Abstract Gestures: A Unifying Concept in Mathematical Music Theory

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Abstract. We present the notion of abstract gestures and show how it encompasses Mazzola's notions of gestures on topological spaces and topological categories, the notion of diagrams in categories, and our notion of gestures on locales. A relation to formulas is also discussed.

1 Introduction

Soon after the accomplishment of the first version of his *The Topos of Music* [9], an enterprise that achieved a topos-theoretic based framework for musicology (a theory of performance included), and that gave a very complete account of the mathematical structures present in music, Mazzola became aware of that his own activity as a free jazz pianist had little to do with the structures and procedures described in his monograph. *Gestures*, rather than formulas, were the essence of his performance. Certainly, improvisation in free jazz is mainly determined by the movements of the body's limbs, that is, by a *dancing of the body*, the classical structures of western music being secondary and auxiliary. Then a rigorous reflection on gestures is necessary, and not only in the case of musical improvisation, but in music in general, since all its power and intensity relies on its realization in bodily terms, even in the western classical tradition.

The point of departure towards a formal definition of gesture is the one given by Hugues de Saint-Victor in the chapter XII of his *De Institutione Novitiorum* [12]:

> Gestus est motus et figuratio membrorum corporis, ad omnen agendi et habendi modum.
> [Gesture is the movement and configuration of the body's limbs, towards all an action and having a modality.][1]

Based on this definition, Mazzola gives the first mathematical definition of a gesture as a *diagram* of curves in a topological space (see the Sect. 3 for the precise definition); here the diagram corresponds to the configuration of the body's limbs and the topological space corresponds to the space-time where the movement occurs. Further, this definition is generalized to topological categories in [11] to include both algebraic and topological information in gestures, and then

[1] Our translation.

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to locales in [1] as a first step to define gestures on generalized notions of space. These different instances of defining gestures belong, though not so strictly, to the topological branch of the theory of gestures.

On the other hand, there is an algebraic counterpart of this. In [10, p. 39], Mazzola defines a formula in a spectroid[2] as a suitable *diagram* in this particular kind of linear category, which is the starting point to develop a mathematical framework for both the theory of nets and Lewin's transformational theory.

It is important to stress that all these different definitions rely on the notion of digraph: both gestures and formulas are morphisms of digraphs with domain a given skeleton. Moreover, following Mazzola's ideas, these instances can be regarded as attempts of reanimation of the implicit movement that the drawing of a digraph by means of arrows and nodes suggest. In Mazzola's own words [10, p. 25]:

> *The gesture is a morphism, where the linkage is a real movement and not only a symbolic arrow without bridging substance.*

Regarding these two branches, there are two main problems. The first one deals with the search for a common universe: that is, the diamond conjecture. The second one corresponds to a gestural representation of categories in which composition of arrows can be manipulated at the level of gestural intuitions, in much the same way as the Yoneda embedding allows the representation of categories in topoi of presheaves. To a great extent, topological categories were introduced in gesture theory so as to construct a bicategory of gestures proposed by Mazzola as a first step to solve these two problems.

It is remarkable that gestures and formulas are at the core of the relation between mathematics and music. Mazzola has proposed a fundamental conceptual *adjunction*

$$\text{formulas} \underset{\text{mathematics}}{\overset{\text{music}}{\rightleftarrows}} \text{gestures}\ ,$$

where the arrows correspond to the activities of the disciplines: mathematicians take gestures (intuitions, mental movements, analogies with reality,...) to produce formulas, musicians take formulas (scores, diagrams, musical notations,...) to produce gestures. The term adjunction refers to a relation that is more profound than a mere inversion or isomorphism, it corresponds to a true dialectic that is grasped formidably by the categorical concept of adjunction between functors. Certainly the diamond conjecture is the search for such an adjunction in precise mathematical terms.

This article is an overview of a general framework for gesture theory that could unify the definitions of gestures on several notions of space (more related to the topological branch of mathematical music theory) and the notions of formulas in spectroids and of diagrams in categories (more related to the algebraic branch

[2] See Sect. 7 or [4, p.29] for the definition of spectroid. Spectroids were introduced by Pierre Gabriel in representation theory of quivers or digraphs; details can be found in [4].

of mathematical music theory). In addition, this framework is flexible enough to introduce gestural ideas in other fields of mathematics given its category-theoretic nature.

The structure of this article is that of a *theme with variations*. We first present the notion of abstract gestures and then proceed to unfold different *realizations* thereof. Justifications for all statements that are not proved in this article will be found in [2].

2 Abstract Gestures

Before giving the definition of gestures we need some basic definitions and fix the notation.

Directed graphs and internal digraphs

Let G_1 be the category with two parallel arrows between two vertices $[0], [1]$ plus the identities; it can be depicted as follows:

$$id \circlearrowright [0] \underset{\varepsilon_0}{\overset{\varepsilon_1}{\rightrightarrows}} [1] \circlearrowleft id \ .$$

A *directed graph* (or *digraph*, for short) is a tuple $\Gamma = (A, V, t, h)$, where A, V are sets and $t, h : A \longrightarrow V$ are functions. Digraphs correspond bijectively to presheaves on the category G_1 so from now on we identify a digraph $\Gamma = (A, V, t, h)$ with its associated presheaf $\Gamma : G_1^{op} \longrightarrow \mathbf{Set}$ defined by $\Gamma([1]) = A, \Gamma([0]) = V, \Gamma(\epsilon_0) = t, \Gamma(\epsilon_1) = h$. In this way, there is a topos of digraphs, namely the Grothendieck topos[3]

$$Digraph := \mathbf{Set}^{G_1^{op}} .$$

Thus, a morphism from $\Gamma_1 = (A_1, V_1, t_1, h_1)$ to $\Gamma_2 = (A_2, V_2, t_2, h_2)$ (that is, a natural transformation) corresponds to a pair of functions (u, v), with $u : A_1 \longrightarrow A_2$ and $v : V_1 \longrightarrow V_2$, satisfying the identities

$$vt_1 = t_2u, \ vh_1 = h_2u.$$

Similarly, if $\mathscr{C}$ is an arbitrary category, a functor $S : G_1^{op} \longrightarrow \mathscr{C}$ can be identified with a tuple (S_1, S_2, e_0, e_1), that is, with the diagram

$$S_1 \underset{e_0}{\overset{e_1}{\rightrightarrows}} S_0$$

of morphism of $\mathscr{C}$ by putting $S_1 = S([1]), S_0 = S([0]), e_0 = S(\epsilon_0), e_1 = S(\epsilon_1)$. A tuple (S_1, S_2, e_0, e_1), where $e_0, e_1 : S_1 \longrightarrow S_0$ are morphisms of $\mathscr{C}$ is called an *internal digraph* in $\mathscr{C}$. In this way, functors $S : G_1^{op} \longrightarrow \mathscr{C}$ can be identified with internal digraphs in $\mathscr{C}$.

