf{w} \\ \mathbf{z} \end{pmatrix} \geq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Farkas's Lemma implies that there exists an n-dimensional vector u and a real number t such that

$$\begin{bmatrix} \mathbf{A}_D & \mathbf{1} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u} \\ t \end{pmatrix} \geq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \text{ with } \left(\mathbf{0}^T \ 1\right) \begin{pmatrix} \mathbf{u} \\ t \end{pmatrix} < 0.$$

Thus $\mathbf{u} \geq \mathbf{0}$, $\mathbf{A}_D\mathbf{u} + t\mathbf{1} \geq \mathbf{0}$ and $t < 0$. Hence $\mathbf{A}_D\mathbf{u} > \mathbf{0}$, so $\frac{1}{\mathbf{1}^T\mathbf{u}}\mathbf{u}$ is the vector associated to a losing distribution, a contradiction. □

We shall now give some properties of losing distributions.

Lemma 2.4.5 *Let D be a digraph and p a losing distribution. If $p(v) > 0$, then $p(N^+(v)) = p(N^-(v))$.*

Proof: We use the notation of the previous proof.

Since p is a losing distribution, then $\mathbf{A}_D\mathbf{w}_p \geq \mathbf{0}$ and $\mathbf{w}_p \geq \mathbf{0}$. Hence $(\mathbf{w}_p)_i(\mathbf{A}_D\mathbf{w}_p)_i \geq 0$ for all i. But, since $\mathbf{A}_D$ is skew-symmetric, $(\mathbf{w}_p)^T\mathbf{A}_D\mathbf{w}_p = 0$, so $(\mathbf{w}_p)_i(\mathbf{A}_D\mathbf{w}_p)_i = 0$ for all i. Therefore if $(\mathbf{w}_p)_i = p(v_i) > 0$, necessarily, $0 = (\mathbf{A}_D\mathbf{w}_p)_i = p(N^+(v_i)) - p(N^-(v_i))$. In other words, if $w(v_i) > 0$, then $p(N^+(v_i)) = p(N^-(v_i))$. □

Lemma 2.4.6 *Let p be a losing distribution on a tournament T. Then $p(N^-(v)) \leq p(N^{--}(v))$ for every vertex v.*

Proof: Let v be a vertex of T. Since p is a losing distribution, $p(N^-(v)) \leq \frac{1}{2}$. If $p(N^{--}(v)) \geq \frac{1}{2}$, then we are done, so we may assume that $p(N^{--}(v)) < \frac{1}{2}$. Set $R = N^-(v) \cup N^{--}(v)$ and $Q = V(T) \setminus R$. We have $p(R) < 1$ and so $p(Q) > 0$.

Now

$$\sum_{w\in Q} p(w)p(N^-_{T\langle Q\rangle}(w)) = \sum_{w\in Q}\sum_{u\in N^-_{T\langle Q\rangle}(w)} p(w)p(u) = \sum_{u\in Q}\sum_{w\in N^+_{T\langle Q\rangle}(u)} p(w)p(u)$$
$$= \sum_{u\in Q} p(u)p(N^+_{T\langle Q\rangle}(u)).$$

Hence, there is a vertex $w \in Q$ with $p(w) > 0$ such that $p(N^-_{T\langle Q\rangle}(w)) \geq p(N^+_{T\langle Q\rangle}(w))$. By Lemma 2.4.5, $p(N^+_T(v)) = p(N^-_T(v))$. Since w is not in $N^{--}(v)$, it is dominated by $N^-_T(v)$. Thus $p(N^-_T(w)) \geq p(N^-_{T\langle Q\rangle}(w)) + p(N^-_T(v))$ and $p(N^+_T(w)) \leq p(N^+_{T\langle Q\rangle}(w)) + p(N^{--}_T(v))$. Hence

$$p(N^-_{T\langle Q\rangle}(w)) + p(N^-_T(v)) \leq p(N^+_{T\langle Q\rangle}(w)) + p(N^{--}_T(v)).$$

Since $p(N^-_{T\langle Q\rangle}(w)) \geq p(N^+_{T\langle Q\rangle}(w)$, we obtain $p(N^-(v)) \leq p(N^{--}(v))$.

□

We are now ready to prove Theorem 2.4.2.

Proof of Theorem 2.4.2: Let T be a tournament. By Theorem 2.4.4, it admits a losing distribution p.

Set $E^+ = \sum_{v\in V(T)} p(v)|N^+(v)|$ and let $E^{++} = \sum_{v\in V(T)} p(v)|N^{++}(v)|$. Since $w \in N^+(v)$ if and only if $v \in N^-(w)$, we have

$$E^+ = \sum_{v\in V(T)} p(v)|N^+(v)| = \sum_{v\in V(T)}\sum_{w\in N^+(v)} p(v) = \sum_{w\in V(T)}\sum_{v\in N^-(w)} p(v)$$
$$= \sum_{w\in V(T)} p(N^-(w)).$$

Similarly, since $w \in N^{++}(v)$ if and only if $v \in N^{--}(w)$, we have

$$E^{++} = \sum_{w \in V(T)} p(N^{--}(w)).$$

Now, as p is a losing distribution, it follows from Lemma 2.4.6 that we have $p(N^-(w)) \leq p(N^{--}(w))$ for every vertex w. Hence $E^+ \leq E^{++}$. Consequently, there must be a vertex v such that $|N^+(v)| \leq |N^{++}(v)|$. □

2.4.2 Proof Using Median Orders

Theorem 2.4.7 ([109]) *Let T be a tournament and $\sigma = (v_1, v_2, \ldots, v_n)$ be a median order of T. Then $|N^+_T(v_n)| \leq |N^{++}_T(v_n)|$.*

Proof: We distinguish two types of vertices of $N^-(v_n)$: a vertex $v_j \in N^-(v_n)$ is **σ-good** if there exists a vertex $v_i \in N^+(v_n)$, with $i < j$, such that $v_i \rightarrow v_j$; otherwise v_j is **σ-bad**. We denote by G_σ the set of σ-good vertices. Observe that $G_\sigma \subseteq N^{++}_T(v_n)$.

We shall prove by induction on n that $|N^+_T(v_n)| \leq |G_\sigma|$ which directly implies the result. The case $n = 1$ holds vacuously. Assume now $n > 1$. If there is no σ-bad vertex, then $G_\sigma = N^-(v_n)$. Moreover, by the property (M2) of Lemma 2.2.11, $|N^+(v_n)| \leq |N^-(v_n)|$, so the conclusion holds. Assume now that there exists a σ-bad vertex. Let i be the smallest integer i such that v_i is σ-bad. Set $T_r = T\langle\{v_{i+1}, \ldots, v_n\}\rangle$. By the property (M1) of Lemma 2.2.11, $\sigma_r = (v_{i+1}, \ldots, v_n)$ is a median order of T_r. By the induction hypothesis, $|N^+_{T_r}(v_n)| \leq |G_{\sigma_r}|$. Since every σ_r-good vertex is also σ-good, we get

$$|N^+_T(v_n) \cap \{v_{i+1}, \ldots, v_n\}| \leq |G_\sigma \cap \{v_{i+1}, \ldots, v_n\}|. \tag{2.3}$$

By the minimality of the index of i, every vertex of $\{v_1, \ldots, v_{i-1}\}$ is either in G_σ or in $N^+(v_n)$. Moreover, since v_i is σ-bad, we have $N^+(v_n) \cap \{v_1, \ldots, v_i\} \subseteq N^+(v_i) \cap \{v_1, \ldots, v_i\}$, so $G_\sigma \cap \{v_1, \ldots, v_i\} \supseteq N^-(v_i) \cap \{v_1, \ldots, v_i\}$. Now by property (M2) of Lemma 2.2.11, $|N^-(v_i) \cap \{v_1, \ldots, v_i\}| \geq |N^+(v_i) \cap \{v_1, \ldots, v_i\}|$. Hence

$$|N^+_T(v_n) \cap \{v_1, \ldots, v_i\}| \leq |N^-(x_i) \cap \{v_1, \ldots, v_i\}| \leq |G_\sigma \cap \{v_1, \ldots, v_i\}| \tag{2.4}$$

Equations (2.3) and (2.4) yield $|N^+_T(v_n)| \leq |G_\sigma|$. □

A natural question is to seek another vertex v **with large second out-neighbourhood**, i.e. such that $|N^+(v)| \leq |N^{++}(v)|$. Obviously, this is not always possible: consider, for instance, a regular tournament dominating a single vertex, or simply a transitive tournament. In both cases, the sole vertex v with $|N^{++}(v)| \geq |N^+(v)|$ is the sink. Still using median orders, Havet and Thomassé [109] proved that a tournament always has two vertices with large second out-neighbourhood, provided that every vertex has out-degree at least 1.

Theorem 2.4.8 ([109]) *A tournament with no sink has at least two vertices v such that $|N^+(v)| \le |N^{++}(v)|$.*

To prove this result, we need the notion of the **sedimentation** of a median order $\sigma = (v_1, \dots, v_n)$ of a tournament T, denoted by $Sed(\sigma)$. If $|N^+(v_n)| < |G_\sigma|$, then $Sed(\sigma) = \sigma$. If $|N^+(v_n)| = |G_\sigma|$, we denote by $b_1, ..., b_k$ the σ-bad vertices and by $w_1, ..., w_{n-1-k}$ the vertices of $N^+(v_n) \cup G_\sigma$, both enumerated in increasing order with respect to σ. In this case, $Sed(\sigma)$ is the order $(b_1, \dots, b_k, v_n, w_1, \dots, w_{n-1-k})$.

Lemma 2.4.9 *If σ is a median order of a tournament T, then $Sed(\sigma)$ is also a median order of T.*

Proof: Let $\sigma = (v_1, \dots, v_n)$ be a median order of T. If $Sed(\sigma) = \sigma$, there is nothing to prove. So we assume it is not the case, that is, $|N^+(v_n)| = |G_\sigma|$.

The proof is by induction on the number k of σ-bad vertices. If $k = 0$, all the vertices are σ-good or in $N^+(v_n)$, in particular $N^-(v_n) = G_\sigma$. Thus, $|N^+(v_n)| = |N^-(v_n)|$ and the order $Sed(\sigma) = (v_n, v_1, \dots, v_{n-1})$ is a median order of T. Assume now that k is a positive integer. Let i be the smallest index (wrt. σ) of a σ-bad vertex.

For convenience, for any set S, we denote by $S[i,j]$ the set $S \cap \{v_i, \dots, v_j\}$. By Equation (2.3), $|G_\sigma[i+1,n]| \ge |N_T^+(v_n)[i+1,n]|$, and by Equation (2.4), $|G_\sigma[1,i]| \ge |N_T^+(v_n)[1,i]|$. Now by assumption, $|G_\sigma| = |N^+(x_n)|$, that is, $|G_\sigma[1,i]| + |G_\sigma[i+1,n]| = |N_T^+(v_n)[1,i]| + |N_T^+(v_n)[i+1,n]|$. Hence $|G_\sigma[1,i]| = |N_T^+(v_n)[1,i]|$ and $|G_\sigma[i+1,n]| = |N_T^+(v_n)[i+1,n]|$. But since x_i is σ-bad, $N^+(v_n)[1,i] \subseteq N^+(v_i)[1,i]$ and so $N^-(v_i)[1,i-1] \subseteq N^-(v_n)[1,i-1]$. Moreover, by property (M2) of Lemma 2.2.11, $|N^+(v_i)[1,i]| \le |N^-(v_i)[1,i]|$ and by defintion of i, $N^-(v_n)[1,i-1] = G_\sigma[1,i-1] = G_\sigma[1,i]$. Hence,

$$|G_\sigma[1,i]| \le |N^+(v_i)[1,i]| \le |N^-(v_i)[1,i]| = G_\sigma[1,i].$$

Thus $|N^+(v_i)[1,i]| \le |N^-(v_i)[1,i]|$, and so $(v_i, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ is a median order of T. Applying the induction hypothesis to the median order $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$, which has one bad vertex less than σ, we obtain the result. □

Proof of Theorem 2.4.8: Let $\sigma = (v_1, \dots, v_n)$ be a median order of T. By Theorem 2.4.7, v_n has a large second neighbourhood, so we need to find another vertex with this property.

Observe that if $(u_1, \dots, u_{n-1})$ is a median order of $T - v_n$, then the order $(u_1, \dots, u_{n-1}, v_n)$ is a median order of T, and consequently $u_{n-1} \to v_n$.

Set $T^* = T - v_n$ Assume first that T^* has a median order $\sigma^* = (u_1, \dots, u_{n-1})$ such that $\sigma^* = Sed(\sigma^*)$. Then

$$|N_T^+(u_{n-1})| = |N_{T^*}^+(u_{n-1})| + 1 \le |G_{\sigma^*}| \le |N_{T^*}^{++}(u_{n-1})| \le |N_T^{++}(u_{n-1})|.$$

Assume now that for every median order σ^* of T^*, $\sigma^* \neq Sed(\sigma^*)$. Define now inductively $\sigma_0 = (v_1, \dots, v_{n-1})$ and $\sigma_{q+1} = Sed(\sigma_q)$. By property (M1) of Lemma 2.2.11, σ_0 is a median order of T^*; Lemma 2.4.9 and an easy induction imply that σ_q is a median order of T^* for every positive integer q. Since T has no dominated vertex, v_n has an out-neighbour v_j. As observed above, for every integer q, the last vertex of σ_q dominates v_n. So v_j is not the last vertex of any σ_q. Observe also that there is a q such that v_j is σ_q-bad, for otherwise the index of x_j would always increase. Let $\sigma_q = (u_1, \dots, u_{n-1})$. We have

$$|N_T^+(u_{n-1})| = |N_{T^*}^+(u_{n-1})| + 1 = |G_{\sigma_q}| + 1.$$

Moreover $u_{n-1} \rightarrow v_n \rightarrow v_j$, so the second neighbourhood of u_{n-1} has at least $|G_{\sigma_q}| + 1$ elements. Hence $|N_T^+(u_{n-1})| \leq |N_T^{++}(u_{n-1})|$. □

2.4.3 Relation with Other Conjectures

One of the most celebrated problems concerning digraphs is the Caccetta–Häggkvist conjecture.

Conjecture 2.4.10 (Caccetta and Häggkvist [54]) *Every digraph D on n vertices and with minimum out-degree at least n/k has a directed cycle of length at most k.*

Since every non-transitive tournament contains a directed 3-cycle, this conjecture easily holds for tournaments. However, little is known about this problem, and, more generally, questions concerning digraphs and involving the minimum out-degree tend to be intractable. As a consequence, many open problems flourished in this area, see [175] for a survey. The Hoàng–Reed conjecture [112] is one of these.

A **directed-cycle-tree** is either a singleton or consists of a set of directed cycles $C_1, \dots, C_k$ such that $|V(C_i) \cap (V(C_1) \cup \dots \cup V(C_{i-1}))| = 1$ for all $i = 2, \dots, k$, where $V(C_j)$ is the set of vertices of C_j. A less explicit, yet concise, definition is simply that a directed-cycle-tree is a digraph in which there exists a unique directed (x, y)-path for every choice of distinct vertices x and y. A vertex-disjoint union of directed-cycle-trees is a **directed-cycle-forest**. When all directed cycles have length 3, we speak of a **triangle-tree**. For short, a k-directed-cycle-forest is a directed-cycle-forest consisting of k directed cycles.

Conjecture 2.4.11 (Hoàng and Reed [112]) *Every digraph D has a $\delta^+(D)$-directed-cycle-forest.*

In the case $\delta^+(D) = 2$, Thomassen proved in [187] that every digraph with minimum out-degree 2 has two directed cycles intersecting on a vertex (i.e. contains a directed-cycle-tree with two directed cycles). Welhan [192] proved Conjecture 2.4.11 for $\delta^+(D) = 3$. The motivation of the Hoàng-Reed conjecture is that it would imply the Caccetta-Häggkvist conjecture, as the reader can easily check.

Havet, Thomassé and Yeo [111] proved Conjecture 2.4.11 for tournaments. This result does not yield a better understanding of Hoàng–Reed conjecture. However, it gives a little bit of insight into the triangle-structure of a tournament T, that is, the 3-uniform hypergraph on the vertex set $V(T)$ whose hyperedges are the directed 3-cycles of T.

Indeed, by the fact that every directed cycle in a tournament induces a strong subtournament that contains a directed 3-cycle through any given vertex, if a tournament T has a $\delta^+(T)$-directed-cycle-forest, then T also has a $\delta^+(T)$-triangle-forest. Observe that a $\delta^+(T)$-triangle-forest spans exactly $2\delta^+(T) + c$ vertices, where c is the number of components of the triangle-forest. When T is a regular tournament with out-degree $\delta^+(T)$, hence with $2\delta^+(T) + 1$ vertices, a $\delta^+(T)$-triangle-forest of T is necessarily a spanning $\delta^+(T)$-triangle-tree. Havet, Thomassé and Yeo [111] established the existence of such a tree for every tournament.

Theorem 2.4.12 ([111]) *Every tournament T has a $\delta^+(T)$-triangle-tree.*

2.5 Disjoint Paths and Cycles

We now turn to results on linkages and weak linkages in semicomplete digraphs. The reader may wish to recall the definitions of these from Section 1.6.

2.5.1 Polynomial Algorithms for Linkage and Weak Linkage

WEAK k-LINKAGE
Input: A digraph $D = (V, A)$ and not necessarily distinct vertices $s_1, \dots, s_k, t_1, \dots, t_k$.
Question: Does D contain a weak k-linkage from $(s_1, \dots, s_k)$ to $(t_1, \dots, t_k)$?

Recall that for general digraphs the WEAK k-LINKAGE problem is $\mathcal{NP}$-complete already when $k = 2$ [84]. Bang-Jensen [16] solved the WEAK k-LINKAGE problem for semicomplete digraphs by giving a polynomial algorithm and a complete characterization of those semicomplete digraphs that do not have a weak linkage from (s_1, s_2) to (t_1, t_2) for given vertices s_1, s_2, t_1, t_2 where we may have $s_{3-i} = t_i$ for $i = 1$ or $i = 2$ but all other vertices are distinct (all the remaining cases are easy for semicomplete digraphs).

Fradkin and Seymour [85] generalized the algorithmic part of these results in two ways: from weak 2-linkage to weak k-linkage for any fixed integer k and from semicomplete digraphs to digraphs of bounded independence number.

Theorem 2.5.1 ([85]) *For every fixed positive integer α the* WEAK k-LINKAGE *problem is polynomially solvable for every fixed k, when we consider digraphs with independence number at most α.*

A key ingredient in the proof of this theorem is the notion of the cutwidth of a digraph. Let $D = (V, A)$ be a digraph and let $\mathcal{O} = (v_1, v_2, \ldots, v_n)$ be an ordering of the vertices of D. We say that $\mathcal{O}$ has **cutwidth** at most θ if for all $j \in \{2, 3, \ldots, n\}$ there are at most θ arcs uv with $u \in \{v_1, \ldots, v_{j-1}\}$ and $v \in \{v_j, \ldots, v_n\}$ and we say that D has cutwidth at most θ if there exists an ordering $\mathcal{O}$ of $V(D)$ which has cutwidth at most θ. The minimum θ such that D has cutwidth at most θ is called the **cutwidth** of D and is denoted by $cw(D)$.

Barbero, Paul and Pilipczuk proved that, even for semicomplete digraphs, cutwidth is not an easy parameter to determine.

Theorem 2.5.2 ([37]) *Determining the cutwidth of a semicomplete digraph is $\mathcal{NP}$-hard.*

Single exponential FPT algorithms were obtained in [82, 152]. Pilipczuk found an approximation algorithm for the cutwidth of semicomplete digraphs.

Theorem 2.5.3 ([152]) *There exists an $O(n^2)$ algorithm for computing an ordering $\mathcal{O}$ of an n-semicomplete digraph D whose cutwidth is at most $O(cw(D)^2)$.*

In fact, it is shown in [152] (see also [153]) that just sorting the vertices according to their out-degrees achieves the bound above. See [153] for a discussion of which properties of a semicomplete digraph forces high cutwidth. One such example is the result that if a semicomplete digraph D contains a set S of $4k + 2$ vertices such that the maximum difference between the out-degrees of any pair of vertices in S is at most k, then $cw(D) \geq k/2$ holds. Many other results on cutwidth of semicomplete digraphs can be found in the paper [81] by Fomin and Pilipczuk and in Pilipczuk's thesis [154].

For tournaments the situation is much better. Barbero, Paul and Pilipczuk proved the following.

Theorem 2.5.4 ([37]) *One can determine the cutwidth of a tournament in polynomial time. Furthermore, if $cw(T) = p$, then T contains a subtournament T' whose number of vertices is linear in p and such that $cw(T) = cw(T')$.*

Fradkin and Seymour also solved the WEAK k-LINKAGE problem for the class of directed pseudographs that one obtains from semicomplete digraphs by adding arcs and loops.

Theorem 2.5.5 (Fradkin and Seymour [85]) *The* WEAK k-LINKAGE *problem is solvable in polynomial time for every fixed k, when we consider directed pseudographs that are obtained from a semicomplete digraph by replacing some arcs with multiple copies of those arcs and adding any number of loops.*

We now turn to vertex-disjoint linkages.

> k-LINKAGE
> **Input:** A digraph $D = (V, A)$ and not necessarily distinct vertices $s_1, \dots, s_k, t_1, \dots, t_k$.
> **Question:** Does D contain a k-linkage from $(s_1, \dots, s_k)$ to $(t_1, \dots, t_k)$?

Below we shall always assume that all the terminals to be linked (that is, $s_1, \dots, s_k, t_1, \dots, t_k$) are distinct. Bang-Jensen and Thomassen solved the 2-linkage problem for semicomplete digraphs.

Theorem 2.5.6 ([34]) *The 2-linkage problem is solvable in time $O(n^5)$ for semicomplete digraphs.*

Bang-Jensen and Thomassen also proved that if k is part of the input, then the k-linkage problem is $\mathcal{NP}$-complete already for tournaments.

Besides the trivial case $k = 1$, the value 2 remained the only k for which the k-linkage problem was solved for semicomplete digraphs until Chudnovsky, Seymour and Scott [62] found a polynomial algorithm for the k-linkage problem for any fixed k in semicomplete digraphs. In fact, their algorithm works for a more general class of digraphs which they call d-path dominant. A digraph $D = (V, A)$ is **d-path-dominant** if, for every minimal path P on d vertices, every vertex $v \in V - V(P)$ is adjacent to at least one vertex of $V(P)$. Thus D is 1-path dominant if and only if it is semicomplete and 2-path dominant if and only if it is semicomplete multipartite. Hence this is a very general class of digraphs.

Theorem 2.5.7 ([62]) *For all fixed d, k there is a polynomial algorithm for the k-linkage problem in d-path-dominant digraphs.*

Following [62], for a given sequence $\mathbf{x} = (x_1, \dots, x_k)$ of positive integers, we say that the digraph D has an **x-linkage** from $(s_1, s_2, \dots, s_k)$ to $(t_1, t_2, \dots, t_k)$ if it has a collection of disjoint paths $P_1, P_2, \dots, P_k$ such that P_i is an (s_i, t_i)-path and has x_i vertices. A sequence $\mathbf{x} = (x_1, \dots, x_k)$ of positive integers is then a **quality** of $(D, s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_k)$ if D has an $\mathbf{x}$-linkage from $(s_1, s_2, \dots, s_k)$ to $(t_1, t_2, \dots, t_k)$. A quality $\mathbf{x}$ of $(D, s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_k)$ is a **key quality** if there is no other quality $\mathbf{y} \neq \mathbf{x}$ with $y_i \leq x_i$ for all $i \in [k]$. The main result of [62] is the following.

Theorem 2.5.8 ([62]) *For all integers $d, k \geq 1$ there exists a polynomial algorithm for the following problem: Given a d-path-dominant digraph $D = (V, A)$*

and vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$, compute the set of key qualities of $(D, s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k)$. The algorithm runs in time $O(n^{6k^2d(k+d)+13k})$.

Corollary 2.5.9 ([62]) *For all integers $d, k \geq 1$ there exists a polynomial algorithm for the following problem: Given a d-path-dominant digraph $D = (V, A)$, vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ and integers $x_1, x_2, \ldots, x_k \geq 1$, decide whether D contains disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is an (s_i, t_i)-path and has at most x_i vertices.*

The proof of Theorem 2.5.7 is long but the main idea is simple: as in the algorithm for k-linkage in acyclic digraphs (see Section 3.4) one can define an auxiliary digraph H with two special vertices s_0, t_0 such that H has an (s_0, t_0)-path if and only if D has the desired k-linkage.

The following problem is open even for $k = 2$ and independence number 2.

Problem 2.5.10 *Determine the complexity of the k-linkage problem for digraphs with bounded independence number.*

A special case of digraphs with independence number at most p is the class of digraphs that have p-partition $(V_1, V_2, \ldots, V_p)$ such that $D[V_i]$ is a semicomplete digraph. For this class Chudnovsky, Scott and Seymour recently found a solution.

Theorem 2.5.11 ([63]) *For every pair of fixed positive integers k, p, the k-linkage problem is polynomially solvable for digraphs which have a p-partition each part of which is semicomplete and provided we are given such a partition as part of the input.*

For an application of that result, see the discussion around Theorem 6.11.3.

