12. Lexicographic Orientation Algorithms

Jing Huang

12.1 Introduction

Graph orientation, which provides a link between graphs and digraphs, is an actively studied area in the theory of graphs and digraphs. One of the fundamental problems asks whether a given graph admits an orientation that satisfies a prescribed property and to find such an orientation if it exists. A celebrated theorem of Robbins [34] which answers a question of this type states that a graph has a strong orientation if and only if it is 2-edge-connected (i.e., has no bridge). It is easy to check whether a graph is 2-edge-connected and to obtain, using the depth-first search algorithm, a strong orientation of a 2-edge-connected graph, cf. [35].

Which graphs have orientations in which the longest directed path has at most k vertices? Answering this question, Gallai, Roy and Vitaver [13, 37, 47] proved that a graph has such an orientation if and only if it is k-colourable. The theorem nicely links orientations and colourings of graphs but it provides little help in finding such orientations. This is due to the fact that the k-colouring problem is NP-complete for each $k \geq 3$, cf. [15].

Given a graph G, an **orientation** of G is a digraph D obtained from G by replacing every edge uv of G with an arc (i.e., a directed edge that is either $u \to v$ or $v \to u$). Since graphs considered in this chapter are all simple (i.e., having no loops or multiple edges), the digraphs resulting from orientations are **oriented graphs**. Let Π be a property of oriented graphs. We say that a graph G is **Π-orientable** if it admits an orientation that has the property Π. For a fixed property Π the Π-ORIENTATION PROBLEM is as follows.

Π-ORIENTATION PROBLEM **Input:** A graph G. **Find:** A Π-orientation of G or certify that G is not Π-orientable.

J. Huang (✉)
Mathematics and Statistics,
University of Victoria, PO BOX 1700 STN CSC,
Victoria, B.C. V8W 2Y2, Canada
e-mail: huangj@uvic.ca

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For instance, an oriented graph D is **transitive** if for any three vertices u, v, w, $u \to v$ and $v \to w$ imply $u \to w$ in D. Thus a graph is **transitively orientable** if it admits an orientation that is a transitive oriented graph. The TRANSITIVE ORIENTATION PROBLEM asks whether a graph is transitively orientable and to find a transitive orientation of the graph if it exists.

Transitively orientable graphs are also known as **comparability graphs**, cf. [18]. Naturally connected to partially ordered sets, comparability graphs are perfect (in Berge's sense) and have been extensively studied, cf. [14, 16–19, 33]. A classical result of Gallai [14] characterizes comparability graphs by forbidden subgraphs (cf. [30] for the English translation). Gallai's characterization however does not immediately imply a polynomial time algorithm for recognizing comparability graphs or finding transitive orientations. But he proved that a graph is a comparability graph if and only if its knotting graph (cf. [14]) is bipartite, and he also gave a procedure for constructing knotting graphs which runs in polynomial time. It follows that comparability graphs can be recognized in polynomial time. Polynomial time algorithms for finding transitive orientations of comparability graphs have been given by Ghouila-Houri [16], Habib, McConnel Paul and Viennot [21], McConnell and Spinrad [32], and Pnueli, Lempel and Even [33].

In [22] Hell and Huang devised a very simple algorithm for determining whether a graph G is a comparability graph and, if it is, finding a transitive orientation of it. The algorithm first constructs the auxiliary graph G^+ of the input graph G. The auxiliary graph G^+ is used to test whether G is a comparability graph and to find, whenever possible, a transitive orientation of G. To test whether G is a comparability graph, the algorithm proceeds to find a 2-colouring of G^+ using a lexicographic scheme. If the 2-colouring scheme fails, G is not a comparability graph. Otherwise a 2-colouring of G^+ is obtained and the algorithm transforms the 2-colouring of G^+ into a transitive orientation of G. The 2-colourability of G^+ alone is sufficient for G to be a comparability graph. Using the lexicographic scheme to find a 2-colouring of G^+ is to guarantee that the orientation of G transformed from the 2-colouring is transitive. The time complexity of this algorithm is $O(m\Delta)$ where m and Δ are the number of edges and the maximum degree of the input graph.

The technique described above for recognizing comparability graphs and obtaining transitive orientations is called the **lexicographic orientation method**. The lexicographic orientation method has also been applied for recognizing several other classes of graphs and finding desired orientations, cf. [22]. An oriented graph D is called a **local tournament** (respectively, **locally transitive local tournament**) if for every vertex v, the in-neighbourhood and the out-neighbourhood of v each induces a tournament (respectively, transitive tournament) in D, cf. [26]. Local tournaments and locally transitive local tournaments naturally generalize tournaments and transitive tournaments, respectively, cf. [1]. Despite the fact that the class of local tournaments properly contains the class of locally transitive local tournaments, it is proved by Hell and Huang [22] that they share the same class of underlying graphs, that is, a graph is local tournament orientable if and only if it is local transitive tournament orientable (see Corollary 12.2.7).

A graph G is called a **circular arc graph** if it is the intersection graph of a family of circular arcs $I_v, v \in V(G)$, on a circle (i.e., two vertices u, v are adjacent in G if and only if I_u, I_v intersect). The family $I_v, v \in V(G)$, is called a **circular arc representation** of G. Circular arc graphs have also been extensively studied by McConnell [31], Spinrad [39], Trotter and Moore [42], and Tucker [43–46].

Circular arc graphs generalize **interval graphs** which are the intersection graphs of intervals on the real line. A circular arc graph (respectively, an interval graph) is called **proper** if the family of circular arcs (respectively, intervals) can be chosen so that none of them is contained in another. Proper circular arc graphs and proper interval graphs are closely related to local tournaments. In fact, as proved by Skrien [38], a connected graph is local tournament orientable if and only if it is a proper circular arc graph (see Corollary 12.2.7). It is proved in [22, 26] that a graph is acyclic local tournament orientable if and only if it is a proper interval graph (see Corollary 12.2.11). Locally transitive local tournament (respectively, acyclic local tournament) orientations are useful in constructing proper circular arc (respectively, proper interval) representations of their underlying graphs, cf. [9]. Thus the lexicographic orientation method simultaneously solves the recognition and the representation problems for proper circular arc graphs and for proper interval graphs.

Let G be a bipartite graph with bipartition (X, Y). Then G is called an **interval containment bigraph** if there is a family of intervals $I_v, v \in X \cup Y$ such that for all $x \in X$ and $y \in Y$, xy is an edge of G if and only if $I_x \supset I_y$. The family of intervals will be referred to as an **interval containment representation** of G. Various characterizations of interval containment bigraphs have been obtained by Feder, Hell and Huang [10], Hell and Huang [23], Huang [25], and Spinrad [39], and Trotter and Moore [42]. Interval containment bigraphs are closely related to circular arc graphs. In fact, the complements of interval containment bigraphs are precisely the circular arc graphs of clique covering number two. The lexicographic orientation method can also be used for recognizing interval containment bigraphs and constructing interval containment representations whenever possible.

The lexicographic orientation method has also been applied by Bang-Jensen, Huang and Zhu in [4] to solve some orientation completion problems. A **partially oriented graph** is a mixed graph which may contain both edges and arcs. We use $Q = (V, E \cup A)$ to denote a partially oriented graph where E consists of edges and A consists of arcs. An **orientation completion** of Q is an oriented graph obtained from Q by replacing every edge in E with an arc. For a fixed property Π of oriented graphs, the Π-ORIENTATION COMPLETION PROBLEM is as follows.

> *Π*-ORIENTATION COMPLETION PROBLEM
> **Input:** A partially oriented graph $Q = (V, E \cup A)$.
> **Find:** An orientation of the edges in E to a set of arcs A' so that $Q = (V, A \cup A')$ has property Π or certify that no such orientation is possible.

Clearly, the *Π*-ORIENTATION COMPLETION PROBLEM generalizes the *Π*-ORIENTATION PROBLEM. Robbins' theorem as stated at the beginning of this chapter provides a polynomial time solution to the STRONG ORIENTATION PROBLEM. A result of Boesch and Tindell [5] implies that a partially oriented graph can be completed to a strong oriented graph if and only if it has no bridge and no directed cut. Either a bridge or a directed cut in a partially oriented graph (if any exists) can be detected in polynomial time. Hence the STRONG ORIENTATION COMPLETION PROBLEM is also polynomial time solvable. The orientation completion problem for local tournaments is polynomial time solvable (see Theorem 12.3.4). By slightly modifying the lexicographic orientation method for the orientation problem for acyclic local tournaments, Bang-Jensen, Huang and Zhu [4] proved that the corresponding orientation completion problem is polynomial time solvable (see Theorem 12.3.5). In contrast they [4] showed that the orientation completion problem for locally transitive local tournaments is NP-complete (see Theorem 12.3.14).

Orientation completion problems generalize certain representation extension problems. For example, the REPRESENTATION EXTENSION PROBLEM for proper interval graphs asks whether it is possible to obtain a proper interval representation of a graph G that includes a proper interval representation of an induced subgraph of G. This problem has been studied by Klavik, Kratochvil, Otachi, Rutter, Saitoh, Saumell and Vystocil in [28]. As mentioned above, a proper interval representation of a proper interval graph corresponds to an acyclic local tournament orientation of the graph. Thus the representation extension problem for proper interval graphs is just the orientation completion problem for acyclic local tournaments where a partial orientation corresponds to an interval representation of an induced subgraph. The representation extension problem for proper interval graphs was shown to be polynomial time solvable, cf. [28]. The lexicographic orientation method can be applied to show that the orientation completion problem for acyclic local tournaments is polynomial time solvable.

The key notion used in the lexicographic method is the concept of lexicographic order. Suppose $(s_1, s_2, \ldots, s_k), (t_1, t_2, \ldots, t_k)$ are two ordered k-tuples over the set $\{1, 2, \ldots, n\}$. We say that $(s_1, s_2, \ldots, s_k)$ is **lexicographically smaller than** $(t_1, t_2, \ldots, t_k)$, provided $s_1 < t_1$ or there exists an f with $1 < f \leq k$ such that $s_f < t_f$ and $s_i = t_i$ for all $i < f$. If S and T are two sets of k elements, we say that S is lexicographically smaller than T provided $(s_1, s_2, \ldots, s_k)$ is lexicographically smaller than $(t_1, t_2, \ldots, t_k)$, where $s_1, s_2, \ldots, s_k$ and $t_1, t_2, \ldots, t_k$ are the elements of S and T listed in increasing order. Suppose S' is a subset of S and T' is lexicographically smaller than S'. Then it is easy to see that $T = (S - S') \cup T'$ is lexicographically smaller than S. Note that lexicographic orders are linear, and hence any subset of a lexicographically ordered set has a smallest element.

12.2 Algorithms for Π-Orientations

We begin by formalizing the generic idea of the lexicographic orientation algorithm for deciding whether a graph is Π-orientable and finding (if one exists) such a Π-orientation of G. Let G be the input graph. Define the **auxiliary graph** G^+ of G as follows: The vertex set of G^+ consists of all ordered pairs (u, v) such that uv is an edge of G. Note that each each edge uv of G gives rise to two vertices $(u, v), (v, u)$ and these two vertices are always adjacent in G^+. Depending on the property Π, G^+ may contain additional edges, which will be defined for each problemin question.

Algorithm 1 Generic lexicographic orientation

Input: A graph G with vertices $1, 2, \ldots, n$.
Output: A Π-orientation of G if one exists.