3 Any category of presheaves on a small category is a Grothendieck topos. In fact, given a category of presheaves on a small category $\mathscr{C}$, it is a category of sheaves if we consider on $\mathscr{C}$ the *trivial topology*, whose unique covering sieve for each object of $\mathscr{C}$ is the maximal sieve.

The category of elements

Given a presheaf $P : \mathscr{C}^{op} \longrightarrow \mathbf{Set}$ on a category $\mathscr{C}$, *the category of elements of* Γ, denoted by $\int \Gamma$ is defined as follows. Its objects are pairs (C, p) where C is an object of $\mathscr{C}$ and $p \in P(C)$, and a morphism from (C', p') to (C, p) is a morphism $u : C' \longrightarrow C$ of $\mathscr{C}$ such that $P(u)(p) = p'$. Also, there is a projection functor $\pi_P : \int P \longrightarrow \mathscr{C}$ sending $u : (C', p') \longrightarrow (C, p)$ to its underlying morphism $u : C' \longrightarrow C$.

In the case when $\mathscr{C} = G_1$, note that the category $\int \Gamma$ of elements of a digraph $\Gamma = (A, V, t, h)$ can be identified with the category whose set of objects is $A \sqcup V$ and whose morphisms are the identities and the pairs of the form $(t(a), a)$ or $(h(a), a)$ where $a \in A$, domains and codomains being the first and second projections respectively. With this identification the projection $\int \Gamma \xrightarrow{\pi_\Gamma} G_1$ sends the vertices in V to $[0]$, the arrows in A to $[1]$, $(t(a), a)$ to ϵ_0, and $(h(a), a)$ to ϵ_1.

2.1 Realizations

As we will see through this article, the concept of *realization* of a digraph is closely related to that of gestures. In fact, *realization and gestures are dual concepts of each other!* (Subsect. 2.2). For simplicity, we start with realization.

Let $\mathscr{C}$ be a category with small hom-sets, $\Gamma : G_1^{op} \longrightarrow \mathbf{Set}$ a digraph, and $T : G_1 \longrightarrow \mathscr{C}$ a functor. We define *the realization of* Γ *respect to* T, denoted by $|\Gamma|_T$, as the colimit in $\mathscr{C}$ of the functor

$$\int \Gamma \xrightarrow{\pi_\Gamma} G_1 \xrightarrow{T} \mathscr{C},$$

whenever it exists.

Since Γ corresponds to a tuple (A, V, t, h) and T can be identified with a pair of morphisms $i_0, i_1 : T_0 \longrightarrow T_1$ of $\mathscr{C}$, the realization $|\Gamma|_T$ is the limit of the following diagram in $\mathscr{C}$: take a copy of T_1 for each $a \in A$, a copy of T_0 for each $x \in V$, a copy of i_0 whenever $t(a) = x$, and a copy of i_1 whenever $h(a) = x$.

If the realization $|\Gamma|_T$ exist for each digraph Γ, then there is a functor

$$|\_|_T : Digraph \longrightarrow \mathscr{C},$$

which is left adjoint[4] to the functor $\mathscr{C}(T, \_)$ that sends each object C of $\mathscr{C}$ to the digraph $\mathscr{C}(T(\_), C)$. This means that for each digraph Γ and each object of C there is a bijection

$$\mathscr{C}(|\Gamma|_T, C) \cong Digraph(\Gamma, \mathscr{C}(T(\_), C)),$$

natural in both arguments Γ and C. As we will see, this adjunction is very useful in the theory of gestures.

[4] See the theorem at [8, p. 47], which holds for cocomplete categories. This theorem remains valid if we only assume the existence of the colimits involved in the definition of L.

2.2 Definition

Let $\mathscr{C}$ be a category with small hom-sets. Given a digraph $\Gamma : G_1^{op} \longrightarrow \mathbf{Set}$ and a functor $S : G_1^{op} \longrightarrow \mathscr{C}$, we define *the object of* $\mathscr{C}$ *of gestures with skeleton* Γ *respect to* S, denoted by $\Gamma @ S$, as the limit of the functor

$$\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G_1^{op} \xrightarrow{S} \mathscr{C},$$

whenever it exists.

Following this definition, since Γ corresponds to a tuple (A, V, t, h) and S can be identified with an internal digraph (S_1, S_0, e_0, e_1) in $\mathscr{C}$, the object of gestures with skeleton Γ respect to S is the limit of the following diagram in $\mathscr{C}$: take a copy of S_1 for each $a \in A$, a copy of S_0 for each $x \in V$, a copy of $e_0 : S_1 \longrightarrow S_0$ whenever $t(a) = x$, and a copy of $e_1 : S_1 \longrightarrow S_0$ whenever $h(a) = x$.

On the other hand, note that this definition is the dual of that of realization. To see this, change $\mathscr{C}$ for $\mathscr{C}^{op}$ in the definition of the realization of Γ respect to T (Subsect. 2.1). In this way, we obtain that the realization of a digraph respect to a functor $T : G_1 \longrightarrow \mathscr{C}^{op}$ (which corresponds uniquely to a functor $S : G_1^{op} \longrightarrow \mathscr{C}$, by applying $(\_)^{op}$) is to be

$$Colim\ \left(\int \Gamma \xrightarrow{\pi_\Gamma} G_1 \xrightarrow{T} \mathscr{C}^{op}\right) = Lim\ \left(\left(\int \Gamma\right)^{op} \xrightarrow{\pi_\Gamma^{op}} G_1^{op} \xrightarrow{S} \mathscr{C}\right) = \Gamma @ S.$$

So we have the following delicate and fundamental fact:

The concept of gestures is the dual of that of realization.

By dualizing the case of the realization functor, if the object of gestures $\Gamma @ S$ exists for each digraph Γ, then there is a functor

$$\_ @ S : Digraph^{op} \longrightarrow \mathscr{C},$$

which is right adjoint to the functor $\mathscr{C}(\_, S)$ that sends each object C of $\mathscr{C}$ to the digraph $\mathscr{C}(C, S(\_))$. This means that for each digraph Γ and each object C of $\mathscr{C}$ there is a bijection

$$Digraph(\Gamma, \mathscr{C}(C, S(\_))) \cong \mathscr{C}(C, \Gamma @ S),$$

natural in both arguments Γ and C.