2.5.2 Sufficient Conditions for a Tournament to be k-Linked

We now turn to sufficient conditions in terms of connectivity for a semicomplete digraph to be k-linked. Bang-Jensen determined the minimum connectivity implying 2-linkedness.

Theorem 2.5.12 ([17]) *Every 5-strong semicomplete digraph is 2-linked. Furthermore, there exists an infinite class of 4-strong tournaments which are not 2-linked (see Figure 2.1).*

We leave it to the reader to check that one can generalize the example in Figure 2.1 to an infinite family of 4-strong semicomplete digraphs none of which is 2-linked (see also [17]).

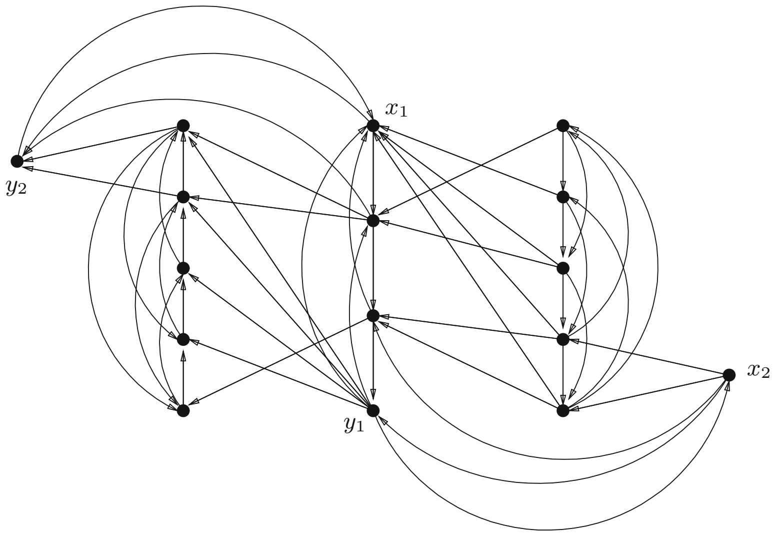

Figure 2.1 A 4-strong non-2-linked semicomplete digraph T. All arcs not shown go from left to right and $x_1y_2x_1, x_2y_1x_2$ are the only 2-cycles in T. There is no pair of disjoint (x_1, y_1)-,(x_2, y_2)-paths in T. The tournament which results from T by deleting the arcs y_2x_1 and y_1x_2 is also 4-strong

Thomassen [179] proved the existence of a function $f(k)$ such that every $f(k)$-strong tournament is k-linked. Clearly $f(1) = 1$ and by Theorem 2.5.12 we have $f(2) = 5$. Thomassen's function $f(k)$ grows exponentially in k. This was first improved to a polynomial in k by Kühn, Lapinskas, Osthus and Patel [124] and recently Pokrovskiy showed that a linear function suffices. We will give the main details in the proof of that result below.

A key ingredient in Pokrovskiy's proof of Theorem 2.5.15 is the following interesting result which illustrates the richness of tournament structure.

Theorem 2.5.13 ([156]) *Let n, p be positive integers satisfying $p \leq n/11$. Every n-tournament contains two disjoint sets of vertices $\{x_1, \ldots, x_p\}$ and $\{y_1, \ldots, y_p\}$ such that for every permutation σ of $[p]$, T contains vertex-disjoint paths $P_1, \ldots, P_p$ such that P_i is an $(x_i, y_{\sigma(i)})$-path.*

For later reference, we call the sets $\{x_1, \ldots, x_p\}$, $\{y_1, \ldots, y_p\}$ in the above theorem an **all-linkable pair**.

We need some more concepts which were introduced by Kühn, Lapinskas, Osthus and Patel in [124]. A set of vertices X **in-dominates** (**out-dominates**) another set Y in a digraph D if every $y \in Y \setminus X$ has an out-neighbour (in-neighbour) in X. The definition implies that any set in-dominates (out-dominates) itself. An **in-dominating** (**out-dominating**) set in D is then a set which in-dominates (out-dominates) $V(D)$.

Below we focus on semicomplete digraphs. Every n-semicomplete digraph contains an in-dominating (out-dominating) set of size $\lceil \log n \rceil$. Such a set X can be constructed from the empty set by repeatedly adding a vertex v of maximum in-degree (out-degree) in the current semicomplete digraph D to X and then deleting v together with its in-neighbourhood (out-neighbourhood) from D.

In a semicomplete digraph a vertex x may be both an in- and an out-neighbour of a vertex v, so we needed to adjust the definition below a bit compared to [124]. For a vertex v of a semicomplete digraph we define the sets $N^{+!}(v), N^{-!}(v)$ as follows: $N^{+!}(v) = V \setminus (N^-(v) \cup \{v\})$ and $N^{-!}(v) = V \setminus (N^+(v) \cup \{v\})$.

A sequence of vertices $(v_1, v_2, \dots, v_k)$ of a semicomplete digraph D is a **partial greedy in-dominating sequence** if v_1 has maximum in-degree in D and for each i, the vertex v_i has maximum in-degree in $D[N^{+!}(v_1) \cap \dots \cap N^{+!}(v_{i-1})]$. Similarly, $(v_1, v_2, \dots, v_k)$ is a **partial greedy out-dominating sequence** if v_1 has maximum out-degree in D and for each i, v_i has maximum out-degree in $D[N^{-!}(v_1) \cap \dots \cap N^{-!}(v_{i-1})]$.

Note that if at some point the set $N^{+!}(v_1) \cap \dots \cap N^{+!}(v_{i-1})$ ($N^{-!}(v_1) \cap \dots \cap N^{-!}(v_{i-1})$) becomes empty, then the sequences above may have less than k vertices. This will not affect the validity of the proof below.

As we saw above, if $k = \lceil \log n \rceil$ then every partial greedy in-dominating (resp. out-dominating) sequence on k vertices is an in-dominating (resp. out-dominating) sequence. The following very nice property of partial greedy in- and out-dominating sequences, which was first observed by Kühn *et al.* [124] and later reformulated by Pokrovskiy [156], shows that already for much smaller values of k, partial greedy dominating sequences are useful (as illustrated in the proof below).

Lemma 2.5.14 ([124, 156]) *Let $X = (v_1, v_2, \dots, v_k)$ be a partial greedy in-dominating (resp. out-dominating) sequence in a semicomplete digraph D. Let Y be the set of vertices which are not in-dominated (resp. out-dominated) by X. Then every $y \in Y$ satisfies $d^+(y) \geq 2^{k-1}|Y|$ ($d^-(y) \geq 2^{k-1}|Y|$).*

We are now ready to state and prove the main result of [156].

Theorem 2.5.15 ([156]) *Every $452k$-strong semicomplete digraph is k-linked.*

Proof: Pokrovskiy did not express his result for semicomplete digraphs, but his proof, which we give below, is also valid for semicomplete digraphs. Let D be a $452k$-strong semicomplete digraph. In particular this means that $\delta^0(D) \geq 452k$. Let $x_1, \dots, x_k, y_1, \dots, y_k$ be an arbitrary collection of $2k$ distinct vertices of D. We shall construct disjoint paths $R_1, \dots, R_k$ so that R_i is an (x_i, y_i)-path for $i \in [k]$. Let $D' = D \setminus \{x_1, \dots, x_k, y_1, \dots, y_k\}$.

Let I_1^- be a partial greedy in-dominating set on two vertices of D' and for each $i = 2, \dots, 55k$, let I_i^- be a partial greedy in-dominating set of $D' \setminus (I_1^- \cup \dots \cup I_{i-1}^-)$. Finally, let $D'' = D' \setminus (I_1^- \cup \dots \cup I_{55k}^-)$. Denote the vertices of I_i^- by u_i^-, v_i^-, $i \in [55k]$, where u_i^- is the first vertex chosen. Note that if at some point the first vertex we choose is already an in-dominating set, then $I_i^- = \{u_i^-\}$ and we let $v_i^- = u_i^-$. Otherwise $I_i^- = \{u_i^-, v_i^-\}$ and u_i^- dominates v_i^-. Now let O_1^+ be a partial greedy out-dominating set on two vertices of D'' and for each $i = 2, \ldots, 55k$ let O_{i-1}^+ be a partial greedy out-dominating set on two vertices of $D'' \setminus (O_1^+ \cup \ldots \cup O_{i-1}^+))$. As above we denote O_i^+ by $\{u_i^+, v_i^+\}$, where possibly $v_i^+ = u_i^+$ and otherwise v_i^+ dominates u_i^+.

Let $X = I_1^- \cup \ldots \cup I_{55k}^- \cup O_1^+ \cup \ldots \cup O_{55k}^+ \cup \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. By construction, $|X| \leq 222k$. Note that we may not have equality since, by the remark above, some of the sets constructed may have size one instead of two. For each $i \in [55k]$ denote by E_i^- (resp. E_i^+) the sets of those vertices of $D - X$ that are not in-dominated by I_i^- (resp. out-dominated by O_i^+). By Lemma 2.5.14, each vertex in $v \in E_i^-$ (resp. $w \in E_i^+$) satisfies $d^+(v) \geq 2|E_i^-|$ (resp. $d^-(w) \geq 2|E_i^+|$).

Let $V^- = \{v_1^-, \ldots, v_{55k}^-\}$ and $V^+ = \{v_1^+, \ldots, v_{55k}^+\}$. By Theorem 2.5.13, applied to $D[V^-]$ (resp. $D[V^+]$), we can find two sets X^-, Y^- (resp. X^+, Y^+) both of order $5k$ in V^- (resp. V^+) which form an all-linkable pair in $D[V^-]$ (resp. $D[V^+]$). Now relabel $I_1^-, \ldots, I_{55k}$ and $O_1^+, \ldots, O_{55k}^+$ so that $X^- = \{v_1^-, \ldots, v_{5k}^-\}$ and $Y^+ = \{v_1^+, \ldots, v_{5k}^+\}$.

By assumption, D is $452k$-strong so Menger's theorem (Theorem 1.5.3) implies that $D[(V - X) \cup Y^- \cup X^+)]$ has $5k$ disjoint paths $Q_1, \ldots, Q_{5k}$ which all start in Y^- and end in X^+. As $|X| \leq 222k$, for each $i \in [k]$ there exist distinct vertices $x_1', \ldots, x_k', y_1', \ldots, y_k' \in V \setminus X$ such that x_i' is dominated by x_i and y_i' dominates y_i for $i \in [k]$. Let $X' = X \cup \{x_1', \ldots, x_k', y_1', \ldots, y_k'\}$.

Now we consider the vertices of E_i^- and E_i^+, $i \in [55k]$. We saw above that each vertex in $v \in E_i^-$ (resp. $w \in E_i^+$) satisfies $d^+(v) \geq 2|E_i^-|$ (resp. $d^-(w) \geq 2|E_i^+|$). We also have $d^+(v) \geq 452k \geq 2|X'| + 4k$ (resp. $d^-(w) \geq 452k \geq 2|X'| + 4k$) so by averaging these two lower bounds we get that $d^+(v) \geq |E_i^-| + |X'| + 2k$ for every $v \in E_i^-$ and similarly $d^-(w) \geq |E_i^+| + |X'| + 2k$ for every $w \in E_i^+$. This implies that every $v \in E_i^-$ (resp. $w \in E_i^+$) has at least $2k$ out-neighbours (resp. in-neighbours) outside of $E_i^- \cup X'$ ($E_i^+ \cup X'$). For each $i \in [k]$ define x_i'' (resp. y_i'') as follows: If $x_i' \notin E_i^-$ (resp. $y_i' \notin E_i^+$), then x_i' dominates (resp. y_i' is dominated by) at least one vertex of I_i^- (O_i^+) and we let $x_i'' = x_i'$ (resp. $y_i'' = y_i'$). Otherwise $x_i' \in E_i^-$ (resp. $y_i' \in E_i^+$) and now we let x_i'' (resp. y_i'') be an out-neighbour of x_i' (resp. y_i') in $D - (E_i^- \cup X')$ ($D - (E_i^+ \cup X')$). By the remark above, we can choose the $2k$ vertices (some of which may not be new) $x_1'', \ldots, x_k'', y_1'', \ldots, y_k''$ so that these are all distinct.

Note that x_i'' dominates (resp. y_i'' is dominated by) at least one of the vertices in I_i^- (resp. O_i^+) for $i \in [k]$. Thus, for each $i \in [k]$ we can take the (x_i'', v_i^-)-path Q_i^- to be either the arc $x_i''v_i^-$ or the path $x_i''u_i^-v_i^-$. Similarly, we can take the (v_i^+, y_i'')-path Q_i^+ to be either the arc $v_i^+y_i''$ or the path $v_i^+u_i^+y_i''$. By construction, all the paths $Q_1^-, \ldots, Q_k^-, Q_1^+, \ldots, Q_k^+$ are disjoint.

At least k of the paths $Q_1, \ldots, Q_{5k}$ do not intersect any of the paths $Q_1^-, \ldots, Q_k^-, Q_1^+, \ldots, Q_k^+$ so fix such a set $Q_1', \ldots, Q_k'$ to be such paths. Since

Q_i^- ends in X^- and Q_i' starts in Y^-, Theorem 2.5.13 implies that we can find disjoint paths $P_1^-, \ldots, P_k^-$ in $D[V^-]$ such that P_i^- starts in v_i^- and ends in the initial vertex of Q_i'. Similarly, we can find disjoint paths $P_1^+, \ldots, P_k^+$ in $D[V^+]$ such that P_i^+ starts in the terminal vertex of Q_i' and ends in v_i^+.

Let $R_i = x_i x_i' Q_i^- P_i^- Q_i' P_i^+ y_i' y_i$ for $i \in [k]$. By the above arguments, $R_1, \ldots, R_k$ form the desired linkage. □

The value $452k$ is probably far from being best possible and the real answer could be close to $2k$. By Theorem 2.5.12, $f(k) > 2k$, at least when $k = 2$.

Proposition 2.5.16 ([156]) *For all $n \geq 6k$, there exists a $(2k-2)$-strong n-tournament T which is not k-linked.*

Note also that Theorem 2.6.15 gives a better bound when $k < 449$ and even guarantees that there is a linkage that spans all vertices of T.

Pokrovskiy conjectures that when the minimum semi-degree is sufficiently high, already $2k$-strong should be sufficient to guarantee a k-linkage for every choice of terminals.

Conjecture 2.5.17 ([156]) *For every k there exists an integer $d = d(k)$ such that every $2k$-strong tournament T with $\delta^0(T) \geq d$ is k-linked.*

2.5.3 The Bermond–Thomassen Conjecture for Tournaments

We now turn to disjoint directed cycles. We only discuss the celebrated Bermond–Thomassen conjecture. For more results on disjoint cycles, see Section 2.8.

Thomassen [42, 180] proved that every digraph D with $\delta^+(D) \geq 3$ has two disjoint cycles. Inspired by this, Bermond and Thomassen posed the following difficult conjecture.

Conjecture 2.5.18 (Bermond–Thomassen [42]) *For every positive integer k, every digraph D with $\delta^+(D) \geq 2k+1$ has k disjoint cycles.*

This difficult conjecture is wide open. Lichiardopol, Pór and Sereni [134] have verified the conjecture for $k = 3$. Alon [4] was the first to prove that a linear bound suffices. He obtained the following result.

Theorem 2.5.19 *There exists an absolute constant C such that $f(k) \leq Ck$ for all k. In particular, $C = 64$ will do.* ◇

We now consider tournaments and semicomplete digraphs. By Moon's Theorem (2.2.7), a tournament has k disjoint cycles if and only if it has k disjoint 3-cycles so the following result, due to Bang-Jensen, Bessy and Thomassé, shows that Conjecture 2.5.18 holds for tournaments.

Theorem 2.5.20 ([19]) *Every tournament T with $\delta^+(T) \geq 2k-1$ has k disjoint 3-cycles.*

Bang-Jensen, Bessy and Thomassé showed how to improve this bound on the minimum out-degree for tournaments with large minimum out-degree. Roughly speaking, a tournament T with $\delta^+(T) > 1.5k$ and k large enough contains k disjoint 3-cycles. More precisely, they proved the following.

Theorem 2.5.21 ([19]) *For every real number $\alpha > 1.5$, there exists a constant k_α such that, for every $k \geq k_\alpha$, every tournament T with $\delta^+(T) \geq \alpha k$ has k disjoint 3-cycles.*

The constant 1.5 is the best possible as shown by the circulant tournaments $CT_{2p+1}(\{1,2,\ldots,p\})$: when $2p+1 \equiv 0 \mod 3$, every vertex has out-degree $p = \lfloor \frac{3}{2}k \rfloor$, where $k = \frac{2p+1}{3}$, and $CT_{2p+1}(\{1,2,\ldots,p\})$ has a cycle factor consisting of k disjoint 3-cycles covering all its vertices [19].

It is important to note that the following obvious idea does not lead to a proof of Conjecture 2.5.18 for tournaments: find a 3-cycle C which is not dominated by any vertex of $V(T) \setminus V(C)$, remove C and apply induction. This approach does not work because of the following.[2]

Proposition 2.5.22 ([19]) *For infinitely many $k \geq 3$ there exists a tournament T with $\delta(T) = 2k-1$ such that every 3-cycle C is dominated by at least one vertex of minimum out-degree.*

Proof: Consider the Paley tournament $\mathbb{P}_{11}$. It has vertex set $V(\mathbb{P}_{11}) = \{1,2,\ldots,11\}$ and arc set $A(\mathbb{P}_{11}) = \{(i, i+p \pmod{11}) \mid i \in [11], p \in \{1,3,4,5,9\}\}$. The possible types of 3-cycles in T are $i \to i+1 \to i+10 \to i, i \to i+1 \to i+6 \to i, i \to i+3 \to i+6 \to i, i \to i+3 \to i+7 \to i$, where the indices are taken modulo 11. These are dominated by the vertices $i-3, i-3, i+2, i+2$, respectively. By substituting an arbitrary tournament for each vertex of $\mathbb{P}_{11}$, we can obtain a tournament with arbitrarily many vertices which has the property that every 3-cycle is dominated by some vertex of minimum out-degree. □

On the other hand, removing a 2-cycle from a digraph D with $\delta^+(D) \geq 2k-1$ clearly results in a new digraph D' with $\delta^+(D') \geq 2(k-1)-1$ and hence, when trying to prove Conjecture 2.5.18, we may always assume that the digraph in question has no 2-cycles. In particular, the following is a direct consequence of Theorem 2.5.20.

Corollary 2.5.23 *Every semicomplete digraph D with $\delta^+(D) \geq 2k-1$ contains k disjoint cycles.* □

[2] See also [9, Section 9.1].

For regular tournaments Lichiardopol proved the following, which strengthens Theorem 2.5.20 when r is larger than 20.

Theorem 2.5.24 ([132]) *Every $(2r-1)$-regular tournament contains at least $\frac{7}{6}r - \frac{7}{3}$ disjoint cycles.*

Lichiardopol posed the following conjecture, which he proved for $k = 2$. The complete digraph on $g(k) + 1$ vertices shows that $g(k) \geq \frac{k^2+3k-2}{2}$.

Conjecture 2.5.25 ([131]) *For every $k \geq 2$, there exists an integer $g(k)$ such that every digraph D with $\delta^+(D) \geq g(k)$ has k disjoint cycles of different lengths.*

Bensmail, Harutyunyan, Le, Li and Lichiardopol [40] confirmed the conjecture for tournaments.

Theorem 2.5.26 ([40]) *Every tournament T with $\delta^+(T) \geq \frac{k^2+4k-3}{2}$ contains k disjoint cycles of different lengths.*

It is natural to ask for the minimum function $g_T(k)$ such that every tournament T with $\delta^+(T) \geq \frac{k^2+4k-3}{2}$ contains k disjoint cycles of different lengths. The regular tournaments on $n = 2g_T(k) + 1$ vertices show that $g_T(k) \geq \frac{k^2+5k-2}{4}$.

Finally, we point out that already for tournaments it is difficult to find the maximum number of disjoint cycles. The following recent result is due to Bessy, Marin and Thiebaut. The authors also showed that there is no polynomial time approximation scheme for the problem unless $\mathcal{P} = \mathcal{NP}$.

Theorem 2.5.27 ([43]) *Finding the maximum number of disjoint 3-cycles in a tournament is $\mathcal{NP}$-hard.*

2.6 Hamiltonian Paths and Cycles

In this section we discuss results on the number of Hamiltonian paths in tournaments, Hamiltonian paths with prescribed end vertices and Hamiltonian cycles containing or avoiding a set of prescribed arcs.

2.6.1 Redei's Theorem

Rédei proved an interesting generalization of Theorem 2.2.4 concerning the parity of the number of Hamiltonian directed paths;

Theorem 2.6.1 (Rédei [158]) *Every tournament contains an odd number of Hamiltonian directed paths.*

The proof of Theorem 2.6.1 is established by means of a proof technique known as the **Inclusion-Exclusion Principle**, or the **Möbius Inversion Formula**, an inversion formula with applications throughout mathematics. We present here a simple version which suffices for our purpose. We refer the interested reader to Chapter 21 of Handbook of Combinatorics by Gessel and Stanley [96].

Lemma 2.6.2 (Inclusion-Exclusion Principle) *Let Z be a finite set and $f : 2^Z \to \mathbb{N}$ a real-valued function defined on the subsets of Z. Define the function $g : 2^Z \to \mathbb{N}$ by $g(X) = \sum_{\{Y | X \subseteq Y \subseteq Z\}} f(Y)$. Then*

$$f(X) = \sum_{\{Y | X \subseteq Y \subseteq Z\}} (-1)^{|Y|-|X|} g(Y).$$

Proof: By the Binomial Theorem,

$$\sum_{\{Y | X \subseteq Y \subseteq W\}} (-1)^{|Y|-|X|} = \sum_{k=|X|}^{|W|} \binom{|W|-|X|}{k-|X|} (-1)^{k-|X|} = (1-1)^{|W|-|X|}$$

which is equal to 0 if $X \subset W$, and to 1 if $X = W$. Therefore,

$$\begin{aligned} f(X) &= \sum_{\{W | X \subseteq W \subseteq Z\}} f(W) \sum_{\{Y | X \subseteq Y \subseteq W\}} (-1)^{|Y|-|X|} \\ &= \sum_{\{Y | X \subseteq Y \subseteq Z\}} (-1)^{|Y|-|X|} \sum_{\{W | Y \subseteq W \subseteq Z\}} f(W) \\ &= \sum_{\{Y | X \subseteq Y \subseteq Z\}} (-1)^{|Y|-|X|} g(Y). \end{aligned}$$

□

Proof of Theorem 2.6.1 Let $T = (V, A)$ be a tournament with vertex set $V = \{1, 2, \ldots, n\}$ and denote by $h(T)$ the number of Hamiltonian paths in T. For any permutation σ of V, let $A_\sigma = A \cap \{\sigma(i)\sigma(i+1) \mid 1 \le i \le n-1\}$. Then A_σ induces a subdigraph of T each of whose components is a directed path.

For any subset X of A, let us define $f(X) = |\{\sigma \in \mathcal{S}_n \mid X = A_\sigma\}|$ and $g(X) = |\{\sigma \in \mathcal{S}_n \mid X \subseteq A_\sigma\}|$. Then $g(X) = \sum_{X \subseteq Y \subseteq A} f(Y)$, so by the Inclusion-Exclusion Principle

$$f(X) = \sum_{X \subseteq Y \subseteq A} (-1)^{|Y|-|X|} g(Y).$$

Observe that $g(Y) = r!$ if and only if the spanning subdigraph of T with arc set Y is the disjoint union of r directed paths. Thus $g(Y)$ is odd if and only if Y induces a Hamiltonian directed path of T. Hence, defining $h(X) =$

$|\{H \in \mathcal{H} \mid X \subseteq A(H)\}|$ with $\mathcal{H}$ the set of Hamiltonian directed paths of T, we obtain

$$f(X) \equiv \sum_{\{H \in \mathcal{H} | X \subseteq A(H)\}} (-1)^{n-1-|X|} \equiv h(X) \pmod 2.$$

The theorem is true for transitive tournaments as there is a unique Hamiltonian directed path. Since any n-tournament may be obtained from the transitive n-tournament by reversing the orientation of appropriate arcs, it suffices to prove that the parity of the number of Hamiltonian directed paths $h(T)$ is unaltered by the reversal of any one arc e.

Let T' be the tournament obtained from T by reversing e. Then $h(T') = h(T) + f(\{e\}) - h(\{e\})$. Since $f(\{e\}) \equiv h(\{e\}) \pmod 2$, we have $h(T') \equiv h(T) \pmod 2$. □

2.6.2 Hamiltonian Connectivity

Recall that an $[x, y]$-path in a digraph $D = (V, A)$ is a directed path which either starts at x and ends at y or oppositely. We say that D is **weakly Hamiltonian-connected** if it has a Hamiltonian $[x, y]$-path (also called an **$[x, y]$-Hamiltonian path**) for every choice of distinct vertices $x, y \in V$. Thomassen found the following characterization of weakly Hamiltonian-connected tournaments.

Theorem 2.6.3 ([184]) *Let $D = (V, A)$ be a tournament and let x_1, x_2 be distinct vertices of D. Then D has an $[x_1, x_2]$-Hamiltonian path if and only if none of the following holds.*

(a) D is not strong and either none of x_1, x_2 belongs to the initial strong component of D or none of x_1, x_2 belongs to the terminal strong component of D.
(b) D is strong and for $i = 1$ or 2, $D - x_i$ is not strong and x_{3-i} belongs to neither the initial nor the terminal strong component of $D - x_i$.
(c) D is isomorphic to one of the two tournaments in Figure 2.2 (possibly after interchanging the names of x_1 and x_2).