Construct the auxiliary graph G^+.
While there exist uncoloured vertices do
  Colour by A the lexicographically smallest uncoloured vertex (u, v)
  Use breadth first search to 2-colour (if possible) the connected component of G^+ which contains (u, v).
  If some component could not be 2-coloured then report that G is not Π-orientable.
For every edge $uv \in E$ orient it as $u \rightarrow v$ if (u, v) obtained colour A and otherwise orient it as $v \rightarrow u$.

The purpose of Algorithm 1 is two-fold. First, it determines whether the input graph G is Π-orientable by verifying the 2-colourability of the auxiliary graph G^+. Second, it constructs a Π-orientation of G in the case when G^+ is 2-colourable. The correctness of Algorithm 1 is validated by the two statements described in the following proposition.

Proposition 12.2.1 *Algorithm 1 is correct if and only if the following two statements hold:*

- *If G is Π-orientable, then G^+ is bipartite.*
- *If G^+ is bipartite, then the orientation of G obtained by Algorithm 1 has the property Π.* □

As a simple example suppose that Π is the property of being acyclic and that G^+ is the auxiliary graph of G as defined above, which contains no other edges except those between (u,v) and (v,u) for edges uv of G. Since every graph is acyclically orientable and G^+ is bipartite for every graph G, the first statement holds vacuously. According to Step 2, vertex (u,v) of G^+ is coloured by A if and only if $u < v$. It follows that the orientation of G obtained by the algorithm is acyclic and hence the second statement holds.

We will show that the above generic lexicographic orientation algorithm can be modified to solve the Π-orientation problem when Π is the property of being a transitive digraph, respectively being a locally transitive local tournament, respectively being an acyclic local tournament. The only modifications involved are on the definition of the auxiliary graph G^+. We will also show that it can be applied to recognize interval containment bigraphs and obtain the desired orientations of their complements.

12.2.1 Comparability Graphs

For the input graph G, we modify the definition of the auxiliary graph G^+ as follows: The vertex set of G^+ is the same as above (i.e., consisting of ordered pairs $(u,v),(v,u)$ for edges uv of G). In G^+, every vertex (u,v) is adjacent to (v,u), to any (w,u) such that v and w are not adjacent in G, and to any (v,w) such that u and w are not adjacent in G. Figure 12.1 shows an example of a graph G and its auxiliary graph G^+.

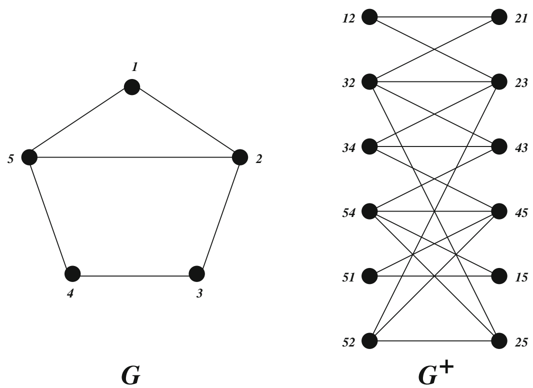

Figure 12.1 A graph G and its auxiliary graph G^+.

Suppose that G^+ is bipartite. Colour G^+ with two colours A, B and orient each edge uv of G as $u \to v$ whenever (u,v) is coloured A. Then for any edges uv, vw with uw being a non-edge of G, (u,v) and (v,w) are adjacent and (w,v) and (v,u) are adjacent in G^+. Thus (u,v) and (v,w) are coloured by opposite colours and (w,v) and (v,u) are coloured by opposite colours in any 2-colouring of G^+. Consequently, we have either $u \to v$ and $w \to v$ or $v \to u$ and $v \to w$. Therefore we obtain an orientation of G which satisfies the property that $u \to v$ and $v \to w$ imply that there is an arc between u and w. An oriented graph which has this property is called **quasi-transitive**, cf. [3] and Chapter 8. On the other hand, any quasi-transitive orientation of G corresponds to a colour class of a 2-colouring of G^+.

Every transitive oriented graph is quasi-transitive and thus every transitively orientable graph is also quasi-transitively orientable. It was first observed by Ghouila-Houri [16] that every quasi-transitively orientable graph is also transitively orientable. Hence comparability graphs are exactly the quasi-transitively orientable graphs. In particular, if G^+ is not bipartite then G is not a comparability graph and hence not transitively orientable. The result of Ghouila-Houri will follow as a byproduct from the lexicographic orientation algorithm, as stated below.

Theorem 12.2.2 ([22]) *Suppose that G is a comparability graph and that D is an orientation of G obtained by the lexicographic orientation algorithm. Then D is a transitive orientation of G.*

Proof: Since G is a comparability graph, G^+ is bipartite. For each vertex (u, v) of G^+, let $C(u, v)$ be the set of all vertices whose distance from (u, v) in G^+ is even. It follows from the definition of G^+ that if $(x, y), (x', y') \in C(u, v)$ then there exist

$$(x_0, y_0), (x_1, y_0), (x_1, y_1), (x_2, y_1), \ldots, (x_k, y_k) \in C(u, v)$$

such that $(x_0, y_0) = (x, y)$ and $(x_k, y_k) = (x', y')$ and for each $i = 0, 1, \ldots, k-1$, $x_i x_{i+1} \notin E(G)$ and $y_i y_{i+1} \notin E(G)$. The following claim, known as "**The Triangle Lemma**, can be found in the book [18] by Golumbic.

Claim. Let $uvwu$ be a 3-cycle in G. Suppose that $C(u, v) \neq C(w, v)$ and $C(u, v) \neq C(u, w)$. Then for any $(u', v') \in C(u, v)$, we must have $(w, v') \in C(w, v)$ and $(u', w) \in C(u, w)$.

Proof of Claim. Since $(u', v') \in C(u, v)$, there exist

$$(u_0, v_0), (u_1, v_0), (u_1, v_1), (u_2, v_1), \ldots, (u_\ell, v_\ell) \in C(u, v)$$

such that $(u_0, v_0) = (u, v)$ and $(u_\ell, v_\ell) = (u', v')$ and for each $= 0, 1, \ldots, \ell-1$, $u_i u_{i+1} \notin E(G)$ and $v_i v_{i+1} \notin E(G)$. We prove by induction on ℓ that $(w, v_\ell) \in C(w, v)$ and $(u_\ell, w) \in C(u, w)$. Assume that $(w, v_{\ell-1}) \in C(w, v)$ and $(u_{\ell-1}, w) \in C(u, w)$. Since $C(u, v) \neq C(w, v) = C(w, v_{\ell-1})$, $wu_\ell \in E(G)$. Since $u_{\ell-1}u_\ell \notin E(G)$, $(u_\ell, w) \in C(u_{\ell-1}, w) = C(u, w)$. Similarly, since $C(u_\ell, v_\ell) \neq C(u_\ell, w)$, $wv \in E(G)$ and since $v_{\ell-1}v_\ell \notin E(G)$, $(w, v_\ell) \in C(w, v_{\ell-1}) = C(w, v)$. □

Suppose to the contrary that D is not transitive. Then there is a triangle $uvwu$ such that $u \to v$, $v \to w$ and $w \to u$ in D. Assume that $\{u, v, w\}$ is the lexicographically smallest amongst all such triangles. Without loss of generality assume that $u > v$ and therefore (u, v) was not the first vertex coloured A in its component of G^+. It follows that there exists $(u', v') \in$

$C(u,v)$ such that $\{u',v'\}$ is lexicographically smaller than $\{u,v\}$. Since $u \to v$, $v \to w$ and $w \to u$, $C(u,v) \neq C((w,v)$ and $C(u,v) \neq C(u,w)$. Hence by the claim above, $(w,v') \in C(w,v)$ and $(u',w) \in C(u,w)$. Since $u \to v$, $v \to w$ and $w \to u$ in D, we must also have $u' \to v'$, $v' \to w$ and $w \to u'$ in D. But $\{u',v',w\}$ is lexicographically smaller than $\{u,v,w\}$, which contradicts the choice of $\{u,v,w\}$. □

For $k \geq 1$, a $(2k+1)$**-asteroid** in a graph is a sequence of $2k+1$ vertices

$$u_0, u_1, \ldots, u_{2k}$$

together with $2k+1$ paths

$$P_0, P_2, \ldots, P_{2k}$$

where P_i is a (u_i, u_{i+1})-path such that u_i has no neighbours in P_{i+k} (subscripts are modulo $2k+1$) for each $i = 0, 1, \ldots, 2k$. A 3-asteroid is also known as an **asteroidal triple**, which is an important concept for characterizing interval graphs, cf. [29]. It is easy to verify that an odd cycle in G^+ corresponds to a $(2k+1)$-asteroid for some k in $\overline{G}$.

Corollary 12.2.3 *The following statements are equivalent for a graph G.*

1. *G is a comparability graph;*
2. *G is transitively orientable;*
3. *G is quasi-transitively orientable;*
4. *G^+ is bipartite;*
5. *$\overline{G}$ contains no asteroid.* □

12.2.2 Proper Circular Arc Graphs

A **round ordering** of a digraph D is a cyclic ordering $\mathcal{O} = v_1, v_2, \ldots, v_n, v_1$ of the vertices of D such that for each vertex v_i we have $N^+(v_i) = \{v_{i+1}, \ldots, v_{d^+(v_i)+i}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, \ldots, v_{i-1}\}$ where indices are modulo n. A digraph which has a round ordering is called **round**. Round digraphs were characterized by Huang in [27]. It is easy to see that if an oriented graph has a round ordering then it is locally transitive. The following theorem, due to Bang-Jensen, asserts that the converse is also true when D is connected.

Theorem 12.2.4 ([1]) *A connected oriented graph D has a round ordering $\mathcal{O} = v_1, v_2, \ldots, v_n, v_1$ of its vertices if and only if D is a locally transitive local tournament. Furthermore, there is a polynomial algorithm for deciding whether a given oriented graph is round and finding a round ordering if one exists.* □

Suppose that G is a proper circular arc graph and that $I_v, v \in V(G)$, is a proper circular arc representation of G. We may assume without loss of generality that if two circular arcs I_u, I_v intersect then either I_u contains the counterclockwise endpoint of I_v or I_v contains the counterclockwise endpoint of I_u (but not both). Orient G in such a way that each edge uv of G is oriented as $u \rightarrow v$ if I_u contains the counterclockwise endpoint of I_v. It is easy to see that this is a locally transitive local tournament orientation of G. A round ordering of the orientation of G corresponds to the clockwise ordering of clockwise endpoints of the circular arcs in the proper circular arc representation of G. Conversely, suppose that D is a connected locally transitive local tournament. Then D has a round ordering by Theorem 12.2.4 and a family of inclusion-free circular arcs $I_v, v \in V(D)$, can be obtained such that $u \rightarrow v$ in D if and only if I_u contains the counterclockwise endpoint of I_v, cf. [22, 26]. Thus the underlying graph of D is a proper circular arc graph.