In particular, if the category $\mathscr{C}$ has a terminal object $\mathbf{1}$, then we obtain a bijection between the set $\mathscr{C}(\mathbf{1}, \Gamma @ S)$ of points of $\Gamma @ S$ and

$$Digraph(\Gamma, \mathscr{C}(\mathbf{1}, S(\_))).$$

The digraph $\mathscr{C}(\mathbf{1}, S(\_))$ is called the *underlying digraph* of the internal digraph S.

2.3 Hypergestures

Let C be an object of $\mathscr{C}$ and $T : G_1 \longrightarrow \mathscr{C}$ a functor whose images T_0, T_1 are exponentiable in $\mathscr{C}$. We define the *internal digraph* S_C of C respect to T as the composite

$$G_1 \xrightarrow{T} \mathscr{C} \xrightarrow{C^{(\_)}} \mathscr{C},$$

which is, of course, a contravariant functor. In this case, given a digraph Γ, we write $\Gamma @ C$ instead of $\Gamma @ S_C$, and call it *the object of gestures with skeleton* Γ *and body in* C, whenever the limit exists. This construction implies that of *hypergestures*: if Γ' is another skeleton, we can construct the object $\Gamma' @ \Gamma @ C$, and so on, depending on the existence of suitable limits in $\mathscr{C}$.

This construction of hypergestures is the main reason for which we have defined the object of gestures $\Gamma @ S$ with skeleton Γ respect to an internal digraph S. In particular, when the internal digraph is S_C we have defined the object of gestures with skeleton Γ and body in C rather than an individual gesture. Certainly, the key point of the construction of hypergetures is that $\Gamma @ C$ is an object of $\mathscr{C}$ again so that can be regarded as a new body for gestures and we can iterate the construction.

2.4 Gestures from External Digraphs

The preceding construction of hypergestures relies on the existence of suitable exponentials. However, the construction of exponentials is not always available so we introduce the following notion of *external digraph* of an object. Besides, this construction allows to give the notion of *a gesture* in contrast to our preceding definition of the object of gestures.

Let $\mathscr{C}$ be a category with small hom-sets, and $T : G_1 \longrightarrow \mathscr{C}$ a functor such that the realization functor $|\_|_T$ exists. Then given an object C of $\mathscr{C}$, we define the *external digraph* s_C of C as the composite

$$G \xrightarrow{T} \mathscr{C} \xrightarrow{\mathscr{C}(\_,C)} \mathbf{Set},$$

which coincides with its underlying digraph (Subsect. 2.2) since it is a functor to **Set**. Therefore, according to Subsect. 2.2, we have a bijection between the points of $\Gamma @ s_C$ (that is, its elements) and the set

$$Digraph(\Gamma, \mathscr{C}(T(\_), C)).$$

Consequently, in this case, we can define *a gesture with skeleton* Γ *and body in* C *respect to the cosimplicial object* T as a morphism

$$\delta : \Gamma \longrightarrow s_C$$

of digraphs. In this way, the set of gestures $\Gamma @ s_C$ is completely determined by all the individual gestures δ, in contrast to the case of the locales of gestures, which need not be characterized by their points (see Sect. 4).

Note that, in turn, s_C coincides with the value at C of the left adjoint to the realization functor (Subsect. 2.1) and hence there is a bijection

$$\mathscr{C}(|\Gamma|_T, C) \cong Digraph(\Gamma, \mathscr{C}(T(\_), C)).$$

Thus, individual gestures with skeleton Γ and body in C correspond bijectively to morphisms from the realization $|\Gamma|_T$ to C.

2.5 An Orientation

Now we proceed to the study of the particular examples. The Fig. 1 offers an orientation for the different variations to be considered. It shows the different incarnations of the functors T and S used in the definition of gestures as well as the respective bodies of the gestures. Note that the gestures related to the columns 2–5 (left to right) come from internal digraphs of objects of the respective categories and hence yield hypergestures. This is not the case for the gestures of the column 6, where S is an external digraph $s_{\mathscr{M}}$. Despite this, as we have observed, it makes sense to construct individual gestures and to say that $\mathscr{M}$ is the body, but in this case, the object of gestures is not enriched as in the preceding ones. The examples from the columns 2–6 correspond to the sections 3–7 of this article, in order-preserving correspondence.

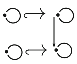
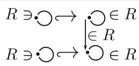
cat.	**Top**	**Loc**	**Cat**(**Top**)	**Cat**	**Cat**$_R$
T	$i_0, i_1 : \{*\} \to I$ endpoint inc.	$\mathcal{O}(i_0), \mathcal{O}(i_1)$	$\alpha, \beta : \mathbf{1} \to \mathbb{I}$ $\mathbb{I}$ cat. of $(I, \leq)$ $\mathbf{1}$ final cat.		
body	X topological space	L locale	$\mathbb{K}$ topological category	$\mathcal{C}$ category	$\mathcal{M}$ linear category
S	S_X $e_0, e_1 : X^I \to X$ endpoint ev.	S_L $e_0, e_1 : L^{\mathcal{O}(I)} \to L$	$S_{\mathbb{K}}$ $e_0, e_1 : \mathbb{K}^{\mathbb{I}} \to \mathbb{K}$	$S_{\mathcal{C}}$ $dom, cod : \mathcal{C}^{\circ \to \circ} \to \mathcal{C}$	$s_{\mathcal{M}} = \mathbf{Cat}_R(T(\_), \mathcal{M})$ external digraph

Fig. 1. Ingredients for defining gestures in different categories.

3 Gestures on Topological Spaces

Let Γ be a digraph, X a topological space, and $I = [0, 1]$ the unit *interval* in $\mathbb{R}$. In the sequel, we will denote the set of opens of the topological space X by $\mathscr{O}(X)$.

First, we construct the space X^I of *paths* in X. In fact, the space I is an exponentiable object in **Top** by Theorem [3, 5.3]: it is a *locally compact space*[5],

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[5] A topological space X is said to be locally compact if for each point $x \in X$ and each open neighborhood U of it, there is a compact neighborhood of x contained in U. In the case when X is a Hausdorff space, this definition is equivalent to saying that each point in X has a compact neighborhood. In this way, every compact Hausdorff space is locally compact.

so $\mathscr{O}(I)$ is a *continuous lattice*[6] by Lemma [6, VII.4.2]. Furthermore, the exponential X^I is the set $\mathbf{Top}(I, X)$ of continuous maps from I to X endowed with the compact-open topology.

The internal digraph in **Top** to be considered in this instance is the *spatial digraph* $\overrightarrow{X}$ of the space X. It is the tuple (X^I, X, e_0, e_1), where e_0 and e_1 are obtained by applying the functor $X^{(\_)}$ to the inclusions $i_0, i_1 : \{*\} \longrightarrow I$ of the endpoints. Note that $\overrightarrow{X}$ corresponds to the functor S_X defined in Subsect. 2.3.

