11. Miscellaneous Digraph Classes

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11.1 Introduction

Obviously, there are countless digraph classes, so that any attempt to give a complete overview is doomed to failure. One has to restrict oneself to a selection. Some will be presented in their own chapter or section, some will only be mentioned for some specific results throughout the book and some won't be mentioned at all. As tournaments (`tou`) are arguably the best studied class of digraphs with a rich library of strong results (see Chapter 2), their prominent place in any selection is a given. Unsurprisingly, several authors have tried to generalize the class in different directions in order to obtain larger classes of digraphs while retaining enough structure that most central results on tournaments still hold. Those classes include semicomplete digraphs (`scd`) (see Chapter 2), multipartite tournaments (`mut`) (see Chapter 7) and local tournaments (`lct`) (see Chapter 6). Results on hypertournaments (`hyt`), an extension of tournaments to directed hypergraphs that is not featured in this book, have been obtained by Q. Guo, Y. Guo, Gutin, Kayibi, Khan, Koh, H. Li, R. Li, S. Li, Lu, Ning, Petrović, Pirzada, Ree, Surmacs, Thomassen, Wang, Yang, Yao, Yeo, K.M. Zhang, X. Zhang and Zhou (see, e.g., [77–79, 100–102, 104, 109–111, 126, 145, 154, 166, 173]).

Several of those tournament generalizations have since been generalized themselves, resulting in an array of tournament-related digraph classes. Locally semicomplete digraphs (`lsd`), round digraphs (`rod`), in/out-round digraphs (`ird`), locally in/out-tournaments (`lit`), locally in/out-semicomplete digraphs (`lis`) and path-mergeable digraphs (`pmd`) are considered in Chapter 6. Chapter 7 is dedicated to semicomplete multipartite digraphs (`smd`). Results on transitive digraphs (`trd`), k-transitive digraphs (`ktd`), quasi-transitive digraphs (`qtd`) and k-quasi-transitive digraphs (`kqt`) can be found in Chapter 8.

In Section 11.8 of this chapter, we will consider another generalization of both semicomplete and semicomplete bipartite digraphs: Arc-locally

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semicomplete digraphs (als). They are themselves generalized by $\mathcal{H}_1$-free digraphs (h1f) and $\mathcal{H}_2$-free digraphs (h2f) in Section 11.9. The related classes of $\mathcal{H}_3$-free digraphs (h3f) and $\mathcal{H}_4$-free digraphs (h4f) are also briefly considered.

Of course, there are also digraph classes (fairly) unrelated to tournaments such as acyclic digraphs (acd), investigated in Chapter 3. Kernel-perfect digraphs (kpd) are mentioned in several results throughout the book, for example in Section 11.7, which is mainly dedicated to perfect digraphs (ped), game-perfect digraphs (gpd) and weakly game-perfect digraphs (wgp).

Many digraph classes appear naturally in applications to other fields such as mathematical logic or computer science. One such class is that of circulant digraphs (cid), which have been considered by such authors as Alspach, Burkard, Çela, Parsons, Van Doorn, Woeginger and Yang (see, e.g., [2, 151, 167]). They include regular round digraphs (rrd) and are themselves included in the class of Cayley digraphs (cad), whose properties have been investigated, for example, by Curran, Gallian, Hamidoune, Parhami, Rankin, Witte, Xiao and Xu (see, e.g., [44, 81, 132, 160–162, 164]).

Two classes which also have applications in the construction of interconnection networks (see [27] for a survey by Bermond, Homobono and Peyrat) are de Bruijn digraphs (dbd) and Kautz digraphs (kad), which we will consider in Sections 11.4 and 11.5, respectively. Both classes can be defined using the line digraph operator, which will be investigated more closely in Section 11.2 on line digraphs (lnd) and Section 11.3 on iterated line digraphs (ild).

The closely related minimal series-parallel digraphs (msp), series-parallel digraphs (spd) and series-parallel partial order digraphs (spo), appear in flow diagrams and dependency charts and have an application to the problem of scheduling under constraints. We will consider them briefly in Section 11.6 on directed cographs (dco), a generalization of series-parallel partial order digraphs.

Figure 11.1 gives a first overview of how the previously mentioned classes relate to one another. For more structure, we also include the subclasses of loopless line digraphs (lld) and loopless iterated line digraphs (lil). Class x is included in class y if the depicted digraph contains an (x, y)-path. Obviously, neither the list of considered digraph classes nor the relations depicted are necessarily exhaustive.

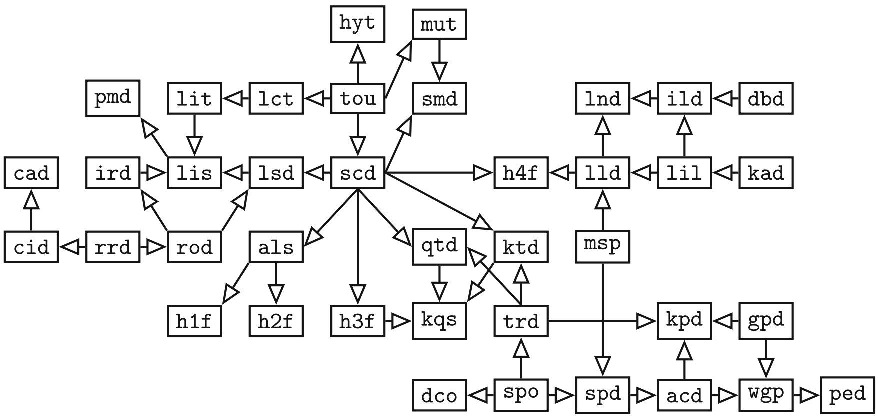


Figure 11.1 Digraph depicting relations between digraph classes.

Note that we omitted certain digraph classes, such as Euler digraphs (see Chapter 4), planar digraphs (see Chapter 5), digraphs with bounded width (see Chapter 9), digraph products (see Chapter 10) and underlying graphs of digraphs (see Chapter 12), mostly because they intersect many others but are not contained in / do not contain other classes. Intersection digraphs on the other hand include all digraphs, as Beineke and Zamfirescu [23] and Sen, Das, Roy and West [139] proved, which makes their inclusion in the figure redundant. For further results on intersection digraphs and their subclass of interval digraphs, however, we also refer to work by Brown, Busch, Dasgupta, Feder, Francis, Hell, Huang, Lundgren, Müller, Rafiey, Sanyal and Talukdar (see, e.g., [35, 45–47, 62, 120, 138, 140, 141, 159, 168]).

11.2 Line Digraphs

Krausz [105] defined the line graph $L(G)$ of a graph $G = (V, E)$ to be the graph with vertex set E and an edge between $e, f \in E$, if and only if e and f are incident in G. Since then, differing generalizations of the concept for directed pseudographs have been introduced. The most common definition for the line digraph $L(D)$ of a directed pseudograph $D = (V, A)$ – and the only one we will consider here – is due to Harary and Norman [82]. Corresponding to the undirected version, the vertex set of $L(D)$ is the arc set A of D. Due to the orientation of arcs, there is the additional choice of when and how to connect two vertices $a, b \in A$ of $L(D)$, which distinguishes the competing concepts of line digraphs. Here, (a, b) is an arc of $L(D)$ if and only if the head of a coincides with the tail of b. In other words, ab is a directed walk of length 2 in D. Note that the line digraph $L(D)$ does not contain multiple arcs, but contains a loop, if and only if D contains a loop. Therefore, technically, the line digraph of a directed pseudograph containing a loop is not a digraph, but again a directed pseudograph.

A directed pseudograph D is called a line digraph if $D = L(D')$ for some directed pseudograph D'.

The first easy observation Harary and Norman [82] then made is the following.

Theorem 11.2.1 ([82]) *Let D be a directed pseudograph. Then,*

$$|V(L(D))| = |A(D)| \quad \textit{and} \quad |A(L(D))| = \sum_{v \in V(D)} d_D^-(v) d_D^+(v).$$

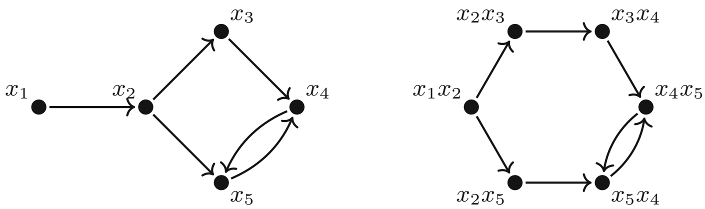
Another nice property that directly follows from the definition is the invariance of the minimum and maximum semi-degree under the line digraph operator.

Proposition 11.2.2 *Let $D = (V, A)$ be a directed pseudograph. Then,*

$$d^+_{L(D)}(xy) = d^+_D(y) \quad \textit{and} \quad d^-_{L(D)}(xy) = d^-_D(x) \quad \textit{for all } xy \in A.$$

Particularly,

$$\delta^0(L(D)) = \delta^0(D) \quad \textit{and} \quad \Delta^0(L(D)) = \Delta^0(D).$$



(a) digraph D (b) line digraph $L(D)$

Figure 11.2 A digraph and its line digraph.

In the following theorem we collect a number of characterizations of line digraphs. Characterization (ii) is among the first results on line digraphs and due to Harary and Norman [82]. Later, Heuchenne [90] found the local criterion (iii) and Richards [137], in (iv) and (v), considered adjacency matrices to determine line digraphs, for which we recall the following definition. For a matrix $M = [m_{ik}] \in \{0,1\}^{n \times n}$, a row i is **orthogonal** to a row j if $\sum_{k=1}^n m_{ik} m_{jk} = 0$. One can give a similar definition of orthogonal columns. Conditions (ii) and (iii) have each been rediscovered by several authors, as Hemminger and Beineke [88] found in their survey on line graphs and line digraphs. The proof presented here is also adapted from that survey.

Theorem 11.2.3 *Let $D = (V, A)$ be a directed pseudograph with vertex set $V = \{1, 2, \ldots, n\}$ and with no multiple arcs and let $M = [m_{ij}]$ be its adjacency matrix (i.e., the $n \times n$-matrix such that $m_{ij} = 1$, if $ij \in A(D)$, and $m_{ij} = 0$, otherwise). Then the following assertions are equivalent:*

(i) D is a line digraph;
(ii) there exist two partitions $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ of $V(D)$ such that

$$A(D) = \bigcup_{i \in I} A_i \times B_i;$$

(iii) if vw, uw and ux are arcs of D, then so is vx;
(iv) any two rows of M are either identical or orthogonal;
(v) any two columns of M are either identical or orthogonal.

Proof: We prove the following implications and equivalences: (i) $\Leftrightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv), (iv) $\Leftrightarrow$ (v), (iv) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii). Let $D = L(H)$. For each $v_i \in V(H)$, let A_i and B_i be the sets of in-coming and out-going arcs at v_i, respectively. Then the arc set of the subdigraph of D induced by $A_i \cup B_i$ equals $A_i \times B_i$. If $ab \in A(D)$, then there is an i such that $a = v_j v_i$ and $b = v_i v_k$. Hence, $ab \in A_i \times B_i$. The result follows.

(ii) $\Rightarrow$ (i). Let Q be the directed pseudograph with ordered pairs (A_i, B_i) as vertices, and with $|A_j \cap B_i|$ arcs from (A_i, B_i) to (A_j, B_j) for each i and j (including $i = j$). Let σ_{ij} be a bijection from $A_j \cap B_i$ to this set of arcs (from (A_i, B_i) to (A_j, B_j)) of Q. Then the function σ defined on $V(D)$ by taking σ to be σ_{ij} on $A_j \cap B_i$ is a well-defined function of $V(D)$ into $V(L(Q))$, since $\{A_j \cap B_i\}_{i,j \in I}$ is a partition of $V(D)$. Moreover, σ is a bijection since every σ_{ij} is a bijection. Furthermore, it is not difficult to see that σ is an isomorphism from D to $L(Q)$.

(ii) $\Rightarrow$ (iii). If vw, uw and ux are arcs of D, then there exist i, j such that $\{u, v\} \subseteq A_i$ and $\{w, x\} \subseteq B_j$. Hence, $(v, x) \in A_i \times B_j$ and $vx \in D$.

(iii) $\Rightarrow$ (iv). Assume that (iv) does not hold. This means that some rows, say i and j, are neither identical nor orthogonal. Then there exist k, h such that $m_{ik} = m_{jk} = 1$ and $m_{ih} = 1, m_{jh} = 0$ (or vice versa). Hence, ik, jk, ih are in $A(D)$ but jh is not. This contradicts (iii).

(iv) $\Leftrightarrow$ (v). Both (iv) and (v) are equivalent to the statement:

$$\text{for all } i, j, h, k, \text{ if } m_{ih} = m_{ik} = m_{jk} = 1, \text{ then } m_{jh} = 1.$$

(iv) $\Rightarrow$ (ii). For each i and j with $m_{ij} = 1$, let $A_{ij} = \{h : m_{hj} = 1\}$ and $B_{ij} = \{k : m_{ik} = 1\}$. Then, by (iv), A_{ij} is the set of vertices in D whose row vectors in M are identical to the ith row vector, whereas B_{ij} is the set of vertices in D whose column vectors in M are identical to the jth column vector (we use the previously proved fact that (iv) and (v) are equivalent).

Thus, $A_{ij} \times B_{ij} \subseteq A(D)$, and moreover $A(D) = \bigcup\{A_{ij} \times B_{ij} : \ m_{ij} = 1\}$. By the orthogonality condition, A_{ij} and A_{hk} are either equal or disjoint, as are B_{ij} and B_{hk}. For a zero row vector i in M, let A_{ij} be the set of vertices whose row vector in M is the zero vector, and let $B_{ij} = \emptyset$. Doing the same with the zero column vectors of M completes the partition as in (ii). □

The characterizations (ii)–(v) all imply polynomial algorithms to verify whether a given directed pseudograph is a line digraph. For an example of an effective polynomial algorithm using (ii) to recognize acyclic line digraphs, see [16, Page 42]. Criterion (iii) can also be reformulated to obtain a characterization of line digraphs in terms of forbidden induced subdigraphs.

Corollary 11.2.4 ([16]) *A directed pseudograph D is a line digraph if and only if D does not contain, as an induced subdigraph, any directed pseudograph that can be obtained from one of the directed pseudographs in Figure 11.3 (dashed arcs are missing) by adding zero or more arcs (other than the dashed ones).*

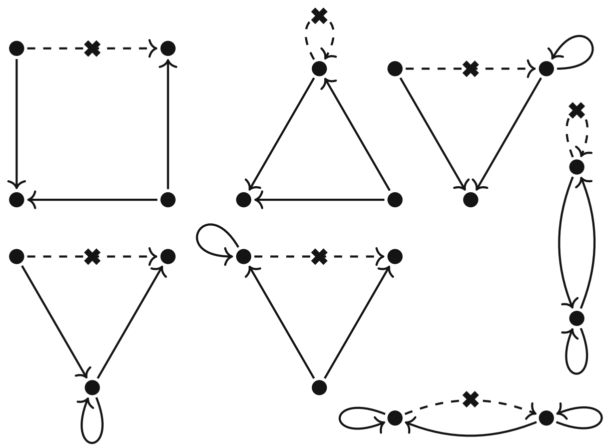


Figure 11.3 Forbidden directed pseudographs of line digraphs.

Observe that the digraph of order 4 in Figure 11.3 corresponds to the case of distinct vertices in Part (iii) of Theorem 11.2.3, and the two directed pseudographs of order 2 correspond to the cases $x = u \neq v = w$ and $u = w \neq v = x$, respectively.

From Corollary 11.2.4 a simpler characterization of the line digraphs of digraphs (i.e. without loops and multiple arcs) is easily obtained by omitting those forbidden induced subdigraphs that imply said loops or parallel arcs.

More details and further characterizations of special classes of line digraphs can be found in the surveys by Hemminger and Beineke [88] and Prisner [131].