For semicomplete digraphs there is also a characterization which can be read out of Theorem 6.7.3 (as every semicomplete digraph is also locally semicomplete).

Corollary 2.6.4 ([184]) *Let D be a strong tournament and let x, y, z be distinct vertices of D. Then D has a Hamiltonian path connecting two of the vertices in the set $\{x, y, z\}$.* □

Corollary 2.6.5 ([184]) *A tournament T with at least three vertices is weakly Hamiltonian-connected if and only if it satisfies (1)–(3) below.*

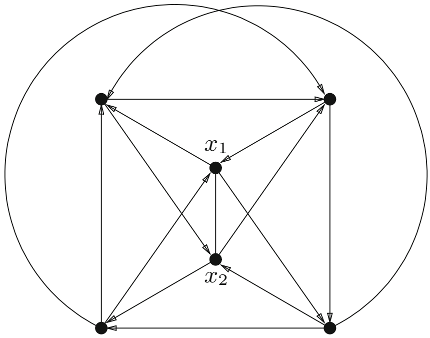

Figure 2.2 The exceptional tournaments in Theorem 2.6.3. The edge between x_1 and x_2 can be oriented arbitrarily

(1) T *is strong.*
(2) For every vertex $v \in V(T)$, $T - v$ *has at most two strong components.*
(3) T *is not isomorphic to any of the two tournaments in Figure 2.2.*

We now turn to Hamiltonian paths with specified initial and terminal vertices. An **(x,y)-Hamiltonian path** is a Hamiltonian path from x to y. A digraph $D = (V, A)$ is **Hamiltonian-connected** if D has an (x,y)-Hamiltonian path for every choice of distinct vertices $x, y \in V$. The following result of Thomassen gives a sufficient condition for a semicomplete digraph to have an (x,y)-Hamiltonian path.

Theorem 2.6.6 (Thomassen [184]) *Let* $D = (V, A)$ *be a* 2*-strong semicomplete digraph with distinct vertices* x, y*. Then* D *contains an* (x,y)*-Hamiltonian path if either (a) or (b) below is satisfied.*

(a) D *contains three internally disjoint* (x,y)*-paths each of length at least* 2*.*
(b) D *contains a vertex* z *which is dominated by every vertex of* $V \setminus \{x\}$ *and* D *contains two internally disjoint* (x,y)*-paths each of length at least* 2*.* □

Theorem 2.6.6 and Menger's theorem (Theorem 1.5.3) immediately imply the following result.

Theorem 2.6.7 ([184]) *If a semicomplete digraph* D *is* 4*-strong, then* D *is Hamiltonian-connected.* □

Thomassen constructed an infinite family of 3-strongly connected tournaments with two vertices x, y for which there is no (x,y)-Hamiltonian path [184]. Hence, from a connectivity point of view, Theorem 2.6.7 is the best possible.

Theorem 2.6.7 has several important consequences. Thomassen has shown in several papers how to use Theorem 2.6.7 to obtain results on spanning collections of paths and cycles in semicomplete digraphs. See, e.g., the papers [179, 181] and also Section 2.6.3.

The next theorem of Bang-Jensen, Manoussakis and Thomassen generalizes Theorem 2.6.6 (when $n \geq 10$). Recall that for specified distinct vertices s, t, an **$(\mathbf{s}, \mathbf{t})$-separator** is a subset $S \subseteq V \setminus \{s, t\}$ such that $D - S$ has no (s, t)-path. An (s, t)-separator is **trivial** if either s has out-degree 0 or t has in-degree 0 in $D - S$.

Theorem 2.6.8 ([32]) *Let D be a 2-strong semicomplete digraph on at least ten vertices and let x, y be vertices of D such that $xy \notin A(D)$. Suppose that both of $D - x$ and $D - y$ are 2-strong. If all (x, y)-separators consisting of two vertices (if any exist) are trivial, then D has an (x, y)-Hamiltonian path.* □

Based on Theorem 2.6.8 and several other structural results on 2-strong semicomplete digraphs Bang-Jensen, Manoussakis and Thomassen proved the following.

Theorem 2.6.9 ([32]) *The (x, y)-Hamiltonian path problem is solvable in polynomial time for semicomplete digraphs.*

The algorithm uses a divide-and-conquer approach and cannot be easily modified to find a longest (x, y)-path in a semicomplete digraph. There also does not seem to be any simple reduction of this problem to the problem of deciding the existence of a Hamiltonian path from x to y. Bang-Jensen and Gutin conjectured that there exists a polynomial algorithm for the problem.

Conjecture 2.6.10 ([23]) *There exists a polynomial algorithm that, given a semicomplete digraph D and two distinct vertices x and y of D, finds a longest (x, y)-path.*

Note that if we ask for the longest $[x, y]$-path in a tournament, then this can be answered using Theorem 2.6.3. We leave the details to the interested reader.

The following result, due to Bang-Jensen, Maddaloni and Simonsen, shows that if we generalize the (x, y)-Hamiltonian path problem in a natural way, we obtain an $\mathcal{NP}$-complete problem.

Theorem 2.6.11 ([31]) *The following problem is $\mathcal{NP}$-complete: given a strong tournament T, a p-partition $(V_1, \ldots, V_p)$ of $V(T)$ and distinct vertices x, y of T; determine whether T has an (x, y)-path which intersects each of the sets V_i, $i \in [p]$.*

2.6.3 Hamiltonian Cycles Containing or Avoiding Prescribed Arcs

We now turn our attention to Hamiltonian cycles in digraphs with the extra condition that these cycles must either contain or avoid all arcs from a prescribed subset A' of the arcs. As we shall see, problems of this type are quite difficult even for semicomplete digraphs. If we have no restriction on the size of A', then we may easily formulate the Hamiltonian cycle problem for arbitrary digraphs as an avoiding problem for semicomplete digraphs. Hence the avoiding problem without any restrictions is certainly $\mathcal{NP}$-complete. Below, we study both types of problems from a connectivity as well as from a complexity point of view. Bang-Jensen and Gutin [24] showed that when the number of arcs to be avoided, respectively, contained in a Hamiltonian cycle, is some constant, then, from a complexity point of view, the avoiding version is no harder than the containing version.

Consider the following problem.

HAMILTONIAN CYCLE THROUGH k-PRESCRIBED ARCS (k-HCA) **Input:** A digraph D and prescribed arcs $e_1, e_2, \ldots, e_k$ **Question:** Does D have a Hamiltonian cycle containing all of these arcs?

Clearly this is $\mathcal{NP}$-complete for general digraphs, but even for semicomplete digraphs this is a difficult problem. For $k = 1$ the k-HCA problem is a special case of the (x, y)-Hamiltonian path problem and hence it is polynomial for semicomplete digraphs by Theorem 2.6.9. The problem is open for semicomplete digraphs for all other values of k.

Based on the evidence from Theorem 2.6.9, Bang-Jensen, Manoussakis and Thomassen posed the following conjecture.

Conjecture 2.6.12 ([32]) *For each fixed k, the k-HCA problem is polynomial time solvable for semicomplete digraphs.*

Bang-Jensen and Thomassen proved that when k is not fixed the k-HCA problem becomes $\mathcal{NP}$-complete even for tournaments [34]. The proof of this result in [34] contains an interesting idea which was generalized by Bang-Jensen and Gutin in [24]. Consider a digraph D containing a set W of k vertices such that $D - W$ is semicomplete. Construct a new semicomplete digraph D_W as follows. First, split every vertex $w \in W$ into two vertices w_1, w_2 such that all arcs entering w now enter w_1 and all arcs leaving w now leave w_2. Let $W_i = \{w_i | w \in W\}$, $i = 1, 2$. For each $w_1 \in W_1, w'_2 \in W_2$ add the arc $w_1 w'_2$ except if the arc $w'_2 w_1$ is already present. Add all edges between distinct vertices of W_i for $i = 1, 2$ and orient these arbitrarily. Finally, add all arcs of the kind $w_1 z$ and $z w_2$, where $w \in W$ and $z \in V(D) - W$. See Figure 2.3. It is easy to show that the following proposition holds:

Proposition 2.6.13 ([24]) *Let W be a set of k vertices of a digraph D such that $D - W$ is a semicomplete digraph. Then D has a cycle of length $c \geq k$ containing all vertices of W if and only if the semicomplete digraph D_W has a cycle of length $c + k$ through the arcs $\{w_1 w_2 : w \in W\}$.*

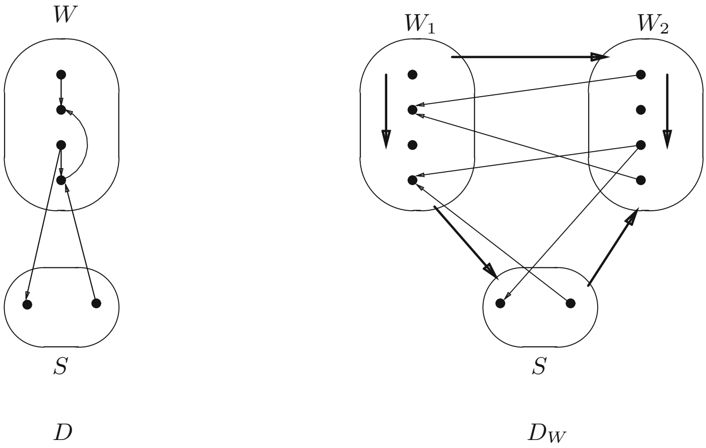

Figure 2.3 The construction of D_W from D and W. The bold arc from W_1 to W_2 indicates that all arcs not already going from W_2 to W_1 (as copies of arcs in D) go in the direction shown. The four other bold arcs indicate that all possible arcs are present in the shown direction

Bang-Jensen and Gutin observed that the following is equivalent to Conjecture 2.6.12.

Conjecture 2.6.14 ([24]) *Let k be a fixed positive integer. There exists a polynomial algorithm to decide if there is a Hamiltonian cycle in a given digraph D which is obtained from a semicomplete digraph by adding at most k new vertices and some arcs.*

The truth of this conjecture when $k = 1$ follows from Proposition 2.6.13 and Theorem 2.6.9. Surprisingly, when $|W| = 2$ the problem already seems to be very difficult.

Using Theorem 2.6.7 Thomassen [179] proved the existence of a function $h(k)$ such that for every $h(k)$-strong semicomplete digraph D and every choice of distinct vertices $x_1, y_1, ..., x_k, y_k$ D has k-path factor $P_1 \cup P_2 \cup ... \cup P_k$ such that P_i is an (x_i, y_i)-path for $i = 1, ..., k$. The function $h(k)$ is super-exponential. Recently Kim, Kühn and Osthus improved this to a polynomial.

Theorem 2.6.15 ([121]) *Let k be a positive integer, and let T be a (k^2+3k)-strong tournament. For any set $\{x_1, y_1, ..., x_k, y_k\}$ of distinct vertices, T has a k-path factor $P_1 \cup P_2 \cup ... \cup P_k$ such that P_i is an (x_i, y_i)-path for $i = 1, ..., k$.*

Note that Theorem 2.6.15 gives a better bound than Theorem 2.5.15 when $k < 449$ and even guarantees a k-linkage that spans all vertices of the tournament.

Corollary 2.6.16 ([121]) *If $a_1, ..., a_k$ are arcs with no common head or tail in a (k^2+3k)-strong tournament T, then T has a Hamiltonian cycle containing $a_1, ..., a_k$ in that cyclic order.*

Pokrovskiy [155] showed that the bound in Theorem 2.6.15 can be replaced by a linear function, thus answering a question of Thomassen from [179]. The constant C below is very large, which is why we also stated Theorem 2.6.15, which gives a better bound as long as k is not very large.

Theorem 2.6.17 ([155]) *There exists a constant C such that for every Ck-strong tournament T and every set $\{x_1, y_1, ..., x_k, y_k\}$ of distinct vertices, T has a k-path-factor $P_1, P_2, \ldots, P_k$ such that P_i is an (x_i, y_i)-path for $i = 1, ..., k$.*

By Theorem 2.3.2, similar results hold for semicomplete digraphs.

Recall that a set of arcs is **independent** if no two of the arcs share a vertex. Combining the ideas of avoiding and containing, Thomassen proved the following (below we have replaced his exponential function by the one from Theorem 2.6.15).

Theorem 2.6.18 ([179]) *Let T be a (k^2+3k)-strong tournament. For any set A_1 of at most k arcs in T and for any set A_2 of at most k independent arcs of $T \setminus A_1$, the digraph $T \setminus A_1$ has a Hamiltonian cycle containing all arcs of A_2.*

Even though tournaments have a lot of structure and the Hamiltonian cycle problem is almost trivial, the situation changes dramatically if we delete just a few arcs from a tournament. For some tournaments, such as the almost transitive tournaments, the answer is that even one missing arc may destroy all Hamiltonian cycles. If there is exactly one arc entering (resp. leaving) a vertex, then deleting that arc clearly suffices to destroy all Hamiltonian cycles. However, it is not just a simple degree question since, for every p, there exists an infinite set $\mathcal{S}$ of strong tournaments in which $\delta^0(T) \geq p$ for every $T \in \mathcal{S}$ and yet there is some arc of T which is on every Hamiltonian cycle of T ([22, Exercise 7.19]). It follows from Theorem 2.6.19 below that all such tournaments are strong but not 2-strong.

Obviously, if a k-strong tournament T has $\delta^0(T) = k$ (this is the smallest possible by the connectivity assumption), we may again kill all Hamiltonian cycles by removing just k arcs. Thomassen [181] conjectured that in a k-strong tournament, k is the minimum number of arcs one can delete in order to destroy all Hamiltonian cycles. The next theorem due to Fraisse and Thomassen answers this in the affirmative.

Theorem 2.6.19 (Fraisse and Thomassen [87]) *For every k-strong tournament T and every set $A' \subset A(T)$ such that $|A'| \leq k-1$, there is a Hamiltonian cycle C in $T \setminus A'$.*

The proof is long and non-trivial; in particular it uses Theorem 2.6.7. Below we describe a stronger result due to Bang-Jensen, Gutin and Yeo [25].

Theorem 2.6.20 ([25]) *Let $T = (V, A)$ be a k-strong n-tournament, and let $X_1, X_2, \ldots, X_p$ ($p \geq 1$) be a partition of V such that $1 \leq |X_1| \leq |X_2| \leq \ldots \leq |X_p|$. Let D be the digraph obtained from T by deleting all arcs which have both head and tail in the same X_i (i.e., $D = T \setminus \bigcup_{i=1}^{p} A(T[X_i])$). If $|X_p| \leq n/2$ and $k \geq |X_p| + \sum_{i=1}^{p-1} \lfloor |X_i|/2 \rfloor$, then D is Hamiltonian. In other words, T has a Hamiltonian cycle which avoids all arcs with both head and tail in some X_i. Furthermore, the bound on k is sharp.*

The proof of Theorem 2.6.20 in [25] uses results on irreducible cycle factors in multipartite tournaments, in particular Yeo's irreducible cycle factor theorem (Theorem 7.3.2).

The main idea of the proof is as follows: By construction (deleting all arcs inside several disjoint sets) D is a multipartite tournament. The goal is to apply Theorem 7.3.2 to D. Hence we need to establish that D is strong and has a cycle factor. Both of these are true and the latter can be proved using Hoffman's circulation theorem. Now we can apply Theorem 7.3.2 to prove that every irreducible cycle factor in D is a Hamiltonian cycle. This last step is non-trivial.

Problem 2.6.21 ([25]) *Which sets B of edges of the complete graph K_n have the property that every k-strong orientation of K_n induces a Hamiltonian digraph on $K_n - B$?*

The Fraisse–Thomassen theorem says that this is the case whenever B contains at most $k - 1$ edges. Theorem 2.6.20 says that a union of disjoint cliques of sizes $r_1, \ldots, r_p$ has the property whenever $\sum_{i=1}^{l} \lfloor r_i/2 \rfloor + \max_{1 \leq i \leq l}\{\lceil r_i/2 \rceil\} \leq k$. As shown in [25] this is the best possible result for unions of cliques.

See [22, pages 293–294] for a proof that Theorem 2.6.20 implies Theorem 2.6.19. Note that if A' induces a tree and possibly some disjoint edges in $UG(T)$, then Theorem 2.6.20 is no stronger than Theorem 2.6.19. In all other cases Theorem 2.6.20 provides a stronger bound.

How easy is it to decide, for a given semicomplete digraph $D = (V, A)$ and a subset $A' \subseteq A$, whether D has a Hamiltonian cycle C which avoids all arcs of A'? As we mentioned earlier, this problem is $\mathcal{NP}$-complete if we pose no restriction on the arcs in A'. In the case when A' is precisely the set of those arcs that lie inside the sets of some partition $X_1, X_2, \ldots, X_r$ of V, then the existence of C can be decided in polynomial time. This follows from the fact that $D \setminus A'$ is a semicomplete multipartite digraph and, by Theorem 7.6.1, the Hamiltonian cycle problem is polynomial for semicomplete multipartite digraphs. The same argument also covers the case when $k = 1$ in the conjecture below.

Conjecture 2.6.22 ([22]) *For every fixed positive integer k, there exists a polynomial algorithm which, for a given semicomplete digraph D and a subset $A' \subseteq A(D)$ such that $|A'| = k$, decides whether D has a Hamiltonian cycle that avoids all arcs in A'.*

At first glance, cycles that avoid certain arcs seem to have very little to do with cycles that contain certain specified arcs. Hence, somewhat surprisingly, if Conjecture 2.6.12 is true, then so is Conjecture 2.6.22 as observed by Thomassen[3]: Suppose that Conjecture 2.6.12 is true. Then it follows from the discussion above on Hamiltonian cycles containing prescribed arcs that Conjecture 2.6.14 also holds. Hence, for fixed k, there is a polynomial algorithm $\mathcal{A}_k$ which, given a digraph D and a subset $W \subseteq V(D)$ for which $D-W$ is semicomplete and $|W| \leq k$, decides whether or not D has a Hamiltonian cycle. Let k be fixed and D be a semicomplete digraph and let A', $|A'| \leq k$, be a prescribed set of arcs in D. Let W be the set of all vertices such that at least one arc of A' has head or tail in W. Then $|W| \leq 2|A'|$ and D has a Hamiltonian cycle avoiding all arcs in A' if and only if the digraph $D \setminus A'$ has a Hamiltonian cycle. By the above remark, we can test this using the polynomial algorithm $\mathcal{A}_r$, where $r = |W|$.

2.7 Oriented Subgraphs of Tournaments

A digraph is n-**unavoidable** if it is contained in every n-tournament and simply **unavoidable** if there exists some n such that it is n-unavoidable. Redei's Theorem states that the directed n-path is n-unavoidable. A natural question is which digraphs are unavoidable? Because the transitive tournaments are acyclic, every digraph containing a directed cycle is not unavoidable. On the other hand, we now prove that every acyclic digraph is unavoidable.

Theorem 2.7.1 (Folklore) *A digraph is unavoidable if and only if it is acyclic. Moreover, every acyclic n-digraph is 2^{n-1}-unavoidable.*

Proof: We already mentioned that every non-acyclic digraph is not unavoidable. Reciprocally, we need to prove that every acyclic digraph is unavoidable, and more precisely that every acyclic n-digraph 2^{n-1}-unavoidable. As every acyclic n-digraph is a subdigraph of the transitive n-tournament TT_n, it suffices to prove the result for TT_n. This follows directly from Proposition 2.2.3. □

Now, for each acyclic (and hence unavoidable) digraph D, it is natural to ask for the minimum integer $\mathrm{unvd}(D)$ such that D is $\mathrm{unvd}(D)$-unavoidable. Since an acyclic n-digraph is contained in TT_n and so $\mathrm{unvd}(D) \leq \mathrm{unvd}(TT_n)$,

[3] private communication, August 1999.

the first interesting case is that of transitive tournaments, which also yields a good estimate of unvd(D) for digraphs D with many arcs. The unavoidability of transitive tournaments is detailed in Subsection 2.7.1. We then study the unavoidability of acyclic digraphs with few arcs, namely oriented paths (Subsection 2.7.2), oriented cycles (Subsection 2.7.3), and oriented trees (Subsection 2.7.4).

2.7.1 Transitive Subtournaments

Erdős and Moser [76] ask for the value of unvd(TT_n).

Problem 2.7.2 ([76]) *What is* unvd(TT_n)*?*

Theorem 2.7.1 yields unvd$(TT_n) \leq 2^{n-1}$. This upper bound is almost tight, as shown by the following result due to Erdős and Moser [76].

Theorem 2.7.3 ([76]) *There exists a tournament on* $2^{(n-1)/2}$ *vertices which contains no* TT_n.

Proof: The proof is probabilistic and uses the First Moment Method. (For more on the Probabilistic Method and in particular the First Moment Method, we refer the reader to the book of Alon and Spencer [8].) Set $N = 2^{(n-1)/2}$ and consider $T = RT_N$, a random tournament on N vertices.

For an ordered n-tuple $(v_1, v_2, \dots, v_n)$ the probability that $T\langle\{v_1, \dots, v_n\}\rangle$ is a transitive tournament with Hamiltonian directed path $v_1v_2 \dots v_n$ is $\left(\frac{1}{2}\right)^{\binom{n}{2}}$. Hence the expected number of transitive n-subtournaments is

$$\frac{N!}{(N-n)!}\left(\frac{1}{2}\right)^{\binom{n}{2}} < N^n \left(\frac{1}{2}\right)^{\binom{n}{2}} \leq 1$$

because $N \leq 2^{(n-1)/2}$. Hence by the First Moment Principle, there exists an N-tournament with less than 1 (i.e. no) n-subtournament. □

In the same way, for every acyclic n-digraph D with m arcs one can show that unvd$(D) > 2^{\frac{m}{n}}$. This gives a meaningful lower bound for digraphs with sufficiently many arcs, namely at least $n \log n$ arcs.

Clearly, unvd$(TT_1) = 1$, unvd$(TT_2) = 2$ and unvd$(TT_3) = 4$. Also unvd$(TT_4) = 8$ because the Paley tournament $\mathbb{P}_7$ contains no TT_4. Moreover, Reid and Parker [162] showed that unvd$(TT_5) = 14$ and unvd$(TT_6) = 28$ and Sanchez-Flores [167] showed unvd$(TT_7) \leq 54$. A similar induction as in the proof of Theorem 2.7.1 yields that unvd$(TT_n) \leq 54 \times 2^{n-7}$ if $n \geq 7$.

In addition, for $1 \leq n \leq 6$ it has been shown [162, 167] that there is a unique tournament of order unvd$(TT_n) - 1$ that contains no TT_n. This leads us to the following conjecture:

Conjecture 2.7.4 (Havet, 2008) *For every n, there is a unique tournament on* unvd$(TT_n) - 1$ *vertices that contains no* TT_n.

2.7.2 Oriented Paths in Tournaments

An **oriented path** is an orientation P of an undirected path $x_1 \cdots x_n$. We say that x_1 is the **origin** of P and x_n is the **terminus** of P. If $x_1 \to x_2$, P is an **out-path**, otherwise P is an **in-path**. The **directed out-path** of order n is the orientation of $x_1 \cdots x_n$ in which $x_i \to x_{i+1}$ for all i, $1 \le i < n$; the dual notion is **directed in-path**. The **length** of a path is its number of arcs. We denote by ${}^*P$ (resp. P^*) the oriented path obtained from P by removing its origin (resp. terminus). The **blocks** of P are the maximal directed subpaths of P. We enumerate the blocks of P from the origin to the terminus. The first block of P is denoted by $B_1(P)$ and its length by $b_1(P)$. Likewise, the ith block of P is denoted by $B_i(P)$ and its length by $b_i(P)$. The path P is totally described by the signed sequence $sgn(P)(b_1(P), b_2(P), \cdots, b_k(P))$ where k is the number of blocks of P and $sgn(P) = +$ if P is an out-path and $sgn(P) = -$ if P is an in-path. An **antidirected** path is an oriented path in which all blocks have length 1.

Let X be a set of vertices of T. The **out-section** generated by X in T is the set of vertices y to which there exists a directed out-path from some $x \in X$; we denote this set by $S^+(X)$ (note that $X \subseteq S^+(X)$ since we allow paths of length zero). We abbreviate $S^+(\{x\})$ to $S^+(x)$ and $S^+(\{x, y\})$ to $S^+(x, y)$. The dual notion, the **in-section**, is denoted by $S^-(X)$. We also write $s^+(X)$ (resp. $s^-(X)$) for the number of vertices of $S^+(X)$ (resp. $S^-(X)$). If $X \subseteq Y \subseteq V$, we write $S^+_Y(X)$ instead of $S^+_{T[Y]}(X)$. An **out-generator** of T is a vertex $x \in T$ such that $S^+(x) = V(T)$, the dual notion is an **in-generator**.