In this way, since the category **Top** of all topological spaces has all small limits, following the definition in Subsect. 2.3, we have the space $\Gamma @ X$ of gestures with skeleton Γ and body in X. However, in [10], Mazzola first defines a gesture as a diagram of curves in the topological space X, that is, a morphism of digraphs

$$\delta : \Gamma \longrightarrow \overrightarrow{X},$$

where $\overrightarrow{X}$ is regarded as a digraph by forgetting the topological structure. This means that the spatial digraph $\overrightarrow{X}$ can be identified with its underlying digraph (Subsect. 2.2) since topological spaces are determined by their points. In this way, the elements of $\Gamma @ X$ correspond bijectively to these individual gestures δ according to our discussion of points of objects of gestures in Subsect. 2.2.

Example 1. Consider the case when $X = \mathbb{R}^2$. In this case, the spatial digraph $\overrightarrow{\mathbb{R}^2}$ of $\mathbb{R}^2$ is the tuple

$$(\mathbf{Top}(I, \mathbb{R}^2), \mathbb{R}^2, e_0, e_1),$$

where e_0 (respectively e_1) sends a continuous curve $c : I \longrightarrow \mathbb{R}^2$ to $c(0)$ (respectively $c(1)$). In this way, the digraph $\overrightarrow{\mathbb{R}^2}$ has as arrows all continuous curves in $\mathbb{R}^2$ and as vertices all points in $\mathbb{R}^2$.

Now suppose that Γ is the digraph of the Fig. 2, that is, $\Gamma = (\{a, b\}, \{x, y\}, t, h)$, where $t(a) = h(a) = t(b) = x$ and $h(b) = y$. Then a gesture $\delta : \Gamma \longrightarrow \overrightarrow{\mathbb{R}^2}$, which can be illustrated with the Fig. 2, is a pair (u, v), where $u : \{a, b\} \longrightarrow \mathbf{Top}(I, \mathbb{R}^2)$ and $v : \{x, y\} \longrightarrow \mathbb{R}^2$ are functions satisfying the conditions $u(a)(0) = u(a)(1) = u(b)(0) = v(x)$ and $u(b)(1) = v(y)$. In words, it is simply a diagram of curves that match according to the configuration of Γ.

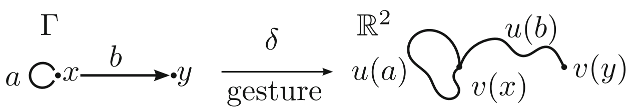


Fig. 2. A topological gesture δ.

On the other hand, the space $\Gamma @ \mathbb{R}^2$ is the limit in **Top** of the diagram

$$\mathbf{Top}(I, \mathbb{R}^2) \underset{e_0}{\overset{e_1}{\rightrightarrows}} \mathbb{R}^2 \xleftarrow{e_0} \mathbf{Top}(I, \mathbb{R}^2) \xrightarrow{e_1} \mathbb{R}^2 .$$

[6] Or core-compact, according to the terminology in [3].

According to the construction of limits (by means of products and equalizers) in **Top**, the space $\Gamma @ \mathbb{R}^2$ is the subspace of the cartesian product (equipped with the Tychonoff topology)

$$\mathbf{Top}(I, \mathbb{R}^2) \times \mathbf{Top}(I, \mathbb{R}^2) \times \mathbb{R}^2 \times \mathbb{R}^2$$

consisting of all tuples (c_a, c_b, p_x, p_y) satisfying the conditions $c_a(0) = c_a(1) = c_b(0) = p_x$ and $c_b(1) = p_y$. Note that such a tuple is essentially the same as a gesture δ. □

Gestures and geometric realization

In the case when the functor $T : G_1 \longrightarrow \mathbf{Top}$ corresponds to the pair of inclusions $i_0, i_1 : \{*\} \longrightarrow I$ of the endpoints, the realization $|\Gamma|$ of a digraph Γ respect to T always exists since **Top** is small cocomplete and is often called the *geometric realization*[7] of Γ.

Example 2. Consider the digraph Γ of the Example 1. The geometric realization $|\Gamma|$ is the colimit in **Top** of the diagram

$$I \underset{i_1}{\overset{i_0}{\leftleftarrows}} \{*\} \xrightarrow{i_0} I \xleftarrow{i_1} \{*\} \; .$$

According to the construction of colimits (via coproducts and coequalizers) in **Top**, the geometric realization $|\Gamma|$ is the quotient of the disjoint union

$$(I \times \{a\}) \cup (I \times \{b\}) \cup \{x\} \cup \{y\}$$

by the relation $\sim$ defined by $(0, a) \sim (1, a) \sim (0, b) \sim x$ and $(1, b) \sim y$. The resulting object is illustrated in Fig. 3. In this way, an open of the quotient topology on $|\Gamma|$ corresponds to a tuple

$$(U_a, U_b, V_x, V_y),$$

where $U_a, U_b \in \mathscr{O}(I)$, $V_x \subseteq \{x\}$, and $V_y \subseteq \{y\}$ satisfying the conditions (i) $0 \in U_a$ iff $1 \in U_1$ iff $0 \in U_b$ iff $x \in V_x$ and (ii) $1 \in U_b$ iff $y \in V_y$. □

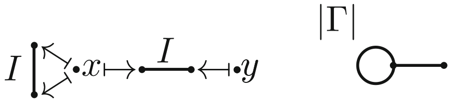


Fig. 3. The way of identifying the points of the disjoint union (left-hand) and the realization of the digraph from Fig. 2 (right-hand)

[7] This name is due to Milnor, who first studied the geometric realization in the context of algebraic topology, though for simplicial sets instead of digraphs. However, in [10], this object is called *spatialization*.

By the associated adjunction to the geometric realization (Subsect. 2.1), we have an isomorphism

$$\mathbf{Top}(|\Gamma|, X) \cong Digraph(\Gamma, \overrightarrow{X}),$$

natural in both arguments Γ, X. Thus, a gesture with skeleton Γ and body in X is essentially a continuous map from $|\Gamma|$ to X; for instance, note that the gesture at Fig. 2 can be interpreted as a continuos map from the geometric realization at Fig. 3 to $\mathbb{R}^2$. Moreover one may ask whether there is a homeomorphism

$$X^{|\Gamma|} \cong \Gamma @ X.$$

The answer is affirmative iff Γ is a locally finite digraph[8], that is, iff $|\Gamma|$ is exponentiable in **Top**; we omit the proof here. The important point is that this result illustrates a basic problem in gesture theory: *the reduction of objects of gestures defined by the procedure in Subsect.* 2.3 *to exponentials*. It is important to stress that isomorphisms of the above type are not always possible; for example, if the digraph has infinitely many arrows with the same tail, the above isomorphism makes no sense. And in some respect, this is what makes topological gestures so interesting from a strictly mathematical viewpoint; if they were reducible to exponentials nothing new is to be studied.