Theorem 12.2.5 ([22, 26]) *A connected graph is a proper circular arc graph if and only if it is orientable as a locally transitive local tournament.* □

Every locally transitive local tournament is a local tournament. Skrien [38] proved that a connected graph is a proper circular arc graph if and only if it is local tournament orientable. Clearly, a graph (not necessarily connected) is local tournament (respectively, locally transitive local tournament) orientable if and only if so is every connected component of the graph. Therefore a graph G is orientable as a locally transitive local tournament if and only if it is orientable as a local tournament. With this in mind we define the edge set of the auxiliary graph G^+ of G as follows: each vertex (u, v) is adjacent to (v, u), to any vertex (u, w) such that v and w are not adjacent in G, and to any vertex (w, v) such that u and w are not adjacent in G. As in the previous subsection, we see that any local tournament orientation of G gives rise to a 2-colouring of G^+ and in case when G^+ is 2-colourable the vertices of one colour in any 2-colouring of G^+ induce a local tournament orientation of G. Not every 2-colouring of G^+ induces a locally transitive local tournament orientation of G. However, the 2-colouring of G^+ produced by the lexicographic orientation algorithm gives a locally transitive local tournament orientation of G.

Theorem 12.2.6 ([22]) *Suppose that G is a proper circular arc graph and that D is an orientation of G obtained by the lexicographic orientation algorithm. Then D is a local transitive tournament orientation of G.*

Proof: Since G is a proper circular arc graph, G^+ is bipartite and hence D is a local tournament. Suppose to the contrary that D is not a locally transitive local tournament. Then there exists a set $\{u, v, w, z\}$ of vertices of D such that u, v, w induce a directed 3-cycle $u \rightarrow v \rightarrow w \rightarrow u$, which either dominates z or is dominated by z. Assume that $\{u, v, w, z\}$ is the lexicographically smallest set with this property. Assume further that z dominates $\{u, v, w\}$. (The situation is symmetric when z is dominated by $\{u, v, w\}$.) Without loss of generality assume that $u > v$ and therefore (u, v) was not the first vertex coloured A in its component of G^+.

Let $C(u, v)$ (respectively, $C(v, u)$) be the set of all vertices in G^+ whose distance from (u, v) in G^+ is even (respectively, odd), and let $(u', v') \in C(u, v)$ be the first vertex coloured A in the component of (u, v). Then $\{u', v'\}$ is lexicographically smaller than $\{u, v\}$ and hence $\{u', v', w, z\}$ is lexicographically smaller than $\{u, v, w, z\}$. We show that the subdigraph of D induced by $\{u', v', w, z\}$ also contains a directed 3-cycle which either dominates the fourth vertex or is dominated by the fourth vertex. This contradicts the choice of $\{u, v, w, z\}$ and therefore D is a locally transitive local tournament.

Since $(u', v') \in C(u, v)$, there exist

$$(u_0, v_0), (u_1, v_1), \ldots, (u_\ell, v_\ell)$$

such that

- $(u_0, v_0) = (u, v)$;
- $(u_i, v_i) \in C(u, v)$ when i is even and $(u_i, v_i) \in C(v, u)$ when i is odd;
- $(u_\ell, v_\ell) = (u', v')$ when ℓ is even and $(u_\ell, v_\ell) = (v', u')$ when ℓ is odd;
- for each $i = 0, 1, \ldots, \ell - 1$, either $u_i = u_{i+1}$ and $v_i v_{i+1} \notin E(G)$ or $v_i = v_{i+1}$ and $u_i u_{i+1} \notin E(G)$.

Let $U_i = \{u_0, u_1, \ldots, u_i\}$ and $V_i = \{v_0, v_1, \ldots, v_i\}$. Note that not all elements in U_i (respectively, V_i) are distinct. We use $||U_i||$ (respectively $||V_i||$) to denote the number of distinct elements in U_i (respectively, V_i). Observe that i and $||U_i|| + ||V_i||$ have the same parity for each i. We claim that in D the following property holds:

- when $||U_i||$ is odd, $\{w, z\} \to u_i \to v$;
- when $||U_i||$ is even, $v \to u_i \to \{w, z\}$;
- when $||V_i||$ is odd, $\{u, z\} \to v_i \to w$;
- when $||V_i||$ is even, $w \to v_i \to \{u, z\}$.

When $i = 0$, we have $||U_0|| = ||V_0|| = 1$ and the property holds. Assume that $i \geq 1$ and the property holds for $i - 1$. We consider only the case when $u_{i-1} = u_i$ and $v_{i-1} v_i \notin E(G)$. (The other case, $v_{i-1} = v_i$ and $u_{i-1} u_i \notin E(G)$, is symmetric.)

Suppose that i is odd. Then $v_i \to u_i = u_{i-1} \to v_{i-1}$. Since i is odd, $||U_i||$ and $||V_i||$ have different parity. Suppose first that $||U_i||$ is odd. Then $||V_{i-1}||$ is also odd. By the inductive hypothesis, $\{w, z\} \to u_{i-1} = u_i \to v$ and $\{u, z\} \to v_{i-1} \to w$. Hence v_i, w, z are in-neighbours of u_i. Since D is a local tournament, v_i is adjacent to both w and z. Since $z \to v_{i-1} \to w$, we must have $w \to v_i \to z$. Hence u, v_i are both out-neighbours of w and must be adjacent. Since $u \to v_{i-1}$ and $v_{i-1} v_i \notin E(G)$, we have $v_i \to u$. Therefore $w \to v_i \to \{u, z\}$. Suppose that $||U_i||$ is even. Then $||V_{i-1}||$ is also even. By the inductive hypothesis, $v \to u_{i-1} = u_i \to \{w, z\}$ and $w \to v_{i-1} \to \{u, z\}$. Since v, v_i are both in-neighbours of u_i, v, v_i are adjacent. Either $v \to v_i$ or $v_i \to v$ in D. Assume that $v \to v_i$. (The case when $v_i \to v$ is again symmetric.).

Then w, v_i are out-neighbours of v and hence are adjacent. Since $w \to v_{i-1}$ and $v_{i-1}v_i \notin E(G)$, $v_i \to w$; thus both v_i and z are in-neighbours of w. Since $v_{i-1} \to z$ and $v_{i-1}v_i \notin E(G)$, $z \to v_i$. Hence u, v_i are both out-neighbours of z and must be adjacent. Since $v_{i-1} \to u$ and $v_{i-1}v_i \notin E(G)$, we have $u \to v_i$. Therefore $\{u, z\} \to v_i \to w$.

Suppose that i is even. Then $v_{i-1} \to u_{i-1} = u_i \to v_i$. Since i is even, $||U_i||$ and $||V_i||$ have the same parity. Suppose first that $||U_i||$ is odd. Then $||V_{i-1}||$ is even. By the induction hypothesis, $\{w, z\} \to u_{i-1} = u_i \to v$ and $w \to v_{i-1} \to \{u, z\}$. Since v, v_i are both out-neighbours of u_i, v, v_i are adjacent. Either $v \to v_i$ or $v_i \to v$ in D. Assume that $v \to v_i$. (The case when $v_i \to v$ is again symmetric.) Then w, v_i are out-neighbours of v and hence are adjacent. Since $w \to v_{i-1}$ and $v_{i-1}v_i \notin E(G)$, $v_i \to w$; thus both v_i and z are in-neighbours of w. Since $v_{i-1} \to z$ and $v_{i-1}v_i \notin E(G)$, we have $z \to v_i$. Hence u, v_i are both out-neighbours of z and must be adjacent. Since $v_{i-1} \to u$ and $v_{i-1}v_i \notin E(G)$, we have $u \to v_i$. Therefore $\{u, z\} \to v_i \to w$. Suppose now that $||U_i||$ is even. Then $||V_{i-1}||$ is odd. By the inductive hypothesis, $v \to u_{i-1} = u_i \to \{w, z\}$ and $\{u, z\} \to v_{i-1} \to w$. Thus v_i, w, z are out-neighbours of u_i. So v_i is adjacent to both w and z. Since $z \to v_{i-1} \to w$ and $v_{i-1}v_i \notin E(G)$, $w \to v_i \to z$. Now u and v_i are both out-neighbours of w and must be adjacent. Since $u \to v_{i-1}$ and $v_{i-1}v_i \notin E(G)$, we must have $v_i \to u$. Therefore $w \to v_i \to \{u, z\}$.

If ℓ is even, then $(u_\ell, v_\ell) = (u', v')$, and $||U_\ell||$ and $||V_\ell||$ have the same parity. When $||U_\ell||$ and $||V_\ell||$ are both odd, $\{u', v', w\}$ induces a directed cycle and is dominated by z; when $||U_\ell||$ and $||V_\ell||$ are both even, $\{w, v', z\}$ induces a directed cycle and is dominated by u'. If ℓ is odd, then $(u_\ell, v_\ell) = (v', u')$, and $||U_\ell||$ and $||V_\ell||$ have different parity. When $||U_\ell||$ is odd and $||V_\ell||$ is even, $\{w, v', z\}$ induces a directed cycle and dominates u'; when $||U_\ell||$ is even and $||V_\ell||$ is odd, $\{u', v', z\}$ induces a directed cycle and dominates w. □

Combining Theorems 12.2.5 and 12.2.6 and the remarks made between the two theorems we have the following:

Corollary 12.2.7 *The following statements are equivalent for a connected graph G.*

1. *G is a proper circular arc graph;*
2. *G is local tournament orientable;*
3. *G is locally transitive local tournament orientable;*
4. *G^+ is bipartite.* □

Through a careful analysis of the structure of proper circular arc graphs, a full description of all local tournament orientations of a proper circular arc graph was obtained in [24]. Let G be a graph and $uv, u'v'$ be two edges of G. We say that $uv, u'v'$ are **implicated** if (u, v) and (u', v') are in the same connected component of G^+. The implication relation is an equivalence relation on the set of edges of G and each equivalence class is called an **implication class** of G. Call an edge uv in G **balanced** if $N[u] = N[v]$

and **unbalanced** otherwise. It follows from the definition that an edge is balanced if and only if it forms an implication class by itself. In general, two edges of G are implicated with each other if and only if the orientation of one uniquely determines the orientation of the other in any local tournament orientation of G.

Theorem 12.2.8 ([24]) *Let G be a connected proper circular arc graph. Suppose that $C_1, C_2, \ldots, C_k$ are the connected components of $\overline{G}$. Then all unbalanced edges of G within a fixed C_i form an implication class and all unbalanced edges between two fixed C_i and C_j $(i \neq j)$ form an implication class. Moreover, if $\overline{G}$ is not bipartite, then $k = 1$ and all unbalanced edges of G form an implication class.* □

12.2.3 Proper Interval Graphs

Proper interval graphs are proper circular arc graphs and hence are locally transitive local tournament orientable. In fact they admit locally transitive local tournament orientations that contain no directed cycles (or equivalently, acyclic local tournament orientations). Indeed, suppose that G is a proper interval graph and that $I_v, v \in V(G)$, is a proper interval representation of G. Orient G in such a way that $u \rightarrow v$ if and only if I_u contains the left endpoint of I_v. This is an acyclic local tournament orientation of G. On the other hand, an acyclic local tournament orientation of G can be efficiently transformed into a proper interval representation of G, cf. [26] and [22]. So acyclic local tournament orientations of proper interval graphs are in a sense an orientation formulation of their proper interval representations.