Redei's Theorem states that the directed n-out-path is n-unavoidable. It is then a natural question to ask whether the other oriented n-paths are also n-unavoidable. Grünbaum [98] proved that this is the case for antidirected paths except for three exceptions, the paths $\pm(1, 1)$ which is not contained in the directed 3-cycle $\vec{C}_3$, $\pm(1, 1, 1, 1)$ which is not contained in the regular 5-tournament R_5, and $\pm(1, 1, 1, 1)$ which is not contained in the Paley 7-tournament $\mathbb{P}_7$. A year later, in 1972, Rosenfeld [165] gave an easier proof of a stronger result: in a tournament on at least 9 vertices, each vertex is the origin of an antidirected Hamiltonian path. He also made the following conjecture: there is an integer $N > 7$ such that every tournament on n vertices, $n \ge N$, contains any orientation of the Hamiltonian path. The condition $N > 7$ results from Grünbaum's counterexamples. Several papers gave partial answers to this conjecture: for paths with two blocks (Alspach and Rosenfeld [13], Straight [174]), and for paths having the ith block of length at least $i + 1$ (Alspach and Rosenfeld [13]); interestingly Forcade [83] proved in a way similar to the proof of Theorem 2.6.1 that there is always an odd number of Hamiltonian paths of any type in tournaments with 2^n vertices. Rosenfeld's conjecture was verified by Thomason, who proved in [176] that N exists and is less than 2^{128}. While he did not make any attempt to sharpen this bound, he wrote that $N = 8$ should be the right value. The problem was finally closed by Havet and Thomassé [110] who proved the following theorem.

Theorem 2.7.5 (Havet and Thomassé [110]**)** *Apart from Grünbaum's exceptions, every n-tournament contains every oriented n-path.*

The proof of Havet and Thomassé relies on sufficient conditions for vertices to be an origin of a given oriented path in a tournament. An easy condition for a vertex x to be an origin of an oriented out-path P is that $s^+(x) \geq b_1(P)+1$. It is sometimes sufficient: for example, this condition says that an origin of a Hamiltonian directed out-path in a tournament must be a out-generator, and one can easily show that it is also sufficient.

Proposition 2.7.6 *In a tournament T, a vertex v is an origin of a Hamiltonian directed out-path in T if and only if v is an out-generator of T.*

In contrast, for other Hamiltonian oriented paths, the condition $s^+(x) \geq b_1(P) + 1$ is not sufficient to guarantee x being an origin of P. However, Havet and Thomassé [110] proved that among two distinct vertices x, y such that $s^+(x, y) \geq b_1(P)+1$, there must be an origin of P except in some exceptional cases that they completely characterized. The proof of this result is by induction and is tedious because of a long case analysis due to the exceptional cases (51 small ones plus 17 infinite families). However, the general idea of the proof is the same as that of the following weaker theorem about oriented n-paths in $(n + 1)$-tournaments.

Theorem 2.7.7 ([110]) *Let T be a tournament of order $n+1$, P an out-path of order n and x, y two distinct vertices of T. If $s^+(x, y) \geq b_1(P)+1$, then x or y is an origin of P in T.*

Proof: We prove the statement and its directional dual (where P is an in-path and $s^-(x, y) \geq b_1(P)+1$) by induction on n, the result holding trivially for $n = 1$. Let x and y be two vertices of a tournament $T = (V, A)$ such that $x \to y$ and $s^+(x, y) \geq b_1(P) + 1$. We distinguish two cases:

Case 1 : $b_1(P) \geq 2$. If $d^+(x) \geq 2$, let $z \in N^+(x)$ be an out-generator of $T\langle S^+(x)\backslash\{x\}\rangle$ and let $t \in N^+(x)$, $t \neq z$. By definition of z, $s^+_{V\backslash\{x\}}(t, z) = s^+(x) - 1 > b_1({}^*P)$. Since ${}^*P$ is an out-path, by the induction hypothesis, either t or z is an origin of ${}^*P$ in $T - x$. Thus x is an origin of P in T.

So we may assume that y is the unique out-neighbour of x. Let z be an out-generator of $T\langle N^+(y)\rangle$ (z exists since $s^+(x, y) \geq 3$). Then $z \to x$ and z is an out-generator of $T\langle S^+(x, y)\backslash\{y\}\rangle$. It follows that $s^+_{V\backslash\{y\}}(x, z) = s^+(x, y) - 1$, so by the induction hypothesis, either x or z is an origin of ${}^*P$ in $T - y$. Since $d^+_{V\backslash\{y\}}(x) = 0$, this origin is certainly z. We conclude that y is an origin of P in T.

Case 2 : $b_1(P) = 1$. Assume first that $d^+(x) \geq 2$. We denote by X the set $S^-_{V\backslash\{x\}}(N^+(x))$. Consider the partition $(X, Y, \{x\})$ of V where $Y = V\backslash(X \cup \{x\})$. We have $Y \to x$, $X \to Y$ and $y \in X$. If $|X| \geq b_2(P) + 1$, then x is an origin of P in T; indeed, let $z \in N^+(x)$ be an in-generator of $T\langle X\rangle$ and let $u \in N^+(x)$ $u \neq z$. By the induction hypothesis, z or u is an origin of ${}^*P$ in $T - x$. Hence x is an origin of P in T. If $|X| \leq b_2(P)$, we have $|Y| > 1$ since $b_2(P) \leq n - 2$ and $|X| + |Y| = n$. Let $w \in Y$ be an in-generator of $T\langle Y\rangle$. Notice that since $d^+(x) > 1$, $S^-_{V\backslash\{y\}}(w) = V\backslash\{y\}$. Let $u \in Y - w$. By the induction hypothesis, u or w is an origin of ${}^*P$ in $T - y$, consequently y is an origin of P in T.

Now assume that $d^+(x) = 1$, thus $N^+(x) = \{y\}$. If $d^+(y) < 2$, then $N^-_{V\backslash\{x\}}(y)$ has at least $n - 2$ vertices. By the induction hypothesis, one can find ${}^{**}P$ in $T\langle N^-_{V\backslash\{x\}}(y)\rangle$, thus x is an origin of P in T. If $d^+(y) \geq 2$, denote $S^-_{V\backslash\{y\}}(N^+(y))$ by Y and consider the partition $(X, Y, \{x\}, \{y\})$ of V with $X = V\backslash(Y \cup \{x, y\})$. By definition, $X \to \{x, y\}$, $Y \to X \cup \{x\}$. If $|Y| \geq b_2(P) + 1$, then y is an origin of P by the previous argument. If $|Y| \leq b_2(P)$, then $b_2(P) \geq d^+(y) \geq 2$. If $|X| \geq 2$, let $z \in X$ be an in-generator of $T - \{x, y\}$ and let $u \in X$ $u \neq z$. Since $b_2(P) \geq 2$, ${}^{**}P$ is an in-path and by the induction hypothesis, z or u is an origin of ${}^{**}P$ in $T - \{x, y\}$. Thus x, (via y) is an origin of P in T. Finally, if $|X| = 1$ then $|Y| = n - 2$ and since $n - 2 \geq b_2(P) \geq |Y|$ we have $b_2(P) = n - 2$. This means that ${}^*P$ is a directed in-path. Since y is an in-generator of $T - x$, x is an origin of P in T. □

The following result, due to Thomason, is an easy consequence of Theorem 2.7.7.

Corollary 2.7.8 ([176]) *Every tournament T of order $n + 1$ contains each oriented path P of order n. Moreover, any subset of $b_1(P)+1$ vertices contains an origin of P. In particular, at least two vertices of T are origins of P .*

2.7.3 Oriented Cycles in Tournaments

As we did for paths, we can seek arbitrary orientations of cycles, i.e. **oriented cycles**. Observe that by Camion's Theorem (2.2.6) a tournament has a directed Hamiltonian cycle if and only if it is strong. A natural equation is then whether every tournament contains all Hamiltonian non-directed cycles. The existence of Grünbaum's exceptions implies the existence of tournaments that do not contain certain Hamiltonian oriented cycles. Indeed $\vec{C}_3$, R_5 and $\mathbb{P}_7$ do not contain the cycle obtained from a Hamiltonian antidirected path by adding an arc between its terminus and its origin. Moreover, the tournaments of order n that have a subtournament on $n - 1$ vertices isomorphic to one of $\vec{C}_3$, R_5 and $\mathbb{P}_7$ do not contain a Hamiltonian antidirected cycle. (Similarly to paths, an **antidirected cycle** is a cycle in which every block has length 1.)

However, as for oriented paths, Rosenfeld [164] conjectured that there is an integer $N > 8$ such that every tournament of order $n \geq N$ contains every non-directed cycle of order n. This was settled by Thomason [176] for tournaments of order $n \geq 2^{128}$. While Thomason made no attempt to sharpen this bound, he indicated that it should be true for tournaments of order at least 9.

Conjecture 2.7.9 (Rosenfeld–Thomason) Every tournament of order $n \geq 9$ contains every non-directed cycle of order n.

Havet [107] improved Thomason's result by showing that this conjecture is true for $n \geq 68$.

Theorem 2.7.10 ([107]) *Every tournament of order $n \geq 68$ contains every non-directed cycle of order n.*

The proof is based on complementary lemmas: Some establish the existence of an oriented cycle in every tournament whose strong connectivity is small compared to the length of its longer block; others show the existence of an oriented cycle in every tournament whose strong connectivity is large compared to the lengths of all blocks. In particular, Conjecture 2.7.9 is true if the tournament is either not 2-strong or 8-strong [107]. The conjecture is also true if the tournament is either 5-strong and of order at least 43 or 4-strong and of order at least 65.

Better results are also known for particular types of directed cycles. Conjecture 2.7.9 has been proved for cycles with a block of length $n-1$ by Grünbaum [98], for antidirected cycles by Thomassen [177] ($n \geq 50$), Rosenfeld [164] ($n \geq 28$) and Petrović [151] ($n \geq 16$), and for cycles with just two blocks by Benhocine and Wojda [39].

2.7.4 Trees in Tournaments

As we did for paths and cycles, we can seek an arbitrary orientation of trees, i.e. **oriented trees**. Observe that an oriented tree of order k is an acyclic digraph and thus it is 2^{k-1}-unavoidable by Theorem 2.7.1. However this bound 2^{k-1} is far from tight as an oriented tree has very few arcs compared to the transitive tournament of the same order.

Conjecture 2.7.11 (Sumner, 1972) Every oriented tree with $k > 1$ vertices is $(2k-2)$-unavoidable.

If true, this conjecture would be tight since the **out-star** S_k^+, which is the out-tree of order k with a root dominating $k-1$ leaves, is not contained in any regular tournament of order $2k-3$, because all vertices of such a tournament have out-degree $k-2$.

The first linear bound was given by Häggkvist and Thomason [104]. Havet and Thomassé [109] proved that the conjecture holds for out-trees (and thus also for in-trees).

Theorem 2.7.12 ([109]) *Every tournament of order $2k-2$ contains every out-tree of order $k>1$.*

Proof: Let $(v_1, v_2, \dots, v_{2k-2})$ be a median order of a tournament T on $2k-2$ vertices, and let A be an out-tree on k vertices. Consider the intervals $(v_1, v_2, \dots, v_i)$, $1 \le i \le 2k-2$. We show, by induction on k, that there is a copy of A in T whose vertex set includes at least half the vertices of any such interval.

This is clearly true for $k=2$. Suppose, then, that $k \ge 3$. Delete a leaf y of A to obtain an out-tree A' on $k-1$ vertices, and set $T' := T - \{v_{2k-3}, v_{2k-2}\}$. By (M1), $(v_1, v_2, \dots, v_{2k-4})$ is a median order of the tournament T', so there is a copy of A' in T' whose vertex set includes at least half the vertices of any interval $v_1, v_2, \dots, v_i$, $1 \le i \le 2k-4$. Let x be the predecessor of y in A. Suppose that x is located at vertex v_i of T'. In T, by (M2), v_i dominates at least half of the vertices $v_{i+1}, v_{i+2}, \dots, v_{2k-2}$, thus at least $k-1-i/2$ of these vertices. On the other hand, A' includes at least $(i-1)/2$ of the vertices $v_1, v_2, \dots, v_{i-1}$, thus at most $k-1-(i+1)/2$ of the vertices $v_{i+1}, v_{i+2}, \dots, v_{2k-2}$. It follows that, in T, there is an out-neighbour v_j of v_i, where $i+1 \le j \le 2k-2$, which is not in A'. Locating y at v_j, and adding the vertex y and the arc xy to A', we now have a copy of A in T. It is readily checked that this copy of A satisfies the required additional property. □

The same method can be easily adapted to prove that every oriented tree of order k is $(4k-4)$-unavoidable. At each step of the induction, we add two vertices to the right and two vertices to the left of the ordering and we insist that at each step for each vertex v at least half of the vertices to the right of v are unused and half of the vertices to the left are unused. El Sahili [73] used it in a clever way to show that every oriented tree of order k is $(3k-3)$-unavoidable. Recently, Kühn, Mycroft and Osthus [125] proved that Sumner's conjecture is true for all sufficiently large k.

Theorem 2.7.13 (Kühn, Mycroft and Osthus [125]**)** *There exists a k_0 such that every oriented tree with $k \ge k_0$ vertices is $(2k-2)$-unavoidable.*

Their complicated proof makes use of the directed version of Szemeredi's Regularity Lemma.

As we mentioned above, Sumner's conjecture is tight for out-stars. On the other hand, it is not tight for paths which are trees with two leaves. Consequently, Havet and Thomassé made the following conjecture, which directly implies Sumner's conjecture because a tree of order n has at most $n-1$ leaves.

Conjecture 2.7.14 (Havet and Thomassé, 1996) *If A is an oriented tree with n vertices and k leaves, then it is $(n+k-1)$-unavoidable.*

If true this conjecture would be tight because of out-stars, but also because of Grünbaum's exceptions. Conjecture 2.7.14 holds for $k=2$, as trees with two leaves are paths, Ceroi and Havet [57] proved it for $k=3$, and it easily holds for $k=n-1$, that is, when the tree is an oriented star. Havet [106] proved that it holds for a large class of oriented trees. Häggkvist and Thomason [104] proved that there is a function g such that every tree with n vertices and k leaves is $(n+g(k))$-unavoidable.

Instead of looking for a fixed oriented tree in tournaments, one may also seek an oriented tree having certain properties. In this vein, Lu [137] proved that there exists an out-branching of height 2, in which all nodes except the root have small out-degree.

Theorem 2.7.15 ([137]) *Every tournament T is has an out-branching of height 2 and whose vertices on level 1 have out-degree at most 2.*

Proof: The proof we give here is due to Bondy [49]. Let x be a vertex of maximal out-degree. By Theorem 2.2.12, x is a king, so $(\{x\}, N^+(x), N^{++}(x))$ is a partition of $V(T)$. Note that, by the choice of x and since in every k-tournament there is a vertex with out-degree at least $\lfloor k/2 \rfloor$, for every $A \subseteq N^{++}(x)$ we have $2|A^- \cap N^+(x)| \geq |A|$. By Hall's theorem, one can cover $N^{++}(x)$ by two directed matchings from $N^+(x)$ to $N^{++}(x)$. This gives the desired out-branching. □

2.7.5 Largest n-Unavoidable Digraphs

Let $\mathrm{lu}(n)$ be the largest m such that there is an n-unavoidable digraph with m arcs. Linial, Saks and Sós [135] showed the following.

Theorem 2.7.16 ([135]) *There exist positive constants c_1 and c_2 such that for all positive integers n, $n \log n - c_1 n \log\log n \leq \mathrm{lu}(n) \leq n \log n - c_2 n$.*

The upper bound comes from a simple counting argument working over all labelled n-tournaments. The lower bound follows from several propositions that allow an inductive construction of an n-unavoidable, weakly connected spanning digraph with $n \log n - c_1 n \log\log n$ arcs.

2.7.6 Generalization to k-Chromatic Digraphs

A tournament is an orientation of a complete graph, and the complete graph K_k is the easiest example of a graph with chromatic number k. Recall that the **chromatic number** of a digraph D, denoted by $\chi(D)$, is the chromatic number of its underlying undirected graph. A digraph is **k-chromatic** if its chromatic number is k. One can then wonder whether some results on tournaments can be extended to digraphs with large chromatic number. This is in particular the case with Rédei's Theorem (2.2.4), which has been generalized to the following theorem, often referred to as the Gallai–Roy Theorem, even if it was independently proved by four researchers: Gallai [94], Hasse [105], Roy [166] and Vitaver [191].

Theorem 2.7.17 (Gallai–Hasse–Roy–Vitaver [94, 105, 166, 191]**)** *Every k-chromatic digraph contains a directed path of order k.*

Theorem 2.7.17 has many proofs. One of them is based on median orders (see [50] Chapter 14). We present here a proof due to El-Sahili and Kouider [74]. It is based on the concept of **out-forests**, which are disjoint unions of out-trees. An out forest of D is spanning if it covers all vertices of D.

Let F be a spanning out-forest of D. The **level** of x is the number of vertices of a longest directed path of F ending at x. For instance, the level 1 vertices are the roots of the out-trees of F. We denote by F_i the set of vertices with level i in F. A vertex y is a **descendant** of x in F if there is a directed path from x to y in F.

If there is an arc xy in D from F_i to F_j, with $i \geq j$, and x is not a descendant of y, then the out-forest F' obtained by adding xy and removing the arc of F with head y (if such exists, that is, if $j > 1$) is called an **elementary improvement** of F. An out-forest F' is an **improvement** of F if it can be obtained from an out-forest F by a sequence of elementary improvements. The key-observation is that if F' is an improvement of F then the level of every vertex in F' is at least its level in F. Moreover, at least one vertex of F has its level in F' strictly greater than its level in F. Thus, one cannot perform infinitely many improvements. A spanning out-forest F is **final** if there is no elementary improvement of F.

The following proposition follows immediately from the definition of a final spanning out-forest:

Proposition 2.7.18 (El Sahili and Kouider [74]**)** *Let D be a digraph and F a final spanning out-forest of D. If a vertex $x \in F_i$ dominates in D a vertex $y \in F_j$ for $j \leq i$, then x is a descendant of y in F. In particular, every level of F is an independent set in D.*

Proof of Theorem 2.7.17**:** Consider a final spanning out-forest of a k-chromatic digraph D. Since every level is an independent set by Proposition 2.7.18, there are at least k levels. Hence D contains a directed path of order at least k. □

More generally, one can ask which digraphs are k**-universal**, i.e. contained in every k-chromatic digraph. A result of Erdős [75] states that for every choice of positive integers k and g, there exist k-chromatic graphs with no cycle of length less than g. Consequently, k-universal digraphs must be oriented trees.

Bondy conjectured the following generalization of Theorem 2.7.5.

Conjecture 2.7.19 (Bondy, 1995) *For sufficiently large k, every oriented path on k vertices is k-universal.*

As support for this conjecture, El Sahili proved [72] that every oriented path of order 4 is 4-universal and that the antidirected path of order 5 is 5-universal. Addario-Berry, Havet, and Thomassé [2] proved that every oriented path of order $k \geq 4$ with two blocks is k-universal. Their proof use the notion of a final spanning out-forest.

Burr [53] generalized Sumner's Conjecture as follows.

Conjecture 2.7.20 (Burr [53], 1980) *Every oriented tree on k vertices is $(2k-2)$-universal.*

Burr [53] showed that every oriented tree of order k is $(k-1)^2$-universal. This was slightly improved by Addario-Berry, Havet, Linhares Sales, Reed, and Thomassé [1].

Theorem 2.7.21 ([1]) *Every oriented tree on k vertices is $(k^2/2-k/2+1)$-universal.*

Addario-Berry *et al.* [1] proved that every oriented tree on k vertices is contained in every acyclic digraph of order n. They also established that every antidirected tree of order $k \geq 3$ is $(5k-9)$-universal. An **antidirected tree** is an oriented tree in which every vertex has either in-degree 0 or out-degree 0.

Finally, Havet and Thomassé generalized Conjecture 2.7.14 about unavoidabiity to universality.

Conjecture 2.7.22 (Havet and Thomassé, 2000) *If A is an oriented tree with n vertices and k leaves, then it is $(n+k-1)$-universal.*

Let us now consider cycles. As we already saw, they cannot be universal because there are digraphs with no cycles of small length having arbitrarily large chromatic number, as stated by a result of Erdős [75]. However, Bondy generalized Camion's Theorem (2.2.6) to digraphs with large chromatic number.

Theorem 2.7.23 (Bondy [48]) *Every strong digraph of chromatic number at least k contains a directed cycle of length at least k.*

A directed cycle of length at least k may be seen as a subdivision of the directed k-cycle $\vec{C}_k$. Recall that a **subdivision** of a digraph D is a digraph obtained from D by replacing each arc ab of D by a directed (a,b)-path. Hence a natural question is to ask whether Theorem 2.7.10 can be generalized, or if at least every non-directed cycle C is k-universal for some large enough k. This was answered in the negative by Cohen, Havet, Lochet, and Nisse.

Theorem 2.7.24 ([65]) *Let C be an oriented cycle. There exist digraphs with arbitrarily large chromatic number that contains no subdivision of C.*

However, they conjectured that, as for the directed cycle, if we require the digraph to be strongly connected, the picture is different.

Conjecture 2.7.25 ([65]) *Let C be an oriented cycle C. There exists a constant $h(C)$ such that every strong digraph with chromatic number at least $h(C)$ contains a subdivision of C.*

As partial evidence, Cohen, Havet, Lochet, and Nisse [65] proved this conjecture for cycles with two blocks and the antidirected cycle of order 4. In particular, they proved that for $C(k, \ell)$ the cycle on two blocks, one of length k and the other of length ℓ, $h(C(k, \ell)) = O((k + \ell)^4)$. This bound on the value was recently improved by Kim, Kim, Ma and Park [122] who proved $h(C(k, \ell)) = O((k + \ell)^2)$.

2.8 Vertex-Partitions of Semicomplete Digraphs

In this section, we consider properties of vertex-partitions of semicomplete digraphs. A **k-partition** of a digraph $D = (V, A)$ is a partition $(V_1, V_2, \dots, V_k)$ of V into k non-empty disjoint sets.

2.8.1 2-Partitions into Strong Semicomplete Digraphs

Being strongly connected is one of the basic properties of a digraph. Hence, it is natural to determine which (semicomplete) digraphs D have a k-partition into strong subdigraphs, that is, a partition $(V_1, \dots, V_k)$ such that $D[V_i]$ is strong for $i = 1, \dots, k$. Bang-Jensen, Cohen and Havet proved [21] that this problem is $\mathcal{NP}$-complete for general digraphs already when $k = 2$. The papers [21, 26] provide a complete characterization of the complexity of a number of related problems where we wish to partition $V(D)$ into two sets such that each of these have prescribed properties (e.g. both are strongly connected).

We now turn to semicomplete digraphs. Recall that a cycle factor is a spanning collection of disjoint cycles. Since every strongly connected semicomplete digraph is Hamiltonian, a semicomplete digraph has a 2-partition into two strong subdigraphs if and only if it has a cycle factor with two cycles. A pair of cycles forming a cycle factor with two cycles is also called a pair of **complementary cycles**.

Reid proved that every 2-strong n-tournament with $n \geq 8$ has a 2-partition into strong subtournaments, one of which has order 3. Song extended this result by showing that there exists such a partition with one subtournament of any fixed order k for any $3 \leq k \leq n-3$.

Theorem 2.8.1 ([161, 171]) *Every 2-strong tournament D on at least 8 vertices has a 2-partition (V_1, V_2) such that $D[V_i]$ is strong for $i = 1, 2$ and $|V_1| = k$ for every $3 \leq k \leq n-3$.*

Theorem 2.8.1 also holds for 2-strong tournaments on 6 vertices and the only exception on 7 vertices is the Paley tournament $\mathbb{P}_7$ (see [161]). Furthermore, there are infinite families of tournaments T with $\kappa(T) = 1$ which do not have complementary cycles. One such example was given in [130] by Li and Shu. Those families show that Theorem 2.8.1 cannot be extended to strong tournaments. However, Li and Shu proved that strong tournaments with sufficiently large minimum in- or out-degree have a partition into strong subtournaments.

Theorem 2.8.2 ([130]) *Let T be a strong tournament on at least 6 vertices. If $\max\{\delta^-(T), \delta^+(T)\} \geq 3$ and T is not isomorphic to the Paley tournament $\mathbb{P}_7$, then T has a 2-partition into strong subtournaments.* □

It follows from Theorem 6.9.2 that Theorem 2.8.1 also holds for semicomplete digraphs. For semicomplete digraphs Bang-Jensen and Nielsen solved the problem from a complexity point of view.

Theorem 2.8.3 ([33]) *There exists a polynomial algorithm that, given semicomplete digraph D, finds a 2-partition (V_1, V_2) such that $D[V_i]$ is strong for $i = 1, 2$, or correctly reports that no such pair exists.*

If we require more structure on the digraphs $D[V_i]$, such as requiring each of these to induce a tournament, then the problem becomes very difficult, even when the input is a semicomplete digraph. The following result is due to Bang-Jensen and Christiansen.

Theorem 2.8.4 ([20]) *It is $\mathcal{NP}$-complete to decide whether a given semicomplete digraph D has a 2-partition (V_1, V_2) such that $D[V_i]$ is a strong tournament for $i = 1, 2$.*

In an attempt to generalize Theorem 2.8.1, Bollobás asked whether every sufficiently large k-strong tournament has a cycle factor with k-cycles or equivalently a k-partition into strong subtournaments (see [161]). This was answered in the positive by Chen, Gould and Li.