As, for every line digraph Q, under '(ii) $\Rightarrow$ (i)' in Theorem 11.2.3 a directed pseudograph D such that $Q = L(D)$ is constructed, it is natural to ask whether D is unique with said property. Harary and Norman [82] answered the question in the negative, but recognized that two directed pseudographs with the same line digraph cannot differ too much, as the following theorem shows.

Theorem 11.2.5 ([82]) *Let D and D' be directed pseudographs such that $L(D) = L(D')$. Then the directed pseudographs obtained from D and D', by deleting all vertices with in-degree 0 and all vertices with out-degree 0, are isomorphic.*

Prisner [130] found that under certain circumstances, even the consideration of the underlying graph may be enough to determine quasi-uniqueness, a generalization of results due to Villar [152].

Theorem 11.2.6 ([130]) *Let D and D' be directed pseudographs without parallel arcs and both of minimum semi-degree at least 2. Then $UG(L(D)) \cong UG(L(D'))$ implies that D is isomorphic to D' or its converse.*

Harary and Norman [82] also gave a partial answer to the related question of which directed pseudographs are isomorphic to their line digraph.

Theorem 11.2.7 ([82]) *Let D be a unilateral (i.e., any two vertices are connected by a directed path in at least one direction) directed pseudograph without multiple arcs. Then D is isomorphic to $L(D)$ if and only if each of its vertices has in-degree 1 or each of its vertices has out-degree 1.*

Aigner [1] then gave a generalization of this result.

Theorem 11.2.8 ([1]) *Let D be a directed pseudograph without isolated vertices. Then D is isomorphic to $L(D)$, if and only if $D \cong D_1 \cup \ldots \cup D_k$, where the D_is are mutually vertex-disjoint and either D_i consists of a directed cycle and a number (possibly zero) of out-trees, each rooted at a vertex of this cycle or D_i is the converse of such a digraph.*

Harary and Norman [82] provided corresponding examples which show that these characterizations do not hold for general directed pseudographs. Therefore, finding a general characterization is still an open problem.

11.2.1 Connectivity

In most applications, connectivity plays a vital role. Thus, Aigner's [1] result that strong connectivity is preserved under the line digraph operator is particularly useful.

Theorem 11.2.9 ([1]) *Let D be a directed pseudograph without isolated vertices. Then D is strongly connected if and only if $L(D)$is strongly connected. Furthermore, $L(D)$ being unilateral implies D is unilateral.*

Several authors then noted the following (see, e.g., [171]).

Proposition 11.2.10 *Let D be a directed pseudograph without parallel arcs. Then,*

$$\kappa(L(D)) \; = \; \lambda(D).$$

Therefore, by the well-known fact that $\kappa(D) \leq \lambda(D) \leq \delta^0(D)$ for any directed pseudograph D without parallel arcs and Proposition 11.2.2, we have

$$\kappa(D) \leq \lambda(D) = \kappa(L(D)) \leq \lambda(L(D)) \leq \delta^0(L(D)) = \delta^0(D).$$

In other words, application of the line digraph operator can only increase the connectivity, which is one of the reasons it has been used in the construction of interconnection networks (see also the following section on iterated line digraphs). In this context, more refined connectivity concepts, as a measure of reliability, such as super connectivity, introduced by Bauer, Boesch, Suffel and Tindell [21], have been considered. A separator (cut) of a directed pseudograph is called **trivial** if its removal yields a strong component of order 1. In other words, all in-neighbours or all out-neighbours (or the corresponding arcs, respectively) of a vertex are contained in the separator (cut). A directed pseudograph D has **super (vertex-)connectivity** k if $\kappa(D) = k$ and every minimum separator is trivial. Analogously, a D has **super arc-connectivity** k if $\lambda(D) = k$ and every minimum cut is trivial. Obviously, super connectivity implies maximum fault tolerance, in some sense.

Although not every cut of D is a separator of $L(D)$, we still get the following natural-feeling result, due to Cheng, Du, Min, Ngo, Ruan, Sun and Wu [38], which was rediscovered by Zhang, Liu and Meng [171] with a more precise proof.

Proposition 11.2.11 ([38]) *Let D be a strongly connected directed pseudograph without parallel arcs. Then, D has super arc-connectivity k if and only if $L(D)$ has super connectivity k.*

Furthermore, Cheng, *et al.* [38] claimed that super arc-connectivity is preserved by the line digraph operator. Their proof is incorrect and the claim false, as, for example, the complete digraph on 3 vertices is super arc-connected, but its line digraph is not. However, Zhang, *et al.* [171] obtained a weaker version of the claim as a corollary of the following theorem.

Theorem 11.2.12 ([171]) *Let D be a strongly connected directed pseudograph without parallel arcs with $\delta^0(D) \geq 3$. Then, if $L(D)$ has super connectivity k, it also has super arc-connectivity k.*

Now, we simply combine Proposition 11.2.11 and Theorem 11.2.12.

Corollary 11.2.13 ([171]) *Let D be a strongly connected directed pseudograph without parallel arcs with $\delta^0(D) \geq 3$. If D has super arc-connectivity k, then $L(D)$ has super arc-connectivity k.*

Lü and Xu [115] and Zhang and Zhu [172] published results on even more refined connectivity measures for line digraphs.

11.2.2 Diameter

In the previous subsection we have seen that strong connectivity is preserved under the line digraph operator. As a consequence, it is natural to ask whether the distances between vertices increase drastically, since the number of vertices of the line digraph may possibly be up to almost the square of the order of the corresponding digraph. In spite of this fact, Aigner [1] was able to prove that the diameter increases by at most 1 under the line digraph operator.

Theorem 11.2.14 ([1]) *Let D be a strongly connected directed pseudograph. Then,*

$$\mathrm{diam}(L(D)) \;=\; \mathrm{diam}(D) + 1,$$

unless $D \cong L(D)$ (i.e., D is a directed cycle).

As we already know that the maximum semi-degree is also invariant, iterated application of the line digraph operator to the right digraphs is predestined to obtain digraphs of high order and comparatively small degree and diameter (cf. Sections 11.4 and 11.5 on de Bruijn and Kautz digraphs, respectively).

11.2.3 Kernels, Solutions and Generalizations

Another popular distance related concept are kernels of digraphs. Introduced in the context of game theory by von Neumann and Morgenstern [153], they have since found a wide array of applications in other fields.

A set N of vertices of a digraph D is called a **kernel** of D if N is independent in D and for every vertex $u \in V(D) \setminus N$, there is a vertex $v \in N$ such that $uv \in A(D)$. A **solution** of D is a kernel of the converse of D.

Since the introduction of the concept, several generalizations of kernels have been considered, many of which can be described as (k,l)-kernels. A set N of vertices of a digraph D is called a (k,l)**-kernel** of D if there is no oriented path of length shorter than k between any two distinct vertices of N in D and for every vertex $u \in V(D) \setminus N$, there is a directed path of length at most l from u to a vertex in N in D. Now, obviously, a $(2,1)$-kernel is a common kernel. Furthermore, a $(k,k-1)$-kernel is also called a k**-kernel**, a $(2,2)$-kernel is called a **quasi-kernel** and a $(k,2(k-1))$-kernel is called a k**-quasi-kernel**.

A (k,l)-semikernel is defined slightly differently. A set N of vertices of a digraph D is called a (k,l)**-semikernel** of D if there is no oriented path of length shorter than k between any two distinct vertices of N in D and for every vertex $u \in V(D) \setminus N$, if there is a directed path of length at most l from a vertex in N to u in D, then there is such a path from u to a vertex in N. A $(k,k-1)$-kernel is also called a k**-semikernel** and a 2-semikernel is also called a **semikernel**.

For all these generalized concepts of kernels, again, a corresponding version of a solution can be defined by considering the converse digraph.

Harminc [83] considered the correlation between solutions of a digraph and its line digraph and found the following.

Theorem 11.2.15 ([83]) *The cardinality of the system of all solutions of a digraph is equal to the cardinality of the system of all solutions of its line digraph.*

More precisely, for a digraph $D=(V,A)$, he proved that $f:\mathcal{K}\to\mathcal{K}',\ S\mapsto \{xy\in A \mid x\in S, y\in V\}$, where $\mathcal{K}$ and $\mathcal{K}'$ are the systems of all solutions of D and its line digraph, respectively, is an injective function. Conversely, $g:\mathcal{K}'\to\mathcal{K},\ H\mapsto X(H)\cup Y(H)$, where $X(H)$ is the set of all tails of arcs in H and $Y(H)$ consists of those vertices of D with out-degree 0 that are not adjacent to any vertices in $X(H)$, is also shown to be injective. Thus, we can easily obtain the kernels of $L(D)$ from the kernels of D and vice versa.

Proof: f **is well-defined**: Let R be a solution of $D=(V,A)$. Suppose that $ab\in A(L(D))$ for $a,b\in f(r)$. Then, by the definition of f, the tails of both a and b are contained in R and they are connected by the arc $a\in A$, a contradiction to the choice of R. Now, let $b\in A\setminus f(R)$. By the definition of f, the tail of b is not in R and is therefore dominated by some vertex of R in D via some arc $a\in f(R)$. Hence, b is dominated by a and, all in all, $f(R)$ is a solution of $L(D)$.

f **is injective**: Let R and S be distinct solutions of D. Without loss of generality, we may assume that there is a vertex $y\in R\backslash S$. Since S is a solution of D, there is a vertex $x\in S$ such that $xy\in A$ and therefore, $xy\in f(S)$. The independence of R implies that $xy\notin f(R)$. Hence, $f(R)\neq f(S)$.

g **is well-defined**: Let R be a solution of $D=(V,A)$. Suppose that there are vertices $x,y\in g(R)$ such that $xy\in A$. If $x\in Y(R)$ or $y\in Y(R)$, the definition of $Y(R)$ immediately implies a contradiction. Thus, we may assume that $x,y\in X(R)$. Consequently, x is the tail of some arc $a\in R$ and y is the tail of some arc $b\in R$. The independence of R implies $xy\notin R$, as xy and b are connected in $L(D)$. Hence, there exists a $c\in R$ that dominates xy in $L(D)$ and, by definition of the line digraph, also dominates $a\in R$, a contradiction to the choice of R. Now, let $y\in V\setminus g(R)$. If y is the head of some arc $b\in A$, then, by the definition of $g(R)$, $b\notin R$. Therefore, b is dominated by some $a=xy\in R$ and hence, y is dominated by $x\in X(R)\subseteq g(R)$. If y has out degree 0, by the definition of $g(R)$, y is dominated by some $x\in X(R)\subseteq g(R)$. All in all, $g(R)$ is a solution of D.

g **is injective**: Let R and S be distinct solutions of $L(D)$. Without loss of generality, we may assume that there is an arc $b = yz \in R \setminus S$. Therefore, $y \in X(R) \subseteq g(R)$. Since $b \notin S$, there is some arc $a = xy \in S$ that dominates b. As $x \in X(S) \subseteq g(S)$, the independence of $g(S)$ implies $y \notin g(S)$. Hence, $g(R) \neq g(S)$. □

As an obvious corollary, we have the following.

Corollary 11.2.16 ([83]) *A digraph has a solution if and only if its line digraph has a solution.*

The easily seen fact that the converse of $L(D)$ is the line digraph of the converse of the digraph D immediately implies the corresponding results on kernels.

Corollary 11.2.17 ([83]) *The cardinality of the system of all kernels of a digraph is equal to the cardinality of the system of all kernels of its line digraph.*

Corollary 11.2.18 ([83]) *A digraph has a kernel if and only if its line digraph has a kernel.*

Since then, utilizing Harminc's functions, several authors have found similar results for the various generalizations of kernels. Galeana-Sánchez, Ramírez and Rincón-Mejía [71] compared the number of semikernels and quasi-kernels of digraphs D with $\delta^-(D) \geq 1$ with the respective numbers of their line digraphs. Galeana-Sánchez and Li [70] proved that Corollary 11.2.18 also holds for semikernels, if $\delta^-(D) \geq 1$, which is a necessary condition, and studied the relationship between the number of (k, l)-kernels of certain digraphs and their line digraphs.

Galeana-Sánchez and Gómez [69] provided, amongst other results, a weaker version of 11.2.17 for (k, l)-semikernels of certain digraphs, with the use of state splittings.

Theorem 11.2.19 ([69]) *Let $k \geq 2$, $l \geq 2$ and let D be a digraph with $g(D) \geq k$ and $\delta^-(D) \geq 1$. Then, the cardinality of the system of all (k, l)-semikernels of D is less than or equal to the cardinality of the system of all (k, l)-semikernels of its line digraphs.*

Shan, Kang and Lu [142], found a generalization of Corollary 11.2.18 for k-semikernels of certain digraphs.

Theorem 11.2.20 ([142]) *Let D be a digraph with $g(D) \geq k \geq 2$ and $\delta^-(D) \geq 1$. Then, D has a k-semikernel if and only if its line digraph has a k-semikernel.*

Lu, Shan and Zhao [116] proved that Harminc's functions are also well-defined and injective on the respective sets of (k,l)-kernels of certain digraphs and thereby obtained the following generalizations of Corollaries 11.2.17 and 11.2.18.

Theorem 11.2.21 ([116]) *Let $k > l \geq 2$ and let D be a digraph with $g(D) \geq k$ and $\delta^-(D) \geq 1$. Then, the cardinality of the system of all (k,l)-kernels of D is equal to the cardinality of the system of all (k,l)-kernels of its line digraphs.*

Theorem 11.2.22 ([116]) *Let $k > l \geq 2$ and let D be a digraph with $g(D) \geq k$ and $\delta^-(D) \geq 1$. Then, D has a (k,l)-kernel if and only if its line digraph has a (k,l)-kernel.*

Some additional results on kernels and related concepts in generalized line digraphs have been found by Balbuena and Guevara [12] and Guevara, Balbuena and Galeana-Sánchez [76].

11.2.4 Branchings

Recall that an in-branching (also called a rooted spanning tree or an arborescence in the literature) is an oriented spanning tree with exactly one vertex (the root) of out-degree 0. For a vertex x of a directed pseudograph D, let $\mathrm{IB}_x(D)$ be the number of in-branchings of D rooted at x.

Knuth [103] proved the following correlation (in a different form) between in-branchings of a directed pseudograph and those of its line digraph algebraically, using Tutte's Matrix Tree Theorem [147]. Orlin [125] gave a combinatorial proof of the theorem in its present form.

Theorem 11.2.23 ([103]) *Let $D = (V, A)$ be a directed pseudograph without isolated vertices. Then,*

$$\mathrm{IB}_{xy}(L(D)) = \begin{cases} \mathrm{IB}_y(D) \cdot F, & \text{if } d^+(y) = 0 \text{ or } d^-(y) = 1 \\ d^+(y)^{-1}\, \mathrm{IB}_x(D) \cdot F, & \text{otherwise,} \end{cases}$$

where $F = \prod_{v \in V} d^+(v)^{d^-(v)-1}$.

Among other results, Levine [108] found a generating function identity for digraphs with minimum in-degree 1, which implies the following formula for the total number of in-branchings of a line digraph.

Corollary 11.2.24 ([108]) *Let $D = (V, A)$ be a directed pseudograph with $\delta^-(D) \geq 1$. Then, the number of in-branchings of $L(D)$ is*

$$b \cdot \prod_{v \in V} d^+(v)^{d^-(v)-1},$$

where b is the number of in-branchings of D.

Bidkhori and Kishore [30] found another proof of the result by constructing an explicit bijection. Furthermore, it can be extended to iterated line digraphs (see Corollary 11.3.9). The following identity due to Orlin [125] implies that Corollary 11.2.24 is also a corollary of Theorem 11.2.23 and also holds for directed pseudographs without isolated vertices.

Proposition 11.2.25 ([125]) *Let $D = (V, A)$ be a directed pseudograph without isolated vertices. Then, for each $y \in V$,*

$$d^+(y)\,\mathrm{IB}_y(D) \;=\; \sum_{x \in V} a_{xy}\,\mathrm{IB}_x(D),$$

where a_{xy} is the number of arcs from x to y in D.