4 Gestures on Locales

The category of *frames*, denoted by **Frm** has as objects the *complete Heyting algebras*, that is, complete lattices L satisfying the infinite distributive law $a \wedge \bigvee_{s\in S} s = \bigvee_{s\in S} a \wedge s$, for all $a \in L$ and $S \subseteq L$. The morphisms of frames are the functions that preserve finite meets including $\mathbf{1}$ and arbitrary joins including $\mathbf{0}$. In particular these functions preserve the order. The category **Loc** of *locales* is the opposite of **Frm**. The category **Loc** is small complete and cocomplete (see [13, II.3]), the terminal object $\mathbf{2} = \{\emptyset, \{*\}\}$ being the locale of opens of the singleton.

Let $\Gamma = (A, V, t, h)$ be a digraph and L a locale. As we have already noted, the locale $\mathscr{O}(I)$ is a continuous lattice. Therefore $\mathscr{O}(I)$ is exponentiable in **Loc** (Theorem [6, VII 4.11]) and we have the *locale* $L^{\mathscr{O}(I)}$ *of paths in* L.

The *localic digraph* $\overrightarrow{L}$ of L is the tuple $(L^{\mathscr{O}(I)}, L, e_0, e_1)$ where e_0, e_1 are obtained by applying the functor $L^{(\_)}$ to the endpoint inclusions $\mathscr{O}(i_0), \mathscr{O}(i_1) : \mathbf{2} \longrightarrow \mathscr{O}(I)$ induced by their analogues in **Top**. Once again, $\overrightarrow{L}$ corresponds to the functor S_L defined in Subsect. 2.3. In this way, since **Loc** has all small limits, we have the locale $\Gamma @ L$ of gestures with skeleton Γ and body in L. This definition coincides with that given in [1].

As in the case of topological spaces, there is a realization induced by the inclusions $\mathscr{O}(i_0), \mathscr{O}(i_1) : \mathbf{2} \longrightarrow \mathscr{O}(I)$, and the realization of a digraph coincides with the locale of opens of the geometric realization in **Top**.

[8] A digraph is locally finite if each vertex is the tail or head of only finitely many arrows.

Example 3. Let Γ the digraph of the Example 1. The realization $|\Gamma|$ in **Loc** corresponds to the locale $\mathscr{O}(|\Gamma|)$, whose elements were already described in the Example 2. □

Also, we have a reduction to exponentials, namely an isomorphism of locales

$$L^{\mathscr{O}(|\Gamma|)} \cong \Gamma@L,$$

natural in L, for each locally finite digraph Γ.

Locales are the objects of study of the pointless topology, an approach to a great extent derived from the vision of Grothendieck of the notion of topos as a generalization of that of topological space. Locales are in some respect residues of topoi, but they exemplify transparently the spatial aspect of topoi. In first instance, locales need not be characterized by their points, and there are examples (complete boolean algebras without atoms) of locales that are non-trivial and without points at all! As a collateral effect, the objects of gestures on these complete boolean algebras are also non-trivial and with no points.

Example 4. Let $\mathscr{O}(\mathbb{R})_{\neg\neg}$ be the sublocale of $\mathscr{O}(\mathbb{R})$ induced by the double negation nucleus. The elements of $\mathscr{O}(\mathbb{R})_{\neg\neg}$ are the opens $U \in \mathscr{O}(\mathbb{R})$ for which $Int(\overline{U}) = U$. The locale $\mathscr{O}(\mathbb{R})_{\neg\neg}$ is a boolean algebra without atoms and hence has no points. In the same way, if Γ is any non-initial digraph, according to [1, Proposition 4], the space of points of $\Gamma@\mathscr{O}(\mathbb{R})_{\neg\neg}$ is homeomorphic to the space of gestures with skeleton Γ and body in the space of points of $\mathscr{O}(\mathbb{R})_{\neg\neg}$, but the latter is the empty space, and hence the space of points of $\Gamma@\mathscr{O}(\mathbb{R})_{\neg\neg}$ is empty. However, it can be shown that $\mathscr{O}(\mathbb{R})_{\neg\neg}$ is a retract of $\Gamma@\mathscr{O}(\mathbb{R})_{\neg\neg}$, and hence $\Gamma@\mathscr{O}(\mathbb{R})_{\neg\neg}$ is not a trivial locale. In particular, if Γ is the digraph $\bullet \rightarrow \bullet$, the locale $\mathscr{O}(\mathbb{R})_{\neg\neg}^{\mathscr{O}(I)} = \Gamma@\mathscr{O}(\mathbb{R})_{\neg\neg}$ of paths has no points. □

This is a fundamental example for abstract gesture theory since it shows that the notion of *an individual gesture* is insufficient if a theory of gestures on generalized spaces is desired. Besides, if we want to define a correct generalization of gestures on locales, then it is impossible to give a satisfactory definition of a gesture with skeleton Γ and body in $\mathscr{O}(\mathbb{R})_{\neg\neg}$ as a morphism of digraphs $\delta : \Gamma \longrightarrow \overrightarrow{\mathscr{O}(\mathbb{R})_{\neg\neg}}$ since both the locale of paths $\mathscr{O}(\mathbb{R})_{\neg\neg}^{\mathscr{O}(I)}$ and $\mathscr{O}(\mathbb{R})_{\neg\neg}$ have no points—the object $\overrightarrow{\mathscr{O}(\mathbb{R})_{\neg\neg}}$ is not a digraph, but an internal digraph in **Loc** whose underlying digraph (Subsect. 2.2) has no vertices and no arrows!

This generalized notion of space (locales) that is concerned with notions of neighborhoods and coverings rather than points should be taken into account to model the space-time in different ways than usual. It is absolutely legitimate to ask whether the euclidean models $\mathbb{R}^n$ and their derivatives (as the interval object I), and even topological spaces, which are essentially characterized by their points, are suitable to describe processes that have to do with wraps and indecomposable movements that occur through non-atomic neighborhoods of space-time, as in the case of the human body (absolutely indecomposable in terms of points!) or the pianist's hand[9]. Probably, it is time for a new topology, closer to Grothendieck's ideas of a tame (moderate) topology and a geometry of shapes.