Theorem 2.8.5 ([59]) *Every k-strong tournament on $n \geq 8k$ vertices has a k-partition into strong subtournaments.*

Furthermore, Kühn, Osthus and Townsend proved that if the tournament is r-strong for r sufficiently high, then one can prescribe the sizes of the strong subtournaments of the k-partition. This answers a question by Song [171].

Theorem 2.8.6 ([127]) *Let T be a tournament on n vertices, let $k \geq 2$ and let $n_1, n_2, \ldots, n_k \geq 3$ satisfy $n = n_1 + n_2 + \ldots + n_k$. If T is $10^{10}k^4 \log k$-strong, then it has a partition $(V_1, \ldots, V_k)$ into strong subtournaments such that $|V_i| = n_i$ for $i \in [k]$.*

2.8.2 Partition into Highly Strong Subtournaments

As a generalization of Theorem 2.8.1, Thomassen (see [161]) conjectured that for all positive integers k_1, k_2 there exists an integer $f(k_1, k_2)$ such that every $f(k_1, k_2)$-strong tournament T has a 2-partition (V_1, V_2) so that $T[V_i]$ is k_i-strong, $i = 1, 2$. This is clearly equivalent to the existence, for all integers k, t, of an integer $g(k, t)$ such that every $g(k, t)$-strong tournament T has a t-partition $(V_1, \ldots, V_t)$ so that $T[V_i]$ is k-strong, $i \in [t]$. The existence of such a $g(k, t)$ was established by Kühn, Osthus and Townsend [127].

Theorem 2.8.7 ([127]) *Let $k, t \geq 1$ be integers. Every tournament T which is $(10^7k^6t^3 \log(kt^2))$-strong has a t-partition $(V_1, \ldots, V_t)$ such that $T[V_i]$ is k-strong for $i \in [t]$.*

Kim, Kühn and Osthus proved that when the connectivity is sufficiently high we can get an even stronger type of 2-partition. For a digraph D and a 2-partition (V_1, V_2), we denote by $D[V_1, V_2]$ the bipartite subdigraph induced by the arcs with one end in V_1 and the other in V_2.

Theorem 2.8.8 ([121]) *Every $10^9k^6 \log(2k)$-strong tournament has a 2-partition (V_1, V_2) such that each of $T[V_1], T[V_2], T[V_1, V_2]$ is a k-strong digraph.*

See Theorem 2.8.18 for a related partition result for out-degrees.

2.8.3 2-Partitions With Prescribed Minimum Degrees

We now turn to 2-partitions where we want a certain minimum out-, in- or semi-degree in each of the parts. E.g. a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition is a 2-partition (V_1, V_2) where the digraph induced by each set has minimum out-degree at least 1. Bang-Jensen, Cohen and Havet proved in [21] that when we want the chosen parameter among $\{\delta^+, \delta^-, \delta^0\}$ to be at least 1 in each side of the partition, then we obtain an $\mathcal{NP}$-complete problem for general digraphs, except in the case of $(\delta^+ \geq 1, \delta^+ \geq 1)$- and $(\delta^- \geq 1, \delta^- \geq 1)$-partitions for which easy polynomial algorithms exist. Furthermore, Bang-Jensen and Christiansen proved that the $(\delta^+ \geq 1, \delta^+ \geq 2)$-partition problem (that is, deciding whether there is a 2-partition (V_1, V_2) of D such that $\delta^+(D[V_i]) \geq i$ for $i = 1, 2$) is already $\mathcal{NP}$-complete [20].

A suprisingly difficult problem is the following conjecture due independently to Alon and Stiebitz.

Conjecture 2.8.9 ([4, 173]) *There exists a function $f(k, \ell)$, where k, ℓ are positive integers, such that every digraph D with $\delta^+(D) \geq f(k, \ell)$ has a $(\delta^+ \geq k, \delta^+ \geq \ell)$-partition.*

It is easy to see that a digraph with minimum out-degree $k + \ell$ has a $(\delta^+ \geq k, \delta^+ \geq \ell)$-partition if and only if it has two disjoint subdigraphs with minimum out-degree at least k and ℓ. Thomassen [180] proved that every digraph D with $\delta^+(D) \geq 3$ has two disjoint cycles, hence Conjecture 2.8.9 holds for $k = \ell = 1$ and $f(1, 1) = 3$ because the second power C^2_{2r+1} of an odd cycle has no $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition. But even the existence of $f(1, 2)$ is still open.

In the remaining part of this section, we shall see that the situation is a lot simpler for semicomplete digraphs: Conjecture 2.8.9 holds for semicomplete digraphs and the problem of deciding whether a semicomplete digraph has a $(\delta^+ \geq k, \delta^+ \geq \ell)$-partition can be solved in polynomial time. A crucial notion here is that of an out-critical set. A set X of vertices of a digraph D is k***-out-critical*** if $\delta^+(D\langle X\rangle) = k$ and for every proper subset $S \subset X$, $\delta^+(D\langle S\rangle) < k$. Let X be a set of vertices in a digraph D. A set $X' \subseteq V(D)$ is called (X, k)***-out-critical*** if $X \subseteq X'$, $\delta^+(D[X']) \geq k$ and $\delta^+(D[Y]) < k$ for every $X \subseteq Y \subset X'$. Note that if $\delta^+(D[X]) \geq k$, then X is the only (X, k)-out-critical set in D. By definition, a digraph of minimum out-degree at least k contains at least one (X, k)-out-critical set for every subset X of vertices (including the empty set). The key fact is that the number of (X, k)-out-critical sets is bounded since their size is bounded. This was first observed by Lichiardopol for tournaments. In fact, his result holds for semicomplete digraphs as well.

Lemma 2.8.10 ([133]) *Let k be a positive integer, let D be a semicomplete digraph with minimum degree at least k, and let $X \subseteq V(D)$. If X' is an (X, k)-out-critical set in D, then $|X'| \leq \frac{k^2+3k+2}{2} + |X|$. In particular, every k-out-critical digraph in D has order at most $\frac{k^2+3k+2}{2}$.*

Proof: By induction on $|V(D)|$. If $|V(D)| \leq \frac{k^2+3k+2}{2} + |X|$ we are done, so assume $|V(D)| > \frac{k^2+3k+2}{2} + |X|$. Let M be the set of vertices that have out-degree k in T and let $m = |M|$.

Since $T[M]$ is semicomplete, we have

$$|N^+[M]| \leq m + mk - \frac{m(m-1)}{2} = -\frac{m^2}{2} + \left(\frac{3}{2} + k\right) m =: P(m).$$

Now $P(m)$ has global maximum at $(3/2+k)$ and maximum for m integer at $k+1$ and $k+2$ with $P(k+1) = P(k+2) = \frac{k^2+3k+2}{2}$. Hence $|N^+[M]| \leq \frac{k^2+3k+2}{2}$ and since $|V(D)| > \frac{k^2+3k+2}{2} + |X|$ there exists a vertex $u \in V(D) \setminus (N^+[M] \cup X)$. Then $\delta^+(T-u) \geq k$ and the result follows by induction. □

Corollary 2.8.11 ([133]) *For any pair of integers $k, \ell \geq 1$, every semicomplete digraph D with $\delta^+(D) \geq (k^2+3k+2)/2 + \ell$ has a $(\delta^+ \geq k, \delta^+ \geq \ell)$-partition. Furthermore, such a partition can be constructed in polynomial time.*

Proof: This follows easily from Lemma 2.8.10 by taking a k-out-critical set V_1 (which has size at most $(k^2+3k+2)/2$) and taking $V_2 = V \setminus V_1$.

Let us describe a polynomial algorithm, due to Bang-Jensen and Christiansen, for deciding whether a given semicomplete digraph has a $(\delta^+ \geq k, \delta^+ \geq \ell)$–partition.

Theorem 2.8.12 ([20]) *For every fixed pair of integers k and ℓ, there exists a polynomial algorithm that either constructs a $(\delta^+ \geq k, \delta^+ \geq \ell)$-partition of a given semicomplete digraph D or correctly outputs that none exists.*

Proof: Let O be the set of vertices with out-degree less than $k+\ell-1$. For a given partition (O_1, O_2) of O we let X be an (O_1, k)-out-critical set such that $X \subseteq V \setminus O_2$ (if no such set exists, we stop considering the pair (O_1, O_2)). The following subalgorithm $\mathcal{B}$ will decide whether there exists a $(\delta^+ \geq k, \delta^+ \geq \ell)$-partition (V_1, V_2) with $X \subseteq V_1, O_2 \subseteq V_2$: Starting from the partition $(V_1, V_2) = (X, V \setminus X)$, and moving one vertex at a time, the algorithm will move vertices v of $V_2 \setminus O_2$ such that $d^+_{T[V_2]}(v) < \ell$ to V_1. If, at any time, this results in a vertex $v \in O_2$ having $d^+_{T[V_2]}(v) < \ell$, or $V_2 = \emptyset$, then there is no $(\delta^+ \geq k, \delta^+ \geq \ell)$-partition with $O_i \subseteq V_i, = 1, 2$ and $\mathcal{B}$ terminates. Otherwise $\mathcal{B}$ will terminate with $O_2 \subseteq V_2 \neq \emptyset$ and hence it has found a $(\delta^+ \geq k, \delta^+ \geq \ell)$-partition (V_1, V_2) with $O_i \subseteq V_i$, $i = 1, 2$.

The correctness of $\mathcal{B}$ follows from the fact that we only move vertices that are not in O and each such vertex has at least $k+\ell-1$ out-neighbours in D. Hence, when moved, a vertex has less than ℓ out-neighbours in V_2, so it has at least k out-neighbours in V_1. Thus $\delta^+(D[V_1]) \geq k$ holds throughout the execution of $\mathcal{B}$.

By Proposition 2.2.2, $|O| \leq 2k+2\ell-3$ and hence the number of (O_1, O_2)-partitions is at most $2^{2k+2\ell-3}$ which is a constant when k and ℓ are fixed. Furthermore, by Lemma 2.8.10, the size of every O_1-critical set is also bounded by a function of k and hence each (O_1, O_2)-partition induces only a polynomial number of O_1-critical sets. Thus we obtain the desired polynomial algorithm by running the subalgorithm $\mathcal{B}$ for (at most) all possible partitions (O_1, O_2) of O and all possible (O_1, k)-out-critical sets. □

Lichiardopol proved an analogue of Corollary 2.8.11 for partitions with prescribed lower bounds on semi-degrees in tournaments. His result can easily be extended to semicomplete digraphs.

Theorem 2.8.13 ([133]) *For any choice of integers $k, \ell \geq 1$, every semicomplete digraph D with $\delta^0(D) \geq (k^2+3k+2)+\ell$ has a $(\delta^0 \geq k, \delta^0 \geq \ell)$-partition. Furthermore, such a partition can be constructed in polynomial time.*

The complexity of finding 2-partitions with prescribed minimum semi-degrees has been studied by Bang-Jensen and Christiansen. Recall that for general digraphs it is $\mathcal{NP}$-complete to decide the existence of a $(\delta^0 \geq k, \delta^0 \geq \ell)$-partition when $k+\ell \geq 2 \rightarrow k, \ell \geq 1$ [21]. Bang-Jensen and Christiansen showed that for semicomplete digraphs the situation is better, at least when $k = \ell = 1$.

Theorem 2.8.14 ([20]) *There exists a polynomial algorithm that given a semicomplete digraph D either finds a $(\delta^0 \geq 1, \delta^0 \geq 1)$-partition of D or correctly returns that none exists.*

Problem 2.8.15 *For any fixed positive integers k, ℓ, what is the complexity of deciding whether a semicomplete digraph has a $(\delta^0 \geq k, \delta^0 \geq \ell)$-partition ?*

One may also study all other possible variants, for example $(\delta^+ \geq k, \delta^- \geq \ell)$-partitions. The associated complexity problem is the following.

Problem 2.8.16 *For any fixed positive integers k, ℓ, what is the complexity of deciding whether a semicomplete digraph has a $(\delta^+ \geq k, \delta^- \geq \ell)$-partition?*

Bang-Jensen, Cohen and Havet proved that Problem 2.8.16 is $\mathcal{NP}$-complete for general digraphs already when $k = \ell = 1$. Bang-Jensen and Christiansen [20] proved that a semicomplete digraph D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition if and only if it has two disjoint cycles. Since one can find such a pair of disjoint cycles if one exists in polynomial time, one can decide in polynomial time whether a semicomplete digraph D has a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition. The following partial result on Problem 2.8.16 was obtained by Bang-Jensen and Christiansen.

Theorem 2.8.17 ([20]) *For every fixed integer $k \geq 1$ there exists a polynomial algorithm that either constructs a $(\delta^+ \geq 1, \delta^- \geq k)$-partition of a given semicomplete digraph D or correctly outputs that none exists.*

2.8.4 2-Partitions with Restrictions Both Inside and Between Sets

For a 2-partition (V_1, V_2) we denote by $D[V_1, V_2]$ the digraph induced by the arcs between V_1 and V_2. We now consider the degree analogue of Theorem 2.8.8, that is, we seek a 2-partition (V_1, V_2) so that each of

$D[V_1], D[V_2], D[V_1, V_2]$ has minimum out-degree at least some prescribed number. The following results are due to Alon, Bang-Jensen and Bessy.

Theorem 2.8.18 ([6]) *Except for the Paley tournament $\mathbb{P}_7$ every semicomplete digraph D with minimum out-degree at least 3 has a 2-partition (V_1, V_2) such that $D[V_1], D[V_2], D[V_1, V_2]$ has minimum out-degree at least one. Furthermore, when $D \neq \mathbb{P}_7$ one can always find such a 2-partition which is balanced, that is, $||V_1| - |V_2|| \leq 1$.*

For higher values of the degree bounds the authors obtained the following.

Theorem 2.8.19 ([6]) *There exist two absolute positive constants c_1, c_2 such that the following holds.*

1. *Let $T = (V, E)$ be a semicomplete digraph with minimum out-degree at least $2k + c_1\sqrt{k}$. Then there is a balanced a 2-partition (V_1, V_2) of V such that $\delta^+(D[V_1]), \delta^+(D[V_2])$ and $\delta^+(D[V_1, V_2])$ are all at least k.*
2. *For infinitely many values of k there is a tournament with minimum out-degree at least $2k + c_2\sqrt{k}$ such that for any 2-partition (V_1, V_2) of V at least one of the quantities $\delta^+(D[V_1]), \delta^+(D[V_2])$ and $\delta^+(D[V_1, V_2])$ is smaller than k.*

We only give the proof of the second part of Theorem 2.8.19. The proof illustrates one of the remarkable properties of the Paley tournaments: They behave almost like random tournaments.

Recall that for a prime q which is congruent to 3 modulo 4, the Paley tournament $\mathbb{P}_q$ is the tournament whose vertices are the integers modulo p where (i, j) is a directed edge if and only if $i - j$ is a quadratic residue modulo q.

Lemma 2.8.20 *Let $\mathbb{P}_q = (V, A)$ be the Paley tournament on q vertices. Then for any function $f : V \to \{-1, 1\}$ there is a vertex $v \in V$ such that $|\sum_{u \in N^+(v)} f(u)| > \frac{1}{2}\sqrt{q}$.*

Proof: It is easy and well known (c.f., e.g., [10], Chapter 9) that every vertex of $\mathbb{P}_q$ has out-degree and in-degree $(q-1)/2$ and any two vertices of it have exactly $(q-3)/4$ common in-neighbours (and out-neighbours). Let $A = A_q$ be the adjacency matrix of $\mathbb{P}_q$, that is, the 0/1 matrix whose rows and columns are indexed by the vertices of $\mathbb{P}_q$, where $A_{ij} = 1$ if and only if (i, j) is an arc. By the above comment, each diagonal entry of $A^t A$ is $(q-1)/2$ and each other entry is $(q-3)/4$. Thus the eigenvalues of $A^t A$ are $(q-1)/2 + (q-1)(q-3)/4 = (q-1)^2/4$ (with multiplicity 1) and $(q-1)/2 - (q-3)/4 = (q+1)/4$ (with multiplicity $(q-1)$). This implies that

$$||Af||_2^2 = f^t A^t A f \geq (q+1)/4||f||_2^2 = q(q+1)/4.$$

It follows that there is an entry of Af whose square is at least $(q+1)/4$, completing the proof. □

Note that, by Lemma 2.8.20, for any partition of the vertices of $\mathbb{P}_q$ into two disjoint (not necessarily nearly equal) sets V_1 and V_2 there is a vertex v of $\mathbb{P}_q$ such that the number of its out-neighbours in V_1 differs from that in V_2 by more than $\sqrt{q}/4$ (if there are x more neighbours in one set than in the other, then the sum in the lemma is $|\sum_{u\in N^+(v)} f(u)| = 2x$). This implies the assertion of part (ii) of Theorem 2.8.19 for infinitely many values of k.

2.8.5 Partitioning into Transitive Tournaments

A k**-dicolouring** of a digraph D is a k-partition $(V_1,\dots,V_k)$ of its vertex set such that $D\langle V_i\rangle$ is acyclic. The **dichromatic number** of D, denoted by $\vec{\chi}(D)$, is the smallest positive integer such that D admits a k-dicolouring. This notion was first treated by Neumann-Lara [148] and was independently introduced by Mohar [142] two decades later. Note that if G is an undirected graph, and D is the symmetric digraph obtained from G by replacing each edge by the pair of oppositely directed arcs joining its end vertices, then $\chi(G) = \vec{\chi}(D)$ since any two adjacent vertices in D induce a directed 2-cycle. Observe, moreover, that the dichromatic number of a tournament T is the minimum integer k such that T can be partitioned into k transitive subtournaments.

Finding the dichromatic number of a tournament is $\mathcal{NP}$-hard. Chen, Hu, and Zhang [61] proved that it is in fact already $\mathcal{NP}$-complete to decide whether a tournament has dichromatic number 2.

Theorem 2.8.21 ([61]) *Deciding whether a tournament has a 2-partition into two transitive subtournaments is $\mathcal{NP}$-complete.*

Proof: The original proof by Chen *et al.* was a reduction from NAE-3-SAT. We present here a simpler reduction from MONOTONE NAE-3-SAT (recall that monotone means that there are no negated variables).

Let $\mathcal{F} = C_1 \wedge \dots \wedge C_m$ be an instance of MONOTONE NAE-3-SAT on n variables $x_1,\dots,x_n$. We construct a tournament $T_{\mathcal{F}}$ as follows. Its vertex set is the union of $X = \{x_1,\dots,x_n\}$, a set $Y = \{y_1,y_2,y_3\}$ and $Z = \bigcup_{j=1}^m Z_j$, where $Z_j = \{z_j^1, z_j^2, z_j^3\}$. For $z\in Z$, we define x_z as follows: let j and ℓ be the indices such that $z = z_j^\ell$, and let i be the index such that x_i is the ℓth literal of C_j; then $x_z = x_i$.

Let σ be the following ordering of $V(T_{\mathcal{F}})$

$$(x_1,\dots,x_n,y_1,y_2,y_3,z_1^1,z_1^2,z_1^3,z_2^1,z_2^2,z_2^3,\dots,z_m^1,z_m^2,z_m^3).$$

All the arcs of $T_{\mathcal{F}}$ agree with σ (i.e. if u precedes v in σ, then $u\to v$) except for a set $B = B_Y \cap B_Z \cap B'$ of backward arcs, where $B_Y = \{y_3y_1\}$, $B_Z = \{z_j^3 z_j^1 \mid 1\le j\le m\}$ and $B' = \{zx_z \mid z\in Z\}$.

Observe that every directed cycle in $T_{\mathcal{F}}$ is either the 3-cycle $y_1y_2y_3y_1$, or the 3-cycle $z_j^1z_j^2z_j^3z_j^1$ for some $1 \leq j \leq m$, or contains an arc in B.

We shall now prove that $\mathcal{F}$ has an NAE-assignment if and only if $T_{\mathcal{F}}$ has a 2-partition (V_1, V_2) such that $T[V_i]$ is transitive.

Let us assume that $\mathcal{F}$ has an NAE-assignment ϕ. Let $X_1 = \{x_i \mid \phi(x_i) = true\}$, $X_2 = \{x_i \mid \phi(x_2) = false\}$, $Z_1 = \{z \mid x_z \in X_2\}$ and $Z_2 = \{z \mid x_z \in X_1\}$. Setting $V_1 = X_1 \cup \{y_1\} \cup Z_1$ and $V_1 = X_2 \cup \{y_2\} \cup Z_2$, one can easily check that (V_1, V_2) is a partition of $T_{\mathcal{F}}$ into two transitive tournaments. Indeed, the arcs of B have their end vertices in different part, each $\{z_j^1, z_j^2, z_j^3\}$ contains at least one vertex in V_1 and one in V_2 because ϕ is an NAE-assignment.

Assume now that $T_{\mathcal{F}}$ admits a partition (V_1, V_2) into two transitive subtournaments. Since Y induces a 3-cycle, at least one vertex of Y is in V_1 and another one is in V_2. Without loss of generality, we may assume $y_1 \in V_1$ and $y_2 \in V_2$. Similarly, each Z_j, $1 \leq j \leq m$ has a vertex in V_1 and a vertex in V_2. Now consider an arc zx_z in B'. The two vertices z and x_z are not in the same V_k ($k \in \{1, 2\}$) for otherwise zx_zy_kz would be a directed 3-cycle. Now one checks easily that the truth assignment ϕ defined by $\phi(x_i) = true$ if $x_i \in V_1$ and $\phi(x_i) = false$ if $x_i \in V_2$ is an NAE-assignment. □

Theorem 2.8.21 implies that it is unlikely to find a characterization of tournaments with dichromatic number k. However, it is interesting to find properties of such tournaments. A natural question, in the same flavour as unavoidablity (see Section 2.7), is to ask which subtournaments must appear in every tournament with sufficiently large dichromatic number. Such a tournament is called a **hero**. Clearly, transitive tournaments are heroes, since every tournament of order n contains a transitive subtournament of order at least $\log_2 n$ by Proposition 2.2.3. Moreover, Theorem 2.2.7 implies that the directed 3-cycle is contained in every tournament of dichromatic number at least 2. Observe moreover that if H is a hero, then every subtournament of H is also a hero.

When P, Q are tournaments, we denote by $C(P, Q)$ the tournament that one obtains from disjoint copies of P and Q, by adding a new vertex x dominating P and dominated by Q, and adding all the arcs from P to Q (thus $C(P, Q) = C_3[P, Q, \{x\}]$).

Let us define the sequence $(A_i)_{i \in \mathbb{N}}$ of tournaments inductively as follows:

- A_1 is the tournament with one vertex and no arcs;
- $A_{i+1} := C(A_i, A_i)$.

The proof of the next proposition is left to the reader.

Proposition 2.8.22 *If $i \geq 1$, then $\vec{\chi}(A_i) = i$.*

The tournaments $(A_i)_{i \in \mathbb{N}}$ imply that strongly connected heroes must have a special form.

Lemma 2.8.23 *Every strongly connected hero is of the form $C(P,Q)$, where P and Q are heroes.*

Proof: Let H be a hero. Then, by Proposition 2.8.22, for i sufficiently large, A_i contains H. Let k be the minimum integer i such that A_i contains H. Let us denote by L and R the copies of A_{k-1} in A_k such that all arcs are from L to R and let x be the vertex of $A_k - (L \cup R)$. By definition of k, neither L nor R contains H, so the copy of H in A_k must contain x. Now, since A_k is strong, H must contain at least one vertex of L and one vertex of R. Therefore H is of the desired form.

In fact, the tournaments that are heroes have been completely characterized by Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé.

Theorem 2.8.24 ([41])

- *A tournament T is a hero if and only if all its strong components are heroes;*
- *a strong tournament T is a hero if and only if $T = C(P, TT_r)$ or $T = C(TT_r, P)$ for some hero P and some $r \geq 1$.*

2.9 Feedback Sets

Feedback sets in a digraph are sets of vertices or arcs whose removal leaves the digraph acyclic. Formally, a **feedback vertex set** in a digraph D is a set S of vertices such that $D - S$ is acyclic, and a **feedback arc set** in a digraph D is a set F of arcs such that $D \setminus F$ is acyclic.

FEEDBACK VERTEX SET	**Parameter:** k
Input: A digraph $D = (V, A)$	
Question: Does D have a vertex set X of size at most k such that $D - X$ is acyclic?	

FEEDBACK ARC SET	**Parameter:** k
Input: A digraph $D = (V, A)$	
Question: Does D have a set of arcs A' of size at most k such that $D \setminus A'$ is acyclic?	

A feedback vertex (resp. arc) set is **minimal** if none of its proper subsets is also a feedback vertex (resp. arc) set. A feedback vertex (resp. arc) set is **minimum** if it is of minimum size. The minimum size of a feedback vertex set (resp. feedback arc set) in D is denoted by $\text{fvs}(D)$ (resp. $\text{fas}(D)$).

We are then interested in the optimization versions of FEEDBACK VERTEX SET and FEEDBACK ARC SET where one wishes to determine $\text{fvs}(D)$

and fas(D), respectively, for a given digraph D, as well as their restriction to tournaments FEEDBACK VERTEX SET IN TOURNAMENT (FVST for short) and FEEDBACK ARC SET IN TOURNAMENT (FAST for short).[4] These problems are very fundamental and have many practical applications. For example, FEEDBACK ARC SET IN TOURNAMENTS models the problem of ranking the teams of a round-robin sport tournament and the problem of clustering webpages (see e.g. the paper [190] by van Zuylen and Williamson).