When the input graph G is a proper interval graph (and hence a proper circular arc graph), the lexicographic orientation algorithm using the same auxiliary graph G^+ as defined in Subsection 12.2.2 will produce a locally transitive local tournament orientation D of G according to Theorem 12.2.6. But this D may not be acyclic. To make sure that D is also acyclic, we use a **perfect elimination ordering** of G (that is, a vertex ordering $1, 2, \ldots, n$ such that for each i the set of neighbours j of i with $j > i$ induce a complete subgraph of G). It is well-known that G, which is a chordal graph, must have such an ordering, which can be obtained in time $O(m + n)$ using the algorithm called **Lexicographic Breadth First Search (LBFS)** devised by Rose, Tarjan and Lueker in [36]. We summarize the lexicographic orientation algorithm for finding an acyclic local tournament orientation of a proper intervalgraph.

The proof of correctness of the algorithm makes use of a full description of implication classes of a proper interval graph obtained in [24]. A vertex in a graph G is called **universal** if it is adjacent to every other vertex in G.

Algorithm 2 Lexicographic acyclic local-tournament-orientation

Input: A graph G.
Output: An acyclic local tournament orientation of G.

Find a perfect elimination ordering $1, 2, \ldots, n$ of G.
  If G does not have a perfect elimination ordering then report that G is not a proper interval graph.
Construct the auxiliary graph G^+.
While there exist uncoloured vertices do
  Colour by A the lexicographically smallest uncoloured vertex (u, v)
  Use breadth first search to 2-colour (if possible) the connected component of G^+ which contains (u, v).
  If some component could not be 2-coloured then report that G is not a proper interval graph.
Orient each edge uv of G as $u \rightarrow v$ if (u, v) obtained colour A and otherwise orient it as $v \rightarrow u$.
Check whether the resulting oriented graph contains a directed cycle.
If it has a directed cycle then report that G is not a proper interval graph.

Theorem 12.2.9 ([24]) *Let G be a connected proper interval graph. Then one of the following statements holds:*

- *if G has no universal vertex, then all unbalanced edges of G form an implication class;*
- *if G has universal vertices, then all unbalanced edges incident with universal vertices form an implication class and all other unbalanced edges form an implication class.* □

Theorem 12.2.10 ([22]) *Suppose that G is a proper interval graph. Then the orientation of G obtained by Corollary 12.2.3 is an acyclic local tournament.*

Proof: Assume without loss of generality that G is connected. Suppose first that G has no universal vertex. Then by Theorem 12.2.9, the vertices of G can be partitioned into complete subgraphs $V_1, V_2, \ldots, V_p$ and G^+ has the following components: For each pair of vertices u, v in the same V_i, there is a separate component consisting of adjacent vertices $(u, v), (v, u)$. In addition, there is one component containing all remaining vertices (u, v) (i.e., $u \in V_i$ and $v \in V_j$ with $i \neq j$). Moreover, in this last component, one colour class contains all vertices (u, v) with $u \in V_i, v \in V_j$ and $i < j$. In this case, the lexicographic orientation algorithm orients each V_i as a transitive tournament and the remaining edges uv as $u \rightarrow v$ either for all $u \in V_i, v \in V_j, i < j$ or for all $u \in V_i, v \in V_j, i > j$. It is clear that the orientation does not contain a directed cycle and hence is an acyclic local tournament.

Suppose now that G has universal vertices and that $1, 2, \ldots, n$ is a perfect elimination ordering of G. Then again by Theorem 12.2.9 the vertices of G

can be partitioned into complete subgraphs $V_1, V_2, \ldots, V_p$ where V_m with $1 < m < p$ consists of all universal vertices that are in V_m and $V_1 \cup V_p$ consists of all simplicial vertices. The components of G^+ are as follows: For each u, v in the same V_i, there is a separate component consisting of adjacent vertices $(u, v), (v, u)$. There is again one component consisting of all vertices (u, v) with $u \in V_i$, $v \in V_j$, $i \neq j$, $i \neq m$, and $j \neq m$. One colour class in this component consists of all (u, v) with $u \in V_i$, $v \in V_j$, $i < j$. Finally, there is, for each vertex $w \in V_m$, a component consisting of all vertices $(v, w), (w, v)$ for all $v \in V_i$ with $i \neq m$. One colour class of this component consists of $(u, w), (w, v)$ for all $u \in V_i$ and $v \in V_j$ with $1 \leq i < m$ and $m < j \leq p$. The simplicial vertex 1 is in V_1 or V_p. The lexicographic orientation algorithm orients each V_i as a transitive tournament and the remaining edges uv as $u \to v$ either for all $u \in V_i, v \in V_j, i < j$ or for all $u \in V_i, v \in V_j, i > j$. The orientation is an acyclic local tournament. □

Corollary 12.2.11 *The following statements are equivalent for a graph G.*

1. *G is a proper interval graph;*
2. *G is acyclic local tournament orientable.* □

12.2.4 Interval Containment Bigraphs

Let G be a bipartite graph with bipartition (X, Y). Recall that G is an interval containment bigraph if there is a family of intervals $I_v, v \in X \cup Y$, such that for all $x \in X$ and $y \in Y$, xy is an edge of G if and only if $I_x \supset I_y$. The family of intervals will be refered to as an interval containment representation of G. See Figure 12.2 for an example of an interval containment bigraph and its interval containment representation.

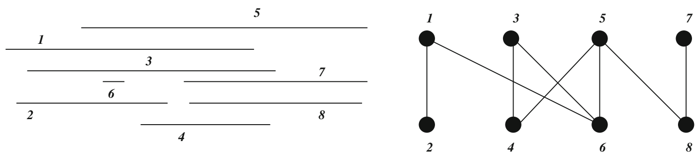

Figure 12.2 An interval containment bigraph and an interval containment representation.

Suppose that G is an interval containment bigraph and that the collection of intervals $I_v = [\ell_v, r_v], v \in X \cup Y$, form an interval containment representation of G. Assume without loss of generality that the ends of the intervals are all distinct. We orient $\overline{G}$ as follows: each edge uv of $\overline{G}$ is oriented as $u \to v$ if $\ell_u < \ell_v$. Clearly, the orientation is acyclic. We claim that it does not contain the digraph in Figure 12.3 as an induced subdigraph.

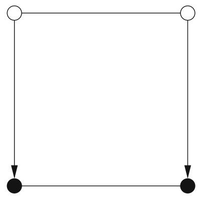

Figure 12.3 White vertices are in X and black vertices are in Y or the other way around. The orientation between white vertices or between black vertices is not specified and may be in either direction.

Indeed, suppose that $u \to u'$ and $v \to v'$ are oriented edges where the four vertices u, v, v', u' induce a 4-cycle $vv'u'u$ in $\overline{G}$. By the way of orientation we must have $\ell_u < \ell_{u'}$ and $\ell_v < \ell_{v'}$. If $u, v \in X$ and $u', v' \in Y$, then $r_u < r'_u$ and $r_v < r_{v'}$ as $uu', vv' \notin E(G)$. Since $uv', vv' \in E(G)$, we have

$$\ell_u < \ell_{v'} < r_{v'} < r_u \quad \text{and} \quad \ell_v < \ell_{u'} < r_{u'} < r_v.$$

Hence we have $\ell_v < \ell_{v'} < r_{v'} < r_u < r_{u'} < r_v$ and so $I_v \supset I_{v'}$, a contradiction to the assumption that $vv' \notin E(G)$. If $u, v \in Y$ and $u', v' \in X$, then

$$\ell_{u'} < \ell_v < r_v < \ell_{u'} \quad \text{and} \quad \ell_{v'} < \ell_u < r_u < r_{v'}.$$

Thus we have $\ell_{v'} < \ell_u < \ell_{u'} < \ell_v < \ell_{v'}$, a contradiction.

Acyclic orientations of the complements of bipartite graphs which do not contain an induced subdigraph in Figure 12.3 may again be viewed as an orientation formulation of interval containment representations of interval containment bigraphs. Thus the recognition and representation problems for interval containment bigraphs become the following:

Problem 12.2.12 *Given a bipartite graph G, does $\overline{G}$ have an acyclic orientation which does not contain one of the digraphs in Figure 12.3 as an induced subdigraph?*

Define the auxiliary graph G^+ of G with bipartition (X, Y) as follows: The vertices of G^+ are ordered pairs $(v, v'), (v', v)$ with $v \in X, v' \in Y$ and $vv' \notin E(G)$. In G^+, each (v, v') is adjacent to (v', v) and for each induced 4-cycle $vv'u'u$ in $\overline{G}$, (v, v') is adjacent to (u, u') and (v', v) is adjacent to (u', u). The above observation simply asserts that if G is an interval containment bigraph then G^+ is bipartite.

Suppose that the auxiliary graph G^+ of G is bipartite. Colour the vertices of G^+ with colours A, B and orient an edge vv' of $\overline{G}$ as $v' \to v$ if (v, v') is coloured A and as $v \to v'$ if (v', v) is coloured A. This is a partial orientation of $\overline{G}$; all edges between X and Y are oriented but none of edges in X or in Y is oriented. The definition of G^+ implies that any completion of this partial orientation to an orientation of $\overline{G}$ will not contain the digraph in Figure 12.3

as an induced subdigraph. However, there may be no acyclic completion. In order for the partial orientation of $\overline{G}$ to have an acyclic completion, particular 2-colourings of G^+ are needed.

We will fix a bipartition (X, Y) of G and use letters without primes for vertices in X and letters with primes for vertices in Y.

Algorithm 3 Lexicographic restricted acyclic orientation

Input: A bipartite graph G with bipartition (X, Y) and vertices $1, 2, \ldots, n$ where vertices of X preceede the vertices of Y.
Output: An acyclic orientation of $\overline{G}$ that does not contain one of the digraphs in Figure 12.3 as an induced subdigraph.

Construct the auxiliary graph G^+ with respect to (X, Y).
While there exist uncoloured vertices do
    Colour by A the lexicographically smallest uncoloured vertex (α, β)
    Use breadth first search to 2-colour (if possible) the component of G^+ which contains (α, β).
    If some component could not be 2-coloured then report that G is not an interval containment bigraph.
Orient the edge vv' of $\overline{G}$ as $v' \rightarrow v$ if (v, v') is coloured A and as $v \rightarrow v'$ otherwise.
Complete the partial orientation obtained in Step 3 to an orientation of $\overline{G}$ as follows: orient each edge uv as $u \rightarrow v$ if $N^-(u) \cap Y \subseteq N^-(v) \cap Y$ and orient each edge $u'v'$ as $u' \rightarrow v'$ if $N^+(u') \cap X \supseteq N^+(v') \cap X$.

The correctness of the algorithm above is ensured by the following reformulation of a theorem of Hell and Huang [22].

Theorem 12.2.13 *Suppose that G is an interval containment bigraph and that D is an orientation of $\overline{G}$ obtained by Theorem 12.2.4. Then D is acyclic and does not contain the digraph in Figure 12.3 as an induced subdigraph.*

Proof: We first prove that for any $u, v \in X$, the following properties hold:

- either $N^-(u) \cap Y \subseteq N^-(v) \cap Y$ or $N^-(u) \cap Y \supseteq N^-(v) \cap Y$;
- either $N^+(u) \cap Y \subseteq N^+(v) \cap Y$ or $N^+(u) \cap Y \supseteq N^+(v) \cap Y$.

We prove it by contradiction. So suppose that one of the properties does not hold for some $u, v \in X$. Let u, v be such vertices with the minimum $u + v$. Assume by symmetry that the first property does not hold for u, v, that is, there are vertices $u', v' \in Y$ such that

- $u' \rightarrow u$ and $v' \rightarrow v$,
- vu' is not an edge of $\overline{G}$ or $v \rightarrow u'$, and
- uv' is not an edge of $\overline{G}$ or $u \rightarrow v'$.