5 Gestures on Topological Categories

Topological categories are internal categories (see [8, V.7] or [5, B2.3.1] for the definition) in **Top**. Roughly speaking, this means that a topological category $\mathbb{K}$ is a tuple (C_1, C_0, e, d, c, m) with C_1, C_0 topological spaces of arrows and objects respectively and e, d, c, m continuous operations of unity, domain, codomain, and composition respectively. Topological categories and internal functors in **Top** (which we call *topological functors*) form a category denoted by **Cat**(**Top**) according to the notation in [5, B2.3.1].

Before explaining the construction of gestures, we mention two basic results on limits and exponentials of internal categories that we will need and whose justification can be found in [1].

Theorem 1. *Let $\mathscr{C}$ be a cartesian category. If $\mathbb{E} = (E_1, E_0, e', d', c', m')$ is an internal category in $\mathscr{C}$ such that E_0, E_1, and the object of composable arrows $E_2 = E_1 \times_{E_0} E_1$ are exponentiable in $\mathscr{C}$, then $\mathbb{E}$ is exponentiable in the category* **Cat**$(\mathscr{C})$ *of internal categories in $\mathscr{C}$.*

Theorem 2. *If $\mathscr{C}$ is a small complete category, then* **Cat**$(\mathscr{C})$ *is small complete.*

Let I be the unit interval in $\mathbb{R}$ and $\mathbb{I} = (E_1, E_0, e', d', c', m')$ the topological category of the poset $(I, \leq)$, that is,

(i) $(E_1, E_0) = (\{(x, y) |\ x \leq y \text{ in } I\}, I)$;
(ii) $e' : E_0 \longrightarrow E_1$ is the diagonal, that is, $e'(x) = (x, x)$;
(iii) $d', c' : E_1 \longrightarrow E_0$ are the first and second projection respectively;
(iv) $E_2 = E_1 \times_{E_0} E_1 = \{((w, z), (x, y)) \in I^2 \times I^2 |\ x \leq y = w \leq z\}$, and $m' : E_2 \longrightarrow E_1$ is defined by $m'((y, z), (x, y)) = (x, z)$; and
(v) the set $E_0 = I$ has the usual topology on I, E_1 is a subspace of $I \times I$ (product topology), and E_2 is a subspace of I^4; so that e' (diagonal), d', c', m' (projections) are continuous.

To show that $\mathbb{I}$ is exponentiable in **Cat**(**Top**) we check the conditions of Theorem 1: in fact, E_0, E_1, E_2 are exponentiable in **Top**, that is locally compact, since they are closed subsets of some finite power of I, the latter being locally compact since finite products of locally compact spaces are locally compact.

Also, we have two endpoint inclusions into $\mathbb{I}$. In fact, note that the terminal category $\mathbf{1} = (\{*\}, \{*\}, id, id, id, !)$ is the terminal object in **Cat**(**Top**). The internal functors $\alpha, \beta : \mathbf{1} \longrightarrow \mathbb{I}$ are defined by $\alpha_0(*) = 0$, $\beta_0(*) = 1$, $\alpha_1(*) = (0, 0)$, and $\beta_1(*) = (1, 1)$.

[9] I borrowed this idea from Octavio Agustín-Aquino.

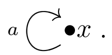
Given a topological category $\mathbb{K}$, the corresponding internal digraph $\overrightarrow{\mathbb{K}}$ of $\mathbb{K}$ in $\mathbf{Cat}(\mathbf{Top})$ is the tuple $(\mathbb{K}^{\mathbb{I}}, \mathbb{K}, e_0, e_1)$, where $\mathbb{K}^{\mathbb{I}}$ is the category of all topological functors from $\mathbb{I}$ to $\mathbb{K}$ with its set of objects P_0 (that is, of topological functors) topologized as a subspace of $C_1^{E_1} \times C_0^I$ and its set of morphisms P_1 (that is, of natural transformations) topologized as a subspace of $P_0 \times P_0 \times C_1^{E_0}$, and

$$\mathbb{K}^{\mathbb{I}} \xrightarrow{\ e_i\ } \mathbb{K}$$

$$\begin{array}{ccc} F & \longmapsto & F(i) \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau_i} \\ G & \longmapsto & G(i), \end{array}$$

for $i = 0, 1$. This internal digraph $\overrightarrow{\mathbb{K}}$ corresponds to the functor $S_{\mathbb{K}}$ defined in Subsect. 2.3, so since $\mathbf{Cat}(\mathbf{Top})$ is small complete by Theorem 2, for each digraph Γ, we have the topological category of gestures $\Gamma@\mathbb{K}$ with skeleton Γ and body in $\mathbb{K}$. This definition is essentially the same given in [11], where applications of gestures on topological categories in mathematical music theory are discussed.

Example 5. Let Γ be a loop digraph as in the picture



Let us make an explicit computation of the topological category $\Gamma@\mathbb{K}$ for any topological category $\mathbb{K} = (C_1, C_0, e, d, c, m)$. First, note that according to the definition of $\Gamma@\mathbb{K}$, it is the equalizer of the diagram

$$\mathbb{K}^{\mathbb{I}} \underset{e_0}{\overset{e_1}{\rightrightarrows}} \mathbb{K} \, .$$

Thus, $\Gamma@\mathbb{K}$ can be described as follows:

(i) Its objects are topological functors $F : \mathbb{I} \longrightarrow \mathbb{K}$, that is, pairs $(F_1, F_0) \in C_1^{E_1} \times C_0^I$ (correspondence on morphisms and on objects) satisfying the functor conditions and $F_0(0) = F_0(1)$. In this way, the set of objects of $\Gamma@\mathbb{K}$ is equipped with the subspace topology of the Tychonoff topology on the product $C_1^{E_1} \times C_0^I$. Here, $C_1^{E_1}$ and C_0^I are function spaces, which are endowed with the compact-open topology.

(ii) A morphism from F to G, where F and G are topological functors as in (i), is a triple (F, G, τ), where $\tau : F \longrightarrow G$ is a natural transformation such that $\tau_0 : F_0(0) \longrightarrow G_0(0)$ and $\tau_1 : F_0(1) \longrightarrow G_0(1)$ are the same morphism. Here we regard τ as a continuous map from I to C_1 satisfying the usual natural transformation conditions. In this way, the set of morphisms of $\Gamma@\mathbb{K}$ is endowed with the subspace topology of the Tychonoff topology on the product

$$C_1^{E_1} \times C_0^I \times C_1^{E_1} \times C_0^I \times C_1^I . \qquad \square$$

6 Diagrams: Gestures on Categories

Let **Cat** be the category of all small categories, which coincides with **Cat**(**Set**). Consider the functor $T : G_1 \longrightarrow \mathbf{Cat}$ identified with the pair of functors F_0, F_1 from the terminal category **1** (just an object and an arrow) to the category of the poset $\{0 < 1\}$, where $F_0(*) = 0$ and $F_1(*) = 1$ (see Fig. 1). Since **Cat** is small cocomplete (Exercise 5 in [7, p. 112]), we know that the realization $|\_|_T$ exists according to Subsect. 2.1, but we require a more explicit presentation. Recall (Subsect. 2.1) that $|\_|_T$ is left adjoint to the functor $\mathbf{Cat}(T, \_) : \mathbf{Cat} \longrightarrow$ *Digraph* which is essentially the forgetful functor! But we know that it has a left adjoint (unique up to isomorphism), namely the free category functor *Path* (see [7, II.7]), so we can assume that $|\_|_T = Path$.