Branchings are not only interesting from a theoretical point of view, but, particularly in line digraphs, as a model of interconnection networks, for their practical use in broadcasting algorithms, that is to say, sending a message from one vertex to all others in an efficient manner. In this context, the sheer number of branchings in a digraph is less important than the number of arc-disjoint or independent branchings (for fault-tolerance) and their depth, which is to say the length of a longest directed path between the root and a leaf (for efficiency). Two out-branchings of a directed pseudograph with root r are called **independent** if, for any vertex x, the unique paths from r to x are internally disjoint. Hasunuma and Nagamochi [85] studied both the number of independent out-branchings and their depths in line digraphs. Applying the following theorem, they were able to prove the well-known Independent Spanning Tree Conjecture (disproved in general by Huck [91]) for line digraphs.

Theorem 11.2.26 ([85]) *Let D be a directed pseudograph without parallel arcs and let r be a vertex of $L(D)$. Suppose that for any vertex $v \neq r$ of $L(D)$, there are k internally disjoint paths from r to v in $L(D)$. Then there are k independent out-branchings rooted at r of $L(D)$.*

Corollary 11.2.27 (Independent Spanning Tree Conjecture [85]) *Let D be a directed pseudograph without parallel arcs. If $L(D)$ is k-strong, then there are k independent out-branchings rooted at any vertex of $L(D)$.*

For considerations of the depth of independent out-branchings, see Theorems 11.3.10 and 11.3.11 in the section on iterated line digraphs.

Du, Lyuu and Hsu [51, 55, 56] introduced the related concept of spreads, prescribing a number of vertex-disjoint paths of certain maximum length between sets of vertices, to combine fault-tolerance and transmission delay considerations in interconnection networks and gave results on (iterated) line digraphs as an example of such networks.

Bermond, Munos and Marchetti-Spaccamela [28] proposed broadcasting algorithms for the (iterated) line digraph of a regular digraph D based on a broadcasting protocol for D.

11.2.5 Cycles and Trails

Aigner [1] was the first to notice the natural relation between Euler trails in a digraph and Hamiltonian cycles in its line digraph.

Theorem 11.2.28 ([1]) *Let D be a directed pseudograph without isolated vertices. Then, $L(D)$ is Hamiltonian if and only if D is Eulerian.*

The well-known characterization of Eulerian directed pseudographs and the definition of line digraphs lead to the following characterization of Eulerian line digraphs.

Theorem 11.2.29 *Let D be a strongly connected directed pseudograph. Then $L(D)$ is Eulerian if and only if $d^-_D(u) = d^+_D(v)$ for each arc uv of D.*

For line graphs of strongly connected regular directed pseudographs, Aardenne-Ehrenfest and de Bruijn [150] determined the number of Euler trails contained, a result that can also be derived from Corollary 11.2.24.

Theorem 11.2.30 ([150]) *Let D be a strongly connected d-regular directed pseudograph of order n. Then, the number of Euler trails of $L(D)$ is*

$$d^{-1}(d!)^{n(d-1)} \cdot t,$$

where t is the number of Euler trails of D.

Hasunuma and Otani [86] noted the following lower bound on the number of arc-disjoint Hamiltonian cycles in a regular line digraph.

Theorem 11.2.31 ([86]) *Let D be a strongly connected d-regular directed pseudograph without parallel arcs. Then there are $\lfloor d/2 \rfloor$ arc-disjoint Hamiltonian cycles in $L(D)$.*

As a generalization of pancyclicity (i.e. containing a cycle of every possible length), Imori, Matsumoto and Yamada [96] introduced the similar property of pancircularity. A directed pseudograph $D = (V, A)$ is called **pancircular** if it contains closed trails of length ℓ for all $3 \le \ell \le |A|$. As a first obvious result, they noted the following consequence of the fact that a cycle in $L(D)$ corresponds to a trail in D.

Proposition 11.2.32 ([96]) *A directed pseudograph is pancircular if and only if its line digraph is pancyclic.*

For regular directed pseudographs, they gave a stronger result.

Theorem 11.2.33 ([96]) *If a regular directed pseudograph is pancircular, then its line digraph is pancircular.*

Note that pancyclicity is not a sufficient condition in Theorem 11.2.33. Furthermore, it can be iterated (see Corollary 11.3.15).

11.2.6 $\mathcal{NP}$-Complete Problems for Line-Digraphs

The following results on $\mathcal{NP}$-completeness were published by Gavril [72]. He proved that several graph problems that are known to be $\mathcal{NP}$-complete on general (di)graphs (see, e.g., [98]), are still $\mathcal{NP}$-complete when restricted to line digraphs. The considered problems are the following.

Simple max cut **Parameter:** k
Input: An undirected graph $G = (V, E)$ and a positive integer k.
Question: Does there exist a set of vertices $S \subseteq V$ such that there are at least k edges between S and $V \setminus S$ in G?

Independent set **Parameter:** k
Input: A digraph $D = (V, A)$ and a positive integer k.
Question: Is there a set of vertices $S \subseteq V$ of size k such that no vertex in S dominates any other vertex in S?

vertex cover **Parameter:** k
Input: A digraph $D = (V, A)$ and a positive integer k.
Question: Is there a set of vertices $S \subseteq V$ of size at most k such that every vertex not in S either dominates or is dominated by a vertex in S?

Feedback vertex set **Parameter:** k
Input: A digraph $D = (V, A)$ and a positive integer k.
Question: Is there a set of vertices $S \subseteq V$ of size at most k such that $D - S$ is acyclic?

Feedback arc set **Parameter:** k
Input: A digraph $D = (V, A)$ and a positive integer k.
Question: Is there a set of arcs $F \subseteq A$ of size at most k such that $D - F$ is acyclic?

The reductions used below are partially based on private communication between Gavril and Knuth.

Lemma 11.2.34 ([72]) Simple Max Cut *is reducible to* Independent Set *for line digraphs.*

Proof: Given an undirected graph $G = (V, E)$ and a positive integer k, we consider the complete biorientation $D = \overleftrightarrow{G}$ of G obtained by replacing each edge $\{x, y\}$ of G with the pair xy, yx of arcs. Now, for a cut $(S, V \setminus S)$ of size at least k of G, the arc set $\{(x, y) \mid x \in S, y \in V \setminus S\}$ is an independent vertex set of size at least k in $L(D)$. Conversely, for an independent vertex set F

of order k of $L(D)$, let $S = \{x \in V \mid (x, y) \in F\}$. Since F is independent in $L(D)$, $y \in V \setminus S$ for all $(x, y) \in F$ and thus, $(S, V \setminus S)$ is a cut of size at least k of G. □

Lemma 11.2.35 ([72]) INDEPENDENT SET *for line digraphs is reducible to* VERTEX COVER *for line digraphs.*

Proof: A set of vertices is independent if and only if its complement is a vertex cover. □

Lemma 11.2.36 ([72]) FEEDBACK VERTEX SET *(FVS) for line digraphs is reducible to* FEEDBACK ARC SET *(FAS) for line digraphs.*

Proof: Let $D = (V, A)$ be a line digraph. By Theorem 11.2.3, there exist two partitions $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ of V such that $A = \cup_{i \in I} A_i \times B_i$. We define the digraph $D' = (V \times \{0, 1\}, A')$ through the partitions $\{A'_i\}_{i \in I'}$ and $\{B'_i\}_{i \in I'}$ of V', where $I' = I \cup V$, $A'_i = \{(x, 1) \mid x \in A_i\}$, $B_i = \{(y, 0) \mid y \in B_i\}$ for $i \in I$ and $A'_i = \{(i, 0)\}$, $B'_i = \{(i, 1)\}$ for $i \in V$, and the arc set $A' = \cup_{i \in I'} A'_i \times B'_i$. Obviously, D' is also a line digraph. Furthermore, a feedback vertex set S of D implies that $\{((x, 0), (x, 1)) \mid x \in S\}$ is a feedback arc set of D'. Conversely, a feedback arc set S of D' implies that $\{y \in V \mid ((x, i), (y, j)) \in S\}$ is a feedback vertex set of D. □

Lemma 11.2.37 ([72]) FEEDBACK ARC SET *(FAS) is reducible to* FEEDBACK VERTEX SET *(FVS) for line digraphs.*

Proof: It is easy to see that an arc set of a digraph is a feedback arc set if and only if it is a feedback vertex set of its line digraph. □

Summarizing the discussion above, we have shown the following.

Theorem 11.2.38 ([72]) INDEPENDENT SET*,* VERTEX COVER*,* FEEDBACK VERTEX SET*,* FEEDBACK ARC SET *are $\mathcal{NP}$-complete for line digraphs.*

Syslo [146] showed that the TRAVELLING SALESMAN PROBLEM (TSP) – the problem of finding a minimum weight Hamiltonian cycle in a weighted digraph – notorious for being $\mathcal{NP}$-complete in the general case, is solvable in polynomial time in terms of the size of the digraph, for line digraphs with constant arc weights.

By Theorem 11.2.3, we know that line digraphs can be recognized in polynomial time. In contrast, the problem of recognizing underlying graphs of line digraphs is $\mathcal{NP}$-complete, as Chvátal and Ebenegger [41] proved. Prisner [130] qualified the result by giving a polynomial-time algorithm to recognize underlying graphs of line digraphs with minimum semi-degree at least 2.

Poljak and Rödl [129] found that the problem of determining the chromatic number of a line digraph is $\mathcal{NP}$-complete.

11.2.7 Independence Number

Since the determination of the independence number of line digraphs is $\mathcal{NP}$-complete by Theorem 11.2.38, Lichiardopol [112] searched for and found an upper bound for the independence number of regular line digraphs.

Proposition 11.2.39 ([112]) *Let D be a d-regular directed pseudograph without parallel arcs, $d \geq 2$. Then,*

$$\alpha(L(D)) \;\leq\; \frac{|V(L(D))|}{2}.$$

He then went on to prove that the ratio can be obtained asymptotically for any regular line digraph, by iterated application of the line digraph operator (see Theorem 11.3.16).

11.2.8 Chromatic Number

As we have seen in Subsection 11.2.6, the exact determination of the chromatic number of a line digraph is $\mathcal{NP}$-complete. However, Harner and Entringer [84] gave bounds on the chromatic number of the line digraph of a digraph D in terms of the chromatic number $\chi(D)$ of D.

Theorem 11.2.40 ([84]) *Let D be a digraph. Then,*

$$\min\{t \mid \chi(D) \leq 2^t\} \;\leq\; \chi(L(D)) \;\leq\; \min\{t \mid \chi(D) \leq \binom{t}{\lfloor \frac{t}{2} \rfloor}\},$$

where the lower bound is sharp.

Iterated application of the line digraph operator eventually leads to a 3-colourable digraph (see Corollary 11.3.18). For more on the chromatic number of certain line digraphs, see the work of Ochem, Pinlou and Sopena [123, 124, 127, 128].

11.3 Iterated Line Digraphs

Since the 1980s, interconnection networks have attracted more and more attention. In their design, for varying technical reasons, it is interesting to find digraphs with certain attributes such as bounded maximum degree, small diameter and good connectivity. Early on, iterated line digraphs were recognized as a potential source to obtain digraphs of large order but fixed degree and diameter that also allow for easy routing, as Fiol, Yebra and Alegre [63] proved.

Iterated line digraphs are, as their name suggests, defined recursively. For some directed pseudograph D, the **first-order line digraph** $L^1(D)$ of

D is the line digraph of D. For an integer $k \geq 1$, the $(k+1)$th-order line digraph $L^{k+1}(D)$ of D is defined as the line digraph of $L^k(D)$. A directed pseudograph is called a **kth-order line digraph** if it is the kth-order line digraph of some directed pseudograph and it is called an iterated line digraph if it is a kth-order line digraph for some integer $k \geq 1$.

It is not difficult to prove by induction that $L^k(D)$ is isomorphic to the digraph Q whose vertex set consists of directed walks of D of length k and a vertex $v_0v_1 \dots v_k$ (which is a directed walk in D) dominates the vertex $v_1v_2 \dots v_kv_{k+1}$ for every $v_{k+1} \in V(D)$ such that $v_kv_{k+1} \in A(D)$. This fact allows for a new perspective that can be useful in proofs and is, for example, the basis for Fiol, Yebra and Alegre's [63] routing algorithm.

While Theorem 11.2.3 provides several concise characterizations of (first-order) line digraphs, the problem is more complicated for higher order iterated line digraphs. Hemminger [87] generalized condition (iii) from Theorem 11.2.3, which he called the **(first) Heuchenne condition**, in the following way. For a positive integer k, a directed pseudograph D satisfies the **kth Heuchenne condition** if, for any vertices $x, y, u, v \in V(D)$ such that there is a directed walk of length k from x to u, from y to u and from y to v, there is also a directed walk of length k from x to v. He then proposed that a directed pseudograph without multiple arcs is a kth-order line digraph if and only if it satisfies the first k Heuchenne conditions. He did not prove his statement, as, at first glance, it seemed to be obvious. Like several other such results on line digraphs, it turned out to be false. While it is true that it is a necessary condition, it is not sufficient, as Beineke and Zamfirescu [23] proved by constructing counterexamples.

They then set out to find further conditions to add to the kth Heuchenne condition to obtain a characterization of iterated line digraphs. With this approach, they were able to characterize the line digraphs that also are second-order line digraphs. Sadly, even for $k = 2$, the necessary conditions are much more complicated than for first-order line digraphs, which is why we will not consider them here in detail and why it seems unlikely that a characterization of kth-order line digraphs for $k > 2$ can be derived in a similar manner. This assumption is furthermore backed by an attempt by Beineke and Zamfirescu [23] to find a characterization of second-order line digraphs via forbidden subgraphs comparable to Corollary 11.2.4, which, again, needed rather complicated additional conditions that could not be stated in the form of forbidden subgraphs. Still, the problem of characterizing higher order iterated line digraphs, probably by different means, remains open.

To be able to give any sort of general characterization of higher order iterated line digraphs, in the following theorem, Beineke and Zamfirescu [23] considered only a restricted set of directed pseudographs. Their proof of the result given below is a nice example of the natural idea of using induction on the order of the iterated line digraph.

Theorem 11.3.1 ([23]) *Let D be a directed pseudograph without multiple arcs and vertices of in-degree or out-degree 0. Then D is a kth-order line digraph if and only if, for $i = 1, \ldots, k$, the following conditions are satisfied.*

(1) For any pair of vertices $x, y \in V(D)$, there is at most one directed walk of length i from x to y.
(2) D satisfies the ith Heuchenne condition.

Proof: We establish the sufficiency of these conditions using induction on k. The result holds for $k = 1$, and we assume it holds for $k = p$. Assume that D satisfies the hypotheses for $k = p + 1$. Then, by induction hypothesis, D is a line digraph. Let Q be a directed pseudograph such that $D = L(Q)$. Since the removal of isolated vertices in Q does not affect $L(Q)$, we may assume that Q contains no isolated vertices. Suppose that there is a vertex x of in-degree 0 in Q. Since x is not isolated, there is an arc $a \in A(Q)$ with x as its tail. Now a, as a vertex of $D = L(Q)$, has in-degree 0, a contradiction. Analogously, Q does not contain vertices of out-degree 0.

Suppose now that, for some $i \leq p$ and a pair of vertices $x, y \in V(Q)$, there are two distinct directed walks P_1 and P_2 of length i from x to y. As there is at least one arc $a \in A(Q)$ whose head coincides with x and one arc $b \in A(Q)$ whose tail coincides with y, P_1 and P_2 can be extended to distinct directed walks P_1' and P_2', respectively, of length $i + 2$, by appending both a and b. Consequently, P_1' and P_2' imply distinct directed walks of length $i+1 \leq p+1$ from a to b in $D = L(Q)$, a contradiction to the choice of D.