Observe that at least one of vu', uv' must be an edge of $\overline{G}$; otherwise (u, u') and (v, v') are adjacent vertices of G^+ of the same colour A, a contradiction. Assume without loss of generality that uv' is an edge of $\overline{G}$. Since $u \rightarrow v'$, the vertex (u, v') was coloured B. Hence there exists a vertex (w, w') of colour A such that $wuv'w'$ is an induced 4-cycle of $\overline{G}$. Since $u \to v'$, $w' \to w$. Now we have $w \to w$, $u' \to u$ and uw' is not an edge of $\overline{G}$. This implies that wu' is an edge of $\overline{G}$. If $w \to u'$, then the four vertices w, u, w', u' can be used in the place of u, v, u', v'. On the other hand, if $u' \to w$, then w, v, u', v' can be used in the place of u, v, u', v'. Therefore we may assume without loss of generality that for the four vertices u, v, u', v', vu' is not an edge of $\overline{G}$. We show that there exist z, z' with $z < u$ such that $z \to u'$ and $v \to z'$. This implies that y, z are two vertices for which one of the above two properties does not hold. This contradicts the choice of u, v because $u + v > z + v$.

Since $u \to v'$, (u, v') was coloured B, which implies that (u, v') is not the lexicographically smallest vertex of its component. Let (z, z') be the lexicographically smallest vertex in the component of (u, v'). Then there are vertices (u_i, v_i'), $i = 1, 2, \ldots, k$, with $(u_1, v_1') = (u, v')$, $(u_k, v_k') = (z, z')$ and each $u_i v_i' v_{i+1}' u_{i+1}$ is an induced 4-cycle in $\overline{G}$. Note that $u_i \to v_i'$ when i is odd and $v_i' \to u_i$ when i is even. In particular, k must be even. We prove by induction on k that $z = u_k < u_1 = u$, $z = u_k \to u'$ and $v \to v_k' = z'$. Note that to show $u_k < u_1 = u$ it suffices to prove $u_k \neq u_1 = u$. When $k = 2$, clearly $u_2 \neq u_1$. As $v_2' \to u_2$, $v_1' = v' \to v$ and $u_2 v_1'$ is not an edge of $\overline{G}$, vz' is an edge of $\overline{G}$. Since $u' \to u_1$ and neither $u_1 v_2'$ nor vu' is an edge of $\overline{G}$, $v \to v_2' = z$. Similarly, as $v_2' \to u_2$, $u' \to u_1$ and $u_1 v_2'$ is not an edge of $\overline{G}$, $u_2 u'$ is an edge of $\overline{G}$. Since $v_1' \to v$ and neither $u_2 v_1'$ nor vu' is an edge of $\overline{G}$, $u_2 \to u'$.

Assume that $k > 2$ and that, by the induction hypothesis, $u_{k-2} \to u'$ and $v \to v_{k-2}'$. If we can show that $v_{k-1}' \to v$ and $u' \to u_{k-1}$, then we can argue exactly as in the case of $k = 2$, to conclude that both $v \to v_k'$ and $u_k \to u'$ and $u_k \neq u$. Thus we can again let $z = u_k, z' = v_k'$ to complete the proof. Since both $u_{k-2} \to u'$ and $u_{k-1} \to v_{k-1}'$ and $u_{k-2} v_{k-1}'$ is not an edge of $\overline{G}$, $u_{k-1} u'$ is an edge of $\overline{G}$. We must have $u' \to u_{k-1}$ as $v \to v_{k-2}'$ and $u_{k-1} v v_{k-2}' u'$ is an induced 4-cycle in $\overline{G}$. Similarly, since $v \to v_{k-2}'$, $u_{k-1} \to v_{k-1}'$ and $u_{k-1} v_{k-2}'$ is not an edge of $\overline{G}$, vv_{k-1}' is an edge of $\overline{G}$. We must have $v_{k-1}' \to v$ as $u_{k-2} \to u'$ and $u_{k-2} u' v_{k-1}' v$ is an induced 4-cycle in $\overline{G}$.

This justifies that the execution of Step 4 of Theorem 12.2.4 is possible. It is easy to verify now that the orientation of $\overline{G}$ obtained by Theorem 12.2.4 is acyclic and does not contain the digraph in Figure 12.3 as an induced subdigraph. □

Corollary 12.2.14 *The following statements are equivalent for a bipartite graph G.*

1. *G is an interval containment bigraph;*
2. *$\overline{G}$ is a circular arc graph of clique covering number two;*
3. *$\overline{G}$ has an acyclic orientation that does not contain as an induced subdigraph the digraph in Figure 12.3;*
4. *G^+ is bipartite.* □

12.3 Orientation Completion Problems

It is easy to see that a partially oriented graph can be completed to an acyclic oriented graph if and only if it does not contain a directed cycle. Algorithm 1 can be adapted to obtain an acyclic orientation completion of the input partially oriented graph that contains no directed cycle.

We have seen in Section 12.2 that the orientation problem is polynomial time solvable for each of the five classes: quasi-transitive oriented graphs, transitive oriented graphs, local tournaments, locally transitive local tournaments, and acyclic local tournaments. The situation changes for the orientation completion problem. We will show that the orientation completion problem is NP-complete for locally transitive local tournaments, while it remains polynomial time solvable for the other classes.

12.3.1 Quasi-transitive and Transitive Orientation Completions

Let $Q = (V, E \cup A)$ be a partially oriented graph. We use $G = UG(Q)$ to denote the underlying graph of Q and G^+ to denote the auxiliary graph of G as defined in Subsection 12.2.1. That is, the vertex set of G^+ consists of all ordered pairs $(u, v), (v, u)$ for edges $uv \in E(G)$ and in G^+ each vertex (u, v) is adjacent to (v, u), to any vertex (v, w) such that u and w are not adjacent in G, and to any vertex (w, u) such that v and w are not adjacent in G. Thus the arc set A of Q corresponds to a subset S of the vertex set of G^+. An orientation completion of Q to a quasi-transitive oriented graph corresponds to a colour class of a 2-colouring of G^+ that contains S. It follows that Q can be completed to a quasi-transitive oriented graph if and only if the following properties hold:

- G^+ is bipartite, and
- no two vertices of S are at an odd distance in G^+.

If G^+ has these two properties, then it can be 2-coloured such that all vertices of S are of the same colour and the colour class that contains S gives rise to a quasi-transitive orientation completion of Q. Finding such a 2-colouring of G^+ (if it exists) can be done in linear time. Therefore we have the following:

Theorem 12.3.1 ([4]) *The orientation completion problem is polynomial time solvable for the class of quasi-transitive oriented graphs.* □

A partially oriented graph that can be completed to a transitive oriented graph cannot contain directed cycles. So the additional assumption of being acyclic is necessary for a partially oriented graph to admit a completion to a transitive oriented graph. But this additional assumption is not sufficient as there are acyclic partially oriented graphs which can be completed to quasi-transitive oriented graphs but not to transitive oriented graphs. Nevertheless, we show that deciding whether a partially oriented graph can be completed to a transitive oriented graph can be done in polynomial time.

A partially oriented graph $Q = (V, E \cup A)$ is called **consentaneous** if the following properties hold: Let G^+ be the auxiliary graph of $UG(Q)$ and S correspond to the arc set A.

- G^+ is bipartite,
- no two vertices of S are at an odd distance in G^+, and
- for any two vertices at an even distance in G^+, either both are in S or neither.

Theorem 12.3.2 *Let $Q = (V, E \cup A)$ be a partially oriented graph. Suppose that $UG(Q)$ is a comparability graph and Q is consentaneous. Then Q can be completed to a transitive oriented graph if and only if Q does not contain a directed cycle.*

Proof: Let σ be a vertex ordering of $UG(Q)$ such that all arcs in A are forward (i.e., $(u, v) \in A$ implies $\sigma^{-1}(u) < \sigma^{-1}(v)$). Obtain an orientation completion of Q using the lexicographic orientation algorithm in Subsection 12.2.1 with respect to σ. By Theorem 12.2.2 the orientation completion of Q is a transitive oriented graph. □

Corollary 12.3.3 *The orientation completion problem for the class of transitive oriented graphs is solvable in polynomial time.*

Proof: Suppose that a partially oriented graph $Q = (V, A \cup E)$ is given. Let $G = UG(Q)$. If G^+ is not bipartite, then the answer is 'no'. Assume that G^+ is bipartite. Obtain the minimal consentaneous partial oriented graph $Q' = (V, A' \cup E')$ from Q by orienting (if needed) some edges in E. If Q' contains a directed cycle, then the answer is again 'no' by Theorem 12.3.2. Otherwise, Q' contains no directed cycle and we can complete Q' to a transitive oriented graph according to Theorem 12.3.2. This transitive oriented graph is also an orientation completion of Q. □

12.3.2 Local and Acyclic Local Tournament Orientation Completions

The orientation completion problem for local tournaments can be solved in a similar way as above for the quasi-transitive orientation completion problem.

Theorem 12.3.4 ([4]) *The orientation completion problem is polynomial time solvable for the class of local tournaments.* □

We consider next the orientation completion problem for the class of acyclic local tournaments. For a partially oriented graph $Q = (V, E \cup A)$, we use G^+ to denote the auxiliary graph of $UG(Q)$ as defined in Subsection 12.2.2 and use S to denote the set of vertices of G^+ corresponding to the arc set A. Again, we call Q **consentaneous** if the following conditions hold:

- G^+ is bipartite,
- no two vertices of S are at an odd distance in G^+, and
- for any two vertices at an even distance in G^+, either both are in S or neither.

Theorem 12.3.5 ([4]) *Let $Q = (V, E \cup A)$ be a partially oriented graph. Suppose that $UG(Q)$ is a proper interval graph and Q is consentaneous. Then Q can be completed to an acyclic local tournament if and only if Q does not contain a directed cycle.*

Proof: If Q contains a directed cycle then it cannot be completed to an acyclic oriented graph and hence not to an acyclic local tournament. For the other direction, we first show that Q admits a perfect elimination ordering $v_1, v_2, \ldots, v_n$ such that all arcs are forward, that is, if (v_i, v_j) is an arc then $i < j$. To obtain such an ordering we apply a modified LBFS beginning with a vertex of out-degree 0, with preferences (in the case of ties) given to vertices having no out-neighbours among unlabeled vertices.

Let $v_1, v_2, \ldots, v_n$ be an ordering obtained by the modified LBFS. According to Rose, Tarjan and Lueker [36], it is a perfect elimination ordering. Suppose that the ordering contains a backward arc. Let $(v_i, v_j) \in A$ be a backward arc having the largest subscript i. Since (v_i, v_j) is backward, we have $i > j$. The choice of v_n implies $n > i$. Since $i > j$, at the time of labeling v_i the vertex v_j is an unlabeled out-neighbour of v_i. The LBFS rule ensures that v_i is a vertex having the lexicographically largest neighbourhood among the vertices $v_n, \ldots, v_{i+1}$. If the neighbourhood of v_i (among the labeled vertices) is lexicographically larger than the neighbourhood of v_j, some vertex v_ℓ with $\ell > i$ is adjacent to v_i but not to v_j in Q. The assumption that Q is consentaneous implies (v_ℓ, v_i) is an arc which is backward with respect to the ordering. This contradicts the choice of (v_i, v_j). Hence v_i and v_j must have the same neighbourhood among the labeled vertices. But then the rule prefers v_j to v_i for the next labeled vertex, unless v_j has an out-neighbour v_k among unlabeled vertices. A similar proof above (when applied to v_j, v_k) implies v_j and v_k must have the same neighbourhood among the labeled vertices. Continuing in this way, we obtain a directed cycle, which contradicts the assumption. Hence $v_1, v_2, \ldots, v_n$ is a perfect elimination ordering of Q that contains no backward arcs.