Now **Cat** is cartesian closed by Theorem 1, the categories of functors being the exponentials, so given a category $\mathscr{C}$, we have the internal digraph $S_{\mathscr{C}}$ from Subsect. 2.3. In this way, we have the category $\Gamma @\mathscr{C}$ of gestures with skeleton Γ and body in $\mathscr{C}$. The interesting fact here is that the reduction to exponentials always holds, that is, we have an isomorphism of categories

$$\Gamma @\mathscr{C} \cong \mathscr{C}^{|\Gamma|_T} = \mathscr{C}^{Path(\Gamma)}$$

for any digraph Γ. Therefore, the category $\Gamma @\mathscr{C}$ of gestures with skeleton Γ and body in $\mathscr{C}$ can be identified with the category of all functors from the free category $Path(\Gamma)$ to $\mathscr{C}$. So diagrams are gestures!

Example 6. Let Γ be the digraph $\bullet x \xrightarrow{a} \bullet y$. Its realization in **Cat** is its free category, which is the category with just an arrow plus identities and can be depicted as

$$id_x \circlearrowright x \xrightarrow{a} y \circlearrowleft id_y \; .$$

Note that this category is isomorphic to the category of the poset $\{0 < 1\}$. Moreover, given a small category $\mathscr{C}$, the category $(\bullet x \xrightarrow{a} \bullet y)@\mathscr{C}$ is precisely the category of functors from the category of the poset $\{0 < 1\}$ to $\mathscr{C}$. Thus, the objects of $(\bullet x \xrightarrow{a} \bullet y)@\mathscr{C}$ are essentially morphisms of $\mathscr{C}$ and a morphisms of $(\bullet x \xrightarrow{a} \bullet y)@\mathscr{C}$ from $f : A \longrightarrow B$ to $g : C \longrightarrow D$ is just a pair of morphisms $(h : A \longrightarrow C, k : B \longrightarrow D)$ such that $kf = gh$. That is, our category of gestures is the *category of morphisms of* $\mathscr{C}$. Note that this also exemplifies the exponential reduction. □

7 Gestures on Linear Categories

Let R be a commutative ring. We define an *R-linear category* to be a category $\mathscr{M}$ with small hom-sets such that for each pair A, B of objects of $\mathscr{M}$ the set of morphisms $\mathscr{M}(A, B)$ is an R-module and such that for each triple A, B, C of objects of $\mathscr{M}$ the composition $\circ : \mathscr{M}(B, C) \times \mathscr{M}(A, B) \longrightarrow \mathscr{M}(A, C)$ is R-bilinear. Given two R-linear categories $\mathscr{M}, \mathscr{N}$, an R-linear functor from $\mathscr{M}$ to $\mathscr{N}$ is a functor $F : \mathscr{M} \longrightarrow \mathscr{N}$ such that for each pair A, B of objects of $\mathscr{M}$ the function $F : \mathscr{M}(A,B) \longrightarrow \mathscr{N}(F(A),F(B))$ is an R-homomorphism of modules. In this way, we have *the category* $\mathbf{Cat}_R$ *of all small* R*-linear categories and* R*-linear functors between them.* On the other hand, an ideal $\mathscr{I}$ of an R-linear category consists of a family of subgroups $\mathscr{I}(A,B) \leqslant \mathscr{M}(A,B)$ indexed by all pairs of objects of $\mathscr{M}$ such that $f \in \mathscr{M}(A,B)$ implies $gfe \in \mathscr{M}(D,C)$ for all $e \in \mathscr{M}(D,A)$ and $g \in \mathscr{M}(B,C)$.

Now let k be a commutative field. We say that a small k-linear category $\mathscr{M}$ is a *spectroid* if the non-invertible morphisms of $\mathscr{M}$ form an ideal $Rad(\mathscr{M})$ of $\mathscr{M}$ and if distinct objects of $\mathscr{M}$ are not isomorphic. It can be shown that the first requirement is equivalent to saying that the k-algebras $\mathscr{M}(A,A)$ are local[10] for all $A \in Ob(\mathscr{M})$.

A construction of free R-linear categories is possible in much the same way that in the case of free modules in $\mathbf{Mod}_R$. In fact, there is a functor $R(\_)$: $\mathbf{Cat} \longrightarrow \mathbf{Cat}_R$ which is left adjoint to the forgetful functor from $\mathbf{Cat}_R$ to $\mathbf{Cat}$. Given a small category $\mathscr{C}$, the R-linear category $R\mathscr{C}$ has as objects the objects of $\mathscr{C}$, for each pair of objects A,B the set $R\mathscr{C}(A,B)$ is defined to be the free module $R^{\mathscr{C}(A,B)}$ on $\mathscr{C}(A,B)$, and the composition is the linear extension of the composition in $\mathscr{C}$. We thus have the functor

$$RT : G_1 \xrightarrow{T} \mathbf{Cat} \xrightarrow{R(\_)} \mathbf{Cat}_R,$$

where T is the functor in Sect. 6; see Fig. 1 for a picture. Further, the realization $|\_|_{RT}$ coincides with $R(\_) \circ Path$ since $|\_|_T = Path$ and $R(\_)$, as a left adjoint, preserves colimits.

Given an R-linear category $\mathscr{M}$, so as to construct gestures, we consider the external digraph of $\mathscr{M}$ (contravariant functor, Subsect. 2.4)

$$s_{\mathscr{M}} : G_1 \xrightarrow{RT} \mathbf{Cat}_R \xrightarrow{\mathbf{Cat}_R(\_,\mathscr{M})} \mathbf{Set},$$

rather than an internal digraph in $\mathbf{Cat}_R$. So since the functor $\mathbf{Cat}_R(\_,\mathscr{M})$ transforms colimits to limits and the functor $R(\_) \circ Path$ is left adjoint to the forgetful functor $U : \mathbf{Cat}_R \longrightarrow \mathbf{Cat} \longrightarrow Digraph$, we have the bijections

$$\Gamma @ s_{\mathscr{M}} \cong \mathbf{Cat}_R(|\Gamma|_{RT},\mathscr{M}) = \mathbf{Cat}_R(RPath(\Gamma),\mathscr{M}) \cong Digraph(\Gamma, U(\mathscr{M})).$$

Note that since right adjoint are unique up to natural isomorphism, the set

$$Digraph(\Gamma, U(\mathscr{M}))$$

is essentially the set of gestures defined in Subsect. 2.4. Moreover, this set of gestures is strongly related to formulas. The difference is that formulas are defined for spectroids $\mathscr{M}$ and that the arrows of the codomain of a formula are only allowed to be non-invertible morphisms of $\mathscr{M}$. A similar result should express formulas as gestures. The better situation would be when the functor Rad (see [10, p. 39]) from spectroids to digraphs has a left adjoint[11]; in such a case, using

[10] That is, local rings: all non-invertible elements form a two-sided ideal.