Finally, let $x, y, u, v \in V(Q)$ be vertices such that there are directed walks of length i from x to u, from y to u and from y to v. Again, we find arcs $a, b, c, d \in A(Q)$ such that the head a coincides with x, the head of b coincides with y, the tail of c coincides with u and the tail of d coincides with v. By appending these arcs to the appropriate directed walks of length i in Q, we find directed walks of length $i + 1 \leq p + 1$ from a to c, from b to c and from b to d in $D = L(Q)$. Since D satisfies the $(p + 1)$th Heuchenne condition, we also obtain a directed walk of length $i + 1$ from a to d in $D = L(Q)$. By the definition of the line digraph operator, a and d, said walk implies a directed walk of length i from x to v in Q and hence, Q satisfies the ith Heuchenne condition.

All in all, by induction hypothesis, Q is a pth-order line digraph and thus, $D = L(Q)$ is a $(p + 1)$th-order line digraph.

The proof of necessity can be derived from Beineke and Zamfirescu's proof [23] of their general characterization of second-order line digraphs. □

Using what they called coreflexive vertex sets, whose definition is tightly linked to the (iterated) Heuchenne condition, Liu and West [113] gave similar characterizations of (iterated) line digraphs, viewed from a new perspective.

Harary and Norman [82] considered the characteristics of high order iterated line digraphs.

Theorem 11.3.2 ([82]) *Let D be a directed pseudograph.*

(i) $L^k(D) = \emptyset$ for sufficiently large k if and only if D contains no directed cycles.
(ii) The order of $L^k(D)$ becomes arbitrarily large for sufficiently large k, if and only if D contains two directed cycles that are connected by a directed path.
(iii) If D contains two cycles, which are not connected by a directed path, then $L^k(D)$ is disconnected for sufficiently large k.

As a corollary, Hemminger and Beineke [88] noted the following.

Corollary 11.3.3 ([88]) *If D is a directed pseudograph such that $D \cong L^k(D)$ for some integer k, then D is a directed cycle and particularly $D \cong L(D)$.*

11.3.1 Connectivity

As mentioned in the previous section, refined connectivity concepts, as a measure of reliability, are of particular importance for interconnection networks and have therefore been studied for line digraphs, in particular. The well-known fact that $\kappa(D) \leq \lambda(D) \leq \delta^0(D)$ for any digraph D, for example, motivated the following definition. A strongly connected digraph D is **maximally connected** if $\kappa(D) = \lambda(D) = \delta^0(D)$. We have already seen in the last section that the line digraph operator does not decrease connectivity and therefore, line digraphs of maximally connected digraphs are again maximally connected. Fàbrega and Fiol [60] proved a stronger result for iterated line digraphs, using the following graph invariant. For a given digraph D, let $l(D)$ be the largest integer such that, for any two (not necessarily distinct) vertices $x, y \in V(D)$, (a) if $d(x, y) < l(D)$, the shortest path from x to y is unique and there is no such path of length $d(x, y) + 1$; (b) if $d(x, y) = l(D)$, there is only one shortest path from x to y. As a corollary of a more general result, they found that the kth-order line graph of any digraph with minimum semi-degree at least 2 is maximally connected, for k sufficiently large.

Theorem 11.3.4 ([60]) *Let D be a digraph with $\delta^0(D) > 1$. Then,*

(a) $\lambda(L^k(D)) = \delta^0(D)$ *if* $k \geq \mathrm{diam}(D) - 2l(D)$;
(b) $\kappa(L^k(D)) = \delta^0(D)$ *if* $k \geq \mathrm{diam}(D) - 2l(D) + 1$.

Fàbrega and Fiol [60] also proved a similar result on super connectivity.

Theorem 11.3.5 ([60]) *Let D be a digraph with $\delta^0(D) \geq 3$. Then,*

(a) $L^k(D)$ is super arc-connected if $k \geq \mathrm{diam}(D) - 2l(D) + 1$;
(b) $L^k(D)$ is super connected if $k \geq \mathrm{diam}(D) - 2l(D) + 2$.

As a corollary of Theorem 11.2.12 and Corollary 11.2.13, Zhang, Liu and Meng [171] obtained a related result.

Corollary 11.3.6 ([171]) *Let D be a strongly connected directed pseudograph without parallel arcs with $\delta^0(D) \geq 3$. If D is super arc-connected, then $L^k(D)$ is super connected and super arc-connected for any positive integer k.*

11.3.2 Diameter

As previously indicated, Theorem 11.2.14 directly implies an iterated version of the result.

Corollary 11.3.7 *Let D be a strongly connected directed pseudograph that is not a cycle. Then, for any positive integer k,*

$$\mathrm{diam}(L^k(D)) \;=\; \mathrm{diam}(D) + k.$$

11.3.3 Branchings

For the number $\mathrm{IB}(D)$ of in-branchings of a regular directed pseudograph D, Zhang, Zhang and Huang [170] gave the following formula.

Theorem 11.3.8 ([170]) *Let D be a d-regular digraph of order n. Then*

$$\mathrm{IB}(L^k(D)) \;=\; d^{(d^k-1)n} \cdot \mathrm{IB}(D).$$

Since line digraphs of d-regular directed pseudographs of order n are d-regular directed pseudographs of order $d^k n$, Theorem 11.3.8 is also an easy corollary of Corollary 11.2.24. Levine [108] was able to extend Corollary 11.2.24 to iterated line digraphs.

Corollary 11.3.9 ([108]) *Let $D = (V, A)$ be a directed pseudograph with $\delta^-(D) \geq 1$. Then,*

$$\mathrm{IB}(L^k(D)) \;=\; \mathrm{IB}(D) \cdot \prod_{v \in V} d^+(v)^{p(k,v)-1},$$

where $p(k, v)$ is the number of directed walks of length k that end in v.

Xu, Zhang, Ning and Li [163] extended Levine's results to directed pseudographs without isolated vertices.

As in the previous section, Hasunuma and Nagamochi [85] studied independent out-branchings of iterated line digraphs. Their proof of Corollary 11.2.27 can be applied iteratively to obtain a corresponding result on iterated line digraphs. But they were able to prove more.

Theorem 11.3.10 ([85]) *Let D be an l-strong directed pseudograph without parallel arcs such that $l < \delta^0(D)$. Let c be an upper bound on the depths of l arc-disjoint out-branchings rooted at any vertex of D. Then there are l independent out-branchings rooted at any vertex of depths at most $k+\log_2 k+c+1$ of $L^k(D)$ such that any vertex except for the root is contained in at most one tree as an internal vertex.*

Theorem 11.3.11 ([85]) *Let D be an l-strong directed pseudograph without parallel arcs such that $l = \delta^0(D) \geq 3$. Let c be an upper bound on the depths of l arc-disjoint out-branchings rooted at any vertex of D. Then there are l independent out-branchings rooted at any vertex of depths at most $k+\log_{\sqrt{3}} k+c+1$ of $L^k(D)$.*

11.3.4 (h, p)-Domination Number

Another concept used in fault-tolerance analysis of interconnection networks is (h, p)-domination. Let $D = (V, A)$ be a directed pseudograph and $S \subset V$. Then S is called an **(h, p)-domination set** if $D[S]$ is h-strong and $|(\{x\} \cup N^-(x)) \cap S| \geq p$ and $|(\{x\} \cup N^+(x)) \cap S| \geq p$ for every vertex $x \in V$. The **(h, p)-domination number** $\gamma_{h,p}(D)$ of D is the minimum cardinality of an (h, p)-domination set of D. (h, p)-domination has been studied for iterated line digraphs by Hasunuma and Otani [86]. Particularly interesting are their results on regular iterated line digraphs, which generalized several results for popular interconnection networks.

Theorem 11.3.12 ([86]) *Let D be a strong d-regular directed pseudograph without parallel arcs and $1 \leq p < d$. Then,*

$$\gamma_{h,p}(L^k(D)) \;=\; pd^{k-1}|V(D)|$$

for all $k \geq 2$ and $0 \leq h \leq \min\{p, \lfloor d/2 \rfloor\}$.

11.3.5 Cycles and Trails

Using Theorem 11.3.8, Zhang, Zhang and Huang [170] were able to calculate the number of Euler trails of regular iterated line digraphs.

Theorem 11.3.13 ([170]) *Let D be a strongly connected d-regular digraph of order n. Then, the number of Euler trails of $L^k(D)$ is*

$$\frac{(d!)^{nd^k}}{nd^{k+n}} \cdot b,$$

where b is the number of in-branchings of D.

By iteratively applying Theorem 11.2.30, we obtain the following corollary of it.

Corollary 11.3.14 *Let D be a strongly connected d-regular directed pseudograph of order n. Then, the number of Euler trails of $L^k(D)$ is*

$$d^{-k}(d!)^{(d^k-1)n} \cdot t,$$

where t is the number of Euler trails of D.

Analogous iteration of Proposition 11.2.32 and Theorem 11.2.33 produces the following corollary.

Corollary 11.3.15 ([96]) *If a regular directed pseudograph D is pancircular, then $L^k(D)$ is pancyclic and pancircular for any positive integer k.*

11.3.6 Independence Number

In addition to the upper bound in Proposition 11.2.39, Lichiardopol [112] also gave a (far more complicated) lower bound for the independence number of regular iterated line digraphs, which implies that approximately half of the vertices of a regular kth-order line digraph are contained in an independent set for k large enough.

Theorem 11.3.16 ([112]) *Let D be a d-regular directed pseudograph without parallel arcs, $d \geq 2$. Then,*

$$\lim_{k\to\infty} \frac{\alpha(L^k(D))}{|V(L^k(D))|} = \frac{1}{2}.$$

11.3.7 Chromatic Number

Duffus, Lefmann and Rödl [58] noted that the second-order line digraph of a 4-colourable digraph is 3-colourable, a result that can be generalized as follows.

Proposition 11.3.17 *Let D be a digraph with $\chi(D) \geq 4$. Then,*

$$\chi(L^2(D)) < \chi(D).$$

Proof: Let $c : V(D) \to \{1, \ldots, \chi(D)\}$ be a proper colouring of D. We then define a colouring $c' : V(L^2(D)) \to \{1, \ldots, \chi(D) - 1\}$ of $L^2(D)$. For $((u, v)(v, w)) \in V(L^2(D))$, let $c'(((u, v)(v, w))) = c(v)$, if $c(v) \neq \chi(D)$, and $c'(((u, v)(v, w))) = i$ for an arbitrary $i \in \{1, \ldots, \chi(D)\} \setminus \{c(u), c(v), c(w)\}$, otherwise. Suppose two adjacent vertices $((u, v)(v, w))$ and $((v, w)(w, x))$

of $L^2(D)$ receive the same colour. Since v and w are adjacent in D, we have $c(v) \neq c(w)$. Therefore, without loss of generality, we may assume that $c(v) \neq \chi(D) = c(w)$. Consequently, $c'(((u,v)(v,w))) = c(v)$ and $c'(((v,w)(w,x))) \in \{1,\ldots,\chi(D)\} \setminus \{c(v),c(w),c(x)\}$, a contradiction. Hence, c' is a proper colouring of $L^2(D)$. □

Proposition 11.3.17 implies that iterated lined digraphs of any digraph eventually become 3-colourable, a fact recognized by Prisner [131].

Corollary 11.3.18 ([131]) *Let D be a digraph. Then $\chi(L^k(D)) \leq 3$, for k sufficiently large.*

For a digraph with large chromatic number, by Theorem 11.2.40, the chromatic number of its iterated line digraphs decrease much faster than suggested by Proposition 11.3.17.

11.4 de Bruijn Digraphs

As previously mentioned, the line digraph operator has been found very useful in the design of interconnection networks because of its specific properties, which are particularly suitable for the following problem: Given positive integers n and d, construct a digraph D of order n and maximum out-degree at most d such that the diameter $\text{diam}(D)$ is as small as possible, while the vertex-strong connectivity $\kappa(D)$ is as large as possible. In general, such 2-objective optimization problems do not necessarily have admissible solutions. In this case, however, solutions which (almost) maximize/minimize both objective functions exist and can be constructed via the line digraph operator. They are presented in this and the following section.

For positive integers d and t, the **de Bruijn digraph** [48] $D_B(d,t)$ can be defined as the directed pseudograph whose vertices are all words of length t from an alphabet of d letters. There is an arc from a vertex x to a vertex y if and only if the last $t-1$ letters of x coincide with the first $t-1$ letters of y (see Figure 11.4). This definition bears a striking similarity to the alternative definition of iterated line digraphs we gave in the previous section. In fact, if K_d° is the complete digraph on d vertices with a loop at each vertex, then

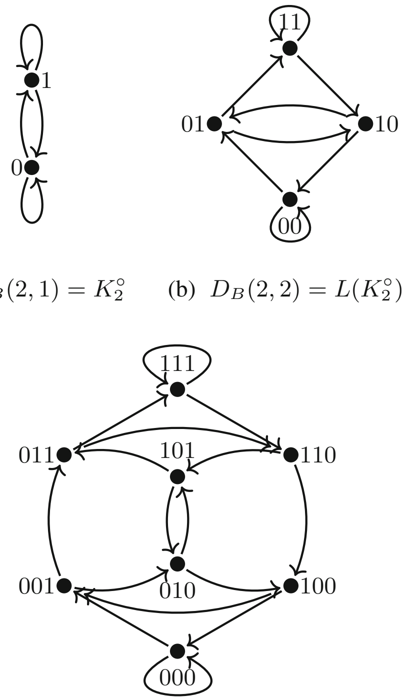
$$D_B(d,t) \;\cong\; L^{t-1}(K_d^\circ).$$

Therefore, all results on iterated line digraphs can be applied to de Bruijn digraphs and many of them have been proven for exactly that purpose. The following proposition is a collection of obvious consequences.

Proposition 11.4.1 *Let d and t be positive integers. Then the de Bruijn digraph $D_B(d,t)$:*

(a) has d^t vertices;

(b) has in- and out-degree d for every vertex (counting loops);
(c) has diameter t;
(d) has no parallel arcs;
(e) has a loop at exactly those vertices represented by repetitions of a single letter;
(f) has $\kappa(D_B(d,t)) = \lambda(D_B(d,t)) = d-1$.



(a) $D_B(2,1) = K_2^\circ$ (b) $D_B(2,2) = L(K_2^\circ)$

(c) $D_B(2,3) = L^2(K_2^\circ)$

Figure 11.4 Construction of de Bruijn digraphs via the line digraph operator.

11.4.1 Connectivity

As, in Proposition 11.4.1, we have seen that $\kappa(D_B(d,t)) = \lambda(D_B(d,t)) = d-1$, de Bruijn digraphs are almost maximally connected. The connectivity is obviously best possible for d-regular digraphs containing a loop.

Furthermore, Soneoka [143] proved that de Bruijn digraphs are super arc-connected by relating the order, degrees and diameter of a de Bruijn digraph.

Theorem 11.4.2 ([143]) *$D_B(d,t)$ is super arc-connected for all integers $d \geq 2$ and $t \geq 1$.*

Zhang, Liu and Meng [171] obtained the same from a result on iterated line digraphs (see Corollary 11.3.6). In the same manner, they were able to prove the super connectivity of de Bruijn digraphs.

Corollary 11.4.3 ([171]) *$D_B(d,t)$ is super connected for all integers $d \geq 2$ and $t \geq 1$.*

Lü and Xu [115] and Cheng, Du, Min, Ngo, Ruan, Sun and Wu [38] obtained the result as a corollary of their own results on iterated line digraphs.

11.4.2 Diameter

By Proposition 11.4.1, we know that $\mathrm{diam}(D_B(d,t)) = t$. The well-known Moore-bound states for any strongly connected digraph on n vertices with maximum out-degree d and diameter t that

$$n \;\leq\; 1 + d + d^2 + \ldots + d^t,$$

where Bridges and Toueg [34] proved that equality is not attained unless $t = 1$ or $d = 1$. The corresponding values for de Bruijn digraphs given in Proposition 11.4.1 now imply the following.