Now we apply the lexicographic orientation algorithm using the perfect elimination ordering to obtain an orientation D of $UG(Q)$. By Theorem 12.2.10 D is an acyclic local tournament. Since the perfect elimination ordering has no backward arc from A, the arc set of D contains A. Hence D is an orientation completion of Q. □

Corollary 12.3.6 *The orientation completion problem for the class of acyclic local tournaments is solvable in polynomial time.*

Proof: Suppose that a partially oriented graph $Q = (V, A \cup E)$ is given. Let $G = UG(Q)$. If G^+ is not bipartite, then the answer is 'no'. Assume that G^+ is bipartite. Obtain the minimal consentaneous partial oriented graph $Q' = (V, A' \cup E')$ from Q by orienting (if needed) some edges in E. If Q' contains a directed cycle, then the answer is again 'no' by Theorem 12.3.5. Otherwise, Q' contains no directed cycle and we can complete Q' to an acyclic local tournament orientation according to Theorem 12.3.5. This acyclic local tournament is also an orientation completion of Q. □

Corollary 12.3.7 ([28]) *The problem of extending partial proper interval representations of proper interval graphs is solvable in polynomial time.*

Proof: We show how to reduce the problem of extending partial proper interval representations of proper interval graphs to the orientation completion problem for the class of acyclic local tournaments which is polynomial time solvable according to Corollary 12.3.6. Suppose that G is a proper interval graph and H is an induced subgraph of G. Given a proper interval representation $I_v, v \in V(H)$, of H (i.e., a partial proper interval representation of G), we obtain an orientation of H in such a way that (u, v) is an arc if and only if I_u contains the left endpoint of I_v. The oriented edges together with the remaining edges in G yield a partial orientation of G. This partial orientation of G can be completed to an acyclic local tournament if and only if the partial representation of H can be extended to a proper interval representation of G. □

12.3.3 Locally Transitive Local Tournament Orientation Completions

A cyclic ordering $\mathcal{O} = v_1, v_2, \ldots, v_n, v_1$ of the vertices of a partially oriented graph $Q = (V, E \cup A)$ is called **excellent** if Q has no pair of arcs $v_i \to v_j$ and $v_s \to v_t$ (with a possibility that $i = t$ or $s = j$) such that the vertices occur as v_i, v_t, v_s, v_j in the cyclic ordering, cf. [4]. Since a round ordering of an oriented graph is excellent, by Theorem 12.2.4, every connected locally transitive local tournament has an excellent cyclic ordering, cf. [24]. Thus, a necessary condition for completing Q to a locally transitive local tournament is that it has an excellent ordering. It turns out, as we will show, that the problem of determining whether a partially oriented graph has an excellent ordering is polynomially equivalent to the orientation completion problem for locally transitive local tournaments and both problems are NP-complete (Theorem 12.3.14). The presentation below follows the paper [4] by Bang-Jensen, Huang and Zhu.

Let $\mathcal{O} = v_1, v_2, \ldots, v_n, v_1$ be a cyclic ordering of the vertices of a partially oriented graph $P = (V, E \cup A)$. An arc $(v_i, v_j) \in A$ **dominates** an arc $(v_s, v_t) \in A$ with respect to $\mathcal{O}$ if the vertices of the two arcs appear in the order v_i, v_s, v_t, v_j in $\mathcal{O}$, where we can have $i = s$ or $j = t$. An arc $(v_i, v_j) \in A$

dominates an edge $v_p v_q$ if both of the vertices v_p, v_q occur in the interval $[v_i, v_j]$ from v_i to v_j according to $\mathcal{O}$. An arc is **maximal** with respect to $\mathcal{O}$ if it is not dominated by any other arc.

Lemma 12.3.8 ([4]) *Suppose $P = (V, E \cup A)$ is a partially oriented graph for which the digraph $D = (V, A)$ induced by its arcs has an excellent cyclic ordering $\mathcal{O} = v_1, \ldots, v_n, v_1$ of its vertices. Then P can be completed to an oriented graph D' for which the same cyclic ordering $\mathcal{O}$ is excellent.*

Proof: Let $P = (V, E \cup A)$ be a partially oriented graph and let $\mathcal{O} = v_1, v_2, \ldots, v_n, v_1$ be an excellent cyclic ordering of D. Let $a_1 = (v_{i_1}, v_{j_1}), a_2 = (v_{i_2}, v_{j_2}), \ldots, a_k = (v_{i_k}, v_{j_k})$ be the maximal arcs of D with respect to $\mathcal{O}$. By the assumption of the lemma, for each arc a_r every arc (v_p, v_q) for which both vertices v_p, v_q occur after in the interval $[v_i, v_j]$ satisfy that the vertices occur in the order $v_{i_r}, v_p, v_q, v_{j_r}$. For each $r \in [k]$ in increasing order and all indices p, q with $v_{i_r}, v_p, v_q, v_{j_r}$ occurring in that order such that $v_p v_q$ is an edge of P, we orient this edge as the arc (v_p, v_q). Let $D^* = (V, A \cup A^*)$ be the oriented graph consisting of the original arcs and those edges which we have oriented so far. By construction of D^*, $\mathcal{O}$ is an excellent ordering of D^*. Hence if no edge of E is still unoriented we are done. It suffices to show that we may orient one of the remaining edges, since then the claim follows by induction on the number of unoriented edges. Let $v_p v_q$ be an edge which was not oriented and orient this as (v_p, v_q). We claim that $\mathcal{O}$ is an excellent ordering of $D^* \cup \{(v_p, v_q)\}$. If not then there is an arc (v_a, v_b) of D^* such that the vertices occur in the order v_p, v_b, v_a, v_q but then the edge $v_p v_q$ is dominated by the arc (v_a, v_b) and hence by one of the arcs $a_1, \ldots, a_k$, contradicting that it was not oriented above. □

Lemma 12.3.9 ([4]) *An oriented graph D has an excellent cyclic ordering $\mathcal{O}$ if and only if it can be extended to a round local tournament D^* by adding new arcs. In particular, every excellent ordering of D is a round ordering of D^* and conversely.*

Proof: Suppose first that D can be extended to a round local tournament D^*. According to Theorem 12.2.4 there is a round ordering $\mathcal{O} = v_1, v_2, \ldots, v_n, v_1$ of $V(D^*) = V(D)$. We claim that this ordering is also excellent. If not, then there are arcs (v_i, v_j) and (v_s, v_t) so that the vertices occur in the order v_i, v_t, v_s, v_j according to $\mathcal{O}$. Since $\mathcal{O}$ is a round ordering, we have that (v_i, v_t) and (v_t, v_j) are arcs of D^* but then the neighbours of v_t do not occur correctly according to $\mathcal{O}$, contradiction. So $\mathcal{O}$ is an excellent ordering of D^* and hence also of the subdigraph D. To prove the only if part let $\mathcal{O} = v_1, v_2, \ldots, v_n, v_1$ be an excellent cyclic ordering of the oriented graph D. It suffices to observe that for every maximal arc (v_i, v_j) with respect to $\mathcal{O}$ and any pair of non-adjacent vertices v_a, v_b in the interval $[v_i, v_j]$ with v_a before v_b we may add the arc (v_a, v_b) and still have an excellent ordering of the resulting oriented graph.

Now the claim follows by induction on the number of such non-adjacent pairs. □

For a given oriented graph D we denote by D^c the partially oriented complete graph obtained from D by adding an edge between each pair of non-adjacent vertices.

Lemma 12.3.10 ([4]) *If D is a round oriented graph, then D^c can be completed to a locally transitive tournament.*

Proof: We prove the statement by induction on the number of vertices in D which are not adjacent to all other vertices. By Theorem 12.2.4, the base case where there is no such vertex is true. So assume that all round oriented graphs on n vertices with at most k vertices as above can be completed to a locally transitive tournament and let D be a round digraph with $k+1$ vertices, each of which has a non-neighbour. Let $\mathcal{O} = v_1, v_2, \ldots, v_n, v_1$ be a round ordering of D. W.l.o.g. the vertex v_1 has a non-neighbour, so we have that $v_{d^+(v_1)+2} \neq v_{n-d^-(v_1)}$. We claim that there is no arc (v_p, v_q) with $1 \leq q < p < n - d^-(v_1)$. Suppose such an arc does exist. Then we have $p > d^+(v_1)+1$ by the choice of $\mathcal{O}$ and we have $q > 1$ since v_p is not adjacent to v_1. But this contradicts the fact that the vertex v_p sees its out-neighbourhood as an interval just after itself according to $\mathcal{O}$ because v_1 is not-adjacent to v_p. Thus if we add all the arcs $(v_1, v_{d^+(v_1)+2}), \ldots, (v_1, v_{n-d^-(v_1)-1})$ to D the order $\mathcal{O}$ is an excellent ordering of the resulting digraph D'. By Lemmas 12.3.8 and 12.3.9 this implies that D' can be extended to a round local tournament D'' by adding new arcs. Now the claim follows by induction since D' has fewer vertices with non-neighbours than D does. □

Combining Lemmas 12.3.8, 12.3.9, and 12.3.10 we have the following:

Lemma 12.3.11 ([4]) *An oriented graph D has an excellent ordering if and only if the partially oriented graph D^c has a completion to a tournament T which is locally transitive. Furthermore, given an excellent ordering of D we can construct T in polynomial time and conversely, given T, we can obtain an excellent ordering of D in polynomial time.* □

The following is easy to check.

Proposition 12.3.12 *Each of the two labellings $X, \bar{X}$ of the same partially oriented complete graph in Figure 12.4 have exactly two completions to a locally transitive tournament. For X these are obtained by orienting the two edges $ab, \alpha\beta$ as either $(b,a), (\beta,\alpha)$ or $(a,b), (\alpha,\beta)$. For $\bar{X}$ they are obtained by orienting the two edges $uv, \alpha\beta$ as either $(v,u), (\alpha,\beta)$ or $(u,v), (\beta,\alpha)$.* □

Lemma 12.3.13 ([4]) *Consider the partially oriented 6-wheel W in Figure 12.5. Let D be an orientation completion of W. Then D does not have an excellent ordering if and only if the three edges $c_{11}c_{12}, c_{21}c_{22}, c_{31}c_{32}$ are oriented as $(c_{11}, c_{12}), (c_{21}, c_{22}), (c_{31}, c_{32})$.*

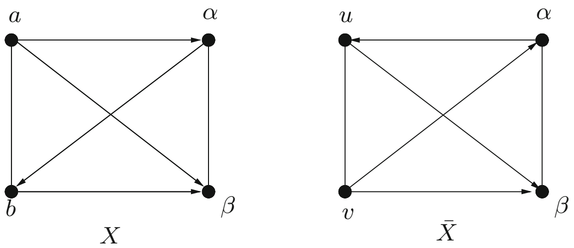

Figure 12.4 Two different labellings of the same partially oriented complete graph on 4 vertices. For later convenience we name these $X, \bar{X}$.