An **ordering associated to** a feedback vertex set S (resp. feedback arc set F) is an acyclic ordering of $D - S$ (resp. $D \setminus F$). Observe that if $(v_1, \dots, v_n)$ is an ordering associated to a feedback arc set F of D, then $\{v_iv_j \in A(D) \mid i > j\}$ is a feedback arc set contained in F. Therefore, every minimum feedback arc set induces an acyclic digraph. In contrast, a feedback vertex set is usually not acyclic: a digraph has an acyclic feedback vertex set if and only if its dichromatic number is at most 2 (see Subsection 2.8.5). Some papers studied feedback vertex sets with a certain property $\mathbb{P}$, this is the same as studying a 2-partition (V_1, V_2) of a digraph D such that $D[V_1]$ has property $\mathbb{P}$ and $D[V_2]$ is acyclic. See e.g. the papers of Bang-Jensen, Cohen and Havet [21, 26].

Proposition 2.9.1 *Let F be a minimum feedback arc set in a digraph D. The digraph obtained from D by reversing all arcs of F is acyclic.*

Proof: Let $(v_1, \dots, v_n)$ be an acyclic ordering associated to F. Observe that every arc a of F is of the form v_iv_j with $i > j$ for otherwise $F \setminus \{a\}$ would also be a feedback arc set with $(v_1, \dots, v_n)$ associated to it, contradicting the minimality of F. Therefore reversing the arcs of F results in an acyclic digraph with acyclic ordering $(v_1, \dots, v_n)$. □

Proposition 2.9.1 implies that fas(D) is the minimum size of a set F of arcs whose reversal yields an acyclic digraph.

2.9.1 Feedback Vertex Sets

FEEDBACK VERTEX SET is one of the the first problems shown to be $\mathcal{NP}$-complete listed by Karp in [118]. Its easy reduction from VERTEX COVER is the following. Let (G, k) be an instance of VERTEX COVER. Let D be the symmetric digraph associated to G, that is, the digraph obtained from G by replacing each edge by a directed 2-cycle. One can easily check that G has a vertex cover of size k if and only if D has a feedback vertex set of size k.

It is also not very hard to show that FVST is $\mathcal{NP}$-complete. this was shown independently by Speckenmeyer [172] and by Bang-Jensen and Thomassen [34]. The proof below is from [34].

[4] For simplicity and because they are polynomially equivalent, we do not distinguish between the decision and the optimization versions of these problems.

Theorem 2.9.2 ([34]) FEEDBACK VERTEX SET IN TOURNAMENT *is $\mathcal{NP}$-complete.*

Proof: Reduction from INDEPENDENT SET which is well-known to be $\mathcal{NP}$-complete [118]. Let G be an undirected graph with vertices $v_1^0, \dots, v_n^0$. Let T be the tournament defined as follows. $V(T) = V(G) \cup \{v_i^j \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n+1\}$ and there is an arc $(v_{i_1}^{j_1}, v_{i_2}^{j_2})$ whenever $i_1 > i_2$ or $i_1 = i_2$ and $j_1 > j_2$, unless $j_1 = j_2 = 0$ and $v_{i_1}^0 v_{i_2}^0$ is an edge of G, in which case T contains the arc $(v_{i_2}^0, v_{i_1}^0)$. One can easily check that a vertex set S is a maximum independent set in G if and only if $V(G) \setminus S$ is a minimum feedback vertex set in T. □

FVST has a trivial 3-approximation algorithm, which proceeds as follows. As long as the tournament T is not transitive, find a directed 3-cycle C, delete its vertices from T and add them to the feedback vertex set S. Cai, Deng, and Zang [55] gave a 5/2-approximation. Recently, a 7/3-approximation was found by Mnich, Vassilevska Williams, and Végh [141].

For general digraphs, no non-trivial upper bound on the number of minimal feedback vertex sets is known. In contrast, we have some bounds for tournaments. Let $\#\text{fvs}(n)$ denote the maximum over all n-tournaments of the number of minimal feedback vertex sets. Note that $\#\text{fvs}(n)$ is also the maximum of the number of maximal transitive subtournaments since in a tournament T, a set S is a feedback vertex set if and only if $T - S$ is transitive. Moon [143] was the first to give bounds on $\#\text{fvs}(n)$. He proved $1.4757^n \leq \#\text{fvs}(n) \leq 1.7170^n$. This was later improved by Gaspers and Mnich [95]

$$1.5548^n < 21^{n/7} \leq \#\text{fvs}(n) \leq 1.6740^n.$$

To get the lower bound, consider the tournament T on $n = 7k$ vertices obtained from a transitive k-tournament by blowing up each vertex into a copy of the Paley tournament $\mathbb{P}_7$ on 7 vertices. The minimal feedback vertex sets of $\mathbb{P}_7$ are also minimum feedback vertex sets and have size 4. Furthermore, there are 21 of them. Hence T has $21^{n/7}$ minimal feedback vertex sets.

The upper bound relies on an enumeration algorithm, based on iterative compression (see the proof of Theorem 2.9.6 for an example of iterative compression), that enumerates in 1.6740^n-time all minimal feedback vertex sets in tournaments. Since a minimum feedback vertex set is also minimal, this algorithm allows us to solve FVST in 1.6740^n-time.

2.9.2 Feedback Arc Sets

FEEDBACK ARC SET is also one of the the first problems known to be $\mathcal{NP}$-Complete listed by Karp in [118]. The easy reduction due to Karp and Lawler is from VERTEX COVER. Given a graph G, let D be the digraph defined by

$V(D) = V(G) \times \{0,1\}$.
$A(D) = \{((v,0),(v,1)) \mid v \in V(G)\} \cup \{((u,1),(v,0)) \mid (u,v) \in E(G)\}$.

We easily check that G has a vertex cover of size k if and only if D has a feedback arc set of size k.

In contrast, FEEDBACK ARC SET IN TOURNAMENTS was conjectured to be $\mathcal{NP}$-complete in 1992 by Bang-Jensen and Thomassen [34]. Ailon, Charikar, and Newman [3] proved it is $\mathcal{NP}$-hard under randomized reductions. Shortly after, it was proved under deterministic reductions independently by Alon in [5] and by Charbit, Thomassé and Yeo in [58].

Theorem 2.9.3 FEEDBACK ARC SET IN TOURNAMENTS *is $\mathcal{NP}$-complete.*

The proofs of Alon [5] and Charbit, Thomassé, and Yeo [58] both use the same reduction based on the existence of bipartite tournaments with large minimum feedback arc sets. Here, by large, we mean close to the trivial upper bound. Indeed, consider a bipartite tournament B with both partite sets R, S of size k. Considering the set of arcs from R to S and the one from S to R, we obtain trivially that $\mathrm{fas}(B) \leq \frac{k^2}{2}$. Hence by a large minimum feedback arc set, we mean a minimum feedback arc set of size close to $\frac{k^2}{2}$.

Lemma 2.9.4 ([58]) *Let ℓ be a positive integer ℓ and set $k = 2^{3\ell}$. There exists a bipartite tournament B_k with both partite sets of size k and $\mathrm{fas}(B_k) \geq \frac{k^2}{2} - 2k^{5/3}$.*

Proof of Theorem 2.9.3 using Lemma 2.9.4: The reduction is from FEEDBACK ARC SET in general digraphs.

Let D be a digraph. We may assume that D has no directed cycle of length at most 2, as deleting such a cycle decreases fas by exactly 1. Let $V(D) = \{v_1, \ldots, v_n\}$ and set $k = 2^{6\lceil 1+\log_2 n\rceil}$. Observe that $k = O(n^6)$ and $k \geq 64n^6$.

Let B_k be the bipartite tournament defined in Lemma 2.9.4, and let $\{r_1, \ldots, r_k\}$ and $\{s_1, \ldots, s_k\}$ be the partite set of B_k.

Let T be the tournament obtained by blowing up every vertex of D by a transitive tournament, and adding copies of B_k between blow-ups of non-adjacent vertices. To be precise, the vertex set of T is $\{w_a^i \mid 1 \leq a \leq n$ and $1 \leq j \leq k\}$ and its arc set is $A_a \cup A_b \cup A_c$, where

$$A_a = \{w_a^i w_a^j \mid 1 \leq a \leq n \text{ and } 1 \leq i < j \leq k\},$$
$$A_b = \{w_a^i w_b^j \mid v_a v_b \in A(D) \text{ and } 1 \leq i,j \leq k\}, \text{and}$$
$$A_c = \{w_a^i w_b^j \mid ab, ba \notin A(D) \text{ and } 1 \leq a < b \leq n \text{ and } r_i s_j \in A(B_k)\}$$
$$\cup\{w_b^j w_a^i \mid ab, ba \notin A(D) \text{ and } 1 \leq a < b \leq n \text{ and } s_j r_i \in A(B_k)\}.$$

Let us now bound $\mathrm{fas}(T)$. Without loss of generality, we may assume that $(v_1, \ldots, v_n)$ is an acyclic ordering associated to a minimum feedback arc set of D. Observe that since a minimum feedback arc set induces an acyclic digraph, Lemma 2.9.4 implies that the arcs of A_c contribute at least $(\binom{n}{2} - |A(D)|)(\frac{k^2}{2} - 2k^{5/3})$ and at most $(\binom{n}{2} - |A(D)|)(\frac{k^2}{2} + 2k^{5/3})$ to $\text{fas}(T)$.

Considering the ordering $(w_1^1 \dots, w_1^k, w_2^1, \dots, w_2^k, w_3^1 \dots, w_n^k)$, we get

$$\text{fas}(T) \leq k^2 \,\text{fas}(D) + \left(\binom{n}{2} - |A(D)|\right)\left(\frac{k^2}{2} + 2k^{5/3}\right). \tag{2.5}$$

Consider now a minimum feedback arc set of T. For any integers $i_1, \dots, i_n$ in $\{1, \dots, k\}$, at least $\text{fas}(D)$ arcs of $T\langle\{w_1^{i_1}, w_2^{i_2}, \dots, w_n^{i_n}\}\rangle$ are in F because this digraph is isomorphic to D. Summing over all possible values of $i_1, \dots, i_n$ we get at least $k^n \,\text{fas}(D)$ arcs, where each arc can be counted at most k^{n2} times. Hence

$$\text{fas}(T) \geq \frac{k^n \,\text{fas}(D)}{k^{n-2}} + \left(\binom{n}{2} - |A(D)|\right)\left(\frac{k^2}{2} - 2k^{5/3}\right). \tag{2.6}$$

Now as $k^{1/3} \geq 64^{1/3} n^2$, we get that $(\binom{n}{2} - |A(D)|) \cdot 2k^{5/3} < \frac{k^2}{2}$. Hence Equations (2.5) and (2.6) imply the following.

$$\text{fas}(D) - \frac{1}{2} < \frac{\text{fas}(T)}{k^2} - \frac{1}{2}\left(\binom{n}{2} - |A(D)|\right) < \text{fas}(D) + \frac{1}{2}.$$

Hence if we could compute $\text{fas}(T)$ in polynomial time, we could also compute $\text{fas}(D)$. □

For general digraphs, the best known approximation algorithm for Feedback arc set has performance guarantee $O(\log n \log\log n)$. The existence of such a feedback arc set is due to Seymour [168] and the algorithmic part is due to Even, Naor, Schieber and Sudan [77]. In contrast, for tournaments van Zuylen and Williamson [190] proposed a 2-approximation. Their algorithm is based on a linear programming relaxation of the problem and a nice rounding procedure. This procedure is a derandomization of the algorithm by Ailon, Charikar and Newman given in [3] based on the so-called 'pivot'.

The dual maximization problem consisting in finding an acyclic spanning subdigraph of a digraph D with the maximum number of arcs is easy to approximate. A trivial 2-approximation consists in considering any ordering $(v_1, \dots, v_n)$ of the vertices of D and the subdigraphs D^+ and D^- with arc set $A^+ = \{v_i v_j \in A(D) \mid i < j\}$ and $A^- = \{v_j v_i \in A(D) \mid i > j\}$. These two digraphs are acyclic, and each of them has at least $|A(D)|/2$ arcs. There exist polynomial time approximation schemes (PTAS) for this problem in tournaments, see the papers [15] by Arora, Frieze and Kaplan and [91] by Frieze and Kannan.

By the above upper bound on $\text{fas}(D)$, for every n-tournament T, we have $\text{fas}(T) \leq \frac{n(n-1)}{4}$. This upper bound is almost tight as shown below.

Theorem 2.9.5 *For every $n \geq 3$, there exists a tournament T of order n such that* $\mathrm{fas}(T) \geq \frac{n(n-1)}{4} - \frac{1}{2}\sqrt{n^3 \log_e n}$.

Proof: Consider a random tournament RT_n on vertices $1, 2, \ldots, n$. Observe that for every pair $i \neq j \in \{1, 2, \ldots, n\}$, $ij \in A(RT_n)$ with probability 1/2.

For every pair $i < j \in \{1, 2, \ldots, n\}$, define the random variable $x_{i,j}$ by

$$x_{i,j} := \begin{cases} +1 \text{ if } ij \in A(RT_n) \\ -1 \text{ otherwise.} \end{cases}$$

Let $N = \binom{n}{2}$. With respect to the ordering $\pi = 1, 2, \ldots, n$, the number of forward arcs minus the number of backward arcs equals

$$\sum_{1 \leq i < j \leq n} x_{i,j} = S_N.$$

Then, $E_\pi := \{|S_N| > a\}$ denotes the event that, in one of the two orderings $\pi = \pi(1), \pi(2), \ldots, \pi(n) (= 1, 2, \ldots, n)$ and $\pi^* = \pi(n), \pi(n-1), \ldots, \pi(1) (= n, n-1, \ldots, 1)$, the number of forward arcs exceeds $n(n-1)/4 + a/2$. On the other hand, S_N is the sum of $\binom{n}{2}$ random independent variables taking values $+1$ and -1, each with probability 1/2. By the Chernoff bound (Corollary A.2 in the book of Alon and Spencer [8]),

$$\mathrm{Prob}(|S_N| > a) \leq 2 \exp\left(-\frac{a^2}{2N}\right), \tag{2.7}$$

for every positive number a.

Observe that the event E that for at least one permutation of $1, 2, \ldots, n$, the number of forward arcs exceeds $n(n-1)/4 + a/2$ equals the union of the events E_π for all permutations π of $1, 2, \ldots, n$, whose total number is $n!$. Put $a = \sqrt{n^3 \log_e n}$. Applying (2.7) we obtain

$$\begin{aligned} \mathrm{Prob}(E) &\leq 2n! \exp(-n \log_e n) \\ &\leq 2n! n^{-n} \\ &< 1 \end{aligned}$$

for every $n \geq 3$. This means that with positive probability the event E does not hold, i.e. for every permutation of $1, 2, \ldots, n$, the number of forward arcs does not exceed $\frac{n(n-1)}{4} + \frac{1}{2}\sqrt{n^3 \log_e n}$. By the definition of RT_n, it follows that there exists a tournament of order n with the above-mentioned property. □

A slightly better result was obtained by de la Vega in [70] who proved that $\sqrt{\log_e n}$ in the inequality of Theorem 2.9.5 can be replaced by a constant.

2.9.3 FPT Algorithms for Feedback vertex set in tournaments

Downey, Langston, Niedermeier, Raman, and Saurabh [157] proved that FVST is FPT by giving a $O(2.42^k \cdot n^{O(1)})$-time algorithm that solves it. This running time was improved by Fernau [78] who gave a $O(2.18^k \cdot n^{O(1)})$-time algorithm to solve FVST. We present below a faster FPT algorithm in $O(2^k \cdot n^{O(1)})$ time due to Dom, Guo, Hüffner, Niedermeier and Truß [71]. Very recently, an even faster FPT algorithm in $O(1.618^k + n^{O(1)})$ time was shown by Kumar and Lokshtanov [128].

Theorem 2.9.6 ([71]) Feedback vertex set in tournaments *can be solved in time* $O(2^k \cdot n^3)$.

Proof: We present an algorithm solving FVST in $O(2^k \cdot n^3)$ time. This algorithm uses the method, called **iterative compression**, which was introduced by Reed, Smith, and Vetta [159]. The key part of this algorithm is a **compression routine** which, given a tournament and a feedback vertex set of size $k+1$, computes a feedback vertex set of size k or proves that none exists.

Using such a compression routine FVST can be solved as follows. Let $\{v_1, \ldots, v_n\} = V(T)$, and let $T_i = T\langle\{v_1, \ldots, v_i\}\rangle$. We start with $S_2 = \emptyset$, which is a minimum feedback vertex set of T_2. Now for $i = 3$ to n, we compute a minimum feedback vertex set of T_i using S_{i-1}. Observe that $S_{i-1} \cup \{v_i\}$ is a feedback vertex set of T_i, so a minimum feedback vertex set of T_i has size $|S_{i-1}|$ or $|S_{i-1}| + 1$. Therefore, using the compression routine, we either find a feedback vertex set S_i of T of size $|S_{i-1}|$, or we prove that none exists, in which case we set $S_i = S_{i-1} \cup \{v_i\}$. At the end, after $n-2$ calls to the compression routine, we obtain S_n, a minimum feedback vertex set of T.

Let us now describe the compression routine running in $O(2^k \cdot n^2))$ time Let T be a tournament and S a feedback vertex set of size $k+1$. By brute-force, we enumerate all $O(2^k)$ partitions $(X, S \setminus X)$ of S, and for each of them we only look for feedback vertex sets that contain all vertices of $S \setminus X$ and none of X.

We delete the vertices of $S \setminus X$, i.e. $T' := T - (S \setminus X)$. Observe that T has a feedback vertex set of size k that contains all vertices of $S \setminus X$ and none of X if and only if T' has a feedback vertex set of size $|X| - 1$ disjoint from X. If $T'\langle X\rangle$ is not acyclic, we stop as there cannot be any feedback vertex set of T' disjoint from X. Hence we may assume that $T'\langle X\rangle$ is acyclic. Note also that $T' - X = T - S$ is acyclic.

We shall now determine the minimum size s of a feedback vertex set of T' disjoint from X. In fact, we compute $|T'| - s$, which is the maximum size of an acyclic subtournament of T' containing all of X. Such a tournament has an acyclic ordering which can be thought of as resulting from the insertion of a subset of $V(T') \setminus X$ into the acyclic ordering $(x_1, \ldots, x_{|X|})$ of X.

We first determine the set P of vertices v that we can insert into X, that are the vertices such that $T'\langle X \cup \{v\}\rangle$ is acyclic. Note that such a vertex has a unique possible position in L: there is an index $i(v) = i$ such that $N^-(v) \cap X = \{x_1, \dots, x_i\}$ and $N^+(v) \cap X = \{x_{i+1}, \dots, x_n\}$. Note that for each vertex $v \in V(T') \setminus X$, deciding if $v \in P$ and, if so, computing $i(v)$ can be done in $O(n^2)$ time. Let L be an acyclic ordering of $T' - X$ (it exists because $T' - X$ is acyclic), and let $R = (r_1, \dots, r_{|P|})$ be the ordering of P in which the vertices are ordered in increasing order of $i(v)$ and according to L as tie-breaker: for any $j < \ell$, either $i(r_j) < i(r_\ell)$, or $i(r_j) = i(r_\ell)$ and r_j is before r_ℓ in L. Now a largest acyclic subtournament of T' containing all of X is obtained from X by adding a longest common subsequence of L and P. Since L and P are permutations of each other, finding a longest common subsequence reduces to finding a longest increasing subsequence of the intersection. This can be done in $O(n \log n)$ time [90]. □

Dom, Guo, Hüffner, Niedermeier and Truß [71] also proved that FVST admits a cubic kernel. The idea of the proof is to transform an instance of FVST (T, k) into an equivalent instance (H, k) of Hitting set, where H is the 3-uniform hypergraph with vertex set $V(T)$ and hyperedge set the sets of 3-cycles in T. Then applying the kernelization algorithm given by Niedermeier and Rossmanith [150] for Hitting set, one can show that the resulting instance has cubic size.

2.9.4 FPT Algorithms for Feedback arc set in tournaments

Downey, Langston, Niedermeier, Raman, and Saurabh [157] proved that FAST is FPT providing a $O(2.42^k \cdot n^{O(1)})$-time algorithm for this problem. Alon, Lokshtanov and Saurabh [7] gave a faster algorithm that runs in $2^{O(\sqrt{k} \log^2 k)} + n^{O(1)}$ time. Their algorithm combines the colour coding technique (initiated in [11]) with a divide-and-conquer algorithm and a quadratic kernel for FAST. The existence of such a kernel was established by Dom, Guo, Hüffner, Niedermeier and Truß [71].

Theorem 2.9.7 ([71]) Feedback arc set in tournaments *admits a quadratic kernel. In particular, it is FPT.*

Proof: Here we use the fact that fas(D) is the minimum size of a set of arcs whose reversal makes the digraph acyclic (see Proposition 2.9.1). The kernelization procedure `FastKer`(T, k) proceeds as follows.

1. If a vertex v is in no directed 3-cycle, then return `FastKer`$(T - v, k)$;
2. If $|T| = 0$, then return a 'Yes' instance;
3. If $k = 0$, then return a 'No' instance;
4. If there is an arc a in more than k directed 3-cycles, then let T' be the tournament obtained from T by reversing a and return `FastKer`$(T', k-1)$;
5. If $|T| \le k(k+1)$, return (T, k), otherwise return a 'No' instance.

Clearly, FastKer(T,k) runs in $O(kn^3)$ time. Clearly, Steps 1 to 4 of FastKer are valid, since a feedback arc set of size k must contain each arc which is in more than k directed 3-cycles. At Step 5, all arcs are in less than k directed 3-cycles. Hence if T has a feedback arc set of size k it has at most $k(k-1)$ directed 3-cycles, spanning at most $k(k+1)$ vertices. Since every vertex is in a directed 3-cycle after Step 1, $|T| \leq k(k+1)$. Hence Step 5 is valid. □

Finally, the existence of a linear-size kernel for FAST has been proved by Cuzzocrea, Taniar, Bessy, Fomin, Gaspers, Paul, Perez, Saurabh, and Thomassé [67].

2.10 Small Certicates for k-(Arc)-Strong Connectivity

By a **certificate** for the k-(arc)-strong connectivity of a digraph D, we mean a spanning k-(arc)-strong subdigraph D' of D. Already for $k = 1$ it is $\mathcal{NP}$-hard to find a certificate with the minimum number of arcs, as this number is $|V(D)|$ if and only if D is Hamiltonian. Since every vertex in a k-(arc)-strong digraph has out-degree at least k, an optimal certificate for k-(arc)-strong connectivity has at least kn arcs.

Together with Edmonds' branching theorem (Theorem 1.8.2) the next result implies that, in polynomial time, one can find a certificate for k-arc-strong connectivity with at most twice the size of an optimal certificate.

Proposition 2.10.1 *Every k-arc-strong digraph contains a spanning k-arc-strong subdigraph with at most $2k(n-1)$ arcs. Furthermore, such a certificate can be constructed in polynomial time.*

Proof: Let $D = (V, A)$ be a k-arc-strong and let $s \in V$ be arbitrary. By Edmonds' branching theorem, D has k arc-disjoint out-branchings $B^+_{s,1}, \dots, B^+_{s,k}$ and k arc-disjoint in-branchings $B^-_{s,1}, \dots, B^-_{s,k}$. The union of the arcs of these $2k$ branchings is clearly k-arc-strong and it has exactly $2k(n-1)$ arcs. The complexity claim follows from Theorem 1.8.2. □

For all $k \geq 1$ and $n \geq 5k+2$, we define $\mathcal{T}_{n,k}$ as the class of tournaments that can be obtained from a transitive tournament $A = TT_k$ on k vertices and two k-arc-strong tournaments B, C as shown in Figure 2.4. It is not difficult to show that each tournament in $\mathcal{T}_{n,k}$ is k-arc-strong.

Let T be any member of $\mathcal{T}_{n,k}$. Observe that every k-arc-strong subdigraph D of T must contain at least $k(k+1)/2$ arcs from B to A and exactly k arcs from C to B (there are no more). Hence we have $[\sum_{x\in B} d^+_D(x)] - [\sum_{x\in B} d^-_D(x)] \geq k(k+1)/2 - k$, implying that $\sum_{x\in B} d^+_D(x) \geq k|B| + k(k-1)/2$. This implies that D has at least $nk + k(k-1)/2$ arcs. Thus the tournaments in $\mathcal{T}_{n,k}$ show the existence of k-arc-strong tournaments for which every certificate has at least $nk + ck^2$ arcs for some constant $c > 0$ and hence the

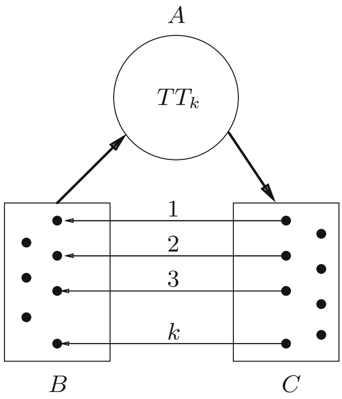

Figure 2.4 The structure of the tournaments in $\mathcal{T}_{n,k}$. The tournament A is the transitive tournament on k vertices, B and C are arbitrary k-arc-strong tournaments. The bold arcs $B \to A, A \to C$ indicate that all possible arcs are present in that direction. There are exactly k arcs from C to B and all other arcs go from B to C

following result of Bang-Jensen, Huang and Yeo is the best possible in terms of the exponent on k.