Proposition 11.4.4 *For all positive integers d and t, the de Bruijn digraph $D_B(d,t)$ achieves the minimum value t of diameter for directed pseudographs of order d^t and maximum out-degree at most d.*

Furthermore, Imase, Soneoka and Okada [94] noted that the diameter of de Bruijn digraphs is fairly robust against deletion of vertices and/or arcs. They proved that, in $D_B(d,t)$, the diameter increases by at most one if fewer than $d-1$ vertices or arcs are deleted. To prove this result we will use the following lemma.

Lemma 11.4.5 ([16]) *Let d and t be positive integers and let x and y be distinct vertices of $D_B(d,t)$ such that $x \to y$. Then, there are $d-2$ internally disjoint (x,y)-paths different from xy, each of length at most $t+1$.*

Proof: Let $x = (x_1, x_2, \ldots, x_t)$ and $y = (x_2, \ldots, x_t, y_t)$. Consider the walk W_k given by $W_k = (x_1, x_2, \ldots, x_t), (x_2, \ldots, x_t, k), (x_3, \ldots, x_t, k, x_2), \ldots, (k, x_2, \ldots, x_t), (x_2, \ldots, x_t, y_t)$, where $k \neq x_1, y_t$. For each k, every internal vertex of W_k has coordinates forming the same multiset $M_k = \{x_2, \ldots, x_t, k\}$. Since for different k, the multisets M_k are different, the walks W_k are internally disjoint. Each of these walks is of length $t+1$. Therefore, $D_B(d,t)$ contains $d-2$ internally disjoint (x,y)-paths P_k with $A(P_k) \subseteq A(W_k)$. Since $k \neq x_1, y_t$, we may form the paths P_k such that none of them coincides with xy. □

The result, due to Imase, Soneoka and Okada [94], now states the following.

Theorem 11.4.6 *[94] For all positive integers d and t, from any vertex to any other in $D_B(d,t)$, there are at least $d-1$ internally-disjoint paths, one of which has length at most t, and $d-2$ have length at most $t+1$.*

Proof: By induction on $t \geq 1$. Clearly, the claim holds for $t = 1$ since $D_B(d,1)$ contains, as spanning subdigraph, $\overset{\leftrightarrow}{K}_d$. For $t \geq 2$, we know that

$$D_B(d,t) = L(D_B(d,t-1)). \tag{11.1}$$

Let x, y be a pair of distinct vertices in $D_B(d,t)$ and let e_x, e_y be the arcs of $D_B(d,t-1)$ corresponding to vertices x, y due to (11.1). Let u be the head of e_x and let v be the tail of e_y.

If $u \neq v$, by the induction hypothesis, $D_B(d,t-1)$ has $d-1$ internally disjoint (u,v)-paths, one of length at most $t-1$ and the others of length at most t. The arcs of these paths together with arcs e_x and e_y correspond to $d-1$ internally disjoint (x,y)-paths in $D_B(d,t)$, one of length at most t and the others of length at most $t+1$.

If $u = v$, we have $x \to y$ in $D_B(d,t-1)$. It suffices to apply Lemma 11.4.5 to see that there are $d-1$ internally disjoint (x,y)-paths in $D_B(d,t)$, one of length one and the others of length at most $t+1$. □

11.4.3 Branchings

Zhang and Lin [169] calculated the total number of in-branchings of de Bruijn digraphs.

Theorem 11.4.7 *[169] For all positive integers d and t, the number of in-branchings of $D_B(d,t)$ is*

$$d^{d^t-1}.$$

Bermond and Fraigniaud [26] and Ge and Hakimi [73] both found $d-1$ independent out-branchings rooted at any vertex of $D_B(d,t)$, while the latter group gave the better estimation of their depths.

Theorem 11.4.8 *[73] For all positive integers d and t, in $D_B(d,t)$, there are $d-1$ independent out-branchings rooted at any vertex of depths at most $\lceil 3t/2 \rceil$.*

As a corollary of Theorem 11.3.10, Hasunuma and Nagamochi [85] obtained the following result.

Corollary 11.4.9 *[85] For all positive integers d and $t \geq 2$, in $D_B(d,t)$, there are $d-1$ independent out-branchings rooted at any vertex of depths at most $t + \log_2(t-1) + 1$ such that any vertex except for the root is contained in at most one tree as an internal vertex.*

11.4.4 (h, p)-Domination Number

As an application of Theorem 11.3.12, Hasunuma and Otani [86] calculated the (h, p)-domination number for certain de Bruijn digraphs.

Theorem 11.4.10 *[86] Let d and p be integers such that $d \geq 2$ and $1 \leq p < d$. Then,*

$$\gamma_{h,p}(D_B(d,t)) \;=\; pd^{t-1}$$

for all $t \geq 3$ and $0 \leq h \leq \min\{p, \lfloor d/2 \rfloor\}$.

11.4.5 Cycles and Trails

Imori, Matsumoto and Yamada [96] obtained the pancyclicity and pancircularity of de Bruijn digraphs as a corollary of their work on iterated line digraphs (see Corollary 11.3.15).

Corollary 11.4.11 *[96] For all positive integers d and t, $D_B(d,t)$ is pancyclic and pancircular.*

Due to Zhang and Lin [169] and, via different method, Zhang, Zhang and Huang [170], we know the exact number of Euler trails contained in de Bruijn digraphs.

Theorem 11.4.12 *[169] For all positive integers d and t, the number of Euler trails of $D_B(d,t)$ is*

$$(d!)^{d^t} d^{-t-1}.$$

Generalizations of de Bruijn digraphs such as **generalized de Bruijn digraphs**, introduced independently by Imase and Itoh [93] and Reddy, Pradhan and Kuhl [134], and **consecutive-d digraphs** suggested by Du, Hsu and Hwang [50] share many of their desirable properties (see, e.g., [36, 49, 53, 54, 95]).

11.5 Kautz Digraphs

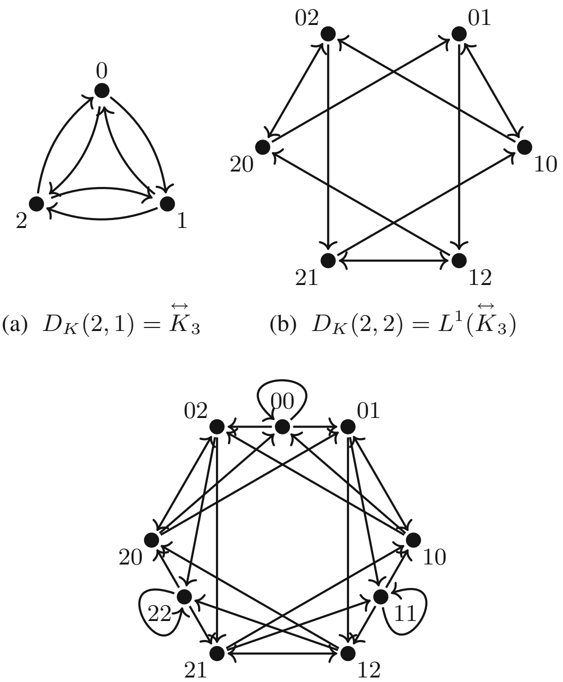
For positive integers d and t, the **Kautz digraph** [99] $D_K(d,t)$ can be obtained from the de Bruijn digraph $D_B(d+1,t)$ by deleting all vertices representing words containing two consecutive identical letters (see Figure 11.5). In particular, Kautz digraphs do not contain loops and are therefore actual digraphs. Fiol, Yebra and Alegre [63] noted that Kautz digraphs, just as de Bruijn digraphs, can be described as iterated line digraphs,

$$D_K(d,t) \;\cong\; L^{t-1}(\overleftrightarrow{K}_{d+1}),$$

where $\overleftrightarrow{K}_{d+1}$ is the complete digraph on $d+1$ vertices. And just as with de Bruijn digraphs, this fact is a widely-used tool in proofs on Kautz digraphs. For example, the following proposition is easily deduced.

Proposition 11.5.1 *Let d and t be positive integers. Then the Kautz digraph $D_K(d,t)$:*

(a) has $d^t + d^{t-1}$ vertices;
(b) has in- and out-degree d for every vertex;
(c) has diameter t.



(a) $D_K(2,1) = \overleftrightarrow{K}_3$ (b) $D_K(2,2) = L^1(\overleftrightarrow{K}_3)$

(c) $D_B(3,2) = L^1(K_3^\circ)$

Figure 11.5 Construction of a Kautz digraph via the line digraph operator or from a de Bruijn digraph.

11.5.1 Connectivity

Reddy, Kuhl, Hosseini and Lee [133], as well as Fàbrega, Fiol and Yebra [61] and Imase, Soneoka and Okada [95] independently noted that Kautz digraphs are maximally connected, which is implied by corresponding results on iterated line digraphs.

Theorem 11.5.2 ([133]) *$D_K(d,t)$ is maximally connected, i.e. $\kappa(D_K(d,t)) = d$.*

In a sense, this result suggests that Kautz digraphs are better than de Bruijn digraphs.

Fàbrega and Fiol [60] obtained the super connectivity and super arc-connectivity of Kautz digraphs as a corollary of their more general results on iterated line digraphs (see Theorem 11.3.5).

Corollary 11.5.3 ([60]) *$D_K(d,t)$ is super connected and super arc-connected for all integers $d \geq 3$ and $t \geq 2$.*

Soneoka [143] independently proved the super arc-connectivity of Kautz digraphs by relating the order, degrees and diameter of a Kautz digraph. Furthermore, Zhang, Liu and Meng [171] and Lü and Xu [115] realized that super connectivity and super arc-connectivity of Kautz digraphs follows from their respective results on iterated line digraphs.

11.5.2 Diameter

By the same reasoning as for Proposition 11.4.4, Reddy, Kuhl, Hosseini and Lee [133] noted that the diameter of Kautz digraphs is minimum for digraphs of their order and degree, making them a solution of the optimization problem mentioned at the beginning of the previous section.

Proposition 11.5.4 *For all positive integers d and t, the Kautz digraph $D_K(d,t)$ achieves the minimum value t of diameter for directed pseudographs of order $d^t + d^{t-1}$ and maximum out-degree at most d.*

Du, Hsu and Lyuu [52] improved the results on diameter vulnerabilities due to Reddy, Kuhl, Hosseini and Lee [133] and Imase, Soneoka and Okada [94].

Theorem 11.5.5 ([52]) *For all positive integers d and t, from any vertex to any other in $D_K(d,t)$, there are at least d internally-disjoint paths, one of which has length at most t, $d-2$ have length at most $t+1$ and one has length at most $t+2$.*

Furthermore, they determined that, in the worst case, the diameter of $D_K(d,t)$ increases by 1, if fewer than $d-1$ vertices are deleted, and by 2, if $d-1$ vertices are deleted, thereby proving their result to be best possible.

11.5.3 Branchings

As an application of Theorem 11.3.13, Zhang, Zhang and Huang [170] gave the number of in-branchings of a Kautz digraph.

Corollary 11.5.6 ([170]) *For all positive integers d and t, the number of in-branchings of $D_K(d,t)$ is*

$$d^{(d+1)d^{t-1}-d-1}(d+1)^d.$$

Just as for de Bruijn digraphs, Ge and Hakimi [73] found the maximum possible number, d, of independent out-branchings rooted at any vertex of $D_K(d,t)$.

Theorem 11.5.7 ([73]) *For all positive integers d and t, in $D_K(d,t)$, there are d independent out-branchings rooted at any vertex of depths at most $\lceil 3t/2 \rceil + 1$.*

As a corollary of Theorem 11.3.11, Hasunuma and Nagamochi [85] obtained the following result.

Corollary 11.5.8 ([85]) *For all positive integers d and $t \geq 2$, in $D_K(d,t)$, there are d independent out-branchings rooted at any vertex of depths at most $t + \log_b t + 1$, where $b = (1+\sqrt{5})/2$, if $d = 2$, and $b = \sqrt{3}$, if $d \geq 3$.*

11.5.4 (h,p)-Domination Number

Hasunuma and Otani [86] used Theorem 11.3.12 to give the (h,p)-domination number for certain Kautz digraphs.

Corollary 11.5.9 ([86]) *Let d and p be integers such that $d \geq 2$ and $1 \leq p < d$. Then,*

$$\gamma_{h,p}(D_K(d,t)) \;=\; p(d^{t-1} + d^{t-2})$$

for all $t \geq 3$ and $0 \leq h \leq \min\{p, \lfloor d/2 \rfloor\}$.

11.5.5 Cycles and Trails

As a consequence of their work on iterated line digraphs, Imori, Matsumoto and Yamada [96] obtained the pancyclicity and pancircularity of Kautz digraphs.

Corollary 11.5.10 ([96]) *For all positive integers d and t, $D_K(d,t)$ is pancyclic and pancircular.*

Zhang, Zhang and Huang [170] calculated the number of Euler trails of Kautz digraphs.

Theorem 11.5.11 ([170]) *For all positive integers d and t, the number of Euler trails of $D_K(d,t)$ is*

$$(d!)^{(d+1)d^{t-1}} d^{-d-t}(d+1)^{d-1}.$$

Generalizations of Kautz digraphs such as **Imase–Itoh digraphs**, introduced by Imase and Itoh [92], and **consecutive-d digraphs** suggested by Du, Hsu and Hwang [50] share many of their desirable properties (see, e.g., [36, 49, 53, 54]).

11.6 Directed Cographs

A **series-parallel partial order** is a partially ordered set $(X, \leq)$ that can be constructed from a single element using the **series composition** and the **parallel composition** operation. For two disjoint series-parallel partial orders $(X_1, \leq)$ and $(X_2, \leq)$, distinct elements $x, y \in X_1 \cup X_2$ of the series composition have the same order they have in X_1 or X_2, respectively, if they are both from the same set, and $x \leq y$, if $x \in X_1$ and $y \in X_2$. Elements $x, y \in X_1 \cup X_2$ of the parallel composition are comparable if and only if they are both in X_1 or both in X_2, and then retain their corresponding order.

A **series-parallel partial order digraph** is a digraph whose vertex set is a series-parallel partial order $(V, \leq)$ and $x \to y$ if and only if $x \neq y$ and $x \leq y$. More commonly, series-parallel partial orders are represented by **(vertex) series-parallel digraphs**, which can be defined as exactly those digraphs whose transitive closure is a series-parallel partial order digraph, i.e. $x \leq y$, if and only if there is an (x, y)-path in the corresponding series-parallel digraph. For some applications it might be desirable to use a particularly sparse representation. A **minimal series-parallel digraph** is a series-parallel digraph for which the removal of any arc alters its transitive closure. Valdes, Tarjan and Lawler [149] defined minimal series-parallel digraphs recursively: The trivial digraph is minimal series-parallel. For two vertex-disjoint minimal series-parallel digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$, $P = (V_1 \cup V_1, A_1 \cup A_2)$ is a minimal series-parallel digraph. Furthermore, if $O_1 \subseteq V_1$ is the set of vertices of out-degree zero in D_1 and $I_2 \subseteq V_2$ is the set of vertices of in-degree zero in D_2, then $S = (V_1 \cup V_1, A_1 \cup A_2 \cup (O_1 \times I_2))$ is a minimal series-parallel digraph. Based on this definition, they defined series-parallel digraphs as exactly those digraphs whose transitive closure equals the transitive closure of a minimal series-parallel digraph.

Among other results, Valdes, Tarjan and Lawler [149] gave a forbidden subdigraph characterization of series-parallel digraphs using the following definition. A digraph is called N-free if it does not contain an induced subdigraph on four vertices $\{u, v, w, x\}$ with the arc set $\{vw, uw, ux\}$.

Theorem 11.6.1 ([148, 149]) *An acyclic digraph is series-parallel, if and only if its transitive closure is N-free.*

Note that the N-free property is fairly reminiscent of the Heuchenne condition in the characterization of line digraphs (cf. Theorem 11.2.3 (iii)). In particular, Theorem 11.6.1 implies that transitive acyclic line digraphs are series-parallel partial order digraphs.

Valdes, Tarjan and Lawler [149] also found a connection in the opposite direction.