[11] The author ignores whether or not such a left adjoint exists.

the same reasoning from above, this left adjoint could be regarded as a realization such that the associated set of gestures with skeleton Γ and body in a spectroid $\mathscr{M}$ would be isomorphic (as in the above isomorphism) to

$$Digraph(\Gamma, Rad(\mathscr{M})),$$

that is, to the set of formulas! Now the functor $R(\_) \circ Path$ is a naive candidate for such adjoint, but the images of the functor $R(\_) \circ Path$ are not spectroids in general as discussed in the following example and hence we discard it.

Example 7. If Γ is a loop (see the Example 5), then the realization $RPath(\Gamma)$ is isomorphic to the polynomial ring $R[x]$ which is never local since $1 - x$ and x are non-invertible with $1 = 1 - x + x$ invertible. This shows that $RPath(\Gamma)$ is not a spectroid.

However, if R is a field k, the quotient algebra $k[x]/\langle x^2 \rangle$, which can be identified with the algebra of dual numbers, is local with ideal of non-invertible elements generated by the equivalence class of x. Thus, $k[x]/\langle x^2 \rangle$, regarded as the set of morphisms of a category with just an object, is an spectroid.

In this way, a gesture with skeleton a loop and body in the linear category $k[x]/\langle x^2 \rangle$ is just the choice of an equivalence class $[a+bx]$ in $k[x]/\langle x^2 \rangle$. In contrast, a formula in the spectroid $k[x]/\langle x^2 \rangle$ is the choice of a class of the form $[bx]$. For instance, the element $[x]$ is a formula, which can be interpreted as the element x subject to the condition $x^2 = 0$; hence the relation with the intuitive idea of a formula. Finally, note that the class of the unit of k is a gesture that is not a formula. □

8 Final Comments

Further generalization

The formal definition of gestures in Subsect. 2.2 was deliberately chosen in this form to illustrate the several possibilities of generalizing it. The category G_1 can be replaced by the semi-simplicial category so that we can define gestures whose skeleta are semi-simplicial sets Γ respect to semi-simplicial objects. In that case we can regard digraphs as particular examples of semi-simplicial sets and hence the resultant theory generalizes that for digraphs. This is not only a mathematical fantasy; these generalization have relevant consequences in the theory of gestures for digraphs. For example, in the case of topological spaces, the space of hypergestures $\Gamma'@\Gamma@X$, where Γ', Γ are locally finite digraphs and X is a space, satisfies

$$\Gamma'@\Gamma@X \cong X^{|\Gamma'|\times|\Gamma|} \cong X^{|\Gamma' \times_g \Gamma|},$$

where $\Gamma' \times_g \Gamma$ is the *geometric product* of the digraphs Γ' and Γ, which is usually a semi-simplicial set rather than a digraph. This fact also exemplifies the *combinatorial nature of topological hypergestures with locally finite skeleta*: they basically depend on the digraphs, not on the particular space! Furthermore, the above formula is also valid for gestures on locales.

Gestures and Kan Extensions

The formulas defining objects of gestures and realizations show that the realization functor $|\_|_T$ and the gesture functor $\_@S$ are left and right Kan extensions respectively. In fact, note first that the category of elements $\int \Gamma$ is isomorphic to the comma category $Y \downarrow \Gamma$ and that $(\int \Gamma)^{op}$ is isomorphic to $\Gamma \downarrow Y^{op}$, where $Y : G_1 \longrightarrow \mathbf{Set}^{G_1^{op}}$ is the Yoneda embedding. Thus, from the definitions of realization and gestures in Subsects. 2.1 and 2.2, we obtain the formulas

$$|\Gamma|_T = Colim\ (Y \downarrow \Gamma \xrightarrow{P} G_1 \xrightarrow{T} \mathscr{C}) = Lan_Y(T)(\Gamma)$$

and

$$\Gamma@S = Lim\ (\Gamma \downarrow Y^{op} \xrightarrow{Q} G_1^{op} \xrightarrow{S} \mathscr{C}) = Ran_{Y^{op}}(S)(\Gamma).$$

This means, according to Theorem 1 in [7, X.3], that

> *the realization functor $|\_|_T$ is the left Kan extension of T along the Yoneda embedding and, dually, the contravariant gesture functor $\_@S$ is the right Kan extension of S along the opposite of the Yoneda embedding.*

This fact helps to locate gesture theory as a particular case of the theory of Kan extensions. Then we have a notion of preservation of gestural structures as shown, for example, by the formula

$$pt(\Gamma@L) \cong \Gamma@pt(L),$$

which says that the space of points of the locale of gestures with skeleton Γ and body in a locale L is homeomorphic to the space of gestures with skeleton Γ and body in the space of points $pt(L)$. Moreover, this viewpoint helps to give a definition of gestures, based exclusively on Kan extensions, that need not deal with limits, that is, there may be objects of gestures that are not pointwise Kan extensions[12].

From the diamond to a category

It is important to make clear that we are not claiming a solution for the so-called 'diamond conjecture', instead we consider that it has not been formulated in a correct way yet. In this way, we hope that the piece of theory presented in this article is useful for giving a more theoretical shape to the diamond diagram [10, p. 43]. In the initial diamond[13], the two vertices related to the category of gestures and the category of formulas should correspond to two particular *realizations* of the category of digraphs (or semi-simplicial objects, if we are more risky).

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12 Though probably the more interesting objects to study are the pointwise Kan extensions and hence the realizations and gesture objects defined by means of (co)limits as above.

13 Which was not precisely a diamond since it is noticed there that there is a possible framework for formulas for each field k.

For gestures it is done, but not for formulas though we are close. Moreover the particular notions of gestures can be compared since we have a notion of preservation of gestural structures, taken from the theory of Kan extensions. Thus we have a category of gestural structures which could be useful to find a precise adjunction between gestures and formulas, allowing us to recover the gestures behind formulas and the formulas behind gestures.

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