Theorem 2.10.2 ([28]) *For any $n \geq 3$ and $k \geq 1$, every k-arc-strong tournament on n vertices T contains a spanning k-arc-strong subdigraph with at most $nk + 136k^2$ arcs.*

The following result can be shown using network flows.

Proposition 2.10.3 ([28]) *Every k-arc-strong tournament contains a spanning subdigraph D on at most $nk + k(k-1)/2$ arcs such that $\delta^0(D) \geq k$.*

By the remark in Theorem 2.10.2, the truth of the following conjecture, due to Bang-Jensen, Huang and Yeo, would imply that the right bound in Theorem 2.10.2 would be $nk + k(k-1)/2$.

Conjecture 2.10.4 ([28]) *For every k-arc-strong tournament T, the minimum number of arcs in a k-arc-strong spanning subdigraph of T is equal to the minimum number of arcs in a spanning subdigraph of T with the property that every vertex has in- and out-degree at least k.*

Bang-Jensen asked [18] whether a result similar to Theorem 2.10.2 would also hold for vertex connectivity. This was confirmed recently by Kang, Kim, Kim and Suh.

Theorem 2.10.5 ([117]) *For $k \geq 1$, every k-strong n-tournament T has a k-strong spanning subdigraph with at most $nk + 750k^2 \log(k+1)$ arcs.*

The proof of this result is long and uses several results on linkages in tournaments. Some of the methods are very similar to those used in [156].

Below we prove a weaker, yet interesting, result from [117] which is used in the proof of Theorem 2.10.5 in [117].

Let t, k be positive integers with $t \geq k$. For a given ordering $\mathcal{O} = (v_1, v_2, \ldots, v_n)$ of the vertices of a digraph $D = (V, A)$ we denote by $F_{\mathcal{O}}$ the set of arcs v_iv_j with $i < j$ and call such arcs **forward** arcs wrt. $\mathcal{O}$. An ordering $\mathcal{O} = (v_1, v_2, \ldots, v_n)$ of the vertices of a digraph D is (k, t)**-good** if $D_{\mathcal{O}} = (V, F_{\mathcal{O}})$ satisfies

(a) $d^+_{D_F}(v_i) \geq k$ for all $i \in [n-t]$,
(b) $d^-_{D_F}(v_j) \geq k$ for all $t+1 \leq j \leq n$.

The following lemma is a special case of a lemma proved by Kang, Kim, Kim and Suh [117].

Lemma 2.10.6 ([117]) *Let $k \geq 1$ be an integer and let T be an n-tournament. Then there exists an ordering $\mathcal{O}$ of $V(T)$ and a spanning subdigraph D' of $T_{\mathcal{O}}$ such that D' is $(k, 2k-1)$-good and $|A(D')| \leq kn - k$.*

The following lemma is similar to Theorem 2.5.13.

Lemma 2.10.7 ([117]) *Let $k \geq 1$ and $n \geq 5k$ be integers. Every n-tournament T contains disjoint sets of vertices X, Y, each of size k such that, for any set S of $k-1$ vertices, the tournament $T - S$ has an (x, y)-path for every choice of $x \in X \setminus S, y \in Y \setminus S$.*

Let v be a vertex of a k-strong digraph D and let $Z = \{z_1, z_2, \ldots, z_k\}$ be a set of k vertices in $V(D) \setminus v$. A $(\mathbf{v}, \mathbf{Z})$**-fan** (resp. $(\mathbf{Z}, \mathbf{v})$**-fan**) is a collection of internally disjoint paths $P_1, \ldots, P_k$ (resp. $Q_1, Q_2, \ldots, Q_k$) such that P_i (resp. Q_i) is a (v, z_i)-path (resp. (z_i, v)-path). It is an easy consequence of Menger's theorem that every k-strong digraph has such a fan for arbitrary v and Z as above. We denote it by $F^+_{v,Z}$ (resp. $F^-_{Z,v}$). Note that it has at most $n-1$ arcs it is an out-tree (resp. in-tree) in D.

Theorem 2.10.8 ([117]) *Let k be a positive integer. Every k-strong n-tournament T contains a k-strong spanning subdigraph D with $|A(D)| \leq (5k-2)n + \binom{5k}{2}$.*

Proof: Set $V := V(T)$. If $n \leq 5k$, we let D be T itself. So assume $n > 5k$ and let $V' \subset V$ be an arbitrary set of $5k$ vertices. By Lemma 2.10.7, we can find two disjoint k-sets X, Y such that for every $S \subset V$ with $|S| = k-1$ and every choice of $x \in X \setminus S, y \in Y \setminus S$ the tournament $T[V' \setminus S]$ has an (x, y)-path. Applying Lemma 2.10.6, we obtain an ordering $\mathcal{O}$ of $V(T)$ and a spanning $(k, 2k-1)$-good subdigraph D' of $D_{\mathcal{O}}$ such that $|A(D')| \leq kn - k$. For each $n - 2k + 2 \leq i \leq n$, let $F_{v_i,X}$ be a (v_i, X)-fan in T, and, for each $1 \leq i \leq 2k-1$, let F_{Y,v_i} be a (Y, v_i)-fan. Now define the spanning digraph $D^* = (V, A^*)$ to be the union of all the arcs in $T[V']$, D', $F_{v_{n-2k+2},X}, \ldots, F_{v_n,X}$, $F_{Y,v_1}, \ldots, F_{Y,v_{2k-1}}$. By the remark on the size of fans above, it is easy to check that $|A(D^*)| \leq (5k-2)n + \binom{5k}{2}$. We now prove that D^* is k-strong. To show this, let S be any subset of $k-1$ vertices and let $u, v \in V \setminus S$ be arbitrary. We need to show that $D^* - S$ has a (u,v)-path. Because D' is $(k, 2k-1)$-good, in $D' - S$ there is a (u, v_i)-path P for some $n-2k+2 \leq i \leq n$ and a (v_j, v)-path P' for some $j \in [2k-1]$ (recall that D' is acyclic so every directed path moves forward in the ordering). After deleting the vertices of S from the fans $F_{v_i,X}$ and F_{Y,v_j} there still remains at least one intact path in each of these (as there are k internally disjoint such paths). Let $x_s \in X, y_s \in Y$ be such that $F_{v_i,X} - S$ contains a (v_i, x_s)-path P_{v_i,x_s} and $F_{Y,v_j} - S$ contains a (y_s, v_j)-path P_{y_s,v_j}. By Lemma 2.10.7, $T[V' \setminus S]$ has an (x_s, y_s)-path P''. Now the subdigraph of $D^* - S$ formed by the arcs of P, P', P'', P_{v_i,x_s} and P_{y_s,v_j} contains a (u,v)-path and we are done. □

2.11 Increasing Connectivity by Adding or Reversing Arcs

In this section we consider the following problems for semicomplete digraphs

(1) Given a digraph $D = (V, A)$ on at least $k+1$ vertices for some positive integer k, find a minimum set F of new arcs such that the digraph $D' = (V, A \cup F)$ is k-strong. Let $a_k(D) = |F|$.
(2) Given a digraph $D = (V, A)$ on at least $2k+1$ vertices for some positive integer k, find a minimum set $F \subset A$ of arcs in D such that the digraph D' obtained from D by reversing every arc in F is k-strong. Let $r_k(D) = |F|$.

Clearly,
$$a_k(D) \leq r_k(D), \tag{2.8}$$
since, instead of reversing arcs in D, we may add exactly those new arcs we would obtain by reversing and keep the original ones.

Frank and Jordán showed that $a_k(D)$ can be computed in polynomial time [88, 89]. The number $r_1(D)$ can be calculated via submodular flows (see e.g. [22, Section 13.1]). For $k \geq 2$, it is not clear how we can decide whether $r_k(D)$ even exists for a given arbitrary digraph D, let alone find an optimal reversal (unless we try all possibilities, which clearly requires exponential time). Indeed, this seems to be a very difficult problem.

We will now show that for semicomplete digraphs D, the function $r_k(D)$ behaves nicely. Note that, since we are dealing with vertex-connectivity, we gain nothing by reversing arcs that are contained in 2-cycles. Hence below we only consider arcs that are not contained in 2-cycles for possible reversal.

Proposition 2.11.1 ([29]) *If a semicomplete digraph D has at least $2k+1$ vertices, then $r_k(D)$ exists and is bounded by a function depending only on k.*

Proof: To see this it suffices to use the following two simple observations; the proof of the first one is left to the reader, and the second one follows directly from Proposition 2.2.2 and its directional dual.

(a) If D is a k-strong digraph and D' is obtained from D by adding a new vertex x and arcs from x to every vertex in a set X of k distinct vertices of D and arcs from every vertex of a set Y of k distinct vertices of D to x, then D' is also k-strong.
(b) If D is a semicomplete digraph on at least $4k-1$ vertices, then D contains a vertex with in-degree and out-degree at least k.

By observations (a) and (b), for every semicomplete digraph D, $r_k(D) \leq r_k(D')$ for some induced subdigraph D' of D with $|V(D')| \leq 4k-2$. We can find such a subdigraph D' as follows: Continue removing vertices as long as the current semicomplete digraph graph has at least $2k+2$ vertices and a vertex of in-degree and out-degree at least k. When this process stops, we have $2k+1 \leq |V(D')| \leq 4k-2$ in the current semicomplete digraph D'. Then we can make D' k-strong by reversing some arcs and add back each of the removed vertices in the reverse order of the deletion. This provides a simple upper bound for $r_k(D)$ (and hence for $a_k(D)$) as a function of k: we need to reverse at most half of the arcs in D', that is, at most $\frac{(4k-2)(4k-3)}{4}$ arcs. □

The process above may not lead to an optimal reversal for the original semicomplete digraph (in terms of the number of arcs to reverse), not even if we reverse optimally in D'.

It is easy to see that $r_k(TT_n) = k(k+1)/2$ when $n \geq 2k+1$. Bang-Jensen conjectured that no other tournament needs more reversals.

Conjecture 2.11.2 (Bang-Jensen [22]) *For every tournament T with $|V(T)| = n \geq 2k+1$, we have $r_k(T) \leq k(k+1)/2$.*

Since every semicomplete digraph contains a spanning tournament, if Conjecture 2.11.2 is true, this implies that the same conclusion holds for semicomplete digraphs on at least $2k+1$ vertices.

Bang-Jensen and Jordán showed that as soon as the number of vertices in the given semicomplete digraph D is sufficiently high (depending only on k), the minimum number of arcs in D we need to reverse in order to achieve a k-strong semicomplete digraph equals the minimum number of new arcs we need to add to D to obtain a k-strong semicomplete digraph.

Theorem 2.11.3 ([29]) *Let $k \geq 2$ be an integer. If D is a semicomplete digraph on at least $3k-1$ vertices, then $a_k(D) = r_k(D)$.*

The idea, which also leads to a polynomial algorithm for finding the desired reversal (see [29]), is to show that $r_k(D) \leq a_k(D)$, by demonstrating that a certain optimal augmenting set F of D has the property that, if we reverse the existing (opposite) arcs of F in D, then we obtain a k-strong semicomplete digraph. It was shown in [29] that $3k-1$ is the best possible bound for semicomplete digraphs. However, in the case when D is a tournament, the question as to whether or not the bound is the best possible was left open and the following conjecture was implicitly formulated.

Conjecture 2.11.4 ([29]) *For every tournament T on at least $2k+1$ vertices, we have $a_k(T) = r_k(T)$.*

Now we turn to arc-strong connectivity, where we shall see that the analogous problem to the one above has been solved.

Let $r_k^{deg}(D)$ be the minimum number of arcs one needs to reverse in a directed multigraph D in order to obtain a directed multigraph D' with $\delta^0(D') \geq k$. If no such reversal exists, we let $r_k^{deg}(D) = \infty$. Determining r_k^{deg} for a given digraph can be formulated as a feasibility flow problem and is thus polynomial (see e.g. [22, Section 14.5.1]). Analogously define $r_k^{arc}(D)$ to be the minimum number of arcs one needs to reverse in D in order to obtain a k-arc-strong directed multigraph.

By the Nash-Williams orientation Theorem [147], $r_k^{arc}(D) < \infty$ precisely when $UMG(D)$ is $2k$-edge-connected and one can calculate $r_k^{arc}(D)$ (including detecting whether $r_k^{arc}(D) = \infty$) in polynomial time using submodular flows (see e.g. [22, Section 11.8]). It follows from the results below that for tournaments the function r_k^{arc} can be calculated just using standard maximum-flow calculations.

The following result by Bang-Jensen and Yeo shows that for tournaments $r_k^{deg}(T)$ and $r_k^{arc}(T)$ are always bounded by a function that depends only on k.

Theorem 2.11.5 ([36]) *Let T be an n-tournament, with $n \geq 2k+1$. The following hold:*

(i) $r_k^{deg}(T) \leq k(k+1)/2$.
(ii) $r_k^{arc}(T) = \max\{k - \lambda(T), r_k^{deg}(T)\}$.

Observe that combining (i) and (ii) of Theorem 2.11.5, we obtain $r_k^{arc}(T) \leq k(k+1)/2$ which provides support to Conjecture 2.11.2. Recall again that the transitive tournaments show that this is the best possible.

The proof in [36] of Theorem 2.11.5 can be turned into a polynomial algorithm for finding a set of q arcs whose reversal makes T k-arc-strong using just maximum-flow calculations.

We now consider the operation of deorienting an arc. Let xy be an arc of a digraph D which is not in a 2-cycle. By **deorienting** xy we mean the operation which adds the arc yx to D. Clearly, deorienting arcs is equivalent to adding new arcs with the restriction that we can only add an arc which is opposite to an existing arc and we cannot create parallel arcs. Hence we may view deorienting arcs as a restricted version of the arc addition operation.

Let $deor_k^{deg}(D)$ denote the minimum number of arcs we need to deorient in D in order to obtain a digraph D' with $\delta^0(D') \geq k$. Using flows one can determine $deor_k^{deg}(D)$ for an arbitrary digraph D ([22, Exercise 14.18]). Clearly $deor_k^{deg}(D) \leq r_k^{deg}(D)$ for every oriented graph, in particular for every tournament. The example in Figure 2.5 shows that this inequality does not always hold for semicomplete digraphs.

Bang-Jensen and Yeo proved that for tournaments deorienting arcs is generally no better than reversing in terms of obtaining a desired minimum degree.

Theorem 2.11.6 ([36]) *Let T be a tournament on at least $2k+1$ vertices. Then $deor_k^{deg}(T) = r_k^{deg}(T)$. In particular, $deor_k^{deg}(T) \leq k(k+1)/2$.*

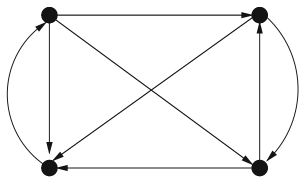

Figure 2.5 A semicomplete digraph D for which $1 = r_2^{arc}(D) < deor_2^{arc}(D) = 2$

Analogously define $deor_k^{arc}(D)$ to be the minimum number of arcs one needs to deorient in D in order to obtain a k-arc-strong digraph. It is easy to see that $deor_k^{arc}(D) < \infty$ if and only if $UG(D)$ is k-edge-connected. Furthermore, if D is an oriented graph (in particular, if D is a tournament), then we have $deor_k^{arc}(D) \leq r_k^{arc}(D)$ since instead of reversing an optimal set A' of arcs we may deorient these arcs and obtain a digraph with minimum semi-degree at least k. Figure 2.5 shows that the inequality above may not hold when D contains 2-cycles.

The following is a corollary of the Lucchesi–Younger theorem [138] about covering of directed cuts in a digraph.

Theorem 2.11.7 *Let D be a non-strong digraph for which $UG(D)$ is 2-edge-connected. Then $deor_1^{arc}(D) = r_1^{arc}(D)$.* □

When $k \geq 2$ and D is an arbitrary digraph, we do not know how to determine $deor_k^{arc}(D)$ efficiently, but as we show below, this is possible when D is a tournament.

One might expect that $deor_k^{arc}(D) < r_k^{arc}(D)$ for most oriented graphs. The next result, due to Bang-Jensen and Yeo, shows that for tournaments the two numbers are equal and hence, with respect to increasing the arc-strong connectivity of a tournament, there is no gain from deorienting arcs rather than reversing arcs.

Theorem 2.11.8 ([36]) *For every tournament T on at least $2k+1$ vertices we have $deor_k^{arc}(T) = r_k^{arc}(T)$.*

Proof: We saw in Theorem 2.11.5 that $r_k^{arc}(T) = \max\{k - \lambda(T), r_k^{deg}(T)\}$. If $r_k^{arc}(T) = r_k^{deg}(T)$, then, by Theorem 2.11.6, we have

$$\begin{aligned} deor_k^{arc}(T) &\leq r_k^{arc}(T) \\ &= r_k^{deg}(T) \\ &= deor_k^{deg}(T) \\ &\leq deor_k^{arc}(T), \end{aligned}$$

implying that $deor_k^{arc}(T) = r_k^{arc}(T)$. So we may assume that $r_k^{arc}(T) = k - \lambda(T)$. Now the claim follows from the easy fact that $deor_k^{arc}(T) \geq k - \lambda(T)$. □

We argued above that, in polynomial time, for a given tournament T, we can find a set of arcs $A' \subset A(T)$ of size $r_k^{arc}(T)$ such that reversing the arcs of A' results in a k-arc-strong tournament. Thus it follows from Theorem 2.11.8 that, in polynomial time, we can determine $deor_k^{arc}(T)$ and find a set of $deor_k^{arc}(T)$ arcs to deorient such that the resulting semicomplete digraph is k-arc-strong. One optimal set of arcs to deorient is simply a set that would form an optimal reversal.

Problem 2.11.9 ([36]) *Let $k \geq 1$ be a fixed integer. Is there a polynomial algorithm for determining the number $deor_k^{arc}(D)$ for a given input D?*

As we saw above the answer is yes if either $k = 1$ or if D is a tournament, but even the case of semicomplete digraphs and $k = 2$ is open. We also do not know whether there exists a polynomial algorithm for general oriented graphs when $k = 2$. Recall that for any digraph D and positive integer k the number $deor_k^{deg}(D)$ can be calculated in polynomial time via flows.

Analogously to the definition of $deor_k^{arc}(D)$ we may define $deor_k(D)$ to denote the minimum number of arcs we need to deorient in D in order to obtain a k-strong digraph. Clearly $deor_k(D) < \infty$ precisely when $UG(D)$ is k-connected. We have $deor_1(D) = deor_1^{arc}(D)$ for every digraph but for higher values of k nothing is known about $deor_k(D)$ for general digraphs. Notice that when D is a semicomplete digraph on at least $3k-1$ vertices we have $deor_k(D) = a_k(D)$ by Theorem 2.11.3.

2.12 Arc-Disjoint Spanning Subdigraphs of Semicomplete Digraphs

Below we discuss results on arc-disjoint Hamiltonian cycles, strong spanning subdigraphs and in- and out-branchings.

2.12.1 Arc-Disjoint Hamiltonian Paths and Cycles

Let T be a non-strong tournament and let $T_1, T_2, \dots, T_k$ be the acyclic ordering of its strong components. Two components T_i, T_{i+1} are called **consecutive** for $i = 1, 2, \dots, k-1$.

Thomassen [181] completely characterized tournaments having a pair of arc-disjoint Hamiltonian paths.

Theorem 2.12.1 ([181]) *A tournament T fails to have two arc-disjoint Hamiltonian paths if and only if T has a strong component which is an almost transitive tournament of odd order or T has two consecutive strong components of order 1.* $\diamond$

Thomassen posed the following problem.

Problem 2.12.2 ([181]) *What is the complexity of deciding whether a tournament has a Hamiltonian path P and a Hamiltonian cycle C which are arc-disjoint?*

Thomassen solved this problem for tournaments that are arc-3-cyclic (that is, every arc is contained in a 3-cycle) [181]. Moon proved that almost all tournaments are arc-3-cyclic [146] so Thomassen's result covers almost all tournaments.

Theorem 2.12.3 ([181]) *Every arc-3-cyclic n-tournament with $n \geq 6$ has a Hamiltonian path and a Hamiltonian cycle which are arc-disjoint.*

Observe that Theorem 2.12.1 implies that every 2-arc-strong tournament has two arc-disjoint Hamiltonian paths. Thomassen [181] conjectured the existence of a function $h(k)$ such that every $h(k)$-strong tournament contains k arc-disjoint Hamiltonian cycles. He proved that $h(2) \geq 3$ and conjectured that equality holds. The existence of $h(k)$ was recently verified by Kühn, Lapinskas, Osthus and Patel [124] who proved that $h(k) \in O(k^2 \log^2(k))$ suffices. They conjectured that $h(k) \in O(k^2)$ would suffice. This was confirmed by Pokrovskiy.

Theorem 2.12.4 ([155]) *There exists a constant C such that every Ck^2-strong tournament contains k arc-disjoint Hamiltonian cycles.*

By Theorem 2.6.19, $h(2) = 3$ would follow from the following conjecture due to Bang-Jensen and Yeo.

Conjecture 2.12.5 ([35]) *Every tournament T either contains two arc-disjoint Hamiltonian cycles or a set A' of at most two arcs such that $T \setminus A'$ has no Hamiltonian cycle.*

Confirming a conjecture of Erdős, Kühn and Osthus proved the following. Here 'almost all' means that the probability of a random n-tournament having the desired property tends to 1 as n tends to infinity.

Theorem 2.12.6 ([126]) *Almost all tournaments have $\delta^0(T)$ arc-disjoint Hamiltonian cycles.*

Now we turn to decompositions into arc-disjoint Hamiltonian cycles. Clearly any digraph which has an arc-decomposition into Hamiltonian cycles must be regular. Tillson characterized when one can decompose the arc set of the complete digraph into arc-disjoint Hamiltonian cycles.

Theorem 2.12.7 ([189]) *The complete digraph $\overleftrightarrow{K}_k$ can be decomposed into arc-disjoint Hamiltonian cycles if and only if $k \neq 4, 6$.*

The following conjecture, due to Kelly (see [146]), is the most famous open problem on tournaments.

Conjecture 2.12.8 (Kelly, 1968) *Every regular n-tournament can be partitioned into $(n-1)/2$ Hamiltonian cycles.*

This conjecture has attracted a lot of attention and a number of partial or closely related results have been obtained, e.g. [42, 103, 113, 119, 181, 183, 196]

The major breakthrough on the Kelly conjecture was made by Kühn and Osthus who proved the conjecture for (very) large n.

Theorem 2.12.9 ([126]) *For k sufficiently large, every k-regular tournament decomposes into k arc-disjoint Hamiltonian cycles.*

The proof in [126] is very long, almost 100 pages. It still remains a major challenge to prove Conjecture 2.12.8 in full.

For k-regular semicomplete digraphs, we do not necessarily have k-arc-disjoint Hamiltonian cycles. For $k = 2$, one such example is obtained from a 4-cycle by adding a 2-cycle between the two pairs of vertices of distance 2 along the cycle.

Problem 2.12.10 *What is the complexity of deciding whether a given regular semicomplete digraph has a decomposition into arc-disjoint Hamiltonian cycles?*

It follows from Theorem 2.12.9 that for regular tournaments there is a polynomial algorithm to decide whether the given tournament has a decomposition into Hamiltonian cycles. Of course, if Kelly's conjecture is true, then there is a trivial algorithm, because the answer will always be 'yes'.

Let T bethe tournament on $n = 4m+2$ vertices obtained from two regular tournaments T_1 and T_2, each on $2m+1$ vertices, by adding all arcs from the vertices of T_1 to T_2. Clearly T is not strong and so has no Hamiltonian cycle. The minimum semi-degree of T is $m = \frac{n-2}{4}$. One can easily prove that every

n-tournament with $\delta^0(T) \geq \frac{n}{4}$ is strongly connected. Bollobás and Häggkvist [45] showed that if we increase the minimum semi-degree slightly, then, not only do we obtain many arc-disjoint Hamiltonian cycles, we also obtain a very structured set of such cycles provided that the tournament has enough vertices.

Theorem 2.12.11 ([45]) *For every $\epsilon > 0$ and every positive integer k, there is an integer $n(\epsilon, k)$ with the following property. If T is a tournament of order $n > n(\epsilon, k)$ such that $\delta^0(T) \geq (\frac{1}{4} + \epsilon)n$, then T contains the kth power of a Hamiltonian cycle.* ◇

2.12.2 Arc-Disjoint Spanning Strong Subdigraphs

In this subsection, we study the decomposition of digraphs into strong subdigraphs. Since adding an arc to a strong digraph results in another strong digraph, a digraph decomposes into k arc-disjoint spanning strong subdigraphs if and only of it contains k arc-disjoint spanning strong subdigraphs.

Bang-Jensen and Yeo posed the following conjecture, which contains the Kelly conjecture (Conjecture 2.12.8) as the special case when $n = 2k + 1$.

Conjecture 2.12.12 (Bang-Jensen and Yeo [35]) *A tournament T can be decomposed into k arc-disjoint spanning strong subdigraphs if and only if T is k-arc-strong.*

They proved this conjecture for $k = 2$ and also characterized the 2-strong semicomplete digraphs that have an arc decomposition into two spanning strong subdigraphs.