Theorem 11.6.2 ([149]) *Every minimal series-parallel digraph is a line digraph.*

In fact, they were able to characterize those directed pseudographs whose line digraphs are minimal series-parallel digraphs, which they used in a linear-time recognition algorithm for series-parallel digraphs.

For further results and applications, see, e.g., the work of Monma and Sidney [119], Lawler [107], Baffi and Petreschi [11], Bertolazzi, Cohen, Di Battista, Tamassia and Tollis [29], Rendl [135], Steiner [144] and Möhring [117].

A **cograph**, short for **complement-reducible graph**, is an undirected graph that, like series-parallel partial order digraphs, can be defined recursively: The trivial graph is a cograph. The complement of a cograph is a cograph. And finally, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are vertex-disjoint cographs, so is their disjoint union $(V_1 \cup V_2, E_1 \cup E_2)$. There are several further equivalent characterizations of cographs. Particularly, Jung [97] showed that cographs are comparability graphs of series-parallel partial orders $(X, \leq)$, i.e. the graph that contains an edge xy between distinct vertices $x, y \in X$ if and only if $x \leq y$ or $y \leq x$. In other words, if we consider a graph to be a symmetric digraph (i.e. each edge is represented by a directed 2-cycle), then cographs can be defined as the family of digraphs that contains the trivial digraph and is closed under the operations of **disjoint union** and **series**, where, for h disjoint digraphs $D_1, \ldots, D_h$, the disjoint union of $D_1, \ldots, D_h$ is the digraph on the vertex set $\bigcup_{1 \leq i \leq h} V(D_i)$ and the arc set $\bigcup_{1 \leq i \leq h} A(D_i)$, while the series composition of $D_1, \ldots, D_h$ is obtained from the disjoint union by adding all possible arcs between vertices of distinct D_i.

Like series-parallel digraphs, cographs can be recognized in linear-time. Corresponding algorithms have been found, e.g., by Corneil, Perl and Stewart [42] and Bretscher, Corneil, Habib and Paul [33].

Finally, we arrive at the eponym of this section. **Directed cographs** generalize both series parallel partial order digraphs and cographs, which is obvious by their recursive definition: The trivial digraph is a directed cograph. Both the disjoint union and the series composition of disjoint directed cographs are directed cographs. Additionally, the **order composition** of h disjoint digraphs $D_1, \ldots, D_h$, which is obtained from the disjoint union by adding the arcs from vertices in D_i to vertices in D_j if and only if $1 \leq i < j \leq h$.

Consistent with the definition of symmetric digraphs, we call an arc $xy \in A(D)$ **symmetric** if $yx \in A(D)$. Otherwise, we call it **asymmetric**.

The **symmetric part** sym(D) of a digraph D is the spanning subdigraph containing exactly the symmetric arcs of D. The **asymmetric part** asym(D) is defined analogously.

Then, a result due to Bechet, de Groote and Retoré [22] implies that the asymmetric part of a directed cograph is a series-parallel partial order digraph and the symmetric part is a cograph. Furthermore, Crespelle and Paul [43] noted that a forbidden subdigraph characterization can be derived from a result due to Ehrenfeucht and Rozenberg [59].

Theorem 11.6.3 ([43]) *A digraph is a directed cograph if and only if it does not contain any of the (connected) digraphs depicted in Figure 11.6 as an induced subdigraph.*

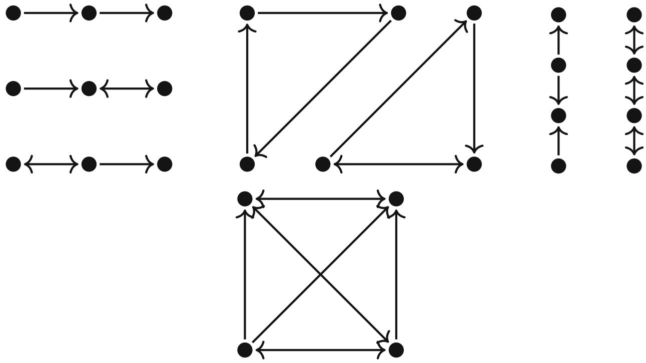


Figure 11.6 Forbidden subdigraphs for directed cographs.

Consequently, the class of directed cographs is hereditary (that is, an induced subdigraph of a directed cograph is a directed cograph) and closed under complementation. Furthermore, by results due to Möhring and Radermacher [118], directed cographs have a unique representation as a modular decomposition tree, also called a **cotree**. The leaves of the cotree are labelled with the vertices of the directed cograph, while the inner nodes are labelled with the respective operation (disjoint union, series, order) connecting its children (see Figure 11.7).

Using the cotree representation, Crespelle and Paul [43] obtained an optimal algorithm for the Dynamic Recognition and Representation Problem for directed cographs. The input of the problem is a directed cograph with its cotree representation and a series of modifications of the following form: adding/deleting a vertex and its incident arcs or adding/deleting an arc or two symmetric arcs, where all modifications must be valid, i.e., a vertex/arc to be deleted must exist, one to be added must not. If the resulting digraph is again a directed cograph, the algorithm provides its representation, if not, it provides a certificate of that fact, i.e. a forbidden subdigraph.

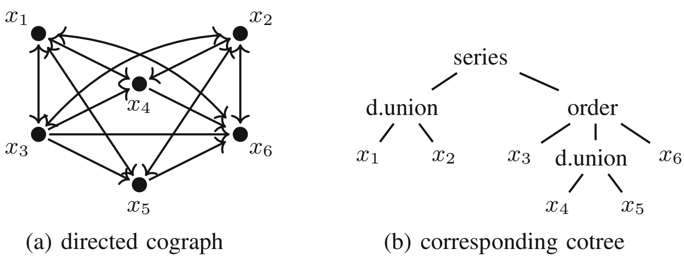


Figure 11.7 Cotree representation of a directed cograph.

Theorem 11.6.4 ([43]) *The* DYNAMIC RECOGNITION AND REPRESENTATION PROBLEM *for directed cographs is solvable in $O(d)$ worst-case time per update, where d is the number of arcs involved in the updating operation. Moreover, if needed, a certificate that the modified digraph is not a directed cograph is provided within the same time complexity.*

For another problem that is solvable in polynomial time, we turn to Bang-Jensen and Maddaloni [19], who considered the WEAK k-LINKAGE PROBLEM for directed cographs.

WEAK k-LINKAGE **Input:** A digraph $D = (V, A)$ and not necessarily distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$. **Question:** Does D contain a weak-k-linkage from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$?

In fact, they proved that the WEAK k-LINKAGE problem is solvable in polynomial time for fixed k for totally Φ-decomposable digraphs, for certain digraph classes Φ.

A digraph D is **totally Φ-decomposable** if either $D \in \Phi$ or $D = P[T_1, \ldots, T_h]$ is composed of a digraph $P \in \Phi$ and pairwise vertex-disjoint totally Φ-decomposable digraphs $T_1, \ldots, T_h$. The recursive definition of totally Φ-decomposable digraphs is valuable in the construction of polynomial algorithms. Of course, the choice of the underlying digraph class Φ is important. It should be chosen large enough as to assure a rich class of totally Φ-decomposable digraphs, while restricted enough to still allow for polynomial algorithms for important problems in Φ itself.

One promising class, Φ_1, was introduced by Bang-Jensen and Gutin [17], who, among other results, proved that totally Φ_1-decomposable digraphs are recognizable in polynomial time, another desirable property.

Φ_1 is the union of all semicomplete bipartite digraphs, all connected extended locally semicomplete digraphs and all acyclic digraphs.

The following result is a special case of a broader one due to Bang-Jensen and Maddaloni [19].

Theorem 11.6.5 ([19]) *For every fixed k there exists a polynomial algorithm for the* WEAK k-LINKAGE PROBLEM *for the totally* Φ_1*-decomposable digraphs.*

All that remains is to realize that directed cographs are in fact totally Φ_1-decomposable digraphs, which is fairly obvious by their recursive definition. The trivial digraph (the initial directed cograph), arcless digraphs (realizing disjoint unions) and transitive tournaments (realizing the order composition) are all acyclic, while complete digraphs (realizing the series composition) are particularly connected locally semicomplete digraphs. Thus, we obtain the following corollary.

Corollary 11.6.6 ([19]) *For every fixed k there exists a polynomial algorithm for the* WEAK k-LINKAGE PROBLEM *for directed cographs.*

For more results on totally Φ-decomposable digraphs, see Chapter 8 and for an application of directed cographs in mathematical logic, we refer to the work of Retoré [136].

11.7 Perfect Digraphs

First, recall that an undirected graph is called perfect if the chromatic number of every induced subgraph equals its clique number. This property is particularly interesting for its impact on complexity results, as several well-known $\mathcal{NP}$-complete problems, such as the determination of the chromatic number, the clique number or the independence number of a graph, are solvable in polynomial time for perfect graphs (cf. Grötschel, Lovász and Schrijver [75]). Furthermore, the results are actually applicable in practice, since several common graph classes, such as bipartite graphs, chordal graphs, triangulated graphs, interval graphs and comparability graphs, are perfect.

The long-standing Strong Perfect Graph Conjecture, due to Berge [24], after inspiring generations to an array of related research, was finally proven after more than four decades by Chudnovsky, Robertson, Seymour and Thomas [40] and is now known as the Strong Perfect Graph Theorem. It states that a graph is perfect if and only if it contains neither odd holes nor odd antiholes as induced subgraphs, where an odd hole is an induced cycle of odd length at least 5 and an odd antihole is the complement of such a graph. Combined with the corresponding result of Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [39] for graphs without odd holes and odd antiholes, the Strong Perfect Graph Theorem furthermore implies that perfect graphs can be recognized in polynomial time.

Motivated by this breakthrough for undirected perfect graphs, Andres and Hochstättler [9] introduced the class of perfect digraphs and, among other results, gave a Strong Perfect Digraph Theorem. The following additional notation is needed. Particularly, in the context of this book, it has to be pointed out that, in their definition of perfect digraphs, instead of the chromatic number for digraphs, as introduced in the first chapter, Andres and Hochstättler used the dichromatic number, as introduced by Neumann-Lara [122]: A k-dicolouring of a digraph is vertex-colouring with k colours such that no directed cycle is monochromatic. The **dichromatic number** $\vec{\chi}(D)$ of a digraph D is the smallest positive integer k such that D admits a k-dicolouring. The **clique number** $\omega(D)$ of a digraph D is the order of a largest complete subdigraph of D. Now, a digraph is called **perfect** if, for any induced subdigraph, the dichromatic number equals its clique number. Note that a graph is perfect if and only if its complete biorientation (where every edge is replaced by a pair of opposing arcs) is perfect. Therefore, the given concept is a natural extension of perfectness to digraphs.

Recall that the **symmetric part** $\text{sym}(D)$ of a digraph D is the spanning subdigraph containing exactly the symmetric arcs of D (see Figure 11.8(b)). The **asymmetric part** $\text{asym}(D)$ is defined analogously (see 11.8(c)). Now we can state the Strong Perfect Digraph Theorem due to Andres and Hochstättler [9] and give their proof.

Theorem 11.7.1 (Strong Perfect Digraph Theorem [9]) *A digraph D is perfect if and only if* $\text{sym}(D)$ *(identified with the corresponding undirected graph) is perfect and D does not contain any directed cycle of length at least* 3 *as an induced subdigraph.*

Proof: Assume that $\text{sym}(D)$ is not perfect. Then there is an induced subgraph $G = (V, E)$ of $\text{sym}(D)$ (identified with the corresponding undirected graph) with $\omega(G) < \chi(G)$. Since $\omega(D\langle V\rangle) = \omega(\text{sym}(D\langle V\rangle))$ and $\text{sym}(D\langle V\rangle)$ is the complete biorientation of G, we conclude that

$$\omega(D\langle V\rangle) = \omega(\text{sym}(D\langle V\rangle)) = \omega(G) < \chi(G) = \chi(\text{sym}(D\langle V\rangle)) \leq \chi(D\langle V\rangle).$$

Therefore, D is not perfect.

If D contains a directed cycle C of length at least 3 as an induced subdigraph, then D is obviously not perfect, since $\omega(D) = 1 < 2 = \chi(C)$.

Now, assume that $\text{sym}(D)$ is perfect, but D is not. It suffices to show that D contains a directed cycle of length at least 3 as an induced subdigraph. Let $D' = (V', A')$ be an induced subdigraph of D such that $\omega(D') < \chi(D')$. As $\text{sym}(D)$ is perfect, there is a k-dicolouring of $\text{sym}(D') = \text{sym}(D)\langle V'\rangle$ with $k = \omega(\text{sym}(D')) = \omega(D')$ colours. By choice of D', this cannot be a k-dicolouring of D'. Hence, there is a (not necessarily induced) monochromatic directed cycle C of minimal length in $\text{asym}(D')$, which automatically implies its length to be at least 3. C cannot have a symmetric chord, since its head and tail would receive the same colour, in contradiction to the definition of a k-dicolouring of $\text{sym}(D')$. By minimality, C cannot have an asymmetric arc. Therefore, C is an induced directed cycle of length at least 3 in D', and thus in D. □

Applying the Strong Perfect Graph Theorem, Theorem 11.7.1 can be restated without undirected perfectness, using the following terminology. A **filled odd hole** is a digraph D such that $\mathrm{sym}(D)$ is the complete biorientation of an odd hole. A **filled odd antihole** is defined analogously.

Corollary 11.7.2 ([9]) *A digraph D is perfect if and only if it does not contain filled odd holes, filled odd antiholes, or any directed cycle of length at least* 3 *as induced subdigraphs.*

Furthermore, Theorem 11.7.1 implies that, for perfect digraphs, the symmetric part determines the validity of a k-dicolouring.

Corollary 11.7.3 ([9]) *If D is a perfect digraph, then every k-dicolouring of* $\mathrm{sym}(D)$ *is a k-dicolouring of D.*

Since the maximum order of an induced acyclic subdigraph of a digraph D also depends solely on $\mathrm{sym}(D)$, as another corollary of Theorem 11.7.1 combined with the respective results on perfect graphs due to Grötschel, Lovász and Schrijver [75], Andres and Hochstättler [9] obtained the following complexity results.

Corollary 11.7.4 ([9]) *For a perfect digraph D, the problems of determining the chromatic number, the clique number and the maximum order of an induced acyclic subdigraph are solvable in polynomial time.*

As a natural follow-up question, Andres and Hochstättler [9] asked whether there are more interesting instances of such problems.

Problem 11.7.5 ([9]) *Are there any other problems that are $\mathcal{NP}$-complete for general digraphs but solvable in polynomial time for perfect digraphs?*

While the results we have considered so far all indicate that the properties of perfect digraphs are as favourable as those of their undirected counterparts, Andres and Hochstättler [9] had to concede that perfect digraphs lack one central virtue: In contrast to the results of Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [39] on perfect graphs, perfect digraphs cannot be recognized in polynomial time (unless $\mathcal{P} = \mathcal{NP}$).

Theorem 11.7.6 ([9]) *Deciding whether a digraph is perfect is a* co-$\mathcal{NP}$-*complete problem.*

Their proof is mainly based on a result of Bang-Jensen, Havet and Trotignon [18] stating the co-$\mathcal{NP}$-completeness of determining whether a given digraph does not contain any directed cycle of length at least 3 as an induced subdigraph (cf. Theorem 11.7.1).

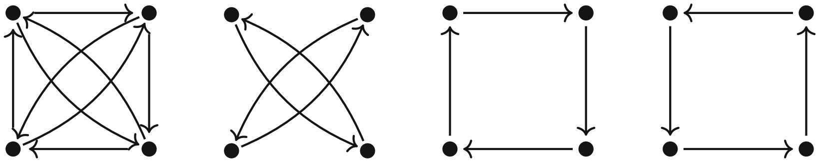


Figure 11.8 A perfect digraph D with imperfect complement $\bar{D}$.

The loss of another nice property of perfectness in the translation from graphs to digraphs is implied by the following results on kernels due to Andres and Hochstättler [9].