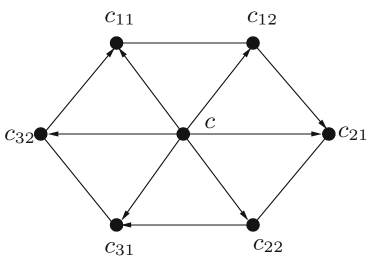

Figure 12.5 A partially oriented wheel W.

Proof: If the three edges $c_{11}c_{12}, c_{21}c_{22}, c_{31}c_{32}$ are oriented as (c_{11}, c_{12}), (c_{21}, c_{22}), (c_{31}, c_{32}) then the vertex c has a directed 6-cycle in its out-neighbourhood and hence D^c has no completion to a locally transitive tournament. By Lemma 12.3.9, D has no excellent ordering. On the other hand, if D contains at least one of the arcs $(c_{12}, c_{11}), (c_{22}, c_{21}), (c_{32}, c_{31})$, then D is acyclic. Clearly D^c can be completed to a transitive tournament and hence by Lemma 12.3.11, D has an excellent ordering. □

Theorem 12.3.14 ([4]) *The following polynomially equivalent problems are NP-complete.*

- *Deciding whether an oriented graph has an excellent ordering.*
- *Deciding whether a given partially oriented complete graph can be completed to a locally transitive tournament.*

Proof: We describe polynomial reductions from 3-SAT to these problems.

Let $\mathcal{F}$ be an instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$, where each clause is of the form $(\ell_1 \vee \ell_2 \vee \ell_3)$ and each ℓ_i is either one of the variables x_j or the negation $\bar{x}_j$ of such a variable.

Let p_i (q_i) be the number of times variable x_i $(\bar{x}_i)$ occurs as a literal in $\mathcal{F}$. The enumeration of the clauses $C_1, \ldots, C_m$ induces an ordering on the occurrences of the same literal in the formula. Guided by this ordering we now construct a partially oriented graph $H' = H'(\mathcal{F})$ as follows:

Let $X, \bar{X}$ be as in Figure 12.4. For each variable x_i we form the partially oriented graph X_i from p_i copies of X and q_i copies of $\bar{X}$ (these $p_i + q_i$ graphs are vertex disjoint) by identifying all the α vertices and all the β vertices and denote these identified vertices by $\alpha(x_i), \beta(x_i)$, respectively. Denote the p_i copies of a, b by $a_{i,1}, \ldots, a_{i,p_i}, b_{i,1}, \ldots, b_{i,p_i}$ and the q_i copies of u, v by $u_{i,1}, \ldots, u_{i,q_i}, v_{i,1}, \ldots, v_{i,q_i}$.

Take m disjoint copies $W_1, W_2, \ldots, W_m$ of the partially oriented 6-wheel from Figure 12.5 where we use $c_i, c^i_{11}, c^i_{12}, c^i_{21}, c^i_{22}, c^i_{31}, c^i_{32}$ to denote the vertices of W_i. Make the following association between the literals of $\mathcal{F}$ and the W_i's: If $C_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ we associate the vertices c^i_{j1}, c^i_{j2} with the literal $\ell_{i,j}$ of C_i, $j \in [3]$.

Now we make the following vertex identifications. For each clause $C_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ we identify the vertices $c^i_{11}, c^i_{12}, c^i_{21}, c^i_{22}, c^i_{31}, c^i_{32}$ with vertices from the union of the graphs $X_1, \ldots, X_n$ as follows: If $\ell_{i,j} = x_r$ and this is the h'th occurrence of variable x_r according to the induced ordering of that literal, then identify c^i_{j1} with $a_{r,h}$ and c^i_{j2} with $b_{r,h}$. If $\ell_{i,j} = \bar{x}_r$ and this is the t'th occurrence of $\bar{x}_r$ according to the induced ordering of that literal, then identify c^i_{j1} with $u_{r,t}$ and c^i_{j2} with $v_{r,t}$. Note that even after these identifications each of the subdigraphs $W_1, \ldots, W_m$ are still vertex disjoint.

Clearly we can construct H' in polynomial time from $\mathcal{F}$. Denote by H the oriented graph obtained from H' by deleting all (unoriented) edges. It is easy to check that the in- and out-neighbourhoods of each vertex in H is acyclic.

By Lemma 12.3.11 it suffices to show that H has an excellent ordering if and only if $\mathcal{F}$ is satisfiable.

First suppose that H has an excellent ordering. By Lemma 12.3.11 this means that the partially oriented complete graph H^c has a completion T as a locally transitive tournament. We claim that the following is a satisfying truth assignment: If the edge $\alpha(x_i)\beta(x_i)$ is oriented in T as $(\alpha(x_i), \beta(x_i))$ then let x_i be false and if it is oriented as $(\beta(x_i), \alpha(x_i))$ then let x_i be true. First observe that, by Proposition 12.3.12, this implies that for each $i \in [n]$ the variable x_i is false if and only if each of the edges $a_{i,j}b_{i,j}$, $j \in [p_i]$, are oriented as $(a_{i,j}, b_{i,j})$ and each of the edges $u_{i,r}v_{i,r}$, $r \in [q_i]$, are oriented as $(v_{i,r}, u_{i,r})$.

We now use this to show that each of the clauses of $\mathcal{F}$ are satisfied by our truth assignment. As T is locally transitive, for each of the induced subdigraphs $T[W_j]$, $j \in [m]$, the out-neighbourhood of c_j is acyclic which implies that at least one of three arcs of H which correspond to the literals of $\mathcal{F}$ is oriented as (c_{j2}, c_{j1}). If this arc corresponds to the literal x_s then, by the identification rule above, this is an arc of the form $(b_{s,t}, a_{s,t})$, so the variable x_s is true and C_j is satisfied. If the arc corresponds to the literal $\bar{x}_s$ then the identification rule implies that this is an arc of the form $(v_{s,t}, u_{s,t})$, implying that $\bar{x}_s$ is true so again C_j is satisfied. Thus we have shown that $\mathcal{F}$

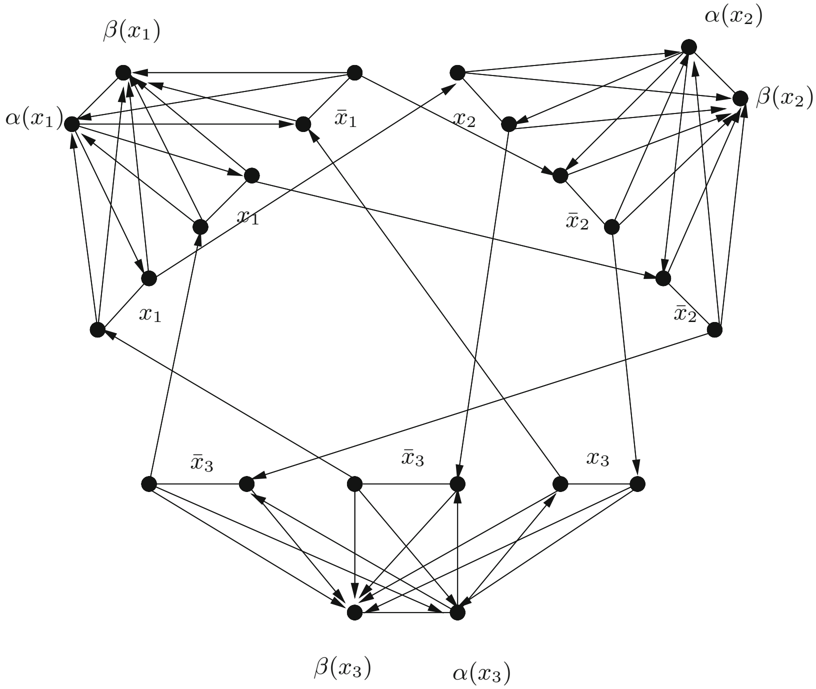

Figure 12.6 Part of the digraph $H'(\mathcal{F})$ when $\mathcal{F} = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$. For better readability the vertices c_1, c_2, c_3 are not shown.

is satisfiable if H^c has a locally transitive completion (H has an excellent ordering).

Now suppose that $t : \{x_1, \ldots, x_n\} \to \{true, false\}$ is a satisfying truth assignment for $\mathcal{F}$. We shall use this truth assignment to construct an excellent ordering of the partially oriented graph H'. Recall that this is also an excellent ordering of the directed part H of H'.

We first orient the edges $\alpha(x_1)\beta(x_1), \ldots \alpha(x_n)\beta(x_n)$ as follows: Orient $\alpha(x_i)\beta(x_i)$ as $(\beta(x_i), \alpha(x_i))$ if $x_i = true$ and as $(\alpha(x_i), \beta(x_i))$ otherwise. Denote by $\hat{H}$ the resulting partially oriented graph. It follows from Proposition 12.3.12, the way we made identifications between vertices of the W_j's and variable vertices and the fact that t is a satisfying truth assignment that we can now orient all the remaining edges of $\hat{H}$ (recall that those correspond to the literals) uniquely so that the resulting full orientation $\overrightarrow{H}$ of H' satisfies that the in- and out-neighbourhood of each vertex is still acyclic.

We now construct an excellent ordering for $\overrightarrow{H}$. Denote by $A(x_i)$ ($B(x_i)$), $i \in [n]$ the set of out-neighbours (in-neighbours) of $\alpha(x_i)$ in $\overrightarrow{H}$. Note that if $t(x_i) = false$, then $A(x_i) = \{b_{i,1}, \ldots, b_{i,p_i}, u_{i,1}, \ldots, u_{i,q_i}, \beta(x_i)\}$, $B(x_i) = \{a_{i,1}, \ldots, a_{i,p_i}, v_{i,1}, \ldots, v_{i,q_i}\}$ and there is no oriented arc from $A(x_i)$ to $B(x_i)$.

Similarly, if $t(x_i) = true$, then $A(x_i) = \{b_{i,1}, \ldots, b_{i,p_i}, u_{i,1}, \ldots, u_{i,q_i}\}$, $B(x_i) = \{a_{i,1}, \ldots, a_{i,p_i}, v_{i,1}, \ldots, v_{i,q_i}, \beta(x_i)\}$ and there is no oriented arc from $B(x_i)$ to $A(x_i)$.

Furthermore, observe that $\beta(x_i)$ has no out-neighbour when $t(x_i) = false$ and precisely one out-neighbour, namely $\alpha(x_i)$ when $t(x_i) = true$. Let $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $1 < j_1 < j_2 < \ldots < j_g \leq n$ denote the indices of the true, respectively the false variables. Consider the following cyclic ordering $\mathcal{O}$ of $V(\overrightarrow{H})$:

$\alpha(x_{i_1}), \alpha(x_{i_2}), \ldots, \alpha(x_{i_k}), c_1, c_2, \ldots, c_m, A(x_{i_1}), \ldots, A(x_{i_k}), B(x_{j_1}), \ldots,$
$B(x_{j_g}), \alpha(x_{j_1}), \ldots, \alpha(x_{j_g}), A(x_{j_1}), \ldots, A(x_{j_g}), B(x_{i_1}), \ldots, B(x_{i_k}), \alpha(x_{i_1}),$
where the ordering inside each $A(x_i), B(x_i)$ is as according to the way we listed those sets above.