Let S_{2k} be the semicomplete digraph which one obtains from two disjoint copies of the complete digraph $\overleftrightarrow{K}_k$ by adding a perfect matching oriented from one copy to the other and adding all remaining arcs in the opposite direction.

Lemma 2.12.13 ([35]) *The semicomplete digraph S_{2k} decomposes into k-arc-disjoint spanning strong subdigraphs except when $k = 2$.*

The following theorem implies that Conjecture 2.12.12 holds for $k = 2$.

Theorem 2.12.14 ([35]) *Let D be a 2-arc-strong semicomplete digraph, on n vertices. Then D decomposes in two arc-disjoint spanning strong subdigraphs if and only if it is not isomorphic to S_4.*

Below we shall give a proof Conjecture 2.12.12 for the class of k-arc-strong tournaments which have a non-trivial k arc-cut (Theorem 2.12.17). The proof, which is due to Bang-Jensen and Yeo, uses Theorem 2.12.7 and Theorem 2.12.16, which can be deduced from the following result of Smetanuik on completion of partial Latin squares.

Theorem 2.12.15 ([170]) *Let B be a complete bipartite graph (undirected), with n vertices in each partite set, and let R be a set of edges in B such that $|R| \leq n-1$. Then we can decompose $E(B)$ into n edge-disjoint matchings $M_1, M_2, ..., M_n$ such that $|M_i \cap R| \leq 1$ for all $i = 1, 2, ..., n$.*

Theorem 2.12.16 ([170]) *Let $B = (X, Y, E)$ be an undirected complete bipartite graph with $|X| = t$, $|Y| = s$ and $t > s$. Let R be a set of edges in B such that $|R| \leq s$. Then we can colour the edges of B by $|R|$ colours in such a way that all edges in R receive distinct colours and every vertex in $X \cup Y$ is incident with all $|R|$ colours.*

Theorem 2.12.17 *Let $k \geq 1$ and let D be a k-arc-strong semicomplete digraph such that there exists a set $S \subset V(D)$ with $2 \leq |S| \leq |V(D)| - 2$ and $d^+(S) = k$. There exist k arc-disjoint strong spanning subgraphs of D except if $D = S_4$.*

Proof: By Lemma 2.12.13 we may assume that D is not isomorphic to S_4.

It is not difficult to show that $k \leq |S| \leq n-k$ (by showing that $|S| \geq k$ and $|V(D) - S| \geq k$, respectively). If $|S| = |V - S| = 2$ then D contains S_4 as a proper spanning subdigraph and it is easy to check that adding any arc to S_4 will result in a digraph with two arc-disjoint strong spanning subdigraphs. Hence we may assume that $n \geq 5$. Let $e_1, e_2, \ldots, e_k$ be the k arcs from S to $V(D) - S$, and let $e_i = x_i y_i$, for $i = 1, 2, \ldots, k$. Let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$. Note that we may have $|X| < k$ or $|Y| < k$ or both. We may assume, by reversing all arcs if necessary, that $|V - S| \geq |S|$.

By Lemma 2.12.13 and the remark above, we may assume that $|V - S| > |S|$ if $|S| = k$. By Theorem 2.12.16 (with $R = \{e_1, e_2, \ldots, e_k\}$) we can colour all arcs between S and $V(D) - S$ with k colours such that the arcs from S to $V(D) - S$ get different colours and every vertex in V is incident with arcs of all k colours. Note that if $|V - S| = |S| > k$ this follows from Theorem 2.12.15.

Assume, without loss of generality, that the arc $x_i y_i$ is coloured with colour i, and let F_i contain all arcs between S and $V(D) - S$ of colour i.

By Theorem 1.8.2 there exists k arc-disjoint out-branchings $U_1, U_2, \ldots, U_k$, in $D[V(D) - S]$ such that U_i is rooted at y_i, for $i = 1, 2, \ldots, k$ (consider k arc-disjoint out-branchings from any vertex in S. Each of these must contain exactly one of the arcs $e_1, e_2, \ldots, e_k$. Thus the out-branching that contains the arc e_i must contain an out-branching from y_i in $D[V(D) - S]$). Analogously, there exists k arc-disjoint in-branchings $V_1, V_2, \ldots, V_k$, in $D[S]$ such that V_i is rooted at x_i, for $i = 1, 2, \ldots, k$. Let $T_i = V_i \cup U_i \cup F_i$, for $i = 1, 2, \ldots, k$. Clearly $T_1, T_2, \ldots, T_k$ are arc-disjoint and spanning. Each T_i is furthermore strong: by the construction of the colouring, every vertex in V is incident to an arc of colour i, every vertex in $V(D) - S - y_i$ is the tail of an arc in T_i into S, and hence every vertex in V can reach y_i (via V_i and the arc $x_i y_i$) and every vertex in $S - x_i$ is the head of an arc from $V(U_i)$ in T_i, implying that in T_i all vertices can be reached by y_i and reach x_i. This completes the proof. □

The following theorem, due to Bang-Jensen and Yeo, implies that we can always obtain about $\frac{1}{37}\lambda(T)$ arc-disjoint spanning strong subdigraphs in any tournament T. Note that in the case when $\lambda(T) < 37k$ the result below follows from Theorem 2.12.17.

Theorem 2.12.18 ([35]) *Let T be a k-arc-strong tournament, with minimum semi-degree $\delta^0(T) \geq 37k$. Then there exists k arc-disjoint spanning strong subgraphs in T.*

2.12.3 Arc-Disjoint In- and Out-Branchings

We now turn to branchings and consider the following problem

> ARC-DISJOINT IN- AND OUT-BRANCHINGS
> **Input:** A digraph D and vertices u, v (not necessarily distinct).
> **Question:** Does D have a pair of arc-disjoint branchings B_u^+, B_v^- such that B_u^+ is an out-branching rooted at u and B_v^- is an in-branching rooted at v?

The following result was proved by Thomassen [16].

Theorem 2.12.19 ARC-DISJOINT IN- AND OUT-BRANCHINGS *is $\mathcal{NP}$-complete for arbitrary digraphs.*

Bang-Jensen and Huang showed that, if the vertex that is to be the root is adjacent to all other vertices in the digraph and is not in any 2-cycle, then the problem becomes polynomially solvable.

Theorem 2.12.20 ([27]) *Let $D = (V, A)$ be a strongly connected digraph and v a vertex of D such that v is not on any 2-cycle and $V(D) = \{v\} \cup N^-(v) \cup N^+(v)$. Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ ($\mathcal{B} = \{B_1, B_2, \dots, B_r\}$) denote the set of terminal (initial) components in $D\langle N^+(v)\rangle$ ($D\langle N^-(v)\rangle$). Then D contains a pair of arc-disjoint branchings B_v^+, B_v^- such that B_v^+ is an out-branching rooted at v and B_v^- is an in-branching rooted at v if and only if there exist two disjoint arc sets $E_{\mathcal{A}}, E_{\mathcal{B}} \subset A$ such that all arcs in $E_{\mathcal{A}} \cup E_{\mathcal{B}}$ go from $N^+(v)$ to $N^-(v)$ and every $A_i \in \mathcal{A}$ ($B_j \in \mathcal{B}$) is incident with an arc from $E_{\mathcal{A}}$ ($E_{\mathcal{B}}$). Furthermore, there exists a polynomial algorithm to find the desired branchings, or demonstrate the non-existence of such branchings.*

This implies the following result due to Bang-Jensen.

Corollary 2.12.21 ([16]) *A tournament $T = (V, A)$ has arc-disjoint branchings B_v^+, B_v^- rooted at a specified vertex $v \in V$ if and only if T is strong and for every arc $a \in A$ the digraph $T - a$ contains either an out-branching or an in-branching with root v.*

When $u \neq v$, Arc-disjoint in- and out-branchings becomes much harder even for semicomplete digraphs. Bang-Jensen [16] found a complete characterization for the case of tournaments. This characterization, which is quite complicated, implies the tournament case of the following Theorem by Bang-Jensen and Yeo.

Theorem 2.12.22 ([35]) *Every 2-arc-strong semicomplete digraph $T = (V, A)$ contains arc-disjoint in- and out-branchings B_r^-, B_s^+ for every choice of vertices $r, s \in V$.*

Proof: This follows from Lemma 2.12.13 since it is easy to show that the semicomplete digraph S_4, which is the unique exception to that theorem, has arc-disjoint in- and out-branchings B_u^-, B_v^+ for every choice of $u, v \in V(S_4)$. □

Bang-Jensen found a polynomial algorithm for Arc-disjoint in- and out-branchings in the case of tournaments.

Theorem 2.12.23 ([16]) *There is a polynomial algorithm for checking whether a given tournament with specified distinct vertices u, v has arc-disjoint branchings B_u^+, B_v^- and finding such branchings if they exist.* □

Thomassen conjectured that every digraph which has sufficiently high arc-strong connectivity has arc-disjoint in- and out-branchings for every choice of roots.

Conjecture 2.12.24 ([178]) *There exists a positive integer N such that every digraph D which is N-arc-strong has arc-disjoint branchings B_v^+, B_v^- for every choice of $v \in V(D)$.*

Bang-Jensen and Yeo generalized this as follows.

Conjecture 2.12.25 *There exists a positive integer N such that every digraph D which is N-arc-strong has two arc-disjoint spanning strong subdigraphs.*

Theorem 2.12.14 implies that the conjecture holds with $N = 3$ for semicomplete digraphs and with $N = 2$ for tournaments. The following consequence of Theorem 2.12.18 verifies a conjecture by Bang-Jensen and Gutin [23].

Theorem 2.12.26 ([35]) *Let T be $74k$-arc-strong tournament. Then T has $2k$ arc-disjoint branchings $B_{v,1}^+, \ldots, B_{v,k}^+, B_{v,1}^-, \ldots, B_{v,k}^-$ such that $B_{v,1}^+, \ldots, B_{v,k}^+$ are out-branchings rooted at v and $B_{v,1}^-, \ldots, B_{v,k}^-$ are in-branchings rooted at v, for every vertex $v \in V(T)$.*

Note that if Conjecture 2.12.12 is true then we may replace $74k$ by $2k$.

Conjecture 2.12.27 ([35]) *Theorem 2.12.26 also holds if we replace* $74k$ *by* $2k$.

2.13 Minors of Semicomplete Digraphs

The most important advance in graph theory in the last few decades is certainly the Robertson–Seymour minor theory and by now the minor relation for graphs is well-established. However it is not clear how it should be extended to digraphs. A **minor** of a graph G is usually defined as a graph that can be obtained from a subgraph of G by contracting edges. Unfortunately, in digraphs, contracting an arc may yield a directed cycle, even when we are starting from an acyclic digraph, and this seems undesirable for a theory of excluded minors. One way to avoid this is to permit the contraction only of certain special arcs; for instance, in the paper [114] by Johnson, Robertson, Seymour and Thomas, an arc uv can be contracted if it is either the only arc with tail u or the only arc with head v. Another way, called **shallow directed minors**, has been introduced by Kreutzer and Tazari in [123]. A third approach comes from the observation that a minor of a graph G can also be defined as a graph that can be obtained from a subgraph of G by contracting connected subgraphs. Therefore Kim and Seymour [120] defined a **minor** of a digraph D as a digraph that can be obtained from a subdigraph of D by contracting strong subdigraphs.

An important property of minors for graphs is that they define a **well quasi-order** as shown by Robertson and Seymour [163]. (Recall that a quasi-order $\leq$ is a reflexive and transitive relation, and that it is a well quasi-order if for every infinite sequence $q_1, q_2, \dots$ there exist $j > i$ such that $q_i \leq q_j$.) The analogous statement is not true for directed minors. For example, a directed cycle is not a minor of a bigger directed cycle, and so if we take an infinite set of directed cycles, all of different lengths, then this set is an infinite antichain under the minor order. However, Kim and Seymour [120] proved that minor containment defines a well quasi-order for the class of all semicomplete digraphs, and therefore the same is true for the class of all tournaments.

Theorem 2.13.1 ([120]) *Minor containment is a well quasi-order on the class of all semicomplete digraphs.*

Kim and Seymour [120] also showed that this result cannot be generalized to larger classes of digraphs. In particular, they showed that minor containment is not a well quasi-order on the class of all digraphs with independence number 2. Indeed, consider the digraphs D_i, $i \geq 2$, defined as follows:

- $V(D_i)$ is the disjoint union of C_i, C'_i, T_i and T'_i;
- $D_i\langle C_i\rangle$ and $D_i\langle C'_i\rangle$ are directed 3-cycles;

- $D_i\langle T_i\rangle$ and $D_i\langle T_i'\rangle$ are transitive tournaments with Hamiltonian directed paths $t_1 \dots t_i$ and $t_i' \dots t_1'$, respectively;
- $C_i \to T_i$ and $T_i' \to C_i'$;
- there is exactly one arc with tail in C_i' and head in C_i;
- for every $1 \leq j \leq i$, $\{t_j, t_{j+1}\} \to t_j'$ with $t_{i+1} = t_1$.

One can check that there do not exist $j > i \geq 2$ such that D_i is a minor of D_j.

In [64], Chudnovsky and Seymour proved that immersion is a well quasi-order on the class of all tournaments, by using the parameter cutwidth (see the definition in Section 2.5.1). This was recently extended to the class of all semicomplete digraphs by Barbero, Paul and Pilipczuk [38].

Kim and Seymour proved Theorem 2.13.1 by using another parameter called path-width. For a digraph D, a sequence $(W_1, \dots, W_r)$ of subsets of $V(D)$ is a **path decomposition** of D if it satisfies the following conditions:

- $\bigcup_{i=1}^{r} W_i = V(D)$;
- for $1 \leq h < i < j \leq r$, $W_h \cap W_j \subseteq W_i$; and
- if $uv \in A(D)$, then $u \in W_i$ and $v \in W_j$ for some $i \geq j$.

The **width** of such a path decomposition is defined to be the number $\max\{|W_i| - 1 \mid 1 \leq i \leq r\}$. The **path-width** of D is the smallest width of a path-decomposition. For example, if $v_1, \dots, v_n$ is an acyclic ordering of an acyclic digraph, then $(\{v_1\}, \dots, \{v_n\})$ is a path-decomposition of this digraph of width 0. Hence every acyclic digraph has path-width 0.

Having bounded path-width is a minor-closed property.

Lemma 2.13.2 ([120]) *If a digraph has path-width at most k, then so do all its minors.*

Proof: Let $(W_1, \dots, W_r)$ be a path-decomposition of a digraph D with width at most k. It is also a path-decomposition of $D \setminus a$ for every arc $a \in A(D)$ and $(W_1 \setminus \{v\}, \dots, W_r \setminus \{v\})$ is a path-decomposition of $D - v$ for every vertex $v \in V(D)$. Hence, it remains to show that for every strong subdigraph H, the digraph D/H obtained from D by contracting H into a vertex w has path-width at most k.

Let $I_H = \{i \mid W_i \cap V(H) \neq \emptyset\}$. One can check that i_H is an interval and that the path-decomposition $(W_1', \dots, W_r')$ defined by $W_i' = (W_i \setminus V(H)) \cup \{w\}$ if $i \in i_H$ and $W_i' = W_i$ otherwise is a path-decomposition of G/H of width at most k. □

The k**-triple** is the digraph T_k defined by

$$\begin{aligned} V(T_k) &= \{a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_k\}, \text{and} \\ E(T_k) &= \{a_i b_j \mid 1 \leq i \leq k, 1 \leq j \leq k\} \cup \{b_i c_j \mid 1 \leq i \leq k, 1 \leq j \leq k\} \cup \\ &\quad \{c_i a_i \mid 1 \leq i \leq k\}. \end{aligned}$$

Observe that every semicomplete digraph with k vertices is a minor of the k-triple T_k. Indeed, set $B = \{b_1, \dots, b_k\}$; then $D\langle\{a_i, b_i, c_i\}\rangle$ is strong for each i. The digraph D' obtained from D by contracting $D\langle\{a_i, b_i, c_i\}\rangle$ to a vertex for each i is the complete symmetric digraph of order k.

A theorem of Fradkin and Seymour [86] says that a semicomplete digraph D has large path-width if and only if it contains a large k-triple.

Theorem 2.13.3 ([86]) *Let $\mathcal{S}$ be a set of semicomplete digraphs. The following two statements are equivalent:*

1. *There exists a positive integer k_1 such that for each $D \in \mathcal{S}$, there is no k_1-triple in D.*
2. *There exists a positive integer k_2 such that each $D \in \mathcal{S}$ has path-width at most k_2.*

Hence in order to prove Theorem 2.13.1, Kim and Seymour [120] proved the following result.

Theorem 2.13.4 ([120]) *Minor containment is a well quasi-order on the class of all semicomplete digraphs with path-width at most k.*

Proof of 2.13.1 assuming Theorem 2.13.4: Let $D_1, D_2, \dots$ be an infinite sequence of semicomplete digraphs. We may assume that D_1 is not a minor of D_i for each $i \geq 2$. Set $k_1 = |D_1|$. By the above observation D_1 is a minor of T_{k_1}, so T_{k_1} is not contained in D_i for each $i \geq 2$. Hence, by Theorem 2.13.3, there exists a k_2 such that every D_i, $i \geq 2$, has path-width at most k_2. Thus, by Theorem 2.13.4, there exists $j > i \geq 2$ such that G_i is a minor of G_j. □

2.14 Miscellaneous Topics

In the next few subsections we briefly survey results and problems on a few further topics on tournaments.

2.14.1 Arc-Pancyclicity

As mentioned earlier, Moon proved that almost all tournaments are arc-3-cyclic [146], so for tournaments this is not a very hard restriction.

Tian, Wu and Zhang characterized all tournaments that are arc-3-cyclic but not arc-pancyclic. See Figure 2.6 for the definition of the classes $\mathcal{D}_6, \mathcal{D}_8$.

Theorem 2.14.1 ([188]) *An arc-3-cyclic tournament is arc-pancyclic unless it belongs to one of the families $\mathcal{D}_6, \mathcal{D}_8$ (in which case the arc yx belongs to no Hamiltonian cycle).*

Corollary 2.14.2 ([188]) *Every arc-3-cyclic tournament has at most one arc which is not in cycles of all lengths $3, 4, \dots, n$.*

The following result due to Wu, Zhang and Zou is also a corollary of Theorem 2.14.1.

Corollary 2.14.3 ([193]) *A tournament is arc-pancyclic if and only if it is arc-3-cyclic and arc-n-cyclic.*

The following result due to Alspach is also an easy corollary:

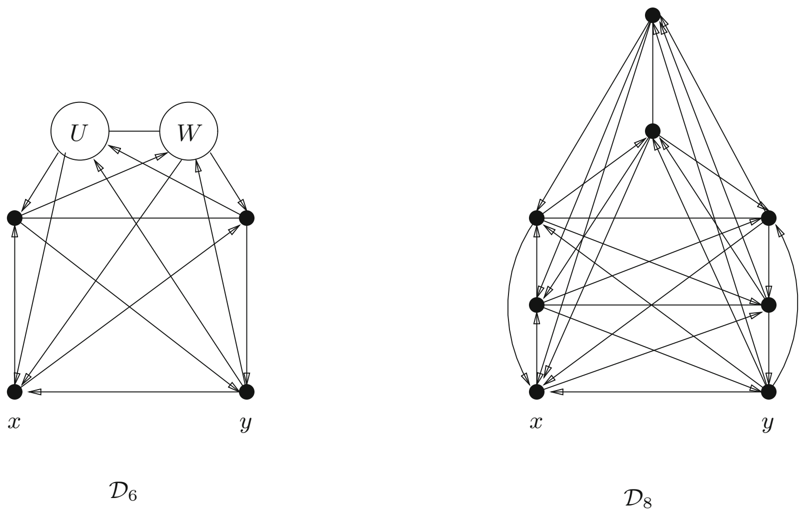

Figure 2.6 The two families of non-arc-pancyclic arc-3-cyclic tournaments. Each of the sets U and W induces an arc-3-cyclic tournament. All edges that are not already oriented may be oriented arbitrarily, but all arcs between U and W have the same direction

Corollary 2.14.4 ([12]) *Every regular tournament is arc-pancyclic.*

Finally, observe that since each tournament in the infinite family $\mathcal{D}_6$ is 2-strong and the arc yx is not in any Hamiltonian cycle we obtain the following result due to Thomassen:

Theorem 2.14.5 ([184]) *There exist infinitely many* 2*-strong tournaments containing an arc which is not in any Hamiltonian cycle.*

Problem 2.14.6 *Characterize arc-pancyclic semicomplete digraphs.*

A partial result on this problem was obtained by Darrah, Liu and Zhang [68].

A vertex u in a digraph D is **out-pancyclic** if every arc whose tail is u is contained in cycles of all lengths $3, 4, \ldots, |V(D)|$. Clearly D is arc-pancyclic if and only if every vertex of D is out-pancyclic and hence it is of interest to study out-pancyclic vertices in tournaments and semicomplete digraphs.

When T is a strong tournament with $\delta^+(T) \geq 2$, Yao, Guo and Zhang [194] call a vertex $v \in V(T)$ a **bridgehead** if there is a 2-partition (V_1, V_2) of $V(T)$ such that $|V_1| \geq 2$, $T[V_1]$ is strong and there is no arc from $V_1 \setminus \{v\}$ to V_2. It is easy to check that every tournament of minimum out-degree at least 2 contains a vertex which is not a bridgehead.

Theorem 2.14.7 ([194]) *Let T be a strong tournament on n vertices and let $u_1, u_2, \ldots, u_n$ be a labelling of its vertices so that $d^+(u_1) \leq d^+(u_2) \leq \ldots \leq d^+(u_n)$. Let u be the vertex u_1 if $d^+(u_1) = 1$ and otherwise u is a vertex of minimum out-degree among those that are not bridgeheads. Then u is an out-pancyclic vertex.*

Corollary 2.14.8 ([194]) *Every strong tournament has an out-pancyclic vertex.*

Yao *et al.* [194] constructed an infinite family of strong tournaments with exactly one out-pancyclic vertex.

Conjecture 2.14.9 ([194]) *Every k-strong tournament has at least k out-pancyclic vertices.*

When $r_i \geq k$, $i \in [3]$ the tournament $C_3[TT_{r_1}, TT_{r_2}, TT_{r_3}]$ is k-strong and has exactly 3 out-pancyclic vertices, namely the vertices with out-degree 0 in each of the three transitive tournaments [195]. Yeo conjectured that every 2-strong tournament contains three out-pancyclic vertices and this was confirmed by Guo, Guo, Li, Li and Zhao.

Theorem 2.14.10 ([100, 101]) *Every strong tournament T with $\delta^+(T) \geq 2$ contains at least three out-pancyclic vertices and this is the best possible.*

See [108, 195] for results and conjectures by Havet and Yeo on the number of pancyclic arcs in tournaments as well as the number of pancyclic arcs contained in the same Hamiltonian cycle.

2.14.2 Critically *k*-Strong Tournaments

A digraph is **critically k-strong** if D is k-strong but $\kappa(D - v) = k - 1$ for all $v \in V$. When $k = 1$ such digraphs are also called **critically strong**. The structure of critically strong digraphs is surprisingly complicated, see the paper [139] by Mader. By Corollary 2.2.10 the only critically strong semicomplete digraph is the 3-cycle. For larger connectivities Thomassen gave a construction which shows that the situation is quite different.

Theorem 2.14.11 (Thomassen [22] Section 5.7) *For every $k \geq 3$, there are infinitely many critically k-strong tournaments.*

See [22, Figure 5.9] for an infinite family of critically 3-strong tournaments. Let us call a tournament T **minimally k-strong** if T is k-strong but no proper subtournament of T is k-strong. We saw above that there are arbitrarily large critically-k-strong tournaments. Lichiardopol conjectured [133] that this is not the case for minimally k-strong tournaments.

Conjecture 2.14.12 ([133]) *For every integer $k \geq 1$ there exists a function $f(k)$ such that every minimally k-strong tournament has at most $f(k)$ vertices.*

2.14.3 Subdivisions and Linkages

A famous conjecture due to Lovász (see e.g. [182, page 262]) states that for every positive integer k there exists an integer $r(k)$ such that for every pair of vertices x, y in a $r(k)$-connected graph G we can find an induced (x, y)-path P such that $G - V(P)$ is k-connected. Thomassen proved the following tournament version of Lovász's conjecture.

Theorem 2.14.13 ([179]) *Let k be a positive integer and let T be a $(k+4)$-strong tournament. Then for every pair of vertices x, y and every shortest (x, y)-path P the tournament $T - V(P)$ is k-strong.*

Kim, Kühn and Osthus generalized this as follows. Theorem 2.14.13 is obtained by taking $d = 2$ and $m = 1$.

Theorem 2.14.14 ([121]) *Let k, d, m be positive integers. Suppose that T is a $(k + m(d + 2))$-strong tournament, that X is a set of d vertices of T, that H is a digraph on d vertices and m arcs and that ϕ is a bijection from $V(H)$ to X. Then T contains a subdivision H^* of H such that*

(i) for each $h \in V(H)$, the branch vertex of H^* corresponding to h is $\phi(h)$,
(ii) $T - V(H^*)$ is k-strong,
(iii) for every arc a of H, the path P_a of H^* is minimal.

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