Theorem 11.7.7 ([9]) *It is $\mathcal{NP}$-complete to decide whether a perfect digraph has a kernel.*

On the other hand, a result due to Boros and Gurvich [31] can be rephrased as follows.

Corollary 11.7.8 ([9]) *The complement of a perfect digraph is kernel-perfect, i.e. every induced subdigraph has a kernel.*

Therefore, Theorem 11.7.7 and Corollary 11.7.8 imply that complements of perfect digraphs are not necessarily perfect (see, e.g., Figures 11.8(a) and 11.8(d), respectively), in contrast to the result that, for undirected graphs, is well-known as the Weak Perfect Graph Theorem due to Lovász [114].

Still, Andres and Hochstättler [9] were able to prove a similar result.

Theorem 11.7.9 ([9]) *A digraph D is perfect if and only if its complement $\bar{D}$ is a biorientation of a perfect graph G such that no vertex set of a cycle in* asym($\bar{D}$) *induces a clique in G.*

Note that, if we identify an undirected graph G with its complete biorientation D, then $\bar{D}$ is the complete biorientation of $\bar{G}$. Furthermore, asym($\bar{D}$) is empty and, in particular, does not contain any cycle. Therefore, Theorem 11.7.9 is a generalization of the Weak Perfect Graph Theorem.

Before we close this section with a variation of perfectness in digraphs, we give another problem posed by Andres and Hochstättler [9].

Problem 11.7.10 ([9]) *Are there any other problems that are $\mathcal{NP}$-complete* (co-*$\mathcal{NP}$-complete, respectively) for general digraphs that remain so for perfect digraphs?*

For several decades, game-variants of certain graph invariants have become increasingly popular. Colouring games and corresponding game chromatic numbers are certainly among the most prominent. For an undirected graph, we define a maker-breaker game, where both players (starting with maker) take turns assigning a colour to a previously uncoloured vertex from a given finite set of colours such that no two adjacent vertices receive the same colour. The game stops, if either the whole graph is coloured properly, in which case maker wins, or none of the remaining uncoloured vertices can be coloured properly, in which case breaker wins. The **game chromatic number** $\chi_g(G)$ of a graph G is the smallest number of colours for maker to have a winning strategy for the colouring game on G, which is well-defined, as $|V(G)|$ colours are obviously sufficient.

Andres [6] extended the concept to digraphs in the following way. The colouring game is now played on a digraph and on each turn a player must choose a vertex to assign a colour to, distinct from the colours of all its in-neighbours. The **game chromatic number** $\chi_g(D)$ of a digraph D is then defined as in the undirected case. Since, for a graph G and its complete biorientation $\overleftrightarrow{G}$, we have

$$\chi_g(G) \;=\; \chi_g(\overleftrightarrow{G}),$$

this definition is natural and well-defined.

Yang and Zhu [165] gave another variant of the game chromatic number for digraphs. In their colouring game, on each turn a player must colour a vertex without creating a monochromatic directed cycle. As their colouring rule is weaker than Andres', they called the smallest number of colours for maker to have a winning strategy for their colouring game on a digraph D the **weak game chromatic number**. For the obvious similarity to the dichromatic number, we prefer the name **game dichromatic number**, which we will denote by $\vec{\chi}_g(D)$. Obviously, the game chromatic number of a graph also equals the game dichromatic number of its complete biorientation.

As with many problems, the directed versions have so far received less attention than the undirected game chromatic number, but in addition to the introductory paper [6], there are some results due to Andres [3, 4, 8], Yang and Zhu [165] and Chan, Shiu, Sun and Zhu [37]. Note that the oriented game chromatic number introduced by Nešetril and Sopena [121], while also based on a digraph colouring game, differs greatly from the game dichromatic number considered here. Particularly, it is only defined for orientations of graphs.

Finally, we can give the definitions that motivated our brief excursion. A digraph D is called **game-perfect** if, for any induced subdigraph, the game chromatic number equals its clique number. Analogously, D is **weakly game-perfect** if, for any induced subdigraph, the game dichromatic number equals its clique number. Note that since

$$\omega(D) \;\leq\; \vec{\chi}(D) \;\leq\; \vec{\chi}_g(D) \;\leq\; \chi_g(D)$$

for every digraph D, game-perfect digraphs are also weakly game-perfect and weakly game-perfect digraphs are perfect.

The following result due to Andres [8], in combination with Theorem 11.7.7, implies that game-perfect digraphs are a proper subclass of perfect digraphs.

Theorem 11.7.11 ([8]) *Game-perfect digraphs are kernel-perfect.*

As a natural consequence of their fairly recent introduction by Andres [5], except for Theorem 11.7.11, there are mostly only basic results on game-perfect digraphs so far and a lot of open questions, the most interesting one arguably being the following.

Problem 11.7.12 ([9]) *Give a characterization of game-perfect digraphs by a set of forbidden induced subdigraphs (analogue to Theorem 11.7.1 and Corollary 11.7.2, respectively).*

For weakly game-perfect digraphs, this problem has been solved by Andres [8].

Theorem 11.7.13 ([8]) *A digraph D is weakly game-perfect if and only if* $\mathrm{sym}(D)$ *(identified with the corresponding undirected graph) is game-perfect and D does not contain any directed cycle of length at least* 3 *as an induced subdigraph.*

Since game-perfect graphs have previously been characterized by a set of forbidden induced graphs [7], we obtain the following characterization of weakly game-perfect digraphs.

Corollary 11.7.14 ([8]) *A digraph D is weakly game-perfect if and only if D does not contain a directed cycle of length at least* 3 *or a C_4, P_4, a triangle star, a Ξ-graph, two double fans, two split* 3*-stars, or one of each (see Figure 11.9, where an edge corresponds to a directed* 2*-cycle) as an induced subgraph.*

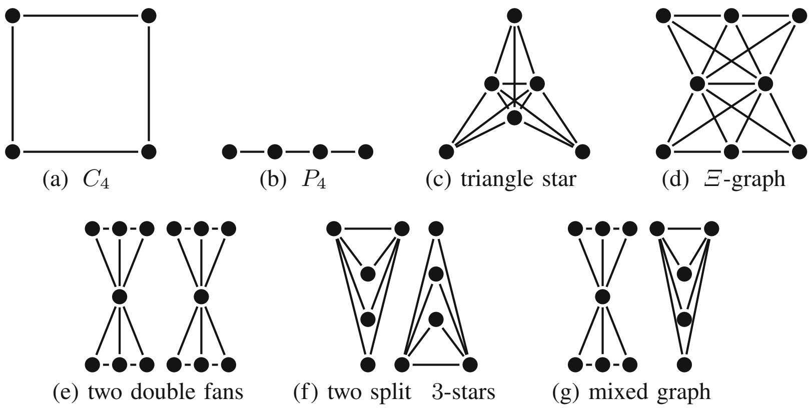


Figure 11.9 Forbidden subgraphs for weak game-perfectness.

The research on perfect and (weakly) game perfect digraphs is very much a young and active field and further results are to be expected.

11.8 Arc-Locally Semicomplete Digraphs

Arc-locally semicomplete digraphs were initially introduced as arc-local tournament digraphs by Bang-Jensen [13] as a natural analogue of locally semicomplete digraphs (cf. Chapter 6) and a generalization of both semicomplete and semicomplete bipartite digraphs. A digraph is called **arc-locally semicomplete** if, for any pair of adjacent vertices x and y, every in-neighbour (out-neighbour, respectively) of x is adjacent to every distinct in-neighbour (out-neighbour, respectively) of y.

Although arc-locally semicomplete digraphs can be quite sparse, directed cycles being among the simplest examples, the first result on the class, given by Bang-Jensen [13], suggests that arc-locally semicomplete digraphs, in general, are fairly dense in some sense.

Lemma 11.8.1 ([13]) *Let D be a connected arc-locally semicomplete digraph and let D' be any non-trivial strong subdigraph of D. Every vertex $x \in V(D) \setminus V(D')$ is adjacent to D'.*

Proof: Suppose some vertex $x \in V(D) \setminus V(D')$ is not adjacent to D'. Let $x = x_1x_2 \dots x_n$, $n \geq 3$, $x_n \in V(D')$, be a shortest path between x and D' in $UG(D)$. Let $u \in V(D')$ ($w \in V(D')$) be some vertex which dominates (is dominated by) x_n in D'. Now, depending on the orientation of the edge $x_{n-2}x_{n-1}$ in D, we conclude that x_{n-2} is adjacent to u or w, contradicting the minimality of the path above. □

A common method of proof relating to arc-locally semicomplete digraphs is to show that the considered arc-locally semicomplete digraph either has some desired property, or it is semicomplete or semicomplete bipartite, respectively. The following lemma, which Bang-Jensen [13] derived from Lemma 11.8.1, is particularly useful.

Lemma 11.8.2 ([13]) *Let D be a connected, non-strong arc-locally semicomplete digraph. If every vertex is on some cycle, then D is semicomplete or semicomplete bipartite.*

Combining Lemma 11.8.2 with the following one, Bang-Jensen [13] provided a characterization of Hamiltonian arc-locally semicomplete digraphs.

Lemma 11.8.3 ([13]) *Every strong arc-locally semicomplete digraph D having two disjoint cycles covering $V(D)$ is Hamiltonian.*

Theorem 11.8.4 ([13]) *A strong arc-locally semicomplete digraph D has a Hamiltonian cycle if and only if it has a directed cycle factor.*

Proof: One direction is clear, so suppose D is a strong arc-local tournament which has a directed cycle factor. We prove by induction on its order n that D is Hamiltonian. The cases $n = 3, 4, 5$ are trivial, so we proceed to the induction step assuming $n \geq 6$. Let $C_1, \dots, C_k$, $k \geq 1$, be a directed cycle factor of D chosen such that k is minimum. We claim that $k = 1$, in which case we are done. So suppose that $k \geq 2$. By Lemma 11.8.3 we must have $k \geq 3$. Now it follows, from the induction hypothesis and the minimality of k, that no proper subset of $C_1, \dots, C_k$ can induce a strong digraph. Thus if we delete the vertices of C_i for any $1 \leq i \leq k$, the remaining digraph $D - C_i$ is a non-strong arc-locally semicomplete digraph in which each vertex lies on a cycle and hence, by Lemma 11.8.2, it is semicomplete or semicomplete bipartite. From this and the fact that no proper subset of $C_1, \dots, C_k$ can induce a strong digraph, we conclude that $k = 3$, and that there is no arc from C_{i+1} to C_i for $i = 1, 2, 3$, indices modulo 3. Now it is easy to see that D has a Hamiltonian cycle, contradicting the choice of $C_1, \dots, C_k$. $\square$

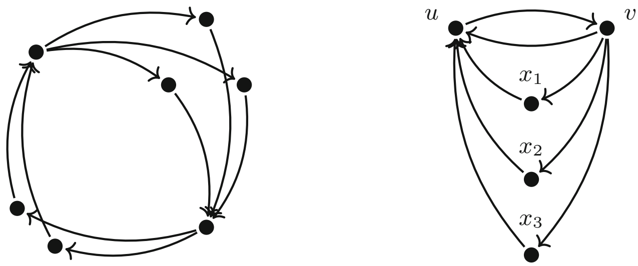
Moreover, Bang-Jensen [13] proved that the problem of deciding the existence of (and finding) a Hamiltonian cycle can be solved in polynomial time. Corresponding complexity results, based on the following theorem, also hold for Hamiltonian paths.

Theorem 11.8.5 ([13]) *A connected arc-locally semicomplete digraph D has a Hamiltonian path if and only if it has a path P (where we allow $V(P) = \emptyset$ or $V(P) = V(D)$) such that $D - V(P)$ has a directed cycle factor.*

For a characterization of strong arc-locally semicomplete digraphs, we need the following additional definitions. Let $E_1, \dots, E_k$ be k disjoint sets of independent vertices, then $\vec{C}_k[E_1, \dots, E_k]$ is the digraph obtained by substituting the vertex x_i for the vertex set E_i in a k-cycle $\vec{C}_k = x_1 \dots x_k x_1$. In other words, $V(\vec{C}_k[E_1, \dots, E_k]) = E_1 \cup \dots \cup E_k$ and

$$xy \in A(\vec{C}_k[E_1, \dots, E_k]) \quad \Leftrightarrow \quad x \in E_i \text{ and } y \in E_{i+1} (\text{modulo } k)$$

for some $i \in \{1, \dots k\}$ (see Figure 11.10(a)). We call $\vec{C}_k[E_1, \dots, E_k]$ an **extended cycle**. Furthermore, for an integer $n \geq 4$, let F_n be the digraph on the vertex set $\{u, v, x_1, \dots, x_{n-2}\}$ and the arc set $\{uv, vu\} \cup \{x_i u, v x_i \mid 1 \leq i \leq n-2\}$ (see Figure 11.10(b)).



(a) Extended $\vec{C}_4$. (b) F_5

Figure 11.10 Example of an extended cycle and an F_n digraph.

Galeana-Sánchez and Goldfeder [65, 74] and, independently, Wang and Wang [158] completed a previously deficient characterization of strong arc-locally semicomplete digraphs due to Bang-Jensen [15].

Theorem 11.8.6 ([65, 74, 158]) *Let D be a strong arc-locally semicomplete digraph, then D is either semicomplete, semicomplete bipartite, an extended cycle or isomorphic to F_n for some $n \geq 4$.*

For the following complete characterization of arc-locally semicomplete digraphs due to Galeana-Sánchez and Goldfeder [66] we note that the concept of extended cycles is easily transferable to paths $P_k = x_1 \ldots x_k$ and transitive tournaments TT_k on k vertices $x_1, \ldots, x_k$ such that $x_i \to x_j$ if and only if $1 \leq i < j \leq k$. Furthermore, we may want to substitute a vertex x_i by a digraph D_i instead of a set of independent vertices E_i. $\vec{C}_k[D_1, \ldots, D_k]$, for example, is obtained from the digraph $\vec{C}_k[V(D_1) \cup \ldots \cup V(D_k)]$ by adding all arcs of $A(D_1) \cup \ldots \cup A(D_k)$.

Theorem 11.8.7 ([66]) *Let D be a connected digraph. Then D is arc-locally semicomplete if and only if it is one of the following:*

(1) a subdigraph of an extended P_2,
(2) $P_3[E_1, D', E_1]$, where D' is a semicomplete digraph,
(3) $TT_3[E_1, E_n, E_1]$, for some positive integer n,
(4) F_n for some $n \geq 4$,
(5) an extended path or an extended cycle,
(6) a semicomplete digraph, or
(7) semicomplete bipartite digraph.

Using Theorem 11.8.7, Arroyo and Galeana-Sánchez [10] verified the Directed Path Partition Conjecture for arc-locally semicomplete digraphs.

Theorem 11.8.8 ([10]) *Let D be an arc-locally semicomplete digraph. If D contains no path with more than λ vertices, then, for every pair a, b of positive integers with $\lambda = a + b$, there exists a partition (A, B) of $V(D)$ such that no path in $D\langle A\rangle$ ($D\langle B\rangle$, respectively) has more than a (b, respectively) vertices.*

One particular instance of the Directed Path Partition Conjecture due to Laborde, Payan and Xuong [106] states that every digraph contains a maximal independent set that intersects every longest path. Galeana-Sánchez and Gómez [68] proved a stronger result for arc-locally semicomplete digraphs concerned with a generalization of longest paths. A path $P = x_1 \dots x_k$ is non-augmentable if P cannot be expanded to a path $Q = x_1 \dots x_i y_1 \dots y_\ell x_{i+1} \dots x_k$, $0 \le i \le k$.

Theorem 11.8.9 ([68]) *Let D be an arc-locally semicomplete digraph. If $\delta^+(D) > 0$, then every maximal independent set intersects every non-augmentable path in D.*

Furthermore, Galeana-Sánchez and Gómez [68] constructed an infinite family of arc-locally semicomplete digraphs containing a maximal independent set that does not intersect at least one non-augmentable path. Thus, the degree condition is necessary. For the general case, they found that there is at least one maximal independent set that intersects a particular subset of non-augmentable paths, a result that was extended by Wang and Wang [157].