We shall prove that the ordering $\mathcal{O}$ is excellent. Suppose for contradiction that there is a pair of arcs (v_i, v_j) and (v_s, v_t) with the vertices occurring in the order v_i, v_t, v_s, v_j according to $\mathcal{O}$.

- We cannot have $v_i = \alpha(x_{i_f})$ for some $f \in [k]$ because there is no backward arc in the interval of $\mathcal{O}$ from $\alpha(x_{i_f})$ to (the end of) $A(x_f)$ ($\alpha(x_{i_f})$ is only adjacent to vertices in $A(x_{i_f})$). Similarly, we cannot have v_i in the interval $[\alpha(x_{j_1}), \alpha(x_{j_g})]$.
- We cannot have $v_i = c_p$ for some $p \in [m]$ because the only arcs incident to c_p are from c_p to the six vertices which correspond to its three literals and we ordered the A and B sets and $\alpha(x_{j_1}), \ldots, \alpha(x_{j_g})$ in such a way that any arc between them goes forward in the ordering. In particular, there are no backwards arcs with respect to the ordering in the interval
$A(x_{i_1}), \ldots, A(x_{i_k}), B(x_{j_1}), \ldots, B(x_{j_g}), \alpha(x_{j_1}), \ldots, \alpha(x_{j_g}), A(x_{j_1}), \ldots,$
$A(x_{j_g}), B(x_{i_1}), \ldots, B(x_{i_k})$.
- We cannot have v_i in the interval $A(x_{i_1}), \ldots, A(x_{i_k})$ since all out-neighbours of those vertices are in the interval $B(x_{i_1}), \ldots, B(x_{i_k})$ and then the remark above implies the claim. Similarly, we cannot have v_i in the interval $A(x_{j_1}), \ldots, A(x_{j_g})$.
- We cannot have v_i in the interval $B(x_{j_1}), \ldots, B(x_{j_g})$ because there are no backward arcs in the interval $B(x_{j_1}), \ldots, B(x_{j_g}), \alpha(x_{j_1}), \ldots, \alpha(x_{j_g}),$
$A(x_{j_1}), \ldots, A(x_{j_g})$ and this contains all out-neighbours of such a v_i.
- Finally we cannot have v_i in the interval $B(x_{i_1}), \ldots, B(x_{i_k})$ because all arcs out of a vertex in this interval remain inside the interval $B(x_{i_1}), \ldots, B(x_{i_k}), \alpha(x_{i_1}), \alpha(x_{i_2}), \ldots, \alpha(x_{i_k})$ and there is no backward arc here.

Thus we have shown that $\mathcal{O}$ is excellent and hence, by Lemma 12.3.11, the partially oriented complete graph H^c has a completion to a locally transitive tournament. $\square$

12.4 Orientation Sandwich Completion Problems

For a fixed property Π of partially oriented graphs, the Π-SANDWICH PROBLEM is defined as follows:

> Π-SANDWICH PROBLEM
> **Input:** A pair of partially oriented graphs $Q_1 = (V, E_1 \cup A_1)$ and $Q_2 = (V, E_2 \cup A_2)$.
> **Question:** Is there a partially oriented graph $Q = (V, E \cup A)$ with $E_1 \subseteq E \subseteq E_2$ and $A_1 \subseteq A \subseteq A_2$ which satisfies Π?

Sandwich problems for partially oriented graphs simultaneously generalize graph sandwich problems and digraph sandwich problems, which have been studied by Golumbic, Kaplan and Shamir in [20]. **Graph sandwich problems** restrict Q_1, Q_2 and Q in the above definition to be graphs, while **digraph sandwich problems** restrict them to be digraphs.

Graph sandwich problems are polynomial time solvable for several graph properties, including being bipartite graphs, threshold graphs, split graphs, cographs and Eulerian graphs, and are NP-complete for properties such as being chordal graphs, interval graphs, circle graphs, circular arc graphs, proper circular arc graphs, comparability graphs, co-comparability graphs, and permutation graphs, cf. [20]. Little is known about digraph sandwich problems but for Eulerian digraphs it is proved to be polynomial time solvable by Ford and Fulkerson in [11].

A partially oriented graph $Q = (V, E \cup A)$ is called **mixed Eulerian** if both (V, E) and (V, A) are Eulerian, that is, in (V, E) every vertex has an even degree and in (V, A) every vertex has its in-degree equal to its out-degree. Although both sandwich problems for Eulerian graphs and digraphs are polynomial time solvable, the sandwich problem for mixed Eulerian partially oriented graphs remains open.

Problem 12.4.1 *Determine the complexity of the sandwich problem for mixed Eulerian partially oriented graphs.*

For a fixed property Π of oriented graphs, we define the Π-ORIENTATION SANDWICH COMPLETION PROBLEM as follows:

> Π-ORIENTATION SANDWICH COMPLETION PROBLEM
> **Input:** A pair of partially oriented graphs $Q_1 = (V, E_1 \cup A_1)$ and $Q_2 = (V, E_2 \cup A_2)$.
> **Question:** Is there a partially oriented graph $Q = (V, E \cup A)$ with $E_1 \subseteq E \subseteq E_2$ and $A_1 \subseteq A \subseteq A_2$ which can be completed to an oriented graph that satisfies Π?

Orientation sandwich completion problems generalize orientation completion problems and hence orientation problems. Orientation sandwich completion problems and sandwich problems for partially oriented graphs are closely related. Let Π be a property of oriented graphs. A partially oriented graph is said to have property Π^* if it can be completed to an oriented graph that has the property Π. Then the Π-orientation sandwich completion problem is just the Π^*-sandwich problem. For instance, suppose that Π is the property of being an Eulerian oriented graph, then a partially oriented graph has property Π^* if and only if it is mixed Eulerian and thus the Π-orientation sandwich completion problem is just Problem 12.4.1. As mentioned above, the Π-orientation completion problem is polynomial time solvable but the Π-orientation sandwich completion problem is open. Special cases of the Π-orientation sandwich completion problem have been studied by de Gevigney, Klein, Nguyen and Szigeti [8].

A property Π of oriented graphs is called **sup-preservable** if $Q_1 = (V, A_1)$ has the property Π and $A_1 \subseteq A_2$ imply that $Q_2 = (V, A_2)$ also has the property Π. As an example, being k-arc-strong is a sup-preservable property for each $k \geq 1$. Let Π be a sup-preservable property of oriented graphs. Then the Π-orientation sandwich completion problem can be reduced to the Π-orientation completion problem. Indeed, suppose that $Q_1 = (V, E_1 \cup A_1)$ and $Q_2 = (V, E_2 \cup A_2)$ form an instance of the Π-orientation sandwich completion problem. In order to have a partially oriented graph $Q = (V, E \cup A)$ satisfying $E_1 \subseteq E \subseteq E_2$ and $A_1 \subseteq A \subseteq A_2$, we must have $E_1 \subseteq E_2$ and $A_1 \subseteq A_2$. For any such Q, Q can be completed to an oriented graph that has the property Π if and only if Q_2 can. Hence the Π-orientation sandwich completion problem reduces to the Π-orientation completion problem. In particular, the k-arc-strong orientation sandwich completion problem reduces to the k-arc-strong orientation completion problem for each $k \geq 1$. Each k-arc-strong-orientation completion problem can be formulated as a feasible submodular flow problem which is polynomial time solvable (cf. [4]). Consequently, we have the following:

Theorem 12.4.2 *For each $k \geq 1$, the k-arc-strong orientation sandwich completion problem is polynomial time solvable.* □

In contrast, the k-strong orientation sandwich completion problem is NP-complete for each $k \geq 3$ as this is shown to be the case for the k-strong orientation problem by de Gevigney [7]. Thomassen [41] proved that a graph G has a 2-strong orientation if and only if G is 4-edge-connected and $G - v$ is 2-edge-connected for every vertex v. This implies that the 2-strong orientation problem is polynomial time solvable.

Theorem 12.4.3 ([7, 41]) *The k-strong orientation problem is polynomial time solvable when $k \leq 2$ and NP-complete when $k \geq 3$.* □

Thus to complete a dichotomy of k-strong orientation completion problems and k-strong orientation sandwich completion problems the only case left open is $k = 2$.

Problem 12.4.4 *Determine the complexity of the 2-strong orientation sandwich completion problem and of the 2-strong orientation completion problem.*

A **directed cycle factor** in a digraph is a spanning subdigraph that is a vertex-disjoint union of directed cycles. The orientation completion problem for the property of having a directed cycle factor is shown to be NP-complete in [4].

Theorem 12.4.5 ([4]) *It is NP-complete to decide whether a partially oriented graph Q has a completion D with a directed cycle factor.*

Proof: It was shown by Bang-Jensen and Casselgren [2] that it is NP-complete to decide whether a bipartite digraph B has a directed cycle-factor consisting of cycles $C_1, C_2, \ldots, C_k$ so that no C_i has length 2. Let B be given and form the partially oriented graph Q from B by replacing the two arcs of each directed 2-cycle by an edge. It is easy to see that Q has a completion with a directed cycle factor if and only if B has a cycle factor with no directed 2-cycle, implying the theorem. □

The complexity of the orientation sandwich completion problem for having directed cycle factors is open.

Problem 12.4.6 *Determine the complexity of the orientation sandwich completion problem for having directed cycle factors.*

Let $\pi = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ be a set of k pairs of distinct vertices in a (di)graph H. A **π-linkage** in H is a collection of k disjoint paths $R_1, \ldots, R_k$ such that R_i starts in s_i and ends in t_i. For a given class $\mathcal{C}$ of digraphs, the **$\mathcal{C}$-π-linkage completion problem** is defined as follows: given a partially oriented graph $Q = (V, E \cup A)$ and a set π of k terminal pairs in V, is it possible to complete the orientation of Q so that the resulting oriented graph is in $\mathcal{C}$ and has a π-linkage?

For general digraphs the π-linkage problem, and hence also the completion version, is NP-complete already when $k = 2$ and even if the digraph is highly connected [12, 40]. Chudnovsky, Scott and Seymour [6] proved that the π-linkage problem is polynomial for semicomplete digraphs (that is, digraphs whose underlying graph is complete). This implies that the **tournament-π-linkage completion** problem is polynomial because such a completion is possible if and only if the digraph that we obtain from the partially oriented graph Q by replacing each undirected edge by a directed 2-cycle is semicomplete and has a π-linkage (no two paths in a linkage intersect).

Problem 12.4.7 *What is the complexity of the local-tournament-π-linkage completion problem when $k \geq 2$ is fixed?*

An oriented graph is called an **in-tournament** if the in-neighbourhood of every vertex induces a tournament. The orientation completion problem for in-tournaments is polynomial time solvable, cf. [4]. The orientation sandwich completion problem for in-tournaments is open.

Problem 12.4.8 *Determine the complexity of the orientation sandwich completion problem for in-tournaments.*

The orientation problem for the class of acyclic in-tournaments is polynomial time solvable. This follows from the fact that chordal graphs are exactly the graphs which admit acyclic in-tournament orientations. However, the orientation completion problem as well as the orientation sandwich completion problem for acyclic in-tournaments remain open.

Problem 12.4.9 *Determine the complexity of the orientation sandwich completion problem for acyclic in-tournaments.*

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