Theorem 11.8.10 ([157]) *Let D be an arc-locally semicomplete digraph. Then there exists a maximal independent set intersecting every non-augmentable path in D.*

Bang-Jensen and Manoussakis [20] considered a somewhat complementary problem. Instead of a set of vertices intersecting a prescribed set of paths, they were interested in a cycle intersecting a prescribed set of vertices. Combining their result, which is for semicomplete bipartite digraphs, and Theorem 11.8.7 one obtains the following.

Theorem 11.8.11 ([20]) *Every k-strong arc-locally semicomplete digraph has a cycle through any set of k given vertices.*

In [80], Häggkvist and Manoussakis gave examples of $(k-1)$-connected bipartite tournaments with no cycle through some set of k vertices, proving Theorem 11.8.11 best possible.

In contrast to locally semicomplete digraphs (cf. Proposition 6.2.4), arc-locally semicomplete digraphs are not necessarily path-mergeable. However, Bang-Jensen [14] gave a sufficient condition for an arc-locally semicomplete digraph to be path-mergeable. A digraph is **2-path-mergeable**, if, for every pair of vertices x and y and every pair of internally disjoint (x, y)-paths P and P' of length at most 2, there is an (x, y)-path P^* such that $V(P^*) = V(P) \cup V(P')$.

Proposition 11.8.12 ([14]) *Every 2-path-mergeable arc-locally semicomplete digraph is path-mergeable.*

A conjecture proposed by Berge and Duchet [25] stating that a graph is perfect, if and only if any normal biorientation is kernel-perfect (where the "only if" part has since been proven by Boros and Gurvich [31] and the "if" part is implied by the Strong Perfect Graph Theorem [32]), inspired Galeana-Sánchez [64] to investigate kernel-perfectness of arc-locally semicomplete digraphs and the relation to perfectness of their underlying graphs.

Adapted from the definition of perfect graphs, a digraph D is called **kernel-perfect** if every induced subdigraph D' contains a kernel, i.e. an independent vertex set $N \subseteq V(D')$ such that, for every $u \in V(D') \setminus N$, there is a $v \in N$ such that $uv \in A(D')$. A digraph is **critical kernel-imperfect** if it is not kernel-perfect, but every induced subdigraph is.

Using the following additional notation, Galeana-Sánchez [64] characterized kernel-perfect arc-locally semicomplete digraphs. A **pseudodiagonal** of a cycle C is an arc whose initial and terminal vertices belong to $V(C)$, but itself is not contained in $A(C)$. A digraph is called **odd-chorded** if every cycle of odd length has at least one pseudodiagonal. Furthermore, we call a digraph **normal** if every semicomplete subdigraph contains a vertex that is a kernel.

Theorem 11.8.13 ([64]) *Let D be an arc-locally semicomplete digraph. Then, D is a kernel-perfect digraph if and only if D is a normal odd-chorded digraph.*

Note that the proof of Theorem 11.8.13 is based on an incomplete characterization of arc-locally semicomplete digraphs, but the missing case is easily added and thus, the result and those based on it still hold. As a corollary, Galeana-Sánchez [64] verified a conjecture due to Meyniel [57], although disproven for general digraphs, for arc-locally semicomplete digraphs.

Corollary 11.8.14 ([64]) *Let D be an arc-locally semicomplete digraph. If each odd cycle has at least two pseudodiagonals, then D is a kernel-perfect digraph.*

Furthermore, Galeana-Sánchez [64] found that critical kernel-imperfect arc-locally semicomplete digraphs have a very specific structure.

Theorem 11.8.15 ([64]) *Let D be an arc-locally semicomplete digraph. Then, D is critical kernel-imperfect if and only if $D \cong C_{2n+1}$, $n \geq 1$ or $D \cong C_n[1, \pm 2, \pm 3, \ldots, \pm\lfloor n/2 \rfloor]$, $n \geq 4$, where $C_n[j_1, \ldots, j_k]$ is the digraph on the vertex set $\{0, \ldots, n-1\}$ and the arc set $\{u, v \mid u - v = j_s \mod n \text{ for } s = 1, \ldots, k\}$.*

The following result of Galeana-Sánchez [64], as a corollary, implies a strong relation between kernel-perfectness of arc-locally semicomplete digraphs and perfectness of their underlying graphs. In fact, kernel-perfectness even implies **strong perfectness**, that is to say, every induced subgraph G' contains an independent vertex set which meets every maximal clique of G'.

Theorem 11.8.16 ([64]) *Let D be an arc-locally semicomplete digraph. If N is a kernel of D, then N is an independent set of $UG(D)$ which meets every maximal clique of $UG(D)$.*

Corollary 11.8.17 ([64]) *Let D be an arc-locally semicomplete digraph. If D is a kernel-perfect digraph, then $UG(D)$ is strongly perfect.*

Galeana-Sánchez [64] was able to extend the result to the following characterization.

Theorem 11.8.18 ([64]) *Let D be an arc-locally semicomplete digraph.*

(i) D is a kernel-perfect digraph if and only if $UG(D)$ is a strongly perfect graph.
(ii) D is a critical kernel-imperfect digraph if and only if $UG(D)$ is a critically imperfect graph.

Building on this result, for underlying graphs of normal arc-locally semicomplete digraphs, Galeana-Sánchez [64] proved a variation of the Strong Perfect Graph Theorem (cf. Section 11.7).

Theorem 11.8.19 ([64]) *Let D be a normal arc-locally semicomplete digraph. Then $UG(D)$ is a strongly perfect graph if and only if it contains no induced subgraph to C_{2n+1}, for $n \geq 2$.*

Unlike underlying graphs of line digraphs (cf. Section 11.2.6), those of arc-locally semicomplete digraphs can be recognized in polynomial time, as Bang-Jensen [13] showed by reducing the problem to 2-SAT.

11.9 $\mathcal{H}_i$-Free Digraphs

Just as locally semicomplete and quasi-transitive digraphs can be characterized by forbidden induced subdigraphs, so can arc-locally semicomplete digraphs, as Bang-Jensen [15] noted. Let $\mathcal{H}$ denote the digraphs on 4 vertices whose underlying graphs contain two non-adjacent vertices x and y that are connected by a path $P = xuvy$ of length 3. We then distinguish four subsets based on the possible orientations of the path P. Let $\mathcal{H}_1$ be those digraphs where P is oriented $x \rightarrow u \leftarrow v \leftarrow y$. $\mathcal{H}_2$ are the digraphs where P is oriented $x \leftarrow u \rightarrow v \rightarrow y$. The subset $\mathcal{H}_3$ contains exactly those digraphs where P is oriented $x \rightarrow u \rightarrow v \rightarrow y$. And finally, let $\mathcal{H}_4$ be the digraphs where P is oriented $x \rightarrow u \leftarrow v \rightarrow y$.

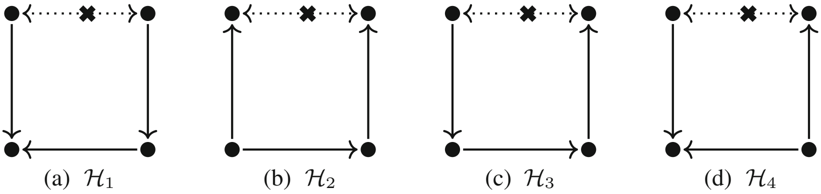


Figure 11.11 Substructures defining $\mathcal{H}_i$ digraphs. The dotted arc with a cross indicates that the two vertices are not adjacent.

Now arc-locally semicomplete digraphs are exactly the $\{\mathcal{H}_1, \mathcal{H}_2\}$-free digraphs, i.e. those which do not contain any induced subdigraph from $\mathcal{H}_1$ or $\mathcal{H}_2$. $\mathcal{H}_1$-free ($\mathcal{H}_2$-free, respectively) digraphs were dubbed **arc-locally in-semicomplete** (**arc-locally out-semicomplete**, respectively) digraphs by Wang and Wang [158]. $\mathcal{H}_3$-free digraphs are also known as **3-quasi-transitive** (see, e.g., [89] or Chapter 8) or **quasi-arc-transitive** (see, e.g., [158]) digraphs and $\mathcal{H}_4$-free digraphs are sometimes called **quasi-antiarc-transitive** (see, e.g., [158]).

In the introductory paper [15], Bang-Jensen conjectured that Theorem 11.8.4 also holds for $\mathcal{H}_i$-free digraphs, $i = 1, \dots, 4$, a conjecture that was the main motivator for further work on $\mathcal{H}_i$-free digraphs and that has since been verified, as we will see in the remainder of this section.

Since an $\mathcal{H}_2$-free digraph is the converse of an $\mathcal{H}_1$-free digraph, we will limit our considerations to $\mathcal{H}_1$-free digraphs and only remark that analogous results obviously hold for $\mathcal{H}_2$-free digraphs. Wang and Wang [158] extended several structural results on arc-locally semicomplete digraphs to $\mathcal{H}_1$-free digraphs aiming at a generalization of Theorem 11.8.6. For their characterization of strong $\mathcal{H}_1$-free digraphs, we need to define another class of digraphs.

A **T-digraph** is a strong digraph $D = (V, A)$ whose vertex set has a partition (called a **T-partition**) (V_1, V_2, V_3, V_4) such that

(i) $|V_2| = 1$ and one of V_3 or V_4 is permitted to be empty,
(ii) $D_4 := D\langle V_4 \rangle$ is semicomplete,
(iii) $A_{\min} := A(D_4) \cup V_1 \times V_2 \cup V_2 \times V_3 \cup (V_3 \cup V_4) \times V_1 \cup V_4 \times V_3 \subseteq A$,
(iv) $A \subset A_{\min} \cup V_4 \times V_2 \cup V_2 \times (V_1 \cup V_4)$, and
(v) every vertex of V_2 is adjacent to every vertex of $V_1 \cup V_4$.

Note that F_n, $n \geq 4$ (cf. Theorem 11.8.6), is a T-digraph with T-partition $(\{u\}, \{v\}, \{x_1, \dots, x_{n-2}\}, \emptyset)$ and the converse of a T-digraph. Now we may give the characterization.

Theorem 11.9.1 ([158]) *Let D be a strong $\mathcal{H}_1$-free digraph. Then D is either semicomplete, semicomplete bipartite, an extended cycle or a T-digraph.*

So, by Theorem 11.8.6, except for T-digraphs, strong $\mathcal{H}_1$-free digraphs are arc-locally semicomplete digraphs, which implies the following corollary.

Corollary 11.9.2 ([158]) *Let D be a 2-strong $\mathcal{H}_1$-free digraph. Then D is an arc-locally semicomplete digraph.*

Furthermore, Wang and Wang [158] used Theorem 11.9.1 to verify Bang-Jensen's [15] conjecture stating that Theorem 11.8.4 also holds for $\mathcal{H}_1$-free digraphs.

Theorem 11.9.3 ([158]) *A strong $\mathcal{H}_1$-free digraph has a Hamiltonian cycle if and only if it has a directed cycle factor.*

Theorem 11.9.1 also implies that the Directed Path Partition Conjecture is true for strong $\mathcal{H}_1$-free digraphs, as Arroyo and Galeana-Sánchez [10] noted.

Theorem 11.9.4 ([10]) *Let D be a strong $\mathcal{H}_1$-free digraph. If D contains no path with more than λ vertices, then, for every pair a, b of positive integers with $\lambda = a + b$, there exists a partition (A, B) of $V(D)$ such that no path in $D\langle A\rangle$ ($D\langle B\rangle$, respectively) has more than a (b, respectively) vertices.*

Finally, Wang and Wang [158] proved Theorem 11.8.10 actually not only for arc-locally semicomplete digraphs, but for the larger class of $\mathcal{H}_1$-free digraphs.

Theorem 11.9.5 ([157]) *Let D be an $\mathcal{H}_1$-free digraph. Then there exists a maximal independent set intersecting every non-augmentable path in D.*

For their results (and others) on $\mathcal{H}_3$-free digraphs, also known as 3-quasi-transitive digraphs, we refer to Chapter 8 and therefore turn directly to $\mathcal{H}_4$-free digraphs, whose structure seems much more elaborate than those of $\mathcal{H}_1$-, $\mathcal{H}_2$- and $\mathcal{H}_3$-free digraphs.

Galeana-Sánchez and Goldfeder [67] and, independently, Wang [155] proved Bang-Jensen's [15] conjecture for $\mathcal{H}_4$-free digraphs.

Theorem 11.9.6 ([67, 155]) *A strong $\mathcal{H}_4$-free digraph has a Hamiltonian cycle if and only if it has a directed cycle factor.*

For strong $\mathcal{H}_i$-free digraphs, $i = 1, 2, 3$, the corresponding theorems were derived from structural results implying a close relation to semicomplete and semicomplete bipartite digraphs. The lack of such results for $\mathcal{H}_4$-free digraphs, particularly of a characterization similar to Theorem 11.9.1, made Galeana-Sánchez and Goldfeder [67] prove Theorem 11.9.6 directly via algebraic methods. Wang [155] on the other hand proved the necessary structure combinatorially.

As a consequence of Theorem 11.9.6, Galeana-Sánchez and Goldfeder [67] obtained that Theorem 11.8.5 also holds for $\mathcal{H}_4$-free digraphs.

Theorem 11.9.7 ([67]) *A connected $\mathcal{H}_4$-free digraph D has a Hamiltonian path if and only if it has a path P (where we allow $V(P) = \emptyset$ or $V(P) = V(D)$) such that $D - V(P)$ has a directed cycle factor.*

While the Directed Path Partition Conjecture remains open for $\mathcal{H}_4$-free digraphs, Galeana-Sánchez and Gómez [68] proved that, in $\mathcal{H}_4$-free digraphs, not only does a maximal independent set of vertices intersecting every non-augmentable path exist (cf. Theorem 11.8.10), but every maximal independent set has this property.

Theorem 11.9.8 ([68]) *Let D be an $\mathcal{H}_4$-free digraph. Then every maximal independent set intersects every non-augmentable path in D.*

As Galeana-Sánchez and Gómez [68] noted, the Heuchenne condition (cf. Theorem 11.2.3 (iii)) implies $x \to y$ for every oriented path $x \to u \leftarrow v \to y$. Thus, $\mathcal{H}_4$-free digraphs are a generalization of line digraphs (without loops).

Corollary 11.9.9 ([68]) *Let D be a line digraph. Then every maximal independent set intersects every non-augmentable path in D.*

Wang [156] considered a cycle analogue of Theorem 11.9.8. To account for the missing structure of $\mathcal{H}_4$-free digraphs, Wang restricted his studies to a subclass mirroring line digraphs. An $\mathcal{H}_4$-free digraph is called an **$\mathcal{H}_4^*$-free digraph** if every oriented path $x \to u \leftarrow v \to y$ implies $y \to x$. Wang then proceeded to show that these digraphs, under certain conditions, again, are closely related to semicomplete and semicomplete bipartite digraphs.

Theorem 11.9.10 ([156]) *Let D be a strong $\mathcal{H}_4^*$-free digraph. If D has a directed cycle factor $C_1, \ldots, C_t$, $t \geq 2$, then D is either semicomplete, semicomplete bipartite or isomorphic to D^* (see Figure 11.12).*

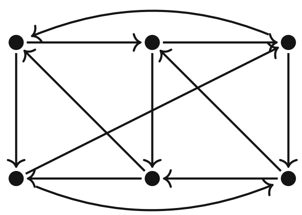


Figure 11.12 Special $\mathcal{H}_4^*$-free digraph D^*.

Finally, Wang's [156] variation of Theorem 11.9.8 reads as follows.

Theorem 11.9.11 ([156]) *Let D be a strong $\mathcal{H}_4^*$-free digraph. Then there exists a maximal independent set intersecting every longest cycle in D.*

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